UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

# Homework #5

## Sam Fleischer

February 21, 2016

## Hunter and Nachtergaele 8.2

*If $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ is an orthogonal direct sum, show that $\mathcal{M}^\perp = \mathcal{N}$ and $\mathcal{N}^\perp = \mathcal{M}$.*

*Proof.* Suppose $x \in \mathcal{M}^\perp$. Then $x \in \mathcal{H} \implies \exists! y, z$ such that $x = y + z$ where $y \in \mathcal{M}$ and $z \in \mathcal{N}$. Then

$$\langle x, x\rangle = \cancelto{0}{\langle x, y\rangle} + \langle x, z\rangle = \langle x, z\rangle \implies \langle x, x - z\rangle = 0 \implies x = z$$

which shows $x \in \mathcal{N}$, i.e. $\mathcal{M}^\perp \subset \mathcal{N}$.
Now suppose $x \notin \mathcal{M}^\perp$. Then $x \in \mathcal{M}$ and $x \neq 0$. Thus $x \notin \mathcal{N}$ since a direct sum implies $\mathcal{N} \cap \mathcal{M} = \{0\}$. Thus $\mathcal{N} \subset \mathcal{M}^\perp \subset \mathcal{N} \implies \mathcal{N} = \mathcal{M}^\perp$.
Switching $\mathcal{N}$ and $\mathcal{M}$ shows $\mathcal{M} = \mathcal{N}^\perp$. □

## Hunter and Nachtergaele 8.3

*Let $\mathcal{M}$, $\mathcal{N}$ be closed subspaces of a Hilbert space $\mathcal{H}$ and $P$, $Q$ the orthogonal projections with* ran $P = \mathcal{M}$, ran $Q = \mathcal{N}$*. Prove that the following conditions are equivalent: (a) $\mathcal{M} \subset \mathcal{N}$; (b) $QP = P$; (c) $PQ = P$; (d) $\|Px\| \leq \|Qx\|$ for all $x \in \mathcal{H}$; (e) $\langle x, Px\rangle \leq \langle x, Qx\rangle$ for all $x \in \mathcal{H}$.*

*Proof.* We will show *(a) → (b) → (c) → (d) → (e) → (a)*, which proves the statements' equivalence.

*(a) → (b).* Let $\mathcal{M} \subset \mathcal{N}$ and let $x \in \mathcal{H}$. Then $Px \in \mathcal{M} \subset \mathcal{N} \implies Q(Px) = Px \implies QP = P$.

*(b) → (c).* Let $QP = P$. Then

$$\langle x, Py\rangle = \langle Px, y\rangle = \langle QPx, y\rangle = \langle Px, Qy\rangle = \langle x, PQy\rangle \implies \langle x, Py - PQy\rangle = 0 \qquad \forall x, y \in \mathcal{H}.$$

Thus $Py - PQy = 0$ for all $y \in \mathcal{H}$, i.e. $PQ = P$.

*(c) → (d).* Let $PQ = P$. First note $\|Px\| \leq \|x\|$ for all $x \in \mathcal{H}$ because

$$\|Px\| = \frac{\|Px\|^2}{\|Px\|} = \frac{\langle Px, Px\rangle}{\|Px\|} = \frac{\langle x, P^2x\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|} \leq \|x\|$$

by the Cauchy-Schwarz inequality. Thus,

$$\|Px\| = \|PQx\| = \|P(Qx)\| \leq \|Qx\| \qquad \forall x \in \mathcal{H}$$

*(d) → (e).* Let $\|Px\| \leq \|Qx\|$.

$$\|Px\|^2 \leq \|Qx\|^2 \implies \langle Px, Px\rangle \leq \langle Qx, Qx\rangle \implies \langle x, P^2x\rangle \leq \langle x, Q^2x\rangle \implies \langle x, Px\rangle \leq \langle x, Qx\rangle \qquad \forall x \in \mathcal{H}$$

*(e) → (a).* Let $\langle x, Px\rangle \leq \langle x, Qx\rangle$ and suppose $x \in M$. Then $Px = x$. Then $\|x\|^2 = \langle x, x\rangle = \langle x, Px\rangle \leq \langle x, Qx\rangle$. However, since $\|Qx\| \leq \|x\|$ for all $x$, then $\langle x, Qx\rangle \leq \|x\|\|Qx\| \leq \|x\|^2$, which shows $\langle x, Qx\rangle = \|x\|^2$. Thus $Qx = x$, which shows $x \in \mathcal{N}$, proving $\mathcal{M} \subset \mathcal{N}$.

