HW #2

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Problem 1

Let (X, d) be a metric space, with $d(x, y) = 1 - \delta_{x,y}$, for all $x, y \in X$. Prove that X is compact if and only if X is a finite set.

"$\implies$"

Let X be compact. Then choose $C = \{B_\epsilon(x) | x \in X\}$ as an open cover of X for any $0 < \epsilon < 1$. Since X is compact, there exists a finite subcover of C, say $\tilde{C} = \{B_\epsilon(x_i) | i = 1 \ldots n\}$. However, if $x \in B_\epsilon(x_i)$, then $d(x, x_i) < \epsilon < 1$. This means $d(x, x_i) = 0$, which implies $x = x_i$. Thus $B_\epsilon(x_i) = \{x_i\}$ for $i = 1, \ldots n$. Then $\tilde{C} = \{x_1, \ldots, n_n\}$. Since this covers X, $X \subset \tilde{C}$. But $\tilde{C}$ is finite, and so X is finite.

"$\Longleftarrow$"

Let $X = \{x_1, \ldots, x_n\}$ be finite, and let $C = \{C_i | i \in I\}$ (where I is an arbitrary indexing set) be an open cover of X. Then $\forall x \in X$, there is at least one $i \in I$ such that $x \in C_i$. Choose one i for each $x \in X$, say $C_{i(x)}$. Then $X \subset \bigcup_{x \in X} C_{i(x)}$, and $\{C_{i(x)} | x \in X\}$ is a finite open subcover. Thus X is compact. $\square$

Problem 2

Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that there is a non-empty closed set $F \subset \mathbb{R}$, with $f(F)$ open.

Trivial solution: Let $f(x) = x$ and let $F = \mathbb{R}$. Then F is a non-empty closed set since $F^C = \emptyset$ is open. Then $f(F) = \mathbb{R}$ is open.
Non-trivial solution: Define f as

$$f(x) = \begin{cases} \dfrac{1}{2^{n+1}}x + \dfrac{2^n - n - 1}{2^n} & , \quad x \in [2n, 2n+1], n = 0, 1, \ldots \\ \dfrac{1}{2^{n+1}} & , \quad x \in [2n+1, 2(n+1)], n = 0, 1, \ldots \end{cases}$$
$$f(-x) = -f(x)$$

and let $F = \bigcup_{n=0}^{\infty} ([2n, 2n+1] \cup [-(2n+1), -2n])$. Then f is continuous on $\mathbb{R}$, F is closed (since its complement is a union of open sets), and $f(F) = (-1, 1)$, which is an open set.

Problem 3

Let (X, d) be a metric space and F and K two non-empty subsets of X. Assume F is closed and K is compact. Define

$$d(K, F) = \inf\{d(x, y) | x \in K, y \in F\}. \tag{1}$$

Prove that $d(K, F) > 0$ if and only if $K \cap F = \emptyset$.

“ $\implies$ ”

Let $d(K, F) > 0$. Then $\forall k \in K$ and $f \in F$, $d(k, f) > 0$ which implies, $\forall k \in K$ and $f \in F$, $k \neq f$. Thus $\forall k \in K$, $k \notin F$, and $\forall f \in F$, $f \notin K$. So $K \cap F = \emptyset$.

“$\impliedby$”

Let $d(K, F) \not> 0$. Then, since $d(K, F) \not< 0$, then $d(K, F) = 0$. Then $\forall \epsilon > 0$, $\exists k \in K, f \in F$ such that $d(k, f) < \epsilon$. Now construct sequences $(k_n) \in K$ and $(f_n) \in F$ such that $d(k_n, f_n) < \frac{1}{n}$. Since K is compact, there is a subsequence (k_{n_l}) that converges to some limit $\tilde{k} \in K$. By the triangle inequality,

$$d(f_{n_l}, \tilde{k}) \leq d(f_{n_l}, k_{n_l}) + d(k_{n_l}, \tilde{k})$$

Since $\lim_{l \to \infty} d(f_{n_l}, k_{n_l}) = 0$ and $\lim_{l \to \infty} d(k_{n_l}, \tilde{k}) = 0$, then $\lim_{l \to \infty} d(f_{n_l}, \tilde{k}) = 0$, which shows the subsequence (f_{n_l}) converges to k. However, F is closed, which means every convergent sequence in F converges to a limit in F, and since limits are unique, $\tilde{k} \in F$. Thus $\tilde{k} \in K \cap F \implies K \cap F \neq \emptyset$. $\square$

Problem 4

Consider the space X of all bounded real-valued functions defined on the interval $[0, 1] \subset \mathbb{R}$. For all $f, g \in X$, define $d(f, g)$ by

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}. \tag{2}$$

a)

Prove that d is a metric on X.

