HW #7

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Problem 1

Let X be a Banach space and consider the map exp : $\mathcal{B}(X) \to \mathcal{B}(X)$*, defined by the series*

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n \tag{1}$$

where, by convention, $T^0 = \mathbb{1}$.

a)

Prove that the map exp *is well-defined by showing that this series (1) converges absolutely for all* $T \in \mathcal{B}(X)$.

Since, by part **b)**, $\|\exp T\| \leq \exp \|T\|$. Then, since

$$\exp \|T\| = \sum_{n=0}^{\infty} \frac{1}{n!} \|T\|^n$$

converges for all finite $\|T\|$, then $\|\exp T\|$ converges absolutely for all $T \in \mathcal{B}(X)$. $\square$

b)

Prove the bound

$$\|\exp(T)\| \leq e^{\|T\|}, \quad T \in \mathcal{B}(X)$$

$$\|\exp T\| = \left\| \sum_{n=0}^{\infty} \frac{1}{n!} T^n \right\| \leq \sum_{n=0}^{\infty} \left\| \frac{1}{n!} T^n \right\| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \|T\|^n = \exp \|T\|$$

$\square$

c)

Prove that exp *is a continuous map when $\mathcal{B}(X)$ is considered with the uniform operator topology.*

We want to show

$$\lim_{T\to T_0} \exp T = \exp T_0$$

for every $T_0 \in \mathcal{B}(X)$. Define $P_N : \mathcal{B}(X) \to \mathcal{B}(X)$ by

$$P_N T = \sum_{n=0}^{N} \frac{1}{n!} T^n$$

$P_N \to \exp$ uniformly because

$$\|P_n - \exp\|_{\mathrm{OP}} = \sup_{T=\|1\|} \|P_N T - \exp T\| = \sup_{\|T\|=1} \left\| \sum_{n=N+1}^{\infty} \frac{1}{n!} T^n \right\| \leq \sup_{\|T\|=1} \sum_{n=N+1}^{\infty} \frac{1}{n!} \|T\|^n = \sum_{n=N+1}^{\infty} \frac{1}{n!}$$

which is arbitrarily small as $N \to \infty$. Thus $P_N \to \exp$ strongly, so $\exp T = \lim_{N\to\infty} P_N T \;\; \forall T \in \mathcal{B}(X)$.

$$\lim_{T\to T_0} \exp T = \lim_{T\to T_0} \lim_{N\to\infty} P_N T = \lim_{N\to\infty} \lim_{T\to T_0} P_N T = \lim_{N\to\infty} P_N \lim_{T\to T_0} T = \lim_{N\to\infty} P_N T_0 = \exp T_0$$

Thus exp is a continuous map when $\mathcal{B}(X)$ is considered with the uniform operator topology. $\square$

Problem 2

Let $\mathcal{B}$ be a Banach space and consider a continuous function $A : \mathbb{R} \to \mathcal{B}$.

a)

For $a < b \in \mathbb{R}$, $\int_a^b A(t)\mathrm{d}t$ can be defined as the limit of Riemann sums. E.g., show that the following limit exists in the uniform topology:

$$\int_a^b A(t)\mathrm{d}t = \lim_{N\to\infty} 2^{-N}(b-a) \sum_{n=1}^{2^N} A(a + n2^{-N}(b-a))$$

For a given N, for $n = 0, 1, \ldots, 2^N$, define $x_n \equiv a + n2^{-N}(b-a)$. Then

$$\int_a^b A(t)\mathrm{d}t = \sum_{n=0}^{2^N-1} \int_{x_n}^{x_{n+1}} A(t)\mathrm{d}t$$

Note that by continuity of A, for each $x \in \mathbb{R}$ and $\epsilon > 0$, $\exists \delta$ such that $|x - y| < \delta \implies \|A(x) - A(y)\| < \epsilon$. Then if we choose N such that $\delta > (b-a)2^{-N}$, then it suffices to show

$$\left\| (b-a)2^{-N} \sum_{n=0}^{2^N-1} A(x_n) - \sum_{n=0}^{2^N-1} \int_{x_n}^{x_{n+1}} A(t)\mathrm{d}t \right\| < \epsilon$$

