HW #4

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Problem 1

Let $a < b \in \mathbb{R}$. Prove that $C([a,b])$ with sup norm is a separable metric space.

Define the set of polynomials with rational coefficients on $[a,b]$ as $\mathbb{P}_{\mathbb{Q}}([a,b])$. We will show this set is countable and dense in $C([0,1])$, proving $C([0,1])$ is separable.

To show $\mathbb{P}_{\mathbb{Q}}([a,b])$ is countable, it suffices to show $\bigcup\limits_{n=0}^{\infty} \mathbb{Q}^n$ is countable, since there is a bijection f between these two sets. Namely,

$$f(q_0, q_1, \ldots, q_k) = q_0 + q_1 x + \cdots + q_k x^k$$

However, the countably infinite union of countable sets is countable. Thus $\bigcup\limits_{n=0}^{\infty} \mathbb{Q}^n$ is countable, whice proves $\mathbb{P}_{\mathbb{Q}}([a,b])$ is countable.

To show $\mathbb{P}_{\mathbb{Q}}([a,b])$ is dense in $C([a,b])$, we will show it is dense in the set of polynomials with real coefficients on $[a,b]$ (denoted $\mathbb{P}_{\mathbb{R}}([a,b])$) and use a diagonal argument to show it is dense in $C([a,b])$.

Lemma (1)**.** *$\mathbb{P}_{\mathbb{Q}}([a,b])$ is dense in $\mathbb{P}_{\mathbb{R}}([a,b])$.*

Proof. Choose $p \in \mathbb{P}_{\mathbb{R}}([a,b])$. Then $p = r_0 + r_1 x + \cdots + r_k x^k$ for some $k \in \mathbb{N}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist sequences $(q_{a,j})_j \in \mathbb{Q}$ such that $q_{a,j} \to r_a$ for $a = 1, \ldots, k$. Then construct a sequence $(q_\ell)_\ell \in \mathbb{P}_{\mathbb{Q}}([a,b])$ by

$$q_\ell = q_{\ell,0} + q_{\ell,1} x + \cdots + q_{\ell,k} x^k$$

Then $\lim\limits_{\ell \to \infty} q_\ell = p$ for each $x \in [a,b]$. Thus q_ℓ converges to p in a pointwise manner. However, since p is continuous, and each q_ℓ is continuous, $q_\ell \to p$ uniformly. Thus $\mathbb{P}_{\mathbb{Q}}([a,b])$ is dense in $\mathbb{P}_{\mathbb{R}}([a,b])$. $\square$

By the Weierstrauss approximation theorem, $\mathbb{P}_{\mathbb{R}}([a,b])$ is dense in $C([a,b])$. Thus for any $f \in C([a,b])$, $\exists (p_n)_n \in \mathbb{P}_{\mathbb{R}}([a,b])$ such that $p_n \to f$ uniformly. Choose a subsequence of p_n (for ease, call this subsequence p_n) such that $\|p_n - f\| < \frac{1}{2n}$ for all $n \in \mathbb{N}^+$.

By Lemma 1, for each p_n in the sequence, $\exists (q_{n,\ell})_\ell \in \mathbb{P}_{\mathbb{Q}}([a,b])$ such that $q_{n,\ell} \to p_n$ uniformly.

Then we can construct a sequence in $\mathbb{P}_{\mathbb{Q}}([a,b])$ which converges to f. Choose $w_1 = q_{1,L_{1/2}}$ where $\ell \geq L_{1/2} \implies \|q_{1,\ell} - p_1\|_{\text{sup}} < \frac{1}{2}$. Then choose $w_2 = q_{2,L_{1/4}}$ where $\ell \geq L_{1/4} \implies$

$\|q_{2,\ell} - p_2\|_{\text{sup}} < \frac{1}{4}$. In general, for all $m \in \mathbb{N}^+$, choose $w_m = q_{m,L_{1/2m}}$ where $\ell \geq L_{1/2m} \implies \|q_{m,\ell} - p_m\|_{\text{sup}} < \frac{1}{2m}$. Then

$$\begin{aligned}\|w_n - f\|_{\text{sup}} &\leq \|w_n - p_n\|_{\text{sup}} + \|p_n - f\|_{\text{sup}} \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n} \to 0 \text{ as } n \to \infty\end{aligned}$$

