HW #2

Sam Fleischer

October 12, 2015

Problem 1

Let (X,d) be a metric space, with $d(x,y) = 1 - \delta_{x,y}$, for all $x, y \in X$. Prove that X is compact if and only if X is a finite set.

" $\implies$ "

Let X be compact. Then choose $C = \bigcup_{x\in X} B_\epsilon(x)$ as an open cover of X for any $0 < \epsilon < 1$. Since X is compact, there exists a finite subcover of C, say $\tilde{C} = \bigcup_{i=1}^n B_\epsilon(x_i)$. However, if $x \in B_\epsilon(x_i)$, then $d(x, x_i) < \epsilon < 1$. This means $d(x, x_i) = 0$, which implies $x = x_i$. Thus $B_\epsilon(x_i) = \{x_i\}$ for $i = 1, \ldots n$. Then $\tilde{C} = \bigcup_{i=1}^n \{x_i\}$. Since this covers X, $X \subset \tilde{C}$. But $\tilde{C}$ is finite, and so X is finite.

"$\Longleftarrow$"

Let $X = \{x_1, \ldots, x_n\}$ be finite, and let $C = \bigcup_{i\in I} C_i$ be an open cover of X. Then $\forall x_k \in X$, there is at least one i such that $x_k \in C_i$. Choose one, say C_k. Then $X \subset \bigcup_{k=1}^n C_k$, which is a finite open subcover. Thus X is compact. $\square$

Problem 2

Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that there is a non-empty closed set $F \subset \mathbb{R}$, with $f(F)$ open.

Trivial solution: Let $f(x) = x$ and let $F = \mathbb{R}$. Then F is a non-empty closed set since $F^C = \emptyset$ is open. Then $f(F) = \mathbb{R}$ is open.
Non-trivial solution: Define f as

$$f(x) = \begin{cases} \dfrac{3}{2^{n+2}}x + \dfrac{2^{n+1} - 3n - 2}{2^{n+1}} & , \quad x \in [2n, 2n+1], n = 0, 1, \ldots \\ \dfrac{-1}{2^{n+2}}x + \dfrac{2^{n+1} + n}{2^{n+1}} & , \quad x \in [2n+1, 2(n+1)], n = 0, 1, \ldots \end{cases}$$
$$f(-x) = -f(x)$$

and let $F = \bigcup_{n=0}^{\infty} ([2n, 2n+1] \cup [-(2n+1), -2n])$. Then f is continuous on $\mathbb{R}$, F is closed (since its complement is a union of open sets), and $f(F) = (-1, 1)$, which is an open set.

Problem 3

Let (X, d) be a metric space and F and K two non-empty subsets of X. Assume F is closed and K is compact. Define

$$d(K, F) = \inf\{d(x, y) | x \in K, y \in F\}. \tag{1}$$

Prove that $d(K, F) > 0$ if and only if $K \cap F = \emptyset$.

"$\implies$"

Let $d(K, F) > 0$. Then $\forall k \in K$ and $f \in F$, $d(k, f) > 0$ which implies, $\forall k \in K$ and $f \in F$, $k \neq f$. Thus $\forall k \in K$, $k \notin F$, and $\forall f \in F$, $f \notin K$. So $K \cap F = \emptyset$.

"$\Longleftarrow$"

Let $d(K, F) \not> 0$. Then, since $d(K, F) \not< 0$, then $d(K, F) = 0$. Then $\forall \epsilon > 0$, $\exists k \in K, f \in F$ such that $d(k, f) < \epsilon$. Now construct sequences $(k_n) \in K$ and $(f_n) \in F$ such that $d(k_n, f_n) < \frac{1}{n}$. Since K is compact, there is a subsequence (k_{n_l}) that converges to some limit $\tilde{k} \in K$. By the triangle inequality,

$$d(f_{n_l}, \tilde{k}) \leq d(f_{n_l}, k_{n_l}) + d(k_{n_l}, \tilde{k})$$

Since $\lim_{l \to \infty} d(f_{n_l}, k_{n_l}) = 0$ and $\lim_{l \to \infty} d(k_{n_l}, \tilde{k}) = 0$, then $\lim_{l \to \infty} d(f_{n_l}, \tilde{k}) = 0$, which shows the subsequence (f_{n_l}) converges to k. However, f is closed, which means every convergent sequence in F converges to a limit in F, and since limits are unique, $k \in F$. Thus $K \cap F \neq \emptyset$. $\square$

Problem 4

Consider the space X of all bounded real-valued functions defined on the interval $[0, 1] \subset \mathbb{R}$. For all $f, g \in X$, define $d(f, g)$ by

$$d(f, g) = \sup\{|f(x) - g(x)| | x \in [0, 1]\}. \tag{2}$$

a)

Prove that d is a metric on X.

b)

Prove that the metric space (X, d) is not separable.

Problem 5

Let (X, d) be a metric space and, for each $i = 1, \ldots, n$, let $K_i \subset X$ be compact.

a)

Prove that $\bigcap_{i=1}^n K_i$ is compact.

b)

Prove that $\bigcup_{i=1}^n K_i$ is compact.

c)

Are the union and intersection of an arbitrary family of compact subsets also compact? Why (not)?

Problem 6

Let $f \in C([0,1])$ be such that $\in_0^1 x^n f(x)dx = 0$ for all integers $n \geq 0$. Prove that $f(x) = 0$, for all $x \in [0,1]$.

Problem 7

Let (p_n) be a sequence of real-valued polynomial functions defined on the interval $[0,1]$ with bounded degree, i.e., there exists $0 \leq D \in \mathbb{Z}$, and sequences of real numbers $(a_n(k))_{n=1}^\infty$, $k = 0, \ldots, D$, such that

$$p_n(x) = a_n(0) + a_n(1)x + \cdots + a_n(D)x^D, \quad x \in [0,1]. \tag{3}$$

a)

Prove that if $\|p_n\|_\infty \to 0$, then $\lim_{n\to\infty} \max_{0\leq k\leq D} a_n(k) = 0$ (Hint: try induction on D).

b)

Show that the assumption of a uniform bound on the degree of p_n is essential for the implication in part a) to hold. Specifically, find a sequence of polynomials $p_n(x) = \sum_{k=0}^{D_n} a_n(k)x^k$, such that $\|p_n\|_\infty \to 0$ and

$$\overline{\lim_n} \max_{0\leq k\leq D_n} |a_n(k)| = 1 \tag{4}$$