HW #1

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Exercise 1.1

Complete the proof of the the Monotone Class Theorem.

Lemma 1. *The arbitrary intersection of monotone classes in a monotone class.*

Proof. Let $\mathcal{S}$ be the arbitrary intersection of monotone classes M_j for $j \in J$, where J is an index set. Then let $S_1 \subset S_2 \subset S_3 \subset \dots$ and $S_i \in \mathcal{S}\ \forall i = 1, 2, \dots$. Then since each $S_i \in M_j$ for each M_j and each M_j is a monotone class, then $\bigcup_{i=1}^{\infty} S_i \in M_j$ for each M_j. Thus $\bigcup_{i=1}^{\infty} S_i \in \mathcal{S}$. Now let $S_1 \supset S_2 \supset S_3 \supset \dots$ and $S_i \in \mathcal{S}\ \forall i = 1, 2, \dots$. Then since $S_i in M_j$ for each M_j and each M_j is a monotone class, then $\bigcap_{i=1}^{\infty} S_i \in M_j$ for each M_j. Thus $\bigcap_{i=1}^{\infty} S_i \in \mathcal{S}$. Thus $\mathcal{S}$ is a monotone class. □

Theorem 1 (Monotone Class Theorem)**.** *Let Ω be a set a let $\mathcal{A}$ be an algebra of subsets of Ω such that $\Omega, \emptyset \in \mathcal{A}$. Then there exists a smallest monotone class $\mathcal{S}$ that contain $\mathcal{A}$. That class, $\mathcal{S}$, is also the smallest sigma-algebra that contains $\mathcal{A}$.*

Proof. Let S be the intersection of all monotone classes M_i that contain $\mathcal{A}$. By Lemma 1, S is a monotone class, and thus the smallest monotone class containing $\mathcal{A}$.

Pick $A \in \mathcal{A}$ and construct $C(A) = \{B \in \mathcal{S} \mid B \cup A \in \mathcal{S}\}$. By construction, $C(A) \subset \mathcal{S}$. Since $\mathcal{A}$ is an algebra, $\mathcal{A}$ is closed under finite unions, and thus $\mathcal{A} \subset C(A)$. Now we show $C(A)$ is a monotone class, which would show $\mathcal{S} \subset C(A)$, implying $C(A) = \mathcal{S}$. Take $B_1 \subset B_2 \subset B_3 \subset \dots$ and $B_i \in C(A)\ \forall i = 1, 2, \dots$. Then $B_i \cup A \in \mathcal{S}\ \forall i = 1, 2, \dots$ and $(B_1 \cup A) \subset (B_2 \cup A) \subset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcup_{i=1}^{\infty}(B_i \cup A) \in \mathcal{S}$, but $\bigcup_{i=1}^{\infty}(B_i \cup A) = (\bigcup_{i=1}^{\infty} B_i) \cup A$. Thus $\bigcup_{i=1}^{\infty} B_i \in C(A)$. Similarly, take $D_1 \supset D_2 \supset \dots$ and $D_i \in C(A)\ \forall i = 1, 2, \dots$. Then $D_i \cup A \in \mathcal{S}\ \forall i = 1, 2, \dots$ and $(D_1 \cup A) \supset (D_2 \cup A) \supset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcap_{i=1}^{\infty}(D_i \cup A) \in \mathcal{S}$, but $\bigcap_{i=1}^{\infty}(D_i \cup A) = (\bigcap_{i=1}^{\infty} D_i) \cup A$. Thus $\bigcap_{i=1}^{\infty} D_i \in C(A)$. This proves that $C(A)$ is a monotone class, and thus $C(A) = \mathcal{S}$.

