Name: ______________________________

MAT201A University of California, Davis Fall 2015

Homework # 4

(Due Monday, October 26)

Problem 1. Let $a < b \in \mathbb{R}$. Prove that $C([a,b])$ with the sup norm is a separable metric space.

Problem 2. Let $k \in C([0,1] \times [0,1])$, and define a map $T : C([0,1]) \to C([0,1])$ by

$$(Tf)(x) = \int_0^1 k(x,y)f(y)dy.$$

Prove that the set $\{Tf \mid \|f\|_{\sup} \leq 1\}$ is equicontinuous.

Problem 3. Let $(X, \mathcal{T})$ be a topological space. If $G \subset X$ is open and $F \subset X$ is closed, prove that $G \setminus F$ is open.

Problem 4. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a non-empty set X.

a) Is $\mathcal{T}_1 \cap \mathcal{T}_2$ a topology on X?

b) Is $\mathcal{T}_1 \cup \mathcal{T}_2$ a topology on X?

Prove your answers.

Problem 5. Give an example of two metric spaces (X_1, d_1) and (X_2, d_2), such that X_1 and X_2 are homeomorphic as topological spaces but X_1 is a complete metric space while X_2 is not.

Problem 6. Two metrics, d_1 and d_2, on the same space X are called *equivalent* if there exist constants $c, C > 0$ such that

$$cd_1(x,y) \leq d_2(x,y) \leq Cd_1(x,y), \text{ for all } x, y \in X.$$

a) Show that the topologies on X defined by two equivalent metrics are identical.

b) Let (X, d) be a metric space. Show that there exists a metric d_b with the property that $d_b(x,y) \leq 1$, for all $x, y \in X$, and such that the topology on X derived from the metric d_b is the same as the one derived from the metric d.

c) Give an example of the situation described in part b) with metrics d and d_b that are *not* equivalent.

Problem 7. Prove Theorem 4.7 of the textbook.