Name:

MAT201A University of California, Davis Fall 2015

Homework # 2

(Due Monday, October 12)

Problem 1. Let (X,d) be a metric space, with $d(x,y) = 1 - \delta_{x,y}$, for all $x, y \in X$. Prove that X is compact if and only if X is a finite set.

Problem 2. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, such that there is a non-empty closed set $F \subset \mathbb{R}$, with $f(F)$ open.

Problem 3. Let (X,d) be a metric space and F and K two non-empty subsets of X. Assume F is closed and K is compact. Define

$$d(K,F) = \inf\{d(x,y) \mid x \in K, y \in F\}.$$

Prove that $d(K,F) > 0$ if and only if $K \cap F = \emptyset$.

Problem 4. Consider the space X of all bounded real-valued functions defined on the interval $[0,1] \subset \mathbb{R}$. For all $f, g \in X$, define $d(f,g)$ by

$$d(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}.$$

a) Prove that d is a metric on X.

b) Prove that the metric space (X,d) is not separable.

Problem 5. Let (X,d) be a metric space and, for each $i = 1, \ldots, n$, let $K_i \subset X$ be compact.

a) Prove that $\bigcap_{i=1}^n K_i$ is compact.

b) Prove that $\bigcup_{i=1}^n K_i$ is compact.

c) Are the union and intersection of an arbitrary family of compact subsets also compact? Why (not)?

Problem 6. Let $f \in C([0,1])$ be such that $\int_0^1 x^n f(x)dx = 0$ for all integers $n \geq 0$. Prove that $f(x) = 0$, for all $x \in [0,1]$.

Problem 7. Let (p_n) be a sequence of real-valued polynomial functions defined on the interval $[0,1]$ with bounded degree, i.e., there exists $0 \leq D \in \mathbb{Z}$, and sequences of real numbers $(a_n(k))_{n=1}^\infty$, $k = 0, \ldots, D$, such that

$$p_n(x) = a_n(0) + a_n(1)x + \cdots + a_n(D)x^D, \quad x \in [0,1].$$

a) Prove that if $\|p_n\|_\infty \to 0$, then $\lim_{n\to\infty} \max_{0\leq k\leq D} a_n(k) = 0$ (Hint: try induction on D).

b) Show that the assumption of a uniform bound on the degree of p_n is essential for the implication in part a) to hold. Specifically, find a sequence of polynomials $p_n(x) = \sum_{k=0}^{D_n} a_n(k)x^k$, such that $\|p_n\|_\infty \to 0$ and

$$\limsup_n \max_{0\leq k\leq D_n} |a_n(k)| = 1$$