Notes for Mathematics 202A - Topology and Analysis

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1 Preface

Mathematics 202A is the first course in a two-semester graduate series on analysis and topology at the University of California, Berkeley.

These notes come from fall 2014 - spring 2015. The course was taught by Professor Marc Rieffel.

It's usually incumbent on a note-taker to avoid making his own mistakes and errors while recording proofs. However, at some sections I've tried to rewrite statements or provide more details to proofs for my own understanding of the material. I was also absent from a lecture covering the Stone-Weierstrass theorem. So any errors or omissions from the original lecture series are certainly my own.

Contents

2 Notation

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	natural numbers, integers, rational numbers, real and complex numbers, resp.
$\mathbb{R}^+$	nonnegative real numbers
$C(X,Y)$	continuous functions of $Y-$valued functions on X
$C(X)$	continuous functions of $\mathbb{R}$ or $\mathbb{C}$-valued functions on X
$C_b(X,M)$	the set of bounded continuous functions from X to M
$\overline{A}$	closure of a subset A of a topological space
$f^{-1}(X)$	the inverse image of a function f and a set X
$\circ$	function composition

3 Preliminary definitions

3.1 Metric spaces

Our motivation for the following sections will be generalizing the concept of continuity of functions from metric spaces.

Definition: A *metric space* is a set X with a function $d : X \times X \to \mathbb{R}^+$ satisfying for all $x, y, z \in X$:

$$\begin{aligned} &(1)\ d(x,y) \geq 0 \\ &(2)\ d(x,y) = d(y,x) \\ &(3)\ d(x,z) \leq d(x,y) + d(y,z) \end{aligned}$$

We will denote a metric space by the pair (X, d). If d satisfies only properties (2) and (3), then d is a *pseudometric.*

Example: The $\mathbb{R}-$valued continuous functions $C([0,1])$ with any of the following functions:

$$\begin{aligned} d_\infty(f,g) &= \sup_{x \in [0,1]} \{|f(x) - g(x)|\} \\ d_1(f,g) &= \int_0^1 |f(x) - g(x)| dx \\ d_2(f,g) &= (\int_0^1 |f(x) - g(x)|^2 dx)^{1/2} \end{aligned}$$

is a metric space. It can be checked that $(C([0,1]), d_\infty)$ is a Cauchy metric space, whereas $C([0,1])$ with d_1 or d_2 is not.

Given metric spaces (X, d_X) and (Y, d_Y), we'd like to identify and study appropriate sets of functions from X to Y.

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is *continuous at* $x \in X$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that for all $z \in X$, if $d_X(z,x) < \delta$, then $d_Y(f(z), f(x)) < \epsilon$. If f is continuous at every point of X, then call f a *continuous* function

We will see that this definition of continuity, which depends on the metric space structure of the codomain and domain, will generalize to arbitrary topological spaces.

We can also restrict our attention to subsets of $C(X, Y)$, such as with:

(1) functions $f : X \to Y$ that preserve distance, i.e.

$$d_Y(f(a), f(b)) = d_X(a, b)$$

for all $a, b \in X$. These are the *isometries.*

(2) functions $f : X \to Y$ that do not increase distances between points, i.e.

$$d_Y(f(a), f(b)) \leq d_X(a, b)$$

for all $a, b \in X$. These are the *contractions.*

(3) Lipschitz functions, i.e. functions $f : X \to Y$ to which there exists $M > 0$ such that for all $a, b \in X$:

$$d_Y(f(a), f(b)) \leq M d_X(a, b)$$

It's readily checked that a function $f : X \to Y$ satisfying any of the properties above is continuous. Each of these classes of functions is also closed under composition.

The metric function gives a simple description of the geometry of the metric space.

Definition: An *open ball* of a metric space (X, d) with center x and radius r is the set:

$$B(x, r) = \{z \in X : d(z, x) < r\}$$

A subset $N \subset X$ is a *neighborhood* of x if there exists some open ball $B(x, r)$ such that $B(x, r) \subset N$.

Definition: If $U \subset X$ is a subset and for all $x \in U$, there exists a neighborhood $N_x \subset U$, then U is an *open* subset.

We can reformulate the definition of continuity in this language:

Definition: A function $f : X \to Y$ of metric spaces is *continuous at $x \in X$* if for any open set $U \subset Y$ containing $f(x)$, there exists an open set $V \subset X$ such that the image set $f(V) \subset U$.

The following sections of notes will develop the useful idea of continuity for sets which are not necessarily metric spaces.

3.2 Topological spaces

We will define continuity of functions whose domains and codomains are *topological* spaces, which are characterized by their collections of *open* sets.

Definition: Let X be a set. A subset $\mathcal{T}$ of the powerset of X is a *topology* of X if $\mathcal{T}$ satisfies the following:

(1) $X, \emptyset \in \mathcal{T}$
(2) if $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary subset of $\mathcal{T}$, then $\bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda \in \mathcal{T}$
(3) if $\{\mathcal{O}_i\}_{i=1}^n$ is a finite subset of $\mathcal{T}$, then $\bigcap_{i=1}^n \mathcal{O}_i \in \mathcal{T}$.

We will refer to a set X and a topology $\mathcal{T}$ of X by the double $(X, \mathcal{T})$, or just X when the context is clear.

Example: The powerset $\mathcal{P}(X)$ of a set X is known as the *discrete* topology.

Example: The set $\{X, \emptyset\}$ is the *indiscrete* topology of X.

Example: Let $\mathcal{T}$ be the collection of sets $V \subset X$ such that the complement $X \setminus V$ is finite. Then $\mathcal{T}$ is a topology known as the *co-finite* topology.

Example: Let $\mathcal{T}$ be the collection of sets $V \subset X$ such that $X \setminus V$ is countable. Then $\mathcal{T}$ is a topology known as the *co-countable* topology.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ a function. Our generalized definition of continuity is, naturally:

Definition: A function $f : X \to Y$ of topological spaces is *continuous at* $x \in X$ if for every open set $U \subset Y$ containing $f(x)$, the preimage $f^{-1}(U)$ is an open subset of X. That is, if $U \in \mathcal{S}$, then $f^{-1}(U) \in \mathcal{T}$. If f is continuous at every $x \in X$, then f is a *continuous* function.

Proposition: A function $f : X \to Y$ is continuous if and only if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

Proof. If $f^{-1}(U) \in \mathcal{T}_X$ for all open $U \subset Y$, then for any $x \in X$ and open neighborhood $\mathcal{O}$ of x, $f^{-1}(\mathcal{O})$ must be open.

Conversely, suppose f is continuous and U is an open subset of Y. If $range(f) \cap U \neq \emptyset$, then $f^{-1}(U) \in \mathcal{T}_X$. And if no point of X maps into U, then $f^{-1}(U) = \emptyset$, which is in $\mathcal{T}_X$ by definition of a topology. ∎

This definition of continuity is well-defined with respect to composition.

Proposition: Let $(Z, \mathcal{T}_Z)$ be a third topological space. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$ is a continuous function.

Proof. By the previous proposition, it is sufficient to check that $(g \circ f)^{-1}(U)$ is in $\mathcal{T}_X$ for all $U \in \mathcal{T}_Z$. But

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

... implying the claim. ∎

It will also be useful to study the complements of open sets.

Definition: A subset A of a topological space X is *closed* if $X \setminus A$ is an open set.

From the definition of open sets and De Morgan's law, it's clear that the arbitrary intersection of closed sets is itself closed, and that the finite union of closed sets is also itself closed. It's also clear that every subset of a topological space is contained in a "smallest" closed set. Precisely,

Definition: Suppose $A \subset X$. Define the *closure* of A as:

$$\overline{A} = \bigcap \{K : K \text{ is closed and } A \subset K\}$$

That is, $\overline{A}$ is the intersection of all closed sets containing A as a subset.

There is a useful characterization of points in the closure of a set, given by the following proposition.

Proposition: Suppose $A \subset X$ is a subset. Then $x \in \overline{A}$ if and only if for every open set U containing x, the intersection $U \cap A \neq \emptyset$.

Proof. First suppose $x \notin \overline{A}$. Then $x \notin K$ for some closed K containing A, implying $x \in X \setminus K$, an open set.

Conversely, if U is open and $x \in U$, and $U \cap A = \emptyset$, then $x \notin X \setminus U$, which is a closed set containing A. So $x \notin \overline{A}$. ∎

The preceding proposition motivates the following important definition:

Definition: A point $x \in X$ is a *limit point* of A if for every open subset U of X containing x, the intersection $U \cap (A \setminus \{x\}) \neq \emptyset$.

Colloquially, x is a limit point of A if for every open neighborhood U of x, there are elements of A distinct from x which are in U.

For the next examples, consider $X = \mathbb{R}$ with its usual topology of open sets comprised of unions of open balls.

Example: If $A = \mathbb{Z}$, then A has no limit points.

Example: If $A = [0, 1]$ then every point of A is a limit point of A. But no point of $\mathbb{R} \setminus A$ is a limit point of A.

Example: If $A = (0, 1)$, then 0 and 1 are the only limit points of A not contained in A.

The last two examples suggest that closures "add missing limit points." The previous two propositions prove the following description of closures:

Proposition: Let $A \subset X$, a topological space. Let $Lim(A)$ denote the set of limit points of A in X. Then $\overline{A} = A \cup Lim(A)$.

4 Construction of topologies

4.1 Subbases and bases of topologies

Let X be a set. It's quickly verified that:

Proposition: The intersection of an arbitrary collection of topologies on X is a topology on X.

This topology is called the *intersection topology* of X.

Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X, we say that $\mathcal{T}_1$ is *stronger* or *finer* or *bigger* than $\mathcal{T}_2$ if $\mathcal{T}_2 \subset \mathcal{T}_1$. In this case, $\mathcal{T}_2$ is *weaker, coarser,* or *smaller.*

In this language, the intersection topology of X is the strongest topology weaker than or as weak as all topologies of X.

Now we turn our attention to subsets S of the powerset of X. Given a set X, by the same argument above, there is a weakest topology that contains S, namely the intersection of all topologies containing S.

Definition: Let $\mathcal{T}$ be a topology on X and $S \subset \mathcal{T}$. Assume $X, \emptyset \in S$. Then S is a *subbasis* for $\mathcal{T}$ if $\mathcal{T}$ is the smallest topology containing S.

Definition: Given a topology $\mathcal{T}$ for X and $B \subset \mathcal{T}$, B is a *basis* for $\mathcal{T}$ if every $U \in \mathcal{T}$ is a union of elements of B.

Example: Let $X = \mathbb{R}$ with its usual metric topology. Then a subbasis and basis, respectively, are:

$$S = \{(-\infty, s), (t, \infty) : s, t \in \mathbb{R}\}$$
$$B = \{(s, t) : s, t \in \mathbb{R}\}$$

The next propositions, left as exercises, describe the relationship between bases and subbases.

Proposition: A collection B of subsets of X is a basis for some topology if

(1) $\bigcup \{U : U \in B\} = X$
(2) for all $U, V \in B$, there exists $W \in B$ such that $W \subset U \cap V$

Proposition: Let S be a subbasis of a topology $\mathcal{T}$. Then the collection of all finite intersections:

$$\left\{ \bigcap_{i=1}^{n} S_i : n \in \mathbb{N}, S_i \in S \text{ for all } i = 1, \dots n \right\}$$

of subbase elements is a basis generating $\mathcal{T}$.

A basis is useful for showing continuity. For example,

Proposition: Let $f : X \to Y$ be a map of topological spaces. Let $B \subset \mathcal{T}_Y$ be a basis for the topology on Y, and suppose for all $U \in B$, $f^{-1}(U) \in \mathcal{T}_X$. Then f is continuous.

Proof. By definition of a basis, if $W \in \mathcal{T}_Y$, then there exists a collection $\{B_\lambda\}_{\lambda \in \Lambda} \subset B$ such that $W = \bigcup_\lambda B_\lambda$. Then:

$$f^{-1}(W) = \bigcup f^{-1}(B_\lambda)$$

And the arbitrary union of open sets is open, hence $f^{-1}(W) \in \mathcal{T}_X$. ■

In fact, it would suffice to show that $f^{-1}(W) \in \mathcal{T}_X$ for all $W \in S$, a subbasis for $\mathcal{T}_Y$, by the same argument.

4.2 Initial topologies

Here is the motivating example for this next section of notes: Let X be a set and $(Y, \mathcal{T}_Y)$ be a topological space. Let $f : X \to Y$ be a function. What is the weakest (i.e. smallest) topology $\mathcal{T}_X$ on X such that f is a continuous function with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$?

Definition: Given a set X, a topological space $(Y, \mathcal{T}_Y)$, and a function $f : X \to Y$, the *initial topology of* f is:

$$\left\{ f^{-1}(U) : U \in \mathcal{T}_Y \right\}$$

The initial topology is the weakest topology on X making f a continuous function.

Example: Let $A \subset Y$. Let $\iota : A \to Y$ be the inclusion map $a \mapsto a \in Y$. Then the initial topology of ι on A is

$$\{U \cap A : U \text{ is open in } Y\}$$

This example is referred to as the *subspace topology* or *induced topology* of A with respect to Y. We note that A is open and closed in its subspace topology, even though A may be neither closed nor open in Y.

