HW #2

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January 22, 2015

Exercise 1.9

Verify the linearity of the integral as given in 1.5(7) by completing the steps outlined below. In what follows, f and g are nonnegative summable functions.

a)

Show that $f+g$ is also summable. In fact, by a simple argument $\int(f+g) \leq 2\big(\int f + \int g\big)$.

To show $\int(f+g) \leq 2\big(\int f + \int g\big)$, first note that

$$S_{f+g}(t) = \{x \ : \ (f+g)(x) > t\} \subset \left\{x \ : \ f(x) > \frac{t}{2}\right\} \cup \left\{x \ : \ g(x) > \frac{t}{2}\right\} = S_f\left(\frac{t}{2}\right) \cup S_g\left(\frac{t}{2}\right)$$

Since if $f(x) \leq \frac{t}{2}$ and $g(x) \leq \frac{t}{2}$ then $(f+g)(x) = f(x) + g(x) \leq t$. By properties of measures,

$$\mu(S_{f+g}(t)) \leq \mu\left(S_f\left(\frac{t}{2}\right) \cup S_g\left(\frac{t}{2}\right)\right) \leq \mu\left(S_f\left(\frac{t}{2}\right)\right) + \mu\left(S_g\left(\frac{t}{2}\right)\right)$$
$$\implies \int_0^\infty \mu(S_{f+g}(t))\mathrm{d}t \leq \int_0^\infty \mu\left(S_f\left(\frac{t}{2}\right)\right)\mathrm{d}t + \int_0^\infty \mu\left(S_g\left(\frac{t}{2}\right)\right)\mathrm{d}t$$

Note the integral on the right hand side can split linearly because it is a Riemann integral. By u-substitution with $u = \frac{t}{2}$, we get

$$\int_0^\infty \mu(S_{f+g}(t))\mathrm{d}t \leq 2\int_0^\infty S_f(t)\mathrm{d}t + 2\int_0^\infty S_g(t)\mathrm{d}t$$

Note the constant 2 can be factored of each integral on the right hand side linearly because they are Riemann integrals. Thus, by definition,

$$\int (f+g) \leq 2\left(\int f + \int g\right)$$

and since f and g are summable, $\int f$ and $\int g$ are finite, which proves $\int(f+g)$ is finite, i.e. $f+g$ is summable. $\square$

b)

For any integer N find two functions f_N and g_N that take only finitely many values, such that $|\int f - \inf f_N| \leq \frac{C}{N}$, $|\int g - \int g_N| \leq \frac{C}{N}$, $|int(f+g) - \int(f_N - g_N)| \leq \frac{C}{N}$ for some constant C independent of N.

c)

Show that for f_N and g_N as above $\int(f_N + g_N) = \int f_N + \int g_N$, thus proving the addivitivity of te integral for nonnegative functions.

d)

In a similar fashion, show that for $f, g \geq 0$, $\int(f-g) = \int f - \int g$.

e)

Now use c) and d) to prove the linearity of the integral.

Exercise 1.12

Find a simple condition for $f_n(x)$ so that

$$\sum_{n=0}^{\infty} \int_{\Omega} f_n(x)\mu(\mathrm{d}x) = \int_{\Omega} \left[\sum_{n=0}^{\infty} f_n(x)\right] \mu(\mathrm{d}x)$$

Exercise 1.13

Let f be the function on $\mathbb{R}^n$ defined by $f(x) = |x|^{-p}\mathcal{X}_{\{|x|<1\}}(x)$. Compute $\int f \mathrm{d}\mathcal{L}^n$ in two ways: (i) Use polar coordinates and compute the integral by the standard calculus method. (ii) Compute $\mathcal{L}^n(\{x \ : \ f(x) > a\})$ and then use Lebesgue's definition.

(i) First note that

$$f(x) = \begin{cases} |x|^{-p} & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}$$

Then note that polar coordinates on $\mathbb{R}^n$ are $(r, \phi, \theta_1, \theta_2, \ldots, \theta_{n-2})$ where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $\theta_i \in [0, \pi)$ for $i = 1, 2, \ldots, n-2$.

$$\int f \mathrm{d}\mathcal{L}^n = \int_0^{\pi} \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} r^{-p} \ \mathrm{d}r \ \mathrm{d}\phi \ \mathrm{d}\theta_1 \ldots \ \mathrm{d}\theta_{n-3} \ \mathrm{d}\theta_{n-2}$$

We can use Fubini's theorem since each of these integrals are Riemann integrals. Thus,

$$\int f \mathrm{d}\mathcal{L}^n = 2\pi^{n-1} \int_0^{\infty} r^{-p} \mathrm{d}r = 2\pi^{n-1} \int_0^1 r^{-p} \mathrm{d}r$$

since we know $f(x) = 0$ whenever $r = |x| \geq 1$. This integral is dependent on p in the following way:

$$\int f \mathrm{d}\mathcal{L}^n = \begin{cases} 2\pi^{n-1}\frac{1}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geq 1 \end{cases}$$

(ii) If $0 < p < 1$, f is a decreasing function of modulus and $f \to \infty$ as $x \to 0$. If $p < 0$, f is an increasing function of modulus and $f \to \infty$ as $|x| \to 1$. Thus it should be intuitive that $f^{-1}(a, \infty)$ is either a smaller n-sphere if $0 < p < 1$ or a shell of an n-sphere if $p < 0$.

$$\begin{aligned} \mathcal{L}^n(\{x \ : \ f(x) > a\}) &= \mathcal{L}^n(\{x \in B_1(0) \ : \ |x|^{-p} > a\}) \\ &= \begin{cases} \mathcal{L}^n(\{x \ \in B_1(0) \ : \ |x| < a^{-\frac{1}{p}}\}) & \text{if } 0 < p < 1 \\ \mathcal{L}^n(\{x \ \in B_1(0) \ : \ |x| > a^{-\frac{1}{p}}\}) & \text{if } p < 0 \end{cases} \\ &= \begin{cases} \mathcal{L}^n(B_{a^{-\frac{1}{p}}}(0)) & \text{if } 0 < p < 1 \\ \mathcal{L}^n(B_1(0)) - \mathcal{L}^n(B_{a^{-\frac{1}{p}}}(0)) & \text{if } p < 0 \end{cases} \end{aligned}$$

But the Lebesgue measures of balls are relatively simple to compute:

$$\mathcal{L}^n(B_r(x)) = \frac{2\pi^{\frac{n}{2}} r^n}{n\Gamma(\frac{n}{2})}$$

Thus,

$$\begin{aligned} \mathcal{L}^n(\{x \ : \ f(x) > a\}) &= \begin{cases} \dfrac{2\pi^{\frac{n}{2}} a^{-\frac{n}{p}}}{n\Gamma(\frac{n}{2})} & \text{if } 0 < p < 1 \\ \dfrac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} - \dfrac{2\pi^{\frac{n}{2}} a^{-\frac{n}{p}}}{n\Gamma(\frac{n}{2})} & \text{if } p < 0 \end{cases} \\ &= \begin{cases} \dfrac{2\pi^{\frac{n}{2}} a^{-\frac{n}{p}}}{n\Gamma(\frac{n}{2})} & \text{if } 0 < p < 1 \\ \dfrac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}\left(1 - a^{-\frac{n}{p}}\right) & \text{if } p < 0 \end{cases} \end{aligned}$$