Now we extend the definition of $C(A)$ to be defined for any $A \in \mathcal{S}$. Pick $A' \in \mathcal{S}$. Then since $A' \in C(A)\ \forall A \in \mathcal{A}$, then $A \in C(A')\ \forall A' \in \mathcal{A}$. Thus $\mathcal{A} \subset C(A')$. Now we show $C(A')$ is a monotone class, which would show $\mathcal{S} \subset C(A')$, implying $C(A') = \mathcal{S}$. Take $B_1 \subset B_2 \subset B_3 \subset \dots$ and $B_i \in C(A')\ \forall i = 1, 2, \dots$. Then $B_i \cup A' \in \mathcal{S}\ \forall i = 1, 2, \dots$ and $(B_1 \cup A') \subset (B_2 \cup A') \subset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcup_{i=1}^{\infty}(B_i \cup A') \in \mathcal{S}$, but $\bigcup_{i=1}^{\infty}(B_i \cup A') = (\bigcup_{i=1}^{\infty} B_i) \cup A'$. Thus $\bigcup_{i=1}^{\infty} B_i \in C(A')$. Similarly, take $D_1 \supset D_2 \supset \dots$ and $D_i \in C(A')\ \forall i = 1, 2, \dots$. Then $D_i \cup A' \in \mathcal{S}\ \forall i = 1, 2, \dots$ and $(D_1 \cup A') \supset (D_2 \cup A') \supset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcap_{i=1}^{\infty}(D_i \cup A') \in \mathcal{S}$, but $\bigcap_{i=1}^{\infty}(D_i \cup A') = (\bigcap_{i=1}^{\infty} D_i) \cup A'$. Thus $\bigcap_{i=1}^{\infty} D_i \in C(A')$. This proves that $C(A')$ is a monotone class, and thus $C(A') = \mathcal{S}$. Thus $\mathcal{S}$ is closed under finite unions.

Now define $C = \{B \in \mathcal{S} \mid B^C \in \mathcal{S}\}$. Since $\mathcal{A}$ is an algebra, $\mathcal{A}$ is closed under complimentation, and thus $\mathcal{A} \subset C$. Now take $B_1 \subset B_2 \subset \ldots$ and $B_i \in C\ \forall i = 1, 2, \ldots$. Then since $B_1^C \supset B_2^C \supset \ldots$ and $B_i^C \in \mathcal{S}\ \forall i = 1, 2, \ldots$, and since $\mathcal{S}$ is a monotone class, then $\bigcap_{i=1}^{\infty} \left(B_i^C\right) \in S$. However, $\bigcap_{i=1}^{\infty} \left(B_i^C\right) = \left(\bigcup_{i=1}^{\infty} B_i\right)^C$, and thus $\bigcup_{i=1}^{\infty} B_i \in C$. Then take $D_1 \supset D_2 \supset \ldots$ and $D_i \in C\ \forall i = 1, 2, \ldots$. Then since $D_1^C \subset D_2^C \subset \ldots$ and $D_i^C \in \mathcal{S}$, and since $\mathcal{S}$ is a monotone class, then $\bigcup_{i=1}^{\infty} \left(D_1^C\right) \in S$. However, $\bigcup_{i=1}^{\infty} \left(D_i^C\right) = \left(\bigcap_{i=1}^{\infty} D_C\right)^C$, and thus $\bigcap_{i=1}^{\infty} D_i \in C$. Thus C is a monotone class containing $\mathcal{A}$, and thus $\mathcal{S} \subset C$, proving $C = \mathcal{S}$. Thus $\mathcal{S}$ is closed under complementation.

Now we show $\mathcal{S}$ is closed under countable unions and intersections. Consider a sequene of sets $\{A_i\}_{i=1}^{\infty} \in \mathcal{S}$. Then form $B_n = \bigcup_{i=1}^{n}$. Since each B_n is a finite union of elements in $\mathcal{S}$, then each $B_n \in \mathcal{S}$. Also, $B_1 \subset B_2 \subset \ldots$. Since $\mathcal{S}$ is a monotone class, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{S}$, but $\bigcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty} B_n$, and thus $\mathcal{S}$ is closed under countable unions. Similarly, form $D_n = \bigcup_{i=1}^{n} A_i^C$. Since each D_i is a finite union of elements in $\mathcal{S}$ ($\mathcal{S}$ is closed under complementation), then each $D_i \in \mathcal{S}$. Also, $D_1 \subset D_2 \subset \ldots$. Since $\mathcal{S}$ is a monotone class, $\bigcup_{n=1}^{\infty} D_n \in S$, but $\bigcup_{n=1}^{\infty} D_n = \bigcup_{i=1}^{\infty} \left(A_i^C\right) = \left(\bigcap_{i=1}^{\infty} A_i\right)^C$. Again, since $\mathcal{S}$ is closed under complementation, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{S}$. Thus $\mathcal{S}$ is closed under countable unions and intersections.

