HW #5

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Problem 1

Let $(X, \mathcal{T})$ be a Hausdorff space and $F, K \subset X$ such that F is closed and K is compact.

a)

Prove that K is closed.

Pick y in K^C. Then for every $x \in K$, choose an open neighborhood of x, U_x, and an open neighborhood of y, V_x, such that $U_x \cap V_x = \emptyset$ for each x. This is possible since X is a Hausdorff space. Clearly, $\{U_x\}_{x \in K}$ is an open cover of K. Since K is compact, $\exists x_1, \dots, x_n$ such that $\{U_{x_i}\}_{i=1}^n$ is an open cover of K. Let $V = \bigcap_{i=1}^n V_{x_i}$. Then V is open since it is a finite intersection of open neighborhoods. Let $v \in V$. Then for $i = 1, \dots, n$, $v \notin U_{x_i}$. Then $v \notin K$, i.e. $v \in K^C$. Thus $V \subset K^C$. Thus K^C contains a neighborhood of each element of K^C, and so $K^C \in \mathcal{T}$. Thus K is closed. $\square$

b)

Prove that $F \cap K$ is compact.

Choose an open cover $\{G_\alpha\}_\alpha$ of $F \cap K$. Since K is compact, it is closed (by part a), and since F is also closed, $F \cap K$ is closed, i.e. $(F \cap K)^C$ is open. Then $\left\{\{G_\alpha\}_\alpha, (F \cap K)^C\right\}$ is an open cover of K. Then since K is compact, there is a finite open subcover, namely $\left\{\{G_{\alpha_i}\}_{i=1}^n, (F \cap K)^C\right\}$. But since $(F \cap K)^C \cap (F \cap K) = \emptyset$, then $\{G_{\alpha_i}\}_{i=1}^n$ is an open cover of $F \cap K$. Since this is a subcover of $\{G_\alpha\}$, then $F \cap K$ is compact. $\square$

Problem 2

Let $(X, \mathcal{T})$ be a topological space and K_1, K_2 two compact subsets of X.

a)

Prove that $K_1 \cup K_2$ is compact.

Let $\{G_\alpha\}_\alpha$ be an open cover of $K_1 \cup K_2$. Then $\{G_\alpha\}_\alpha$ is an open cover of both K_1 and K_2. Then there are finite subcovers $\{G_{\alpha_i}\}_{i=1}^n$ and $\{G_{\alpha_j}\}_{j=1}^m$ of K_1 and K_2, respectively. Then $\left\{\{G_{\alpha_i}\}_{i=1}^n, \{G_{\alpha_j}\}_{j=1}^m\right\}$ is a finite cover of $K_1 \cup K_2$, and is a subcover of $\{G_\alpha\}_\alpha$. Thus every open cover has a finite subcover, proving $K_1 \cup K_2$ is compact. □

b)

Assuming $(X, \mathcal{T})$ is Hausdorff, proce that $K_1 \cap K_2$ is compact.

By part 1.a), the compactness of K_1 implies its closure. Thus by part 1.b), $K_1 \cap K_2$ is compact. □

Problem 3

If A is a subset of a toplogical space, then the interior A° *of A is the union of all open sets contained in A, the* closure $\overline{A}$ *of A is the intersection of all closed sets that contain A, and the* boundary ∂A *of A is defined by $\partial A = \overline{A} \cap \overline{A^C}$.*

Lemma 1. $\overline{A^C} = (A^\circ)^C$

Proof. Let $\{C_\alpha\}$ be the set of all closed sets containing A^C. Then by the definition of closure, $\overline{A^C} = \bigcap_\alpha C_\alpha$. Since $A^C \subset C_\alpha$ for all α, then $C_\alpha^C \subset A$ for all α. Also, since C_α is closed for all α, C_α^C is open for all α. In addition, if G is an open set contained in A, then $G = C_\alpha^C$ for some C_α. Then by the definition of interior, $A^\circ = \bigcup_\alpha C_\alpha^C$. Thus,

$$(A^\circ)^C = \left(\cup_\alpha C_\alpha^C\right)^C = \cap_\alpha \left(C_\alpha^C\right)^C = \cap_\alpha C_\alpha = \overline{A^C}$$

□

Lemma 2. $\overline{A}^C = \left(A^C\right)^\circ$

Proof. Let $B = A^C$. Then by Lemma 1, $\overline{B^C} = (B^\circ)^C$. Then $\overline{(A^C)^C} = \left(\left(A^C\right)^\circ\right)^C$. Thus $\overline{A} = \left(\left(A^C\right)^\circ\right)^C$. Thus $\overline{A}^C = \left(A^C\right)^\circ$ □

a)

Show that a set is closed if and only if it contains its boundary.

