

---

# Homework #5

---

Sam Fleischer

February 21, 2016

<b>Hunter and Nachtergaele 8.2 . . . . .</b>	<b>2</b>
<b>Hunter and Nachtergaele 8.3 . . . . .</b>	<b>2</b>
<b>Hunter and Nachtergaele 8.4 . . . . .</b>	<b>3</b>
<b>Hunter and Nachtergaele 8.6 . . . . .</b>	<b>3</b>
<b>Hunter and Nachtergaele 8.7 . . . . .</b>	<b>3</b>
<b>Hunter and Nachtergaele 8.8 . . . . .</b>	<b>4</b>
<b>Hunter and Nachtergaele 8.9 . . . . .</b>	<b>4</b>
<b>Hunter and Nachtergaele 8.11 . . . . .</b>	<b>5</b>
<b>Hunter and Nachtergaele 8.12 . . . . .</b>	<b>6</b>

---

## Hunter and Nachtergaele 8.2

---

If  $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$  is an orthogonal direct sum, show that  $\mathcal{M}^\perp = \mathcal{N}$  and  $\mathcal{N}^\perp = \mathcal{M}$ .

*Proof.* Suppose  $x \in \mathcal{M}^\perp$ . Then  $x \in \mathcal{H} \implies \exists! y, z$  such that  $x = y + z$  where  $y \in \mathcal{M}$  and  $z \in \mathcal{N}$ . Then

$$\langle x, x \rangle = \langle \cancel{x, y} \rangle + \langle x, z \rangle = \langle x, z \rangle \implies \langle x, x - z \rangle = 0 \implies x = z$$

which shows  $x \in \mathcal{N}$ , i.e.  $\mathcal{M}^\perp \subset \mathcal{N}$ .

Now suppose  $x \notin \mathcal{M}^\perp$ . Then  $x \in \mathcal{M}$  and  $x \neq 0$ . Thus  $x \notin \mathcal{N}$  since a direct sum implies  $\mathcal{N} \cap \mathcal{M} = \{0\}$ . Thus  $\mathcal{N} \subset \mathcal{M}^\perp \subset \mathcal{N} \implies \mathcal{N} = \mathcal{M}^\perp$ .

Switching  $\mathcal{N}$  and  $\mathcal{M}$  shows  $\mathcal{M} = \mathcal{N}^\perp$ . □

---

## Hunter and Nachtergaele 8.3

---

Let  $\mathcal{M}, \mathcal{N}$  be closed subspaces of a Hilbert space  $\mathcal{H}$  and  $P, Q$  the orthogonal projections with  $\text{ran } P = \mathcal{M}$ ,  $\text{ran } Q = \mathcal{N}$ . Prove that the following conditions are equivalent: (a)  $\mathcal{M} \subset \mathcal{N}$ ; (b)  $QP = P$ ; (c)  $PQ = P$ ; (d)  $\|Px\| \leq \|Qx\|$  for all  $x \in \mathcal{H}$ ; (e)  $\langle x, Px \rangle \leq \langle x, Qx \rangle$  for all  $x \in \mathcal{H}$ .

*Proof.* We will show  $(a) \rightarrow (b) \rightarrow (c) \rightarrow (d) \rightarrow (e) \rightarrow (a)$ , which proves the statements' equivalence.

(a)  $\rightarrow$  (b). Let  $\mathcal{M} \subset \mathcal{N}$  and let  $x \in \mathcal{H}$ . Then  $Px \in \mathcal{M} \subset \mathcal{N} \implies Q(Px) = Px \implies QP = P$ .

(b)  $\rightarrow$  (c). Let  $QP = P$ . Then

$$\langle x, Py \rangle = \langle Px, y \rangle = \langle QPx, y \rangle = \langle Px, Qy \rangle = \langle x, PQy \rangle \implies \langle x, Py - PQy \rangle = 0 \quad \forall x, y \in \mathcal{H}.$$

Thus  $Py - PQy = 0$  for all  $y \in \mathcal{H}$ , i.e.  $PQ = P$ .

(c)  $\rightarrow$  (d). Let  $PQ = P$ . First note  $\|Px\| \leq \|x\|$  for all  $x \in \mathcal{H}$  because

$$\|Px\| = \frac{\|Px\|^2}{\|Px\|} = \frac{\langle Px, Px \rangle}{\|Px\|} = \frac{\langle x, P^2x \rangle}{\|Px\|} = \frac{\langle x, Px \rangle}{\|Px\|} \leq \|x\|$$

by the Cauchy-Schwarz inequality. Thus,

$$\|Px\| = \|PQx\| = \|P(Qx)\| \leq \|Qx\| \quad \forall x \in \mathcal{H}$$

(d)  $\rightarrow$  (e). Let  $\|Px\| \leq \|Qx\|$ .