Thus any arbitrary $f \in C([a,b])$ is the limit of a sequence of polynomials in the countable set $\mathbb{P}_{\mathbb{Q}}([a,b])$ under the supremum norm. Thus $C([a,b])$ is a separable metric space. $\square$

A SIMPLER PROOF

Lemma (2)**.** *If A is dense in B and B is dense in C, then A is dense in C.*

Proof. Since A is dense in B, then $\overline{A} = B$. Since B is dense in C, then $\overline{B} = C$. Then $\overline{A} = \overline{\overline{A}} = \overline{B} = C$. Thus A is dense in C. $\square$

By lemmas 1 and 2 and the Weierstrauss Approximation Theorem, $\mathbb{P}_{\mathbb{Q}}$ is dense in $C([0,1])$. Since $\mathbb{P}_{\mathbb{Q}}([a,b])$ is countable, $C([0,1])$ is separable. $\square$

Problem 2

Let $k \in C([0,1] \times [0,1])$, and define a map $T : C([0,1]) \to C([0,1])$ by

$$(Tf)(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y$$

Prove that the set $\{Tf \mid \|f\|_{sup} \leq 1\}$ is equicontinuous.

Pick $x \in [0,1]$ and let $\varepsilon > 0$. Then the continuity of k implies $\exists\delta > 0$ such that $d((x,y),(x_0,y_0)) < \delta \implies d(k(x,y),k(x_0,y_0)) < \varepsilon$. Consider $\tilde{x} \in [0,1]$ and assume $d(x,\tilde{x}) < \delta$. Then $d((x,y),(\tilde{x},y)) < \delta$ for any $y \in [0,1]$. Then $d(k(x,y),k(\tilde{x},y)) < \varepsilon$. Now choose $g \in \{Tf \mid \|f\|_{\text{sup}} \leq 1\}$. Then

$$\begin{aligned}d(g(x),g(\tilde{x})) &= \left|\int_0^1 (k(x,y) - k(\tilde{x},y)f(y)\mathrm{d}y\right| \quad \text{for some } f \text{ such that } \|f\|_{\text{sup}} \leq 1 \\ &\leq \int_0^1 |k(x,y) - k(\tilde{x},y)|\,|f(y)|\mathrm{d}y \\ &< \varepsilon \int_0^1 |f(y)|\mathrm{d}y \\ &< \varepsilon \int_0^1 \mathrm{d}y \quad \text{since } \|f\|_{\text{sup}} < 1 \\ &= \varepsilon\end{aligned}$$

Thus $\{Tf \mid \|f\|_{\text{sup}} \leq 1\}$ is equicontinuous. $\square$

Problem 3

Let $(X, \mathcal{T})$ be a topological space. If $G \subset X$ is open and $F \subset X$ is closed, prove that $G \setminus F$ is open.

Since F in closed, F^C is open. Also, $G \setminus F = G \cap F^C$. Since the finite intersection of open sets in open, $G \setminus F$ is open. $\square$

Problem 4

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a non-empty set X.

a)

Is $\mathcal{T}_1 \cap \mathcal{T}_2$ is topology on X?

Yes.

Let $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_\in$. Since $\emptyset$ and X are elements of all topologies, they are elements of the arbitrary intersection of topologies. Thus $\emptyset, X \in \mathcal{T}$.

Consider $\{G_\alpha \mid \alpha \in I\}$ where each $G_\alpha \in \mathcal{T}$. Then each $G_\alpha \in \mathcal{T}_1$ and each $G_\alpha \in \mathcal{T}_2$. Then $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}_1$ and $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}_2$. Thus $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}$.

Consider $\{G_i \mid i = 1, \ldots, N\}$ where each $G_i \in \mathcal{T}$. Then each $G_i \in \mathcal{T}_1$ and each $G_i \in \mathcal{T}_2$. Then $\bigcap_{i=1}^n G_i \in \mathcal{T}_1$ and $\bigcap_{i=1}^n G_i \in \mathcal{T}_2$. Thus $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

Thus $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X. $\square$

b)

Is $\mathcal{T}_1 \cup \mathcal{T}_2$ is topology on X?