"$\Longrightarrow$" Let A be closed. Then $A = \overline{A}$. Then $\partial A = \overline{A} \cap \overline{A^C} \subset \overline{A} = A$. Then A contains its boundary.

"$\Longleftarrow$" Let A contain its boundary, i.e. $\partial A = \overline{A} \cap \overline{A^C} \subset A$. We want to show A^C is open, i.e. $A^C = \left(A^C\right)^\circ$. Obviously, $\left(A^C\right)^\circ \subset A^C$. Let $x \in A^C$. Then $x \notin A$. Since $\partial A \subset A$, $x \notin \partial A$. Then either $x \notin \overline{A}$ or $x \notin \overline{A^C}$, i.e. either $x \in \overline{A}^C$ or $x \in \overline{A^C}^C$. By Lemmas 1 and 2, either $x \in \left(A^C\right)^\circ$ or $x \in A^\circ$. But since $x \notin A$, $x \notin A^\circ$. Thus $x \in \left(A^C\right)^\circ$. Then $A^C = \left(A^C\right)^\circ$. Thus A^C is open, proving A is closed. □

b)

Show that a set is open if any only if it is disjoint from its boundary.

"$\Longrightarrow$" Let A be open. Then $A = A^\circ$. Then $A \cap \partial A = A \cap \left(\overline{A} \cap \overline{A^C}\right) = A \cap \left(\overline{A} \cap (A^\circ)^C\right)$ (by Lemma 1) and thus $A \cap \partial A = A \cap \left(\overline{A} \cap A^C\right) = \left(A \cap A^C\right) \cap \overline{A} = \emptyset \cap \overline{A} = \emptyset$. Thus A is disjoint from its boundary.

"$\Longleftarrow$" Let $A \cap \partial A = \emptyset$, and choose $x \in A$. Then $x \notin \partial A$. Thus $x \notin \overline{A}$ or $x \notin \overline{A^C}$. Since $x \in A$, $x \in \overline{A}$. Thus $x \notin \overline{A^C}$. By Lemma 1, $x \notin (A^\circ)^C$. Thus $x \in A^\circ$. Since A° is open, there is a neighborhood G of x such that $G \subset A^\circ$. But $A^\circ \subset A$. Thus A is open. $\square$

c)

What are the closure, interior, and boundary of the Cantor set, considered as a subset of $\mathbb{R}$ with its usual topology? The Cantor set is defined in Example 1.40 of the textbook.

Define the function f whose domain is closed intervals of $\mathbb{R}$ by

$$f([a,b]) = \left\{\left[a, a + \frac{b-a}{3}\right], \left[b - \frac{b-a}{3}, b\right]\right\}$$

Define G_n as follows:

$$\begin{aligned} G_0 &= \{[0,1]\} \\ G_1 &= \left\{\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right]\right\} \\ &\vdots \\ G_n &= \bigcup_{[a,b] \in G_{n-1}} f([a,b]) \\ &\vdots \end{aligned}$$

and define $F_n \equiv \bigcup_{[a,b] \in G_n} [a,b]$. Finally, define the Cantor set $\mathcal{C} = \bigcap_{n=0}^{\infty} F_n$. Since for each n, $|G_n| = 2^n$, label each element of G_n as $G_{n,k}$ for $k = 1, \ldots, 2^n$. Note that for each $G_{n,k}$, $\sup\{|x_1 - x_2| \mid x_1, x_2 \in G_{n,k}\} = 3^{-n}$. Next we will show $\mathcal{C}^\circ = \emptyset$, which will show $\mathcal{C} = \overline{\mathcal{C}}$ and $\partial\mathcal{C} = \mathcal{C}$.

Let $x \in \mathcal{C}^\circ$. Then since $\mathcal{C}^\circ$ is open, there is some open neighborhood U such that $x \in U \subset \mathcal{C}^\circ$. Since U is an open neighborhood, $\exists \epsilon > 0$ such that $x \in B_\epsilon(x) \subset U \subset \mathcal{C}^\circ$. Since $\mathcal{C}^\circ \subset \mathcal{C} = \bigcap_{n=0}^{\infty} F_n$, then $\forall n$, $\exists k$ such that $B_\epsilon(x) \subset G_{n,k}$. Thus $\forall n$, $\sup\{|y_1 - y_2| \mid y_1, y_2 \in B_\epsilon(x)\} = 2\epsilon < 3^{-n}$, which is a contradiction. Thus $\mathcal{C}^\circ = \emptyset$, and $\overline{\mathcal{C}} = \mathcal{C}$. Finally $\partial\mathcal{C} = \overline{\mathcal{C}} \cap \overline{\mathcal{C}^C} = \mathcal{C} \cap (\mathcal{C}^\circ)^C = \mathcal{C} \cap \mathbb{R} = \mathcal{C}$.
$\square$

Problem 4

A topological space is connected *if it is not the union of two disjoint non-empty open sets. A subset Y of a topological space $(X, \mathcal{T})$ is called connected if Y is a connected topological space with respect to the relative topology.*

a)

Describe the connected subsets of $(\mathbb{R}, |\cdot|)$.