Non-negativity

$$\forall f \in X, \quad d(f,f) = \sup\{|f(x) - f(x)| \mid x \in [0,1]\} = \sup\{0\} = 0$$

$$\forall f, g \in X, \quad d(f,g) = \sup\{|f(x) - g(x)| \, |x \in [0,1]\} \geq 0 \quad \text{since } |a| \geq 0 \text{ for any } a.$$

Symmetry
Since $|f(x) - g(x)| = |g(x) - f(x)|$,

$$\begin{aligned} \forall f, g \in X, \quad d(f,g) &= \sup\{|f(x) - g(x)||x \in [0,1]\} \\ &= \sup\{|g(x) - f(x)||x \in [0,1]\} = d(g,f) \end{aligned}$$

Trangle Inequality

$$\begin{aligned} \forall f, g, h \in X, \quad d(f,g) &= \sup\{|f(x) - g(x)| \, |x \in [0,1]\} \\ &= \sup\{|f(x) - h(x) + h(x) - g(x)| \, |x \in [0,1]\} \\ &\leq \sup\{|f(x) - h(x)| + |h(x) - g(x)| \, |x \in [0,1]\} \\ &\qquad \text{(by the triangle inequality in } \mathbb{R}\text{)} \\ &= \sup\{|f(x) - h(x)| \, |x \in [0,1]\} + \sup\{|h(x) - g(x)| \, |x \in [0,1]\} \\ &\qquad \text{(by properties of supremum)} \\ &= d(f,h) + d(g,h) \end{aligned}$$

Thus d is a metric on X. $\square$

b)

Prove that the metric space (X, d) is not separable.

Solution 1: Consider the set $A = \{f_\alpha\}_{\alpha \in [0,1]}$ where

$$f_\alpha(x) = \begin{cases} 0 & , \quad x \neq \alpha \\ 1 & , \quad x = \alpha \end{cases}$$

Then $\forall \alpha, \beta \in [0,1]$ with $\alpha \neq \beta$, $d(f_\alpha, f_\beta) = 1$. Now consider a countable subset $S \subset X$ and assume S is dense. Then $\forall f \in X$, $\exists (s_n)_n \in S$ such that $s_n \to f$, i.e. $\forall \epsilon > 0 \exists s \in S$ such that $d(s,f) < \epsilon$. In particular, $\forall a \in A$, $\exists (s_n)_n \in S$ such that $s_n \to a$, i.e. $\forall a \in A$, $\exists s \in S$ (dependent on a) such that $d(s,a) < \epsilon$. However, if we pick any $\epsilon < 1/2$, then for each $a \in A$ we must choose a different $s \in S$ to satisfy $d(s,a) < \epsilon$. This implies there is a one-to-one correspondence between A and S, which is a contradiction since A is countable. Thus there is no countable dense subset of X, proving X is not separable. $\square$

Problem 5

Let (X, d) be a metric space and, for each $i = 1, \dots, n$, let $K_i \subset X$ be compact.

a)

Prove that $\bigcap_{i=1}^n K_i$ is compact.

Let $(k_n)_n$ be a sequence in $\bigcap_{i=1}^n K_i$. Then $(k_n)_n$ is a sequence in K_i for each $i = 1, \dots, n$. Since each K_i is compact, then each K_i is sequentially compact, and thus there is a subsequence $(k_{n_\ell})_\ell$ that converges to a limit $\tilde{k} \in K_i$ for each $i = 1, \dots, n$. Then $\tilde{k} \in \bigcap_{i=1}^n K_i$. Thus any sequence in $\bigcap_{i=1}^n K_i$ has a convergent subsequence that converges to a limit in $\bigcap_{i=1}^n K_i$. Thus $\bigcap_{i=1}^n K_i$ is sequentially compact, and therefore compact. $\square$

b)

Prove that $\bigcup_{i=1}^n K_i$ is compact.

