Name:

MAT201A University of California, Davis Fall 2015

Homework # 1

(Due Monday, October 5)

Problem 1. Let (X, d) be a metric space, and let $x, y, w, z \in X$.

a) Prove that

$$d(x, y) \geq |d(x, z) - d(z, y)|.$$

b) Prove that

$$d(x, y) + d(z, w) \geq |d(x, z) - d(y, w)|.$$

c) Let (x_n) and (y_n) be converging sequences in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Prove that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Problem 2. Show that the limit of a convergent sequence in a metric space is unique. I.e., if, for a sequence (x_n) in a metric space (X, d), and $x, y \in X$, $x_n \to x$ and $x_n \to y$, then $x = y$.

Problem 3. Let (a_n) be a sequence in $\mathbb{R}$.

a) Prove that there exists a subsequence of $(a_{n_k})_{k=1}^{\infty}$ of (a_n) such that $\lim_k a_{n_k} = \liminf_n a_n$.

b) Prove that (a_n) converges to $a \in \mathbb{R}$ if and only if $\liminf_n a_n = \limsup_n a_n = a$.

Problem 4. Let (X, d) be a metric space. Prove the statements in Proposition 1.37 in the textbook:

a) The empty set $\emptyset$ and the set X itself are both open and closed sets in (X, d).

b) The intersection of a finite collection of open sets is open.

c) The union of an arbitrary collection of open sets is open.

d) The union of a finite collection of closed sets is closed.

e) The intersection of an arbitrary collection of closed sets is closed.

Problem 5. Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$ a continuous function, and $B \subset Y$ a closed set. Prove that A defined by

$$A = \{x \in X \mid f(x) \in B\}$$

is a closed set.

Problem 6. Let X be a Banach space and let (x_n) be a sequence in X such that $\sum_{n=1}^{\infty} \|x_n\| = 1$.

a) Prove that the series $\sum_{n=1}^{\infty} x_n$ converges to a limit $x \in X$.

b) Prove that for any subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n), the series $\sum_{k=1}^{\infty} x_{n_k}$ also converges and that the norm of its limit is bounded by 1.