Name: ________________________________________

MAT201A University of California, Davis Fall 2015

# Homework # 3

(Due Monday, October 19)

**Problem 1.** Let $(f_n)$ be a sequence in $C([0,1])$ converging uniformly to the function $f(x) = -x\log x$, on $[0,1]$. Define

$$A = \{f_n \mid n \geq 1\} \cup \{f\}.$$

Is $A$ *compact*, or *precompact but not compact*, or *not precompact*, considered as a subset of $(C([0,1]), \|\cdot\|_{\sup})$? Justify your answer.

**Problem 2.** Let $f \in C([a,b])$. Prove that

$$\left|\int_a^b f(x)dx\right| \leq |b-a|^{1/2}\left(\int_a^b f(x)^2 dx\right)^{1/2}.$$

**Problem 3.** For $M > 0$, define $A_M \subset C([a,b])$ as follows:

$$A_M = \{f \in C([a,b]) \mid f' \in C([a,b]), f(a) = f(b) = 0, \text{ and } \int_a^b f'(x)^2 dx \leq M\}.$$

Prove that $A_M$ is precomnpact in $(C([a,b]), \|\cdot\|_{\sup})$.

**Problem 4.** Consider functions $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \quad x \in [0,1], \tag{1}$$

where for all $n \geq 1$, $a_n \in \mathbb{R}$, and such that $\sum_{n=1}^{\infty} |a_n| < +\infty$.

**a)** Prove that $f \in C([0,1])$.

**b)** Prove that the set $A$ defined by

$$A = \{f \in C([0,1]) \mid f \text{ is of the form (1) and } \|f\|_{\sup} \leq 1\},$$

is not precompact in $(C([0,1]), \|\cdot\|_{\sup})$.

**c)** Prove that the set $B$ define by

$$B = \{f \in C([0,1]) \mid f \text{ is of the form (1) and } \sum_{n=1}^{\infty} n^2|a_n|^2 \leq 1\},$$

is precompact in $(C([0,1]), \|\cdot\|_{\sup})$.