UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #7

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Hunter and Nachtergaele 9.1

Prove that $\rho(A^*) = \overline{\rho(A)}$, *where* $\overline{\rho(A)}$ *is the set* $\{\lambda \in \mathbb{C} \mid \overline{\lambda} \in \rho(A)\}$.

Proof. First note

$$(A^* - \lambda I) = (A^* - (\overline{\lambda} I)^*) = (A - \overline{\lambda} I)^*,$$

and since $(A - \overline{\lambda} I) \in \mathscr{B}(\mathscr{H})$, then $(A - \overline{\lambda} I)$ is invertible if and only if $(A - \overline{\lambda} I)^*$ is invertible. Thus

$$\begin{aligned} \lambda \in \rho(A^*) &\iff (A^* - \lambda I) \text{ invertible} \\ &\iff (A - \overline{\lambda} I)^* \text{ invertible} \\ &\iff (A - \overline{\lambda} I) \text{ invertible} \\ &\iff \overline{\lambda} \in \rho(A) \\ &\iff \lambda \in \overline{\rho(A)} \end{aligned}$$

Thus, $\rho(A^*) = \overline{\rho(A)}$. □

Hunter and Nachtergaele 9.3

Suppose that A *is a bounded linear operator on a Hilbert space and* $\lambda, \mu \in \rho(A)$. *Prove that the resolvent* R_λ *of* A *satisfies the* resolvent equation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu.$$

Proof. If the resolvent R_λ is defined as $R_\lambda = (\lambda I - A)^{-1}$, then

$$\begin{aligned} &\big(\mu I - A\big) - (\lambda I - A) = (\mu - \lambda) I \\ \implies &(\lambda I - A)^{-1}\big[\big(\mu I - A\big) - (\lambda I - A)\big]\big(\mu I - A\big)^{-1} = (\lambda I - A)^{-1}\big[(\mu - \lambda) I\big]\big(\mu I - A\big)^{-1} \\ \implies &(\lambda I - A)^{-1} - \big(\mu I - A\big)^{-1} = \big(\mu - \lambda\big)(\lambda I - A)^{-1}\big(\mu I - A\big)^{-1} \\ \implies &R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \end{aligned}$$

□

Hunter and Nachtergaele 9.4

Prove that the spectrum of an orthogonal projection P *is either* $\{0\}$, *in which case* $P = 0$, *or* $\{1\}$, *in which case* $P = I$, *or else* $\{0, 1\}$.

Proof. Let λ be an eigenvalue. Then $Px = \lambda x$ for some nonzero vector x. Clearly if $P \equiv 0$, then $0 = Px = \lambda x$ for some $x \neq 0$, which implies $\lambda = 0$. Also, if $P \equiv I$, then $x = Px = \lambda x \implies (1-\lambda)x = 0$ for some $x \neq 0$. Thus $\lambda = 1$. In general, suppose $P \not\equiv 0$ and $P \not\equiv 1$, then for $x \in \text{ran } P$, $x = Px = \lambda x \implies \lambda = 1$. For $x \notin \text{ran } P$, then since P is an orthogonal projection, $x = y + z$ for some $y \in \text{ran } P$ (i.e. $Py = y$) and $z \in \ker P$ (i.e. $Pz = 0$). Thus $Py + Pz = y = \lambda x$. Since $x \notin \text{ran } P$, $\lambda x \in \text{ran } P$ only if $\lambda = 0$. Thus the only eigenvalues of P are 0 and 1 (i.e. the point spectrum of P is contained in $\{0, 1\}$).
Since orthogonal projections are bounded and self adjoint, then the residual specturm of P is empty.
Let $a \in \text{ran } P$. Then $(1-\lambda)a \in \text{ran } (P-\lambda I)$ (since $(P-\lambda I)a = Pa-\lambda a = (1-\lambda)a$). If $\lambda \neq 1$ then $a \in \text{ran } (P-\lambda I)$ since $\text{ran } (P-\lambda I)$ is closed under scalar multiplication. Let $b \in \ker P$. Then $-\lambda b \in \text{ran } (P-\lambda I)$ (since $(P-\lambda I)b = Pb-\lambda b = -\lambda b$). If $\lambda \neq 0$, then $b \in \text{ran } (P-\lambda I)$ since $\text{ran } (P-\lambda I)$ is closed under scalar multiplication. Thus for $\lambda \in \mathbb{C} \setminus \{0, 1\}$,