$\square$

---

## Hunter and Nachtergaele 8.4

---

*Suppose that $(P_n)$ is a sequence of orthogonal projections on a Hilbert space $\mathscr{H}$ such that*

$$\operatorname{ran} P_{n+1} \supset \operatorname{ran} P_n, \qquad \bigcup_{n=1}^{\infty} \operatorname{ran} P_n = \mathscr{H}.$$

*Prove that $(P_n)$ converges strongly to the identity operator $I$ as $n \to \infty$. Show that $(P_n)$ does not converge to the identity operator with respect to the operator norm unless $P_n = I$ for all sufficiently large $n$.*

*Proof.* Let $x \in \mathscr{H}$. Then $\exists N$ such that $x \in \operatorname{ran} P_N$. Thus $x \in \operatorname{ran} P_n$ for all $n \geq N$. Since each $P_n$ is an orthogonal projection, then $x = P_n x$ for all $n \geq N$. Thus

$$\lim_{n\to\infty} P_n x = \lim_{n\to\infty} x = x = Ix$$

where $I$ is the identity operator. Thus $(P_n)$ converges strongly to $I$.
If $P_n = I$ for all sufficiently large $n$, then obviously

$$\lim_{n\to\infty} \|P_n - I\| = \lim_{n\to\infty} \|I - I\| = \|0\| = 0$$

Thus $P_n$ converges to $I$ with respect to the operator norm. If it is not true that $P_n = I$ for all sufficiently large $n$, then $\operatorname{ran} P_n \subset P_{n+1}\ \forall n \implies P_n \not\equiv I$ for any $n$. Then $\forall n$, $\ker P_n \neq \{0\}$, i.e. $\dim \ker P_n > 0$, and so $\exists e_n \in \ker P_n$ with $\|e_n\| = 1$. Then $P_n e_n = 0$ and $\forall n$,

$$\|P_n - I\| \geq \|(P_n - I)e_n\| = \|e_n\| = 1 \implies \lim_{n\to\infty} \|P_n - I\| \geq 1$$

which shows $P_n$ does not converge to the identity operator with respect to the operator norm. $\square$

---

## Hunter and Nachtergaele 8.6

---

*Show that a linear operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ is unitary if and only if it is an isometric isomorphism of normed linear spaces. Show that an invertible linear map is unitary if and only if its inverse is.*

*Proof.* Let $U : \mathscr{H}_1 \to \mathscr{H}_2$ be unitary. Then $U$ is invertible and $\left\langle Ux, Uy \right\rangle_{\mathscr{H}_2} = \left\langle x, y \right\rangle_{\mathscr{H}_1}$ for all $x, y \in \mathscr{H}_1$. Thus

$$\|Ux\|^2_{\mathscr{H}_2} = \langle Ux, Ux \rangle_{\mathscr{H}_2} = \langle x, x \rangle_{\mathscr{H}_1} = \|x\|^2_{\mathscr{H}_1}$$

Thus $U$ preserves norms and is thus an isometric isomorphism.

Now suppose $U : \mathscr{H}_1 \to \mathscr{H}_2$ is an isometric isomorphism. By the definition of isomorphism, $U^{-1}$ exists and $\|Ux\|_{\mathscr{H}_2} = \|x\|_{\mathscr{H}_1}$ (or $\langle Ux, Ux\rangle = \langle x, x\rangle$) for all $x \in \mathscr{H}_1$. Also,

$$\begin{aligned}
\langle x, y\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)\\
&= \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle - i\langle x+iy, x+iy\rangle + i\langle x-iy, x-iy\rangle\right)\\
&= \frac{1}{4}\left(\langle Ux+Uy, Ux+Uy\rangle - \langle Ux-Uy, Ux-Uy\rangle - i\langle Ux+iUy, Ux+iUy\rangle + i\langle Ux-iUy, Ux-iUy\rangle\right)\\
&= \frac{1}{4}\Bigg(\left[\|Ux\|^2 + \langle Ux, Uy\rangle + \underline{\langle Uy, Ux\rangle} + \|Uy\|^2\right] - \left[\|Ux\|^2 - \langle Ux, Uy\rangle - \underline{\langle Uy, Ux\rangle} + \|Uy\|^2\right]\\
&\qquad - i\left[\|Ux\|^2 + i\langle Ux, Uy\rangle - i\underline{\langle Uy, Ux\rangle} - \|Uy\|^2\right] + i\left[\|Ux\|^2 - i\langle Ux, Uy\rangle + i\underline{\langle Uy, Ux\rangle} - \|Uy\|^2\right]\Bigg)\\
&= \frac{1}{4}\left(4\langle Ux, Uy\rangle\right)\\
&= \langle Ux, Uy\rangle
\end{aligned}$$