However,

$$\begin{aligned}
\left\|(b-a)2^{-N}\sum_{n=0}^{2^N-1}A(x_n)-\sum_{n=0}^{2^N-1}\int_{x_n}^{x_{n+1}}A(t)\mathrm{d}t\right\| &= \left\|\sum_{n=0}^{2^N-1}\left[(b-a)2^{-N}A(x_n)-\int_{x_n}^{x_{n+1}}A(t)\mathrm{d}t\right]\right\| \\
&\leq \sum_{n=0}^{2^N-1}\left\|(b-a)2^{-N}[A(x_n)-A(y_n)]\right\| \\
&\quad \text{for some } y_n\in[x_n,x_{n+1}] \text{ by the intermediate} \\
&\quad \text{value theorem} \\
&= (b-a)2^{-N}\sum_{n=0}^{2^N-1}\|A(x_n)-A(y_n)\| \\
&< (b-a)2^{-N}2^N\frac{\epsilon}{b-a} \\
&= \epsilon
\end{aligned}$$

Thus $\int_a^b A(t)\mathrm{d}t$ is well defined as the limit of Riemann sums. □

b)

$A(\cdot)$ is called differentiable on an interval I if for all $t\in I$, the following limit exists in the uniform topology:

$$A'(t):=\frac{\mathrm{d}}{\mathrm{d}t}A(t):=\lim_{h\to 0}\frac{A(t+h)-A(t)}{h}$$

and the function $t\mapsto A'(t)$ is continuous on I. Let X be a Banach space, $A\in\mathcal{B}(X)$, and consider the map $t\mapsto\exp(tA)$. Show that A is differentiable and that its derivative is given by the following two expressions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA)=A\exp(tA)=\exp(tA)A$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA) &= \lim_{h\to 0}\frac{\exp((t+h)A)-\exp(tA)}{h} \\
&= \lim_{h\to 0}\frac{\exp(tA+hA)-\exp(tA)}{h}
\end{aligned}$$

But the operators tA and hA commute since

$$tAhA=thA^2=htA^2=hAtA$$

thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA) &= \lim_{h\to 0}\frac{\exp(tA)\exp(hA)-\exp(tA)}{h} \\
&= \lim_{h\to 0}\frac{\exp(tA)[\exp(hA)-I]}{h}
\end{aligned}$$

$$
\begin{aligned}
&= \exp(tA) \lim_{h\to 0} \frac{\left[I + hA + \dfrac{h^2A^2}{2!} + \dfrac{h^3A^3}{3!} + \dots\right] - I}{h} \\
&= \exp(tA) \lim_{h\to 0} \left[A + \frac{h^2A}{2!} + \frac{h^3A^2}{3!} + \dots\right] \\
&= \exp(tA) \lim_{h\to 0} A \\
&= \exp(tA)A
\end{aligned}
$$

However, $\exp(tA)$ and A are commutative since tA and A are commutative.

$$
\begin{aligned}
\exp(tA)A &= \left[I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \dots\right]A \\
&= A + tA^2 + \frac{t^2A^3}{2!} + \frac{t^3A^4}{4!} + \dots \\
&= A\left[I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \dots\right] \\
&= A\exp(tA)
\end{aligned}
$$

c)

Let $A(t)$ be a differentiable function $\mathbb{R} \supset [0,1] \to \mathcal{B}(X)$. Show that $\exp(A(t))$ is differentiable and that its derivative satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(A(t)) = \int_0^1 \exp(s(A(t)))A'(t)\exp((1-s)A(t))\mathrm{d}s$$

(Hint: consider $B(s) = \exp(s(A(t+h)))\exp((1-s)A(t))$ and note that $A(t+h) - A(t) = B(1) - B(0)$).