This proves $\mathcal{S}$ is a σ-algebra. However, every σ-algebra is a monotone class, and thus $\mathcal{S}$ must be the smallest σ-algebra containing $\mathcal{A}$ since it is defined as the smallest monotone class containing $\mathcal{A}$. □

Exercise 1.2

With regard to the remark about continuous functions in Section 1.5, show that f is continuous (in the sense of the usual ε, δ definition) if and only if f is both upper and lower semicontinuous. Show that f is upper semicontinuous at x if and only if, for every sequence $x_1, x_2, \ldots$ converging to x, we have $f(x) \geq \overline{\lim}_{n\to\infty} f(x_n)$.

Definition Consider $f : \Omega \to \mathbb{R}$, and define $L_f(t) = \{x \in \Omega \mid f(x) > t\}$ and $U_f(t) = \{x \in \Omega \mid f(x) < t\}$. Then f is *lower semicontinuous* if $L_f(t)$ is open $\forall t \in \mathbb{R}$ and f is *upper semicontinous* if $U_f(t)$ is open $\forall t \in \mathbb{R}$.

Theorem 2. *Let $f : \Omega \to \mathbb{R}$. Then f is continuous if and only if f is both upper and lower semincontinuous.*

Proof. Let f a continuous function. Then $\forall x \in \Omega$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Fix $t \in \mathbb{R}$ and let $x_L \in L_f(t)$. Then $f(x_L) = t + \ell$ for some $\ell > 0$. Now take $\varepsilon = \ell$. Then by the continuity of f, $\exists \delta_\ell$ such that $f(B_{\delta_\ell}(x_L)) \subset B_\ell(t + \ell)$. But since $t_0 \in B_\ell(t+\ell) \implies t_0 > t$, then $B_{\delta_\ell}(x_L) \subset L_f(t)$. Thus $L_f(t)$ is open. Now let $x_U \in U_f(t)$. Then $f(x_U) = t - u$ for some $u > 0$. Again, take $\varepsilon = u$, and again by the continuity of f, $\exists \delta_u$ such that $f(B_{\delta_u}(x_U)) \subset B_u(f(x_U))$. But since $t_0 \in B_u(t-u) \implies t_0 < t$, then $B_{\delta_u}(x_U) \subset U_f(t)$. Thus $U_f(t)$ is open. Thus f is both upper and lower semicontinuous.

Now let f be both upper and lower semicontinuous. Thus $\forall t \in \mathbb{R}$, $L_f(t)$ and $U_f(t)$ are open. Then pick $x \in \Omega$ and let $t = f(x)$. Choose $\varepsilon > 0$ and let $t_1 = t - \varepsilon$ and $t_2 = t + \varepsilon$. Then $x \in L_f(t_2)$ and $x \in U_f(t_1)$. Since $L_f(t_2)$ and $U_f(t_1)$ are open, then $\exists \delta_L$ and δ_U such that $f(B_{\delta_L}(x)) \subset B_\varepsilon(t_2)$ and $f(B_{\delta_U}(x)) \subset B_\varepsilon(t_1)$. Choose $\delta = \min(\delta_L, \delta_U)$ and let $x_0 \in B_\delta(x)$. Then $t_1 < f(x_0) < t_2$, which shows $f(x_0) \in B_\varepsilon(t)$, and thus f is continuous.

Thus f is continuous if and only if f is both lower and upper semicontinuous. □

Theorem 3. *Let $f : \Omega \to \mathbb{R}$. Then f is upper semicontinuous at x if and only if for every sequence $\{x_i\} \in \Omega$ such that $x_i \to x$, we have $f(x) \geq \overline{\lim}_{i\to\infty} f(x_i)$.*

Proof. Fix $x \in \Omega$ and let f by upper semicontinuous at x. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_\Omega(x,y) < \delta \implies f(y) - f(x) < \varepsilon$, i.e. $f(y) < f(x) + \varepsilon$. Now consider a sequence $\{x_i\} \in \Omega$ such that $x_i \to x$. Then consider a sequence ε_k such that $\varepsilon_k \to 0$. By the upper semicontinuity of f, then for each $k = 1, 2, \ldots$, $\exists \delta_k$ such that $d_\Omega(x,y) < \delta \implies f(y) < f(x) + \varepsilon$. But by the convergence of $\{x_i\}$, $\exists I_k$ such that $\forall i \geq I_k$, $d_\Omega(x, x_i) < \delta_k$ for each $k = 1, 2, \ldots$. This is equivalent to $\sup_{i \geq I_k}\{f(x_i)\} \leq f(x) + \varepsilon_k$ for each $k = 1, 2, \ldots$. Since we cannot guarantee $I_k \to \infty$, form a sequence $L_k = \max\{k, I_k\}$ and note $L_k \to \infty$, $L_k \geq I_k$, and $L_k \geq k$ for each $k = 1, 2, \ldots$. Then

