HW #8

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November 30, 2015

Problem 1

Consider the Banach space $C([0,1])$ with the supremum norm. For $x \in [0,1]$ let δ_x denote the linear functional on $C([0,1])$ given by

$$\delta_x(f) = f(x), \qquad \text{for all } f \in C([0,1])$$

a)

Show that $\|\delta_x\| = 1$.

$$\|\delta_x\| = \sup_{\|f\|=1} |\delta_x(f)| = \sup_{\|f\|=1} |f(x)| \leq 1$$

but if $f \equiv 1$, then $|\delta_x(f)| = f(x) = 1$, and thus

$$\|\delta_x\| = 1$$

$\square$

b)

Show that there does not exist a Riemann integrable function $k : [0,1] \to \mathbb{R}$, such that

$$\delta_x(f) = \int_0^1 k(y)f(y)\mathrm{d}y, \qquad \text{for all } f \in C([0,1])$$

Fix $z \in [0,1]$. Then define $\delta_z(f) = f(z)$ and assume there exists $k : [0,1] \to \mathbb{R}$ such that

$$\delta_z(f) = \int_0^1 k(y)f(y)\mathrm{d}y$$

for all $f \in C([0,1])$. For $n = 1, 2, \ldots$, define $f_k \in C([0,1])$ as

$$f_n(x) = \begin{cases} 2^n x + (1 - 2^n x_0) & , \ x \in [z - \frac{1}{2^n}, z] \\ -2^n x + (1 + 2^n z) & , \ x \in [z, z + \frac{1}{2^n}] \\ 0 & , \ \text{else} \end{cases}$$

These are tent functions centered at z with $f_n(z) = 1$ for all n. Then

$$1 = f_n(z) = \delta_z(f) = \int_0^1 k(y) f_n(y) \mathrm{d}y = \int_{\max\left\{0, z - \frac{1}{2^n}\right\}}^{\min\left\{1, z + \frac{1}{2^n}\right\}} k(y) f_n(y) \mathrm{d}y$$

Note $\|f_n\| = 1$ for all n, and let K denote $\|k\|$ (K is finite since k is continuous on a compact set). Then

$$1 = f_n(z)\delta_z(f) \le K \int_{\max\left\{0, z - \frac{1}{2^n}\right\}}^{\min\left\{1, z + \frac{1}{2^n}\right\}} \mathrm{d}y = K \frac{1}{2^{n-1}}$$

Taking the limit as $n \to \infty$ yields $1 \le \lim_{n\to\infty} K \frac{1}{2^{n-1}} = 0$, a contradiction.

Thus, there does not exist a Riemann integrable function k such that

$$\delta_z(f) = \int_0^1 k(y) f(y) \mathrm{d}y$$

for all $f \in C([0,1])$. $\square$

Problem 2

Prove that there does not exist an inner product on $C([0,1])$ such that the supremum norm is derived from this inner product.

Take $f(x) = x$ and $g(x) = 1$. Then $\|f\|_\infty = 1$ and $\|g\|_\infty = 1$. Also, $\|f + g\|_\infty = 2$ and $\|f - g\|_\infty = 1$. Then

$$5 = \|f + g\|_\infty^2 + \|f - g\|_\infty^2 \neq 2\|f\|_\infty^2 + 2\|g\|_\infty^2 = 4$$

Thus $\|\cdot\|_\infty$ cannot be derived from an inner product on $C([0,1])$. $\square$

Problem 3

Let $\mathcal{H}$ be a Hilbert space and let M be a subset of $\mathcal{H}$.

a)

Prove that $M^\perp$ is a closed linear subspace of $\mathcal{H}$.

First we show $M^\perp$ is a linear subspace. Let $x, y \in M^\perp$ and $\lambda, \mu \in \mathbb{C}$. Then for each $m \in M$,

$$\langle m, \lambda x + \mu y \rangle = \lambda \langle m, x \rangle + \mu \langle m, y \rangle = 0$$

Thus $\lambda x + \mu y \in M^\perp$. Thus $M^\perp$ is a linear subspace of $\mathcal{H}$. Next, let (x_n) be a convergent sequence in $M^\perp$, and let $x_n \to x$. Then for each $m \in M$,

$$\langle x, m \rangle = \langle \lim_{n\to\infty} x_n, m \rangle$$

but since $\langle \cdot, \cdot \rangle$ is continuous,

$$\langle \lim_{n\to\infty} x_n, m \rangle = \lim_{n\to\infty} \langle x_n, m \rangle = \lim_{n\to\infty} 0 = 0$$

Thus $\langle x, m \rangle = 0$, which shows $x \in M^\perp$, proving $M^\perp$ is closed. $\square$

b)

Prove that $M \cap M^{\perp} \subset \{0\}$.

