HW #6

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November 16, 2015

Problem 1

Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Define the map $T : C([0,1]) \to C([0,1])$ by

$$(Tf)(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y, \quad \text{for all } f \in C([0,1])$$

a)

Let $\|T\|$ denote the operator norm of T. Prove

$$\|T\| = \sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y \tag{1}$$

First, we show $\|T\| \leq \sup\limits_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y$.

$$\begin{aligned}
\|T\| = \sup_{\|f\|=1} \|Tf\| = \sup_{\|f\|=1} \left\| \int_0^1 k(\cdot,y)f(y)\mathrm{d}y \right\| &= \sup_{x\in[0,1]} \left(\sup_{\|f\|=1} \left| \int_0^1 k(x,y)f(y)\mathrm{d}y \right| \right) \\
&\leq \sup_{x\in[0,1]} \left(\sup_{\|f\|=1} \int_0^1 |k(x,y)||f(y)|\mathrm{d}y \right) \\
&\leq \sup_{x\in[0,1]} \left(\sup_{\|f\|=1} \int_0^1 |k(x,y)|\|f\|\mathrm{d}y \right) \\
&= \sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y
\end{aligned}$$

Note that since $\int_0^1 |k(x,y)|\mathrm{d}y$ is a continuous functions of x, then $\exists x^* \in [0,1]$ such that

$$\sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y = \int_0^1 |k(x^*,y)|\mathrm{d}y$$

Thus,

$$\|T\| \le \int_0^1 |k(x^*, y)| \mathrm{d}y$$

Next, we show $\|T\| \ge \int_0^1 |k(x^*, y)| \mathrm{d}y$. Define a sequence of functions $(f_n)_n$

$$f_n(y) = \frac{k(x^*, y)}{\frac{1}{n} + |k(x^*, y)|}$$

Note $f_n(y) \to \mathrm{sign}(k(x^*, y))$ for each $y \in [0,1]$, i.e. $f_n \to \mathrm{sign}(k(x^*, \cdot))$ pointwise.

$$\begin{aligned}\sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y &= \int_0^1 |k(x^*,y)|\mathrm{d}y \\ &= \int_0^1 k(x^*,y)\mathrm{sign}(k(x^*,y))\mathrm{d}y \\ &= \int_0^1 k(x^*,y) \lim_{n\to\infty} f_n(y)\mathrm{d}y\end{aligned}$$

Since $\|f_n\|_\infty \le 1$ for all n, we can employ Lebesgue's Dominated Convergence Theorem to pull the limit out of the integral:

$$\int_0^1 k(x^*,y) \lim_{n\to\infty} f_n(y)\mathrm{d}y = \lim_{n\to\infty} \int_0^1 k(x^*,y) f_n(y)\mathrm{d}y$$

Thus,

$$\begin{aligned}\sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y &= \lim_{n\to\infty} \int_0^1 k(x^*,y) f_n(y)\mathrm{d}y \\ &= \lim_{n\to\infty} (Tf_n)(x^*) \\ &\le \lim_{n\to\infty} \|Tf_n\|_\infty\end{aligned}$$

But since $\|f_n\| \le 1$ for all n,

$$\sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y \le \lim_{n\to\infty} \|Tf_n\|_\infty \le \lim_{n\to\infty} \sup_{\|f\|\le 1} \|Tf\| = \lim_{n\to\infty} \|T\| = \|T\|$$

Thus,

$$\|T\| \le \sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y \le \|T\| \implies \|T\| = \sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y$$

$\square$

b)

Argue that the sup *in* (1) *is attained in some* $x \in [0,1]$.

As argued in part **a)**, since k is a continuous function of x and y, then $\int_0^1 |k(x,y)|\mathrm{d}y$ is a continuous function of x. Thus $\int_0^1 |k(x,y)|\mathrm{d}y$ must reach its maximum on a compact set. Since $[0,1]$ is compact, $\exists x^* \in [0,1]$ such that $\sup\limits_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y = \int_0^1 |k(x^*,y)|\mathrm{d}y$.

c)

Is it possible that $\|T\| = 1$ but $\|T^2\| = 0$? Prove your answer.