Thus $U$ is unitary.
Suppose $U$ is an invertible, unitary map. Then $\langle Ux, Uy\rangle = \langle x, y\rangle$ $\forall x, y \in \mathscr{H}$. The invertibility of $U$ implies $\langle x, y\rangle = \langle U(U^{-1}(x)), U(U^{-1}(y))\rangle = \langle U^{-1}(x), U^{-1}(y)\rangle$ $\forall x, y \in \mathscr{H}$. Thus $U^{-1}$ is unitary. Similarly, suppose $U$ is invertible and $U^{-1}$ is unitary. Then $\langle x, y\rangle = \langle U^{-1}(U(x)), U^{-1}(U(y))\rangle = \langle Ux, Uy\rangle$. Then $U$ is unitary. Thus, an invertible linear map is unitary if and only if its inverse is. □

---

# Hunter and Nachtergaele 8.7

---

*If $\phi_y$ is the bounded linear functional defined in (8.5),*

$$\phi_y(x) = \langle y, x\rangle \tag{8.5}$$

*prove that $\|\phi_y\| = \|y\|$.*

*Proof.* First we prove $\|\phi_y\|$ is bounded above by $\|y\|$.

$$\|\phi_y\| = \sup_{\|x\|=1}\|\phi_y(x)\| = \sup_{\|x\|=1}\left|\langle y, x\rangle\right| \le \sup_{\|x\|=1}\|y\|\|x\| = \|y\|$$

Next, consider $x = \frac{y}{\|y\|}$ (note $\|x\| = 1$):

$$\|\phi_y(x)\| = \|\langle y, x\rangle\| = \left\|\frac{\langle y, y\rangle}{\|y\|}\right\| = \left\|\frac{\|y\|^2}{\|y\|}\right\| = \|y\|$$

and thus $\|\phi_y\| \ge \|y\|$, which proves $\|\phi_y\| = \|y\|$. □

---

# Hunter and Nachtergaele 8.8

---

*Prove that $\mathcal{H}^*$ is a Hilbert space with the inner product defined by*

$$\langle \phi_x, \phi_y \rangle_{\mathcal{H}^*} = \langle y, x \rangle_{\mathcal{H}}.$$

*Proof.* First note that for $\phi_{y_1}, \phi_{y_2} \in \mathcal{H}^*$, $\|\phi_{y_1} + \phi_{y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$ where $y_1$ and $y_2$ are the associated vectors in $\mathcal{H}$ guaranteed in the Riesz Representation Theorem. This is true because $\phi_{y_1}(x) = \langle y_1, x \rangle_{\mathcal{H}}$ and $\phi_{y_2}(x) = \langle y_2, x \rangle_{\mathcal{H}}$ imply

$$\phi_{y_1+y_2}(x) = \langle y_1 + y_2, x \rangle_{\mathcal{H}} = \langle y_1, x \rangle_{\mathcal{H}} + \langle y_2, x \rangle_{\mathcal{H}} = \phi_{y_1}(x) + \phi_{y_2}(x)$$

and since $\|\phi_{y_1+y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$ by (Hunter and Nachtergaele 8.7), then $\|\phi_{y_1} + \phi_{y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$. Let $(\phi_n)$ be a Cauchy sequence in $\mathcal{H}^*$. Then $\forall \varepsilon > 0$, $\exists N$ such that $\|\phi_m - \phi_n\|_{\mathcal{H}^*} < \varepsilon$ for $m, n \geq N$. By the Riesz Representation Theorem, $\exists (y_n)_n \in \mathcal{H}$ such that for every $n$, $\phi_n(x) = \langle y_n, x \rangle$ $\forall x \in \mathcal{H}$. $(y_n)_n$ is Cauchy since given $\varepsilon > 0$ we can find $N$ such that $\|y_n - y_m\|_{\mathcal{H}} = \|\phi_n - \phi_m\|_{\mathcal{H}^*} < \varepsilon$ for $m, n \geq N$. Since $\mathcal{H}$ is a Hilbert space, then $(y_n)_n$ is convergent to some $y \in \mathcal{H}$. By the Riesz Representation Theorem, $\exists \phi \in \mathcal{H}^*$ such that $\phi(x) = \langle y, x \rangle$ $\forall x \in \mathcal{H}$. Then $(\phi_n)_n$ converges to $\phi$ because $\|\phi_n - \phi\|_{\mathcal{H}^*} = \|y_n - y\|_{\mathcal{H}}$, which can be made arbitrary small by the definition of convergence. Thus $\mathcal{H}^*$ is complete. Also, $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$ is a well-defined inner product since $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is a well-defined inner product (i.e. the properties of inner product hold). Thus $\mathcal{H}^*$ is a Hilbert space. $\square$