$$\text{ran } P \cup \ker P \subset \text{ran } (P-\lambda I)$$

Since P is an orthogonal projection,

$$\mathscr{H} \subset \text{ran } (P-\lambda I) \subset \mathscr{H}$$

and thus $\text{ran } (P-\lambda I)$ is closed. Thus the continuous spectrum of P is empty.
Thus, $\sigma(P) = \{0, 1\}$. □

Hunter and Nachtergaele 9.5

Let A be a bounded, nonnegative operator on a complex Hilbert space. Prove that $\sigma(A) \subset [0, \infty)$.

Proof. Since A is nonnegative, then $\langle x, Ax \rangle \geq 0$ for all $x \in \mathscr{H}$ and $A = A^*$. Since A is self-adjoint, its eigenvalues are real and $\sigma(A) \subset [-\|A\|, \|A\|]$. Let λ be an eigenvalue. Then for some $x \neq 0$,

$$0 \leq \langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle = \lambda \|x\|^2$$
$$\implies 0 \leq \lambda$$

Thus all eigenvalues are positive (i.e. the point spectrum is contained in $[0, \infty)$). Let $\lambda < 0$. Then if $(A-\lambda I)x_1 = (A-\lambda I)x_2$, then $(A-\lambda I)(x_1 - x_2) = 0$. If $x_1 - x_2 \neq 0$, then λ is an eigenvalue, but this is not possible since $\lambda < 0$. Thus $x_1 - x_2 = 0$, or $x_1 = x_2$. This shows $(A-\lambda I)$ is one-to-one. show that $(A-\lambda I)$ is onto for $\lambda < 0$. □

Hunter and Nachtergaele 9.6

Let G be a multiplication operator on $L^2(\mathbb{R})$ defined by

$$Gf(x) = g(x)f(x),$$

where g is continuous and bounded. Prove that G is a bounded linear operator on $L^2(\mathbb{R})$ and that its spectrum is given by

$$\sigma(G) = \overline{\{g(x) : x \in \mathbb{R}\}}.$$

Can an operator of this form have eigenvalues?

Proof. Let $\lambda \notin \overline{\{g(x) : x \in \mathbb{R}\}}$. Then $\exists \varepsilon$ such that $|g(x) - \lambda| > \varepsilon$ for all $x \in \mathbb{R}$. Thus $\frac{1}{g(x)-\lambda}$ is well-defined and we can define the inverse of $(G - \lambda I)$ as

$$\big((G-\lambda I)^{-1}f\big)(x) = \frac{1}{g(x)-\lambda}f(x)$$

because $\forall f \in L^2(\mathbb{R})$,

$$\big((G-\lambda I)(G-\lambda I)^{-1}\big)f = \big((G-\lambda I)^{-1}(G-\lambda I)\big)f = f$$

Thus $\lambda \in \rho(G)$, i.e. $\lambda \notin \sigma(G)$, which shows

$$\sigma(G) \subset \overline{\{g(x) : x \in \mathbb{R}\}}$$

Next, consider $\lambda \in \{g(x) : x \in \mathbb{R}\}$. Then $\exists x_0 \in \mathbb{R}$ such that $g(x_0) = \lambda$. Then consider the characteristic function on $\{x : |x - x_0| < 1\}$,

$$\mathscr{X}(x) = \begin{cases} 1 & \text{if } |x - x_0| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathscr{X} \notin \operatorname{ran}(G - \lambda I)$ since the only candidate function $\mathscr{C}$ to map to $\mathscr{X}$ is

