HW #2

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Exercise 1.9

Verify the linearity of the integral as given in 1.5(7) by completing the steps outlined below. In what follows, f and g are nonnegative summable functions.

Definition: Simple Function Let (Ω, Σ, μ) be a measure space. A *simple function* is a function $\phi : \Omega \to [0, \infty)$ given by

$$\phi = \sum_{i=1}^{N} c_i \mathcal{X}_{E_i}$$

where $c_i \geq 0$, $E_i \in \Sigma$ for $i = 1, \dots, N$, and $E_i \cap E_j = \emptyset$ for $i \neq j$. The integral of a simple function is given by

$$\int_\Omega \phi \mathrm{d}\mu = \sum_{i=1}^{N} c_i \mu(E_i)$$

Define $\mathcal{S}_\Omega$ as the set of all simple functions on Ω, i.e.

$$\mathcal{S}_\Omega = \{\phi \in [\Omega \to [0, \infty)] \ : \ \phi \text{ is a simple function } \}$$

Definition 1: Lebesgue Integral Let $f : \Omega \to [0, \infty)$ be a positive, measurable real-valued function on a measure space (Ω, Σ, μ). Then

$$\int_\Omega f \mathrm{d}\mu = \int_0^\infty F_f(t) \mathrm{d}t$$

where $F_f(t) = \mu(f^{-1}(t, \infty))$.

Definition 2: Lebesgue Integral Let $f : \Omega \to [0, \infty)$ be a positive, measurable real-valued function on a measure space (Ω, Σ, μ). Then

$$\int_\Omega f \mathrm{d}\mu = \sup_{\substack{0 \leq \phi \leq f \\ \phi \in \mathcal{S}_\Omega}} \left\{ \int_\Omega \phi \mathrm{d}\mu \right\}$$

Lemma 1 (Equivalence of Definitions for Simple Functions)**.** *Let Φ be a simple function on a measure space (Ω, Σ, μ). Then*

$$\int_0^\infty F_\Phi(t)\mathrm{d}t = \sup_{\substack{0\le\phi\le\Phi \\ \phi\in\mathcal{S}_\Omega}} \left\{ \int_\Omega \phi \mathrm{d}\mu \right\}$$

where $F_\Phi(t) = \mu(\Phi^{-1}(t,\infty))$. In other words, the two definitions for Lebesgue integral are equivalent for simple functions.

Proof. Since Φ is a simple function, then

$$\sup_{\substack{0\le\phi\le\Phi \\ \phi\in\mathcal{S}_\Omega}} \left\{ \int_\Omega \phi \mathrm{d}\mu \right\} = \int_\Omega \Phi \mathrm{d}\mu = \sum_{i=1}^N c_i \mu(E_i)$$

where c_i and E_i, $i = 1, \ldots, N$ are defined for Φ. Since $E_i \cap E_j = \emptyset$ for $i \ne j$, then the maximum of Φ is the maximum of the coefficients $\{c_i\}$, denoted c_Φ.

$$c_\Phi = \max_{x\in\Omega} \Phi(x) = \max_{1\le i\le N} \{c_i\}$$

This shows that $\Phi^{-1}(t,\infty) = \emptyset$ for $t \ge c_\Phi$, and thus

$$\int_0^\infty F_\Phi(t)\mathrm{d}t = \int_0^{c_\Phi} F_\Phi(t)\mathrm{d}t$$

We can construct a set $\{d_i\}$ such that $\{d_i\} = \{c_i\}$ but $d_1 \le d_2 \le \cdots \le d_N = c_\Phi$. In other words, the set $\{d_i\}$ is simply the set $\{c_i\}$ ordered from least to greatest. Then

$$\int_0^{c_\Phi} F_\Phi(t)\mathrm{d}t = \int_0^{d_1} F_\Phi(t)\mathrm{d}t + \int_{d_1}^{d_2} F_\Phi(t)\mathrm{d}t + \cdots + \int_{d_{N-1}}^{d_N} F_\Phi(t)\mathrm{d}t = \sum_{k=1}^N \int_{d_{k-1}}^{d_k} F_\Phi(t)\mathrm{d}t$$

where, for ease, we define $d_0 = 0$. We can easily form the set $\{D_i\}$ such that D_i corresponds to d_i. In other words, we can write $\int_\Omega \Phi \mathrm{d}\mu = \sum_{i=1}^N d_i \mu(D_i)$. Note that for $t \in (0, d_1)$ we have

