HW #2

Sam Fleischer

January 29, 2015

Lieb and Loss Exercise 1.9

Verify the linearity of the integral as given in 1.5(7) by completing the steps outlined below. In what follows, f and g are nonnegative summable functions.

a)

Show that $f+g$ is also summable. In fact, by a simple argument $\int(f+g) \leq 2\big(\int f + \int g\big)$.

To show $\int(f+g) \leq 2\big(\int f + \int g\big)$, first note that

$$S_{f+g}(t) = \{x \ : \ (f+g)(x) > t\} \subset \left\{x \ : \ f(x) > \frac{t}{2}\right\} \cup \left\{x \ : \ g(x) > \frac{t}{2}\right\} = S_f\left(\frac{t}{2}\right) \cup S_g\left(\frac{t}{2}\right)$$

Since $f(x) \leq \frac{t}{2}$ and $g(x) \leq \frac{t}{2}$ implies $(f+g)(x) = f(x) + g(x) \leq t$. By properties of measures,

$$\begin{aligned}\mu(S_{f+g}(t)) &\leq \mu\left(S_f\left(\frac{t}{2}\right) \cup S_g\left(\frac{t}{2}\right)\right) \leq \mu\left(S_f\left(\frac{t}{2}\right)\right) + \mu\left(S_g\left(\frac{t}{2}\right)\right) \\ \implies \int_0^\infty \mu(S_{f+g}(t))\mathrm{d}t &\leq \int_0^\infty \mu\left(S_f\left(\frac{t}{2}\right)\right)\mathrm{d}t + \int_0^\infty \mu\left(S_g\left(\frac{t}{2}\right)\right)\mathrm{d}t\end{aligned}$$

Note the integral on the right hand side can split linearly because it is a Riemann integral. By u-substitution with $u = \frac{t}{2}$, we get

$$\int_0^\infty \mu(S_{f+g}(t))\mathrm{d}t \leq 2\int_0^\infty S_f(t)\mathrm{d}t + 2\int_0^\infty S_g(t)\mathrm{d}t$$

Note the constant 2 can be factored of each integral on the right hand side linearly because they are Riemann integrals. Thus, by definition,

$$\int(f+g) \leq 2\left(\int f + \int g\right)$$

and since f and g are summable, $\int f$ and $\int g$ are finite, which proves $\int(f+g)$ is finite. Next we confirm $S_{f+g}(t) \in \Sigma$. Construct a function $A \ : \ \Omega \to \mathbb{R}^2$ by

$$A(x) = (f(x), g(x))$$

and a function $B : \mathbb{R}^2 \to \mathbb{R}$ by

$$B(x, y) = x + y$$

Since A and B are measurable, then $B \circ A$ is measurable (since the composition of measurable functions is measurable). Thus $\{x : (f + g)(x) > t\} = \{x : B(A(x)) > t\}$ is measurable, and hence $S_{f+g}(t) \in \Sigma$. Thus $f + g$ is summable. $\square$

b)

For any integer N find two functions f_N and g_N that take only finitely many values, such that $|\int f - \int f_N| \leq \frac{C}{N}$, $|\int g - \int g_N| \leq \frac{C}{N}$, $|\int(f+g) - \int(f_N - g_N)| \leq \frac{C}{N}$ for some constant C independent of N.

c)

Show that for f_N and g_N as above $\int(f_N + g_N) = \int f_N + \int g_N$, thus proving the addivitivity of te integral for nonnegative functions.

Since f_N and g_N are simple functionsm they take on finitely many values, i.e.

$$f_N = \sum_{i=1}^{M} c_i \mu(E_i) \text{ and}$$
$$g_N = \sum_{j=1}^{M} d_j \mu(D_j)$$

Note both summations can be written with the same limit since we can always add finitely many terms where either c_i or d_j are zero.

$$\int_\Omega f_N \mathrm{d}\mu = \int_0^\infty F_{f_N} \mathrm{d}t = \sum_{i=1}^{M} c_i \mu(E_i) \text{ and}$$
$$\int_\Omega g_N \mathrm{d}\mu = \int_0^\infty F_{g_N} \mathrm{d}t = \sum_{j=1}^{M} d_j \mu(D_j)$$

Then

$$\int_\Omega (f_N + g_N) = \int_\Omega \sum_{i=1}^{M} (c_i \mu(E_i) + d_i \mu(D_i))$$

d)

In a similar fashion, show that for $f, g \geq 0$, $\int(f - g) = \int f - \int g$.

e)

Now use c) and d) to prove the linearity of the integral.

$$\begin{aligned}
\int (f+g) &= \int (f^+ - f^- + g^+ - g^-) \\
&= \int ((f^+ + g^+) - (f^- + g^-)) \\
&= \int (f^+ + g^+) - \int (f^- + g^-) && \text{by part } \mathbf{d)} \\
&= \int f^+ + \int g^+ - \int f^- - \int g^- && \text{by approximation and part } \mathbf{c)} \\
&= \int (f^+ - f^-) + \int (g^+ - g^-) \\
&= \int f + \int g
\end{aligned}$$

Lieb and Loss Exercise 1.10

Prove that when we add and subtract the subsets of sets of zero measure to the sets of a sgma-algebra then the result is again a sigma-algebra and the extended measure is again a measure.

