HW #8

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Problem 1

Consider the Banach space $C([0,1])$ with the supremum norm. For $x \in [0,1]$ let δ_x denote the linear functional on $C([0,1])$ given by

$$\delta_x(f) = f(x), \qquad \text{for all } f \in C([0,1])$$

a)

Show that $\|\delta_x\| = 1$.

b)

Show that there does not exist a Riemann integrable function $k : [0,1] \to \mathbb{R}$, such that

$$\delta_x(f) = \int_0^1 k(y)f(y)\mathrm{d}y, \qquad \text{for all } f \in C([0,1])$$

Problem 2

Prove that there does not exist an inner product on $C([0,1])$ such that the supremum norm is derived from this inner product.

Problem 3

Let $\mathcal{H}$ be a Hilbert space and let M be a subset of $\mathcal{H}$.

a)

Prove that $M^\perp$ is a closed linear subspace of $\mathcal{H}$.

First we show $M^\perp$ is a linear subspace. Let $x, y \in M^\perp$ and $\lambda, \mu \in \mathbb{C}$. Then for each $m \in M$,

$$\langle m, \lambda x + \mu y\rangle = \lambda\langle m, x\rangle + \mu\langle m, y\rangle = 0$$

Thus $\lambda x + \mu y \in M^\perp$. Thus $M^\perp$ is a linear subspace of $\mathcal{H}$. Next, let (x_n) be a convergent sequence in $M^\perp$, and let $x_n \to x$. Then for each $m \in M$,

$$\langle x, m \rangle = \langle \lim_{n\to\infty} x_n, m \rangle$$

but since $\langle \cdot, \cdot \rangle$ is continuous,

$$\langle \lim_{n\to\infty} x_n, m \rangle = \lim_{n\to\infty} \langle x_n, m \rangle = \lim_{n\to\infty} 0 = 0$$

Thus $\langle x, m \rangle = 0$, which shows $x \in M^\perp$, proving $M^\perp$ is closed. $\square$

b)

Prove that $M \cap M^\perp \subset \{0\}$.

Let $x \in M \cap M^\perp$. Then by the definition of $M^\perp$,

$$\langle x, x \rangle = 0$$

Then $\|x\| = 0$, which shows $x = 0$. Thus $M \cap M^\perp \subset \{0\}$. $\square$

c)

If M is a linear subspace of $\mathcal{H}$, prove that $(M^\perp)^\perp = \overline{M}$.

Assume $x \in \overline{M}$. Then there is a sequence $x_n \in M$ such that $x_n \to x$. Then $\langle x_n, y \rangle = 0$ for every $y \in M^\perp$. Then by continuity of $\langle \cdot, \cdot \rangle$, $\langle x, y \rangle = 0$ for every $y \in M^\perp$. Then $x \in (M^\perp)^\perp$ by the definition of $(M^\perp)^\perp$. Thus $\overline{M} \subset (M^\perp)^\perp$.

Now assume $x \notin \overline{M}$. Since $\overline{M}$ is closed, then by the Projection Theorem, $\exists y \in \overline{M}$ such that $(x - y) \perp \overline{M}$. Since $y \in \overline{M}$, $\langle x - y, y \rangle = 0$. Since $x \neq y$ ($x \notin \overline{M}$ and $y \in \overline{M}$), then $\langle x - y, x - y \rangle \neq 0$. However, $\langle x - y, x - y \rangle = \langle x - y, x \rangle - \langle x - y, y \rangle = \langle x - y, x \rangle$. Since $x - y \perp \overline{M}$, then $x - y \perp M$. So $x - y \in M^\perp$. Then since $\langle x - y, x \rangle \neq 0$, then $x \notin (M^\perp)^\perp$. Then $(M^\perp)^\perp \subset \overline{M}$.

Thus, $\overline{M} = (M^\perp)^\perp$. $\square$

Problem 4

Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. If $\langle x, Ay \rangle = 0$ for all $x, y, \in \mathcal{H}$, prove $A = \mathbb{0}$.