Let $x \in M \cap M^{\perp}$. Then by the definition of $M^{\perp}$,

$$\langle x, x \rangle = 0$$

Then $\|x\| = 0$, which shows $x = 0$. Thus $M \cap M^{\perp} \subset \{0\}$. $\square$

c)

If M is a linear subspace of $\mathcal{H}$, prove that $(M^{\perp})^{\perp} = \overline{M}$.

Assume $x \in \overline{M}$. Then there is a sequence $x_n \in M$ such that $x_n \to x$. Then $\langle x_n, y \rangle = 0$ for every $y \in M^{\perp}$. Then by continuity of $\langle \cdot, \cdot \rangle$, $\langle x, y \rangle = 0$ for every $y \in M^{\perp}$. Then $x \in (M^{\perp})^{\perp}$ by the definition of $(M^{\perp})^{\perp}$. Thus $\overline{M} \subset (M^{\perp})^{\perp}$.

Now assume $x \notin \overline{M}$. Since $\overline{M}$ is closed, then by the Projection Theorem, $\exists y \in \overline{M}$ such that $(x - y) \perp \overline{M}$. Since $y \in \overline{M}$, $\langle x - y, y \rangle = 0$. Since $x \neq y$ ($x \notin \overline{M}$ and $y \in \overline{M}$), then $\langle x - y, x - y \rangle \neq 0$. However, $\langle x - y, x - y \rangle = \langle x - y, x \rangle - \langle x - y, y \rangle = \langle x - y, x \rangle$. Since $x - y \perp \overline{M}$, then $x - y \perp M$. So $x - y \in M^{\perp}$. Then since $\langle x - y, x \rangle \neq 0$, then $x \notin (M^{\perp})^{\perp}$. Then $(M^{\perp})^{\perp} \subset \overline{M}$.

Thus, $\overline{M} = (M^{\perp})^{\perp}$. $\square$

Problem 4

Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. If $\langle x, Ay \rangle = 0$ for all $x, y, \in \mathcal{H}$, prove $A = \mathbb{0}$.

Since $\langle x, Ay \rangle = 0$ for all $x, y \in \mathcal{H}$, then in particular, take $x = Ay$, and so $\langle Ay, Ay \rangle = 0$ for all $y \in \mathcal{H}$. Then $Ay = 0$ for all $y \in \mathcal{H}$. Thus $A = \mathbb{0}$. $\square$

Problem 5

Let $\mathcal{H}$ be a Hilbert space and P and Q two orthogonal projections on $\mathcal{H}$.

a)

Prove that PQ is an orthogonal projection if and only if $PQ - QP = 0$, i.e., if and only if P and Q commute.

First note that

$$\langle PQx, y \rangle = \langle Qx, Py \rangle = \langle x, QPy \rangle \tag{1}$$

Assume PQ is an orthogonal projection. Then by the definition of orthogonal projection, and by (1), $\langle PQx, y \rangle = \langle x, PQy \rangle$. Then for all $x, y \in \mathcal{H}$,

$$\begin{aligned} &\langle x, QPy \rangle = \langle x, PQy \rangle \\ \implies &\langle x, (QP - PQ)y \rangle = 0 \\ \implies &QP - PQ = \mathbb{0} \\ \implies &QP = PQ \end{aligned}$$

Thus P and Q commute.

Now assume $PQ = QP$. Then $(PQ)^2 = PQPQ = PPQQ = PQ$ since P and Q are orthogonal projections. Also, by (1), $\langle PQx, y\rangle = \langle x, QPy\rangle = \langle x, PQy\rangle$. Thus PQ is an orthogonal projection. $\square$

b)

Prove that for commuting orthogonal projections P and Q, one has $\mathrm{ran(PQ)} = \mathrm{ran(P)} \cap \mathrm{ran(Q)}$.

Let $x \in \mathrm{ran(PQ)}$. Then $\exists y$ such that $PQy = x$. Then P maps Qy on to x. Then $x \in \mathrm{ran(P)}$. However, since P and Q commute, then $QPy = x$, and thus Q maps Py on to x, and so $x \in \mathrm{ran(Q)}$. Thus $x \in \mathrm{ran(P)} \cap \mathrm{ran(Q)}$. So $\mathrm{ran(PQ)} \subset \mathrm{ran(P)} \cap \mathrm{ran(Q)}$.