Consider a measure space (Ω, Σ, μ) and let $\mathcal{A}$ be the collection of measurable sets of measure zero:

$$\mathcal{A} = \{A \in \Sigma \ : \ \mu(A) = 0\}$$

For each $A \in \mathcal{A}$, let $\mathbb{P}(A)$ be the power set of A, i.e.

$$\mathbb{P}(A) = \{a \ : \ a \subset A\}$$

Next, let $\overline{\Sigma}$ be a superset of Σ, consisting of the "addition" and "subtraction" of the subsets of sets of measure to each set:

$$\overline{\Sigma} = \Sigma \cup \Sigma^+ \cup \Sigma^-$$

where

$$\begin{aligned}
\Sigma^+ &= \{\sigma \cup a \ : \ \sigma \in \Sigma \text{ and } a \in \mathbb{P}(A) \text{ for some } A \in \mathcal{A}\} \\
\Sigma^- &= \{\sigma \setminus a \ : \ \sigma \in \Sigma \text{ and } a \in \mathbb{P}(A) \text{ for some } A \in \mathcal{A}\}
\end{aligned} \tag{1}$$

Let $\overline{\mu}$ map sets in $\overline{\Sigma}$ to the nonnegative reals, including infinity, i.e. $\mu \ : \ \overline{\Sigma} \to \mathbb{R}_0^+ \cup \{\infty\}$, by extending the measure μ.

$$\overline{\mu}(\overline{\sigma}) = \begin{cases} \mu(\overline{\sigma}) & \text{if } \overline{\sigma} \in \Sigma \\ \mu(\sigma) & \text{if } \overline{\sigma} \in \Sigma^+ \cup \Sigma^- \text{ where } \sigma \text{ is defined in (1)} \end{cases}$$

We want to show $(\Omega, \overline{\Sigma}, \overline{\mu})$ is a measure space. To do this, we must show $\overline{\Sigma}$ is a σ-algebra and $\overline{\mu}$ is a measure on $\overline{\Sigma}$. First we will show $\overline{\Sigma} = \Sigma^+$ by showing (i) $\Sigma \subset \Sigma^+$ and (ii) $\Sigma^- \subset \Sigma^+$.

(i) Since $\emptyset$ is a subset of all sets, then for $\sigma \in \Sigma$, $\sigma = \sigma \cup \emptyset \in \Sigma^+$. Thus $\Sigma \subset \Sigma^+$.

(ii) Let $\sigma \setminus a \in \Sigma^-$. Then $a \subset A$ for some $A \in \mathcal{A}$. Also, $A \setminus a \subset A$, and

$$\begin{aligned}\sigma \setminus a &= \sigma \cap (a^C) \\ &= \sigma \cap (A^C \cup (A \setminus a)) \\ &= (\sigma \cap A^C) \cup (\sigma \cap (A \setminus a))\end{aligned}$$

But Σ is a σ-algebra, which implies it is closed under finite intersections, and thus σ, $A^C \in \Sigma$ implies $\hat{\sigma} = \sigma \cap A^C \in \Sigma$. Also, $\hat{a} = \sigma \cap (A \setminus a) \subset A \setminus a \subset A$. Thus

$$\sigma \setminus a = \hat{\sigma} \cup \hat{a} \in \Sigma^+$$

which proves $\Sigma^- \subset \Sigma^+$.

This shows

$$\overline{\Sigma} = \Sigma^+$$

Next we will show $\overline{\Sigma}$ is a σ-algebra. To do this we must show (i) it is closed under complementation, (ii) it is closed under countable unions, and (iii) $\Omega \in \overline{\Sigma}$.

(i) Let $x \in \overline{\Sigma} = \Sigma^+$. Then $x = \sigma \cup a$ for some $\sigma \in \Sigma$ and $a \in \mathbb{P}(A)$. Since $\sigma^C \in \Sigma$, then

$$x^C = (\sigma \cup a)^C = \sigma^C \cap a^C = \sigma^C \setminus a \in \Sigma^- \subset \overline{\Sigma}$$

(ii) Let $x = \bigcup_{n=1}^{\infty} \overline{\sigma}_n$ where $\overline{\sigma}_n \in \overline{\Sigma} = \Sigma^+$ for $n = 1, 2, \ldots$. Each $\overline{\sigma}_n$ can be written as

$$\overline{\sigma}_n = \sigma_n \cup a_n$$

for some $\sigma_n \in \Sigma$ and $a_n \subset A_n$ for some $A_n \in \mathcal{A}$. This means that by the commutativity of unions, we can write x as

$$x = \bigcup_{n=1}^{\infty} \overline{\sigma}_n = \bigcup_{n=1}^{\infty} (\sigma_n \cup a_n) = \bigcup_{n=1}^{\infty} \sigma_n \cup \bigcup_{n=1}^{\infty} a_n = \hat{\sigma} \cup \bigcup_{n=1}^{\infty} a_n$$

where $\hat{\sigma} = \bigcup_{n=1}^{\infty} \sigma_n \in \Sigma$ since Σ is closed under countable unions. Since μ is a measure, it has the property of countable additivity, which means

$$\mu\Big(\bigcup \mathcal{A}\Big) = \mu\left(\bigcup_{A \in \mathcal{A}} A\right) = \sum_{A \in \mathcal{A}} \mu(A) = \sum_{A \in \mathcal{A}} 0 = 0$$

This means $\bigcup \mathcal{A} \in \mathcal{A}$ since it has measure zero. Thus

$$\bigcup_{n=1}^{\infty} a_n \subset \bigcup_{n=1}^{\infty} A_n \subset \bigcup \mathcal{A} \in \mathcal{A}$$

Defining $b = \bigcup_{n=1}^{\infty} a_n \in \mathbb{P}(\bigcup \mathcal{A})$, we can write

$$\bigcup_{n=1}^{\infty} (\sigma_n \cup a_n) = \hat{\sigma} \cup \bigcup_{n=1}^{\infty} a_n = \hat{\sigma} \cup b \in \Sigma^+$$

Thus $\overline{\Sigma} = \Sigma^+$ is closed under countable unions.

(iii) Since Σ is a σ-algebra, $\Omega \in \Sigma \subset \overline{\Sigma}$.