More generally, let Λ be an index set, and for each $\lambda \in \Lambda$, assign a topological space $(Y_\lambda, \mathcal{T}_\lambda)$ and function $f_\lambda : X \to Y_\lambda$. Then a subbase for the initial topology making all the $\{f_\lambda\}$ continuous, is given by the subbase:

$$\left\{f_\lambda^{-1}(U) : \lambda \in \Lambda,\ U \in \mathcal{T}_\lambda\right\}$$

Example: Let $X = C([0,1])$, and $\Lambda = \mathbb{R}$. Then for each $t \in [0,1]$, assign $Y_t = \mathbb{R}$ and $f_t : X \to \mathbb{R}$ by $f_t(\phi) = \phi(t)$. Then a subbase for the initial topology from the $\{f_t\}_{t\in[0,1]}$ is given by:

$$\{\{\phi \in C([0,1]) : \phi(t) \in U\} : t \in \mathbb{R},\ U \text{ is open in } \mathbb{R}\}$$

In fact, for any normed vector space V, we can consider the initial topologies from subsets of the functions in V^*, its dual of continuous linear functionals. Then the weakest topology on V making all elements of V^* continuous is called the *weak* topology. We will return to the subject of weak topologies in our discussion of locally convex topological vector spaces and Alaoglu's theorem.

4.3 Final and quotient topologies

Definition: Let $(X, \mathcal{T}_X)$ be a topological space and $f : X \to Y$ a function into some set Y. The *final* topology of f is the strongest topology on Y making f a continuous function. This topology is exactly equal to:

$$\left\{A \subset Y : f^{-1}(A) \in \mathcal{T}_X\right\}$$

A particular kind of final topology is the following:

Definition: Let $(X, \mathcal{T}_X)$ be a topological space and Y a set. Let $f : X \to Y$ be a surjective function. Then the final topology from f on Y is the *quotient* topology.

The next example of a quotient topology will have important applications.

Example: Let X be a set and $\sim$ an equivalence relation. Let $\pi : X \to X/\sim$ be the projection of X onto its equivalence classes modulo $\sim$. Then π induces a quotient topology on $X/\sim$ where $U \subset X/\sim$ is open if and only if $\pi^{-1}(U)$ is open in X.

Example: Let $X = [0, 1]$ with the usual subspace topology. Let $\sim$ glue X by:

$$0 \sim 1 \text{ and for all } r, s \in (0, 1),\ r \sim s \text{ if and only if } r = s$$

If $U \subset X/\sim$ is an open set not containing $[0] = [1]$, then $\pi^{-1}(U)$ is an open set of $[0, 1]$ contained in the interior $(0, 1)$. If U contains $[0]$, then $\pi^{-1}(U)$ is an open set of $[0, 1]$ containing open neighborhoods of 0 and 1. This quotient topology describes a topology of S^1, the circle.

The following proposition is a useful continuity result.

Proposition: Let $(X, \mathcal{T}_X)$ be a topological space and $(Y, \mathcal{T}_Y)$ have the quotient topology from a surjective map f. Let $(Z, \mathcal{T}_Z)$ be another topological space and $g : Y \to Z$ be any map. Then g is continuous if and only if $g \circ f : X \to Z$ is continuous.

Proof. f is continuous by definition of the quotient topology. So if g is continuous, then $g \circ f$ is a continuous composition.

Conversely suppose $g \circ f$ is continuous. Then for any $U \in \mathcal{T}_Z$

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{T}_Y$$

But this is if and only if $g^{-1}(U) \in \mathcal{T}_X$, by the quotient topology. ∎

5 Product topologies I.

5.1 Product spaces

First, consider the cases when (X, d_X) and (Y, d_Y) are metric spaces with their metric topologies, i.e. topologies whose bases are the open balls with respect to their metrics.

Then there are several metric functions that can be put on $X \times Y$. For example,

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

is a possible metric which would induce a metric topology on $X \times Y$.

More generally, for two topological spaces, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we can use the initial topology from projections to find a natural topology on $X \times Y$. Let $\pi_X : X \times Y \to X$ be the natural projection which maps $(x, y) \mapsto x$ (respectively for $\pi_Y : X \times Y \to Y$). Then the initial topology from π_X and π_Y has as a basis:

$$\{U \times V : U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y\}$$

For any finite product $X_1 \times X_2 \times \cdots \times X_n$, consider the initial topology from the projections $\pi_k : \prod X_j \to X_k$ to get the basis:

$$\{U_1 \times \cdots \times U_n : U_j \in \mathcal{T}_j \text{ for all } j = 1, \ldots n\}$$

This shows that, for finite products of topological spaces, there is a clear choice of product topology. The infinite product case is less clear. For example, we must carefully define what is meant by a product of spaces indexed by an infinite set.

Definition: Let Λ be an index set. For each $\lambda \in \Lambda$ assign a set X_λ, nonempty. Define the *infinite product* of the X_λ as the set of all functions:

$$\prod_{\lambda \in \Lambda} X_\lambda = \left\{ x : \Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda \ \middle|\ \text{for all } \lambda \in \Lambda,\ x(\lambda) \in X_\lambda \right\}$$

Example: Let $\Lambda = \mathbb{N}$. For $n \in \Lambda$, let $X_n = \{0, 1\}$. Then

$$\prod_{j=1}^{n} X_n = \prod_{j=1}^{n} \{0, 1\}$$

is the set of index functions f such that $x(n) = 1$ or $x(n) = 0$ for each $n \in \mathbb{N}$.

As a note, we avoid any question of whether $\prod_\lambda X_\lambda$ is nonempty by accepting the axiom of choice.

Now, suppose each $(X_\lambda, \mathcal{T}_\lambda)$ is a topological space. Let $\prod_{\lambda\in\Lambda} X_\lambda$ be defined as the projection functions:

$$\pi_{\lambda_0} : \prod X_\lambda \to X_{\lambda_0}$$

with the action $\pi_{\lambda_0}(x) = x(\lambda_0)$. Then put on $\prod X_\lambda$ the initial topology from the collection of functions $\{\pi_\lambda\}_{\lambda\in\Lambda}$. That is, a typical subbasis element is given by:

$$\left\{x \in \prod X_\lambda : \pi_\lambda(x) \in U_\lambda, \text{ for fixed open set } U_\lambda \in \mathcal{T}_\lambda\right\}$$

and a basis is given by finite intersections of these subbasis elements.

Definition: The topology on $\prod X_\lambda$ generated by the basis above is the *product* topology.

For future use, we'll record this proposition concerning projection maps:

Proposition: For any $\lambda_0 \in \Lambda$, $\pi_{\lambda_0} : \prod X_\lambda \to X_\lambda$ is an open map.

Proof. The claim is that if $\mathcal{O}$ is an open set of the product topology, then $\pi_{\lambda_0}(\mathcal{O})$ is open in X_{λ_0}. It suffices to show this for the case when $\mathcal{O}$ is a basis element. Then $\mathcal{O}$ has the form, for some finite collection $\{\lambda_1, \ldots, \lambda_n\}$ and open sets $U_{\lambda_i} \subset X_{\lambda_i}$ for $i = 1, \ldots n$:

$$\mathcal{O} = \{f : f(\lambda_i) \in U_{\lambda_i} \text{ for } i = 1, \ldots, n\}$$

But then $\pi_{\lambda_0}(\mathcal{O})$ is either all of X_{λ_0} or a specific open set U_{λ_0}, depending on whether $\lambda_0 = \lambda_i$ for any i. ∎

Our motivation for studying product topologies is to answer the following question: Suppose each X_λ is a compact topological space, i.e. every open cover of X_λ admits a finite open subcover. Is $\prod X_\lambda$ compact? To answer this question, we will need to develop some properties of compact spaces.

6 Compact spaces

6.1 Definitions and examples

To make the previous discussion precise, we will review a few definitions and properties of compact and totally bounded sets.

Definition: Given a set X and a collection $\mathcal{C}$ of subsets of X, $\mathcal{C}$ *covers* X if $X \subset \bigcup\{C : C \in \mathcal{C}\}$.

Definition: If X is a topological space, then $\mathcal{C}$ is an *open* cover if $\mathcal{C}$ covers X and every element of $\mathcal{C}$ is an open subset of X.

Definition: A $\mathcal{C}$-*subcover* of X is a subset of $\mathcal{C}$ that covers X. When the context is clear, we will simply say a subcover of X.

Definition: A topological space X is *compact* if every open cover of X has a finite subcover. That is, if $\{C_\lambda : \lambda \in \Lambda\}$ is an open cover of X for some index set Λ, then there exist a finite collection $\lambda_1, \ldots, \lambda_n$ such that

$$X \subset \bigcup_{i=1}^{n} X_{\lambda_i}$$

The following example illustrates a direct proof that a given space is compact. We can appeal to Heine-Borel to verify the result of our proof.

Proposition: The unit interval $[0,1]$ is compact for the usual subspace topology.

Proof. Let $\mathcal{C}$ be any open cover of $[0,1]$. Consider the set

$$L = \{t \in [0,1] : [0,t] \text{ can be covered by a finite subset of } \mathcal{C}\}$$

L is nonempty since $0 \in L$. Let $\alpha = \sup L$. Then $\alpha \leq 1$, so α is necessarily in $[0,1]$.

Now, suppose for contradiction that $0 \leq \alpha < 1$. Then $[0,\alpha]$ is covered by some finite collection $\{U_1, \ldots, U_n\} \subset \mathcal{C}$.

But $\alpha \in B(\alpha,\epsilon) \subset U$ for some small $\epsilon < 0$ and $U \in \mathcal{C}$. So we can choose $z \in B(\alpha,\epsilon)$ such that $\alpha < z < 1$. Then $\{U_1, \ldots, U_n, U\} \subset \mathcal{C}$ is a finite cover of $[0,z]$, contradicting the definition of α as the supremum of L. Conclude $\alpha = 1$ and that $[0,1]$ is compact. ∎

6.2 Compactness for metric spaces

There are many classical results about determining whether a topological space is compact. One such result is the Heine-Borel theorem, which characterizes compact subspaces of Euclidean space as exactly the closed and bounded sets. Our motivation for this section of notes is to prove the following theorem:

Theorem: Let (X, d) be a metric space, totally bounded and complete, Then (X, d) is compact.

To prove this theorem, we develop a few related properties of metric spaces

Suppose (X, d) is a compact metric space. Let $\epsilon > 0$ be given. Consider the open cover:

$$\mathcal{C} = \{B(x, \epsilon) : x \in X\}$$

By compactness, we can find a finite subcover of open balls of X. This property is useful enough to have its own definition:

Definition: Let (X, d) be a metric space. X is *totally bounded* if, given $\epsilon > 0$, there exists a finite collection $\{x_1, \ldots, x_n\} \subset X$ such that

$$X \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$$

This property is strictly weaker than compactness. Consider the space $[0, 1] \cap \mathbb{Q}$ with the usual subspace (metric) topology. Unlike $[0, 1]$, this space is not compact. But it is totally bounded, using a density argument and the fact that $[0, 1]$ is compact.

Proposition: If (X, d) is a metric space and $A \subset X$ is a totally bounded subset, then $\overline{A}$ is a totally bounded subset.

Proof. Given $\epsilon > 0$, let $B(a_1, \epsilon/2), \ldots, B(a_n, \epsilon/2)$ be a finite cover of open balls of A. Let $x \in \overline{A}$. Then $B(x, \epsilon/2) \cap A \neq \emptyset$. Let $a \in B(x, \epsilon/2) \cap A$. So a belongs to some $B(a_j, \epsilon/2)$. By the triangle inequality,

$$d(x, a_j) \leq d(x, a) + d(a, a_j) < \epsilon$$

Hence $x \in B(a_j, \epsilon)$. It follows that $B(a_1, \epsilon), \ldots, B(a_n, \epsilon)$ is an open cover of ϵ-balls of $\overline{A}$. ∎

Proposition: If (X, d) is a metric space and $A \subset X$, then A is totally bounded.

Proof. Given $\epsilon > 0$, by total boundedness of X, consider an open cover of $\epsilon/2$-balls $B(x_1, \epsilon/2), \ldots, B(x_n, \epsilon/2)$ of A, where the $x_i \in X$. Without loss of generality, each $\epsilon/2$-ball has nonempty intersection with A. So choose

$$a_j \in B(x_j, \epsilon/2) \cap A$$

Then it is easily seen that $\{B(a_j, \epsilon) \cap A\}_{j=1}^{n}$ is an open cover of ϵ-balls of A, since each $B(x_j, \epsilon/2) \subset B(a_j, \epsilon)$. ■

The previous proposition will be used in the proof of the motivating theorem:

Theorem: Let (X, d) be a metric space, totally bounded and complete, Then (X, d) is compact for the metric topology.

Proof. Let $\mathcal{C}$ be an open cover of X. For contradiction, suppose $\mathcal{C}$ has no finite subcover.

Using total boundedness, find balls $B_1, \ldots, B_k$ of radius $\frac{1}{2}$ that cover X. There must be at least one of these balls that is not covered by a finite number of elements of C. Choose one of these balls and call it $\mathcal{B}_1$.

Since $\mathcal{B}_1$ is totally bounded, find a finite number of balls of radius $\frac{1}{2}^2$ covering $\mathcal{B}_1$. There must be at least one $\mathcal{B}_2$ of these balls such that $\mathcal{B}_2$ cannot be finitely covered by $\mathcal{C}$.

Inductively, for $\mathcal{B}_{n+1}$, find a cover of $\mathcal{B}_n$ of $\frac{1}{2}^{n+1}$ radius balls and repeat the same process. The centers of the $\mathcal{B}_n$ form a Cauchy sequence.

By completeness, this sequence converges to some $x_0 \in X$. Since $\mathcal{C}$ covers X, there exists an open set $U \in \mathcal{C}$ such that $x_0 \in U$. Then there is an $r > 0$ such that $B(x_0, r) \subset U$.

