

HW #4

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Problem 1

Let $a < b \in \mathbb{R}$. Prove that $C([a, b])$ with sup norm is a separable metric space.

Define the set of polynomials with rational coefficients on $[a, b]$ as $\mathbb{P}_{\mathbb{Q}}([a, b])$. We will show this set is countable and dense in $C([0, 1])$, proving $C([0, 1])$ is separable.

To show $\mathbb{P}_{\mathbb{Q}}([a, b])$ is countable, it suffices to show $\bigcup_{n=0}^{\infty} \mathbb{Q}^n$ is countable, since there is a bijection f between these two sets. Namely,

$$f(q_0, q_1, \dots, q_k) = q_0 + q_1x + \dots + q_kx^k$$

However, the countably infinite union of countable sets is countable. Thus $\bigcup_{n=0}^{\infty} \mathbb{Q}^n$ is countable, which proves $\mathbb{P}_{\mathbb{Q}}([a, b])$ is countable.

To show $\mathbb{P}_{\mathbb{Q}}([a, b])$ is dense in $C([a, b])$, we will show it is dense in the set of polynomials with real coefficients on $[a, b]$ (denoted $\mathbb{P}_{\mathbb{R}}([a, b])$) and use a diagonal argument to show it is dense in $C([a, b])$.

Lemma (1). $\mathbb{P}_{\mathbb{Q}}([a, b])$ is dense in $\mathbb{P}_{\mathbb{R}}([a, b])$.

Proof. Choose $p \in \mathbb{P}_{\mathbb{R}}([a, b])$. Then $p = r_0 + r_1x + \dots + r_kx^k$ for some $k \in \mathbb{N}$. Since \mathbb{Q} is dense in \mathbb{R} , there exist sequences $(q_{a,j})_j \in \mathbb{Q}$ such that $q_{a,j} \rightarrow r_a$ for $a = 1, \dots, k$. Then construct a sequence $(q_{\ell})_{\ell} \in \mathbb{P}_{\mathbb{Q}}([a, b])$ by

$$q_{\ell} = q_{\ell,0} + q_{\ell,1}x + \dots + q_{\ell,k}x^k$$

Then $\lim_{\ell \rightarrow \infty} q_{\ell} = p$ for each $x \in [a, b]$. Thus q_{ℓ} converges to p in a pointwise manner. However, since p is continuous, and each q_{ℓ} is continuous, $q_{\ell} \rightarrow p$ uniformly. Thus $\mathbb{P}_{\mathbb{Q}}([a, b])$ is dense in $\mathbb{P}_{\mathbb{R}}([a, b])$. \square

By the Weierstrauss approximation theorem, $\mathbb{P}_{\mathbb{R}}([a, b])$ is dense in $C([a, b])$. Thus for any $f \in C([a, b])$, $\exists (p_n)_n \in \mathbb{P}_{\mathbb{R}}([a, b])$ such that $p_n \rightarrow f$ uniformly. Choose a subsequence of p_n (for ease, call this subsequence p_n) such that $\|p_n - f\| < \frac{1}{2^n}$ for all $n \in \mathbb{N}^+$.

By Lemma 1, for each p_n in the sequence, $\exists (q_{n,\ell})_{\ell} \in \mathbb{P}_{\mathbb{Q}}([a, b])$ such that $q_{n,\ell} \rightarrow p_n$ uniformly.

Then we can construct a sequence in $\mathbb{P}_{\mathbb{Q}}([a, b])$ which converges to f . Choose $w_1 = q_{1,L_{1/2}}$ where $\ell \geq L_{1/2} \implies \|q_{1,\ell} - p_1\|_{\sup} < \frac{1}{2}$. Then choose $w_2 = q_{2,L_{1/4}}$ where $\ell \geq L_{1/4} \implies$

$\|q_{2,\ell} - p_2\|_{\sup} < \frac{1}{4}$. In general, for all $m \in \mathbb{N}^+$, choose $w_m = q_{m,L_{1/2m}}$ where $\ell \geq L_{1/2m} \implies \|q_{m,\ell} - p_m\|_{\sup} < \frac{1}{2m}$. Then

$$\begin{aligned}\|w_n - f\|_{\sup} &\leq \|w_n - p_n\|_{\sup} + \|p_n - f\|_{\sup} \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty\end{aligned}$$