Next we will show $\overline{\mu}$ is a measure on $\overline{\Sigma} = \Sigma^+$. We have to show (i) $\overline{\mu}(\emptyset) = 0$, (ii) $\overline{\mu}$ has the property of countable additivity, and (iii) $\overline{\mu}$ is well-defined.

(i) Since $\emptyset \in \Sigma$, then $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$ since μ is a measure on Σ.

(ii) Let $\overline{\sigma}_1, \overline{\sigma}_2, \dots$ be a sequence of disjoint sets in $\overline{\Sigma}$. Then for each n,

$$\overline{\sigma}_n = \sigma_n \cup a_n$$

for some $\sigma_n \in \Sigma$ and $a_n \subset A_n$ for some $A_n \in \mathcal{A}$. This means that $\{\sigma_n\}_n$ is a sequence of disjoint sets and $\{a_n\}_n$ is a sequence of disjoint sets. We showed earlier that

$$\bigcup_{n=1}^{\infty} a_n \subset \bigcup \mathcal{A}$$

so denote $b = \bigcup_{n=1}^{\infty} a_n$ and note

$$\bigcup_{n=1}^{\infty} \sigma_n \in \Sigma$$

Then

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} \overline{\sigma}_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} (\sigma_n \cup a_n)\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} \sigma_n \cup \bigcup_{n=1}^{\infty} a_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} \sigma_n \cup b\right) = \mu\left(\bigcup_{n=1}^{\infty} \sigma_n\right)$$

by the definition of $\overline{\mu}$. Thus by the countable additivity of μ,

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} \overline{\sigma}_n\right) = \mu\left(\bigcup_{n=1}^{\infty} \sigma_n\right) = \sum_{n=1}^{\infty} \mu(\sigma_n) = \sum_{n=1}^{\infty} \overline{\mu}(\sigma_n \cup a_n) = \sum_{n=1}^{\infty} \overline{\mu}(\overline{\sigma}_n)$$

This shows $\overline{\mu}$ has countable additivity.

(iii) Let $\overline{\sigma} \in \overline{\Sigma}$ be represented in two arbitrary ways:

$$\overline{\sigma} = \sigma_1 \cup a_1 = \sigma_2 \cup a_2$$

for $a_1 \subset A_1$ and $a_2 \subset A_2$ for some $A_1, A_2 \in \mathcal{A}$. Then let $A = A_1 \cup A_2$ and note that $\sigma_1 \subset A$ and $\sigma_2 \subset A$. Note

$$A = [(A \setminus A_1) \cup (A \setminus A_2)] \cup (A_1 \cap A_2)$$

Then

$$\sigma_1 \cup [(A_1 \setminus A_2) \cup (A_1 \cup A_2)] = \sigma_1 \cup A_1 = \sigma_2 \cup A_2 = \sigma_2 \cup [(A_2 \setminus A_1) \cup (A_2 \cup A_1)]$$

but

$$\begin{aligned} A_1 \setminus A_2 &\subset (A \setminus A_1) \cup (A \setminus A_2) \quad \text{and} \\ A_2 \setminus A_1 &\subset (A \setminus A_1) \cup (A \setminus A_2) \end{aligned}$$

and thus

$$\sigma_1 \cup A = \sigma_1 \cup [(A \setminus A_1) \cup (A \setminus A_2)] \cup (A_1 \cap A_2) = \sigma_2 \cup [(A \setminus A_1) \cup (A \setminus A_2)] \cup (A_1 \cap A_2) = \sigma_2 \cup A$$

This implies

$$\mu(\sigma_1) = \mu(\sigma_1) + \mu(A) = \mu(\sigma_1 \cup A) = \mu(\sigma_2 \sup A) = \mu(\sigma_2) + \mu(A) = \mu(\sigma_2)$$

This shows that $\overline{\mu}(\overline{\sigma})$ well-defined regardless of how it is represented.

Thus $\overline{\mu}$ is a measure on $\overline{\Sigma}$.

Lieb and Loss Exercise 1.12

Find a simple condition for $f_n(x)$ so that

$$\sum_{n=0}^{\infty}\int_{\Omega} f_n(x)\mu(\mathrm{d}x) = \int_{\Omega}\left[\sum_{n=0}^{\infty} f_n(x)\right]\mu(\mathrm{d}x)$$

Let f_{n_n} be a sequence of positive functions. Then let

$$g_n = \sum_{i=0}^{n} f_i$$

be the nth partial sum of $\sum_{n=0}^{\infty} f_n$. Then g_n is an increasing sequence of functions that converges pointwise to $\sum_{n=0}^{\infty} f_n$ in Ω. Then by the monotone convergence theorem,

$$\int_{\Omega}\sum_{n=0}^{\infty} f_n \mathrm{d}\mu = \lim_{n\to\infty}\int_{\Omega} g_n \mathrm{d}\mu$$

and thus

$$\int_{\Omega}\left[\sum_{n=0}^{\infty} f_n\right]\mathrm{d}\mu = \lim_{n\to\infty}\int_{\Omega}\left[\sum_{i=0}^{n} f_i\right]\mathrm{d}\mu = \lim_{n\to\infty}\sum_{i=0}^{n}\int_{\Omega} f_i \mathrm{d}\mu = \sum_{n=0}^{\infty}\int_{\Omega} f_n \mathrm{d}\mu$$

which proves the result.

Lieb and Loss Exercise 1.13

Let f be the function on $\mathbb{R}^n$ defined by $f(x) = |x|^{-p}\mathcal{X}_{\{|x|<1\}}(x)$. Compute $\int f \mathrm{d}\mathcal{L}^n$ in two ways: (i) Use polar coordinates and compute the integral by the standard calculus method. (ii) Compute $\mathcal{L}^n(\{x : f(x) > a\})$ and then use Lebesgue's definition.