Define $k(x,y) : [0,1] \times [0,1] \to \mathbb{R}$ as

$$k(x,y) = \begin{cases} 0, & \text{if } y \le x + \dfrac{1}{2} \\ 4(2y - 2x - 1), & \text{else} \end{cases}$$

Then if T is defined using k, then

$$\begin{aligned} \|T\| &= \sup_{x \in [0,1]} \int_0^1 |k(x,y)| \mathrm{d}y \\ &= 4 \sup_{x \in [0,1]} \int_{x+\frac{1}{2}}^1 |2y - 2x - 1| \mathrm{d}y \\ &= 4 \sup_{x \in [0,1]} \left(y^2 - (2x+1)y\right)\Big|_{x+\frac{1}{2}}^1 \\ &= 4 \sup_{x \in [0,1]} \left(x - \frac{1}{2}\right)^2 = 4 \cdot \frac{1}{4} = 1 \end{aligned}$$

However,

$$\begin{aligned} \left\|T^2\right\| &= \sup_{\|f\|=1} \left\|T^2 f\right\| \\ &= \sup_{\|f\|=1} \int_0^1 k(x,y) \left(\int_0^1 k(y,s) f(s) \mathrm{d}s \right) \mathrm{d}y \\ &= \sup_{\|f\|=1} \int_0^1 \int_0^1 k(x,y) k(y,s) f(s) \mathrm{d}s \mathrm{d}y \end{aligned}$$

However, if $y \le \frac{1}{2} \implies k(x,y) = 0$ and $y \ge \frac{1}{2} \implies k(y,s) = 0$, and thus $\forall y \in [0,1]$, $k(x,y)k(y,s) = 0$. Thus,

$$\begin{aligned} \left\|T^2\right\| &= \sup_{\|f\|=1} \int_0^1 \int_0^1 k(x,y) k(y,s) f(s) \mathrm{d}s \mathrm{d}y \\ &= \sup_{\|f\|=1} \int_0^1 \int_0^1 0 \ \mathrm{d}s \mathrm{d}y = 0 \end{aligned}$$

So it is possible for $\|T\| = 1$ and $\|T^2\| = 0$. $\square$

Problem 2

Study Section 5.4 of the textbook.

Problem 3

Let X be the Banach space $\ell^2(\mathbb{N})$, defined by

$$\ell^2(\mathbb{N}) = \left\{ z = (z_n)_{n=1}^{\infty} \mid z_n \in \mathbb{C},\ \sum_{n=1}^{\infty} |z_n|^2 < \infty \right\}$$

For $m = 1, 2, \dots,$ define $e_m \in X$ to be the sequence with elements $(e_m)_n = \delta_{n,m}$, and define $P_m : X \to X$ by $P_m z = z_m e_m$, for all $z \in X$.

a)

Prove that $P_m \in \mathcal{B}(X)$, for all $m \geq 1$.

Note $\|e_m\| = 1$ for all m. Now, fix $m \geq 1$.

$$\|P_m\| = \sup_{\|z\|=1} \|P_m z\| = \sup_{\|z\|=1} \|z_m e_m\| = \sup_{\|z\|=1} |z_m|\|e_m\| = \sup_{\|z\|=1} |z_m| \tag{2}$$

Since $\|z\| = 1$, then $|z_m| \leq 1$ for all m. Thus $\sup_{\|z\|=1} |z_m| \leq 1$, which implies $\|P_m\| \leq 1$. Thus P_m is a bounded linear operator on X, i.e $P_m \in \mathcal{B}(X)$. $\square$

b)

Verify $P_m P_n = \delta_{n,m} P_m$, for all $n, m \geq 1$.

$$(P_m P_n)(z) = P_m(P_n z) = P_m(z_n e_n) = z_n P_m(e_n) = z_n (e_{n,m}) e_m = z_n \delta_{n,m} e_m$$

If $n \neq m$, then $\delta_{n,m} = 0$ and

$$(P_m P_n)(z) = z_n \delta_{n,m} e_m = 0 = 0 P_m(z) = \delta_{n,m} P_m(z) = (\delta_{n,m} P_m)(z)$$

If $n = m$, then $z_m = z_n$ and

$$(P_m P_n)(z) = z_n \delta_{n,m} e_m = \delta_{n,m} z_m e_m = \delta_{n,m} P_m(z) = (\delta_{n,m} P_m)(z)$$

In either case, $P_m P_n = \delta_{n,m} P_m$. $\square$

c)

Prove that $\|P_m\| = 1$, for all $m \geq 1$.

