HW #3

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Problem 1

Let (f_n) be a sequence in $C([0,1])$ converging uniformly to the function $f(x) = -x\log x$ on $[0,1]$. Define

$$A = \{f_n \mid n \geq 1\} \cup \{f\}$$

Is A compact, or precompact but not compact, or not precompact, considered as a subset of $(C([0,1]), \|\cdot\|_{sup})$? Justify your answer.

Let $G = \{G_i \mid i \in I\}$ for some index set I be an open cover of A. Then $\exists i_0 \in I$ such that $f \in G_{i_0}$. Since G_{i_0} is open, $\exists \epsilon > 0$ such that $B_\epsilon(f) \subset G_{i_0}$. Since f_n converges uniformly to f, $\exists N \in \mathbb{N}$ such that $n \geq N \implies f_n \in B_\epsilon(f)$ (and thus $f_n \in G_{i_0}$). Then there are only finitely many functions in A which are potentially not elements of $B_\epsilon(f)$, specifically, $f_1, \dots, f_{N-1}$. Since G is an open cover of A, $\exists i_1, \dots, i_{N-1} \in I$ such that $f_1 \in G_{i_1}, \dots, f_{N-1} \in G_{i_{N-1}}$. Thus $\widetilde{G} = \{G_{i_0}, G_{i_1}, \dots, G_{i_{N-1}}\}$ is a finite open cover of A. Thus A is compact. $\square$

Problem 2

Let $f \in C([a,b])$. Prove that

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leq |b-a|^{1/2}\left(\int_a^b f(x)^2\mathrm{d}x\right)^{1/2}$$

Define the inner product $< f, g >$ on $C([a,b])$ to be the $\boldsymbol{L}^2$ inner product, or

$$< f, g >= \int_a^b f(x)g(x)\mathrm{d}x$$

Then consider Cauchy-Schwarz inequality: $| < f, g > | \leq \|f\| \cdot \|g\|$, $\forall f, g \in C([a,b])$. Pick $g(x) \equiv 1$. Then,

$$| < f, 1 > | = \left|\int_a^b f(x)\mathrm{d}x\right|$$

$$
\begin{aligned}
&\leq \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2} \left(\int_a^b 1^2 \ \mathrm{d}x\right)^{1/2} && \text{by the Cauchy-Schwarz inequality} \\
&= \left(x\Big|_a^b\right)^{1/2} \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2} \\
&= |b-a|^{1/2}\left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2} && b-a = |b-a| \text{ since } b \geq a.
\end{aligned}
$$

□

Problem 3

For $M > 0$, define $A_M \subset C([a,b])$ as follows:

$$
A_M = \{f \in C([a,b]) \mid f' \in C([a,b]), f(a) = f(b) = 0, \text{ and } \int_a^b f'(x)^2 \mathrm{d}x \leq M\}
$$

Prove that A_M is precompact in $(C([a,b]), \|\cdot\|_{sup})$.

Let $\tilde{x} \in [a,b]$ and $f \in A_M$. Then,

$$
\begin{aligned}
|f(\tilde{x})| = |f(\tilde{x}) - 0| = |f(\tilde{x}) - f(a)| \\
&= \left|\int_a^{\tilde{x}} f'(x) \mathrm{d}x\right| && \text{by the Fundamental Theorem of Calculus} \\
&\leq |\tilde{x} - a|^{1/2}\left(\int_a^{\tilde{x}} f'(x)^2 \mathrm{d}x\right)^{1/2} && \text{by Problem 2} \\
&\leq |b-a|^{1/2}\left(\int_a^{\tilde{x}} f'(x)^2 \mathrm{d}x\right)^{1/2} && \text{since } \tilde{x} \in [a,b] \\
&\leq |b-a|^{1/2}\sqrt{M} && \text{by the definition of } A_M
\end{aligned}
$$

Thus $f(\tilde{x})$ is uniformly bounded by $|b-a|^{1/2}\sqrt{M}$ for any $\tilde{x} \in [a,b]$ and any $f \in A_M$. Thus A_M is bounded.

