HW #1

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Problem 1

Let (X, d) be a metric space, and let $x, y, w, z \in X$.

a)

Prove that $d(x, y) \geq |d(x, z) - d(z, y)|$.

By the triangle inequality, $d(x, z) \leq d(x, y) + d(z, y)$ and $d(z, y) \leq d(x, y) + d(x, z)$. These are equivalent to $d(x, y) \geq d(x, z) - d(z, y)$ and $d(x, y) \geq d(z, y) - d(x, z)$. Thus, $d(x, y) \geq |d(x, z) - d(z, y)|$. $\square$

b)

Prove that $d(x, y) + d(z, w) \geq |d(x, z) - d(y, w)|$.

By the triangle inequality, $d(x, z) \leq d(x, w) + d(z, w)$ and $d(x, w) \leq d(x, y) + d(y, w)$. By substitution, $d(x, y) + d(y, w) + d(w, z) \geq d(x, z)$, or

$$d(x, y) + d(z, w) \geq d(x, z) - d(y, w) \tag{1}$$

Again by the triangle inequality, $d(y, z) \leq d(x, y) + d(x, z)$ and $d(y, z) \leq d(y, z) + d(w, z)$. By substitution, $d(x, y) + d(z, w) + d(x, z) \geq d(y, w)$, or

$$d(x, y) + d(z, w) \geq d(y, w) - d(x, z) \tag{2}$$

Thus, combining (1) and (2),

$$d(x, y) + d(z, w) \geq |d(x, z) - d(y, w)| \tag{3}$$

c)

Let (x_n) and (y_n) be converging sequences in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Prove that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

By the definition of limits, $\forall \frac{\epsilon}{2} > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n > N_1 \implies d(x_n, x) < \frac{\epsilon}{2}$ and $n > N_2 \implies d(y_n, y) < \frac{\epsilon}{2}$. Then for $n > \max\{N_1, N_2\}$, and by the triangle inequality applied twice, $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y)$, or

$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{4}$$

Again, by the triangle inequality applied twice, $d(x, y) \leq d(x_n, x) + d(x_n, y_n) + d(y_n, y)$, or

$$d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{5}$$

Thus, combining (4) and (5),

$$|d(x_n, y_n) - d(x, y)| < \epsilon \tag{6}$$

which proves $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$. $\square$

Problem 2

Show that the limit of a convergent sequence in a metric space is unique. I.e., if, for a sequence (x_n) in a metric space (X, d), and $x, y \in X$, $x_n \to x$ and $x_n \to y$, then $x = y$.

Assume $x \neq y$. Then $d(x, y) > 0$. By the definition of limits, $\forall \epsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n > N_1 \implies d(x_n, x) < \epsilon$ and $n > N_2 \implies d(x_n, y) < \epsilon$. Now suppose $\epsilon = \frac{1}{2} d(x, y)$. Then if $n > \max\{N_1, N_2\}$, then $d(x_n, x) < \frac{1}{2} d(x, y)$ and $d(x_n, y) < \frac{1}{2} d(x, y)$. Adding these inequalities yields

$$d(x_n, x) + d(x_n, y) < d(x, y) \tag{7}$$

which contradicts the triangle inequality. Thus, $x = y$, i.e. the limit of a convergent sequence in a metric space is unique. $\square$

Problem 3

Let (a_n) be a sequence in $\mathbb{R}$.

a)

Prove that there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ of (a_n) such that $\lim_{k\to\infty} a_{n_k} = \underline{\lim}\ a_n$.

There are three cases, either $\underline{\lim}\ a_n = \infty$, $-\infty$, or $L \in \mathbb{R}$. It suffices to show we can construct a suitable subsequence in each case.

Case 1

Suppose $\underline{\lim}\ a_n = -\infty$. First, choose an arbitrary a_{n_1}. Next, choose a_{n_2} such that $a_{n_2} < -2$. Then choose a_{n_3} such that $a_{n_3} < -3$, and so on such that $a_{n_k} < -k$. Then (a_{n_k}) diverges to $-\infty$, i.e. $\lim_{k\to\infty} a_{n_k} = -\infty = \underline{\lim}\ a_n$.

Case 2

Suppose $\underline{\lim}\ a_n = \infty$. First, choose an arbitrary a_{n_1}. Next, choose a_{n_2} such that $a_{n_2} \geq a_{n_1}$. Then choose a_{n_3} such that $a_{n_3} \geq a_{n_2}$, and so on such that $a_{n_1} \leq a_{n_2} \leq \cdots \leq a_{n_k} \leq \ldots$. This sequence (a_{n_k}) does not have a real limit, since that would contradict $\underline{\lim}\ a_n = \infty$. Thus (a_{n_k}) diverges to ∞, i.e. $\lim_{k\to\infty} a_{n_k} = \infty = \underline{\lim}\ a_n$.