$$\begin{aligned}
&\sup_{i \geq L_k}\{f(x_i)\} \leq \sup_{i \geq I_k}\{f(x_i)\} \leq f(x) + \varepsilon_k \\
\implies &\lim_{k\to\infty}\left(\sup_{i \geq L_k}\{f(x_i)\}\right) \leq \lim_{k\to\infty}\left(f(x) + \varepsilon_k\right) \\
\implies &\lim_{k\to\infty}\left(\sup_{i \geq k}\{f(x_i)\}\right) \leq f(x) + 0 = f(x) \\
\implies &\overline{\lim}_{i\to\infty} f(x_i) \leq f(x)
\end{aligned}$$

Now assume f is not upper semicontinuous at x. Then $\exists \varepsilon$ such that $\forall \delta$, $\exists y$ such that $d_\Omega(x,y) < \delta$ and $f(x) + \varepsilon \leq f(y)$. Consider $\{\delta_k\}$ such that $\delta_k \to 0$. For each δ_k, pick x_k such that $d_\Omega(x, x_k) < \delta_k$ but $f(x) + \varepsilon \leq f(x_k)$. Then clearly $x_k \to x$ but $\lim_{k\to\infty} f(x_k) \geq f(x) + \varepsilon$. Then

$$\begin{aligned}
&\lim_{k\to\infty} f(x_k) \geq f(x) + \varepsilon \\
\implies &\overline{\lim}_{k\to\infty} f(x_k) \geq f(x) + \varepsilon > f(x)
\end{aligned}$$

Thus, by contrapositive, for every sequence $\{x_i\} \in \Omega$ such that $x_i \to x$, we have $f(x) \geq \overline{\lim}_{k\to\infty} f(x_k)$. □

Exercise 1.3

Prove the assertion made in Section 1.5 that for any Borel set $A \subset \mathbb{R}$ and any σ-algebra Σ the set $\{x \mid f(x) \in A\} = f^{-1}(A)$ is Σ-measurable whenever the function f is Σ-measurable.

Definition Consider $f : \Omega \to \mathbb{R}$ and let Σ be a σ-algebra on Ω. We say that f is a *measurable function* (with respect to Σ) if the set $L_f(t) = \{x \in \Omega \mid f(x) > t\} = f^{-1}((t, \infty))$ is measurable, i.e. $L_f(t) \in \Sigma$, for every $t \in \mathbb{R}$.

Theorem 4. *Let $f : \Omega \to \mathbb{R}$. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and let $A \in \mathcal{B}$. Then the set $P_A = \{x \mid f(x) \in A\} = f^{-1}(A)$ is Σ-measurable whenever f is Σ-measurable.*

Proof. Consider the interval $(a, b) \subset \mathbb{R}$ $(a < b)$. Note that

$$\begin{aligned}
(a, b) &= (-\infty, b) \cap (a, \infty) \\
&= [b, \infty)^C \cap (a, \infty) \\
&= \left[\bigcap_{i=1}^{\infty} (b - 2^{-i}, \infty)\right]^C \cap (a, \infty)
\end{aligned}$$

This shows that any open ball in $\mathbb{R}$ is countably σ-algebraicly generated from sets of the form (t, ∞). Due to the properties of preimages,

$$\begin{aligned} f^{-1}((a,b)) &= f^{-1}\left(\left[\bigcap_{i=1}^{\infty}(b-2^{-i},\infty)\right]^C \cap (a,\infty)\right) \\ &= \left[\bigcap_{i=1}^{\infty} f^{-1}(b-2^{-i},\infty)\right]^C \cap f^{-1}((a,\infty)) \\ &= \left[\bigcap_{i=1}^{\infty} L_f(b-2^{-i})\right]^C \cap L_f(a) \end{aligned}$$

Thus, since Σ is closed under countable unions, countable intersections, and complements, $f^{-1}((a,b)) \in \Sigma$.

Now consider the set $L = \{B \in \mathcal{B} \mid f^{-1}(B) \in \Sigma\}$. Next we show L is a monotone class.

