UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #6

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Hunter and Nachtergaele 8.1

If M is a linear subspace of a linear space X, then the quotient space *X/M is the set $\{x+M \mid x+y \in M\}$ of affine spaces*

$$x+M = \{x+y \mid y \in M\}$$

parallel to M.

(a) Show that X/M is a linear space with respect to the operations

$$\lambda(x+M) = \lambda x + M, \qquad (x+M)+(y+M) = (x+y)+M.$$

Proof. Since X is a linear space, then $\alpha x + \beta y \in X$ for every $x, y \in X$, $\alpha, \beta \in \mathbb{F}$. Then

$$\alpha(x+M) + \beta(x+M) = (\alpha x + \beta y) + M \in X/M$$

Define the "zero" vector in X/M by $0+M$ where 0 is the "zero" vector in X. Then

$$(0+M)+(x+M) = (0+x)+M = x+M = (x+0)+M = (x+M)+(0+M)$$

Also, the "one" in $\mathbb{F}$ (1) is the "one" in X/M since

$$1(x+M) = 1x+M = x+M$$

Thus X/M is a vector space. □

(b) Suppose that $X = M \oplus N$. Show that N is linearly isomorphic to X/M.

Proof. Define $T : N \to X/M$ by

$$Tn = n+M$$

For any $x, y \in N$, then if $Tx = Ty$, then $x+M = y+M \implies (x-y)+M = 0+M \implies x-y \in M$. But since N is a vector space, then $x-y \in N$. Since $X = M \oplus N$, then $M \cap N = \{0\}$, which means $x = y$. Thus T is injective. Now choose $x+M \in X/M$. Then note $P_N x \in N$ and

$$T(P_N x) = P_N x + M = (P_N x + M) + (P_M x + M) = (P_N x + P_M x) + M = x + M$$

Thus T is surjective. Thus T is a bijection. Also, T is a linear map since

$$T(\alpha x + \beta y) = (\alpha x + \beta y) + M = \alpha(x+M) + \beta(y+M) = \alpha Tx + \beta Ty$$

Thus N is linearly isomorphic to X/M. □

(c) The codimension *of M in X is the dimension of X/M. Is a subspace of a Banach space with finite codimension necessarily closed?*

Proof. □

Hunter and Nachtergaele 8.10

Let $\{u_\alpha\}$ be an orthonormal basis of $\mathscr{H}$. Prove that $\{\phi_{u_\alpha}\}$ is an orthnormal basis of $\mathscr{H}^*$.

Proof. First note $\{\phi_{u_\alpha}\}$ is an orthonormal set since

$$\left\langle \phi_{u_1}, \phi_{u_2} \right\rangle = \langle u_2, u_1 \rangle = \delta_{u_2,u_1} = \begin{cases} 1 & \text{if } u_1 = u_2 \\ 0 & \text{if } u_1 \neq n_2 \end{cases}$$

Next let $\phi \in \mathscr{H}^*$. By the Riesz Representation Theorem, $\exists u \in \mathscr{H}$ such that $\phi(x) = \langle x, u \rangle$ for all $x \in \mathscr{H}$. Then since $\{u_\alpha\}$ is an orthonormal basis of $\mathscr{H}$, then $\exists \{c_\alpha\}$ such that $\sum_\alpha |c_\alpha|^2 < \infty$ and $u = \sum_\alpha c_\alpha u_\alpha$. Then

$$\phi(x) = \langle x, u \rangle = \left\langle x, \sum_\alpha c_\alpha u_\alpha \right\rangle = \sum_\alpha c_\alpha \langle x, u_\alpha \rangle = \sum_\alpha c_\alpha \phi_{u_\alpha}$$

where ϕ_{u_α} is the functional in $\mathscr{H}^*$ such that $\phi_{u_\alpha}(x) = \langle x, u_\alpha \rangle$ for all $x \in \mathscr{H}$. Thus $\{\phi_{u_\alpha}\}$ spans $\mathscr{H}^*$, and hence $\{\phi_{u_\alpha}\}$ is an orthonormal basis of $\mathscr{H}^*$. □