Consider

$$B(s) = \exp(s(A(t+h)))\exp((1-s)A(t))$$

and note that

$$B(1) - B(0) = \exp(A(t+h)) - \exp(A(t))$$

Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\exp(A(t)) &= \lim_{h\to 0} \frac{\exp(A(t+h)) - \exp(A(t))}{h} \\
&= \lim_{h\to 0} \frac{B(1) - B(0)}{h} \\
&= \lim_{h\to 0} \frac{\displaystyle\int_0^1 B'(s)\mathrm{d}s}{h}
\end{aligned}
$$

by the Fundamental Theorem of Calculus for Banach Spaces. Note that

$$B'(s) = [A(t+h) - A(t)]B(s)$$

Thus,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\exp(A(t)) &= \lim_{h\to 0}\frac{\displaystyle\int_0^1 [A(t+h)-A(t)]B(s)\mathrm{d}s}{h} \\
&= \int_0^1 \lim_{h\to 0}\frac{A(t+h)-A(t)}{h}B(s)\mathrm{d}s \\
&= \int_0^1 \lim_{h\to 0}\frac{A(t+h)-A(t)}{h}\exp(s(A(t+h)))\exp((1-s)A(t))\mathrm{d}s \\
&= \int_0^1 A'(t)\exp(s(A(t)))\exp((1-s)A(t))\mathrm{d}s \\
&= \int_0^1 \exp(s(A(t)))A'(t)\exp((1-s)A(t))\mathrm{d}s
\end{aligned}
$$

□

Problem 3

Let X be a Banach space and consider the $T \in \mathcal{B}(X)$ such that $\|T\| < 1$. Prove that $\mathbb{1}+T$ is invertible and that its inverse is given by the following uniformly convergent series:

$$(\mathbb{1}+T)^{-1} = \sum_{n=0}^{\infty}(-1)^n T^n$$

Let $\|T\| = 0$. Then $T = \mathbb{0}$. Then

$$(\mathbb{1}+\mathbb{0})^{-1} = \mathbb{1}^{-1} = \mathbb{1} = \mathbb{0}^0 = \sum_{n=0}^{\infty} 0^n = \sum_{n=0}^{\infty}(-1)^n\mathbb{1}^n$$

Now let $0 < \|T\| < 1$. Then assume $x \in \ker(\mathbb{1}+T)$ and $x \neq 0$. Then

$$(\mathbb{1}+T)x = 0 \quad\Longrightarrow\quad x + Tx = 0 \quad\Longrightarrow\quad Tx = -x \quad\Longrightarrow\quad \|T\| \geq 1 \Longrightarrow\!\Longleftarrow$$

Thus $x \neq 0$ and $\ker(\mathbb{1}+T) = \{0\}$. Thus $\mathbb{1}+T$ is invertible and

$$
\begin{aligned}
(\mathbb{1}+T)\sum_{n=0}^{\infty}(-1)^n T^n &= (\mathbb{1}+T)\left[\mathbb{1} - T + T^2 - T^3 + \dots\right] \\
&= \left[\mathbb{1} - T + T^2 - T^3 + \dots\right] + \left[T - T^2 + T^3 - T^4\right]
\end{aligned}
$$

However, $\mathbb{1} + T + T^2 + T^3 + \dots$ is absolutely convergent, and thus we can rearrange the terms of the series.

$$(\mathbb{1}+T)\sum_{n=0}^{\infty}(-1)^n T^n = \mathbb{1} + (T-T) + (T^2 - T^2) + \cdots = \mathbb{1}$$

Similarly, $\left[\sum_{n=0}^{\infty}(-1)^n T^n\right](\mathbb{1}+T) = \mathbb{1}$. Thus

$$(\mathbb{1}+T)^{-1} = \sum_{n=0}^{\infty}(-1)^n T^n$$

□