Name: ______________________________

MAT201A University of California, Davis Fall 2015

Homework # 5

(Due Monday, November 9)

Problem 1. Let $(X, \mathcal{T})$ be a Hausdorff space and $F, K \subset X$ such that F is closed and K is compact.

a) Prove that K is closed.

b) Prove that $F \cap K$ is compact.

Problem 2. Let $(X, \mathcal{T})$ be a topological space and K_1 and K_2 two compact subsets of X.

a) Prove that $K_1 \cup K_2$ is compact.

b) Assuming $(X, \mathcal{T})$ is Hausdorff, prove that $K_1 \cap K_2$ is compact.

Problem 3. If A is a subset of a topological space, then the *interior* A° of A is the union of all open sets contained in A, the *closure* $\overline{A}$ of A is the intersection of all closed sets that contain A, and the *boundary* ∂A of A is defined by $\partial A = \overline{A} \bigcap \overline{A^c}$.

a) Show that a set is closed if and only if it contains its boundary.

b) Show that a set is open if and only if it is disjoint from its boundary.

c) What are the closure, interior, and boundary of the Cantor set, considered as a subset of $\mathbb{R}$ with its usual topology? The Cantor set is defined in Example 1.40 of the textbook.

Problem 4. A topological space is *connected* if it is not the union of two disjoint non-empty open sets. A subset Y of a topological space $(X, \mathcal{T})$ is called connected if Y is a connected topological space with respect to the relative topology.

a) Describe the connected subsets of $(\mathbb{R}, |\cdot|)$.

b) Show that $(\mathbb{R}, |\cdot|)$ is homeomorphic to the open interval $(0,1) \subset \mathbb{R}$ with the relative topology.

c) Show that $(\mathbb{R}, |\cdot|)$ is not homeomorphic to $(\mathbb{R}^2, \|\cdot\|)$, where $\|\cdot\|$ is the Euclidean norm.

Problem 5. Prove that the sequence (f_n) defined in Example 5.11 in the textbook is a Schauder basis of $(C([0,1]), \|\cdot\|_\infty)$.

Problem 6. For $1 \le p \le \infty$, consider the Banach space $\ell^p(\mathbb{N})$ defined in Example 5.5 of the texbook. The set $\ell_c(\mathbb{N})$ of all sequences of the form $(x_1, x_2, \dots, x_n, 0, 0, \dots)$ whose terms vanish from some point onwards is an infinite-dimensional linear subspace of $\ell^p(\mathbb{N})$ for any $1 \le p \le \infty$.

a) Show that $\ell_c(\mathbb{N})$ is not closed in $\ell^p(\mathbb{N})$, so it is not a Banach space with respect to the norm of $\ell^p(\mathbb{N})$.

b) Show that $\ell_c(\mathbb{N})$ is dense in $\ell^p(\mathbb{N})$ for $1 \le p < \infty$.

c) Find the closure of $\ell_c(\mathbb{N})$ in $\ell^\infty(\mathbb{N})$.