$$\|Px\|^2 \leq \|Qx\|^2 \implies \langle Px, Px \rangle \leq \langle Qx, Qx \rangle \implies \langle x, P^2x \rangle \leq \langle x, Q^2x \rangle \implies \langle x, Px \rangle \leq \langle x, Qx \rangle \quad \forall x \in \mathcal{H}$$

(e)  $\rightarrow$  (a). Let  $\langle x, Px \rangle \leq \langle x, Qx \rangle$  and suppose  $x \in \mathcal{M}$ . Then  $Px = x$ . Then  $\|x\|^2 = \langle x, x \rangle = \langle x, Px \rangle \leq \langle x, Qx \rangle$ . However, since  $\|Qx\| \leq \|x\|$  for all  $x$ , then  $\langle x, Qx \rangle \leq \|x\|\|Qx\| \leq \|x\|^2$ , which shows  $\langle x, Qx \rangle = \|x\|^2$ . Thus  $Qx = x$ , which shows  $x \in \mathcal{N}$ , proving  $\mathcal{M} \subset \mathcal{N}$ .

□

---

## Hunter and Nachtergaele 8.4

---

Suppose that  $(P_n)$  is a sequence of orthogonal projections on a Hilbert space  $\mathcal{H}$  such that

$$\operatorname{ran} P_{n+1} \supset \operatorname{ran} P_n, \quad \bigcup_{n=1}^{\infty} \operatorname{ran} P_n = \mathcal{H}.$$

Prove that  $(P_n)$  converges strongly to the identity operator  $I$  as  $n \rightarrow \infty$ . Show that  $(P_n)$  does not converge to the identity operator with respect to the operator norm unless  $P_n = I$  for all sufficiently large  $n$ .

*Proof.* Let  $x \in \mathcal{H}$ . Then  $\exists N$  such that  $x \in \operatorname{ran} P_N$ . Thus  $x \in \operatorname{ran} P_n$  for all  $n \geq N$ . Since each  $P_n$  is an orthogonal projection, then  $x = P_n x$  for all  $n \geq N$ . Thus

$$\lim_{n \rightarrow \infty} P_n x = \lim_{n \rightarrow \infty} x = x = Ix$$

where  $I$  is the identity operator. Thus  $(P_n)$  converges strongly to  $I$ . If  $P_n = I$  for all sufficiently large  $n$ , then obviously

$$\lim_{n \rightarrow \infty} \|P_n\| = \lim_{n \rightarrow \infty} \|I\| = \|I\|$$

□

---

## Hunter and Nachtergaele 8.6

---

Show that a linear operator  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is unitary if and only if it is an isometric isomorphism of normed linear spaces. Show that an invertible linear map is unitary if and only if its inverse is.

*Proof.* Let  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be unitary. Then  $U$  is invertible and  $\langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1}$  for all  $x, y \in \mathcal{H}_1$ . Thus

$$\|Ux\|_{\mathcal{H}_2}^2 = \langle Ux, Ux \rangle_{\mathcal{H}_2} = \langle x, x \rangle_{\mathcal{H}_1} = \|x\|_{\mathcal{H}_1}^2$$

Thus  $U$  preserves norms and is thus an isometric isomorphism.

Now suppose  $U$  is an isometric isomorphism. Then  $\|Ux\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}$  for all  $x \in \mathcal{H}_1$ . □

---

## Hunter and Nachtergaele 8.7

---

If  $\phi_y$  is the bounded linear functional defined in (8.5),

$$\phi_y(x) = \langle y, x \rangle \quad (8.5)$$

prove that  $\|\phi_y\| = \|y\|$ .

*Proof.* First we prove  $\|\phi_y\|$  is bounded above by  $\|y\|$ .

$$\|\phi_y\| = \sup_{\|x\|=1} \|\phi_y(x)\| = \sup_{\|x\|=1} \langle y, x \rangle \leq \sup_{\|x\|=1} \|y\| \|x\| = \|y\|$$

Next, consider  $x = \frac{y}{\|y\|}$  (note  $\|x\| = 1$ ):

$$\|\phi_y(x)\| = \|\langle y, x \rangle\| = \left\| \frac{\langle y, y \rangle}{\|y\|} \right\| = \left\| \frac{\|y\|^2}{\|y\|} \right\| = \|y\|$$

and thus  $\|\phi_y\| \geq \|y\|$ , which proves  $\|\phi_y\| = \|y\|$ . □

---

## Hunter and Nachtergaele 8.8

---

Prove that  $\mathcal{H}^*$  is a Hilbert space with the inner product defined by

$$\langle \phi_x, \phi_y \rangle_{\mathcal{H}^*} = \langle y, x \rangle_{\mathcal{H}}.$$

