

# HW #2

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## Problem 1

Let  $(X, d)$  be a metric space, with  $d(x, y) = 1 - \delta_{x,y}$ , for all  $x, y \in X$ . Prove that  $X$  is compact if and only if  $X$  is a finite set.

“ $\implies$ ”

Let  $X$  be compact. Then choose  $C = \{B_\epsilon(x) | x \in X\}$  as an open cover of  $X$  for any  $0 < \epsilon < 1$ . Since  $X$  is compact, there exists a finite subcover of  $C$ , say  $\tilde{C} = \{B_\epsilon(x_i) | i = 1 \dots n\}$ . However, if  $x \in B_\epsilon(x_i)$ , then  $d(x, x_i) < \epsilon < 1$ . This means  $d(x, x_i) = 0$ , which implies  $x = x_i$ . Thus  $B_\epsilon(x_i) = \{x_i\}$  for  $i = 1, \dots, n$ . Then  $\tilde{C} = \{x_1, \dots, x_n\}$ . Since this covers  $X$ ,  $X \subset \tilde{C}$ . But  $\tilde{C}$  is finite, and so  $X$  is finite.

“ $\impliedby$ ”

Let  $X = \{x_1, \dots, x_n\}$  be finite, and let  $C = \{C_i | i \in I\}$  (where  $I$  is an arbitrary indexing set) be an open cover of  $X$ . Then  $\forall x \in X$ , there is at least one  $i \in I$  such that  $x \in C_i$ . Choose one  $i$  for each  $x \in X$ , say  $C_{i(x)}$ . Then  $X \subset \bigcup_{x \in X} C_{i(x)}$ , and  $\{C_{i(x)} | x \in X\}$  is a finite open subcover. Thus  $X$  is compact.  $\square$

## Problem 2

Give an example of a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that there is a non-empty closed set  $F \subset \mathbb{R}$ , with  $f(F)$  open.

**Trivial solution:** Let  $f(x) = x$  and let  $F = \mathbb{R}$ . Then  $F$  is a non-empty closed set since  $F^C = \emptyset$  is open. Then  $f(F) = \mathbb{R}$  is open.

**Non-trivial solution:** Define  $f$  as

$$f(x) = \begin{cases} \frac{1}{2^{n+1}}x + \frac{2^n - n - 1}{2^n} & , \quad x \in [2n, 2n+1], n = 0, 1, \dots \\ \frac{1}{2^{n+1}} & , \quad x \in [2n+1, 2(n+1)], n = 0, 1, \dots \end{cases}$$

$$f(-x) = -f(x)$$

and let  $F = \bigcup_{n=0}^{\infty} ([2n, 2n+1] \cup [-(2n+1), -2n])$ . Then  $f$  is continuous on  $\mathbb{R}$ ,  $F$  is closed (since its complement is a union of open sets), and  $f(F) = (-1, 1)$ , which is an open set.

### Problem 3

Let  $(X, d)$  be a metric space and  $F$  and  $K$  two non-empty subsets of  $X$ . Assume  $F$  is closed and  $K$  is compact. Define

$$d(K, F) = \inf\{d(x, y) | x \in K, y \in F\}. \quad (1)$$

Prove that  $d(K, F) > 0$  if and only if  $K \cap F = \emptyset$ .

“ $\implies$ ”

Let  $d(K, F) > 0$ . Then  $\forall k \in K$  and  $f \in F$ ,  $d(k, f) > 0$  which implies,  $\forall k \in K$  and  $f \in F$ ,  $k \neq f$ . Thus  $\forall k \in K$ ,  $k \notin F$ , and  $\forall f \in F$ ,  $f \notin K$ . So  $K \cap F = \emptyset$ .

“ $\impliedby$ ”

Let  $d(K, F) \not> 0$ . Then, since  $d(K, F) \not\leq 0$ , then  $d(K, F) = 0$ . Then  $\forall \epsilon > 0$ ,  $\exists k \in K, f \in F$  such that  $d(k, f) < \epsilon$ . Now construct sequences  $(k_n) \in K$  and  $(f_n) \in F$  such that  $d(k_n, f_n) < \frac{1}{n}$ . Since  $K$  is compact, there is a subsequence  $(k_{n_l})$  that converges to some limit  $\tilde{k} \in K$ . By the triangle inequality,

$$d(f_{n_l}, \tilde{k}) \leq d(f_{n_l}, k_{n_l}) + d(k_{n_l}, \tilde{k})$$

Since  $\lim_{l \rightarrow \infty} d(f_{n_l}, k_{n_l}) = 0$  and  $\lim_{l \rightarrow \infty} d(k_{n_l}, \tilde{k}) = 0$ , then  $\lim_{l \rightarrow \infty} d(f_{n_l}, \tilde{k}) = 0$ , which shows the subsequence  $(f_{n_l})$  converges to  $\tilde{k}$ . However,  $F$  is closed, which means every convergent sequence in  $F$  converges to a limit in  $F$ , and since limits are unique,  $\tilde{k} \in F$ . Thus  $\tilde{k} \in K \cap F \implies K \cap F \neq \emptyset$ .  $\square$