---

# Hunter and Nachtergaele 8.9

---

*Let $A \subset \mathcal{H}$ be such that*

$$\mathcal{M} = \{x \in \mathcal{H} \mid x \text{ is a finite linear combination of elements in } A\}$$

*is a dense linear subspace of $\mathcal{H}$. Prove that any bounded linear functional on $\mathcal{H}$ is uniquely determined by its values on $A$. If $\{u_\alpha\}$ is an orthonormal basis, find a necessary and sufficient condition on a family of complex numbers $c_\alpha$ for there to be a bounded linear functional $\phi$ such that $\phi(u_\alpha) = c_\alpha$.*

*Proof.* Suppose $\phi_1$ and $\phi_2$ are two bounded linear functionals such that $\phi_1(a) = \phi_2(a)$ for all $a \in A$. Let $x \in \mathcal{H}$. By denisty of $\mathcal{M}$, $\exists m_i \in \mathcal{M}$ such that $m_i \to x$. By linearity of $\phi_1$ and $\phi_2$, $\phi_1(m_i) = \phi_2(m_i)$ for all $i = 1, 2, \ldots$. By the Riesz Representation Theorem, $\exists y_1, y_2$ such that $\phi_1(x) = \langle y_1, x \rangle$ and $\phi_2(x) = \langle y_2, x \rangle$ for all $x \in \mathcal{H}$. Thus by continuity of inner products,

$$\phi_1(x) = \langle y_1, x \rangle = \lim_{i\to\infty} \langle y_1, m_i \rangle = \lim_{i\to\infty} \phi_1(m_i) = \lim_{i\to\infty} \phi_2(m_i) = \lim_{i\to\infty} \langle y_2, m_i \rangle = \langle y_2, x \rangle = \phi_2(x) \qquad \forall x \in \mathcal{H}.$$

Thus $\phi_1 \equiv \phi_2$, i.e. bounded linear functionals are uniqueley determined by their values on $A$.
Let $\{u_\alpha\}$ is an orthonormal basis on $\mathcal{H}$. Then if $\exists \phi \in \mathcal{H}^*$ such that $\phi(u_\alpha) = c_\alpha$ then by the Riesz Representation Theorem, $\exists y \in \mathcal{H}$ such that $\langle y, u_\alpha \rangle = c_\alpha$. Then $y = \sum_\alpha c_\alpha u_\alpha$. Thus the necessary condition $\sum_\alpha |c_\alpha|^2 < \infty$.

Suppose $\sum_\alpha |c_\alpha|^2 < \infty$. Then define $y = \sum_\alpha c_\alpha u_\alpha$. Since $\{u_\alpha\}$ is an orthonormal basis, then $c_\alpha = \langle y, u_\alpha \rangle$. Then by the Riesz Representation Theorem, $\exists \phi \in \mathcal{H}^*$ such that $\phi(x) = \langle y, x \rangle \ \forall x \in \mathcal{H}$. In particular, $\phi(u_\alpha) = c_\alpha$.
Thus, given a family of complex numbers $\{c_\alpha\}$,

$$\sum_\alpha |c_\alpha|^2 < \infty \iff \exists \phi \in \mathcal{H}^* \text{ such that } \phi(u_\alpha) = c_\alpha$$

$\square$

---

# Hunter and Nachtergaele 8.11

---

*Prove that if* $A : \mathcal{H} \to \mathcal{H}$ *is a linear map and* $\dim \mathcal{H} < \infty$, *then*

$$\dim \ker A + \dim \operatorname{ran} A = \dim \mathcal{H}.$$

*Prove that if* $\dim \mathcal{H} < \infty$, *then* $\dim \ker A = \dim \ker A^*$. *In particular,* $\ker A = \{0\}$ *if and only if* $\ker A^* = \{0\}$.