$$\mathscr{C}(x) = \frac{\mathscr{X}(x)}{g(x)-\lambda}$$

but this function is not square-integrable, i.e. $\mathscr{C} \notin L^2(\mathbb{R})$. Thus $(G - \lambda I)$ is not surjective, which shows $\lambda \in \sigma(G)$. Thus

$$\{g(x) : x \in \mathbb{R}\} \subset \sigma(G)$$

However, since $\sigma(G)$ is closed (all spectrums are closed), then any closure of a subset of $\sigma(G)$ is also a subset of $\sigma(G)$. Thus

$$\overline{\{g(x) : x \in \mathbb{R}\}} \subset \sigma(G)$$

which shows

$$\overline{\{g(x) : x \in \mathbb{R}\}} = \sigma(G)$$

It is possible for G to have eigenvalues. Consider a function g and $\lambda \in \mathbb{R}$ such that $\mu\big(\{x : g(x) = \lambda\}\big) > 0$. Then λ is an eigenvalue of G and any function f such that $\operatorname{supp} f \subset \{x : g(x) = \lambda\}$ is an eigenvector with respect to λ since

$$(Gf)(x) = g(x)f(x) = \mathscr{X}_{\operatorname{supp} f}\, g(x)f(x) + \cancelto{0}{\mathscr{X}_{\mathbb{R}\setminus\operatorname{supp} f}\, g(x)f(x)} = \lambda f(x)$$

$\square$

Hunter and Nachtergaele 9.7

Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\,\mathrm{d}y.$$

a) *Find the adjoint operator K^*.*

Proof.

$$\begin{aligned}
\langle f, Kg\rangle &= \int_0^1 \overline{f}(x)\int_0^x g(y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \overline{f}(x)\int_0^1 g(y)\mathscr{X}_{0<y<x<1}\mathrm{d}y\mathrm{d}x, \qquad \text{where } \mathscr{X} \text{ is the characteristic function} \\
&= \int_0^1\int_0^1 \overline{f}(x)g(y)\mathscr{X}_{0<y<x<1}\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 g(y)\int_0^1 \overline{f}(x)\mathscr{X}_{0<y<x<1}\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \left[\int_y^1 \overline{f}(x)\mathrm{d}x\right] g(y)\mathrm{d}y \\
&= \int_0^1 \overline{\left[\int_y^1 f(x)\mathrm{d}x\right]} g(y)\mathrm{d}y \\
&= \langle K^* f, g\rangle
\end{aligned}$$

Thus,

$$K^* f(x) = \int_x^1 f(y)\mathrm{d}y$$

□

b) *Show that $\|K\| = \frac{2}{\pi}$.*

Proof. □

c) *Show that the spectral radius of K is equal to zero.*

Proof. □

d) *Show that 0 belongs to the continuous spectrum of K.*

Proof. K is not onto since K is the integral operator, and thus the range of K is equal to the set of differentiable functions. However, not all functions in L^2 are differentiable. Thus K is not onto. However, differentiable functions are dense in L^2, and thus 0 is in the continuous spectrum of K. □

Hunter and Nachtergaele 9.8

Define the right shift operator S on $\ell^2(\mathbb{Z})$ by

$$S(x)_k = x_{k-1} \qquad \textit{for all } k \in \mathbb{Z},$$

where $x = (x_k)_{k=-\infty}^{\infty}$ is in $\ell^2(\mathbb{Z})$. Prove the following facts.

a) The point spectrum of S is empty.