No. We form a counterexample:

Let $X = \{1, 2, 3\}$, and let $\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{2\}, X\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{1\}, \{2\}, X\}$ is not a topology since $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. $\square$

Problem 5

Give an example of two metric spaces (X_1, d_1) and (X_2, d_2), such that X_1 and X_2 are homeomorphic as topological spaces but X_1 is a complete metric space while X_2 is not.

Let $X_1 = [1, \infty)$ and $X_2 = (0, 1]$. Then choose $f : X_1 \to X_2$ by $f(x) = \frac{1}{x}$. f is clearly bijective and continuous, and $f^{-1} : X_2 \to X_1$ by $f^{-1}(x) = \frac{1}{x}$ is also continuous. Thus X_1 and X_2 are homeomorphic as topological spaces, but X_1 is a complete metric while X_2 is not. $\square$

Problem 6

Two metrics, d_1 and d_2, on the same space X are called equivalent if there exist constants $c, C > 0$ such that

$$cd_1(x,y) \leq d_2(x,y) \leq Cd_1(x,y), \quad \text{for all } x, y \in X$$

a)

Show that the topologies on X defined by two equivalent metrics are identical.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the topologies defined by the open sets as defined by the metrics d_1 and d_2, respectively. Denote open balls, with respect to the metric d_i, of radius ε around x as $B_{i,\varepsilon}(x)$ for $i = 1, 2$.

Let $G \in \mathcal{T}_1$. Then G is open with respect to d_1. Then $\forall x \in G$, $\exists \varepsilon$ such that $B_{1,\varepsilon}(x) \in G$. Note that $d_1(x,y) < \varepsilon$ for each $y \in B_{1,\varepsilon}(x)$. Now consider $B_{2,c\varepsilon}(x)$. If $y \in B_{2,c\varepsilon}(x)$, then $d_2(x,y) < c\varepsilon \iff \frac{1}{c}d_2(x,y) < \varepsilon$. But since $d_1(x,y) \leq \frac{1}{c}d_2(x,y)$ for all $x, y \in X$, this implies $d_1(x,y) < \varepsilon$, which then implies $y \in B_{1,\varepsilon}(x)$. Thus $B_{2,c\varepsilon}(x) \subset B_{1,\varepsilon}(x)$, and so $B_{2,c\varepsilon}(x) \subset G$. Thus there is an open ball with respect to the metric d_2 around any point x in G (in particular $B_{2,c\varepsilon}(x)$). Thus G is open with respect to the metric d_2, which shows $G \in \mathcal{T}_2$. Thus $\mathcal{T}_1 \subset \mathcal{T}_2$.

Lemma (3)**.** *Metric equivalence is an equivalence relation.*

Proof. If d_1 is a metric on X, then $1 \cdot d_1(x,y) \leq d_1(x,y) \leq 1 \cdot d_1(x,y)$. Thus d_1 is equivalent to d_1. If d_1 is equivalent to d_2, then $\exists c, C > 0$ such that $cd_1(x,y) \leq d_2(x,y) \leq Cd_1(x,y)$ for all $x, y \in X$. But this implies $\frac{1}{C}d_2(x,y) \leq d_1(x,y) \leq \frac{1}{c}d_2(x,y)$. Since $\frac{1}{C}$ and $\frac{1}{c}$ are greater than 0, this shows d_2 is equivalent to d_1. If d_1 is equivalent to d_2 and d_2 is equivalent to d_3, then $\exists c_1, C_1 > 0$ such that $c_1 d_1(x,y) \leq d_2(x,y) \leq C_1 d_1(x,y)$ for all $x, y \in X$, and $\exists c_2, C_2 > 0$ such that $c_2 d_2(x,y) \leq d_3(x,y) \leq C_2 d_2(x,y)$ for all $x, y \in X$.

$$c_1 c_2 d_1(x,y) \leq c_2 d_2(x,y) \leq d_3(x,y) \leq C_2 d_2(x,y) \leq C_1 C_2 d_1(x,y)$$

and since $c_1 c_2$ and $C_1 C_2$ are greater than 0, this shows d_1 is equivalent to d_3. Thus metric equivalence is an equivalence relation. $\square$

By Lemma 3, we can exchange $\mathcal{T}_1$ and $\mathcal{T}_2$ to show that $\mathcal{T}_2 \subset \mathcal{T}_1$, which, combined with the fact $\mathcal{T}_1 \subset \mathcal{T}_2$, proves $\mathcal{T}_1 = \mathcal{T}_2$. Thus, the topologies on X defined by two equivalent metrics are identical. $\square$

b)

Let (X, d) be a metric space. Show that there exists a metric d_b with the property that $d_b(x,y) \leq 1$, for all $x, y \in X$, such that the topology on X derived from the metric d_b is the same as the one derived from the metric d.

