Name: ____________________

MAT201A | University of California, Davis | Fall 2015

Homework # 7

(Due Monday, November 23)

Problem 1. Let X be a Banach space and consider the map $\exp : \mathcal{B}(X) \to \mathcal{B}(X)$, defined by the series

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n, \tag{1}$$

where, by convention, $T^0 = \mathbb{1}$.

a) Prove that the map exp is well-defined by showing that this series (1) converges absolutely for all $T \in \mathcal{B}(X)$.

b) Prove the bound

$$\| \exp(T) \| \leq e^{\|T\|}, \quad T \in \mathcal{B}(X).$$

c) Prove that exp is a continuous map when $\mathcal{B}(X)$ is considered with the uniform operator topology.

Problem 2. Let $\mathcal{B}$ be a Banach space and consider a continuous function $A : \mathbb{R} \to \mathcal{B}$.

a) For $a < b \in \mathbb{R}$, $\int_a^b A(t)dt$ can be defined as the limit of Riemann sums. E.g., show that the following limit exists in the uniform topology:

$$\int_a^b A(t)dt = \lim_{N\to\infty} 2^{-N}(b-a) \sum_{n=1}^{2^N} A(a + n2^{-N}(b-a)).$$

b) $A(\cdot)$ is called differentiable on an interval I if for all $t \in I$, the following limit exists in the uniform topology:

$$A'(t) := \frac{d}{dt} A(t) := \lim_{h\to 0} \frac{A(t+h) - A(t)}{h},$$

and the function $t \mapsto A'(t)$ is continuous on I. Let X be a Banach space, $A \in \mathcal{B}(X)$, and consider the map $t \mapsto \exp(tA)$. Show that A is differentiable and that its derivative is given by the following two expressions:

$$\frac{d}{dt} \exp(tA) = A \exp(tA) = \exp(tA)A.$$

c) Let $A(t)$ be a differentiable function $\mathbb{R} \supset [0,1] \to \mathcal{B}(X)$. Show that $\exp(A(t))$ is differentiable and that its derivative satisfies:

$$\frac{d}{dt} \exp(A(t)) = \int_0^1 \exp(s(A(t))A'(t) \exp((1-s)A(t))ds.$$

(Hint: consider $B(s) = \exp(s(A(t+h)) \exp((1-s)A(t))$ and note that $A(t+h) - A(t) = B(1) - B(0)$).

Problem 3. Let X be a Banach space and consider the $T \in \mathcal{B}(X)$ such that $\|T\| < 1$. Prove that $\mathbb{1}+T$ is invertible and that its inverse is given by the following uniformly convergent series:

$$(\mathbb{1} + T)^{-1} = \sum_{n=0}^{\infty} (-1)^n T^n.$$