HW #1

Sam Fleischer

October 5, 2015

Problem 1

Let (X, d) be a metric space, and let $x, y, w, z \in X$.

a)

Prove that $d(x, y) \geq |d(x, z) - d(z, y)|$.

By the triangle inequality, $d(x, z) \leq d(x, y) + d(z, y)$ and $d(z, y) \leq d(x, y) + d(x, z)$. These are equivalent to $d(x, y) \geq d(x, z) - d(z, y)$ and $d(x, y) \geq d(z, y) - d(x, z)$. Thus, $d(x, y) \geq |d(x, z) - d(z, y)|$. $\square$

b)

Prove that $d(x, y) + d(z, w) \geq |d(x, z) - d(y, w)|$.

By the triangle inequality, $d(x, z) \leq d(x, w) + d(z, w)$ and $d(x, w) \leq d(x, y) + d(y, w)$. By substitution, $d(x, y) + d(y, w) + d(w, z) \geq d(x, z)$, or

$$d(x, y) + d(z, w) \geq d(x, z) - d(y, w) \tag{1}$$

Again by the triangle inequality, $d(y, z) \leq d(x, y) + d(x, z)$ and $d(y, z) \leq d(y, z) + d(w, z)$. By substitution, $d(x, y) + d(z, w) + d(x, z) \geq d(y, w)$, or

$$d(x, y) + d(z, w) \geq d(y, w) - d(x, z) \tag{2}$$

Thus, combining (1) and (2),

$$d(x, y) + d(z, w) \geq |d(x, z) - d(y, w)| \tag{3}$$

c)

Let (x_n) and (y_n) be converging sequences in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Prove that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

By the definition of limits, $\forall \frac{\epsilon}{2} > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n > N_1 \implies d(x_n, x) < \frac{\epsilon}{2}$ and $n > N_2 \implies d(y_n, y) < \frac{\epsilon}{2}$. Then for $n > \max\{N_1, N_2\}$, and by the triangle inequality applied twice, $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y)$, or

$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{4}$$

Again, by the triangle inequality applied twice, $d(x, y) \leq d(x_n, x) + d(x_n, y_n) + d(y_n, y)$, or

$$d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{5}$$

Thus, combining (4) and (5),

$$|d(x_n, y_n) - d(x, y)| < \epsilon \tag{6}$$

which proves $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$. $\square$

Problem 2

Show that the limit of a convergent sequence in a metric space is unique. I.e., if, for a sequence (x_n) in a metric space (X, d), and $x, y \in X$, $x_n \to x$ and $x_n \to y$, then $x = y$.

Assume $x \neq y$. Then $d(x, y) > 0$. By the definition of limits, $\forall \epsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n > N_1 \implies d(x_n, x) < \epsilon$ and $n > N_2 \implies d(x_n, y) < \epsilon$. Now suppose $\epsilon = \frac{1}{2}d(x, y)$. Then if $n > \max\{N_1, N_2\}$, then $d(x_n, x) < \frac{1}{2}d(x, y)$ and $d(x_n, y) < \frac{1}{2}d(x, y)$. Adding these inequalities yields

$$d(x_n, x) + d(x_n, y) < d(x, y) \tag{7}$$

which contradicts the triangle inequality. Thus, $x = y$, i.e. the limit of a convergent sequence in a metric space is unique. $\square$

Problem 3

Let (a_n) be a sequence in $\mathbb{R}$.

a)

Prove that there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ of (a_n) such that $\lim_{k\to\infty} a_{n_k} = \underline{\lim}\ a_n$.

There are three cases, either $\underline{\lim}\ a_n = \infty$, $-\infty$, or $L \in \mathbb{R}$. It suffices to show we can construct a suitable subsequence in each case.

Case 1

Suppose $\underline{\lim}\ a_n = -\infty$. First, choose an arbitrary a_{n_1}. Next, choose a_{n_2} such that $a_{n_2} < -2$. Then choose a_{n_3} such that $a_{n_3} < -3$, and so on such that $a_{n_k} < -k$. Then (a_{n_k}) diverges to $-\infty$, i.e. $\lim_{k\to\infty} a_{n_k} = -\infty = \underline{\lim}\ a_n$.

Case 2

Suppose $\underline{\lim}\ a_n = \infty$. First, choose an arbitrary a_{n_1}. Next, choose a_{n_2} such that $a_{n_2} \geq a_{n_1}$. Then choose a_{n_3} such that $a_{n_3} \geq a_{n_2}$, and so on such that $a_{n_1} \leq a_{n_2} \leq \cdots \leq a_{n_k} \leq \ldots$. This sequence (a_{n_k}) does not have a real limit, since that would contradict $\underline{\lim}\ a_n = \infty$. Thus (a_{n_k}) diverges to ∞, i.e. $\lim_{k\to\infty} a_{n_k} = \infty = \underline{\lim}\ a_n$.

Case 3

Suppose $\underline{\lim}\ a_n = L \in \mathbb{R}$. First, choose an arbitrary a_{n_1}. Next, choose a_{n_2} such that $|L - a_{n_2}| < \frac{1}{2}$. Then choose a_{n_3} such that $|L - a_{n_3}| < \frac{1}{3}$ and so on such that $|L - a_{n_k}| < \frac{1}{k}$. Then by the Archimedian principle, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $k > N \implies |L - a_{n_k}| < \epsilon$, i.e. $\lim_{k\to\infty} a_{n_k} = L = \underline{\lim}\ a_n$.

$\square$

b)

Prove that (a_n) converges to $a \in \mathbb{R}$ if and only if $\underline{\lim}\ a_n = \overline{\lim}\ a_n = a$.

$\implies$

Let $(a_n) \to a$. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - a| < \epsilon$. Then

$$\begin{aligned}
&a - \epsilon < \inf\{a_k | k \geq n\} < a + \epsilon \\
\iff &-\epsilon < \inf\{a_k | k \geq n\} - a < \epsilon \\
\iff &|\inf\{a_k | k \geq n\} - a| < \epsilon \\
\iff &\lim_{n\to\infty} (\inf\{a_k | k \geq n\}) = a \\
\iff &\underline{\lim}\ a_n = a
\end{aligned}$$

Similarly,

$$\begin{aligned}
&a - \epsilon < \sup\{a_k | k \geq n\} < a + \epsilon \\
\iff &-\epsilon < \sup\{a_k | k \geq n\} - a < \epsilon \\
\iff &|\sup\{a_k | k \geq n\} - a| < \epsilon \\
\iff &\lim_{n\to\infty} (\sup\{a_k | k \geq n\}) = a \\
\iff &\overline{\lim}\ a_n = a
\end{aligned}$$

Thus, $\underline{\lim}\ a_n = \overline{\lim}\ a_n = a$

$\impliedby$