Define d_b as

$$d_b(x,y) = \begin{cases} d(x,y) & , \text{ if } d(x,y) \leq 1 \\ 1 & , \text{ if } d(x,y) > 1 \end{cases}$$

Lemma (5)**.** *d_b is a metric on X.*

Proof. **Non-negativity**: If $x = y$, then $d(x, y) = 0$ since d is a metric on X, and since $0 \leq 1$, $d_b(x, y) = d(x, y) = 0$. If $d_b(x, y) = 0$, then $0 = d_b(x, y) = d(x, y)$. But since d is a metric on X, this implies $x = y$. **Symmetry**: If $d_b(x, y) = 1$, then $d(x, y) \geq 1$. Thus $d(y, x) \geq 1$ since d is a metric on X, and thus $d_b(y, x) = 1$. If $d_b(x, y) < 1$, then $1 > d_b(x, y) = d(x, y) = d(y, x) = d_b(y, x)$. **Triangle Inequality**: If $d_b(x, y) < 1$, then $d_b(x, y) = d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in X$. If either $d(x, y) > 1$ or $d(y, z) > 1$, then $d_b(x, y) < 1 \leq d_b(x, z) + d_b(z, y)$. Otherwise, $d_b(x, y) = d(x, y) \leq d(x, z) + d(z, y) = d_b(x, z) + d_b(z, y)$. If $d_b(x, y) = 1$, then $d_b(x, y) = 1 \leq d(x, y) \leq d(x, z) + d(z, y) = d_b(x, z) + d_b(z, y)$ if $d(x, z) < 1$ and $d(z, y) < 1$. However, if $d(x, z) > 1$ or $d(z, y) > 1$, then $d_b(x, z) = 1$ or $d_b(z, y) = 1$. Thus $d_b(x, y) = 1 \leq d_b(x, z) + d_b(z, y)$. In all cases, the triangle inequality holds. Thus d_b is a metric on X. □

Let $\mathcal{T}$ be the topology defined by the open sets as defined by the metric d, and let $\mathcal{T}_\lfloor$ be the topology defined by the open sets as defined by the metric d_b. Also, denote open balls, with respect to the metric d or d_b, of radius ε around x as $B_\varepsilon(x)$ or $B_{b,\varepsilon}(x)$, respectively. In this proof, we wish to show $T = T_b$.

Let $G \in T$. Then $\forall x \in G$, $\exists \varepsilon$ such that $B_\varepsilon(x) \subset G$. Choose $\hat{\varepsilon} < \min\{\varepsilon, 1\}$. Then $B_{\hat{\varepsilon}}(x) \subset B_\varepsilon(x) \subset G$. Since $\hat{\varepsilon} < 1$, then $B_{\hat{\varepsilon}}(x) = B_{b,\hat{\varepsilon}}(x)$. Thus $B_{b,\hat{\varepsilon}} \subset G$. Therefore $\forall x \in G$, $\exists \hat{\varepsilon}$ such that $B_{b,\hat{\varepsilon}}(x) \subset G$. Thus G is open with respect to d_b, showing $G \in T_b$, which then shows $T \subset T_b$.

Let $G \in T_b$. Then $\forall x \in G$, $\exists \varepsilon$ such that $B_{b,\varepsilon}(x) \in G$. Again, choose $\hat{\varepsilon} < \min\{\varepsilon, 1\}$. Then $B_{b,\hat{\varepsilon}}(x) \subset B_{b,\varepsilon}(x) \subset G$. Since $\hat{\varepsilon} < 1$, then $B_{\hat{\varepsilon}}(x) = B_{b,\hat{\varepsilon}}(x)$. Thus $B_{\hat{\varepsilon}}(x) \subset G$. Therefore $\forall x \in G$, $\exists \hat{\varepsilon}$ such that $B_{b,\hat{\varepsilon}} \subset G$. Thus G is open with respect to d, showing $G \in T$, which then shows $T_b \in T$. Thus $T_b \subset T$, and by the result above, $T = T_b$. □

c)