(i) First note that

$$f(x) = \begin{cases} |x|^{-p} & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}$$

Then note that polar coordinates on $\mathbb{R}^n$ are $(r, \phi, \theta_1, \theta_2, \ldots, \theta_{n-2})$ where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $\theta_i \in [0, \pi)$ for $i = 1, 2, \ldots, n-2$. When transforming rectangular coordinates to polar coordinates in n dimensions, we multiply by the determinant of the Jacobian matrix, and so

$$\int f \mathrm{d}\mathcal{L}^n = \int_0^{\pi}\int_0^{\pi}\ldots\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} r^{-p}\left[\frac{1}{r^{1-n}}\prod_{k=1}^{n-2}\sin^{n-k-1}(\theta_k)\right]\mathrm{d}r\ \mathrm{d}\phi\ \mathrm{d}\theta_1\ldots\ \mathrm{d}\theta_{n-3}\ \mathrm{d}\theta_{n-2}$$

The integrand is separable, and thus

$$\int f \mathrm{d}\mathcal{L}^n = \int_0^{\infty}\frac{1}{r^{p-n+1}}\mathrm{d}r\left[\prod_{k=1}^{n-2}\int_0^{\pi}\sin^{n-k-1}(\theta_k)\mathrm{d}\theta_k\right]\int_0^{2\pi}\mathrm{d}\phi$$

$$= 2\pi \left[\prod_{k=1}^{n-2} \int_0^{\pi} \sin^{n-k-1}(\theta_k) \mathrm{d}\theta_k \right] \int_0^{\infty} \frac{1}{r^{p-n+1}} \mathrm{d}r$$

But

$$\left|\mathbb{S}^{n-1}\right| = 2\pi \prod_{k=1}^{n-2} \int_0^{\pi} \sin^{n-k-1}(\theta_k) \mathrm{d}\theta_k$$

where $|\mathbb{S}^{n-1}|$ is the surface area of an n-dimensional sphere of radius 1. Thus,

$$\int f \mathrm{d}\mathcal{L}^n = |\mathbb{S}^{n-1}| \int_0^{\infty} \frac{1}{r^{p-n+1}} \mathrm{d}r = |\mathbb{S}^{n-1}| \int_0^{1} \frac{1}{r^{p-n+1}} \mathrm{d}r = \begin{cases} |\mathbb{S}^{n-1}| \frac{1}{n-p} & , \ n > p \\ +\infty & , \ n \le p \end{cases}$$

(ii) First note the Lebesgue measure of $f^{-1}(t, \infty)$ for a fixed t.

$$\begin{aligned} \mathcal{L}^n(f^{-1}(t,\infty)) &= \mathcal{L}^n(\{x \ : \ f(x) > t\}) \\ &= \mathcal{L}^n(\{x \in B_1(0) \ : \ |x| < t^{-1/p}\}) \\ &= \begin{cases} \mathcal{L}^n(B_1(0)) & \text{if } t^{-1/p} \ge 1 \\ \mathcal{L}^n(B_{t^{-1/p}}(0)) & \text{if } t^{-1/p} < 1 \end{cases} \\ &= \begin{cases} \frac{1}{n}|\mathbb{S}^{n-1}| & \text{if } t^{-1/p} \ge 1 \\ \frac{1}{n}|\mathbb{S}^{n-1}| t^{-n/p} & \text{if } t^{-1/p} < 1 \end{cases} \end{aligned}$$

Now integrate over $t \in [0, \infty]$:

$$\begin{aligned} \int_0^{\infty} \mathcal{L}^{n-1}(f^{-1}(t,\infty)) &= \int_0^1 \left[\frac{1}{n}|\mathbb{S}^{n-1}|\right] \mathrm{d}t + \int_1^{\infty} \left[\frac{1}{n}|\mathbb{S}^{n-1}| t^{-n/p}\right] \mathrm{d}t \\ &= \frac{1}{n}|\mathbb{S}^{n-1}| \left(1 + \left[\frac{t^{-\frac{n}{p}+1}}{-\frac{n}{p}+1}\right]_1^{\infty}\right) \end{aligned}$$

If $p \ge n$,

$$\left(1 + \left[\frac{t^{-\frac{n}{p}+1}}{-\frac{n}{p}+1}\right]_1^{\infty}\right) = \infty$$

but if $p < n$,

$$\left(1 + \left[\frac{t^{-\frac{n}{p}+1}}{-\frac{n}{p}+1}\right]_1^{\infty}\right) = \frac{p}{p-n}$$

Thus,

$$\int_0^{\infty} \mathcal{L}^{n-1}(f^{-1}(t,\infty)) = \begin{cases} \infty & \text{if } p \ge n \\ \frac{1}{n-p} & \text{if } p < n \end{cases}$$

which matches with our answer in part (i).

Lieb and Loss Exercise 1.17

Show that the infimum of a family of continuous functions is upper semi-continuous.

Let $\mathcal{F} = \{f_i \in [\Omega \to \mathbb{R}] \;:\; f_i \text{ is continuous, and } i \in I\}$ where I is some index set. Then define $f \in [\Omega \to \mathbb{R}]$ by

$$f(x) = \inf_{i \in I} f_i(x)$$

Assume f is not upper semi-continuous at x. Then there is a sequence $\{x_n\}_n \to x$ such that

$$\limsup_{x_n \to x} f(x_n) > f(x)$$

So there is some ε such that $\limsup_{x_n \to x} f(x_n) = f(x) + \varepsilon$. By definition of the infimum, there is a function f_i such that $f_i(x) < f(x) + \frac{\varepsilon}{2}$. The continuity of f_i implies $\exists \delta$ such that $|x - x_0| < \delta \implies |f_i(x) - f_i(x_0)| < \frac{\varepsilon}{2}$. Then for $|x_n - x| < \delta$, $|f_i(x_n) - f_i(x)| < \frac{\varepsilon}{2}$. But $f_i(x) < f(x) + \frac{\varepsilon}{2}$, and thus

$$\begin{aligned} f_i(x_n) - \left(f(x) + \frac{\varepsilon}{2}\right) < f_i(x_n) - f_i(x) &< \frac{\varepsilon}{2} \\ \implies f_i(x_n) - f(x) &< \varepsilon \end{aligned}$$

However, by the definition of f,

$$f(x_n) \leq f_i(x_n) < f(x) + \varepsilon$$

which implies

$$\limsup_{x_n \to x} f(x_n) < f(x) + \varepsilon$$

which is a contradiction since $\limsup_{x_n \to x} f(x_n) = f(x) + \varepsilon$. Thus f is upper semi-continuous at x. Since x was arbitrary, f is upper semi-continuous on its domain.

