UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #4

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February 12, 2016

Hunder and Nachtergaele 7.9

Suppose that $u(t,x)$ is a smooth solution of the one-dimensional wave equation,

$$u_{tt} - c^2 u_{xx} = 0.$$

Prove that

$$(u_t^2 + c^2 u_x^2)_t - (2c^2 u_t u_x)_x = 0.$$

If $u(0,t) = u(1,t) = 0$ for all t, deduce that

$$\int_0^1 |u_t(x,t)|^2 + c^2 |u_x(x,t)|^2 \mathrm{d}x = \textit{constant}.$$

Proof.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ \iff 2u_t u_{tt} &= 2c^2 u_t u_{xx} \\ \iff 2u_t u_{tt} + 2c^2 u_x u_{tx} &= 2c^2 u_t u_{xx} + 2c^2 u_x u_{tx} \\ \iff \left(u_t^2 + c^2 u_x^2\right)_t &= (2c^2 u_t u_x)_x \end{aligned}$$

Since $u(0,t) = u(1,t) = 0$ for all t, then $u(0,t)_t = u(1,t) = 0$ for all t. Thus

$$\begin{aligned} 0 = 2c^2(u_t(1,t)u_x(1,t) - u_t(0,t)u_x(0,t)) = (2c^2 u_t u_x)\Big|_{x=0}^1 & \\ = \int_0^1 (2c^2 u_t u_x)_x \mathrm{d}x & \\ = \int_0^1 (u_t^2 + c^2 u_x^2)_t \mathrm{d}x & \\ = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (u_t^2 + c^2 u_x^2) \mathrm{d}x & \\ \iff \int_0^1 (u_t^2 + c^2 u_x^2) \mathrm{d}x = \text{constant}. & \end{aligned}$$

□

Hunder and Nachtergaele 7.10

Show that

$$u(x,t) = f(x+ct) + g(x-ct)$$

is a solution of the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

for arbitrary functions f and g. This solution is called d'Alembert's solution.

$$\begin{aligned} u(x,t) &= f(x+ct)+g(x-ct) \\ \implies u_t(x,t) &= c(f'(x+ct)-g'(x-ct)) \\ \implies u_{tt}(x,t) &= c^2(f''(x+ct)+g''(x-ct)) \end{aligned}$$

Also,

$$\begin{aligned} u(x,t) &= f(x+ct)+g(x-ct) \\ \implies u_x(x,t) &= f'(x+ct)+g'(x-ct) \\ \implies u_{xx}(x,t) &= f''(x+ct)+g''(x-ct) \end{aligned}$$

Thus,

$$u_{tt}(x,t) = c^2(f''(x+ct)+g''(x-ct)) = c^2 u_{xx}(x,t)$$

Hunder and Nachtergaele 7.14

Consider the logistic map

$$x_{n+1} = 4\mu x_n(1-x_n),$$

where $x_n \in [0,1]$, and $\mu = 1$. Show that the solutions may be written as $x_n = \sin^2\theta_n$ where $\theta_n \in \mathbb{T}$, and

$$\theta_{n+1} = 2\theta_n.$$

What can you say about the orbits of the logistic map, the exitence of an invariant measure, and the validity of an ergodic theorem?

Let $x_n = \sin^2\theta_n$ and $\theta_{n+1} = 2\theta_n$. Then

$$\begin{aligned} \theta_{n+1} &= 2\theta_n \\ \implies \sin^2(\theta_{n+1}) &= \sin^2(2\theta_n) \\ \implies x_{n+1} &= 4\sin^2\theta_n\cos^2\theta_n \\ \implies x_{n+1} &= 4\sin^2\theta_n(1-\sin^2\theta_n) \\ \implies x_{n+1} &= 4x_n(1-x_n) \end{aligned}$$

Thus

$$x_n = \sin^2\theta_n \quad \text{where} \quad \theta_{n+1} = 2\theta_n$$

satisfies the logistic map.

Hunder and Nachtergaele 7.15

Consider the dynamical system on $\mathbb{T}$,

$$x_{n+1} = \alpha x_n \mod 1,$$

where $\alpha = (1+\sqrt{5})/2$ is the golden ration. Show that the orbit with initial value $x_0 = 1$ is not equidistributed on the circle, meaning that it does not satisy (7.39).
HINT. Show that

$$u_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

satisfies the difference equation

$$u_{n+1} = u_n + u_{n-1}$$

and hence s an integer for every $n \in \mathbb{N}$. Then use the fact that

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0 \quad as\ n \to \infty.$$

Hunder and Nachtergaele 7.17

Let B_n and V_n be defined in (7.46) and (7.47). Prove that $\bigcup_{n=0}^N B_n$ in an orthonormal basis of V_N.
HINT. Prove that the set is orthonormal and count its elements.

Hunder and Nachtergaele 7.18

Suppose that $B = \{e_n(x)\}_{n=1}^\infty$ is an orthonormal basis for $L^2([0,1])$. Prove the following:

(a) *For any $a \in \mathbb{R}$, the set $B_a = \{e_n(x-a)\}_{n=1}^\infty$ is an orthonormal basis for $L^2([a,a+1])$.*

(b) *For any $c > 0$, the set $B^C = \{\sqrt{c}e_n(cx)\}_{n=1}^\infty$ is an orthonormal basis for $L^2([0,c^{-1]}])$.*

(c) *With the convention that functions are extended to a larger domain than their original domain by setting them equal to 0, prove that $B \cup B_1$ is an orthonormal basis for $L^2([0,1])$.*

(d) *Prove that $\bigcup_{k\in\mathbb{Z}} B_k$ is an orthonormal basis for $L^2(\mathbb{R})$*