Choose $N > 0$ such that $n > N$ implies $x_n \in B(x_0, r/2)$. For some $k > N$, $B(x_k, \frac{1}{2}^k) \subset B(x_0, r) \subset U$, contradicting the choice of x_k. ■

Here is an immediate application. Suppose (X, d) is a complete metric space and $A \subset X$ is totally bounded. Then $\overline{A}$ is a complete metric space. By the proposition above, $\overline{A}$ is also totally bounded. So the theorem implies $\overline{A}$ is compact.

6.3 Compactness for general topologies

First, we'll see closed subsets of compact spaces are compact for the subspace topology.

Proposition: Suppose $(X, \mathcal{T})$ is a compact topological space and $A \subset X$ is a closed subset. Then A is compact for the subspace topology.

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A for the subspace topology of A. By definition of the subspace topology, each U_λ is equal to:

$$U_\lambda = V_\lambda \cap A$$

where $V_\lambda \in \mathcal{T}$. Then $\{V_\lambda, X \setminus A : \lambda \in \Lambda\}$ is an open cover of X. So there is a finite subcover

$$\{V_{\lambda_i}, X \setminus A : i = 1, \ldots, n\}$$

of X. Then the U_{λ_i} form an open and finite subcover of A. ∎

Proposition: Conversely, suppose X is a Hausdorff topological space and $A \subset X$ is a compact subspace. Then A is closed in X.

Proof. Let $x \in X \setminus A$. For every $a \in A$, there are disjoint open neighborhoods $U_a \ni x$ and $V_a \ni a$. Then $\{V_a \cap A : a \in A\}$ is an open cover of A. By compactness, there exists a finite collection $a_1, \ldots, a_n \in A$ such that $\{V_{a_i \cap A} : i = 1, \ldots, n\}$ covers A. And so $U_x := \bigcap_{i=1}^n U_{a_i}$ is an open neighborhood of x disjoint from A

Then, finding such U_x for all $x \in X \setminus A$, see: $X \setminus A = \bigcup_{x \in X} U_x$, which is an open set. Conclude A is closed. ∎

Corollary: Let (X, d) be a metric space. Then any compact subset of X is closed and bounded.

Also, compact subspaces are mapped to compact subspaces by continuous maps.

Proposition: Let $(X, \mathcal{T}_X)$ be a compact topological space and $(Y, \mathcal{T}_Y)$ be a topological space. Let $f : X \to Y$ be a continuous map. Then $f(X) \subset Y$ is a compact space.

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $f(X)$. Then, by continuity, $\left\{f^{-1}(U_\lambda) : \lambda \in \Lambda\right\}$ is an open cover of X. Using compactness of X, there is a finite subcover

$$\left\{f^{-1}(U_{\lambda_i}) : i = 1, \ldots, n\right\}$$

which gives a finite open subcover of $f(X)$. ■

We can use this fact to prove the following useful proposition:

Proposition: Let X be a compact space and Y a Hausdorff space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof. First, see that f is a closed map; if $C \subset X$ is a closed subset, then C is compact. So $f(C) \subset Y$ is compact. Since Y is Hausdorff, $f(C)$ is closed.

But f is a bijection,which implies f is an open map. This in turn implies f^{-1} is continuous. ■

An easy corollary to the previous proposition is that if $(X, \mathcal{T})$ is a compact Hausdorff space, then there is no strictly smaller Hausdorff topology on X than $\mathcal{T}$.

With our discussions on compactness, we are now almost in a position to prove the first "big" theorem of this course: Tychonoff's theorem, which states:

Theorem: (Tychonoff) Let Λ be some index set, and for each $\lambda \in \Lambda$, let $(X_\lambda, \mathcal{T}_\lambda)$ be a compact topological space. Denote:

$$Y = \prod_{\lambda \in \Lambda} X_\lambda$$

Then Y is compact for the product topology.

But first, we need to discuss partial orders and Zorn's lemma.

6.4 Partial orders and Zorn's lemma

Definition: A *partial order* on a set X is a relation $\leq$ satisfying:
(1) transitivity: if $a \leq b$ and $b \leq c$, then $a \leq c$
(2) reflexivity: $a \leq a$

Some people will also require that if $a \leq b$ and $b \leq a$, then $a = b$.

Example: Let X be a set. Put a partial order on the powerset $\mathcal{P}(X)$ by $A \leq B$ if $A \subset B$.

Given a partial order on X, for any subset W of X, we can put on W the relative partial order, i.e. for $a, b \in W$, $a \leq_W b$ if $a \leq_X b$ in X.

Definition: A *total* order is a partial order such that for all $a, b \in X$, either $a \leq b$ or $b \leq a$.

Definition: Let X be partially ordered. A *chain* $C \subset X$ is a subset such that the relative partial order on C is a total order.

Definition: An *upper bound* of $W \subset X$ is an element $x \in X$ such that for all $w \in W$, $w \leq x$.

Definition: A *maximal* element of a partially ordered set X is some $M \in X$ such that if $M \leq a$, then $a = M$.

Note that the definition does not claim existence or uniqueness of maximal elements of a given partially ordered set.

Definition: A partially ordered set X is *inductively ordered* if every chain of X has a maximal element.

Given these definitions, we now accept the following fact as true:

Theorem: (Zorn) If X is inductively ordered, then X has a maximal element.

Zorn's lemma has many classic applications, such as showing every vector space has a basis and that every ideal of a unital ring is contained in a maximal ideal.

7 Product topologies II.

7.1 Tychonoff's theorem

Surprisingly, the arbitrary product of compact spaces is compact.

Theorem: (Tychonoff) For an index set Λ, let $(X_\lambda, \mathcal{T}_\lambda)$ be a compact topological space for all $\lambda \in \Lambda$. Let $Y = \prod_{\lambda \in \Lambda} X_\lambda$. Then Y is compact for the product topology $\mathcal{T}_Y$.

Proof. We'll show that any collection of open subsets with no finite subcover of Y, does not cover Y.

Suppose $\mathcal{U} \subset \mathcal{T}_Y$ has no finite subcover of Y (denoted NFS). Consider the set:

$$N_{\mathcal{U}} = \{\mathcal{V} \subset \mathcal{T}_Y : U \subset \mathcal{V} \text{ and } \mathcal{V} \text{ has NFS}\}$$

with the partial order by inclusion. The claim is that $N_{\mathcal{U}}$ is inductively ordered.

Let $\mathcal{C}$ be a chain in $N_{\mathcal{U}}$. Set:

$$V_{\mathcal{C}} = \bigcup \{\mathcal{V} \in \mathcal{C}\} \subset \mathcal{T}_Y$$

Then $U \subset V_{\mathcal{C}}$ and $V_{\mathcal{C}} \in N_{\mathcal{U}}$. Furthermore, $V_{\mathcal{C}}$ will be an upper bound of $\mathcal{C}$, provided we can show $V_{\mathcal{C}}$ has no finite subcover. But if $\{W_1, \ldots, W_n\} \subset V_{\mathcal{C}}$, then for each j, there exists $\mathcal{V}_j \in \mathcal{C}$ with $U_j \in \mathcal{V}_j$. By the total order, the W_k belong to the largest of the $\mathcal{V}_j$, implying the W_k do not cover Y.

So $\mathcal{C}$ has an upper bound, proving the first claim. By Zorn's lemma, $N_{\mathcal{U}}$ has a maximal element. Denote this maximal element by $\mathcal{U}^*$. And $\mathcal{U} \subset \mathcal{U}^*$.

Now, some preliminary work before the next claim. Suppose $P \in \mathcal{T}_Y$ such that $P \notin \mathcal{U}^*$. Then

$$\mathcal{U}^* \subsetneq \mathcal{U}^* \cup \{P\}$$

By maximality, $\mathcal{U}^* \cup \{P\}$ must contain a finite subcover of Y, i.e. there exist W_i in $\mathcal{U}^*$ for $i = 1, \ldots, n$ such that $\bigcup W_i \cup P = Y$.

Furthermore, it's easy to see that if $P, Q \in \mathcal{T}_Y \setminus \mathcal{U}^*$, then $P \cap Q \in \mathcal{T}_Y \setminus \mathcal{U}^*$ (look at the finite subcovers from $\mathcal{U}^* \cup \{P\}$ and $\mathcal{U}^* \cup \{Q\}$.

Lastly, recall a subbase for the product topology $\mathcal{T}_Y$ is given by:

$$S = \left\{\pi_\lambda^{-1}(W) : \lambda \in \Lambda,\ W \in \mathcal{T}_\lambda\right\}$$

The next claim is that

$$\bigcup\{W \in \mathcal{U}^*\} = \bigcup\{W \in S \cap \mathcal{U}^*\}$$

The containment from right to left is clear. So suppose $x \in \bigcup\{W \in \mathcal{U}^*\}$. There is a $P \in \mathcal{U}^*$ such that $x \in P$. So there are subbase elements $S_1, \dots, S_k$ such that $x \in S_1 \cap \cdots \cap S_k \subset P$. If $S_j \notin \mathcal{U}^*$ for all $j = 1, \dots, k$, then $S_1 \cap \cdots \cap S_k \notin \mathcal{U}^*$, by the previous discussion. And so there exist open $W_1, \dots, W_l \in \mathcal{U}^*$ such that

$$(W_1 \cup \cdots \cup W_l) \cup (S_1 \cap \cdots \cap S_k) = Y$$

But $S_1 \cap \cdots \cap S_k \subset P$, and so $\{W_i\}_{i=1}^{l} \cup \{S_1 \cap \cdots \cap S_k\}$ cannot be an open cover of Y. Conclude there must be some $S_{j_0} \in \mathcal{U}^*$, i.e. $x \in \bigcup\{W \in S \cap \mathcal{U}^*\}$.

For notation, let $V = S \cap \mathcal{U}^*$ from now on. The last claim, which will prove the theorem, is that V does not cover Y - notice V has NFS.

For any given $\lambda \in \Lambda$, let

$$W_\lambda = \left\{U \in \mathcal{T}_\lambda : \pi_\lambda^{-1}(U) \in V\right\}$$

Notice

$$V = \bigcup_{\lambda \in \Lambda} \left\{\pi_\lambda^{-1}(U) : U \in W_\lambda\right\}$$

For each λ, W_λ does not cover X_λ, because if W_λ covers X_λ, then by compactness of X_λ, there are $Q_1, \dots, Q_n \in W_\lambda$ that cover X_λ. Then $\left\{\pi_\lambda^{-1}(Q_j) : j = 1, \dots, n\right\}$ covers Y, contradicting the fact that V has NFS. Thus, each W_λ does not cover X_λ.

Thus for each $\lambda \in \Lambda$, we can choose $x_\lambda \in X_\lambda$ with $x_\lambda \notin \bigcup\{U \in W_\lambda\}$. Then let:

$$Z = \{x_\lambda\}_{\lambda \in \Lambda} \in Y$$

See that $Z \notin \mathcal{U}$, and that $V = S \cap \mathcal{U}^*$ does not cover Y. So U does not cover Y. ■

Our main application of Tychonoff's theorem will be in proving Alaoglu's theorem. But now, we will turn our attention to results about sequences, continuous functions on normal and regular topological spaces, and the space of continuous functions on compact Hausdorff spaces.

8 Sequences and nets

Definition: A sequence $\{a_n\}$ of a topological space $(X, \mathcal{T})$ has limit a_* if for every open neighborhood U of a_*, there exists N such that $n > N$ implies $a_n \in U$.

Our motivation for this section of notes will come from an example highlighting a "problem" with sequences.

Proposition: There exists a topological space Y and a subset $S \subset Y$ such that not every limit point $x \in \overline{S}$ is a sequential limit point of S.

Proof. An easy example of this is $Y = \mathbb{R}$ with the co-countable topology and $S = [0, 1]$. Here, we will work out a less trivial example:

For each $t \in \mathbb{R}$, let $[0, 1]_t$ be a copy of the unit interval with the usual topology. Let $Y = \prod_{t \in \mathbb{R}} [0, 1]_t$. Let

$$S = \{f \in Y : f(t) = 1 \text{ except at a finite set of points}\}$$

Let $h \in Y$ defined so that $h(t) = 0$ for all $t \in \mathbb{R}$. Suppose U is an open neighborhood of h. There exist open neighborhoods $U_1, \ldots, U_n$ of 0 and a basis element V of the product topology on Y such that, for corresponding $t_1, \ldots, t_n$:

$$V = \{f \in Y : f(t_i) \in U_i \text{ for } i = 1, \ldots n\}$$

and $V \subset U$ is an open neighborhood of h. Then the function $g : \mathbb{R} \to [0, 1]$ defined $g(r) = 0$ for $r = t_1, \ldots, t_n$ and $g(r) = 1$ otherwise is an element of S in V. Conclude $h \in \overline{S}$.

Now let $\{f_n\} \subset S$ be any sequence. Consider the set $R \subset \mathbb{R}$ of real numbers such that if $r \in R$, then there exists $n \in \mathbb{N}$ with $f_n(r) = 0$. By definition of S, R is a countable set. So there is some t_* such that $f_n(t_*) = 1$ for all $n \in \mathbb{N}$. Then

$$\left\{ g \in Y : g(t_*) < \frac{1}{2} \right\}$$

is an open neighborhood of h disjoint from $\{f_n\}$. ∎

However, limit points of a set will correspond to sequential limits given that the ambient topological space has a countable basis for its neighborhood system.

Definition: Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. The *neighborhood system* at x is the set of all open neighborhoods of x.