Pick $\epsilon > 0$ and $x_1 \in [a,b]$. Assume $d(x_1, x_2) < \frac{\epsilon^2}{M}$. Then

$$
\begin{aligned}
|d(f(x_1), f(x_2))| = |f(x_1) - f(x_2)| \\
&= \left|\int_{x_2}^{x_1} f'(x) \mathrm{d}x\right| && \text{by the Fundamental Theorem of Calculus} \\
&\leq |x_2 - x_1|^{1/2}\left(\int_{x_2}^{x_1} f'(x)^2\right)^{1/2} && \text{by Problem 2} \\
&< \sqrt{\frac{\epsilon^2}{M}}\sqrt{M} && \text{by assumption and the definition of } A_M \\
&= \epsilon
\end{aligned}
$$

Thus A_M is equicontinuous. By the Arzelà-Ascoli Theorem, A_M is precompact. □

Problem 4

Consider functions $f : [0,1] \to \mathbb{R}$ *defined by*

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \quad x \in [0,1] \tag{1}$$

where for all $n \geq 1$, $a_n \in \mathbb{R}$, *and such that* $\sum_{n=1}^{\infty} |a_n| < +\infty$.

a)

Prove that $f \in C([0,1])$.

Consider the sequence $f_k(x) = \sum_{n=1}^{k} a_n \sin(n\pi x)$. Clearly, $\lim_{k\to\infty} f_k = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = f(x)$. Since $\sin(n\pi x) \in C([0,1])$ for $n = 1, 2, \ldots$, and linear combinations of continuous functions are continuous, $f_k \in C([0,1])$ for $k = 1, 2, \ldots$. It suffices to show that f_k is a Cauchy sequence. Then, the completeness of $C([0,1])$ will imply that the limit of f_k is in $C([0,1])$.

First, since $\lim_{k\to\infty} \sum_{n=1}^{k} |a_n| = L < \infty$, then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $i > j \geq N \implies \sum_{n=1}^{i} |a_n| - \sum_{n=1}^{j} |a_n| = \sum_{n=j+1}^{i} |a_n| < \epsilon$. Now pick $\epsilon > 0$, and assume $i > j \geq N$, such that $\sum_{n=j+1}^{i} |a_n| < \epsilon$. Then,

$$\begin{aligned}
d(f_i, f_j) = \|f_i - f_j\|_{\sup} &= \left\| \sum_{n=j+1}^{i} a_n sin(n\pi x) \right\|_{\sup} \\
&\leq \sum_{n=j+1}^{i} |a_n| \, |\sin(n\pi x)| \qquad \text{by the Triangle Inequality} \\
&\leq \sum_{n=j+1}^{i} |a_n| \qquad \text{since } |\sin(n\pi x)| \leq 1 \text{ for any } n. \\
&< \epsilon
\end{aligned}$$

Thus f_k is a Cauchy seqeunce, and since $C([0,1])$ is complete, the limit of f_k, which is f, must be an element of $C([0,1])$. □

b)

Prove that the set A *defined by*

$$A = \{f \in C([0,1]) \mid f \text{ is of the form (1) and } \|f\|_{\sup} \leq 1\}$$

is not precompact in $(C([0,1]), \|\cdot\|_{sup})$.

Consider the family of functions $\mathcal{F} = \{\sin(n\pi x)\}$ for $n = 1, 2, \ldots$, and choose any $\delta > 0$. The choose $N > \frac{2}{\delta}$. Then the period of $f_N = \sin(N\pi x)$ is $\frac{2}{N} < \delta$. Then choose a minimum $x_{\min}$ and a maximum $x_{\max}$ of f_N such that $|x_{\min} - x_{\max}| < \delta$ (this can be done since the period of f_N is less than δ). Since $f_N(x_{\min}) = -1$ and $f_N(x_{\max}) = 1$, then $|f_N(x_{\min}) - f_N(x_{\max})| = 2$. Thus the family $\mathcal{F}$ is not equicontinuous, and since $\mathcal{F} \subset A$, A is not equicontinuous. By the Arzelà-Ascoli Theorem, A is not precompact. □

c)

Prove that the set B deinfed by

$$B = \{f \in C([0,1]) \mid f \text{ is of the form (1) and } \sum_{n=1}^{\infty} n^2 |a_n|^2 \leq 1\}$$

is precompact in $(C([0,1]), \|\cdot\|_{sup})$.