Thus any arbitrary $f \in C([a, b])$ is the limit of a sequence of polynomials in the countable set $\mathbb{P}_{\mathbb{Q}}([a, b])$ under the supremum norm. Thus $C([a, b])$ is a separable metric space. \square

A SIMPLER PROOF

Lemma (2). *If A is dense in B and B is dense in C , then A is dense in C .*

Proof. Since A is dense in B , then $\overline{A} = B$. Since B is dense in C , then $\overline{B} = C$. Then $\overline{A} = \overline{\overline{A}} = \overline{B} = C$. Thus A is dense in C . \square

By lemmas 1 and 2 and the Weierstrauss Approximation Theorem, $\mathbb{P}_{\mathbb{Q}}$ is dense in $C([0, 1])$. Since $\mathbb{P}_{\mathbb{Q}}([a, b])$ is countable, $C([0, 1])$ is separable. \square

Problem 2

Let $k \in C([0, 1] \times [0, 1])$, and define a map $T : C([0, 1]) \rightarrow C([0, 1])$ by

$$(Tf)(x) = \int_0^1 k(x, y)f(y)dy$$

Prove that the set $\{Tf \mid \|f\|_{\sup} \leq 1\}$ is equicontinuous.

Pick $x \in [0, 1]$ and let $\varepsilon > 0$. Then the continuity of k implies $\exists \delta > 0$ such that $d((x, y), (x_0, y_0)) < \delta \implies d(k(x, y), k(x_0, y_0)) < \varepsilon$. Consider $\tilde{x} \in [0, 1]$ and assume $d(x, \tilde{x}) < \delta$. Then $d((x, y), (\tilde{x}, y)) < \delta$ for any $y \in [0, 1]$. Then $d(k(x, y), k(\tilde{x}, y)) < \varepsilon$. Now choose $g \in \{Tf \mid \|f\|_{\sup} \leq 1\}$. Then

$$\begin{aligned}d(g(x), g(\tilde{x})) &= \left| \int_0^1 (k(x, y) - k(\tilde{x}, y))f(y)dy \right| \quad \text{for some } f \text{ such that } \|f\|_{\sup} \leq 1 \\ &\leq \int_0^1 |k(x, y) - k(\tilde{x}, y)| |f(y)| dy \\ &< \varepsilon \int_0^1 |f(y)| dy \\ &< \varepsilon \int_0^1 dy \quad \text{since } \|f\|_{\sup} < 1 \\ &= \varepsilon\end{aligned}$$

Thus $\{Tf \mid \|f\|_{\sup} \leq 1\}$ is equicontinuous. \square

Problem 3

Let (X, \mathcal{T}) be a topological space. If $G \subset X$ is open and $F \subset X$ is closed, prove that $G \setminus F$ is open.

Since F is closed, F^C is open. Also, $G \setminus F = G \cap F^C$. Since the finite intersection of open sets is open, $G \setminus F$ is open. \square

Problem 4

Let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on a non-empty set X .

a)

Is $\mathcal{T}_1 \cap \mathcal{T}_2$ a topology on X ?

Yes.

Let $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2$. Since \emptyset and X are elements of all topologies, they are elements of the arbitrary intersection of topologies. Thus $\emptyset, X \in \mathcal{T}$.

Consider $\{G_\alpha \mid \alpha \in I\}$ where each $G_\alpha \in \mathcal{T}$. Then each $G_\alpha \in \mathcal{T}_1$ and each $G_\alpha \in \mathcal{T}_2$. Then $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}_1$ and $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}_2$. Thus $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}$.

Consider $\{G_i \mid i = 1, \dots, N\}$ where each $G_i \in \mathcal{T}$. Then each $G_i \in \mathcal{T}_1$ and each $G_i \in \mathcal{T}_2$. Then $\bigcap_{i=1}^n G_i \in \mathcal{T}_1$ and $\bigcap_{i=1}^n G_i \in \mathcal{T}_2$. Thus $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

Thus $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X . \square

b)

Is $\mathcal{T}_1 \cup \mathcal{T}_2$ a topology on X ?