Definition: Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. Then a *basis for the neighborhood system* at x is a collection $\mathcal{B}$ of open neighborhoods of x such that if $U \in \mathcal{T}$ contains x, then there is some $V \in \mathcal{B}$ such that $V \subset U$.

Example: For a metric space (X, d), the open balls with center at $x \in X$ form a basis for the neighborhood system at x.

Definition: A point $x \in X$ has a *countable basis* for its neighborhood system if it admits a countable basis for the neighborhood system at x.

Example: For the metric space example, consider open balls of rational radius.

Definition: A topological space $(X, \mathcal{T})$ is *first countable* if every point in X has a countable basis for its neighborhood system.

In the language, any metric space is first countable. Another example of a first countable topology is any set with the discrete topology.

Proposition: Let (X, T) be a first countable topology and $A \subset X$. Then, if $b \in \overline{A}$, then b is the limit of some sequence in A.

Proof. Enumerate a countable basis for the neighborhood system at b by: $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$. Then form a set of sequential intersections:

$$U_1 = \mathcal{O}_1$$
$$U_n = \bigcap_{k=1}^{n} \mathcal{O}_k$$

Then choose $a_n \in U_n \cap A$ and form the sequence $\{a_n\} \subset A$. Then it's quickly checked that $\lim a_n = b$. ∎

Corollary: $Y = \prod_{t\in\mathbb{R}} [0, 1]_t$ is not a first countable topology.

For sequences that are not first countable, we can use a generalized notion of sequence to detect limit points.

Definition: A *directed set* is a set Λ with a partial order such that, if $\lambda_1, \lambda_2 \in \Lambda$, then there exists $\lambda_3 \in \Lambda$ such that $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$.

Example: A neighborhood system ordered by reverse inclusion is a directed set, i.e. $U \geq V$ if $U \subset V$.

Definition: A *net* in a set X is a function $f : \Lambda \to X$

Definition: A net $\{x_\lambda\}_{\lambda \in \Lambda}$ *converges* to a point x_* if, given an open neighborhood U of x_*, there exists $\lambda \in \Lambda$ such that $\lambda' \geq \lambda$ implies $\lambda' \in U$.

Proposition: Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Let $b \in \overline{A}$. Then there is a net in A converging to b.

Proof. Let Λ be the neighborhood system at b ordered by reverse inclusion and let the net $\{a_\lambda\}$ be defined by choosing $a_\lambda \in \lambda \cap A$. Then it's quickly verified that $\{a_\lambda\}$ is a net converging to b. ∎

Nets have many familiar properties that sequences enjoy. For example,

Proposition: Let $(X, \mathcal{T})$ be a Hausdorff topological space. Then limits of nets are unique.

Proof. Suppose $\{a_\lambda\}_{\lambda \in \Lambda}$ has limits a_1 and a_2. There exist disjoint neighborhoods U_1 and U_2 of a_1 and a_2, respectively. But there exist $\lambda_i \in \Lambda$ for $i = 1, 2$ such that $\lambda \geq \lambda_i$ implies $a_\lambda \in U_i$. Since Λ is a directed set, there exists λ_3 greater than λ_1 and λ_2. And for all $\lambda \geq \lambda_3$, $a_\lambda \in U_1 \cap U_2$, a contradiction. ∎

8.1 Nets in compact spaces

Now, we will show that nets in compact topological spaces admit accumulation points. Compare this result to the fact that sequences in compact spaces admit convergent subsequences.

Definition: Let $(X, \mathcal{T})$ be a topological space. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in X. Given an open set $U \in \mathcal{T}$, say that the net is *frequently in U* if for any $\lambda \in \Lambda$, there is a $\lambda_1 \geq \Lambda$ such that $x_{\lambda_1} \in U$.

Say that x_* is an *accumulation point* of the net if for any open neighborhood U of x_*, the net is frequently in U.

For the next proposition, it will be useful to phrase compactness in terms of closed sets.

Definition: Given any collection $\mathcal{C}$ of subsets in X, say that $\mathcal{C}$ has the *finite intersection property* if any intersection of a finite subcollection of elements of $\mathcal{C}$ is not the emptyset.

Then in this language, it's easily checked that if $(X, \mathcal{T})$ is compact, then whenever a family $\mathcal{F}$ of closed subsets of X has the finite intersection property, it's true that $\bigcap \{C \in \mathcal{F}\} \neq \emptyset$. We'll use this in our next proposition:

Proposition: Let $(X, \mathcal{T})$ be a compact topological space. Then every net in X has an accumulation point.

Proof. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in X. For each λ, let $A_\lambda = \{x_{\lambda'} : \lambda' \geq \lambda\}$.

Given $\lambda_1, \ldots, \lambda_n$, there exists λ' such that $\lambda' \geq \lambda_j$ for all $j = 1, \ldots, n$. And so $x_{\lambda'} \in A_{\lambda_1} \cap \cdots \cap A_{\lambda_n}$. So the set of all A_λ:

$$\{A_\lambda : \lambda \in \Lambda\}$$

has the finite intersection property. Then the set: $\left\{\overline{A_\lambda} : \lambda \in \Lambda\right\}$ has the finite intersection property. Then, since $(X, \mathcal{T})$ is compact, $\bigcap_\lambda A_\lambda \neq \emptyset$. So there is a $x_* \in \bigcap_\lambda A_\lambda$.

Now, if $x_* \in U$ an open set, then for given λ, $U \cap \overline{A_\lambda}$, implying $U \cap A_\lambda \neq \emptyset$. Hence, by definition of A_λ, there exists $\lambda' \geq \lambda$ such that $x_{\lambda'} \in U \cap A_\lambda$. ■

8.2 Nets and continuity

Proposition: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ a continuous map. Then, if $\{x_\lambda\}$ is a net in X with limit x_*, then $\{f(x_\lambda)\}$ is a net with limit $f(x_*)$.

Proof. This follows from the fact that for any open neighborhood U of $f(x_*)$, $f^{-1}(U)$ is an open neighborhood of x_*. ■

Proposition: Conversely, if $\lim_\lambda f(x_\lambda) = f(\lim_\lambda x_\lambda)$ for any convergent net $\{x_\lambda\}$, then f is continuous.

Proof. Call $\lim x_\lambda = x_*$. Suppose that f is not continuous. Then there exists an open neighborhood V of $f(x_*)$ such that for every open neighborhood U of x_*, $f(U) \not\subset V$. So to each such U, there is x_U such that $f(x_U) \notin V$.

Then the net $\{x_U\}_{U \in \Lambda}$, where Λ is the neighborhood system at x_* ordered by reverse inclusion, converges to x_*. But $f(x_U) \notin V$ for all $U \in \Lambda$. ■

9 $\mathbb{R}$-valued continuous functions on topological spaces

9.1 Urysohn's lemma

Our motivation for this section of notes is to prove Urysohn's lemma, which states that a certain class of topological spaces admits many continuous functions. In fact, the proof of Urysohn's lemma will be constructive and provide functions which "separate" disjoint nonempty closed sets by taking distinct values on them.

Let (X, T) be a topological space, and suppose $f : X \to \mathbb{R}$ is continuous and maps to at least two distinct values $t_1 > t_2$. Then the sets:

$$A_1 = \{x : f(x) \geq t_1\} \quad A_2 = \{x : f(x) \leq t_2\}$$

are closed and disjoint. Then there exists $t_1 > s_1 > s_2 > t_2$ such that the open sets:

$$U_1 = \{x : f(x) > s_1\} \quad U_2 = \{x : f(x) < s_2\}$$

are disjoint open sets containing U_1 and U_2, respectively. We generalize this feature of X in the following definition:

Definition: $(X, \mathcal{T})$ is a *normal* topological space if for any closed disjoint subsets A, B of X, there are open disjoint sets U, V such that $A \subset U$ and $B \subset V$.

Qualitatively, the condition of normality describes how finely a topology separates subsets of X from each other by open sets.

Example: If $(X, \mathcal{T})$ is a compact Hausdorff topological space, then it is also a normal topological space.

Example: Any metric space is a normal space.

To prove Urysohn's lemma, we will also need the following proposition:

Proposition: Let $(X, \mathcal{T})$ be a normal space, and let C be a nonempty closed subset. Then, if $U \in \mathcal{T}$ contains C, then there exists $V \in T$ such that:

$$C \subset V \subset \overline{V} \subset U$$

Proof. Assume, without loss of generality, that U is a proper subset. Then C and $X \setminus U$ are nonempty disjoint closed sets, and there exist disjoint $P, Q \in \mathcal{T}$ such that $C \subset P$ and $X \setminus U \subset Q$. But then $P \subset X \setminus Q \subset U$. So :

$$C \subset P \subset \overline{P} \subset X \setminus Q \subset U$$

∎

Theorem: (Urysohn) Suppose $(X, \mathcal{T})$ is a normal topological space. Let $C_0, C_1 \subset X$ be disjoint nonempty closed subsets. Then there exists a continuous function $f : X \to [0,1]$ such that $f(C_0) = \{0\}$ and $f(C_1) = \{1\}$.

Proof. By normality, there is an open set $U_{\frac{1}{2}}$ with $C_0 \subset U_{\frac{1}{2}}$ and $U_{\frac{1}{2}} \subset X \setminus C_1$. Think of $U_{\frac{1}{2}}$ as points where our constructed function will have values less than or equal to $\frac{1}{2}$. Then, by the previous proposition, there exist open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that:

$$C_0 \subset U_{\frac{1}{4}} \subset \overline{U_{\frac{1}{4}}} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_{\frac{3}{4}} \subset \overline{U_{\frac{3}{4}}} \subset X \setminus C_1$$

Consider the dense set $D = \left\{ \frac{k}{2^n} : n \in \mathbb{N}, k = 1, \dots, n \right\} \subset [0,1]$. Proceeding inductively, for each $r \in D$, there is an open U_r with the property that if $s \in D$ and $r < s$, then $U_r \subset \overline{U_r} \subset U_s$ (note how the fact that $\overline{U_r}$ is closed, is used in the proof; let $U_1 = X$).

Now, for $r \neq 1$, $\overline{U_r} \subset X \setminus C_1$. Define $f : X \to \mathbb{R}$ by:

$$f(x) = \inf \{ r \in D : x \in U_r \}$$

If $x \in C_1$ then $f(x) = 1$. If $x \in C_0$, then $f(x) = 0$ since $x \in U_r$ for all $r \in D$. All that is left is to show that f is continuous. Consider the subbase $\{(-\infty, a) : a \in \mathbb{R}\}$ for the usual topology on $\mathbb{R}$. We'll show $f^{-1}((-\infty, a))$ is open. It is sufficient to show that for any $x \in f^{-1}((-\infty, a))$, there exists an open set $\mathcal{O}$ such that $x \in \mathcal{O} \subset f^{-1}((-\infty, a))$.

Now,

$$x \in f^{-1}((-\infty, a)) \text{ if and only if } f(x) = \inf \{ r \in D : x \in U_r \} < a$$

There exist $\epsilon > 0$ and $s \in D$ such that the following inequality holds:

$$f(x) < s < f(x) + \epsilon < a$$

Then if $y \in U_s$, it's true that $f(y) < a$, implying $U_s \subset f^{-1}((-\infty, a))$. Lastly, U_s is an open neighborhood of x. x being an arbitrary point in the preimage, conclude $f^{-1}((-\infty, a))$ is open.

To complete the proof, consider that the image of $f : X \to \mathbb{R}$ lies completely in $[0, 1]$. ■

It should be noted that Urysohn's lemma is often used to construct continuous functions on compact Hausdorff spaces. We will also return to a stronger version of Urysohn's lemma in our discussion of locally compact Hausdorff spaces.

9.2 The Tietze extension theorem

The Tietze extension theorem is a direct application of Urysohn's lemma. To prove this theorem, we will need a lemma:

Proposition: Let $(X, \mathcal{T})$ be a normal space and $C \subset X$ a closed subset. Given $r > 0$, suppose $f : C \to [0, r]$ is a continuous function. Let $C_0 = \left\{x \in C : f(x) \leq \frac{1}{3}r\right\}$ and $C_1 = \left\{x \in C : f(x) \geq \frac{2}{3}r\right\}$. Then there is a continuous map $g : X \to [0, \frac{1}{3}r]$ such that $g|_{C_0} \equiv 0$ and $g|_{C_1} \equiv \frac{1}{3}r$. Furthermore,

$$g \leq f \text{ and } 0 \leq (f - g)|_C \leq \frac{2}{3}r \text{ on } C_1 \cup C_0$$

Proof. Urysohn's lemma.

Theorem: (Tietze) Let X be a normal topological space. Let C be a closed subset of X and let $f : C \to [0, 1]$ be a continuous map. Then there is a $g : X \to [0, 1]$ such that

$$g|_C \equiv f$$

i.e. g extends f to C.

Proof. For our proof, we will inductively produce sequences $\{f_n\}$ and $\{g_n\}$ of functions and set $g = \sum_{i=1}^{\infty} g_i$. Set $f_1 \equiv f$. By the lemma above, there exists a continuous $g_1 : X \to [0, \frac{1}{3}]$ such that

$$0 \leq (f - g)|_C \leq \frac{2}{3}$$

Inductively set $f_{n+1} = (f_n - g_n)|_C$ to get $f_{n+1} : C \to [0, (\frac{2}{3})^n]$ and a continuous $g_k : X \to [0, \frac{1}{3}(\frac{2}{3})^n]$ such that $0 \leq (f_{n+1} - g_{n+1})|_C \leq (\frac{2}{3})^{n+1}$.