$$F_\Phi(t) = \mu\left(\bigcup_{i=1}^N D_i \right) = \sum_{i=1}^N \mu(D_i)$$

In general, for $t \in (d_{k-1}, d_k)$ we have

$$F_\Phi(t) = \mu\left(\bigcup_{i=k}^N D_i \right) = \sum_{i=k}^N \mu(D_i)$$

Thus,

$$\begin{aligned} \int_0^{c_\Phi} F_\Phi(t)\mathrm{d}t &= \sum_{k=1}^N \int_{d_{k-1}}^{d_k} F_\Phi(t)\mathrm{d}t \\ &= \sum_{k=1}^N \int_{d_{k-1}}^{d_k} \sum_{i=k}^N \mu(D_i)\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} \sum_{i=k}^{N} \left[\mu(D_i) \int_{d_{k-1}}^{d_k} \mathrm{d}t \right] \qquad \text{by linearity of Riemann integrals} \\
&= \sum_{k=1}^{N} \sum_{i=k}^{N} [\mu(D_i)(d_k - d_{k-1})] \\
&= \sum_{k=1}^{N} (d_k - d_{k-1}) \sum_{i=k}^{N} \mu(D_i) \\
&= (d_1 - 0)[\mu(D_1) + \cdots + \mu(D_N)] + (d_2 - d_1)[\mu(D_2) + \cdots + \mu(D_N)] + \ldots \\
&\quad \cdots + (d_{N-1} - d_{N-2})[\mu(D_{N-1}) + \mu(D_N)] + (d_N - d_{N-1})\mu(D_N) \\
&= d_1\mu(D_1) + d_2\mu(D_2) + \cdots + d_N\mu(D_N) \qquad \text{by combining like-terms} \\
&= \sum_{i=1}^{N} c_i \mu(E_i)
\end{aligned}
$$

which completes the proof. □

Lemma 2 (Equivalence of Definitions for Arbitrary Functions)**.** *Let f be a measurable function on a measure space (Ω, Σ, μ). Then*

$$
\int_0^\infty F_f(t)\mathrm{d}t = \sup_{\substack{0 \le \phi \le f \\ \phi \in \mathcal{S}_\Omega}} \left\{ \int_\Omega \phi \mathrm{d}\mu \right\}
$$

where $F_f(t) = \mu(f^{-1}(t, \infty))$. In other words, the two definitions for Lebesgue integral are equivalent.

Proof. Consider the set $E_{n,k} = \left\{ x \ : \ f(x) > \frac{k}{2^n} \right\}$ for $n \in \{1, 2, \dots\}$ and $k \in \{1, 2, \dots, 4^n\}$. Define $\{\phi_n\}_{n=1}^\infty$ by

$$
\phi_n = \frac{1}{2^n} \sum_{k=1}^{4^n} \mathcal{X}_{E_{n,k}}
$$

Note that $E_{n,a} \subset E_{n,b}$ if $a > b$. Then note that for any x, either $x \in E_{n,4^n}$ or $\exists \ell$ such that $x \in E_{n,\ell}$ but $x \notin E_{n,\ell+1}$.

If $x \in E_{n,4^n}$, then by its definition, $f(x) > \frac{4^n}{2^n} = 2^n$ and $\phi_n(x) = \frac{1}{2^n} \sum_{k=1}^{4^n} 1 = 2^n$. If $x \in E_{n,\ell}$ but $x \notin E_{n,\ell+1}$, then by its definition $\frac{\ell+1}{2^n} > f(x) > \frac{\ell}{2^n}$, but $\phi_n(x) = \frac{\ell}{2^n}$. In either case, $\phi_n \le f$ for each $n = 1, 2, 3, \dots$.

Next we show $\phi_{n+1} \ge \phi_n$. Suppose $f(x) < 2^n$. Then $\exists \ell \in \{1, 2, \dots, 4^n\}$ such that $\frac{\ell}{2^n} \le f(x) < \frac{\ell+1}{2^n}$. Then either $\frac{2\ell}{2^{n+1}} < \frac{2\ell+1}{2^{n+1}} \le f(x) < \frac{2\ell+2}{2^{n+1}}$ (in which case $\phi_{n+1}(x) = \frac{2\ell+1}{2^{n+1}} > \phi_n$) or $\frac{2\ell}{2^{n+1}} \le f(x) \le \frac{2\ell+1}{2^{n+1}} < \frac{2\ell+2}{2^{n+1}}$ (in which case $\phi_{n+1}(x) = \frac{2\ell}{2^{n+1}} = \phi_n(x)$).

Suppose $f(x) = 2^n$. Then $\phi_{n+1}(x) = 2^n = \phi_n(x) = f(x)$.

Suppose $f(x) > 2^n = \phi_n(x)$. Then either $f(x) \ge 2^n + \frac{1}{2^{n+1}}$ or $f(x) < 2^n + \frac{1}{2^{n+1}}$. If $f(x) \ge 2^n + \frac{1}{2^{n+1}}$, the $\phi_{n+1}(x) \ge 2^n + \frac{1}{2^{n+1}} > \phi_n(x)$. If $f(x) < 2^n + \frac{1}{2^{n+1}}$, then $\phi_{n+1}(x) = 2^n = \phi_n(x)$.