First we show B is equicontinuous. Pick $\epsilon > 0$ and $x_1 \in [0,1]$. Assume $d(x_1, x_2) < \left(\frac{2\epsilon}{\pi}\right)^2$. Then,

$$\begin{aligned}
|d(f(x_1), f(x_2))| &= |f(x_1) - f(x_2)| \\
&= \left| \sum_{n=1}^{\infty} a_n (\sin(n\pi x_1) - \sin(n\pi x_2)) \right| \\
&= \left| \int_{x_2}^{x_1} \sum_{n=1}^{\infty} a_n \pi n \cos(n\pi x) \mathrm{d}x \right| \qquad \text{by the Fundamental Theorem of Calculus} \\
&= \left| \sum_{n=1}^{\infty} a_n \pi n \int_{x_2}^{x_1} \cos(n\pi x) \mathrm{d}x \right| \\
&= \pi \left| \lim_{N\to\infty} \left(\sum_{n=1}^{N} n a_n \int_{x_2}^{x_1} \cos(n\pi x) \mathrm{d}x \right) \right| \\
&= \pi \left| \lim_{N\to\infty} \int_{x_2}^{x_1} \sum_{n=1}^{N} n a_n \cos(n\pi x) \mathrm{d}x \right| \\
&\leq \pi \left| \lim_{N\to\infty} |x_1 - x_2|^{1/2} \left(\int_{x_2}^{x_1} \left(\sum_{n=1}^{N} n a_n \cos(n\pi x) \right)^2 \mathrm{d}x \right)^{1/2} \right| \qquad \text{by Problem 2} \\
&\leq \pi \sqrt{|x_1 - x_2|} \left| \lim_{N\to\infty} \left(\int_{0}^{1} \left(\sum_{n=1}^{N} n a_n \cos(n\pi x) \right)^2 \mathrm{d}x \right)^{1/2} \right|
\end{aligned}$$

We can change the limits of integration since the integrand is positive, and since $|x_1 - x_2| < 1$. Note that

$$\begin{aligned}
\int_0^1 \left(\sum_{n=1}^{N} n a_n \cos(n\pi x) \right)^2 \mathrm{d}x &= \int_0^1 \sum_{n=1}^{N} n^2 a_n^2 \cos^2(n\pi x) \mathrm{d}x \\
&\qquad \text{since } \int_0^1 \cos(n\pi x)\cos(m\pi x)\mathrm{d}x = 0 \text{ for } m \neq n \\
&= \sum_{n=1}^{N} n^2 a_n^2 \int_0^1 \cos^2(n\pi x) \mathrm{d}x \\
&= \sum_{n=1}^{N} n^2 a_n^2 \left(\frac{1}{2}\right)
\end{aligned}$$

$$\text{since } \int_0^1 \cos^2(n\pi x) = \frac{1}{2} \text{ for every integer } n$$

$$= \frac{1}{2}\sum_{n=1}^{N} n^2 a_n^2$$

Thus,

$$\begin{aligned}
|d(f(x_1), f(x_2))| &\leq \pi\sqrt{|x_1 - x_2|}\left|\lim_{N\to\infty}\left(\int_0^1\left(\sum_{n=1}^{N} na_n\cos(n\pi x)\right)^2 \mathrm{d}x\right)^{1/2}\right| \\
&= \pi\sqrt{|x_1 - x_2|}\left|\lim_{N\to\infty}\left(\frac{1}{2}\sum_{n=1}^{N} n^2 a_n^2\right)^{1/2}\right| \\
&= \frac{\pi}{2}\sqrt{|x_1 - x_2|}\left|\lim_{N\to\infty}\left(\sum_{n=1}^{N} n^2 a_n^2\right)^{1/2}\right| \\
&= \frac{\pi}{2}\sqrt{|x_1 - x_2|}\left|\left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{1/2}\right| \\
&\leq \frac{\pi}{2}\sqrt{|x_1 - x_2|}\left(\sum_{n=1}^{\infty} n^2 |a_n|^2\right)^{1/2} \qquad \text{by the Triangle Inequality} \\
&\leq \frac{\pi}{2}\sqrt{|x_1 - x_2|} \qquad \text{by the definition of the set } B \\
&< \epsilon \qquad \text{by assumption}
\end{aligned}$$

Thus B is equicontinuous.

Next we show boundedness. From the above calculation for equicontinuity, take $x_2 = 0$, and let $x_1 \in [0, 1]$. Then

$$\begin{aligned}
|f(x_1)| = |f(x_1) - 0| &= |f(x_1) - f(x_2)| \\
&\leq \frac{\pi}{2}\sqrt{|x_1 - x_2|} \\
&\leq \frac{\pi}{2}\sqrt{|x_1|} \\
&\leq \frac{\pi}{2}\sqrt{|1|} \qquad \text{since } x_1 \leq 1 \qquad\qquad = \frac{\pi}{2}
\end{aligned}$$

Thus each function in B is bounded by $\frac{\pi}{2}$, and so B is bounded. Since it is also equicontinuous, then by the Arzelà-Ascoli Theorem, B is precompact. $\square$