Hunter and Nachtergaele 8.13

Prove that an orthonormal set of vectors $\{u_\alpha \mid \alpha \in \mathscr{A}\}$ is a Hilbert space $\mathscr{H}$ is an orthonormal basis if and only if

$$\sum_{\alpha \in \mathscr{A}} u_\alpha \otimes u_\alpha = I.$$

Proof. Let $\{u_\alpha\}$ be an orthonormal basis of $\mathscr{H}$. Then $\forall x \in \mathscr{H}$, $x = \sum_\alpha \langle u_\alpha, x \rangle u_\alpha$. However, the projection P_{u_α} is defined as

$$P_{u_\alpha} x = \langle u_\alpha, x \rangle u_\alpha$$

and hence, for every $x \in \mathscr{H}$, $Ix = x = \sum_\alpha \langle u_\alpha, x \rangle u_\alpha = \sum_\alpha P_{u_\alpha} x = \sum_\alpha (u_\alpha \otimes u_\alpha) x$. In other words, $I = \sum_\alpha u_\alpha \otimes u_\alpha$. Now let $\sum_\alpha u_\alpha \otimes u_\alpha = I$. Then $x = \sum_\alpha P_{u_\alpha} x = \sum_\alpha \langle u_\alpha, x \rangle u_\alpha$. Thus $\{u_\alpha\}$ is an orthonormal basis of $\mathscr{H}$. □

Hunter and Nachtergaele 8.14

Suppose that $A, B \in \mathscr{B}(\mathscr{H})$ satisfy

$$\left\langle x, Ay \right\rangle = \left\langle x, By \right\rangle \qquad \textit{for all } x, y \in \mathscr{H}.$$

Prove that $A = B$. Use a polarization-type identity to prove that if $\mathscr{H}$ is a complex Hilbert space and

$$\langle x, Ax\rangle = \langle x, Bx\rangle \qquad \textit{for all } x \in \mathscr{H},$$

then $A = B$. What can you say about A and B for real Hilbert spaces?

Proof. If $\langle x, Ay\rangle = \langle x, By\rangle$, then $\langle x, (A-B)y\rangle = 0$ for all $x, y \in \mathscr{H}$. Then $A - B = 0$, i.e. $A = B$.
Let $\langle x, Ax\rangle = \langle x, Bx\rangle$ for all $x \in \mathscr{H}$. Then

$$0 = \langle x, (A-B)x\rangle = \frac{1}{4}\left[\|x+(A-B)x\|^2 - \|x-(A-B)x\|^2 - i\|x+i(A-B)x\|^2 + i\|x-i(A-B)x\|^2\right]$$

Since norms are always positive, this implies the real and imaginary parts of the right hand side must each equal zero. Thus,

$$\begin{aligned}
&\|x+(A-B)x\|^2 = \|x-(A-B)x\|^2 \\
\implies &\langle x+(A-B)x, x+(A-B)x\rangle = \langle x-(A-B)x, x-(A-B)x\rangle \\
\implies &\|x\|^2 + \langle x,(A-B)x\rangle + \langle (A-B)x, x\rangle + \|(A-B)x\|^2 = \|x\|^2 - \langle x,(A-B)x\rangle - \langle (A-B)x, x\rangle + \|(A-B)x\|^2 \\
\implies &\langle x,(A-B)x\rangle + \langle (A-B)x, x\rangle = 0 \\
\implies &\langle x,(A-B)x\rangle + \overline{\langle x,(A-B)x\rangle} = 0 \\
\implies &\operatorname{Re}\left[\langle x,(A-B)x\rangle\right] = 0
\end{aligned}$$