Now, $\sum_{i=1}^{\infty} g_i$ converges uniformly to some g by the Weierstrass $M-$test, since:

$$0 \leq g(x) \leq \sum_{i=1}^{\infty} \frac{1}{3}(\frac{2}{3})^{i-1} < \infty$$

Let g be the uniform limit of the series. Since each partial sum of $\sum_{i=1}^{\infty} g_i$ is continuous, g is continuous. And it is easily checked by examining the partial sums that $f \equiv g|_C$. ■

9.3 The space $C(X, M)$ of continuous functions

Our motivation for this section of notes is to explore the metric space structure of $C(X, M)$ when M is a metric space and X is a topological space. We will see that discussion will be particularly profitable when X is assumed to be compact.

Recall that if (M, d) is a metric space, and $\{x_n\}$ is a sequence in M, then $\{x_n\}$ is *Cauchy* if for any $\epsilon > 0$, there exists $N > 0$ such that $n \geq m > N$ implies $d(x_n, x_m) < \epsilon$. Recall also that (M, d) is *complete* if every Cauchy sequence in X has a limit in X.

For a topological space $(X, \mathcal{T})$ and a metric space (M, d), let $C(X, M)$ denote the set of all continuous functions from X to M with respect to the metric topology on M.

For example, in the case when X is normal, Urysohn's lemma produces many elements of $C(X, \mathbb{R})$.

Now, from the metric $d : M \times M \to \mathbb{R}^+$, define the *restricted metric* $d_* : M \times M \to \mathbb{R}^+$ by:

$$d_*(x, y) = \min \{d(x, y), 1\}$$

As the name suggests, the restricted metric from d is a metric on M. Furthermore, d_* generates the same basis of open balls as d does on M, hence the metric topologies from d_* and d are identical. This leads to the definition:

Definition: For $f, g \in C(X, M)$, define the *metric for uniform convergence* d_∞ on $C(X, M)$ by:

$$d_\infty(f, g) = \sup \{d_*(f(x), g(x)) : x \in X\}$$

It is quickly verified that d_∞ is indeed a metric on $C(X, M)$, and that convergence of a sequence $\{f_n\}$ to f with respect to this metric implies f_n converges uniformly to f. As a side note, the definition of the restricted metric prevents d_∞ from taking on the value ∞; when X is compact, then the continuous function $d(f(x), g(x))$ will achieve a maximum value on X, and we may simply take d_* in the definition of the metric for uniform convergence to be exactly the metric d.

Now, a natural question is to ask how the metric space structure on $C(X, M)$ depends on M. We will answer this question for the property of completeness.

Theorem: If (M, d) is a complete metric space, then $(C(X, M), d_\infty)$ is a complete metric space.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C(X, M)$. Fix $x \in X$. Then the sequence $\{f_n(x)\} \subset M$ is a Cauchy sequence, since:

$$d(f_n(x), f_m(x)) \leq d_\infty(f_n, f_m)$$

Since M is complete, the $f_n(x)$ limit to some point of M. Define the function $f : X \to M$ by $f(x) = \lim f_n(x)$.

Now the claim is that the f_n converge uniformly to f. Given $0 < \epsilon < 1$, there exists $N > 0$ such that $m \geq n \geq N$ implies $d_\infty(f_n, f_m) < \epsilon$. Now, by definition of f, for arbitrary $x \in X$, there exists $n_x > N$ such that $d(f(x), f(n_x)) < \epsilon$. This implies by the triangle inequality:

$$d(f_n(x), f(x)) \leq d(f_n(x), f_{n_x}(x)) + d_\infty(f_{n_x}, f) < 2\epsilon$$

x being arbitrary, the f_n converge uniformly to f and f is necessarily a continuous function Hence $f \in C(X, M)$ and $C(X, M)$ is a complete metric space. ∎

The following is an important example of the function space $C(X, M)$.

Example: For notation, denote $C_b(X, M)$ as the set of all bounded functions, i.e. $f \in C_b(X, M)$ if and only if f is a continuous function such that the range of f is contained in some open ball.

When $M = \mathbb{R}$ with the usual metric, then:

$$d_\infty(f, g) = \sup \{|f(x) - g(x)| : x \in X\} < \infty$$

This metric defines a norm on $C_b(X, \mathbb{R})$ by:

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\}$$

Then $d(f, g) = \|f - g\|_\infty$, the uniform norm. And by the previous proposition, the metric space $(C_b(X, \mathbb{R}), \| \cdot \|_\infty)$ is a complete metric space. In fact, $C_b(X, \mathbb{R})$ is a vector space by pointwise operations over $\mathbb{R}$. Hence $C_b(X, \mathbb{R})$ is an example of the following:

Definition: A normed vector space with a complete metric topology from the norm is a *Banach space*.

We will return to the subject of Banach spaces as we develop integration theory and topics in functional analysis.

9.4 Note on regular spaces

We note that there are examples of normal Hausdorff topological spaces which have a subset that is not normal for the relative topology. Contrast this with the following class of topological space:

Definition: A Hausdorff topological space $(X, \mathcal{T})$ is *completely regular* if for any closed subset A of X and any point $b \notin A$, there is a function $f \in C(X, [0, 1])$ with $f(b) = 0$ and $f(A) = 1$.

Proposition: A subset A of a completely regular space $(X, \mathcal{T})$ is completely regular with respect to the relative topology.

Proof. Let $C \subset A$ be a closed set in the relative topology and $b \in A \setminus C$. There exists a closed set K of X such that $C = A \cap K$ and $b \notin K$. Use the completely regular property of X to find $f : X \to [0, 1]$ separating b and K, then restrict f to A. ■

This leads to a useful corollary about subsets of normal spaces.

Corollary: A subset of a normal Hausdorff space is completely regular.

More generally, we can require a regularity condition on topological spaces.

Definition: A topological space $(X, \mathcal{T})$ is *regular* if for all closed sets A and points b of X, there are open disjoint sets U and V such that $A \subset U$ and $b \in V$.

9.5 The Stone-Weierstrass theorem

(Notetaker's note: I did not record the proofs given in lecture for the following formulations of the Stone-Weierstrass theorem. For completeness, I will record the statement here, but for proof, I recommend Bass' *Real Analysis for Graduate Students*, which contains motivation and detailed proofs for several versions of Stone-Weierstrass, including a lattice formulation of the result.

It is also somewhat interesting trivia to note that Marshall Stone is the Ph.D. thesis adviser of Professor Rieffel's own thesis advisor, Richard Kadison.)

Theorem: (Stone-Weierstrass) Let X be a compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C(X, \mathbb{R})$. Suppose $\mathcal{A}$ separates points, i.e. if $x, y \in X$ and $x \neq y$, then there is an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Suppose further that $\mathcal{A}$ separates points from 0, i.e. for each $x \in X$, there exists $g \in \mathcal{A}$ such that $g(x) \neq 0$. Then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ for $\|\cdot\|_\infty$.

Example: The set of all polynomial functions on $[a, b]$ is dense in $C([a, b], \mathbb{R})$.

Theorem: (Complex S.W.) Let X be a compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C(X, \mathbb{C})$ such that $\mathcal{A}$ separates points and separates them from 0. Suppose further that if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$, where $\bar{\cdot}$ denotes pointwise complex conjugation. Then $\mathcal{A}$ is dense in $C(X, \mathbb{C})$ for $\|\cdot\|_\infty$.

9.6 The Arzela-Ascoli theorem

Our next section of notes answers the question: What are the totally bounded subsets of $C(X, \mathbb{C})$? We'll find a sufficient condition for a subset to be totally bounded with respect to the d_∞ metric.

Let (M, d) be a metric space and $(X, \mathcal{T})$ a topological space. Consider $C_b(X, M)$, the space of bounded continuous functions with the metric d_∞.

Suppose $\mathcal{F} \subset C_b(X, M)$ and $\mathcal{F}$ is totally bounded with respect to d_∞. By definition, given $\epsilon > 0$, there is some collection $\{f_1, \ldots, f_n\} \subset \mathcal{F}$ such that if $g \in \mathcal{F}$, then $d_\infty(f_k, g) < \epsilon$ for at least one $k = 1, \ldots, n$. We will define this property as being $\epsilon-$dense.

Definition: Let (M, d) be a metric space and $A \subset M$ a subset. Let $\epsilon > 0$. Then A is $\epsilon-$*dense* in M if for all $m \in M$, there exists $a \in A$ such that $d(m, a) < \epsilon$.

As a note, in the above situation, we see that the set $\{f_j(x) : j = 1, \dots, n,\ x \in X\} \subset M$ is ϵ-dense in $\{g(x) : g \in \mathcal{F}\}$. We will use this fact in a proposition; but first we need another property of $\mathcal{F}$.

Definition: A subset $\mathcal{F} \subset C_b(X, M)$ is *equicontinuous at* $x \in X$ if for all $\epsilon > 0$, there exists an open neighborhood $\mathcal{O}$ of x such that if $y \in \mathcal{O}$, then for all $f \in \mathcal{F}$, it is true that $d(f(x), f(y)) < \epsilon$. If $\mathcal{F}$ is equicontinuous at every $x \in X$, then $\mathcal{F}$ is an *equicontinuous* subset of $C_b(X, M)$.

Proposition: If $\mathcal{F}$ is totally bounded, then $\mathcal{F}$ is equicontinuous.

Proof. Let $x \in X$. Given $\epsilon > 0$, there exists an ϵ-dense subset $\{f_1, \dots, f_n\}$ of $C_b(X, M)$. Since each f_j is continuous, there exists an open neighborhood $\mathcal{O}_j$ of x such that if $y \in \mathcal{O}_j$, then $d(f_j(x), f_j(y)) < \epsilon$. Define $\mathcal{O}_x = \bigcap \mathcal{O}_j$.

Then if $g \in \mathcal{F}$, there exists f_k such that:

$$d_\infty(f_k, g) < \epsilon$$

So for x and all $y \in \mathcal{O}_x$:

$$d(g(x), g(y)) \leq d(g(x), f_k(x)) + d(f_k(x), f_k(y)) + d(f_k(y), g(y)) < 3\epsilon$$

■

We are now in a position to prove the following theorem.

Theorem: (Arzela-Ascoli) Let X be a compact Hausdorff space. Let (M, d) be a metric space. Let $\mathcal{F}$ be a nonempty subset of $C(X, M)$ and assume:

(1) For each $x \in X$, $\{f(x) : f \in \mathcal{F}\}$ is totally bounded
(2) $\mathcal{F}$ is equicontinuous.

Then $\mathcal{F}$ is totally bounded for d_∞.

Proof. Let $\epsilon > 0$ be given. Let $\{\mathcal{O}_x : x \in X\}$ be a collection of neighborhoods such that if $y \in \mathcal{O}_x$ for any x, then for all $g \in \mathcal{F}$, $d(g(x), g(y)) < \epsilon$.

Since X is compact, there is a finite subcollection $\{\mathcal{O}_{x_i} : i = 1, \dots, n\}$ such that $X = \bigcup_{i=1}^n \mathcal{O}_{x_i}$.

For each j, the set $\{f(x_j) : f \in \mathcal{F}\}$ is totally bounded. So let $\mathcal{S}_j$ be a finite subset of $\{f(x_j) : f \in \mathcal{F}\}$ that is ϵ-dense.

Let $S = \bigcup_{j=1}^{n} \mathcal{S}_j$, a finite subset of M. Define:

$$\Phi = \{\phi : \{1, 2, \ldots, n\} \to S\}$$

... the set of all set functions from $\{1, ..., n\}$ to S. For each $\phi \in \Phi$, let:

$$F_\phi = \{f \in \mathcal{F} : d(f(x_j, \phi(j)) < \epsilon \text{ for all } j = 1, \ldots, n\}$$

Given $f \in \mathcal{F}$, for each j, there is a $s \in \mathcal{S}_j$ such that $d(f(x_j, s) < \epsilon$. Hence there is a ϕ with $f \in F_\phi$, and so $\mathcal{F} = \bigcup \{F_\phi : \phi \in \Phi\}$. Now, it's left to show that each F_ϕ is sufficiently "small." But if $g, h \in F_\phi$ and $x \in X$, then there is a j such that $x \in \mathcal{O}_j$ and:

$$d(g(x), h(x)) \leq d(g(x), g(x_j)) + d(g(x_j), h(x_j)) + d(h(x_j), h(x))$$

And

$$d(g(x_j), h(x_j)) \leq d(g(x_j), \phi(j)) + d(\phi(j), h(x_j))$$

which gives that $d(g(x), h(x)) \leq 4\epsilon$. Then the diameter of each F_ϕ is bounded:

$$diam(F_\phi) = \sup_{f,g} \{d_\infty(f, g)\} \leq 4\epsilon$$

So each F_ϕ is contained in a ball of radius 4ϵ. ■

10 Brief notes on compactifications and metrizability

10.1 The Stone-Cech compactification

Here, we will discuss several results about injective maps from a general topological space $(X, \mathcal{T})$ into compact spaces. We will refrain from proving the main result of this section and instead refer to the text TEXT.

Proposition: Let X be a topological space. Let $\mathcal{F}$ be a family of functions on X into various spaces, i.e. set Y_f to be the target space for $f : X \to Y_f$. For each Y_f, let $(Y_f, \mathcal{T}_f)$ be a topological structure making f continuous.

Let $Y = \prod_{f \in \mathcal{F}} Y_f$ with the product topology. Let $e : X \to Y$ be the evident evaluation map $e(x)(f) = f(x)$. Then:

(1) e is continuous
(2) If the functions $\{f \in \mathcal{F}\}$ separate the points of X , then e is injective.
(3) If $\mathcal{F}$ separates points from closed subsets of X, then e is open onto its image with the relative topology.

