

HW #1

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Problem 1

Let (f_n) be a sequence in $C([0, 1])$ converging uniformly to the function $f(x) = -x \log x$ on $[0, 1]$. Define

$$A = \{f_n \mid n \geq 1\} \cup \{f\}$$

Is A compact, or precompact but not compact, or not precompact, considered as a subset of $(C([0, 1]), \|\cdot\|_{\text{sup}})$? Justify your answer.

Let $G = \{G_i \mid i \in I\}$ for some index set I be an open cover of A . Then $\exists i_0 \in I$ such that $f \in G_{i_0}$. Since G_{i_0} is open, $\exists \epsilon > 0$ such that $B_\epsilon(f) \subset G_{i_0}$. Since f_n converges uniformly to f , $\exists N \in \mathbb{N}$ such that $n \geq N \implies f_n \in B_\epsilon(f)$ (and thus $f_n \in G_{i_0}$). Then there are only finitely many functions in A which are potentially not elements of $B_\epsilon(f)$, specifically, f_1, \dots, f_{N-1} . Since G is an open cover of A , $\exists i_1, \dots, i_{N-1} \in I$ such that $f_1 \in G_{i_1}, \dots, f_{N-1} \in G_{i_{N-1}}$. Thus $\tilde{G} = \{G_{i_0}, G_{i_1}, \dots, G_{i_{N-1}}\}$ is a finite open cover of A . Thus A is compact. \square

Problem 2

Let $f \in C([a, b])$. Prove that

$$\left| \int_a^b f(x) dx \right| \leq |b - a|^{1/2} \left(\int_a^b f(x)^2 dx \right)^{1/2}$$

Define the inner product $\langle f, g \rangle$ on $C([a, b])$ to be the \mathbf{L}^2 inner product, or

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

Then consider Cauchy-Schwarz inequality: $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$, $\forall f, g \in C([a, b])$. Pick $g(x) \equiv 1$. Then,

$$|\langle f, 1 \rangle| = \left| \int_a^b f(x) dx \right|$$

$$\begin{aligned}
&\leq \left(\int_a^b f(x)^2 dx \right)^{1/2} \left(\int_a^b 1^2 dx \right)^{1/2} && \text{by the Cauchy-Schwarz inequality} \\
&= \left(x \Big|_a^b \right)^{1/2} \left(\int_a^b f(x)^2 dx \right)^{1/2} \\
&= |b-a|^{1/2} \left(\int_a^b f(x)^2 dx \right)^{1/2} && b-a = |b-a| \text{ since } b \geq a.
\end{aligned}$$

□

Problem 3

For $M > 0$, define $A_M \subset C([a, b])$ as follows:

$$A_M = \{f \in C([a, b]) \mid f' \in C([a, b]), f(a) = f(b) = 0, \text{ and } \int_a^b f'(x)^2 dx \leq M\}$$

Prove that A_M is precompact in $(C([a, b]), \|\cdot\|_{sup})$.

Let $\tilde{x} \in [a, b]$ and $f \in A_M$. Then,

$$\begin{aligned}
|f(\tilde{x})| &= |f(\tilde{x}) - 0| = |f(\tilde{x}) - f(a)| \\
&= \left| \int_a^{\tilde{x}} f'(x) dx \right| && \text{by the Fundamental Theorem of Calculus} \\
&\leq |\tilde{x} - a|^{1/2} \left(\int_a^{\tilde{x}} f'(x)^2 dx \right)^{1/2} && \text{by Problem 2} \\
&\leq |b - a|^{1/2} \left(\int_a^{\tilde{x}} f'(x)^2 dx \right)^{1/2} && \text{since } \tilde{x} \in [a, b] \\
&\leq |b - a|^{1/2} \sqrt{M} && \text{by the definition of } A_M
\end{aligned}$$

Thus $f(\tilde{x})$ is uniformly bounded by $|b-a|^{1/2}\sqrt{M}$ for any $\tilde{x} \in [a, b]$ and any $f \in A_M$. Thus A_M is bounded.