No. We form a counterexample:

Let $X = \{1, 2, 3\}$, and let $\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{2\}, X\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{1\}, \{2\}, X\}$ is not a topology since $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. \square

Problem 5

Give an example of two metric spaces (X_1, d_1) and (X_2, d_2) , such that X_1 and X_2 are homeomorphic as topological spaces but X_1 is a complete metric space while X_2 is not.

Let $X_1 = [1, \infty)$ and $X_2 = (0, 1]$. Then choose $f : X_1 \rightarrow X_2$ by $f(x) = \frac{1}{x}$. f is clearly bijective and continuous, and $f^{-1} : X_2 \rightarrow X_1$ by $f^{-1}(x) = \frac{1}{x}$ is also continuous. Thus X_1 and X_2 are homeomorphic as topological spaces, but X_1 is a complete metric space while X_2 is not.

Problem 6

Two metrics, d_1 and d_2 , on the same space X are called equivalent if there exist constants $c, C > 0$ such that

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y), \text{ for all } x, y \in X$$

a)

Show that the topologies on X defined by two equivalent metrics are identical.

Let \mathcal{T}_1 and \mathcal{T}_2 be the topologies defined by the open sets as defined by the metrics d_1 and d_2 , respectively. Denote open balls, with respect to the metric d_i , of radius ε around x as $B_{i,\varepsilon}(x)$ for $i = 1, 2$.

Let $G \in \mathcal{T}_1$. Then G is open with respect to d_1 . Then $\forall x \in G, \exists \varepsilon$ such that $B_{1,\varepsilon}(x) \subset G$. Note that $d_1(x, y) < \varepsilon$ for each $y \in B_{1,\varepsilon}(x)$. Now consider $B_{2,c\varepsilon}(x)$. If $y \in B_{2,c\varepsilon}(x)$, then $d_2(x, y) < c\varepsilon \iff \frac{1}{c}d_2(x, y) < \varepsilon$. But since $d_1(x, y) \leq \frac{1}{c}d_2(x, y)$ for all $x, y \in X$, this implies $d_1(x, y) < \varepsilon$, which then implies $y \in B_{1,\varepsilon}(x)$. Thus $B_{2,c\varepsilon}(x) \subset B_{1,\varepsilon}(x)$, and so $B_{2,c\varepsilon}(x) \subset G$. Thus there is an open ball with respect to the metric d_2 around any point x in G (in particular $B_{2,c\varepsilon}(x)$). Thus G is open with respect to the metric d_2 , which shows $G \in \mathcal{T}_2$. Thus $\mathcal{T}_1 \subset \mathcal{T}_2$.

Lemma (3). *Metric equivalence is an equivalence relation.*

Proof. If d_1 is a metric on X , then $1 \cdot d_1(x, y) \leq d_1(x, y) \leq 1 \cdot d_1(x, y)$. Thus d_1 is equivalent to d_1 . If d_1 is equivalent to d_2 , then $\exists c, C > 0$ such that $cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)$ for all $x, y \in X$. But this implies $\frac{1}{C}d_2(x, y) \leq d_1(x, y) \leq \frac{1}{c}d_2(x, y)$. Since $\frac{1}{C}$ and $\frac{1}{c}$ are greater than 0, this shows d_2 is equivalent to d_1 . If d_1 is equivalent to d_2 and d_2 is equivalent to d_3 , then $\exists c_1, C_1 > 0$ such that $c_1d_1(x, y) \leq d_2(x, y) \leq C_1d_1(x, y)$ for all $x, y \in X$, and $\exists c_2, C_2 > 0$ such that $c_2d_2(x, y) \leq d_3(x, y) \leq C_2d_2(x, y)$ for all $x, y \in X$.

$$c_1c_2d_1(x, y) \leq c_2d_2(x, y) \leq d_3(x, y) \leq C_2d_2(x, y) \leq C_1C_2d_1(x, y)$$

and since c_1c_2 and C_1C_2 are greater than 0, this shows d_1 is equivalent to d_3 . Thus metric equivalence is an equivalence relation. \square