Lieb and Loss Exercise 1.18

Simple facts about measure:

a)

Show that the condition $\{x \;:\; f(x) > a\}$ is measureiable for all $a \in \mathbb{R}$ holds if and only if it holds for all rational a.

Suppose $\{x \;:\; f(x) > a\} \in \Sigma$ for all $a \in \mathbb{Q}$. Then for $a \in \mathbb{R} \setminus \mathbb{Q}$, let $\{a_i\}_i$ be an increasing sequence in $\mathbb{Q}$ such that $\{a_i\} \to a$. Then

$$\{x \;:\; f(x) > a\} = \bigcap_{n=1}^{\infty} \{x \;:\; f(x) > a_n\} \in \Sigma$$

because Σ is closed under countable intersections.

b)

For rational a, show that

$$\{x \ : \ f(x) + g(x) > a\} = \bigcup_{b \in \mathbb{Q}} (\{x \ : \ f(x) > b\} \cap \{x \ : \ g(x) > a - b\})$$

For ease, define $A = \{x \ : \ f(x)+g(x) > a\}$ and $B = \bigcup_{b\in\mathbb{Q}} (\{x \ : \ f(x) > b\} \cap \{x \ : \ g(x) > a-b\})$.

Suppose $x \in A$. Then $f(x)+g(x) > a$. Then $\exists \varepsilon > 0$ such that $f(x)+g(x) = a+\varepsilon$. Now choose $b \in \mathbb{Q}\cap(f(x)-\varepsilon, f(x))$. Then $f(x)-\varepsilon < b < f(x)$, i.e. $f(x) < b+\varepsilon < f(x)+\varepsilon$. If $g(x) \le a-b$, then $f(x)+g(x) \le f(x)+a-b < b+\varepsilon+a-b = a+\varepsilon$, which is a contradiction since $f(x)+g(x) = a+\varepsilon$. Thus $x \in B$, showing $A \subset B$.

Suppose $x \in B$. Then $\exists b \in \mathbb{Q}$ such that $f(x) > b$ and $g(x) > a - b$. Then $f(x) + g(x) > b + a - b = a$, and thus $x \in A$, showing $B \subset A$.

Thus,

$$\{x \ : \ f(x) + g(x) > a\} = \bigcup_{b \in \mathbb{Q}} (\{x \ : \ f(x) > b\} \cap \{x \ : \ g(x) > a - b\})$$

c)

In a similar way, prove that fg is measurable if f and g are measurable.

We want to show if $f^{-1}(t,\infty) \in \Sigma$ and $g^{-1}(t,\infty) \in \Sigma$, then $(fg)^{-1}(t,\infty) \in \Sigma$. We will show this for $t \in \mathbb{Q}$, but by part **b)**, this is equivalent to showing it for $t \in \mathbb{R}$.

To show $(fg)^{-1}(t,\infty) \in \Sigma$, we will show

$$(fg)^{-1}(t,\infty) = \bigcup_{b\in\mathbb{Q}} \left(f^{-1}(b,\infty) \cap g^{-1}\left(\frac{a}{b},\infty\right)\right)$$

For ease, define $A = (fg)^{-1}(t,\infty)$ and $B = \bigcup_{b\in\mathbb{Q}} \left(f^{-1}(b,\infty) \cap g^{-1}\left(\frac{a}{b},\infty\right)\right)$.

Suppose $x \in A$. Then $f(x) + g(x) > a$. Then $\exists \varepsilon > 0$ such that $fg(x) = a(1+\varepsilon)$. Now choose $b \in \mathbb{Q} \cap \left(\dfrac{f(x)}{1+\varepsilon}, f(x)\right)$. Then $\dfrac{f(x)}{1+\varepsilon} < b < f(x)$, i.e. $f(x) < b(1+\varepsilon) < f(x)(1+\varepsilon)$. If $g(x) \le \dfrac{a}{b}$, then $(fg)(x) \le \dfrac{af(x)}{b} < \dfrac{ab(1+\varepsilon)}{b} = a(1+\varepsilon)$, which is a contradiction since $(fg)(x) = a(1+\varepsilon)$. Thus $x \in B$, showing $A \subset B$.

Suppose $x \in B$. Then $\exists b \in \mathbb{Q}$ such that $f(x) > b$ and $g(x) > \dfrac{a}{b}$. Then $(fg)(x) > \dfrac{ba}{b} = a$, and thus $x \in A$, showing $B \subset A$.

Thus, for $t \in \mathbb{Q}$,

$$(fg)^{-1}(t,\infty) = \bigcup_{b\in\mathbb{Q}} \left(f^{-1}(b,\infty) \cap g^{-1}\left(\frac{a}{b},\infty\right)\right)$$

Then since $\mathbb{Q}$ is countable and $(fg)^{-1}(t,\infty)$ is a countable union and intersection of elements in Σ, then $(fg)^{-1}(t,\infty) \in \Sigma$. By part **a)**, this shows the above holds for $t \in \mathbb{R}$ and thus f, g measurable imply fg is measurable.