And (2) and (3) guarantee that e is a homeomorphism of X onto $e(X)$ with the relative topology.

We can rephrase the above proposition as a statement about completely regular topological spaces being homeomorphic to subsets of compact Hausdorff spaces.

Corollary: Every completely regular topological space is homeomorphic to a subset of a compact Hausdorff space.

Proof. Let X be a completely regular topological space. For each $f : X \to [0, 1]$, let $Y_f = [0, 1]$. Let $Y = \prod Y_f$. Let $e : X \to Y$ be the evaluation map. Then e is a homeomorphism onto $e(X)$ with the relative topology, by the previous proposition. ∎

Corollary: Any compact Hausdorff space is homeomorphic to a closed subset of $\prod_{\lambda \in \Lambda} [0, 1]_\lambda$ for some index set Λ.

Now, when $\mathcal{F} = C_b(X, \mathbb{R})$ and $Y_f = [-\|f\|_\infty, \|f\|_\infty]$ for each $f \in C_b(X, \mathbb{R})$, then $e : X \to Y$ is a homeomorphism of X onto $e(X)$, and $\overline{e(X)}$ is compact, by Tychonoff's theorem. $\overline{e(X)}$ is the *Stone-Cech* compactification of X, denoted βX.

Generally, we can define a compactification of a topological space in this way:

Definition: Let X be a topological space and Z a compact Hausdorff space. Let $\phi : X \to Z$ be a homeomorphism onto a subset of Z such that $\phi(X)$ is dense in Z. Then (Z, ϕ) is called a *compactification* of X.

Example: Let $X = \mathbb{R}$ and $Z = \mathbb{R} \cup \{\infty\}$.

Example: Let $X = \mathbb{R}$ and $Z = \mathbb{R} \cup \{-\infty, \infty\}$.

Now, let $\pi_f : Y \to Y_f = [-\|f\|_\infty, \|f\|_\infty]$ be the canonical projection. See that $\pi_f \circ e \equiv f$.

Let $\tilde{f}$ be the restriction of π_f to $\overline{e(X)} = \beta X$. Then by the above calculation, the restriction of $\tilde{f}$ to $e(X)$ "is" f. Conversely, any continuous function on βX restricts to a function in $C_b(X)$. Thus,

$$C(\beta X) \text{``}=\text{''} C_b(X)$$

Example: Let $X = \mathbb{N}$ with the discrete topology. Then

$$C_b(\mathbb{N}) = \{\text{all bounded sequences in } \mathbb{R}\}$$

So $C(\beta\mathbb{N})$ is $C_b(\mathbb{N})$.

10.2 Metrizability

Our motivation for this section of notes is to determine sufficient conditions for a topology to be a metric topology. We start with an important proposition.

Proposition: Let $(X, \mathcal{T})$ be a normal topological space with a countable base. Then X is homeomorphic to a subset of $\prod_{j=1}^{\infty}[0,1]$.

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be an enumeration of the countable basis of $\mathcal{T}$. Our first claim is that if $x \in X$ and $\mathcal{O} \in \mathcal{T}$ is an open neighborhood of x, then there exists a basis element B_n such that $x \in \overline{B_n} \subset \mathcal{O}$.

This is true since $x \notin X \setminus \mathcal{O}$, so by regularity, there are B_m and an open U such that $B_m \cap U = \emptyset$ and $x \in B_m$, $X \setminus \mathcal{O} \subset U$. And:

$$\overline{B_m} \subset X \setminus U \subset \mathcal{O}$$

Let $\Lambda = \{(m,n) : \overline{B_m} \subset B_n\}$. For each $(m,n) \in \Lambda$, we have $\overline{B_m}$ and $X \setminus B_n$ as disjoint closed sets. So by Urysohn's lemma, we can choose $f_{mn} : X \to [0,1]$ such that $f_{mn}(\overline{B_m}) = \{0\}$ and $f_{mn}(X \setminus B_n) = \{1\}$.

Let $e : X \to \prod_{(m,n)\in\Lambda}[0,1]$. The family $\{f_{mn}\}_{(m,n)\in\Lambda}$ separates points from closed sets because, given $x \in X$, $C \subset X$ is closed, and $x \notin C$. So there is B_n such that $x \in B_n \subset X \setminus C$.

By the claim above, there exists B_m such that $x \in B_m \subset \overline{B_n} \subset B_n$, so $f_{mn}(x) = 0$ and $f_{mn}(C) = \{1\}$. Thus, by the main result from the compactification section of notes, the evaluation e is a homeomorphism of X onto $e(X)$ with the relative topology. ■

Proposition: Let $Y = \prod_{n=1}^{\infty}[0,1]$ with the product topology. Then Y is metrizable with topology induced by the metric:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n| \quad \text{where } x(n) = x_n,\ y(n) = y_n$$

Proof. Let $\mathcal{T}_p$ be the compact product topology on Y. Let $\mathcal{T}_m$ be the metric topology on Y induced by d. It will be sufficient to show that the identity map

$$\iota : (Y, \mathcal{T}_p) \to (Y, \mathcal{T}_m)$$

is continuous. So let $r > 0$ and $y \in Y$, and consider $B(y,r) \subset (Y, \mathcal{T}_m)$. Choose N such that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{r}{2}$. Then choose $s \in \mathbb{R}$ such that $s(\sum_{n=1}^{N-1} \frac{1}{2^n}) < \frac{r}{2}$. For each $n < N$, consider $U_n = (y_n - s, y_n + s) \cap [0,1]$, an open set of $[0,1]$.

Let $\pi_n : Y \to [0,1]$ be the canonical projection $y \mapsto y(n) = y_n$. Then $\pi_n^{-1}(U_n)$ is a subbase element for the product topology. Let $\mathcal{O} = \bigcap_{n<N} \pi_n^{-1}(U_n) \in \mathcal{T}_p$. So if $x \in \mathcal{O}$, then $x \in B(y,r)$ since:

$$\begin{aligned} d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n| &= \sum_{n<N} \frac{1}{2^n}|x_n - y_n| + \sum_{n=N}^{\infty} \frac{1}{2^n}|x_n - y_n| \\ &\le s \sum_{n<N} \frac{1}{2^n} + \frac{r}{2} \le r \end{aligned}$$

■

11 Locally compact Hausdorff spaces

Our last section of notes before measure and integration theories will discuss properties of locally compact Hausdorff spaces and their continuous functions. The main result of this section will be a stronger version of Urysohn's lemma for locally compact Hausdorff spaces.

Definition: A topological space $(X, \mathcal{T})$ is *locally compact* if for every point $x \in X$, there is an open set $\mathcal{O}$ such that $x \in \mathcal{O}$ and $\overline{\mathcal{O}}$ is compact.

Example: An n-dimensional topological manifold is locally compact.

Proposition: Let $(X, \mathcal{T})$ be a locally compact Hausdorff space and suppose $x \in X$. Then the neighborhood system at x has a compact basis.

Proof. Let $\mathcal{O}$ be an open neighborhood of x. Let U be an open neighborhood of x such that $\overline{U}$ is compact. Then $U \cap \mathcal{O}$ is an open set and subset of U. We will show that $U \cap \mathcal{O}$ contains a compact neighborhood of x; so without loss of generality, $\mathcal{O} \subset U$.

Since X is Hausdorff, for all $y \in \overline{U} \setminus \{x\}$, there exist disjoint open sets V_y and W_y such that $y \in V_y$ and $x \in W_y$.

Then $\{V_y : y \in \overline{U} \setminus \{x\}\} \cup \{\mathcal{O}\}$ is an open cover of $\overline{U}$. By compactness, there exist $y_1, \ldots, y_n$ such that $\{V_{y_1}, \ldots, V_{y_n}, \mathcal{O}\}$ covers $\overline{U}$. Then:

$$(\bigcap_{i=1}^{n} W_{y_i}) \cap \mathcal{O} \subset \overline{U} \setminus (\bigcup_{i=1}^{n} V_{y_i}) \subset \mathcal{O}$$

Hence $Z_{\mathcal{O}} := (\bigcap_{i=1}^{n} W_{y_i}) \cap \mathcal{O}$ is an open neighborhood of x such that $\overline{Z_{\mathcal{O}}} \subset \mathcal{O}$ is compact. Since $\mathcal{O}$ is arbitrary, the set of compact neighborhoods:

$$\{\overline{Z_{\mathcal{O}}} : \mathcal{O} \text{ is an open neighborhood of } x\}$$

is a compact basis for the neighborhood system at x. ■

This leads to an immediate consequence:

Proposition: Let C be a compact subset of X, and let W be an open set such that $C \subset W$. Then there exists an open set U such that $C \subset U \subset \overline{U} \subset W$ and $\overline{U}$ is compact.

Proof. For every $x \in C$, there exists a compact neighborhood $\overline{U_x} \subset W$ of x. Then $\{U_x : x \in C\}$ is an open cover of C, and by compactness, there exists an open subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Let $U = \bigcup_{i=1}^{n} U_{x_i}$. ∎

The next theorem constructs a compactly supported continuous function on an arbitrary locally compact Hausdorff space. That is to say, this function will be nonzero only on a compact subset and zero elsewhere.

Theorem: Let $C \subset X$ be compact and W an open set such that $C \subset W$. Then there is a continuous function $f : X \to [0, 1]$ such that $f(C) = \{1\}$ and $f(X \setminus W) = \{0\}$, and f has compact support.

Proof. There exist open sets U and $\mathcal{O}$ such that $\overline{U}$ and $\overline{\mathcal{O}}$ are compact and:

$$C \subset U \subset \overline{U} \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset W$$

Since C and $\overline{\mathcal{O}} \setminus U$ are compact (hence closed) subsets of $\overline{\mathcal{O}}$, by Urysohn's lemma, there is a function $f : \overline{\mathcal{O}} \to [0, 1]$ such that $f(C) = \{1\}$ and $f(\overline{\mathcal{O}} \setminus U) = \{0\}$.

Then define $f(z) = 0$ for all $z \in X \setminus \overline{\mathcal{O}}$. In particular, $f(X \setminus W) = \{0\}$ and the support of f is contained in $\overline{\mathcal{O}}$. Lastly, it's easily checked that this extended f is continuous on X, since f is continuous on $\overline{\mathcal{O}}$. ∎

12 Positive Borel measures

12.1 Motivation

Our motivation for this section of notes is generalizing the familiar Lebesgue measure of the real number line. While a background in basic measure theory will be helpful, these notes will develop and review important preliminary results, such as Carathéodory's construction of measures, before discussing more advanced topics.

Our starting data will be a non-decreasing left-continuous function $\alpha : \mathbb{R} \to \mathbb{R}^+$. Left-continuity requires that α satisfy, for all $t_0 \in \mathbb{R}$:

$$\lim_{t_n \to t_0} \alpha(t_n) = \alpha(t_0)$$

for every sequence $\{t_n\}$ increasing to t_0.

Example: Let $\alpha(x) = \mathrm{ceil}(x)$.

Denote by P the set of half-closed, half-open intervals $\{[a, b) : a \leq b\}$.

According to basic measure theory, the Lebesgue measure of $[a, b)$ is $b - a$. More generally, we may define a function $\mu_\alpha : P \to \mathbb{R}^+$ by:

$$\mu_\alpha([a, b)) = \alpha(b) - \alpha(a)$$

Then it is evident that when α is the identity function, that μ_α is exactly the Lebesgue measure on P. It's also clear by inspection that μ_α is a nonnegative function. The function μ_α is also countably additive on P.

Definition: Let $\bigoplus$ denote the union of pairwise disjoint sets. Let X be a set and Q a subset of the powerset of X. A function $f : Q \to \mathbb{R}$ is *countably additive* if for every sequence $\{E_n\} \subset Q$ of pairwise disjoint sets,

$$f(\bigoplus_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} f(E_n)$$

We will prove shortly the following theorem:

Theorem: The function μ_α is countably additive on P.

But first, we will need an intermediate lemma.

Proposition: Let $\alpha : \mathbb{R} \to \mathbb{R}^+$ be a non-decreasing left-continuous function. For $c, d \in \mathbb{R}$ and $a_j, b_j \in \mathbb{R}$ for $j = 1, \ldots, m$, if $[c, d] \subset \bigcup_{j=1}^m (a_j, b_j)$, then

$$\alpha(d) - \alpha(c) \leq \sum_{j=1}^{m} \alpha(b_j) - \alpha(a_j)$$

Proof. Relabel the (a_j, b_j) such that $a_1 < c$. If $b_1 \geq d$, stop. Otherwise, choose a_2 so that $b_1 > a_2$. If $b_2 > d$, stop. Continue until $b_k > d$. Then:

$$a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k$$

and $[c, d] \subset \bigcup_{j=1}^m (a_j, b_j)$. Discard the remaining a_i and b_i which do not appear in this union.

Then

$$\begin{aligned}
\alpha(d) - \alpha(c) &\leq \alpha(b_k) - \alpha(a_1) \\
&\leq \alpha(b_k) - \alpha(a_1) + \sum_{i=1}^{k-1} \alpha(b_j) - \alpha(a_{j+1}) \\
&= \sum_{j=1}^{k} \alpha(b_k) - \alpha(a_k) \\
&\leq \sum_{j=1}^{m} \alpha(b_j) - \alpha(a_j)
\end{aligned}$$

... where the last inequality follows from monotonicity. ∎

Theorem: μ_α is countably additive on P.