Now let $x \in \mathrm{ran(P)} \cap \mathrm{ran(Q)}$. Then $x \in \mathrm{ran(P)}$ and $x \in \mathrm{ran(Q)}$. So $\exists y_1, y_2$ such that $Py_1 = Qy_2 = x$. Thus, $PQy_2 = P^2y_1 = Py_1 = x$, and thus $x \in \mathrm{ran(PQ)}$. So $\mathrm{ran(P)} \cap \mathrm{ran(Q)} \subset \mathrm{ran(PQ)}$.

Thus, $\mathrm{ran(PQ)} = \mathrm{ran(P)} \cap \mathrm{ran(Q)}$. $\square$

c)

Prove that $P + Q$ is an orthogonal projection if and only if $PQ = \mathbb{0}$.

Assume $PQ = \mathbb{0}$. Then $\langle PQx, y\rangle = 0$ for all $x, y \in \mathcal{H}$. But by (1), $\langle x, QPy\rangle = 0$ for all $x, y \in \mathcal{H}$. Thus $QP = \mathbb{0}$. Then $(P + Q)^2 = P^2 + PQ + QP + Q^2 = P^2 + \mathbb{0} + \mathbb{0} + Q^2 = P + Q$ since P and Q are orthogonal projections. Also,

$$\begin{aligned}
\langle (P+Q)x, y\rangle &= \langle Px + Qx, y\rangle \\
&= \langle Px, y\rangle + \langle Qx, y\rangle \\
&= \langle x, Py\rangle + \langle x, Qy\rangle \\
&= \langle x, Py + Qy\rangle \\
&= \langle x, (P+Q)y\rangle
\end{aligned}$$

Thus $P + Q$ is an orthogonal projection.

Assume $P + Q$ is an orthogonal projection. Then $(P + Q)^2 = P + Q$, but $(P + Q)^2 = P^2 + PQ + QP + Q^2 = P + PQ + QP + Q$. Thus $PQ + QP = \mathbb{0}$, i.e. $PQ = -QP$.

Assume $x \in \mathrm{ran(P)} \cap \mathrm{ran(Q)}$ and note $0 = (PQ + QP)x = PQx + QPx$. Since $x \in \mathrm{ran(P)}$, $Px = x$. Also, since $x \in \mathrm{ran(Q)}$, $Qx = x$. Then $PQx = Px = x$ and $QPx = Qx = x$. So $0 = PQx + QPx = 2x$. Thus $x = 0$, which proves $\mathrm{ran(P)} \cap \mathrm{ran(Q)} = \{0\}$.

Now, take any $x \in \mathcal{H}$, then certainly $PQx \in \mathrm{ran(P)}$ and since $PQx = -QPx = Q(-Px)$, then $PQx \in \mathrm{ran(Q)}$. Then $PQx = 0$ by the paragraph above, and thus $PQ = \mathbb{0}$, i.e. $\mathrm{ran(PQ)} = \{0\}$.

Thus, $P + Q$ is an orthogonal projection if and only if $\mathrm{ran(PQ)} = \{0\}$. $\square$

d)

Prove that if $PQ = \mathbb{0}$, we have $\mathrm{ran(P+Q)} = \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$.

Let $PQ = \mathbb{0}$ and assume $y \in \mathrm{ran(P+Q)}$. Then $\exists x \in \mathcal{H}$ such that $Px + Qx = y$. Then y is the sum of an element in $\mathrm{ran(P)}$ and an element in $\mathrm{ran(Q)}$. Thus $y \in \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$. Assume

$y \in \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$. Then $\exists x_1, x_2 \in \mathcal{H}$ such that $y = Px_1 + Qx_2$. Then $Py = P^2x_1 + PQx_2 = Px_1$ and $Qy = QPx_1 + Q^2x_2 = Qx_2$ since $QP = \mathbb{0}$. Thus $y = Px_1 + Qx_2 = Py + Qy = (P+Q)y$. Thus $y \in \mathrm{ran(P+Q)}$, which shows $\mathrm{ran(P+Q)} = \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$. $\square$

Problem 6

Let $\mathcal{H}$ be a Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ such that $P^2 = P$ and $\dim \mathrm{ran(P)} = 1$.

a)

Show that $\|P\| \geq 1$.