*Proof.* First note that for  $\phi_{y_1}, \phi_{y_2} \in \mathcal{H}^*$ ,  $\|\phi_{y_1} + \phi_{y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$  where  $y_1$  and  $y_2$  are the associated vectors in  $\mathcal{H}$  guaranteed in the Riesz Representation Theorem. This is true because  $\phi_{y_1}(x) = \langle y_1, x \rangle_{\mathcal{H}}$  and  $\phi_{y_2}(x) = \langle y_2, x \rangle_{\mathcal{H}}$  imply

$$\phi_{y_1+y_2}(x) = \langle y_1 + y_2, x \rangle_{\mathcal{H}} = \langle y_1, x \rangle_{\mathcal{H}} + \langle y_2, x \rangle_{\mathcal{H}} = \phi_{y_1}(x) + \phi_{y_2}(x)$$

and since  $\|\phi_{y_1+y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$  by (Hunter and Nachtergaele 8.7), then  $\|\phi_{y_1} + \phi_{y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$ . Let  $(\phi_n)$  be a Cauchy sequence in  $\mathcal{H}^*$ . Then  $\forall \varepsilon > 0$ ,  $\exists N$  such that  $\|\phi_m - \phi_n\|_{\mathcal{H}^*} < \varepsilon$  for  $m, n \geq N$ . By the Riesz Representation Theorem,  $\exists (y_n)_n \in \mathcal{H}$  such that for every  $n$ ,  $\phi_n(x) = \langle y_n, x \rangle_{\mathcal{H}} \forall x \in \mathcal{H}$ .  $(y_n)_n$  is Cauchy since given  $\varepsilon > 0$  we can find  $N$  such that  $\|y_n - y_m\|_{\mathcal{H}} = \|\phi_n - \phi_m\|_{\mathcal{H}^*} < \varepsilon$ . Since  $\mathcal{H}$  is a Hilbert space, then  $(y_n)_n$  is convergent to some  $y \in \mathcal{H}$ . By the Riesz Representation Theorem,  $\exists \phi \in \mathcal{H}^*$  such that  $\phi(x) = \langle y, x \rangle_{\mathcal{H}} \forall x \in \mathcal{H}$ . Then  $(\phi_n)_n$  converges to  $\phi$  because  $\|\phi_n - \phi\|_{\mathcal{H}^*} = \|y_n - y\|_{\mathcal{H}}$ , which can be made arbitrary small by the definition of convergence. Thus  $\mathcal{H}^*$  is complete. Also,  $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$  is a well-defined inner product since  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is a well-defined inner product (i.e. the properties of inner product hold). Thus  $\mathcal{H}^*$  is a Hilbert space. □

---

## Hunter and Nachtergaele 8.9

---

Let  $A \subset \mathcal{H}$  be such that

$$\mathcal{M} = \{x \in \mathcal{H} \mid x \text{ is a finite linear combination of elements in } A\}$$

is a dense linear subspace of  $\mathcal{H}$ . Prove that any bounded linear functional on  $\mathcal{H}$  is uniquely determined by its values on  $A$ . If  $\{u_\alpha\}$  is an orthonormal basis, find a necessary and sufficient condition on a family of complex numbers  $c_\alpha$  for there to be a bounded linear functional  $\phi$  such that  $\phi(u_\alpha) = c_\alpha$ .

*Proof.* Suppose  $\phi_1$  and  $\phi_2$  are two bounded linear functionals such that  $\phi_1(a) = \phi_2(a)$  for all  $a \in A$ . Let  $x \in \mathcal{H}$ . By density of  $\mathcal{M}$ ,  $\exists m_i \in \mathcal{M}$  such that  $m_i \rightarrow x$ . By linearity of  $\phi_1$  and  $\phi_2$ ,  $\phi_1(m_i) = \phi_2(m_i)$  for all  $i = 1, 2, \dots$ . By the Riesz Representation Theorem,  $\exists y_1, y_2$  such that  $\phi_1(x) = \langle y_1, x \rangle$  and  $\phi_2(x) = \langle y_2, x \rangle$  for all  $x \in \mathcal{H}$ . Thus by continuity of inner products,

$$\phi_1(x) = \langle y_1, x \rangle = \lim_{i \rightarrow \infty} \langle y_1, m_i \rangle = \lim_{i \rightarrow \infty} \phi_1(m_i) = \lim_{i \rightarrow \infty} \phi_2(m_i) = \lim_{i \rightarrow \infty} \langle y_2, m_i \rangle = \langle y_2, x \rangle = \phi_2(x)$$

Thus  $\phi_1 \equiv \phi_2$ , i.e. bounded linear functionals are uniquely determined by their values on  $A$ .  $\square$

---

## Hunter and Nachtergaele 8.11

---

Prove that if  $A : \mathcal{H} \rightarrow \mathcal{H}$  is a linear map and  $\dim \mathcal{H} < \infty$ , then

$$\dim \ker A + \dim \operatorname{ran} A = \dim \mathcal{H}.$$

Prove that if  $\dim \mathcal{H} < \infty$ , then  $\dim \ker A = \dim \ker A^*$ . In particular,  $\ker A = \{0\}$  if and only if  $\ker A^* = \{0\}$ .