Also,

$$\begin{aligned}
&\|x-i(A-B)x\|^2 = \|x+i(A-B)x\|^2 \\
\implies &\|x\|^2 - i\langle x,(A-B)x\rangle + i\langle (A-B)x, x\rangle - \|(A-B)x\|^2 = \|x\|^2 + i\langle x,(A-B)x\rangle - i\langle (A-B)x, x\rangle - \|(A-B)x\|^2 \\
\implies &i\left[\langle x,(A-B)x\rangle - \overline{\langle x,(A-B)x\rangle}\right] = 0 \\
\implies &\langle x,(A-B)x\rangle - \overline{\langle x,(A-B)x\rangle} = 0 \\
\implies &\operatorname{Im}\left[\langle x,(A-B)x\rangle\right] = 0
\end{aligned}$$

Thus $\langle x, (A-B)x\rangle = 0$ for all $x \in \mathscr{H}$. Thus $A = B$. □

Hunter and Nachtergaele 8.15

Prove that for all $A, B \in \mathscr{B}(\mathscr{H})$, and $\lambda \in \mathbb{C}$, we have (a) $A^{**} = A$; (b) $(AB)^* = B^*A^*$; (c) $(\lambda A)^* = \overline{\lambda}A^*$; (d) $(A+B)^* = A^* + B^*$; (e) $\|A^*\| = \|A\|$.

Proof. (a) For all $x, y \in \mathscr{H}$, $\langle x, Ay\rangle = \langle A^*x, y\rangle = \langle x, (A^*)^*y\rangle$. Thus $\langle x, (A - A^{**})y\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $A = A^{**}$.

(b) For all $x, y \in \mathscr{H}$, $\langle x, (AB)^*y\rangle = \langle ABx, y\rangle = \langle Bx, A^*y\rangle = \langle x, B^*A^*y\rangle \implies \langle x, ((AB)^* - B^*A^*)y\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $(AB)^* = B^*A^*$.

(c) For all $x, y \in \mathscr{H}$, $\langle x, (\lambda A)^*y\rangle = \langle \lambda Ax, y\rangle = \overline{\lambda}\langle Ax, y\rangle = \overline{\lambda}\langle x, A^*y\rangle \implies \left\langle x, ((\lambda A)^* - \overline{\lambda}A^*)y\right\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $(\lambda A)^* = \overline{\lambda}A^*$.

(d) For all $x, y \in \mathscr{H}$, $\langle x, (A+B)^* y\rangle = \langle (A+B)x, y\rangle = \langle Ax, y\rangle + \langle Bx, y\rangle = \langle x, A^* y\rangle + \langle x, B^* y\rangle = \langle x, (A^* + B^*) y\rangle \implies \langle x, ((A+B)^* - (A^* + B^*)) y\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $(A+B)^* = A^* + B^*$.

(e) First define $M \in \mathscr{H}^*$ by $Mx = \langle y, Ax\rangle$. Then M is a bounded linear functional since

$$M(ax_1 + bx_2) = \langle y, A(ax_1 + bx_2)\rangle = \langle y, aAx_1\rangle + \langle y, bAx_2\rangle = a\langle y, Ax_1\rangle + b\langle y, Ax_2\rangle = aMx_1 + bMx_2$$

and

$$\|M\| = \sup_{\|x\|=1} \langle y, Ax\rangle \le \sup_{\|x\|=1} \|y\| \|Ax\| \le \|y\| \|A\|$$

and since $A \in \mathscr{B}(\mathscr{H})$, then $\|M\| < \infty$. Then since $M \in \mathscr{H}^*$, The Riesz Representation Theorem guarantees a unique vector $v \in \mathscr{H}$ such that

$$Mx = \langle v, x\rangle = \langle y, Ax\rangle = \langle A^* y, x\rangle$$

Thus $v = A^* y$. Finally,

$$\|A^* y\| = \sup_{\|x\|=1} |\langle v, x\rangle| = \sup_{\|x\|=1} |\langle y, Ax\rangle| \le \sup_{\|x\|=1} \|y\| \|Ax\| \le \sup_{\|x\|=1} \|y\| \|A\| \|x\| = \|y\| \|A\|$$

Thus $\|A^*\| \le \|A\|$. This also implies $\|A\| = \|A^{**}\| = \|(A^*)^*\| \le \|A^*\|$. Thus, $\|A\| = \|A^*\|$.