Proof. Suppose λ is in the point specturm of S. The for $Sx = \lambda x$ for some nonzero $x \in \ell^2(\mathbb{Z})$. If $\lambda = 0$, the $x \equiv 0$, which is a contradiction. If $\lambda = 1$, then $x_k = e_j$ for all $k, j \in \mathbb{Z}$, i.e. x is constant. However constant bi-infinite sequences are not in ℓ^2 unless they are uniformly 0. This is a contradiction since eigenvectors are nonzero. If $|\lambda| > 1$ and $0 < |\lambda| < 1$, then for all $kin\mathbb{Z}$, x_k, $Sx_k = \lambda x_{k-1}$, and thus for all $n \in \mathbb{Z}$,

$$x_k = \lambda^{k-n} x_n, \qquad \forall n \in \mathbb{Z}.$$

Thus x_k can be made arbitrarily large, which is a contradiction since this is true for all $k \in \mathbb{Z}$. Thus there are no eigenvalues of S (i.e. the point spectrum is empty). □

b) ran $(\lambda I - S) = \ell^2(\mathbb{Z})$ *for every* $\lambda \in \mathbb{C}$ *with* $|\lambda| > 1$.

Proof. If $\|(x)_n\| = 1$, then $\|S(x)_n\| = \|(x)_{n+1}\| = \|(x)_n\| = 1$. Thus $\|S\| = 1$. Then any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$ has $\lambda \in \rho(S)$, and thus $\lambda I - S$ is bijective. Thus ran $(\lambda I - S) = \ell^2(\mathbb{Z})$. □

c) ran $(\lambda I - S) = \ell^2(\mathbb{Z})$ *for every* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$.

Proof. Let $(y_n) \in \ell^2(\mathbb{Z})$. Then since $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})$, let $\mathscr{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be an isomorphism. So $\exists (a_n)_n$ such that $\mathscr{F}((y_n)) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$. Then

$$\mathscr{F}(S(y)_n) = \tilde{S}\left(\sum_{n\in\mathbb{Z}} a_n e^{inx}\right) = e^{ix}\sum_{n\in\mathbb{Z}} a_n e^{inx} = \sum_{n\in\mathbb{Z}} a_n e^{i(n+1)x}$$

where $\tilde{S}$ is the shift operator in $L^2(\mathbb{T})$ ($\tilde{S} = \mathscr{F} \circ S$). Let $|\lambda| < 1$. Then $\left(\tilde{S} - \lambda I\right)g = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ where g is defined as

$$g = \frac{\sum_{n\in\mathbb{Z}} a_n e^{inx}}{e^{ix} - \lambda}$$

since

$$\left(\tilde{S} - \lambda I\right)\left(\frac{\sum_{n\in\mathbb{Z}} a_n e^{inx}}{e^{ix} - \lambda}\right) = \frac{e^{ix}\sum_{n\in\mathbb{Z}} a_n e^{inx} - \lambda \sum_{n\in\mathbb{Z}} a_n e^{inx}}{e^{ix} - \lambda} = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

Thus $(\tilde{S} - \lambda I)$ is surjective, which shows $(S - \lambda I)$ is surjective. □

d) The spectrum of S consists of the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *and is purely continuous.*

Proof. □

Hunter and Nachtergaele 9.18

Let $P_1, \ldots, P_N$ be orthogonal projections with orthogonal ranges. Let

$$A = \sum_{n=1}^{N} \lambda_n P_n$$

be a finite linear combination of these projections. Let $\tilde{f} : \sigma(A) \to \mathbb{C}$ be a continuous function and define $f : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ by

$$f(A) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_n) P_n. \tag{9.23}$$

Suppose that A is a compact self-adjoint operator. Let $f \in C(\sigma(A))$ and consider $f(A)$ defined by (9.23). Prove that

$$\|f(A)\| = \sup\{|\tilde{f}(\lambda_n) \mid n \in \mathbb{N}\}.$$

Let $(\tilde{q}_N)$ be a sequence of polynomials of degree N, converging uniformly to $\tilde{f}$ on $\sigma(A)$. The existence of such a sequence is a consequence of the Weierstrass approximation theorem. Prove that $(q_N(A))$ converges in norm, and that its limit equals $f(A)$ as defined in (9.23).

Proof. $\square$