Lemma 3. *The connected subsets of $\mathbb{R}$ are intervals.*

Proof. "$\Longrightarrow$" Suppose $G \subset \mathbb{R}$ is not an interval. Then $\exists x, \epsilon$ such that $x \notin G$ but $x - \epsilon \in G$ and $x + \epsilon \in G$. Then pick $U_1 = (-\infty, x) \cap G$ and $U_2 = (x, \infty) \cap G$. Then U_1 and U_2 are open in the relative topology on G and $U_1 \cap U_2 = \emptyset$. Thus G is not connected.

"$\Longleftarrow$" Suppose $G \subset \mathbb{R}$ is not connected. Then $G = U \cup V$ where $U, V \in \mathcal{T}$ and $U \cap V = \emptyset$. $U \in \mathcal{T} \implies U = (\cup_{\alpha \in I}(a_\alpha, b_\alpha)) \cap G$, where $a_\alpha \neq b_\alpha$ for every α in the index set I. Similarly, $V \in \mathcal{T} \implies V = (\cup_{\beta \in J}(c_\beta, d_\beta)) \cap G$ where $c_\beta \neq d_\beta$ for every β in the index set J. Let $\epsilon = \inf\{|u - v| \mid u \in U, v \in V\}$.

If $\epsilon = 0$, then pick $(u_n)_n \in U$ and $(v_n)_n \in V$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = L$. If $L \in U$, then $\exists \tilde{\epsilon} > 0$ such that $B_{\tilde{\epsilon}}(L) \subset U$. But since $\lim_{n\to\infty} v_n = L$, then $\exists N$ such that $n \geq N \implies v_n \in B_{\tilde{\epsilon}}(L)$, which is a contradiction since $U \cap V = \emptyset$. Thus $L \notin U$. Similarly, $L \notin V$. Thus $L \notin G$. However, $\exists \bar{\epsilon}$ such that $L \pm \bar{\epsilon} \in U \subset G$ and $L \mp \bar{\epsilon} \in V \subset G$. Thus G is not an interval.

If $\epsilon > 0$, then pick $(u_n)_n \in U$ and $(v_n)_n \in V$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n \pm \epsilon$. Then let $L = \lim_{n\to\infty} u_n \pm \epsilon/2$. Then $L \notin U$ and $L \notin V$ (thus $L \notin G$) but $\exists \bar{\epsilon}$ such that $L \pm \bar{\epsilon} \in U \subset G$ and $L \mp \bar{\epsilon} \in V \subset G$. Thus G is not an interval. $\square$

b)

Show that $(\mathbb{R}, |\cdot|)$ *is homeomorphic to the open interval* $(0, 1) \subset \mathbb{R}$ *with the relative topology.*

Define the function f as

$$f(x) = \frac{\tan^{-1}(x) + \frac{\pi}{2}}{\pi}$$

Then since $\tan^{-1}(x)$ is a continuous bijection from $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$, then since $f(x)$ is a translation of $\tan^{-1}(x)$, then $f(x)$ is a continuous bijection from $\mathbb{R}$ to $(0, 1)$. In addition,

$$f^{-1}(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$$

is a continuous bijection from $(0, 1)$ to $\mathbb{R}$. Thus $(\mathbb{R}, |\cdot|)$ is homeomorphic to $(0, 1)$.

c)

Show that $(\mathbb{R}, |\cdot|)$ is not homeomorphic to $(\mathbb{R}^2, \|\cdot\|)$, where $\|\cdot\|$ is the Euclidean norm.