Since $\langle x, Ay \rangle = 0$ for all $x, y \in \mathcal{H}$, then in particular, take $x = Ay$, and so $\langle Ay, Ay \rangle = 0$ for all $y \in \mathcal{H}$. Then $Ay = 0$ for all $y \in \mathcal{H}$. Thus $A = \mathbb{0}$. $\square$

Problem 5

Let $\mathcal{H}$ be a Hilbert space and P and Q two orthogonal projections on $\mathcal{H}$.

a)

Prove that PQ is an orthogonal projection if and only if $PQ - QP = 0$, i.e., if and only if P and Q commute.

First note that

$$\langle PQx, y\rangle = \langle Qx, Py\rangle = \langle x, QPy\rangle \tag{1}$$

Assume PQ is an orthogonal projection. Then by the definition of orthogonal projection, and by (1), $\langle PQx, y\rangle = \langle x, PQy\rangle$. Then for all $x, y \in \mathcal{H}$,

$$\begin{aligned}
&\langle x, QPy\rangle = \langle x, PQy\rangle \\
\implies &\langle x, (QP - PQ)y\rangle = 0 \\
\implies &QP - PQ = \mathbb{0} \\
\implies &QP = PQ
\end{aligned}$$

Thus P and Q commute.

Now assume $PQ = QP$. Then $(PQ)^2 = PQPQ = PPQQ = PQ$ since P and Q are orthogonal projections. Also, by (1), $\langle PQx, y\rangle = \langle x, QPy\rangle = \langle x, PQy\rangle$. Thus PQ is an orthogonal projection. □

b)

Prove that for commuting orthogonal projections P and Q, one has $\mathrm{ran(PQ)} = \mathrm{ran(P)} \cap \mathrm{ran(Q)}$.

Let $x \in \mathrm{ran(PQ)}$. Then $\exists y$ such that $PQy = x$. Then P maps Qy on to x. Then $x \in \mathrm{ran(P)}$. However, since P and Q commute, then $QPy = x$, and thus Q maps Py on to x, and so $x \in \mathrm{ran(Q)}$. Thus $x \in \mathrm{ran(P)} \cap \mathrm{ran(Q)}$. So $\mathrm{ran(PQ)} \subset \mathrm{ran(P)} \cap \mathrm{ran(Q)}$.

Now let $x \in \mathrm{ran(P)} \cap \mathrm{ran(Q)}$. Then $x \in \mathrm{ran(P)}$ and $x \in \mathrm{ran(Q)}$. So $\exists y_1, y_2$ such that $Py_1 = Qy_2 = x$. Thus, $PQy_2 = P^2y_1 = Py_1 = x$, and thus $x \in \mathrm{ran(PQ)}$. So $\mathrm{ran(P)} \cap \mathrm{ran(Q)} \subset \mathrm{ran(PQ)}$.

Thus, $\mathrm{ran(PQ)} = \mathrm{ran(P)} \cap \mathrm{ran(Q)}$. □

c)

Prove that $P + Q$ is an orthogonal projection if and only if $PQ = \mathbb{0}$.

Assume $PQ = \mathbb{0}$. Then $\langle PQx, y\rangle = 0$ for all $x, y \in \mathcal{H}$. But by (1), $\langle x, QPy\rangle = 0$ for all $x, y \in \mathcal{H}$. Thus $QP = \mathbb{0}$. Then $(P + Q)^2 = P^2 + PQ + QP + Q^2 = P^2 + \mathbb{0} + \mathbb{0} + Q^2 = P + Q$ since P and Q are orthogonal projections. Also,

$$\begin{aligned}
\langle (P + Q)x, y\rangle &= \langle Px + Qx, y\rangle \\
&= \langle Px, y\rangle + \langle Qx, y\rangle \\
&= \langle x, Py\rangle + \langle x, Qy\rangle \\
&= \langle x, Py + Qy\rangle \\
&= \langle x, (P + Q)y\rangle
\end{aligned}$$

Thus $P+Q$ is an orthogonal projection.

