Name: ______

MAT201A University of California, Davis Fall 2015

Homework # 6

(Due Monday, November 16)

Problem 1. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Define the map $T : C([0,1] \to C([0,1])$ by

$$(Tf)(x) = \int_0^1 k(x,y)f(y)dy, \text{ for all } f \in C([0,1]). \tag{1}$$

a) Let $\|T\|$ denote the operator norm of T. Prove

$$\|T\| = \sup_{x\in[0,1]} \int_0^1 |k(x,y)|dy. \tag{2}$$

b) Argue that the sup in (2) is attained in some $x \in [0,1]$.

c) Is it possible that $\|T\| = 1$ but $\|T^2\| = 0$? Prove your answer.

Problem 2. Study Section 5.4 of the textbook.

Problem 3. Let X be the Banach space $l^2(\mathbb{N})$, defined by

$$l^2(\mathbb{N}) = \{z = (z_n)_{n=1}^\infty \mid z_n \in \mathbb{C}, \sum_{n=1}^\infty |z_n|^2 < \infty\},$$

and with the norm given by

$$\|z\| = (\sum_{n=1}^\infty |z_n|^2)^{1/2}.$$

For $m = 1, 2, \ldots$, define $e_m \in X$ to be the sequence with elements $(e_m)_n = \delta_{n,m}$, and define $P_m : X \to X$ by $P_m z = z_m e_m$, for all $z \in X$.

a) Prove that $P_m \in \mathcal{B}(X)$, for all $m \geq 1$.

b) Verify $P_m P_n = \delta_{n,m} P_m$, for all $n, m \geq 1$.

c) Prove that $\|P_m\| = 1$, for all $m \geq 1$.

d) For $m \geq 1$, define S_m by

$$S_m = \sum_{k=1}^m P_k.$$

Calculate $S_m S_n$, for all $n, m \geq 1$.

e) Show that $\|S_m\| = 1$, for all $n \geq 1$.