Since each K_i is compact, each K_i is sequentially compact. Choose any sequence in $\bigcup_{i=1}^n K_i$. There must be infinitely many points in at least one K_i, say K_ℓ. Then choose the subsequence of $(k_{n_m})_m$ such that each $k_{n_m} \in K_\ell$. Since K_ℓ is sequentially compact, there is a convergent subsequence $(k_{n_{m_p}})_p$ in K_ℓ that converges to a limit $\tilde{k} \in K_\ell$. Thus the original sequence $(k_n)_n$ has a convergent subsequence $(k_{n_{m_p}})_p$ that converges to $\tilde{k} \in K_\ell \subset \bigcup_{i=1}^n K_i$. Thus $\bigcup_{i=1}^n K_i$ is sequentially compact, and therefore compact. $\square$

c)

Are the union and intersection of an arbitrary family of compact subsets also compact? Why (not)?

The union of an arbitrary family of compact subsets is not necessarily compact. Consider $X = \mathbb{R}$ and $K_i = [-i, i]$ for $i \in \mathbb{N}$. Then $\bigcup_{i \in \mathbb{N}} K_i = \mathbb{R}$, which is not compact. However, the intersection of an arbitrary family of compact subsets is compact. The following is a generalization of the proof given in part a).

Let $(k_n)_n$ be a sequence in $\bigcap_{i \in I} K_i$ where I is an indexing set. Then $(k_n)_n$ is a sequence in K_i for every $i \in I$. Since each K_i is compact, then each K_i is sequentially compact, and thus there is a subsequence $(k_{n_\ell})_\ell$ that converges to a limit $\tilde{k}$, where $\tilde{k} \in K_i$ for every $i \in I$. Then $\tilde{k} \in \bigcap_{i \in I} K_i$. Thus any sequence in $\bigcap_{i \in I} K_i$ has a convergent subsequence that converges to a limit in $\bigcap_{i \in I} K_i$. Thus $\bigcap_{i \in I} K_i$ is sequentially compact, and therefore compact. $\square$

Problem 6

Let $f \in C([0, 1])$ be such that $\int_0^1 x^n f(x)dx = 0$ for all integers $n \geq 0$. Prove that $f(x) = 0$, for all $x \in [0, 1]$.

By the linearity of integrals, $\int_0^1 p(x)f(x)dx = 0$ for any polynomial $p(x)$. The set of polynomials is dense in $C([0,1])$ by the Weierstrauss Approximation Theorem, and thus there is some sequence of polynomials $\{p_n\}$ such that $p_n \to f$.

$$\int_0^1 f^2(x)dx = \int_0^1 f(x)f(x) = \int_0^1 \lim_{n\to\infty} p_n(x)f(x)dx$$

We can pull the limit out of the integral since f and p are continuous, and so their product is continuous. Since $p(x)f(x)$ is continuous on a compact set, $p(x)f(x)$ is uniformly continuous. Thus,

$$\int_0^1 f^2(x)dx = \int_0^1 f(x)f(x) = \int_0^1 \lim_{n\to\infty} p_n(x)f(x)dx = \lim_{n\to\infty} \int_0^1 p_n(x)f(x) = \lim_{n\to\infty} 0 = 0$$

Since $f^2 \geq 0$ for $x \in [0,1]$ and $\int_0^1 f^2(x)dx = 0$, then $f^2(x) = 0$ for all $x \in [0,1]$. Thus, $f(x) = 0$ for all $x \in [0,1]$. $\square$

Problem 7

Let (p_n) be a sequence of real-valued polynomial functions defined on the interval $[0,1]$ with bounded degree, i.e., there exists $0 \leq D \in \mathbb{Z}$, and sequences of real numbers $(a_n(k))_{n=1}^{\infty}$, $k = 0, \dots, D$, such that

$$p_n(x) = a_n(0) + a_n(1)x + \cdots + a_n(D)x^D, \quad x \in [0,1]. \tag{3}$$

a)

Prove that if $\|p_n\|_\infty \to 0$, then $\lim_{n\to\infty} \max_{0\leq k\leq D} a_n(k) = 0$ (Hint: try induction on D).

Base Case: Let $\{p_n\}$ be a sequence of polynomials of degree 0, i.e. each polynomial is a constant function, or $p_n(x) = a_n(0)$. Assume $\|p_n\|_\infty \to 0$. Then $\|a_n(0)\|_\infty \to 0$. But $\|a_n(0)\|_\infty = a_n(0) = \max_{0\leq k\leq 0} a_n(k)$. Thus $\lim_{n\to\infty} \max_{0\leq k\leq 0} a_n(k) = \lim_{n\to\infty} \|a_n(0)\|_\infty = 0$.