*Proof.* Let $\dim \mathcal{H} < \infty$ (say $\dim \mathcal{H} = n$). Then $\dim \ker A < \infty$ and $\dim \operatorname{ran} A < \infty$ since $\ker A$ and $\operatorname{ran} A$ are subspaces of $\mathcal{H}$. Let $\dim \ker A = m \le n$ and let $\{u_1, u_2, \ldots, u_m\}$ be a basis of $\ker A$. Since $\ker A$ is a subspace of $\mathcal{H}$, this basis can be extended to a basis $\mathcal{U}$ of $\mathcal{H}$: $\mathcal{U} = \{u_1, u_2, \ldots, u_m, v_{m+1}, v_{m+2}, \ldots, v_n,\}$. Let $x \in \mathcal{H}$. Then

$$\begin{aligned}
& x = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m + a_{m+1} v_{m+1} + a_{m+2} v_{m+2} + \cdots + a_n v_n \\
\implies & Ax = \cancelto{0}{a_1 A u_1} + \cancelto{0}{a_2 A u_2} + \cdots + \cancelto{0}{a_m A u_m} + a_{m+1} A v_{m+1} + a_{m+2} A v_{m+2} + \cdots + a_n A v_n \\
& \quad = a_{m+1} A v_{m+1} + a_{m+2} A v_{m+2} + \cdots + a_n A v_n
\end{aligned}$$

since $u_i \in \ker A$ for $i = 1, 2, \ldots, m$. Thus, $\{Av_{m+1}, Av_{m+2}, \ldots, Av_n\}$ spans $\operatorname{ran} A$. However it is also linearly independent since

$$\begin{aligned}
& c_{m+1} A v_{m+1} + c_{m+2} A v_{m+2} + \cdots + c_n A v_n = 0 \\
\implies & A(c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \cdots + c_n v_n) = 0 \\
\implies & c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \cdots + c_n v_n \in \ker A \\
\implies & c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \cdots + c_n v_n = d_1 u_1 + d_2 u_2 + \cdots + d_m u_m \qquad \text{for some } d_i \in \mathbb{C} \\
\implies & d_1 = d_2 = \cdots = d_m = c_{m+1} = c_{m+2} = \cdots = c_n = 0 \qquad \text{since } \mathcal{U} \text{ is a basis}
\end{aligned}$$

Thus $\{Av_{m+1}, Av_{m+2}, \ldots, Av_n\}$ is linearly independent. Since it also spans $\operatorname{ran} A$ then it is a basis of $\operatorname{ran} A$. Thus $\dim \operatorname{ran} A = n - m$. Thus, since $m + (n - m) = n$, then

$$\dim \ker A + \dim \operatorname{ran} A = \dim \mathcal{H}.$$

Let $x \in \mathcal{H}$. Then

$$\begin{aligned}
x \in \ker A^* & \iff A^* x = 0 \\
& \iff \langle y, A^* x \rangle = 0 \qquad \forall y \in \mathcal{H} \\
& \iff \langle Ay, x \rangle = 0 \qquad \forall y \in \mathcal{H}
\end{aligned}$$

$$\begin{aligned}
&\iff x \perp Ay \qquad \forall y \in \mathscr{H} \\
&\iff x \perp (\text{ran } A) \\
&\iff x \in (\text{ran } A)^\perp
\end{aligned}$$

Thus $\ker A^* = (\text{ran } A)^\perp$ and $\dim\ker A^* = \dim(\text{ran } A)^\perp$. However since $\dim\ker A + \dim\text{ran } A = \dim\mathscr{H}$ and $\dim\text{ran } A + \dim(\text{ran } A)^\perp = \dim\mathscr{H}$, then $\dim(\text{ran } A)^\perp = \dim\ker A$. Thus,

$$\dim\ker A = \dim\ker A^*.$$

$\square$

---

# Hunter and Nachtergaele 8.12

---

*Suppose that $A : \mathscr{H} \to \mathscr{H}$ is a bounded, self-adjoint linear operator such that there is a constant $c > 0$ with*

$$c\|x\| \leq \|Ax\| \qquad \textit{for all } x \in \mathscr{H}.$$

*Prove that there is a unique solution $x$ of the equation $Ax = y$ for every $y \in \mathscr{H}$.*

*Proof.* Let $x \neq 0$. Then $\|x\| > 0$. Then $\frac{\|Ax\|}{c} \geq \|x\| > 0 \implies \|Ax\| > 0$, which shows $Ax \neq 0$, and thus $\ker A = \{0\}$. Since $A$ is self-adjoint, then $A = A^*$ and thus $\ker A^* = \{0\}$. Since $\langle y, 0\rangle = 0\ \forall y \in \mathscr{H}$, then $y \perp \ker A^*$. Then by Theorem 8.18 in Hunter Nachtergaele, $\exists x \in \mathscr{H}$ such that $y = Ax$. Suppose $y = Ax_1 = Ax_2$. Then $A(x_1 - x_2) = 0$, which implies $x_1 - x_2 \in \ker A = \{0\}$, thus $x_1 = x_2$, i.e. the solution to $y = Ax$ is unique. $\square$