Consider $L_1 \subset L_2 \subset \dots$ where $L_i \in L$ for $i = 1, 2, \dots$. Then $f^{-1}(L_i) \in \Sigma$ for each i. Since Σ is a σ-algebra, $\bigcup_{i=1}^{\infty}[f^{-1}(L_i)] \in \Sigma$. But by the properties of preimages, $f^{-1}(\bigcup_{i=1}^{\infty} L_i) = \bigcup_{i=1}^{\infty}[f^{-1}(L_i)]$, and thus $f^{-1}(\bigcup_{i=1}^{\infty} L_i) \in \Sigma$, showing $\bigcup_{i=1}^{\infty} L_i \in L$.

Now consider $L_1 \supset L_2 \supset \dots$ where $L_i \in L$ for $i = 1, 2, \dots$. Then $f^{-1}(L_i) \in \Sigma$ for each i. Since Σ is a σ-algebra, $\bigcap_{i=1}^{\infty}[f^{-1}(L_i)] \in \Sigma$. But by the properties of preimages, $f^{-1}(\bigcap_{i=1}^{\infty} L_i) = \bigcap_{i=1}^{\infty}[f^{-1}(L_i)]$, and thus $f^{-1}(\bigcap_{i=1}^{\infty} L_i) \in \Sigma$, showing $\bigcap_{i=1}^{\infty} L_i \in L$.

Thus, L is a monotone class. By the above calculation, we know $B_r(x) \in L$ for all $r > 0$ and $x \in \mathbb{R}$. Thus L contains all open intervals. But $\mathcal{B}$ is, by definition, the smallest σ-algebra (and hence smallest monotone class) containing the open intervals. Thus $L = \mathcal{B}$.

This shows $f^{-1}(A)$ is measurable whenever f is measurable. □

Problem 4

Let $\phi : \mathbb{C} \to \mathbb{C}$ be a Borel measurable function and let the complex-valued function f be Σ-measurable. Prove that $\phi(f(x))$ is Σ-measurable.

Lemma 2. *Let $f : \Omega \to \mathbb{C}$. Let $\mathcal{B}_{\mathbb{C}}$ be the Borel σ-algebra on $\mathbb{C}$ and let $A \in \mathcal{B}_{\mathbb{C}}$. Then the set $P_A = \{x \mid f(x) \in A\} = f^{-1}(A)$ is Σ-measurable whenever f is Σ-measurable.*

Proof. Assume f is Σ-measurable and let $f_{\mathbb{R}} + i f_{\mathbb{C}} = f$ where $f_{\mathbb{R}} : \Omega \to \mathbb{R}$ and $f_{\mathbb{C}} : \Omega \to \mathbb{R}$. By the definition of measurable complex-valued functions, $f_{\mathbb{R}}$ and $f_{\mathbb{C}}$ are measurable, and thus $\forall t \in \mathbb{R}$, $f_{\mathbb{R}}^{-1}((t,\infty)) \in \Sigma$ and $f_{\mathbb{C}}^{-1}((t,\infty)) \in \Sigma$.

Consider the set $L = \{A \in \mathcal{B}_{\mathbb{C}} : f^{-1}(A) \in \Sigma\}$ and an arbitrary rectangle $R_{a,b,c,d} = \{x + iy \in \mathbb{C} : x \in A_1 \text{ and } y \in A_2\}$ where $A_1, A_2 \in \mathcal{B}$. Since $f_{\mathbb{R}}$ and $f_{\mathbb{C}}$ are measurable, $f_{\mathbb{R}}^{-1}(A_1) \in \Sigma$ and $f_{\mathbb{C}}^{-1}(A_2) \in \Sigma$. Then $f^{-1}(R_{a,b,c,d}) = f_{\mathbb{R}}^{-1}((a,b)) \cup f_{\mathbb{C}}^{-1}((c,d)) \in \Sigma$ since Σ is closed under countable unions. Thus L contains all rectangles in $\mathcal{B} \times \mathcal{B}$.

Now consider $L_1 \subset L_2 \subset \dots$ where $L_i \in L$ for each i. Then $f^{-1}(L_i) \in \Sigma$ for each i. Then $f^{-1}[\bigcup_{i=1}^{\infty} L_i] = \bigcup_{i=1}^{\infty} f^{-1}(L_i) \in \Sigma$. Thus $\bigcup_{i=1}^{\infty} L_i \in L$.