In all cases, $\phi_{n+1}(x) \ge \phi_n(x)$ for all x and n. Thus, we that *(i)* f is the pointwise limit of ϕ_n, and *(ii)* $\{\phi_n\}_n$ is a non-decreasing sequence of functions. These are the two hypotheses of the monotone convergence theorem, and so

$$
\int_\Omega f \mathrm{d}\mu = \lim_{n \to \infty} \int_\Omega \phi_n \mathrm{d}\mu
$$

where the above integrals are defined using the second defintion of Lebesgue integrals.

Now note that

$$f^{-1}(t,\infty) = \bigcup_{n=1}^{\infty}\{\phi_n^{-1}(t,\infty)\}$$

since $f > \phi_n$ for all n and $\phi_n \to f$, and

$$\bigcup_{n=1}^{\infty}\{\phi_n^{-1}(t,\infty)\} = \lim_{n\to\infty} \phi_n^{-1}(t,\infty)$$

since $\{\phi_n\}_n$ is an increasing function. Thus,

$$F_f(t) = \mu(f^{-1}(t,\infty)) = \mu(\lim_{n\to\infty} \phi_n^{-1}(t,\infty)) = \lim_{n\to\infty} F_{\phi_n}(t,\infty)$$

However, $\{F_{\phi_n}(t,\infty)\}_n$ is monotone increasing to $F_f(t,\infty) \in \mathbb{R}$, and thus again by the monotone convergence theorem,

$$\int_0^\infty F_f(t)\mathrm{d}t = \lim_{n\to\infty} \int_0^\infty F_{\phi_n}(t)\mathrm{d}t$$

By Lemma 1, the two definitions of Lebesgue integration are equivalent for simple functions, and thus

$$\int_0^\infty F_{\phi_n}(t)\mathrm{d}t = \int_\Omega \phi_n(t)$$

for all n. Then by the result above,

$$\lim_{n\to\infty}\left[\int_0^\infty F_{\phi_n}(t)\mathrm{d}t\right] = \lim_{n\to\infty}\left[\int_\Omega \phi_n(t)\right]$$
$$\implies \int_0^\infty F_f(t)\mathrm{d}t = \sup_{\substack{0\le\phi\le f\\ \phi\in S_\Omega}}\left\{\int_\Omega \phi\,\mathrm{d}\mu\right\}$$

□

a)

Show that $f+g$ is also summable. In fact, by a simple argument $\int(f+g) \le 2\big(\int f + \int g\big)$.

Suppose f and g are summable. Thus

$$\int f < \infty \qquad \text{and} \qquad \int g < \infty$$

By the simple function definition of Lebesgue integrals,

$$\sup_{\substack{0\le\phi\le f\\ \phi\in S_\Omega}}\left\{\int \phi\right\} < \infty \qquad \text{and} \qquad \sup_{\substack{0\le\psi\le g\\ \psi\in S_\Omega}}\left\{\int \psi\right\} < \infty$$

Thus construct two sequences of simple functions $\{\phi_n\}_n$ and $\{\psi_n\}_n$ such that $\int \phi_n \to \int f$ and $\int \psi_n \to \int g$. Then choose any arbitrary ε. Then

$$\exists N_\phi, N_\psi \quad \text{such that if} \quad n \geq \max\{N_\phi, N_\psi\}, \quad \text{then} \quad \int f - \int \phi_n < \frac{\varepsilon}{2} \quad \text{and} \quad \int g - \int \psi_n < \frac{\varepsilon}{2}$$

By the definition of the integration of simple functions, there are disjoint sets $\{E_i\}_{i=1}^{N_E}$ and $\{F_j\}_{j=1}^{N_F}$ such that

$$\int \phi_n = \sum_{i=1}^{N_E} c_i \mathcal{X}_{E_i} \qquad \text{and} \qquad \int \psi_n = \sum_{j=1}^{N_F} d_j \mathcal{X}_{F_j}$$

However we can construct the set $\{G_{i,j}\} = \{E_i \cap F_j \;:\; 1 \leq i \leq N_E, 1 \leq j \leq N_F\}$, and thus

$$\int \phi_n = \sum_{i=1}^{N_E} \sum_{j=1}^{N_F} c_i \mathcal{X}_{G_{i,j}} \qquad \text{and} \qquad \int \psi_n = \sum_{j=1}^{N_F} \sum_{i=1}^{N_E} d_j \mathcal{X}_{G_{i,j}}$$

By linearity of finite sums, omg finish up

b)

For any integer N find two functions f_N and g_N that take only finitely many values, such that $|\int f - \inf f_N| \leq \frac{C}{N}$, $|\int g - \int g_N| \leq \frac{C}{N}$, $|int(f+g) - \int(f_N - g_N)| \leq \frac{C}{N}$ for some constant C independent of N.

c)

Show that for f_N and g_N as above $\int(f_N + g_N) = \int f_N + \int g_N$, thus proving the addivitivity of te integral for nonnegative functions.

d)

In a similar fashion, show that for $f, g \geq 0$, $\int(f - g) = \int f - \int g$.

e)

Now use c) and d) to prove the linearity of the integral.

