HW #2

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Hunter and NachterGaele 7.1

Let ϕ_n be the functions defined in (7.7)

$$\phi_n(x) = c_n(1 + \cos x)^n$$

where c_n is chosen such that

$$\int_{\mathbb{T}} \phi_n(x)\mathrm{d}x = 1$$

for all n.

(a) *Prove (7.5).*

$$\lim_{n\to\infty} \int_{\delta\leq|x|\leq\pi} \phi_n(x)\mathrm{d}x = 0$$

for every $\delta > 0$.

Let $\delta > 0$ and for ease, define $\mathbb{D} = [-\pi, -\delta] \cup [\delta, \pi]$.

$$\int_{\mathbb{D}} \phi_n(x)\mathrm{d}x = \frac{\int_{\mathbb{D}}(1 + \cos x)^n\mathrm{d}x}{\int_{\mathbb{T}}(1 + \cos x)^n\mathrm{d}x}$$

since

$$c_n = \frac{1}{\int_{\mathbb{T}}(1 + \cos x)^n\mathrm{d}x}$$

Note that

$$\phi_n'(x) = -nc_n(1 + \cos x)^{n-1}\sin x$$

which is positive on $[-\pi, 0)$ and negative on $(0, \pi]$, and thus

$$\max_{x\in\mathbb{D}} \phi_n(x) = \phi_n(\delta)$$

So,

$$\int_{\mathbb{D}} \phi_n(x)\mathrm{d}x = \frac{\int_{\mathbb{D}}(1 + \cos x)^n\mathrm{d}x}{\int_{\mathbb{T}}(1 + \cos x)^n\mathrm{d}x} \leq \frac{2\pi(1 + \cos\delta)^n}{\int_{\mathbb{E}}(1 + \cos x)^n\mathrm{d}x}$$

where $\mathbb{E} = [-\frac{\delta}{2}, \frac{\delta}{2}]$. Again, since ϕ_n is decreasing on $\left(0, \frac{\pi}{2}\right]$ and ϕ is an even function,

$$\min_{x \in \mathbb{E}} \phi_n(x) = \phi_n\left(\frac{\delta}{2}\right)$$

Thus,

$$\int_{\mathbb{D}} \phi_n(x)\mathrm{d}x \leq \frac{2\pi(1+\cos\delta)^n}{\int_{\mathbb{E}}(1+\cos x)^n \mathrm{d}x} \leq \frac{2\pi}{\delta}\left(\frac{1+\cos\delta}{1+\cos\frac{\delta}{2}}\right)^n$$

but

$$\frac{1+\cos\delta}{1+\cos\frac{\delta}{2}} < 1$$

since cos is a decreasing function on $[0, \pi]$. Thus,

$$\lim_{n\to\infty} \frac{2\pi}{\delta}\left(\frac{1+\cos\delta}{1+\cos\frac{\delta}{2}}\right)^n = 0$$

and by the comparison test,

$$\lim_{n\to\infty} \int_{\mathbb{D}} \phi_n(x)\mathrm{d}x = 0$$

(b) *Prove that if the set $\mathcal{P}$ of trigonometric polynomials is dense in the space of periodic continuous functions on $\mathbb{T}$ with the uniform norm, then $\mathcal{P}$ is dense in the space of all continuous functions on $\mathbb{T}$ with the L^2-norm.*

(c) *Is $\mathcal{P}$ dense in the space of all continuous functions on $[0, 2\pi]$ with the uniform norm?*

Hunter and NachterGaele 7.2

Suppose that $f : \mathbb{T} \to \mathbb{C}$ is a continuous function, and

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

is the N^{th} partial sum of its Fourier seriers.

(a) *Show that $S_N = D_N * f$, where D_N is the* Dirichlet kernel

$$D_N(x) = \frac{1}{2\pi}\frac{\sin\left[(N+\frac{1}{2})x\right]}{\sin\left(\frac{x}{2}\right)}.$$

(b) *Let T_N be the mean of the first $N+1$ partial sums,*

$$T_N = \frac{1}{N+1}.$$

Show that $T_N = F_N * f$, where F_N is the Fejér kernel

$$F_N(x) = \frac{1}{2\pi(N+1)}\left(\frac{\sin\left[(N+1)\frac{x}{2}\right]}{\sin\left(\frac{x}{2}\right)}\right)^2.$$

(c) *Which of the families (D_N) and (F_N) are approximate identities as $N \to \infty$? What can you say about the uniform convergence of the partial sums S_N and the averaged partial sums T_N to f?*

Hunter and NachterGaele 7.3

Prove that the sets $\{e_n \mid n \geq 1\}$ defined by

$$e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx,$$

and $\{f_n \ : \ n \geq 1\}$ defined by

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \qquad f_n(x) = \sqrt{\frac{2}{\pi}} \cos nx \qquad \textit{for } n \geq 1,$$

are both orthonormal bases of $L^2([0,\pi])$.