Then consider $L_1 \supset L_2 \supset \dots$ where $L_i \in L$ for each i. Then $f^{-1}(L_i) \in \Sigma$ for each i. Then $f^{-1}[\bigcap_{i=1}^{\infty} L_i] = \bigcap_{i=1}^{\infty} f^{-1}(L_i) \in \Sigma$. Thus $\bigcap_{i=1}^{\infty} L_i \in L$.

Thus L is a monotone class. Since L is a subset of $B_{\mathbb{C}}$, which is defined as the smallest σ-algebra containing rectangles, and since the rectangles are contained in L, then by the Monotone Class Theorem, $L = \mathcal{B}_{\mathbb{C}}$.

This shows $f^{-1}(A)$ is Σ-measurable whenever f is Σ-measurable. $\square$

Theorem 5. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a Borel measurable function and let $f : \Omega \to \mathbb{C}$ be Σ-measurable. Prove that $\phi(f(x))$ is Σ-measurable.*

Proof. Pick $A \in \mathcal{B}_{\mathbb{C}}$. Then since ϕ is a Borel measurable function, $\phi^{-1}(A) \in \mathcal{B}_{\mathbb{C}}$. Then, by Lemma 2, $f^{-1}(\phi^{-1}(A)) \in \Sigma$. However, $f^{-1}(\phi^{-1}(A)) = [\phi f]^{-1}(A)$. Thus ϕf is Σ-measurable. $\square$

Problem 5

Prove eauation (2) in Theorem 1.6 (monotone convergence)

Theorem 6. *Let $f_1, f_2, \ldots$ be an increasing sequence of summable functions on (Ω, Σ, μ), with f equal to the pointwise limit of f_j ($f(x) = \lim_{j\to\infty} f_j(x)$) and $I = \lim_{j\to\infty} I_j = \lim_{j\to\infty} \int_\Omega f_j \mathrm{d}\mu$. Then f is measurable and, moreover, I is finite if and only if f is summable, in which case $I = \int_\Omega f \mathrm{d}\mu$.*

Show the following:

$$\lim_{j\to\infty} \int_0^\infty F_{f_j}(t)\mathrm{d}t = \int_0^\infty F_f(t)\mathrm{d}t \tag{2}$$

where

$$F_{f_j}(t) = \mu(f_j^{-1}(t,\infty))$$

Proof. Consider the case where $\int_\Omega f \mathrm{d}\mu < \infty$. Then $\forall \varepsilon > 0$, $\exists B_\varepsilon \in \mathbb{R}$, $B_\varepsilon > 0$, such that $F_f(B_\varepsilon) = \mu(f^{-1}(B_\varepsilon, \infty)) < \varepsilon$. However, since $f_j(x) \leq f(x)$ for each j, $F_{f_j}(B_\varepsilon) = \mu(f_j^{-1}(B_\varepsilon,\infty)) \leq F_f(B_\varepsilon) < \varepsilon$ for each j. Since F_f, and each F_{f_j}, is non-negative and monotone decreasing, $F_{f_j}(t) \leq F_f(t) < \varepsilon$ $\forall t \geq B_\varepsilon$. Thus,

$$\int_\Omega f \mathrm{d}\mu = \int_0^\infty F_f(t)\mathrm{d}t = \int_0^{B_\varepsilon} F_f(t)\mathrm{d}t + \int_{B_\varepsilon}^\infty F_f(t)\mathrm{d}t, \text{ and, for each } j,$$
$$\int_\Omega f_j \mathrm{d}\mu = \int_0^\infty F_{f_j}(t)\mathrm{d}t = \int_0^{B_\varepsilon} F_{f_j}(t)\mathrm{d}t + \int_{B_\varepsilon}^\infty F_{f_j}(t)\mathrm{d}t$$

Since $\int_\Omega f \mathrm{d}\mu < \infty$, then $\int_{B_\varepsilon}^\infty F_{f_j}(t)\mathrm{d}t < \infty$ and $\int_{B_\varepsilon}^\infty F_f(t)\mathrm{d}t < \infty$. In fact, as $\varepsilon \to 0$, then $\int_{B_\varepsilon}^\infty F_{f_j}(t)\mathrm{d}t \to 0$ and $\int_{B_\varepsilon}^\infty F_f(t)\mathrm{d}t \to 0$.