Exercise 1.12

Find a simple condition for $f_n(x)$ so that

$$\sum_{n=0}^{\infty} \int_\Omega f_n(x)\mu(\mathrm{d}x) = \int_\Omega \left[\sum_{n=0}^{\infty} f_n(x)\right] \mu(\mathrm{d}x)$$

Exercise 1.13

Let f be the function on $\mathbb{R}^n$ defined by $f(x) = |x|^{-p}\mathcal{X}_{\{|x|<1\}}(x)$. Compute $\int f\mathrm{d}\mathcal{L}^n$ in two ways: (i) Use polar coordinates and compute the integral by the standard calculus method. (ii) Compute $\mathcal{L}^n(\{x \;:\; f(x) > a\})$ and then use Lebesgue's definition.

(i) First note that

$$f(x) = \begin{cases} |x|^{-p} & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}$$

Then note that polar coordinates on $\mathbb{R}^n$ are $(r, \phi, \theta_1, \theta_2, \ldots, \theta_{n-2})$ where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $\theta_i \in [0, \pi)$ for $i = 1, 2, \ldots, n-2$.

$$\int f\mathrm{d}\mathcal{L}^n = \int_0^\pi \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \int_0^\infty r^{-p} \;\mathrm{d}r \;\mathrm{d}\phi \;\mathrm{d}\theta_1 \ldots \;\mathrm{d}\theta_{n-3} \;\mathrm{d}\theta_{n-2}$$

We can use Fubini's theorem since each of these integrals are Riemann integrals. Thus,

$$\int f\mathrm{d}\mathcal{L}^n = 2\pi^{n-1} \int_0^\infty r^{-p}\mathrm{d}r = 2\pi^{n-1} \int_0^1 r^{-p}\mathrm{d}r$$

since we know $f(x) = 0$ whenever $r = |x| \geq 1$. This integral is dependent on p in the following way:

$$\int f\mathrm{d}\mathcal{L}^n = \begin{cases} 2\pi^{n-1}\frac{1}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geq 1 \end{cases}$$

(ii) If $0 < p < 1$, f is a decreasing function of modulus and $f \to \infty$ as $x \to 0$. If $p < 0$, f is an increasing function of modulus and $f \to \infty$ as $|x| \to 1$. Thus it should be intuitive that $f^{-1}(a, \infty)$ is either a smaller n-sphere if $0 < p < 1$ or a shell of an n-sphere if $p < 0$.

$$\begin{aligned} \mathcal{L}^n(\{x \;:\; f(x) > a\}) &= \mathcal{L}^n(\{x \in B_1(0) \;:\; |x|^{-p} > a\}) \\ &= \begin{cases} \mathcal{L}^n(\{x \;\in B_1(0) \;:\; |x| < a^{-\frac{1}{p}}\}) & \text{if } 0 < p < 1 \\ \mathcal{L}^n(\{x \;\in B_1(0) \;:\; |x| > a^{-\frac{1}{p}}\}) & \text{if } p < 0 \end{cases} \\ &= \begin{cases} \mathcal{L}^n(B_{a^{-\frac{1}{p}}}(0)) & \text{if } 0 < p < 1 \\ \mathcal{L}^n(B_1(0)) - \mathcal{L}^n(B_{a^{-\frac{1}{p}}}(0)) & \text{if } p < 0 \end{cases} \end{aligned}$$

But the Lebesgue measures of balls are relatively simple to compute:

$$\mathcal{L}^n(B_r(x)) = \frac{2\pi^{\frac{n}{2}} r^n}{n\Gamma(\frac{n}{2})}$$

Thus,

$$\mathcal{L}^n(\{x \;:\; f(x) > a\}) = \begin{cases} \dfrac{2\pi^{\frac{n}{2}} a^{-\frac{n}{p}}}{n\Gamma(\frac{n}{2})} & \text{if } 0 < p < 1 \\ \dfrac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} - \dfrac{2\pi^{\frac{n}{2}} a^{-\frac{n}{p}}}{n\Gamma(\frac{n}{2})} & \text{if } p < 0 \end{cases}$$

$$= \begin{cases} \dfrac{2\pi^{\frac{n}{2}} a^{-\frac{n}{p}}}{n\Gamma(\frac{n}{2})} & \text{if } 0 < p < 1 \\ \dfrac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}\left(1 - a^{-\frac{n}{p}}\right) & \text{if } p < 0 \end{cases}$$