Give an example of the situation in part b) with the metrics d and d_b that are not *equivalent.*

Let X be the normed linear space $\mathbb{R}$, and let d be the standard metric on $\mathbb{R}$. Define d_b as in Problem 6b:

$$d_b(x, y) = \begin{cases} d(x, y) & , \text{ if } d(x, y) \leq 1 \\ 1 & , \text{ if } d(x, y) > 1 \end{cases}$$

Since d is unbounded on $\mathbb{R}$, then there is no $c > 0$ such that $d(x, y) < c d_b(x, y)$ for all $x, y \in X$, because if there was, then we could choose $y = x + c + 1$, and $d(x, y) = c + 1 < c d_b(x, y) = c$, which is a contradiction. Thus d and d_b are not equivalent metrics. □

Problem 7

Prove Theorem 4.7 of the textbook.

Lemma (5)**.** *Let $(X, \mathcal{T})$ be a topological space. Then $G \in \mathcal{T}$ if and only if G is a neighborhood of x for each $x \in G$.*

Proof. Let $G \in \mathcal{T}$. Then since $G \subset G$, then G contains an open set which contains every $x \in G$ (namely, G). Then G is a neighborhood of x for each $x \in G$. Now let H be a neighborhood of x for each $x \in H$. Then $\exists H_x \subset H$ such that $x \in H_x$ and $H_x \in \mathcal{T}$ for each $x \in X$. Then since $H_x \subset H$ for each $x \in H$, then $\bigcup_{x \in H} H_x \subset H$. However, since $x \in H_x$ for every $x \in H$, then $H \subset \bigcup_{x \in H} H_x$. Thus $\bigcup_{x \in H} H_x = H$. Then since H is an arbitrary union of open sets, then H is open, i.e. $H \in \mathcal{T}$. $\square$

Lemma (6)**.** *Let $(X, \mathcal{T})$ be a topological space. Then $G \in \mathcal{T}$ if and only if G contains a neighborhood of x for each $x \in G$.*

Proof. Let $G \in \mathcal{T}$. Then by Lemma 5, G is a neighborhood of x for each $x \in G$, but since $G \subset G$, then G contains a neighborhood of x for each $x \in G$. Now let H contain a neighborhood of x for each $x \in H$. Then for each $x \in H$, $\exists H_x \subset H$ such that H_x is a neighborhood of x. Then $\exists G_x \subset H_x$ such that $x \in G_x$ and $G_x \in \mathcal{T}$ for each $x \in H$. Then for each $x \in H$, $G_x \subset H_x \subset H$. Thus H is a neighborhood of x for each $x \in H$. Then by Lemma 5, $H \in \mathcal{T}$. $\square$

Theorem (4.7)**.** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces and $f : X \to Y$. Then f is continuous on X if and only if $f^{-1}(G) \in \mathcal{T}$ for every $G \in S$.*

Proof. Suppose f is continuous. Then by the definition of continuity, for each $x \in X$, for each neighborhood W of $f(x)$, there is a neighborhood V of x such that $f(V) \subset W$. Now choose $G \in S$. Then for each $x \in f^{-1}(G)$, G is a neighborhood of each $f(x)$ by Lemma 5. Since f is continuous, there is a neighborhood H of x such that $f(H) \subset G$, which implies $H \subset f^{-1}(G)$. Thus $f^{-1}(G)$ contains a neighborhood of each x in $f^{-1}(G)$, thus by Lemma 6, $f^{-1}(G) \in \mathcal{T}$. Now suppose $f^{-1}(G) \in \mathcal{T}$ for every $G \in S$. Then pick $x \in X$ and a neighborhood W of $f(x)$. By the definition of neighborhood, $\exists H \in \mathcal{S}$ such that $f(x) \in H$ and $H \subset W$. By assumption, $f^{-1}(H) \in \mathcal{T}$. Since $f(x) \in H$, then $x \in f^{-1}(H)$. Then by Lemma 5, $f^{-1}(H)$ is a neighborhood of x. Also, $f(f^{-1}(H)) = H \subset W$. Thus f is continuous. $\square$