Hunter and Nachtergaele Exercise 6.1

Prove that a closed, convex subset of a Hilbert space has a unique point of minimum norm.

Let A be a closed and convex subset of a Hilbert space $\mathcal{H}$. Let d be the distance of $\vec{0}$ from A,

$$d = \inf_{x \in A} \{\|x\|\}$$

First we prove that there is a closest point $z \in A$ at which this infimum is attained. From the definition of d, there is a sequence of elements $z_n \in A$ such that

$$\lim_{n \to \infty} \|z_n\| = d$$

Thus $\forall \varepsilon$, $\exists N_\varepsilon$ such that

$$\|z_n\| \leq d + \varepsilon \qquad \text{when } n \geq N_\varepsilon$$

Next we will show $\{z_n\}_n$ is Cauchy. Let $n, m \geq N_\varepsilon$. The parallelogram law implies

$$\|z_m - z_n\|^2 + \|z_m + z_n\|^2 = 2\|z_n\|^2 + 2\|z_m\|^2$$

Since A is convex, $\frac{z_m + z_n}{2} \in A$, and thus

$$\left\| \frac{z_m + z_n}{2} \right\| \leq d$$

by the definition of d. Thus,

$$\|z_m + z_n\|^2 \leq 4d^2$$

which implies

$$\begin{aligned} \|z_m - z_n\|^2 + 4d^2 &\leq 2\|z_m\|^2 + 2\|z_n\|^2 \\ \implies \|z_m - z_n\|^2 &\leq 2(d+\varepsilon)^2 + 2(d+\varepsilon)^2 - 4d^2 \\ &= 4\varepsilon(2d + \varepsilon) \end{aligned}$$

which is arbitrarily small as $\varepsilon \to 0$. Thus $\{z_n\}_n$ is Cauchy. The completeness of Hilbert spaces implies $\{z_n\}_n$ converges to a limit, but since A is closed, $\lim_{n\to\infty} z_n = z \in A$. By the continuity of $\|\cdot\|$,

$$\|z\| = \left\| \lim_{n\to\infty} z_n \right\| = \lim_{n\to\infty} \|z_n\| = d$$

Thus there is a point at which A achieves minimum norm. Next, we prove uniqueness. Suppose $\|z_1\| = \|z_2\| = 0$. Then by the parallelogram law,

$$2\|z_1\|^2 + 2\|z_2\|^2 = \|z_1 + z_2\|^2 + \|z_1 - z_2\|^2$$

Again, the convexity of A implies $\frac{z_1 + z_2}{2} \in A$, and thus

$$\|z_1 - z_2\|^2 = 4d^2 - 4\left\| \frac{z_1 + z_2}{2} \right\|^2 \leq 4d^2 - 4d^2 = 0$$

But norm is non-negative, i.e. $\|z_1 - z_2\| = 0$. Thus $z_1 = z_2$. Thus the point of minimum norm is unique.

Hunter and Nachtergaele Exercise 6.3

If A is a subset of a Hilbert space, prove that

$$A^{\perp} = \overline{A}^{\perp},$$

where $\overline{A}$ is the closure of A. If $\mathcal{M}$ is a linear subspace of a Hilbert space, prove that

$$\mathcal{M}^{\perp\perp} = \overline{\mathcal{M}}.$$

Let $x \in A^{\perp}$ and choose any $y \in \overline{A}$. Then $\exists \{y_n\}_n \in A$ such that $y_n \to y$. But since $y_n \in A$, $x \perp y_n$ for all n. Thus, by the continuity of inner products,

$$\langle x, y\rangle = \langle x, \lim_{n\to\infty} y_n\rangle = \lim_{n\to\infty} \langle x, y_n\rangle = \lim_{n\to\infty} 0 = 0$$

and thus $x \perp y$, which shows $x \in \overline{A}^{\perp}$, and hence $A^{\perp} \subset \overline{A}^{\perp}$.

Now let $x \in \overline{A}^{\perp}$. Then $x \perp y$ $\forall y \in \overline{A}$. But $A \subset \overline{A}$, and thus trivially, $x \perp y$ $\forall y \in A$, i.e. $x \in A^{\perp}$. Hence $\overline{A}^{\perp} \subset A^{\perp}$.

Thus, $A^{\perp} = \overline{A}^{\perp}$.

Let $\mathcal{M}$ be a linear subspace of $\mathcal{H}$. Assume $x \in \overline{\mathcal{M}}$. Then there is a sequence $x_n \in \mathcal{M}$ such that $x_n \to x$. Then $\langle x_n, y\rangle = 0$ $\forall y \in \mathcal{M}^{\perp}$. Then by continuity of inner products,

$$\langle x, y\rangle = \langle \lim_{n\to\infty} x_n, y\rangle = \lim_{n\to\infty} \langle x_n, y\rangle = \lim_{n\to\infty} 0 = 0 \qquad \forall y \in \mathcal{M}^{\perp}$$

Then $x \in \mathcal{M}^{\perp\perp}$, which shows $\overline{\mathcal{M}} \subset \overline{M}^{\perp\perp}$.

Now assume $x \notin \overline{\mathcal{M}}$. Since $\overline{\mathcal{M}}$ is closed, the by the Projection Theorem, $\exists y \in \overline{\mathcal{M}}$ such that $(x-y) \perp \overline{\mathcal{M}}$. Since $y \in \overline{\mathcal{M}}$, $\langle x-y, y\rangle = 0$. Since $x \neq y$ ($x \notin \overline{\mathcal{M}}$ and $y \in \overline{\mathcal{M}}$), then $\langle x-y, x-y\rangle \neq 0$. However, $\langle x-y, x-y\rangle = \langle x-y, x\rangle - \langle x-y, y\rangle = \langle x-y, x\rangle$. Since $x-y \perp \overline{\mathcal{M}}$, then $x-y \perp \mathcal{M}$, i.e. $x-y \in \mathcal{M}^{\perp}$. Then since $\langle x-y, x\rangle \neq 0$, then $x \notin \overline{\mathcal{M}}^{\perp\perp} = \mathcal{M}^{\perp\perp}$, which shows $\mathcal{M}^{\perp\perp} \subset \overline{\mathcal{M}}$.