Inductive Hypothesis: Assume that if $\{p_n\}_n$ is a sequence of degree D polynomials such that $\|p_n\|_\infty \to 0$, then $\lim_{n\to\infty} \max_{0\leq k\leq D} a_n(k) = 0$.

Inductive Step: Now let $\{p_n\}_n$ be a sequence of degree $D+1$ polynomials, i.e. $p_n(x) = a_n(0) + a_n(1)x + \cdots + a_n(D+1)x^{D+1}$. Assume $\|p_n\|_\infty \to 0$. Since every polynomial in the sequence has degree $D+1$, then $\|p_n'\|_\infty \to 0' = 0$ (note that in general, $\|f_n - f\|_\infty \to 0 \not\Longrightarrow \|f_n' - f'\|_\infty \to 0$. However, if f_n is a sequence of polynomials of uniformly bounded degree, then the implication does hold). But $\{p_n'\}$ is a sequence of degree D polynomials, and thus the Induction Hypothesis implies $\lim_{n\to\infty} \max_{1\leq k\leq D+1} a_n(k) = 0$. This is equivalent to each $a_n(i) \to 0$ as $n \to \infty$ for $i = 1, \dots, D+1$. Thus $\|p_n\|_\infty \to 0 \implies \|a_n(0)\|_\infty \to 0$. Clearly, this implies $a_n(0) \to 0$. Thus each $a_n(i) \to 0$ as $n \to \infty$ for $i = 0, \dots, D+1$, which is equivalent to $\lim_{n\to\infty} \max_{0\leq k\leq D+1} a_n(k) = 0$. $\square$

b)

Show that the assumption of a uniform bound on the degree of p_n is essential for the implication in part a) to hold. Specifically, find a sequence of polynomials $p_n(x) = \sum_{k=0}^{D_n} a_n(k)x^k$, such that $\|p_n\|_\infty \to 0$ and

$$\overline{\lim_n} \max_{0\le k\le D_n} |a_n(k)| = 1 \tag{4}$$

Let $\{q_i\}_i \in (0,1)\cap\mathbb{Q}$ be an enumeration of the rational numbers between 0 and 1. Then define the sequence $\{q_n\}_n$ as

$$\begin{aligned} q_n(x) &= x(x-1)\prod_{i=1}^{n}(x-q_i) \\ &= b_n(0) + b_n(1)x + \cdots + b_n(n+2)x^{n+2} \end{aligned}$$

For ease, denote $B_n = \max_{0\le k\le n+2} |b_n(k)|$. We know $b_n(n+2) = 1\ \forall n \ge 1$. Thus $B_n \ge 1\ \forall n \ge 1$. Then define the sequence $\{p_n\}$ as

$$\begin{aligned} p_n(x) &= \frac{q_n(x)}{B_n} \\ &= a_n(0) + a_n(1)x + \cdots + a_n(n+2)x^{n+2} \end{aligned}$$

For ease, denote $A_n = \max_{0\le k\le n+2} |a_n(k)|$. Then $A_n = 1\ \forall n \ge 1$, which shows equation (4) holds.

Let x_n be the value at which $|p_n|$ attains its maximum, i.e. $x_n \in [0,1]$ such that

$$\begin{aligned} \max_{x\in[0,1]} |p_n| &= |p_n(x_n)| \\ &= \left| \frac{x_n(x_n-1)\prod_{i=1}^{n}(x_n-q_i)}{A_n} \right| \\ &= \frac{|x||x_n-1|\prod_{i=1}^{n}|x_n-q_i|}{A_n} \end{aligned}$$

Note that $|x_n| \le 1$, $|x_n - 1| \le 1$, and $|x_n - q_i| < 1$ for each $i = 1, \ldots, n$. Then their product $q_n(x_n) < 1$. Moreover, as $n \to \infty$, the product eventually approaches 0 since each $x_n - q_i < 1$. Then since $\max_{x\in[0,1]} |p_n| = \sup_{x\in[0,1]} |p_n| = \|p_n\|_\infty$, then $\|p_n\|_\infty \to 0$.