Assume $P+Q$ is an orthogonal projection. Then $(P+Q)^2 = P+Q$, but $(P+Q)^2 = P^2+PQ+QP+Q^2 = P+PQ+QP+Q$. Thus $PQ+QP = \mathbb{0}$, i.e. $PQ = -QP$.

Assume $x \in \mathrm{ran(P)} \cap \mathrm{ran(Q)}$ and note $0 = (PQ+QP)x = PQx+QPx$. Since $x \in \mathrm{ran(P)}$, $Px = x$. Also, since $x \in \mathrm{ran(Q)}$, $Qx = x$. Then $PQx = Px = x$ and $QPx = Qx = x$. So $0 = PQx + QPx = 2x$. Thus $x = 0$.

Now, take any $x \in \mathcal{H}$, then certainly $PQx \in \mathrm{ran(P)}$ and since $PQx = -QPx = Q(-Px)$, then $PQx \in \mathrm{ran(Q)}$. Then $PQx = 0$ by the paragraph above, and thus $PQ = \mathbb{0}$, i.e. $\mathrm{ran(PQ)} = \{0\}$.

Thus, $P+Q$ is an orthogonal projection if and only if $\mathrm{ran(PQ)} = \{0\}$. □

d)

Prove that if $PQ = \mathbb{0}$, we have $\mathrm{ran(P+Q)} = \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$.

Let $PQ = \mathbb{0}$ and assume $y \in \mathrm{ran(P+Q)}$. Then $\exists x \in \mathcal{H}$ such that $Px + Qx = y$. Then y is the sum of an element in ran(P) and an element in ran(Q). Thus $y \in \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$. Assume $y \in \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$. Then $\exists x_1, x_2 \in \mathcal{H}$ such that $y = Px_1 + Qx_2$. Then $Py = P^2x_1 + PQX_2 = Px_1$ and $Qy = QPx_1 + Q^2x_2 = Qx_2$. Thus $y = Px_1 + Qx_2 = Py + Qy = (P+Q)y$. Thus $y \in \mathrm{ran(P+Q)}$, which shows $\mathrm{ran(P+Q)} = \mathrm{ran(P)} \oplus \mathrm{ran(Q)}$. □

Problem 6

Let $\mathcal{H}$ be a Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ such that $P^2 = P$ and $\dim \mathrm{ran(P)} = 1$.

a)

Show that $\|P\| \geq 1$.

Let $x \in \mathrm{ran(P)}$ such that $\|x\| = 1$. Then $Px = x$, and so $\|Px\| = \|x\| = 1$. Thus $\|P\| \geq 1$. □

b)

Suppose $\dim \mathcal{H} \geq 2$. Find

$$\sup \left\{ \|P\| \mid P \in \mathcal{B}(\mathcal{H}),\ P^2 = P,\ \dim \mathrm{ran(P)} = 1 \right\}$$

Problem 7

Let $\{e_1, e_2, \dots\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$.

a)

Let $(a_n) \in \ell^1(\mathbb{N})$. Show that $\sum_{n=1}^{\infty} a_n e_n$ converges absolutely to a limit in $\mathcal{H}$.

b)

Let $\alpha \in (0, \infty)$ *and define* $a_n = n^{-\alpha}$, $n \geq 1$. *For which values of* α *does* $\sum_{n=1}^{\infty} a_n e_n$ *converge unconditionally but not absolutely?*

Problem 8

Define the Legendre polynomials P_n *by*

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

a)

Show that the Legendre polynomials are orthogonal in $L^2([-1, 1])$, *and that they are obtained by Gram-Schmidt orthogonalization of the monomials.*

b)

Show that

$$\int_{-1}^{1} P_n(x)^2 \mathrm{d}x = \frac{2}{2n+1}$$

c)

Prove that the Legendre polynomials form an orthogonal basis of $L^2([-1, 1])$.

d)

Prove that the Legendre polynomial P_n *is an eigenfunction of the differential operator*

$$L = -\frac{\mathrm{d}}{\mathrm{d}x}(1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x}$$

with eigenvalue $\lambda_n = n(n+1)$, meaning that

$$LP_n = \lambda_n P_n$$