Case 3

Suppose $\underline{\lim}\ a_n = L \in \mathbb{R}$. First, choose an arbitrary a_{n_1}. Next, choose a_{n_2} such that $|L - a_{n_2}| < \frac{1}{2}$. Then choose a_{n_3} such that $|L - a_{n_3}| < \frac{1}{3}$ and so on such that $|L - a_{n_k}| < \frac{1}{k}$. Then by the Archimedian principle, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $k > N \implies |L - a_{n_k}| < \epsilon$, i.e. $\lim_{k\to\infty} a_{n_k} = L = \underline{\lim}\ a_n$.

$\square$

b)

Prove that (a_n) converges to $a \in \mathbb{R}$ if and only if $\underline{\lim}\ a_n = \overline{\lim}\ a_n = a$.

"$\implies$"

Let $(a_n) \to a$. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - a| < \epsilon$. Then

$$\begin{aligned}
& a - \epsilon < \inf\{a_k | k \geq n\} < a + \epsilon \\
\iff & -\epsilon < \inf\{a_k | k \geq n\} - a < \epsilon \\
\iff & |\inf\{a_k | k \geq n\} - a| < \epsilon \\
\iff & \lim_{n\to\infty} (\inf\{a_k | k \geq n\}) = a \\
\iff & \underline{\lim}\ a_n = a
\end{aligned}$$

Similarly,

$$\begin{aligned}
& a - \epsilon < \sup\{a_k | k \geq n\} < a + \epsilon \\
\iff & -\epsilon < \sup\{a_k | k \geq n\} - a < \epsilon \\
\iff & |\sup\{a_k | k \geq n\} - a| < \epsilon \\
\iff & \lim_{n\to\infty} (\sup\{a_k | k \geq n\}) = a \\
\iff & \overline{\lim}\ a_n = a
\end{aligned}$$

Thus, $\underline{\lim}\ a_n = \overline{\lim}\ a_n = a$

"$\impliedby$"

Let $\underline{\lim}\ a_n = \overline{\lim}\ a_n = a$. Then $\forall \epsilon > 0$, $\exists N_1, N_2$ such that $n \geq N_1 \implies |\inf\{a_k | k \geq n\} - a| < \epsilon$ and $n \geq N_2 \implies |\sup\{a_k | k \geq n\} - a| < \epsilon$. Let $\epsilon > 0$ and $n \geq \max\{N_1, N_2\}$. For ease, define $K \equiv \{a_k | k \geq n\}$ Since the infimum of a set is always less than or equal to the supremum of that set, we can write

$$-\epsilon < \inf K - a \leq \sup K - a < \epsilon$$

$$\iff -\epsilon + a < \inf K \leq \sup K < \epsilon + a$$

By the definition of infimum and supremum,

$$\begin{aligned} -\epsilon + a < \inf K \leq a_n &\leq \sup K < \epsilon + a \\ \implies \epsilon + a < a_n &< \epsilon + a \\ \iff |a_n - a| &< \epsilon \end{aligned}$$

Thus, $(a_n) \to a$.

Thus, $(a_n) \to a \iff \underline{\lim}\ a_n = \overline{\lim}\ a_n = a$. $\square$

Problem 4

Let (X, d) be a metric space. Prove the statements in Proposition 1.37 in the textbook:

a)

The empty set $\emptyset$ and X itself are both open and closed sets in (X, d).

It is vacuously true that the empty set $\emptyset$ is open. Thus X is closed. Let $x \in X$ and choose $r \in \mathbb{R}$. Then $y \in B_r(x) \implies y \in X$ since $B_r(x) \subseteq X$. Thus X is open and the empty set $\emptyset$ is closed. $\square$

b)

The intersection of a finite collection of open sets is open.

Let $A = \bigcap\limits_{k=1}^{n} A_k$ be the intersection of a finite collection of open sets, and let $x \in A$. Then $\exists r_1, \ldots r_n$ such that $y \in B_{r_i}(x) \implies y \in A_i$ for $i = 1, \ldots, n$. Then let $r = \min\{r_1, \ldots r_n\}$. Then $y \in B_r(x) \implies y \in B_{r_i}(x)$ for $i = 1, \ldots n$. Thus $y \in A$. Thus the intersection of a finite collection of open sets is open. $\square$

c)

The union of an arbitrary collection of open sets is open.

Let $A = \bigcup\limits_{i \in I} A_i$ be the union of an arbitrary collection of open sets, and let $x \in A$. Then $x \in A_k$ for some $k \in I$. Since A_k is open, $\exists r$ such that $y \in B_r(x) \implies y \in A_k$. But since $A_k \subseteq A$, $y \in A$. Thus the union of an arbitrary collection of open sets is open. $\square$

d)

The union of a finite collection of closed sets is closed.

Let $A = \bigcup_{k=1}^{n} A_k$ be the union of a finite collection of closed sets. By De Morgan's Law in Set Theory, $A^C = \bigcap_{k=1}^{n} A_k^C$. A^C is open since each A_k^C is open and the intersection of a finite collection of open sets is open. Since A_C is open, A is closed. Thus the union of a finite collection of closed sets is closed. $\square$

e)

The intersection of an arbitrary collection of closed sets is closed.