### Problem 4

Consider the space  $X$  of all bounded real-valued functions defined on the interval  $[0, 1] \subset \mathbb{R}$ . For all  $f, g \in X$ , define  $d(f, g)$  by

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}. \quad (2)$$

a)

Prove that  $d$  is a metric on  $X$ .

$$\forall f \in X, \quad d(f, f) = \sup\{|f(x) - f(x)| \mid x \in [0, 1]\} = \sup\{0\} = 0$$

$$\forall f, g \in X, \quad d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\} \geq 0 \quad \text{since } |a| \geq 0 \text{ for any } a.$$

Since  $|f(x) - g(x)| = |g(x) - f(x)|$ ,

$$\begin{aligned}\forall f, g \in X, \quad d(f, g) &= \sup\{|f(x) - g(x)| \mid x \in [0, 1]\} \\ &= \sup\{|g(x) - f(x)| \mid x \in [0, 1]\} = d(g, f)\end{aligned}$$

$$\begin{aligned}\forall f, g, h \in X, \quad d(f, g) &= \sup\{|f(x) - g(x)| \mid x \in [0, 1]\} \\ &= \sup\{|f(x) - h(x) + h(x) - g(x)| \mid x \in [0, 1]\} \\ &\leq \sup\{|f(x) - h(x)| + |h(x) - g(x)| \mid x \in [0, 1]\} \\ &\quad (\text{by the triangle inequality in } \mathbb{R}) \\ &= \sup\{|f(x) - h(x)| \mid x \in [0, 1]\} + \sup\{|h(x) - g(x)| \mid x \in [0, 1]\} \\ &\quad (\text{by properties of supremum}) \\ &= d(f, h) + d(g, h)\end{aligned}$$

Thus  $d$  is a metric on  $X$ . □

**b)**

*Prove that the metric space  $(X, d)$  is not separable.*

**Solution 1:** Consider the set  $A = \{f_\alpha\}_{\alpha \in [0,1]}$  where

$$f_\alpha(x) = \begin{cases} 0 & , \quad x \neq \alpha \\ 1 & , \quad x = \alpha \end{cases}$$

Then  $\forall \alpha, \beta \in [0, 1]$  with  $\alpha \neq \beta$ ,  $d(f_\alpha, f_\beta) = 1$ . Now consider a countable subset  $S \subset X$  and assume  $S$  is dense. Then  $\forall f \in X$ ,  $\exists (s_n)_n \in S$  such that  $s_n \rightarrow f$ , i.e.  $\forall \epsilon > 0 \exists s \in S$  such that  $d(s, f) < \epsilon$ . In particular,  $\forall a \in A$ ,  $\exists (s_n)_n \in S$  such that  $s_n \rightarrow a$ , i.e.  $\forall a \in A$ ,  $\exists s \in S$  (dependent on  $a$ ) such that  $d(s, a) < \epsilon$ . However, if we pick any  $\epsilon < 1/2$ , then for each  $a \in A$  we must choose a different  $s \in S$  to satisfy  $d(s, a) < \epsilon$ . This implies there is a one-to-one correspondence between  $A$  and  $S$ , which is a contradiction since  $A$  is countable. Thus there is no countable dense subset of  $X$ , proving  $X$  is not separable. □

## Problem 5

*Let  $(X, d)$  be a metric space and, for each  $i = 1, \dots, n$ , let  $K_i \subset X$  be compact.*

**a)**

*Prove that  $\bigcap_{i=1}^n K_i$  is compact.*

Let  $(k_n)_n$  be a sequence in  $\bigcap_{i=1}^n K_i$ . Then  $(k_n)_n$  is a sequence in  $K_i$  for each  $i = 1, \dots, n$ . Since each  $K_i$  is compact, then each  $K_i$  is sequentially compact, and thus there is a subsequence  $(k_{n_\ell})_\ell$  that converges to a limit  $\tilde{k} \in K_i$  for each  $i = 1, \dots, n$ . Then  $\tilde{k} \in \bigcap_{i=1}^n K_i$ . Thus any sequence in  $\bigcap_{i=1}^n K_i$  has a convergent subsequence that converges to a limit in  $\bigcap_{i=1}^n K_i$ . Thus  $\bigcap_{i=1}^n K_i$  is sequentially compact, and therefore compact. □

b)

Prove that  $\bigcup_{i=1}^n K_i$  is compact.