Pick $\epsilon > 0$ and $x_1 \in [a, b]$. Assume $d(x_1, x_2) < \frac{\epsilon^2}{M}$. Then

$$\begin{aligned}
|d(f(x_1), f(x_2))| &= |f(x_1) - f(x_2)| \\
&= \left| \int_{x_2}^{x_1} f'(x) dx \right| && \text{by the Fundamental Theorem of Calculus} \\
&\leq |x_2 - x_1|^{1/2} \left(\int_{x_2}^{x_1} f'(x)^2 dx \right)^{1/2} && \text{by Problem 2} \\
&< \sqrt{\frac{\epsilon^2}{M}} \sqrt{M} && \text{by assumption and the definition of } A_M \\
&= \epsilon
\end{aligned}$$

Thus A_M is equicontinuous. By the Arzelà-Ascoli Theorem, A_M is precompact. □

Problem 4

Consider functions $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \quad x \in [0, 1] \quad (1)$$

where for all $n \geq 1$, $a_n \in \mathbb{R}$, and such that $\sum_{n=1}^{\infty} |a_n| < +\infty$.

a)

Prove that $f \in C([0, 1])$.

Consider the sequence $f_k(x) = \sum_{n=1}^k a_n \sin(n\pi x)$. Clearly, $\lim_{k \rightarrow \infty} f_k = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = f(x)$. Since $\sin(n\pi x) \in C([0, 1])$ for $n = 1, 2, \dots$, and linear combinations of continuous functions are continuous, $f_k \in C([0, 1])$ for $k = 1, 2, \dots$. It suffices to show that f_k is a Cauchy sequence. Then, the completeness of $C([0, 1])$ will imply that the limit of f_k is in $C([0, 1])$.

First, since $\lim_{k \rightarrow \infty} \sum_{n=1}^k |a_n| = L < \infty$, then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $i > j \geq N \implies \sum_{n=1}^i |a_n| - \sum_{n=1}^j |a_n| = \sum_{n=j+1}^i |a_n| < \epsilon$. Now pick $\epsilon > 0$, and assume $i > j \geq N$, such that $\sum_{n=j+1}^i |a_n| < \epsilon$. Then,

$$\begin{aligned} d(f_i, f_j) &= \|f_i - f_j\|_{\sup} = \left\| \sum_{n=j+1}^i a_n \sin(n\pi x) \right\|_{\sup} \\ &\leq \sum_{n=j+1}^i |a_n| |\sin(n\pi x)| \quad \text{by the Triangle Inequality} \\ &\leq \sum_{n=j+1}^i |a_n| \quad \text{since } |\sin(n\pi x)| \leq 1 \text{ for any } n. \\ &< \epsilon \end{aligned}$$

Thus f_k is a Cauchy sequence, and since $C([0, 1])$ is complete, the limit of f_k , which is f , must be an element of $C([0, 1])$. \square

b)

Prove that the set A defined by

$$A = \{f \in C([0, 1]) \mid f \text{ is of the form (1) and } \|f\|_{\sup} \leq 1\}$$

is not precompact in $(C([0, 1]), \|\cdot\|_{\sup})$.

Consider the family of functions $\mathcal{F} = \{\sin(n\pi x)\}$ for $n = 1, 2, \dots$, and choose any $\delta > 0$. Choose $N > \frac{2}{\delta}$. Then the period of $f_N = \sin(N\pi x) = \frac{2}{N} < \delta$. Then choose a minimum x_{\min} and a maximum x_{\max} of f_N such that $|x_{\min} - x_{\max}| < \delta$ (this can be done since the period of f_N is less than δ). Since $f_N(x_{\min}) = -1$ and $f_N(x_{\max}) = 1$, then $|f_N(x_{\min}) - f_N(x_{\max})| = 2$. Thus the family \mathcal{F} is not equicontinuous, and since $\mathcal{F} \subset A$, A is not equicontinuous. By the Arzelà-Ascoli Theorem, A is not precompact. \square

c)

Prove that the set B defined by

$$B = \{f \in C([0, 1]) \mid f \text{ is of the form (1) and } \sum_{n=1}^{\infty} n^2 |a_n|^2 \leq 1\}$$

is precompact in $(C([0, 1]), \|\cdot\|_{sup})$.

By the mean value theorem, $\forall x, y \in \mathbb{R}$, $\exists c \in [x, y]$ such that $(\sin x - \sin y) = (\cos c)(x - y)$, and since $|\cos c|$ is bounded by 1, then $|\sin x - \sin y| \leq |x - y|$.