Proof. Let $[a_0, b_0)$ be in P and $\{[a_j, b_j) : j \in \mathbb{N}\}$ be a given subset of P such that

$$[a_0, b_0) = \bigoplus_{j=1}^{\infty} [a_j, b_j)$$

First, it's clear that $\sum_{j=1}^m \mu_\alpha([a_j, b_j)) \leq \mu_\alpha([a_0, b_0))$ for any m, since, by monotonicity:

$$\begin{aligned}
\sum_{j=1}^{m} \mu_\alpha([a_j, b_j)) &= \sum_{j=1}^{m} \alpha(b_j) - \alpha(a_j) \\
&= \alpha(b_m) + (\alpha(b_{m-1}) - \alpha(a_m) + \alpha(b_{m-2}) - \alpha(a_{m-1}) + \ldots) - \alpha(a_1) \\
&\leq \alpha(b_m) - \alpha(a_1) \\
&\leq \alpha(b_0) - \alpha(a_0)
\end{aligned}$$

Hence $\sum_{j\in\mathbb{N}} \mu_\alpha([a_j, b_j)) \le \mu_\alpha([a_0, b_0))$. For the reverse direction of the inequality, it suffices to show that for all $\epsilon > 0$, there is m such that

$$\sum_{j=1}^{m} \alpha(b_j) - \alpha(a_j) \ge \alpha(b_0) - \alpha(a_0) - \epsilon$$

Choose $b_0' \le b_0$ such that $\alpha(b_0') \ge \alpha(b_0) - \frac{\epsilon}{2}$, by left-continuity of α. Choose $\epsilon_j \ge 0$ such that $\sum_{j=1}^{\infty} \epsilon_j$, e.g. $\epsilon_j = (\frac{1}{2^j})\epsilon$.

For each j, choose $\alpha_j' < a_j$ such that $\alpha(a_j') \ge \alpha(a_j) - \epsilon_j$. Notice $[a_0, b_0'] \subset [a_0, b_0)$ and $[a_j, b_j) \subset (a_j', b_j)$. So

$$[a_0, b_0'] \subset \bigcup_{j=1}^{\infty} (a_j', b_j)$$

Hence

$$\begin{aligned}
\alpha(b_0) - \alpha(a_0) \le \alpha(b_0') + \frac{\epsilon}{2} - \alpha(a_0) &\le \sum_{j=1}^{m} \alpha(b_j) - \alpha(a_j') + \frac{\epsilon}{2} \\
&\le \sum_{j=1}^{\infty} (\alpha(b_j) - \alpha(a_j) + \epsilon_j) + \frac{\epsilon}{2} \\
&\le \sum \alpha(b_j) - \alpha(a_j) + \epsilon
\end{aligned}$$

■

The function μ_α is an example of a premeasure on P, a concept which we will rigorously define in the next section of notes. Colloquially, a premeasure on subsets of $\mathbb{R}$ (or, as we will see, an arbitrary set X) may lead to a measure on a generated object and subset of the powerset of $\mathbb{R}$ (resp. the powerset of X). Then, we may ask the standard questions of existence, uniqueness, etc.

12.2 Semi-rings and rings

Let P be defined as in the previous section of notes as the collection of half-open, half-closed intervals of $\mathbb{R}$. Consider a few properties of P:

(1) If $[a, b)$ and $[c, d)$ are in P, then $[a, b) \cap [c, d) \in P$.
(2) If $[a, b)$ and $[c, d)$ are in P, then $[a, b) \setminus [c, d) = \bigoplus_{j=1}^{m} [x_j, y_j)$ for some disjoint collection of $[x_j, y_j) \in P$.

It will be profitable to define a general object that satisfies these properties.

Definition: A collection $\mathcal{P}$ of subsets of a set X is a *semi-ring* if

(1) If $E, F \in \mathcal{P}$ then $E \cap F \in \mathcal{P}$
(2) If $E, F \in \mathcal{P}$, then there exist $G_1, \ldots, G_j \in \mathcal{P}$ such that $E \setminus F = \bigoplus_{j=1}^{m} G_j$

We note that, unlike topologies, the intersection of semi-rings is not necessarily a semi-ring.

Proposition: Let $\mathcal{P}$ be a semi-ring. Then for $E \in \mathcal{P}$ and $F_1, \ldots, F_n \in \mathcal{P}$, there exists a disjoint collection $\{G_j\}_{j=1}^{k} \subset \mathcal{P}$ such that

$$(\ldots((E \setminus F_1) \setminus F_2) \setminus \ldots) \setminus F_n = E \setminus (\bigcup_{i=1}^{n} F_i) = \bigoplus_{j=1}^{k} G_j$$

Proof. The proof is by induction. We see that if $E \setminus (\bigcup_{i=1}^{n-1}) = \bigoplus H_j$, then:

$$(E \setminus (\bigcup_{i=1}^{n-1} F_i)) \setminus F_n = (\bigoplus H_j) \setminus F_n = \bigoplus (H_j \setminus F_n)$$

∎

Definition: A collection $\mathbb{R}$ of subsets of a set X is a *ring* if

(1) If $E, F \in \mathcal{R}$ then $E \cup F \in \mathcal{R}$
(2) If $E, F \in \mathcal{R}$ then $E \setminus F \in \mathcal{R}$.

We note that the ring $\mathcal{R}$ is closed under finite intersections and symmetric difference by rewriting these set operations in terms of set difference and union.

As the name suggests, a semi-ring can be "completed" and form its own ring.

Proposition: Let $\mathcal{P}$ be a semi-ring. Consider the set

$$R(\mathcal{P}) = \left\{ \bigoplus_{j=1}^{m} E_j : m \in \mathbb{N}, E_j \in \mathcal{P} \text{ for } j = 1, \ldots, m \right\}$$

i.e. the set of all finite disjoint elements of semi-ring elements. Then $R(\mathcal{P})$ is a ring, called the *ring generated by $\mathcal{P}$*.

Proof. The proof is a straightforward calculation checking the properties of a ring and is left to inspection by the reader. ∎

12.3 Premeasures on semi-rings

Now that we have seen the algebraic objects of semi-rings and rings of some set X, we would like to explore functions defined on these objects which may be able to describe the "size" of some subset of X.

Broadly, given a subset E of X, we would like to approximate E by sequences of subsets whose sizes we can calculate. To do this, we will define premeasures on semi-rings and extend these premeasures to their generated rings. This program will lead to the construction of measures on σ-rings and σ-fields.

Definition: Let $\mathcal{P}$ be a semi-ring. A nonnegative function $\mu : \mathcal{P} \to [0, \infty]$ is a *premeasure* if it is a countably additive function.

Example: Let $\mathcal{P} = P$, the set of half-open, half-closed intervals of $\mathbb{R}$, and $\rho \equiv \mu_\alpha$ for some nonnegative, nondecreasing, left-continuous α, as before.

The next several propositions will work towards the extension of a premeasure μ on a semi-ring $\mathcal{P}$ to a countably additive function on the generated ring $R(\mathcal{P})$.

Proposition: Let μ be a premeasure on a semi-ring $\mathcal{P}$. Let $E \in \mathcal{P}$, and let $\{E_j\}_{j=1}^m \subset \mathcal{P}$ be a disjoint sequence such that $\bigoplus_{j=1}^m E_j \subset E$. Then

$$\sum_{j=1}^m \mu(E_j) \leq \mu(E)$$

Proof. $E = (\bigoplus (E_j \cap E)) \oplus (\bigoplus (E_j \setminus E))$. Since $\mathcal{P}$ is a semi-ring, there exists a disjoint finite collection of G_j in $\mathcal{P}$ such that

$$\bigcup (E_j \setminus E) = \bigoplus G_j$$

Then by countable additivity and nonnegativity of μ:

$$\mu(E) = \mu(\bigoplus G_j) + \mu(\bigoplus E_j \cap E) \geq \mu(\bigoplus E_j \cap E) = \mu(\bigoplus E_j) = \sum_{j=1}^m \mu(E_j)$$

■

Corollary: If $E \subset \bigoplus_{j=1}^{\infty} E_j$, then

$$\mu(E) \geq \sum_{j=1}^{\infty} E_j$$

$\blacksquare$

Proposition: If $\bigoplus_{i=1}^{n} F_i \subset \bigoplus_{j=1}^{m} E_j$, then

$$\mu(\bigoplus^{n} F_i) \leq \mu(\bigoplus^{m} E_j)$$

Proof. For each i,

$$F_i = \bigoplus_{j=1}^{n} F_i \cap E_j$$

So $\mu(F_i) = \sum_{j=1}^{n} \mu(F_i \cap E_j)$, implying:

$$\begin{aligned}
\mu(\bigoplus_{i=1}^{m} F_i) &= \sum_{i} \sum_{j} \mu(F_i \cap E_j) \\
&= \sum_{j} \mu(\bigoplus_{i=1}^{m} F_i \cap E_j) \\
&\leq \sum_{j=1}^{n} \mu(E_j) = \mu(\bigoplus_{j=1}^{n} E_j)
\end{aligned}$$

$\blacksquare$

Corollary: If $\bigoplus_{j=1}^{m} E_j = \bigoplus_{i=1}^{n} F_i$, then

$$\mu(\bigoplus E_j) = \mu(\bigoplus F_i)$$

$\blacksquare$

Define an extension of μ, denoted $\tilde{\mu} : R(\mathcal{P}) \to [0, \infty]$, by:

$$\tilde{\mu}(\bigoplus_{j=1}^{n} E_j) = \sum_{j=1}^{n} \mu(E_j)$$

By the previous corollary, this function is well defined, and it is easily checked that $\tilde{\mu}$ restricted to $\mathcal{P}$ as a subset of $R(\mathcal{P})$ is exactly μ. Lastly, it's checked that $\tilde{\mu}$ is finitely additive, i.e.

$$\tilde{\mu}(\bigoplus_{j=1}^{n} E_j \oplus \bigoplus_{i=1}^{m} F_i) = \tilde{\mu}(\bigoplus E_j) + \tilde{\mu}(\bigoplus F_i)$$

In fact, $\tilde{\mu}$ is countably additive. This proposition:

Proposition: $\tilde{\mu}$ is countably additive on $R(\mathcal{P})$.

... depends on the (finite) additivity of $\tilde{\mu}$ and is left as an exercise to the reader.

12.4 Hereditary σ-rings

We have seen semi-rings and rings. Now, we recall another collection of sets closed under union, intersection, and difference, called the σ-ring.

Definition: Let $\mathcal{A}$ be a collection of subsets of some set X. $\mathcal{A}$ is a *σ-ring* if

(1) $\{E_n\}_{n=1}^{\infty} \subset \mathcal{A}$ implies $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$ and $\bigcap_{n=1}^{\infty} E_n \in \mathcal{A}$
(2) E and F in $\mathcal{A}$ implies $E \setminus F \in \mathcal{A}$.

By basic measure theory, we know that every collection A of subsets of X is contained in its own generated σ-ring, realized as the intersection of all σ-rings containing A. For a review of these facts, the lecturer recommended Halmos' classic text *Measure Theory.*

Here, we define a particular type of σ-ring:

Definition: Let X be a set and A a collection of subsets of X. The *hereditary* σ-ring $\mathcal{H}(A)$ generated by A is the set:

$$\mathcal{H}(A) = \left\{U \subset X : \text{ there exists a sequence } \{E_n\}_{n=1}^{\infty} \subset A \text{ such that } U \subset \bigcup E_n\right\}$$

That is, $\mathcal{H}(A)$ is the set of all subsets of X that can be covered by a countable collection of elements of A.

Example: If $P = \{[a, b) : a \leq b\}$, then $\mathcal{H}(P)$ is the powerset of $\mathbb{R}$.

Example: If X is an uncountable set, and A is the set of all finite subsets of X, then $\mathcal{H}(A)$ is the set of all countable subsets of X.

The next concept shows how to extend a premeasure on a semi-ring to a function on the hereditary σ-ring of the semi-ring.

Definition: Examine the case when $\mathcal{P}$ is a semi-ring with a premeasure μ. Consider the hereditary σ-ring $\mathcal{H}(\mathcal{P})$. For any $E \in \mathcal{H}(\mathcal{P})$, define the *outer measure* μ^* by:

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} E_j : E \subset \bigcup E_j,\ E_j \in \mathcal{P} \right\}$$

It's quickly verified that $\mu^* : \mathcal{H}(\mathcal{P}) \to [0, \infty]$ is an extension of the premeasure μ from $\mathcal{P}$ to $\mathcal{H}(\mathcal{P})$. Suppose $E \in \mathcal{P}$. Then $\{E\}$ is a cover of E, so

$$\mu^*(E) \leq \mu(E)$$

But, given $\epsilon > 0$, there exists a cover $\{E_n\}$ of E such that

$$\sum \mu(E_n) \leq \mu^*(E) + \epsilon$$

Without loss of generality, this cover is pairwise disjoint, since we may inductively define a new sequence with the same total measure: $A_1 = E_1$ and $A_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$. Then $E = \bigoplus_{n=1}^{\infty}(E \cap E_n)$ and:

$$\mu(E) = \mu(\bigoplus E_n \cap E) \leq \sum_{n=1}^{\infty} \mu(E_n \cap E) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

This shows $\mu(E) \leq \mu(E)^* + \epsilon$ for arbitrary $\epsilon > 0$; hence $\mu(E) = \mu^*(E)$. We summarize this finding in the following proposition:

Proposition: The outer measure μ^* is an extension of $\mathcal{P}$ to $\mathcal{H}(\mathcal{P})$.