So we need only consider the integrals with finite limits, and thus we can approximate $\int_\Omega f \mathrm{d}\mu$ with Riemann sums. Denote the upper Riemann sums of a function g under partition P as $R_U(g, P)$ and the lower Riemann sums of a function g under partition P as $R_L(g, P)$.

Now, choose any $\varepsilon > 0$ and a partition P_ε of $[0, B]$ such $R_U(F_f, P_\varepsilon) - R_L(F_f, P_\varepsilon) < \varepsilon$. Then consider the integral

$$\int_0^B F_f(t) - F_{f_j}(t)\mathrm{d}t$$

Note that since $\{f_j\}_j$ is an increasing sequence for each $x \in \Omega$, that $\{F_{f_j}\}_j$ is an increasing sequence of t for each $t \in [0, B]$. Thus,

$$\int_0^B F_f(t) - F_{f_j}(t)\mathrm{d}t \leq R_U(F_f, P_\varepsilon) - R_L(F_{f_j}, P_\varepsilon)$$

$$\implies \lim_{j\to\infty}\left[\int_0^B F_f(t)-F_{f_j}(t)\mathrm{d}t\right] \leq \lim_{j\to\infty}\left[R_U(F_f,P_\varepsilon)-R_L(F_{f_j},P_\varepsilon)\right]$$

$$\implies \lim_{j\to\infty}\left[\int_0^B F_f(t)-F_{f_j}(t)\mathrm{d}t\right] \leq R_U(F_f,P_\varepsilon)-R_L(F_f,P_\varepsilon), \text{ by the linearity of limits}$$

$$\implies \lim_{j\to\infty}\left[\int_0^B F_f(t)-F_{f_j}(t)\mathrm{d}t\right] < \varepsilon$$

Since ε was arbitrary,

$$\lim_{j\to\infty}\left[\int_0^B F_f(t)-F_{f_j}(t)\right] = 0$$

$$\implies \lim_{j\to\infty}\int_0^B F_f(t)\mathrm{d}t = \lim_{j\to\infty}\int_0^B F_{f_j}(t)\mathrm{d}t$$

$$\implies \lim_{j\to\infty}\int_0^B F_{f_j}(t)\mathrm{d}t = \int_0^B F_f(t)\mathrm{d}t$$

Thus,

$$\lim_{j\to\infty}\int_\Omega f_j\mathrm{d}\mu = \lim_{j\to\infty}\int_0^B F_{f_j}(t)\mathrm{d}t = \int_0^B F_f(t)\mathrm{d}t = \int_\Omega f\mathrm{d}\mu$$

Now consider the case when $\int_\Omega f\mathrm{d}\mu = \infty$. Since F_{f_j} is nonnegative,

$$\int_0^n F_{f_j}(t)\mathrm{d}t \leq \int_0^\infty F_{f_j}(t)\mathrm{d}t$$

$$\implies \lim_{j\to\infty}\int_0^n F_{f_j}(t)\mathrm{d}t \leq \lim_{j\to\infty}\int_0^\infty F_{f_j}(t)\mathrm{d}t$$

By the first case, limits can pass through integrals with finite limits.

$$\int_0^n F_f(t)\mathrm{d}t \leq \lim_{j\to\infty}\int_0^\infty F_{f_j}(t)\mathrm{d}t$$

$$\implies \lim_{n\to\infty}\left[\int_0^n F_f(t)\mathrm{d}t\right] \leq \lim_{n\to\infty}\left[\lim_{j\to\infty}\int_0^\infty F_{f_j}(t)\mathrm{d}t\right]$$

$$\implies \int_0^\infty F_f(t)\mathrm{d}t \leq \lim_{j\to\infty}\int_0^\infty F_{f_j}(t)\mathrm{d}t$$

$$\implies \infty = \int_0^\infty F_f(t)\mathrm{d}t \leq \lim_{j\to\infty}\int_0^\infty F_{f_j}(t)\mathrm{d}t$$

$$\implies \lim_{j\to\infty}\int_0^\infty F_{f_j}(t)\mathrm{d}t = \infty = \int_0^\infty F_f(t)\mathrm{d}t$$

Thus,

$$\lim_{j\to\infty}\int_\Omega f_j\mathrm{d}\mu = \lim_{j\to\infty}\int_0^\infty F_{f_j}(t)\mathrm{d}t = \int_0^\infty F_f(t)\mathrm{d}t = \int_\Omega f\mathrm{d}\mu$$

$\square$