Let $A = \bigcap_{i \in I} A_i$ be the intersection of an arbitrary collection of closed sets. By De Morgan's Law in Set Theory, $A^C = \bigcup_{i \in I} A_i^C$. A^C is open since each A_i^C is open and the union of an arbitrary collection of open sets is open. Since A^C is open, A is closed. Thus the intersection of an arbitrary collection of closed sets is closed. $\square$

Problem 5

Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$ a ontinuous function, and $B \subset Y$ a closed set. Prove that A defined by

$$A = \{x \in X | f(x) \in B\}$$

is a closed set.

Let $a \in A^C$. Then $f(a) \notin B$. Then $f(a) \in B^C$. Since B is closed, B^C is open, and thus $\exists \epsilon$ such that $y \in B_\epsilon(f(a)) \implies y \in B^C$. By the definition of continuous functions, $\exists \delta$ such that $x \in B_\delta(a) \implies f(x) \in B_\epsilon(f(a))$, which then implies $f(x) \in B^C$. Thus, $f(x) \notin B \implies x \notin A \implies x \in A^C$. Thus A^C is open, which implies A is closed. $\square$

Problem 6

Let X be a Banach space and let (x_n) be a sequence in X such that $\sum_{n=1}^{\infty} \|x_n\| = 1$.

a)

Prove that the series $\sum_{n=1}^{\infty} x_n$ converges to a limit $x \in X$.

Convergence of a series is equivalent to convergence of the sequence of partial sums, so since $\sum_{n=1}^{\infty} \|x_n\| = 1$, then $\lim_{k\to\infty} \sum_{n=1}^{k} \|x_n\| = 1$. The sequence is Cauchy since the real numbers are complete, so $\forall \epsilon > 0$, $\exists N$ such that $a \geq b \geq N \implies \left| \sum_{n=1}^{a} \|x_n\| - \sum_{n=1}^{b} \|x_n\| \right| < \epsilon$, or $\left| \sum_{n=b+1}^{a} \|x_n\| \right| < \epsilon$. Since the norm is always positive, the sum of norms is positive. Thus, it can also be written without the absolute value: $\sum_{n=b+1}^{a} \|x_n\| < \epsilon$.

Now let $\epsilon > 0$ and $a \geq b \geq N$. Then

$$\begin{aligned}\left\|\sum_{n=1}^{a} x_n - \sum_{n-1}^{b} x_n\right\| &= \left\|\sum_{n=b+1}^{a} x_n\right\| \\ &\leq \sum_{n=b+1}^{a} \|x_n\| \quad \text{(by the triangle inequality of normed metric spaces)} \\ &< \epsilon\end{aligned}$$

Thus $\left(\sum_{n=1}^{k} x_n\right)_k$ is a Cauchy sequence in X, and since all Banach spaces are complete, $\left(\sum_{n=1}^{k} x_n\right)_k$ converges to a limit $x \in X$. Any sequence of partial sums can be written as a series, and so $\sum_{n=1}^{\infty} x_n$ converges to a limit $x \in X$. $\square$

b)

Prove that for any subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n), the series $\sum_{k=1}^{\infty} x_{n_k}$ also converges and that the norm of its limit is bounded by 1.

By part a), $\sum_{n=1}^{\infty} x_n$ is convergent, and therefore Cauchy. Thus $\forall \epsilon > 0$, $\exists N$ such that $a \geq b \geq N \implies \left|\sum_{n=b+1}^{a} x_n\right| < \epsilon$. Then let $\epsilon > 0$ and pick $c, d \in \mathbb{N}$ such that $n_c \geq n_d \geq N$. Then $\left|\sum_{k=d+1}^{c} x_{n_k}\right| < \epsilon$. Thus, the subsequence $(x_{n_k})_{k=1}^{\infty}$ is a Cauchy sequence in X, and since all Banach spaces are complete, $(x_{n_k})_{k=1}^{\infty}$ is convergent.

Now consider the norm of the limit of partial sums of the subsequence: $\left\|\lim_{l\to\infty} \sum_{k=1}^{l} x_{n_k}\right\|$. By the triangle inequality of normed metric spaces,

$$\left\|\lim_{l\to\infty} \sum_{k=1}^{l} x_{n_k}\right\| \leq \lim_{l\to\infty} \sum_{k=1}^{l} \|x_{n_k}\|$$

Since all norms are positive

$$\begin{aligned}\lim_{l\to\infty} \sum_{k=1}^{l} \|x_{n_k}\| &\leq \lim_{k\to\infty} \sum_{n=1}^{k} \|x_{n_k}\| \\ &= \sum_{n=1}^{\infty} \|x_{n_k}\| = 1\end{aligned}$$

Thus, the norm of the limit of the subseries is bounded by 1. $\square$