Suppose f is a homeomorphism from $(\mathbb{R}, |\cdot|)$ to $(\mathbb{R}^2, \|\cdot\|)$. Then $\mathbb{R}$ and $\mathbb{R}^2$ are indistinguishable as topological spaces. Then $\mathbb{R} \setminus \{x_0\}$ and $\mathbb{R}^2 \setminus \{f(x_0)\}$ are indistinguishable as topological spaces. This is a contradiction since $\mathbb{R} \setminus \{x_0\}$ is not connected, but $\mathbb{R}^2 \setminus \{f(x_0)\}$ is connected. Thus $(\mathbb{R}, |\cdot|)$ is not homeomorphic to $(\mathbb{R}^2, \|\cdot\|)$. $\square$

Problem 5

Prove that the sequence (f_n) defined in Example 5.11 in the textbook is a Schauder basis of $(C([0,1]), \|\cdot\|_\infty)$.

Let f be a continuous function on $[0,1]$. Since $[0,1]$ is compact, f is uniformly continuous, and thus $\forall \epsilon$, $\exists \delta$ such that $\forall x, x_0 \in [0,1]$, $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$.

Now choose $x \in [0,1]$ and k such that $2^{-k} < \delta$, and find m such that

$$x_m = \frac{m}{2^k} < x \leq \frac{m+1}{2^k} = x_{m+1}$$

By the definition of the sequence f_n, we can find c_n such that

$$f(x_m) - \sum_{n=0}^{\infty} c_n f_n(x_m) = 0 \qquad \text{and} \qquad f(x_{m+1}) - \sum_{n=0}^{\infty} c_n f_n(x_{m+1}) = 0$$

However, since $x_m = \frac{m}{2^k}$ and $x_{m+1} = \frac{m+1}{2^k}$, then $\forall \ell > k$, $f_\ell(x_m) = f_\ell(x_{m+1}) = 0$. Thus,

$$f(x_m) - \sum_{n=0}^{k} c_n f_n(x_m) = 0 \qquad \text{and} \qquad f(x_{m+1}) - \sum_{n=0}^{k} c_n f_n(x_{m+1}) = 0$$

Then

$$\begin{aligned}\left|f(x) - \sum_{n=0}^{k} c_n f_n(x)\right| &\leq |f(x) - f(x_m)| + \left|f(x_m) - \sum_{n=0}^{k} c_n f_n(x_m)\right| + \left|\sum_{n=0}^{k} c_n f_n(x_m) - \sum_{n=0}^{k} c_n f_n(x)\right| \\ &< \frac{\epsilon}{2} + 0 + \left|\sum_{n=0}^{k} c_n f_n(x_m) - \sum_{n=0}^{k} c_n f_n(x)\right|\end{aligned}$$

Since $\sum_{n=0}^{k} c_n f_n$ is a linear on $[x_m, x_{m+1}]$,

$$\left|\sum_{n=0}^{k} c_n f_n(x_m) - \sum_{n=0}^{k} c_n f_n(x)\right| \leq \left|\sum_{n=0}^{k} c_n f_n(x_m) - \sum_{n=0}^{k} c_n f_n(x_{m+1})\right|$$

and thus

$$\left|f(x) - \sum_{n=0}^{k} c_n f_n(x)\right| < \frac{\epsilon}{2} + \left|\sum_{n=0}^{k} c_n f_n(x_m) - \sum_{n=0}^{k} c_n f_n(x_{m+1})\right|$$

$$
\begin{aligned}
&\leq \frac{\epsilon}{2} + \left|\sum_{n=0}^{k} c_n f_n(x_m) - f(x_m)\right| + |f(x_m) - f(x_{m+1})| + \left|f(x_{m+1}) - \sum_{n=0}^{k} c_n f_n(x_{m+1})\right| \\
&< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} + 0 \\
&= \epsilon
\end{aligned}
$$

since $|x_m - x_{m+1}| < \delta$. Thus $\sum_{n=0}^{\infty} c_n f_n(x) = f(x)$ for each $x \in [0,1]$, i.e. $\sum_{n=0}^{\infty} c_n f_n$ converges pointwise to f. Since f is continuous, $\sum_{n=0}^{\infty} c_n f_n$ converges uniformly to f. Thus (f_n) is a Schauder basis of $C([0,1], \|\cdot\|_\infty)$. $\square$

Problem 6

For $1 \leq p \leq \infty$, consider the Banach space $\ell^p(\mathbb{N})$ defined in Example 5.5 of the textbook. The set $\ell_c(\mathbb{N})$ is all sequences of the form $(x_1, x_2, \dots, x_n, 0, 0, \dots)$ whose terms vanish from some point onwards is an infinite-dimensional linear subspace of $\ell^p(\mathbb{N})$ for any $1 \leq p \leq \infty$.

a)

Show that $\ell_c(\mathbb{N})$ is not closed in $\ell^p(\mathbb{N})$, so it is not a Banach space with respect to the norm of $\ell^p(\mathbb{N})$.