By part **a)**, $\|P_m\| \leq 1$. However,

$$\|P_m e_m\| = \|e_{m,m} e_m\| = |e_{m,m}|\|e_m\|$$

But $e_{m,m} = \delta_{m,m} = 1$ and $\|e_m\| = 1$, thus $\|P_m e_m\| = 1$. This implies $\|P_m\| \geq 1$, and so $\|P_m\| = 1$. $\square$

d)

For $m \geq 1$, define S_m by

$$S_m = \sum_{k=1}^{m} P_k$$

Calculate $S_m S_n$, for all $n, m \geq 1$.

$$\begin{aligned}
(S_m S_n)(z) &= S_m(S_n z) \\
&= S_m\left(\sum_{k=1}^{n} P_k z\right) \\
&= S_m\left(\sum_{k=1}^{n} z_k e_k\right) \\
&= S_m(z_1, z_2, \dots, z_n, 0, 0 \dots) \\
&= \sum_{j=1}^{m} P_k(z_1, \dots, z_n, 0, 0, \dots)
\end{aligned}$$

If $n \leq m$,

$$\begin{aligned}
(S_m S_n)(z) &= P_1(z_1, \dots, z_n, 0, 0, \dots) + \cdots + P_n(z_1, \dots, z_n, 0, 0, \dots) \\
&\quad + P_{n+1}(z_1, \dots, z_n, 0, 0, \dots) + \cdots + P_m(z_1, \dots, z_n, 0, 0, \dots) \\
&= z_1 e_1 + \cdots + z_n e_n + 0 e_{n+1} + \dots 0 e_m \\
&= (z_1, z_2, \dots, z_n) \\
&= S_n z
\end{aligned}$$

So $S_m S_n = S_n$. If $n \geq m$,

$$\begin{aligned}
(S_m S_n)(z) &= P_1(z_1, \dots, z_n, 0, 0, \dots) + \cdots + P_m(z_1, \dots, z_n, 0, 0, \dots) \\
&= z_1 e_1 + \cdots + z_m e_m \\
&= (z_1, z_2, \dots, z_m) \\
&= S_m z
\end{aligned}$$

So $S_m S_n = S_m$. In either case, $S_m S_n = S_{\min\{m,n\}}$. □

e)

Show that $\|S_m\| = 1$, for all $m \geq 1$.

Fix $m \geq 1$. First we show $\|S_m\|$ is bounded by 1.

$$\|S_m\| = \left\| \sum_{k=1}^{m} P_k \right\|$$

$$
\begin{aligned}
&= \sup_{\|z\|=1} \left\| \left(\sum_{k=1}^{m} P_k \right) z \right\| \\
&= \sup_{\|z\|=1} \left\| \sum_{k=1}^{m} P_k z \right\| \\
&= \sup_{\|z\|=1} \left\| \sum_{k=1}^{m} z_k e_k \right\| \\
&= \sup_{\|z\|=1} \|(z_1, z_2, \dots, z_m, 0, 0, \dots)\| \\
&\le \sup_{\|z\|=1} \|z\| = 1
\end{aligned}
$$

However,

$$
\begin{aligned}
\|S_m e_1\| &= \left\| \left(\sum_{k=1}^{m} P_k \right) e_1 \right\| \\
&= \left\| \sum_{k=1}^{m} P_k e_1 \right\| \\
&= \left\| \sum_{k=1}^{m} e_{1,k} e_k \right\| \\
&= \left\| \sum_{k=1}^{m} \delta_{1,k} e_k \right\| \\
&= \|(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{1,m}, 0, 0, \dots)\| \\
&= \|(1, 0, 0, \dots)\| = 1
\end{aligned}
$$

Thus $\|S_m\| \ge 1$, implying $\|S_m\| = 1$ $\square$