Let $x \in \mathrm{ran(P)}$ such that $\|x\| = 1$. Then $Px = x$, and so $\|Px\| = \|x\| = 1$. Thus $\|P\| \geq 1$. $\square$

b)

Suppose $\dim \mathcal{H} \geq 2$. Find

$$\sup \left\{ \|P\| \mid P \in \mathcal{B}(\mathcal{H}),\ P^2 = P,\ \dim \mathrm{ran(P)} = 1 \right\}$$

Let $\dim \mathcal{H} \geq 2$ and let $\{e_\alpha\}_{\alpha \in I}$ be an orthonormal basis. Then pick two distinct basis elements, e_{α_1} and e_{α_2}, and define

$$P_n(x) = (x_{\alpha_1} + n x_{\alpha_2}) e_{\alpha_1}$$

for $n = 1, 2, \dots$, and where x_{α_i} is the defined as the coefficient on e_{α_i} in the sum

$$x = \sum_{\alpha \in I} x_\alpha e_\alpha$$

Then clearly $\dim \mathrm{ranP_n} = 1$ since $P_n x = a e_{\alpha_1}$ where a is the only degree of freedom (i.e. no vector in the range of P_n is linearly independent from e_{α_1}). Also,

$$\begin{aligned} P_n^2 x &= P_n((x_{\alpha_1} + n x_{\alpha_2}) e_{\alpha_1}) \\ &= (x_{\alpha_1} + n x_{\alpha_2} + n(0)) e_{\alpha_1} \\ &= P_n x \end{aligned}$$

Lastly,

$$\|P_n\| = \sup_{\|x\|=1} \|P_n x\| = \sup_{\|x\|=1} |x_{\alpha_1} + n x_{\alpha_2}|$$

However, since $\|x\| = \sqrt{\langle x, x \rangle} = 1$, and $x = \sum_{\alpha \in I} x_\alpha e_\alpha$, then

$$1 = \langle x, x \rangle = \left\langle \sum_{\alpha \in I} x_\alpha e_\alpha, \sum_{\alpha \in I} x_\alpha e_\alpha \right\rangle = \sum_{\alpha \in I} |x_\alpha|^2$$

Thus each x_α is at most 1, and so $|x_{\alpha_1}| \leq 1$ and $|x_{\alpha_2}| \leq 1$. Thus

$$\|P_n\| = \sup_{\|x\|=1} |x_{\alpha_1} + n x_{\alpha_2}| \leq 1 + n$$

Note, however, $\|P_n e_{\alpha_2}\| = n$, and thus $\|P_n\| \geq n$. In summary,

$$n \leq \|P_n\| \leq n+1$$

So $P_n \in \mathcal{B}(\mathcal{H})$, but as $n \to \infty$, $\|P_n\| \to \infty$. Thus,

$$\sup\left\{\|P\| \mid P \in \mathcal{B}(\mathcal{H}),\ P^2 = P,\ \dim\operatorname{ran}(\mathrm{P}) = 1\right\} = \infty$$

$\square$

Problem 7

Let $\{e_1, e_2, \dots\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$.

a)

Let $(a_n) \in \ell^1(\mathbb{N})$. Show that $\sum_{n=1}^{\infty} a_n e_n$ converges absolutely to a limit in $\mathcal{H}$.

$$\sum_{n=1}^{\infty} \|a_n e_n\| = \sum_{n=1}^{\infty} |a_n|\|e_n\| = \sum_{n=1}^{\infty} |a_n| < \infty$$

since $(a_n) \in \ell^1(\mathbb{N})$. Thus $\sum_{n=1}^{\infty}$ converges absolutely to a limit in $\mathcal{H}$. $\square$

b)

Let $\alpha \in (0, \infty)$ and define $a_n = n^{-\alpha}$, $n \geq 1$. For which values of α does $\sum_{n=1}^{\infty} a_n e_n$ converge unconditionally but not absolutely?

