Name:

MAT201A University of California, Davis Fall 2015

Homework # 8

(Due Monday, November 30)

Problem 1. Consider the Banach space $C([0,1])$ with the supremum norm. For $x \in [0,1]$ let δ_x denote the linear functional on $C([0,1])$ given by

$$\delta_x(f) = f(x), \qquad \text{for all } f \in C([0,1]).$$

a) Show that $\|\delta_x\| = 1$.

b) Show that there does not exist a Riemann integrable function $k : [0,1] \to \mathbb{R}$, such that

$$\delta_x(f) = \int_0^1 k(y) f(y) dy, \qquad \text{for all } f \in C([0,1]).$$

Problem 2. Prove that there does not exist an inner product on $C([0,1])$ such that the supremum norm is derived from this inner product.

Problem 3. Let $\mathcal{H}$ be a Hilbert space and let M be a subset of $\mathcal{H}$.

a) Prove that $M^\perp$ is a closed linear subspace of $\mathcal{H}$.

b) Prove that $M \cap M^\perp \subset \{0\}$.

c) If M is a linear subspace of $\mathcal{H}$, prove that $(M^\perp)^\perp = \overline{M}$.

Problem 4. Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. If $(x, Ay) = 0$ for all $x, y \in \mathcal{H}$, prove that $A = 0$.

Problem 5. Let $\mathcal{H}$ be a Hilbert space and P and Q two orthogonal projections on $\mathcal{H}$.

a) Prove that PQ is an orthogonal projection if and only if $PQ - QP = 0$, i.e., if and only if P and Q commute.

b) Prove that for commuting orthogonal projections P and Q, one has ran $PQ =$ ran $P \cap$ran Q.

c) Prove that $P + Q$ is an orthogonal projection if and only if $PQ = 0$.

d) Prove that if $PQ = 0$, we have ran $(P+Q) =$ ran $P \oplus$ ran Q.

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ such that $P^2 = P$ and $\dim \operatorname{ran} P = 1$.

a) Show that $\|P\| \geq 1$.

b) Suppose $\dim \mathcal{H} \geq 2$. Find

$$\sup \{\|P\| \mid P \in \mathcal{B}(\mathcal{H}), P^2 = P, \dim \operatorname{ran} P = 1\}.$$

Problem 7. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$.

a) Let $(a_n) \in \ell^1(\mathbb{N})$. Show that $\sum_{n=1}^{\infty} a_n e_n$ converges absolutely to a limit in $\mathcal{H}$.

b) Let $\alpha \in (0, \infty)$ and define $a_n = n^{-\alpha}$, $n \geq 1$. For which values of α does $\sum_{n=1}^{\infty} a_n e_n$ converge unconditionally but not absolutely?

Problem 8. Define the Legendre polynomials P_n by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n .$$

a) Show that the Legendre polynomials are orthogonal in $L^2([-1, 1])$, and that they are obtained by Gram-Schmidt orthogonalization of the monomials.

b) Show that

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1}.$$

c) Prove that the Legendre polynomials form an orthogonal basis of $L^2([-1, 1])$.

d) Prove that the Legendre polynomial P_n is an eigenfunction of the differential operator

$$L = -\frac{d}{dx} \left(1 - x^2\right) \frac{d}{dx}$$

with eigenvalue $\lambda_n = n(n+1)$, meaning that

$$LP_n = \lambda_n P_n.$$