Thus $\overline{\mathcal{M}} = \mathcal{M}^{\perp\perp}$.

Hunter and Nachtergaele Exercise 6.5

Suppose that $\{\mathcal{H}_n \,:\, n \in \mathbb{N}\}$ is a set of orthogonal closed subspaces of a Hilbert space $\mathcal{H}$. We define the infinite direct sum

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \left\{ \sum_{n=1}^{\infty} x_n \mid x_n \in \mathcal{H}_n \quad \textit{and} \quad \sum_{n=1}^{\infty} \|x_n\|^2 < +\infty \right\}.$$

Prove that $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a closed linear subspace of $\mathcal{H}$.

First we show $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is linear. Consider $x, y \in \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ where

$$x = \sum_{n=1}^{\infty} x_n \qquad \text{and} \qquad y = \sum_{n=1}^{\infty} y_n$$

Then since each $\mathcal{H}_n$ is linear, then $c_n = ax_n + by_n \in \mathcal{H}_n$ for each n. Thus

$$ax + by = a\sum_{n=1}^{\infty} x_n + b\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} (ax_n + by_n) = \sum_{n=1}^{\infty} c_n$$

Now we need to show $\sum_{n=1}^{\infty} \|c_n\|^2 < \infty$. Consider $x^{(N)}$ and $y^{(N)}$ where

$$x^{(N)} = \sum_{n=1}^{N} x_n \qquad \text{and} \qquad y^{(N)} = \sum_{n=1}^{N} y_n$$

Then

$$\left\|ax^{(N)} + by^{(N)}\right\|^2 = \left\|\sum_{n=1}^{N} (ax_n + by_n)\right\|^2 = \left\|\sum_{n=1}^{N} c_n\right\|^2 = \sum_{n=1}^{N} \|c_n\|^2$$

by the pythagorean theorem. However, since the norm is continuous,

$$\lim_{N\to\infty} \left\|ax^{(N)} + by^{(N)}\right\|^2 = \|ax + by\|^2 = \sum_{n=1}^{\infty} \|c_n\|^2$$

Since $ax + by \in \mathcal{H}$, then $\|ax + by\| \in \mathbb{R}$ by the definition of norm. Thus $\|ax + by\|^2 \in \mathbb{R}$ and hence $< \infty$. Thus $ax + by \in \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, which shows $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a linear subspace.

Hunter and Nachtergaele Exercise 6.8

Let $\mathcal{X} = \{x_n \ : \ n \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space. Show that the sum $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n}$ converges unconditionally but not absolutely.

Let $y_n = \dfrac{x_n}{n}$ and let $\mathcal{Y} = \{y_n \ : \ n \in \mathbb{N}\}$. Since each y_n is a scalar multiple of x_n for all n, and since $\mathcal{X}$ is an orthonomal set, then $\mathcal{Y}$ is an orthogonal set. Thus by the Pythagorean Theorem, $\sum_{n=1}^{\infty} y_n$ converges unconditionally if and only if $\sum_{n=1}^{\infty} \|y_n\|^2$ converges. But

$$\sum_{n=1}^{\infty} \|y_n\|^2 = \sum_{n=1}^{\infty} \frac{\|x_n\|^2}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

by the p-series test. Thus $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{x_n}{n}$ converges unconditionally. However,

$$\sum_{n=1}^{\infty} \left\|\frac{x_n}{n}\right\| = \sum_{n=1}^{\infty} \frac{\|x_n\|}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$$

And so $\sum_{n=1}^{\infty} \frac{x_n}{n}$ does not converge absolutely.

Hunter and Nachtergaele Exercise 6.12

Define the Legendre polynomials P_n *by*

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

(a) *Compute the first few Legendre polynomials, and compare with what you get by Gram-Schmidt orthogonalization of the monomials* $\{1, x, x^2, \dots\}$ *in* $L^2([-1,1])$.

$$\begin{aligned}
P_0(x) &= \frac{1}{2^0 0!}(x^2-1)^0 = 1\\
P_1(x) &= \frac{1}{2^1 1!}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)^1 = \frac{1}{2}2x = x\\
P_2(x) &= \frac{1}{2^2 2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 = \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^4-2x^2+1) = \frac{1}{8}(12x^2-4) = \frac{3}{2}x^2-\frac{1}{2}\\
P_3(x) &= \frac{1}{2^3 3!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3 = \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^6-3x^4+3x^2-1) = \frac{1}{48}(120x^3-72x) = \frac{5}{2}x^3-\frac{3}{2}x
\end{aligned}$$

These polynomials are scalar multiples of the results of the Gram-Schmidt orthogonalization of the monomials $\{1, x, x^2, \dots\}$ in $L^2([-1,1])$.

(b) *Show that the Legendre polynomials are orthogonal in* $L^2([-1,1])$, *and that they are obtained by Gram-Schmidt orthongonalization of the monomials*

Fix n and pick $m < n$. Then

$$\begin{aligned}
\langle x^m, P_n\rangle &= \int_{-1}^{1} x^m P_n \mathrm{d}x\\
&= \int_{-1}^{1} x^m \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \mathrm{d}x\\
\implies 2^n n!\langle x^m, P_n\rangle &= \int_{-1}^{1} x^m \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \mathrm{d}x\\
&= (-1)^m m! \int_{-1}^{1} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}(x^2-1)\mathrm{d}x \quad \text{through integration by parts } m \text{ times}\\
&= (-1)^m m! \left[\frac{\mathrm{d}^{n-m-1}}{\mathrm{d}x^{n-m-1}}(x^2-1)^n\right]_{-1}^{1}\\
&= 0
\end{aligned}$$

because x^2-1 is a factor of $\frac{\mathrm{d}^{n-m-1}}{\mathrm{d}x^{n-m-1}}(x^2-1)^n$. Thus $x^m \perp P_n$ for all $m < n$. However P_m is a linear combination of elements from $\{1, x, \dots, x^m\}$, and thus $P_m \perp P_n$. Thus the Legendre polynomials are orthogonal in $L^2([-1,1])$.