*Proof.* Let  $\dim \mathcal{H} < \infty$  (say  $\dim \mathcal{H} = n$ ). Then  $\dim \ker A < \infty$  and  $\dim \operatorname{ran} A < \infty$  since  $\ker A$  and  $\operatorname{ran} A$  are subspaces of  $\mathcal{H}$ . Let  $\dim \ker A = m \leq n$  and let  $\{u_1, u_2, \dots, u_m\}$  be a basis of  $\ker A$ . Since  $\ker A$  is a subspace of  $\mathcal{H}$ , this basis can be extended to a basis  $\mathcal{U}$  of  $\mathcal{H}$ :  $\mathcal{U} = \{u_1, u_2, \dots, u_m, v_{m+1}, v_{m+2}, \dots, v_n\}$ . Let  $x \in \mathcal{H}$ . Then

$$\begin{aligned} x &= a_1 u_1 + a_2 u_2 + \dots + a_m u_m + a_{m+1} v_{m+1} + a_{m+2} v_{m+2} + \dots + a_n v_n \\ \Rightarrow Ax &= \underbrace{a_1 A u_1}_{\rightarrow 0} + \underbrace{a_2 A u_2}_{\rightarrow 0} + \dots + \underbrace{a_m A u_m}_{\rightarrow 0} + a_{m+1} A v_{m+1} + a_{m+2} A v_{m+2} + \dots + a_n A v_n \\ &= a_{m+1} A v_{m+1} + a_{m+2} A v_{m+2} + \dots + a_n A v_n \end{aligned}$$

since  $u_i \in \ker A$  for  $i = 1, 2, \dots, m$ . Thus,  $\{A v_{m+1}, A v_{m+2}, \dots, A v_n\}$  spans  $\operatorname{ran} A$ . However it is also linearly independent since

$$\begin{aligned} c_{m+1} A v_{m+1} + c_{m+2} A v_{m+2} + \dots + c_n A v_n &= 0 \\ \Rightarrow A(c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \dots + c_n v_n) &= 0 \\ \Rightarrow c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \dots + c_n v_n &\in \ker A \\ \Rightarrow c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \dots + c_n v_n &= d_1 u_1 + d_2 u_2 + \dots + d_m u_m \quad \text{for some } d_i \in \mathbb{C} \\ \Rightarrow d_1 = d_2 = \dots = d_m = c_{m+1} = c_{m+2} = \dots = c_n &= 0 \quad \text{since } \mathcal{U} \text{ is a basis} \end{aligned}$$

Thus  $\{A v_{m+1}, A v_{m+2}, \dots, A v_n\}$  is linearly independent. Since it also spans  $\operatorname{ran} A$  then it is a basis of  $\operatorname{ran} A$ . Thus  $\dim \operatorname{ran} A = n - m$ . Thus, since  $m + (n - m) = n$ , then

$$\dim \ker A + \dim \operatorname{ran} A = \dim \mathcal{H}.$$

Let  $x \in \mathcal{H}$ . Then

$$\begin{aligned}
 x \in \ker A^* &\iff A^* x = 0 \\
 &\iff \langle y, A^* x \rangle = 0 \quad \forall y \in \mathcal{H} \\
 &\iff \langle Ay, x \rangle = 0 \quad \forall y \in \mathcal{H} \\
 &\iff x \perp Ay \quad \forall y \in \mathcal{H} \\
 &\iff x \perp (\operatorname{ran} A) \\
 &\iff x \in (\operatorname{ran} A)^\perp
 \end{aligned}$$

Thus  $\ker A^* = (\operatorname{ran} A)^\perp$  and  $\dim \ker A^* = \dim(\operatorname{ran} A)^\perp$ . However since  $\dim \ker A + \dim \operatorname{ran} A = \dim \mathcal{H}$  and  $\dim \operatorname{ran} A + \dim(\operatorname{ran} A)^\perp = \dim \mathcal{H}$ , then  $\dim(\operatorname{ran} A)^\perp = \dim \ker A$ . Thus,

$$\dim \ker A = \dim \ker A^*.$$

□

---

## Hunter and Nachtergaele 8.12

---

Suppose that  $A : \mathcal{H} \rightarrow \mathcal{H}$  is a bounded, self-adjoint linear operator such that there is a constant  $c > 0$  with

$$c\|x\| \leq \|Ax\| \quad \text{for all } x \in \mathcal{H}.$$

Prove that there is a unique solution  $x$  of the equation  $Ax = y$  for every  $y \in \mathcal{H}$ .