By Lemma 3, we can exchange \mathcal{T}_1 and \mathcal{T}_2 and show that $\mathcal{T}_2 \subset \mathcal{T}_1$, which proves $\mathcal{T}_1 = \mathcal{T}_2$. Thus, the topologies on X defined by two equivalent metrics are identical. \square

b)

Let (X, d) be a metric space. Show that there exists a metric d_b with the property that $d_b(x, y) \leq 1$, for all $x, y \in X$, such that the topology on X derived from the metric d_b is the same as the one derived from the metric d .

c)

Give an example of the situation in part b) with the metrics d and d_b that are not equivalent.

Problem 7

Prove Theorem 4.7 of the textbook.

Lemma (4). *Let (X, \mathcal{T}) be a topological space. Then $G \in \mathcal{T}$ if and only if G is a neighborhood of x for each $x \in G$.*

Proof. Let $G \in \mathcal{T}$. Then since $G \subset G$, then G contains an open set which contains every $x \in G$ (namely, G). Then G is a neighborhood of x for each $x \in G$. Now let H be a neighborhood of x for each $x \in H$. Then $\exists H_x \subset H$ such that $x \in H_x$ and $H_x \in \mathcal{T}$ for each $x \in H$. Then since $H_x \subset H$ for each $x \in H$, then $\bigcup_{x \in H} H_x \subset H$. However, since $x \in H_x$ for every $x \in H$, then $H \subset \bigcup_{x \in H} H_x$. Thus $\bigcup_{x \in H} H_x = H$. Then since H is an arbitrary union of open sets, then H is open, i.e. $H \in \mathcal{T}$. \square

Lemma (5). *Let (X, \mathcal{T}) be a topological space. Then $G \in \mathcal{T}$ if and only if G contains a neighborhood of x for each $x \in G$.*

Proof. Let $G \in \mathcal{T}$. Then by Lemma 4, G is a neighborhood of x for each $x \in G$, but since $G \subset G$, then G contains a neighborhood of x for each $x \in G$. Now let H contain a neighborhood of x for each $x \in H$. Then for each $x \in H$, $\exists H_x \subset H$ such that H_x is a neighborhood of x . Then $\exists G_x \subset H_x$ such that $x \in G_x$ and $G_x \in \mathcal{T}$ for each $x \in H$. Then for each $x \in H$, $G_x \subset H_x \subset H$. Thus H is a neighborhood of x for each $x \in H$. Then by Lemma 4, $H \in \mathcal{T}$. \square

Theorem (4.7). *Let (X, \mathcal{T}) and (Y, \mathcal{S}) be two topological spaces and $f : X \rightarrow Y$. Then f is continuous on X if and only if $f^{-1}(G) \in \mathcal{T}$ for every $G \in \mathcal{S}$.*

Proof. Suppose f is continuous. Then by the definition of continuity, for each $x \in X$, for each neighborhood W of $f(x)$, there is a neighborhood V of x such that $f(V) \subset W$. Now choose $G \in \mathcal{S}$. Then for each $x \in f^{-1}(G)$, G is a neighborhood of each $f(x)$ by Lemma 4. Since f is continuous, there is a neighborhood H of x such that $f(H) \subset G$, which implies $H \subset f^{-1}(G)$. Thus $f^{-1}(G)$ contains a neighborhood of each x in $f^{-1}(G)$, thus by Lemma 5, $f^{-1}(G) \in \mathcal{T}$. Now suppose $f^{-1}(G) \in \mathcal{T}$ for every $G \in \mathcal{S}$. Then pick $x \in X$ and a neighborhood W of $f(x)$. By the definition of neighborhood, $\exists H \in \mathcal{S}$ such that $f(x) \in H$ and $H \subset W$. By assumption, $f^{-1}(H) \in \mathcal{T}$. Since $f(x) \in H$, then $x \in f^{-1}(H)$. Then by Lemma 4, $f^{-1}(H)$ is a neighborhood of x . Also, $f(f^{-1}(H)) = H \subset W$. Thus f is continuous. \square