Since each  $K_i$  is compact, each  $K_i$  is sequentially compact. Choose any sequence in  $\bigcup_{i=1}^n K_i$ . There must be infinitely many points in at least one  $K_i$ , say  $K_\ell$ . Then choose the subsequence of  $(k_{n_m})_m$  such that each  $k_{n_m} \in K_\ell$ . Since  $K_\ell$  is sequentially compact, there is a convergent subsequence  $(k_{n_{m_p}})_p$  in  $K_\ell$  that converges to a limit  $\tilde{k} \in K_\ell$ . Thus the original sequence  $(k_n)_n$  has a convergent subsequence  $(k_{n_{m_p}})_p$  that converges to  $\tilde{k} \in K_\ell \subset \bigcup_{i=1}^n K_i$ . Thus  $\bigcup_{i=1}^n K_i$  is sequentially compact, and therefore compact.  $\square$

c)

Are the union and intersection of an arbitrary family of compact subsets also compact? Why (not)?

The union of an arbitrary family of compact subsets is not necessarily compact. Consider  $X = \mathbb{R}$  and  $K_i = [-i, i]$  for  $i \in \mathbb{N}$ . Then  $\bigcup_{i \in \mathbb{N}} K_i = \mathbb{R}$ , which is not compact. However, the intersection of an arbitrary family of compact subsets is compact. The following is a generalization of the proof given in part a).

Let  $(k_n)_n$  be a sequence in  $\bigcap_{i \in I} K_i$  where  $I$  is an indexing set. Then  $(k_n)_n$  is a sequence in  $K_i$  for every  $i \in I$ . Since each  $K_i$  is compact, then each  $K_i$  is sequentially compact, and thus there is a subsequence  $(k_{n_\ell})_\ell$  that converges to a limit  $\tilde{k}$ , where  $\tilde{k} \in K_i$  for every  $i \in I$ . Then  $\tilde{k} \in \bigcap_{i \in I} K_i$ . Thus any sequence in  $\bigcap_{i \in I} K_i$  has a convergent subsequence that converges to a limit in  $\bigcap_{i \in I} K_i$ . Thus  $\bigcap_{i \in I} K_i$  is sequentially compact, and therefore compact.  $\square$

## Problem 6

Let  $f \in C([0, 1])$  be such that  $\int_0^1 x^n f(x) dx = 0$  for all integers  $n \geq 0$ . Prove that  $f(x) = 0$ , for all  $x \in [0, 1]$ .

## Problem 7

Let  $(p_n)$  be a sequence of real-valued polynomial functions defined on the interval  $[0, 1]$  with bounded degree, i.e., there exists  $0 \leq D \in \mathbb{Z}$ , and sequences of real numbers  $(a_n(k))_{n=1}^\infty$ ,  $k = 0, \dots, D$ , such that

$$p_n(x) = a_n(0) + a_n(1)x + \dots + a_n(D)x^D, \quad x \in [0, 1]. \quad (3)$$

a)

Prove that if  $\|p_n\|_\infty \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} \max_{0 \leq k \leq D} a_n(k) = 0$  (Hint: try induction on  $D$ ).

b)

Show that the assumption of a uniform bound on the degree of  $p_n$  is essential for the implication in part a) to hold. Specifically, find a sequence of polynomials  $p_n(x) = \sum_{k=0}^{D_n} a_n(k)x^k$ , such that  $\|p_n\|_\infty \rightarrow 0$  and

$$\overline{\lim}_n \max_{0 \leq k \leq D_n} |a_n(k)| = 1 \quad (4)$$

Let  $\{q_i\}_i \in (0, 1) \cap \mathbb{Q}$  be an enumeration of the rational numbers between 0 and 1. Then define the sequence  $\{p_n\}_n$  as

$$p_n(x) = x(x-1) \prod_{i=1}^n (x - q_i)$$

Then  $\|p_n\|_\infty \rightarrow 0$  since the number of roots between 0 and 1 approaches infinity, there are no roots outside of the interval  $[0, 1]$ , and polynomials are smooth. Also, the leading coefficient is always equal to 1, and all other coefficients are less than 1 since each  $q_i < 1$ . Thus  $\overline{\lim}_n \max_{0 \leq k \leq D_n} |a_n(k)| = 1$ , as required.  $\square$