(c) *Show that*

$$\int_{-1}^{1} P_n(x)^2 \mathrm{d}x = \frac{2}{2n+1}.$$

$$
\begin{aligned}
\int_{-1}^{1} P_n(x)^2 \mathrm{d}x &= \int_{-1}^{1} \left(\frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2-1)^n \right)^2 \mathrm{d}x \\
&= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2-1)^n \right)^2 \mathrm{d}x \\
&= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}} (x^2-1)^n \mathrm{d}x \qquad \text{through integration by parts } n \text{ times} \\
&= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \mathrm{d}x \qquad \text{through integration by parts } 2n \text{ times}
\end{aligned}
$$

Now just consider the integral

$$
\begin{aligned}
\int_{-1}^{1} (x^2-1)^n \mathrm{d}x &= \int_{-1}^{1} (x-1)^n (x+1)^n \mathrm{d}x \\
&= \frac{(n!)^2(-1)^n}{(2n)!} \int_{-1}^{1} (x+1)^{2n} \mathrm{d}x \qquad \text{through integration by parts } n \text{ times} \\
&= \frac{(n!)^2(-1)^n}{(2n)!} \left[\frac{(x+1)^{2n+1}}{2n+1} \right]_{-1}^{1} \\
&= \frac{(n!)^2(-1)^n 2^{2n+1}}{(2n)!(2n+1)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_{-1}^{1} P_n(x)^2 \mathrm{d}x &= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \\
&= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \cdot \frac{(n!)^2(-1)^n 2^{2n+1}}{(2n)!(2n+1)} \\
&= \frac{2}{2n+1}
\end{aligned}
$$

(d) *Prove that the Legendre polynomials form an orthogonal basis of $L^2([-1,1])$. Suppose that $f \in L^2([-1,1])$ is given by*

$$
f(x) = \sum_{n=0}^{\infty} c_n P_n(x).
$$

Compute c_n and say explicitly in what sense the series converges.

Since $\{P_n\}_n$ can be obtained using the Gram-Schmidt from an orthogonal basis (namely the monomials $\{1, x, x^2, \dots\}$), the $\{P_n\}_n$ is an orthogonal basis of $L^2([-1,1])$.

Bessel's inequality says that since $\{P_n\}_n$ is an orthogonal basis, then

$$
c_n = \left\langle \frac{P_n}{\|P_n\|}, f \right\rangle
$$

(e) *Prove that the Legendre polynomial P_n is an eigenfunction of the differential operator*

$$L = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}$$

with eigenvalue $\lambda_n = n(n+1)$, meaning that

$$LP_n = \lambda_n P_n.$$

Let $u(x) = (x^2-1)^n$ and let D be the differential operator. Then note that

$$(x^2-1)Du = (x^2-1)n(x^2-1)^{n-1} \cdot 2x = 2nxu$$

Apply D^{n+1} to both sides and use Liebnitz's Rule for $(fg)^{(n)}$ to acheive

$$\begin{aligned} \frac{(n+1)n}{2} \cdot 2 \cdot D^{n-1}Du + (n+1)2xD^nDu + (x^2-1)D^{n+1}Du &= 2n(n+1)D^nu + 2nxD^{n+1}u \\ \implies 2xD^{n+1}u + (x^2-1)D^{n+2}u &= n(n+1)D^nu \\ \implies LD^nu &= n(n+1)D^nu \end{aligned}$$

which shows D^n is an eigenfunction of L with eigenvalue $\lambda_n = n(n-1)$. Since $2^n n! P_n = D^n u$ (i.e. P_n is linearly dependent on D^n), then P_n is an eigenfunction of L with eigenvalue $\lambda_n = n(n+1)$.

Extra Problem: Convolution is Continuous

Prove that the convolution of two continuous functions on the unit circle in continuous.

Let $\varepsilon > 0$. g is continuous on a compact set $[0, 2\pi]$, which implies g is uniformly continuous. Thus $\exists\delta$ such that

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \varepsilon$$

for $x, x_0 \in [0, 2\pi]$. Then let $|x - x_0| < \delta$ (which implies $|(x-y) - (x_0 - y)| < \delta$). Then

$$\begin{aligned} |(f*g)(x) - (f*g)(x_0)| &= \left|\int_0^{2\pi} f(y)g(x-y) - f(y)g(x_0-y)\mathrm{d}y\right| \\ &= \left|\int_0^{2\pi} f(y)[g(x-y) - g(x_0-y)]\mathrm{d}y\right| \\ &\leq \int_0^{2\pi} |f(y)|\ |g(x-y) - g(x_0-y)|\mathrm{d}y \\ &< \int_0^{2\pi} |f(y)|\varepsilon\mathrm{d}x \end{aligned}$$

But by the continuity of f, f is bounded on $[0, 2\pi]$ since $[0, 2\pi]$ is compact. Thus $|f(y)| \leq C$ for some $C \in \mathbb{R}^+$. Thus

$$\begin{aligned} |(f*g)(x) - (f*g)(x_0)| &< C\varepsilon \int_0^{2\pi} \mathrm{d}x \\ &= 2\pi C\varepsilon \end{aligned}$$

Since ε was arbitrary, this shows that $f * g$ is continuous.