If $\alpha > 1$, then

$$\sum_{n=1}^{\infty} \|a_n e_n\| = \sum_{n=1}^{\infty} |a_n|\|e_n\| = \sum_{n=1}^{\infty} |n^{-\alpha}| < \infty$$

by the p-series test of calculus. Now consider the norm of the proposed summation:

$$\left\|\sum_{n=1}^{\infty} a_n e_n\right\| = \sqrt{\left\langle \sum_{n=1}^{\infty} a_n e_n, \sum_{n=1}^{\infty} a_n e_n \right\rangle} = \sqrt{\sum_{n=1}^{\infty} |a_n|^2 \langle e_n, e_n\rangle} = \sqrt{\sum_{n=1}^{\infty} n^{-2\alpha}}$$

which converges if $2\alpha > 1$, i.e. if $\alpha > \frac{1}{2}$. Thus if $\alpha \in (\frac{1}{2}, 1]$, then $\sum_{n=1}^{\infty}$ converges unconditionally but not absolutely. $\square$

Problem 8

Define the Legendre polynomials P_n by

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

a)

Show that the Legendre polynomials are orthogonal in $L^2([-1,1])$, and that they are obtained by Gram-Schmidt orthogonalization of the monomials.

Fix n, and pick $m < n$. Then

$$\begin{aligned}
\langle x^m, P_n \rangle &= \int_{-1}^{1} x^m P_n \mathrm{d}x \\
&= \int_{-1}^{1} x^m \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \mathrm{d}x \\
\implies 2^n n! \langle x^m, P_n \rangle &= \int_{-1}^{1} x^m \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \mathrm{d}x \\
&= (-1)^m m! \int_{-1}^{1} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}(x^2-1)^n \mathrm{d}x \quad \text{through integration by parts } m \text{ times}
\end{aligned}$$

b)

Show that

$$\int_{-1}^{1} P_n(x)^2 \mathrm{d}x = \frac{2}{2n+1}$$

$$\begin{aligned}
\langle P_n, P_n \rangle^2 &= \int_{-1}^{1} P_n(x)^2 \mathrm{d}x \\
&= \int_{-1}^{1} \left(\frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \right)^2 \mathrm{d}x \\
&= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \right)^2 \mathrm{d}x \\
&= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(x^2-1)^n \mathrm{d}x \qquad \text{through integration by parts } n \text{ times} \\
&= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \mathrm{d}x \qquad \text{through integration by parts } 2n \text{ times}
\end{aligned}$$

Now just consider the integral

$$\begin{aligned}
\int_{-1}^{1} (x^2-1)^n \mathrm{d}x &= \int_{-1}^{1} (x-1)^n (x+1)^n \mathrm{d}x \\
&= \frac{(n!)^2(-1)^n}{(2n)!} \int_{-1}^{1} (x+1)^{2n} \mathrm{d}x \qquad \text{through integration by parts } n \text{ times} \\
&= \frac{(n!)^2(-1)^n}{(2n)!} \left[\frac{(x+1)^{2n+1}}{2n+1} \right]_{-1}^{1} \\
&= \frac{(n!)^2(-1)^n \; 2^{2n+1}}{(2n)!(2n+1)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle P_n, P_n \rangle^2 &= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \cdot \frac{(n!)^2(-1)^n\ 2^{2n+1}}{(2n)!(2n+1)} \\
&= \frac{2}{2n+1}
\end{aligned}$$

□

c)

Prove that the Legendre polynomials form an orthogonal basis of $L^2([-1,1])$.

In part **a)**, we used the Gram-Schmidt process to generate the Legendre polynomials from the basis of monomials. The Gram-Schmidt process creates an orthogonal basis from any basis. Thus the Legendre polynomials form an orthogonal basis of $L^2([-1,1])$. □

d)

Prove that the Legendre polynomial P_n is an eigenfunction of the differential operator

$$L = -\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right]$$

with eigenvalue $\lambda_n = n(n+1)$, meaning that

$$LP_n = \lambda_n P_n$$

Let $u(x) = (x^2-1)^n$ and note that

$$(x^2-1)Du = (x^2-1)n(x^2-1)^n \cdot 2x = 2nxu$$

Apply D^{n+1} to both sides, and use Liebnitz's Rule for n^{th} derivative of fg to acheive

$$\begin{aligned}
\frac{(n+1)n}{2} \cdot 2 \cdot D^{n-1}Du + (n+1)2xD^nDu + (x^2-1)D^{n+1}Du &= 2n(n+1)D^nu + 2nxD^{n+1}u \\
\implies 2xD^{n+1}u + (x^2-1)D^{n+2}u &= n(n+1)D^nu \\
\implies LD^nu &= n(n+1)D^nu
\end{aligned}$$

which shows D^n is an eigenfunction of L with eigenvalue $\lambda_n = n(n+1)$. Since $2^n n! P_n = D^n$ (i.e. P_n is linearly dependent on D^n), then P_n is an eigenfunction of L with eigenvalue $\lambda_n = n(n+1)$. □