12.5 Measurable sets and Carathéodory's theorem

The next several sections of notes will prove two theorems. The first is Carathéodory's theorem, which constructively proves the existence of measures. The second theorem shows that every measure defined on the *Borel* subsets of $\mathbb{R}$ is characterized by a nondecreasing left-continuous function, in the way we previously defined the μ_α function. To work towards these results, we introduce the definition of measurable sets.

Let $\mathcal{P}$ be a semi-ring with premeasure μ. Let μ^* be the outer measure defined on the hereditary σ-ring $\mathcal{H}(\mathcal{P})$.

Definition: An element $E \in \mathcal{H}(\mathcal{P})$ is a μ^*-*measurable set* if for any $A \in \mathcal{H}(\mathcal{P})$, the following equality holds:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

Denote the set of all μ^*-measurable sets by $M(\mathcal{H}(\mathcal{P}), \mu^*)$, or, if the σ-ring is clear, $M(\mu^*)$.

We leave the proof of the following proposition to any measure theory textbook, such as Halmos' *Measure Theory*.

Proposition: $M(\mu^*)$ is a σ-ring.

From the definition, it is not immediately clear how measurable sets are useful towards constructing measures, or if nontrivial measurable sets (i.e. other than $\emptyset$ or the whole set) exist. We will answer the latter question with the following theorem.

Theorem: Let $\mathcal{P}$ be a semi-ring and μ a premeasure. Let μ^* be its outer measure on $\mathcal{H}(\mathcal{P})$. Then $\mathcal{P} \subset M(\mu^*)$.

Proof. First, let E and F be in $\mathcal{P}$. Then $E \setminus F = \bigoplus_{j=1}^{k} G_j$ for some G_j in $\mathcal{P}$. Calculating:

$$\mu(E) = \mu(E \cap F) + \sum_{j=1}^{k} \mu(G_j) \geq \mu(E \cap F) + \mu^*(E \setminus F) \geq \mu(E)$$

Hence $\mu(E) = \mu(E \cap F) + \mu^*(E \setminus F)$. Then for any $A \in \mathcal{H}(\mathcal{P})$ and given $\epsilon > 0$, there exists a sequence $\{F_j\}_{j=1}^{\infty} \subset \mathcal{P}$ such that $A \subset \bigcup F_j$ and

$$\mu^*(A) + \epsilon \geq \sum_{j=1}^{\infty} \mu^*(F_j)$$

But $A \cap E \subset \bigcup F_j \cap E$, and $A \setminus E \subset \bigcup (F_j \setminus E)$. So

$$\begin{aligned}\mu^*(A \cap E) + \mu^*(A \setminus E) &\leq \sum_{j=1}^{\infty} \mu(F_j \cap E) + \sum_{j=1}^{\infty} \mu^*(F_j \setminus E) \\ &= \sum_{j=1}^{\infty} \mu(F_j \cap E) + \mu^*(F_j \setminus E) \\ &\leq \sum_{j=1}^{\infty} \mu(F_j) \leq \mu^*(A) + \epsilon\end{aligned}$$

■

We recall that if $\mathcal{A}$ is a σ-ring, then $\mu : \mathcal{A} \to \mathbb{R}^+$ is a *measure* if μ is countably additive.

Theorem: (Carathéodory) Let $S(\mathcal{P})$ be the σ-ring generated by $\mathcal{P}$. Then $S(\mathcal{P}) \subset M(\mu^*)$. So $\mu^*|_{S(\mathcal{P})}$ is a measure on $S(\mathcal{P})$ that extends μ.

Proof. The proof is immediate by the previous theorem. ■

The preceding theorem is typically referred to as Carathéodory's construction of the measure from premeasure.

Consider the particular case when the ambient set is $\mathbb{R}$ and $\mathcal{P} = \{[a, b) : a \leq b\}$, the semi-ring of half-open, half-closed intervals. Let α be a nonnegative non-decreasing left-continuous function on $\mathbb{R}$. Using Carathéodory's construction, we obtain a measure μ_α on $\mathcal{H}(\mathcal{P})$. If we do not restrict μ_α, i.e. let it have domain $\mathcal{H}(\mathcal{P})$, then μ_α is a *Borel-Stieltjes* measure. If we do restrict the domain to the measurable sets $M(\mu^*)$, then μ_α is a *Lebesgue-Stieltjes* measure.

12.6 Uniqueness result for Carathéodory's theorem

In the previous section, we proved that a premeasure μ on a semi-ring $\mathcal{P}$ induces an outer measure μ^* on $\mathcal{H}(\mathcal{P})$, which in turn restricts to a measure on the μ^*-measurable sets $M(\mu^*)$.

In this section, we determine a sufficient condition to make this extension of μ to μ^* unique. This condition is σ-finiteness.

Definition: Let $\mathcal{P}$ be a semi-ring and μ a premeasure. Then μ is a *σ-finite* premeasure of $\mathcal{P}$ if, given any subset $A \in \mathcal{P}$, there exists a countable cover $\{E_j\}_{j=1}^{\infty} \subset \mathcal{P}$ of A (i.e. $A \subset \bigcup E_j$ such that $\mu(E_j) < \infty$.

This definition easily generalizes to the case of a σ-ring and measure. In that case, we say μ is a σ-finite measure of the σ-ring.

Theorem: Let μ be a premeasure on a semi-ring $\mathcal{P}$. If μ is a σ-finite premeasure on $\mathcal{P}$, then for any S σ-ring such that $\mathcal{P} \subseteq S \subseteq M(\mu^*)$, the restricted measure $\mu^*|_S$ is the unique extension of μ to S.

Proof. Let $\nu : S \to [0, \infty]$ be another measure on S such that $\nu|_{\mathcal{P}} \equiv \mu$. Let $G \in S$.

First, assume that there is a $F \in \mathcal{P}$ such that $G \subset F$ and $\mu(F) < \infty$. Then:

$$\begin{aligned} \mu(F) = \mu^*(F) &= \mu^*(F \cap G) + \mu^*(F \setminus G) \\ &\geq \nu(F \cap G) + \nu(F \setminus G) = \nu(F) = \mu(F) \end{aligned}$$

Note where we use the claim that μ^* is the largest measure on $M(\mu^*)$ coming from the premeasure μ, that is, $\mu^*(A) \geq \mu(A)$ for all $A \in M(\mu^*)$. Hence,

$$\mu^*(F \cap G) + \mu^*(F \setminus G) - \nu(F \cap G) + \nu(F \setminus G)$$

But $\mu^*(F \cap G) \geq \nu(F \cap G)$ and $\mu^*(F \setminus G) \geq \nu(F \setminus G)$. This implies $\mu^*(F \cap G) = \nu(F \cap G)$, and so $\mu^*(G) = \nu(G)$.

Now, in the general case, by σ-finiteness, for any $G \in S \subset \mathcal{H}(\mathcal{P})$, there exists a sequence $\{E_j\} \subset \mathcal{P}$ such that $G \subset \bigcup E_j$ and $\mu(E_j) < \infty$. Without loss of generality, the E_j are pairwise disjoint. Then $G = \bigoplus (G \cap E_j)$ for $\mu^*(G \cap E_j) < \infty$; apply the first case to see that $\nu(G) = \mu^*(G)$. ∎

12.7 Nullsets and complete measure spaces

Before moving on to our last result in measure theory, we take a detour to talk about nullsets, completeness, and properties of measures. Here, we discuss the former two topics.

Definition: Let S be a σ-ring of a set X along with a measure $\mu : S \to [0, \infty]$. The pair (X, S) is a *measurable space.* The tuple (X, S, μ) is a *measure space.*

Definition: Let $\mathcal{A} \subset S$ be a subset of the σ-ring S. $\mathcal{A}$ is *hereditary* if $G \in \mathcal{A}$ implies all subsets of G are in $\mathcal{A}$.

Example: The set

$$\mathcal{N}(\mu) = \{A \subset X : \text{there exists } E \in S \text{ such that } A \subset E,\ \mu(E) = 0\}$$

is a hereditary set. It's quickly verified that $\mathcal{N}(\mu)$ is even a hereditary σ-ring.

Proposition: The set:

$$\tilde{S} = \{E \cup A : E \in S,\ A \in \mathcal{N}(\mu)\}$$

is a σ-ring.

Proof. Inspection.

Then we may attempt to define a function $\tilde{\mu} : \tilde{S} \to [0, \infty]$ by

$$\tilde{\mu}(E \cup A) = \mu(E)$$

for $E \in S$ and $A \in \mathcal{N}(\mu)$. The details that prove this function is a measure is left to the reader, and the resulting measure space $(X, \tilde{S}, \tilde{\mu})$ is called the *complete* measure space from (X, S, μ). If $\tilde{S} = S$, then $\tilde{\mu} \equiv \mu$ and μ is said to be a *complete* measure.

12.8 Properties of measures

The next two results are known as continuity results for the measure. For notation, if $\{E_j\}$ is a sequence of sets, and E is a set such that $E = \bigcup E_j$ and $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, then say the E_j *increases to* E, and write $E_j \uparrow E$. Similarly, if $E_{n+1} \subset E_n$ for all n and $E = \bigcap E_j$, then say the E_j *decreases to* E, and write $E_j \downarrow E$.

Let (X, S, μ) be a measure space.

Proposition: (Continuity from below) If $\{E_j\}_{j=1}^{\infty} \subset S$ and $E_j \uparrow E$, then

$$\mu(E) = \lim_{n \to \infty} \mu(n_j)$$

Proof. Observe that :

$$E = E_1 \oplus (E_2 \setminus E_1) \oplus \cdots \oplus (E_{j+1} \setminus E_j) \oplus \cdots$$

By countable additivity

$$\mu(E) = \lim_{n\to\infty}(\mu(E_1) + \sum_{j=1}^{n} \mu(E_{j+1} \setminus E_j) = \lim \mu(E_n)$$

■

Proposition: (Continuity from below) If $E_j \downarrow E$ and there is a j such that $\mu(E_j) < \infty$, then

$$\mu(E) = \lim \mu(E_n)$$

Proof. Assume that $\mu(E_1) < \infty$. Consider $E_1 \setminus E_j \uparrow E_1 \setminus E$. So:

$$\mu(E_1 \setminus E) = \lim \mu(E_1 \setminus E_j) = \lim(\mu(E_1) - \mu(E_j))$$

which implies, since $\mu(E_1) < \infty$, that $\mu(E) = \lim \mu(E_n)$. ■

Lastly, a word about finite measures:

Proposition: If $\mu(E) < \infty$ for all $E \in S$, then there is a $k \in \mathbb{R}^+$ such that $\mu(E) \leq k$ for all $E \in S$.

Proof. If not, then for each k there exists E_k such that $\mu(E_k) > k$. Then $\mu(\bigcup E_k) = \infty$ by continuity. ■

12.9 Characterization of positive Borel measures

Now, we further examine the case when $\mathcal{P} = \{[a,b) : a \leq b\}$. We assume an important fact:

Claim: Let $\mathcal{T}_\mathbb{R}$ be the standard metric topology on $\mathbb{R}$. Then the generated σ-ring of $\mathcal{P}$ is exactly equal to the generated σ-ring of $\mathcal{T}_\mathbb{R}$. Denote this σ-ring as $\mathcal{B}_\mathbb{R}$, the Borel σ-ring of $\mathbb{R}$.

... the proof of which can be found in any measure theory textbook.

In fact, for the purpose of generality, we may define the Borel subsets of an arbitrary topological space.

Definition: Let $(X, \mathcal{T})$ be a topological space. Then the *Borel subsets* of $(X, \mathcal{T})$ are the generated σ-ring from the open sets $\mathcal{T}$. Denote this measurable space by $(X, \mathcal{B}_X)$.

A measure is said to be a *Borel* measure if it is defined on $\mathcal{B}_{\mathbb{R}}$ and $\mu(C) < \infty$ for any compact $C \subset \mathbb{R}$.

Returning to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, let $\alpha : \mathcal{P} \to [0, \infty]$ be an arbitrary nondecreasing left-continuous function. Recall the premeasure μ_α defined:

$$\mu_\alpha([a, b)) = \alpha(b) - \alpha(a)$$

By an abuse of notation, let μ_α also refer to the measure from the premeasure μ_α by Carathéodory's theorem, defined on all of $\mathcal{B}_{\mathbb{R}}$.

Theorem: Every positive Borel measure on $\mathbb{R}$ is of the form μ_α for some nondecreasing left-continuous $\alpha : \mathbb{R} \to \mathbb{R}^+$.

Proof. Let μ be a positive Borel measure on $\mathbb{R}$. Define α by

$$\alpha(t) = \begin{cases} \mu([0, t)) & \text{for } t > 0 \\ -\mu([t, 0)) & \text{for } t \leq 0 \end{cases}$$

It's easy to see that α is nondecreasing. Now, let $\{t_n\}$ be a sequence increasing to some $t' > 0$; then continuity from below of μ implies $\alpha(t_n) \to \alpha(t')$ as $n \to \infty$. And if the t_n increase to $t' < 0$, then continuity from above shows $\lim \alpha(t_n) = \alpha(t')$ (keeping in mind that $\mu([t, 0)) < \infty$ and $\mu([0, t)) < \infty$ since μ is a Borel measure).

Lastly, check that $\alpha(0) = 0$. So we can define the measure μ_α on $\mathcal{B}_{\mathbb{R}}$ by Carathéodory's construction and see, for all $0 < a \leq b$:

$$\mu([a, b)) = \mu([0, b) \setminus [0, a)) = \mu([0, b)) - \mu([0, a)) = \mu_\alpha([a, b))$$

Hence μ_α and μ are σ-finite measures that agree on $\mathcal{P}$; so by uniqueness, μ_α is exactly μ. ∎