First we show $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$ are orthonormal. Suppose $n \neq m$. Then

$$\begin{aligned}
\langle e_n, e_m \rangle &= \int_0^{\pi} e_n(x) e_m(x) \mathrm{d}x \\
&= \frac{2}{\pi} \int_0^{\pi} \sin(nx)\sin(mx)\mathrm{d}x \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}[\cos(nx-mx) - \cos(nx+mx)]\mathrm{d}x \\
&= \frac{1}{\pi} \int_0^{\pi} \cos(x(n-m))\mathrm{d}x - \frac{1}{\pi}\int_0^{\pi} \cos(x(n+m)) \\
&= \frac{1}{\pi}\left[\frac{\sin(x(n-m))}{n-m} - \frac{\sin(x(n+m))}{n+m}\right]_0^{\pi} \\
&= 0
\end{aligned}$$

Also,

$$\begin{aligned}
\langle e_n, e_n \rangle &= \frac{2}{\pi}\int_0^{\pi} \sin^2(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} 1 - \cos(2nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left[\pi - \frac{1}{2n}\sin(2n\pi)\right] \\
&= \frac{1}{\pi}\pi \\
&= 1
\end{aligned}$$

Thus $\{e_n\}_{n=1}^{\infty}$ is orthonormal.

Let $f \in L^2([0,1])$. Then extend f to its odd expansion $\tilde{f} \in L^2([-1,1])$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in (0,1) \\ 0 & \text{if } x = 0 \\ -f(x) & \text{if } x \in (-1,0) \end{cases}$$

Then $\{e_n\}_{n=1}^{\infty} \cup \{f_n\}_{n=0}^{\infty}$.

Hunter and NachterGaele 7.4

Let $T, S \in L^2(\mathbb{T})$ be the triangular and square wave, respectively, defined by

$$T(x) = |x|, \qquad \text{if } |x| \leq \pi, \qquad S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \end{cases}$$

(a) *Compute the Fourier series of T and S.*

Since T is an even function, we can represent T with a cosine series

$$T(x) = \frac{1}{2}\hat{T}_0 + \sum_{n=1}^{\infty} \hat{T}_n \cos(nx)$$

where

$$\begin{aligned} \hat{T}_0 &= \frac{1}{\pi}\int_{\mathbb{T}} T(x)\mathrm{d}x \quad \text{and} \\ \hat{T}_n &= \frac{1}{\pi}\int_{\mathbb{T}} T(x)\cos(nx)\mathrm{d}x, \quad n = 1, 2, \ldots \end{aligned}$$

Because cos is even and T is even, $T\sin$ is even, and so

$$\hat{T}_0 = \frac{1}{\pi}\int_{\mathbb{T}} T(x)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\mathrm{d}x = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

and for $n = 1, 2, \ldots$,

$$\hat{T}_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x$$

Utilizing integration by parts, we find

$$\begin{aligned} \hat{T}_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x \\ &= \frac{2}{\pi}\left[\left(\frac{x}{n}\sin(nx)\right)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\mathrm{d}x\right] \\ &= \frac{2}{\pi}\left[\frac{1}{n^2}\cos(nx)\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Thus,

$$\boxed{T(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2}\cos((2n-1)x)\right]}$$

Since S is an odd function, we can represent S with a sin series

$$S(x) = \sum_{n=1}^{\infty} \hat{S}_n \sin(nx)$$

where

$$\hat{S}_n = \frac{1}{\pi}\int_{\mathbb{T}} S(x)\sin(nx)\mathrm{d}x, \quad n = 1, 2, \ldots$$