Consider the sequence in $\ell_c(\mathbb{N})$:

$$
\begin{aligned}
x_1 &= \left(\frac{1}{2}, 0, 0, 0, \dots\right) \\
x_2 &= \left(\frac{1}{2}, \frac{1}{4}, 0, 0, \dots\right) \\
x_3 &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, \dots\right) \\
&\vdots \\
x_n &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, 0, 0, \dots\right) \\
&\vdots
\end{aligned}
$$

Then $x = \lim_{i\to\infty} x_i = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right) \notin \ell_c(\mathbb{N})$, but $x \in \ell^p(\mathbb{N})$ since $\sum_{i=1}^{\infty} (2^{-i})^p < \infty$. So there is a limit point x of $\ell_c(\mathbb{N})$ that is not contained in $\ell^p(\mathbb{N})$, but is contained in $\ell^p(\mathbb{N})$. Thus $\ell_c(\mathbb{N})$ is not closed in $\ell^p(\mathbb{N})$. $\square$

b)

Show that $\ell_c(\mathbb{N})$ is dense in $\ell^p(\mathbb{N})$ for $1 \leq p \leq \infty$.

Let $y = (y_1, y_2, \dots) \in \ell^p(\mathbb{N})$. Then construct the sequence

$$x_1 = (y_1, 0, 0, 0, \dots)$$

$$
\begin{aligned}
x_2 &= (y_1, y_2, 0, 0, \dots) \\
x_3 &= (y_1, y_2, y_3, 0, \dots) \\
&\vdots \\
x_n &= (y_1, y_2, \dots, y_n, 0, 0, \dots) \\
&\vdots
\end{aligned}
$$

Clearly $x_i \in \ell_c(\mathbb{N})$. Since $\sum_{i=1}^{\infty} |y_i| < \infty$, then $y_i \to 0$. Then $\forall \epsilon$, $\exists N$ such that $|y_n| < \epsilon$ for each $n \geq N$. Then $x_i \to y$. Thus $\ell_c(\mathbb{N})$ is dense in $\ell^p(\mathbb{N})$ for $1 \leq p \leq \infty$.

c)

Find the closure of $\ell_c(\mathbb{N})$ in $\ell^\infty(\mathbb{N})$.

Let $y = (y_1, y_2, \dots) \in \ell^\infty(\mathbb{N})$ such that $\lim_{i\to\infty} |y_i| = 0$. Then construct the sequence $(x_n)_n \in \ell_c(\mathbb{N})$:

$$
\begin{aligned}
x_1 &= (y_1, 0, 0, 0, \dots) \\
x_2 &= (y_1, y_2, 0, 0, \dots) \\
x_3 &= (y_1, y_2, y_3, 0, \dots) \\
&\vdots \\
x_n &= (y_1, y_2, \dots, y_n, 0, 0, \dots) \\
&\vdots
\end{aligned}
$$

Then choose $\epsilon > 0$. Since $y_i \to 0$, then $\exists N$ such that $n \geq N \implies |y_i| < \epsilon$. Also, if $n \geq N$, then $\|x_n - y\|_\infty = \|y\|_\infty < \epsilon$. So y is a limit point of a sequence in $\ell_c(\mathbb{N})$, and thus $y \in \overline{\ell_c(\mathbb{N})}$.

Now let $y = (y_1, y_2, \dots) \in \ell^\infty(\mathbb{N})$ such that $\lim_{i\to\infty} |y_i| \neq 0$. Then $\exists \epsilon > 0$ such that $\forall M > 0$, $\exists i > M$ such that $|y_i| > \epsilon$. Assume there is a sequence $(x_n)_n \in \ell_c(\mathbb{N})$ such that $x_n \to y$ (i.e. $x_n - y \to 0$). Denote each x_n as

$$x_n = (x_{n,1}, x_{n,2}, x_{n,3}, \dots)$$

Then $\exists N$ such that $n \geq N \implies \|x_n - y\|_\infty < \epsilon$. However, by the definition of $\ell_c(\mathbb{N})$, for each n, $\exists M_n$ such that $m \geq M_n \implies x_{n,m} = 0$. Then for each n, $\|x_n - y\|_\infty > \epsilon$, which is a contradiction. Thus there is no sequence in $\ell_c(\mathbb{N})$ that converges to y, and thus $y \notin \overline{\ell_c(\mathbb{N})}$.

Thus $\overline{\ell_c(\mathbb{N})} = \{(a_n) \in \ell_c(\mathbb{N}) \mid \lim_{i\to\infty} |a_i| = 0\}$. $\square$