$\square$

Hunter and Nachtergaele 8.16

Let $U : L^2(\Omega, P) \to L^2(\Omega, P)$ by

$$Uf = f \circ T \tag{8.16}$$

where $T : (\Omega, P) \to (\Omega, P)$ is measure preserving, i.e. $P(A) = P(T^{-1}A)$ $\forall$ measurable $A \subset \Omega$. Prove that the operator U defined in (8.16) is unitary.

Proof. Since T is measure-preserving, then T is bijective (by definition) and for any $f \in L^2(\Omega, P)$, we have $\mathscr{X} f = \mathscr{X} f \circ T$ (where $\mathscr{X}$ is the characteristic function), or

$$\int_\Omega f \mathrm{d}P = \int_\Omega f \circ T \mathrm{d}P$$

Then since $\overline{f} g \in L^2(\Omega, P)$, then

$$\int_\Omega \overline{f} g \mathrm{d}P = \int_\Omega (\overline{f} g) \circ T \mathrm{d}P$$

Thus,

$$\langle Uf, Ug\rangle = \int_\Omega \overline{f(T(\omega))} g(T(\omega)) \mathrm{d}P(\omega) = \int_\Omega \left(\left(\overline{f} g\right) \circ T\right)(\omega) \mathrm{d}P(\omega) = \int_\Omega \left(\overline{f} g\right)(\omega) \mathrm{d}P(\omega) = \langle f, g\rangle$$

Also, since T is bijective, T^{-1} exists and $U^{-1} f$ can be defined as

$$U^{-1} f = f \circ T^{-1}$$

Clearly

$$U^{-1}(Uf) = U^{-1}(f \circ T) = (f \circ T) \circ T^{-1} = f \circ (T \circ T^{-1}) = f \circ \mathbb{1} = f$$

and

$$U(U^{-1}f) = U(f \circ T^{-1}) = (f \circ T^{-1}) \circ T = f \circ (T^{-1} \circ T) = f \circ \mathbb{1} = f$$

$\square$

Hunter and Nachtergaele 8.17

Prove that strong convergence implies weak convergence. Also prove that strong and weak convergence are equivalent in a finite-dimensional Hilbert space.

Proof. Let $x_n \to x$ strongly, i.e. $\|x_n - x\| \to 0$. Then

$$\langle x_n, y\rangle - \langle x, y\rangle = \langle x_n - x, y\rangle \leq \|x_n - x\| y \to 0 \qquad \forall y \in \mathscr{H}$$

Then $x_n \to x$ weakly. Suppose $\dim \mathscr{H} = n < \infty$ and $x_n \to x$ weakly. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathscr{H}$. Then $x = \sum_{i=1}^n c_i e_i$ where $c_i = \langle e_i, x\rangle$. Next, define the ℓ^1 norm by

$$\|x\|_1 = \sum_{i=1}^n |c_i|$$

Since $x_n \to x$ weakly, then $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathscr{H}$. This implies $\langle e_i, x_n\rangle \to \langle e_i, x\rangle$ for each $i = 1, \ldots, n$. Also, $x_n - x = \sum_{i=1}^n \langle e_i, x_n - x\rangle e_i$, and thus

$$\|x_n - x\|_1 = \sum_{i=1}^n |\langle e_i, x_n - x\rangle| = \sum_{i=1}^n |\langle e_i, x_n\rangle - \langle e_i, x\rangle| \to 0$$

However, $\|\cdot\|_1 \equiv \|\cdot\|_{\mathscr{H}}$ since all norms are equivalent in finite-dimensional spaces, and thus $x_n \to x$ strongly. $\square$

Hunter and Nachtergaele 8.18

Let (u_n) be a sequence of orthonormal vectors in a Hilbert space. Prove that $u_n \rightharpoonup 0$ weakly.

Proof. $\square$

Hunter and Nachtergaele 8.19

Prove that a strongly lower-semicontinuous convex function $f : \mathscr{H} \to \mathbb{R}$ on a Hilbert space $\mathscr{H}$ is weakly lower-semicontinuous.

Proof. □