Because sin is odd and S is odd, $\sin S$ is even, and thus

$$\begin{aligned}
\hat{S}_n &= \frac{1}{\pi}\int_{\mathbb{T}} S(x)\sin(nx)\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\mathrm{d}x \\
&= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} \\
&= -\frac{2}{\pi n}((-1)^n - 1) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Thus,

$$\boxed{S(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)}\sin((2n-1)x)\right]}$$

(b) *Show that $T \in H^1(\mathbb{T})$ and $T' = S$.*

(c) *Show that $S \notin H^1(\mathbb{T})$.*

Hunter and NachterGaele 7.5

Consider $f : \mathbb{T}^d \to \mathbb{C}$ defined by

$$f(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{in\cdot x},$$

where $x = (x_1, x_2, \ldots, x_d)$, $n = (n_1, n_2, \ldots, n_d)$, and $n \cdot x = n_1x_1 + n_2x_2 + \cdots + n_dx_d$. Prove that if

$$\sum_{n \in \mathbb{Z}^d} |n|^{2k}|a_n|^2 < \infty$$

for some $k > \frac{d}{2}$, then f is continuous.

Hunter and NachterGaele 7.6

Suppose that $f \in H^1([a,b])$ *and* $f(a) = f(b) = 0$. *Prove the* Poincaré inequality

$$\int_a^b |f(x)|^2 \mathrm{d}x \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \mathrm{d}x.$$

Hunter and NachterGaele 7.7

Solve the following initial-boundary value problem for the heat equation,

$$\begin{array}{ll} u_t = u_{xx}, & \\ u(0,t) = 0, \quad u(L,t) = 0 & \text{for } t > 0 \\ u(x,0) = f(x) & \text{for } 0 \leq x \leq L \end{array}$$

Suppose $u(x,t) = F(x)G(t)$ is a solution. Then

$$\begin{aligned} u_t &= u_{xx} \\ \implies F(x)G'(t) &= F''(x)G(t) \\ \implies \frac{F''(x)}{F(x)} &= \frac{G'(t)}{G(t)} \end{aligned}$$

Since the left hand side is a function of x and the right hand side is a function of t, they can only be equal if they are both constant, i.e.

$$\frac{F''(x)}{F(x)} = C = \frac{G'(t)}{G(t)}$$

for some $C \in \mathbb{R}$. Thus,

$$G'(t) - CG(t) = 0, \quad \text{and} \tag{1}$$

$$F''(x) - CF(x) = 0 \tag{2}$$

The solutions of (1) are

$$G(t) = c_1 e^{Ct}$$

Let $\lambda = \sqrt{C}$. If $C \neq 0$, the solutions of (2) are

$$F(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

The initial condition

$$u(0,t) = 0 \implies F(0)G(t) = 0 \implies F(0) = 0$$

provided u is not the trivial solution. Similarly,

$$F(L) = 0$$

If $C > 0$,

$$F(0) = 0 \implies 0 = c_1 + c_2 \implies F(x) = c_1\big(e^{\lambda x} - e^{-\lambda x}\big)$$

Also,

$$F(L) = 0 \implies 0 = c_1\big(e^{\lambda L} - e^{-\lambda L}\big) \implies c_1 = 0$$

Thus u is the trivial solution. If $C = 0$, then either $F'' = 0$ or $F \equiv 0$, but regardless, if $F'' = 0$, the initial conditions imply that $F \equiv 0$. So let $C < 0$ and define $\lambda = \sqrt{-C}$. Then

$$F(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x)$$

Then

$$F(0) = 0 \implies 0 = c_2 \implies F(x) = c_1 \sin(\lambda x)$$

Also,

$$F(L) = 0 \implies 0 = c_1 \sin(\lambda L) \implies \lambda L = \pi n$$

for integer values n. Thus $\lambda = \frac{n\pi}{L}$ for $n = \pm 1, \pm 2, \ldots$. Note $n \neq 0$ since that would imply $\lambda^2 = 0 = C$. Thus,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2}{L^2} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

The initial condition $u(0,x) = f(x)$ implies

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$$

This is a Fourier series, and thus the coefficients c_n are given by

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) \mathrm{d}x$$

Thus the full solution is

$$u(t,x) = \sum_{n=1}^{\infty} \left(\exp\left(-\frac{n^2\pi^2}{L^2} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right) \cdot \frac{2}{L} \int_0^L \left[f(x) \sin\left(\frac{n\pi}{L} x\right) \right] \mathrm{d}x \right)$$