UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #3

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Hunter and Nachtergaele 7.1

Let ϕ_n be the functions defined in (7.7)

$$\phi_n(x) = c_n(1+\cos x)^n$$

where c_n is chosen such that

$$\int_{\mathbb{T}} \phi_n(x)\mathrm{d}x = 1$$

for all n.

(a) *Prove (7.5).*

$$\lim_{n\to\infty}\int_{\delta\le|x|\le\pi}\phi_n(x)\mathrm{d}x = 0$$

for every $\delta > 0$.

Let $\delta > 0$ and for ease, define $\mathbb{D} = [-\pi,-\delta]\cup[\delta,\pi]$.

$$\int_{\mathbb{D}}\phi_n(x)\mathrm{d}x = \frac{\int_{\mathbb{D}}(1+\cos x)^n\mathrm{d}x}{\int_{\mathbb{T}}(1+\cos x)^n\mathrm{d}x}$$

since

$$c_n = \frac{1}{\int_{\mathbb{T}}(1+\cos x)^n\mathrm{d}x}$$

Note that

$$\phi_n'(x) = -nc_n(1+\cos x)^{n-1}\sin x$$

which is positive on $[-\pi,0)$ and negative on $(0,\pi]$, and thus

$$\max_{x\in\mathbb{D}}\phi_n(x) = \phi_n(\delta)$$

So,

$$\int_{\mathbb{D}}\phi_n(x)\mathrm{d}x = \frac{\int_{\mathbb{D}}(1+\cos x)^n\mathrm{d}x}{\int_{\mathbb{T}}(1+\cos x)^n\mathrm{d}x} \le \frac{2\pi(1+\cos\delta)^n}{\int_{\mathbb{E}}(1+\cos x)^n\mathrm{d}x}$$

where $\mathbb{E} = [-\frac{\delta}{2},\frac{\delta}{2}]$. Again, since ϕ_n is decreasing on $\left(0,\frac{\pi}{2}\right]$ and ϕ is an even function,

$$\min_{x\in\mathbb{E}}\phi_n(x) = \phi_n\left(\frac{\delta}{2}\right)$$

Thus,

$$\int_{\mathbb{D}}\phi_n(x)\mathrm{d}x \le \frac{2\pi(1+\cos\delta)^n}{\int_{\mathbb{E}}(1+\cos x)^n\mathrm{d}x} \le \frac{2\pi}{\delta}\left(\frac{1+\cos\delta}{1+\cos\frac{\delta}{2}}\right)^n$$

but

$$\frac{1+\cos\delta}{1+\cos\frac{\delta}{2}} < 1$$

since cos is a decreasing function on $[0,\pi]$. Thus,

$$\lim_{n\to\infty}\frac{2\pi}{\delta}\left(\frac{1+\cos\delta}{1+\cos\frac{\delta}{2}}\right)^n = 0$$

and by the comparison test,

$$\lim_{n\to\infty}\int_{\mathbb{D}}\phi_n(x)\mathrm{d}x = 0$$

(b) *Prove that if the set $\mathscr{P}$ of trigonometric polynomials is dense in the space of periodic continuous functions on $\mathbb{T}$ with the uniform norm, then $\mathscr{P}$ is dense in the space of all continuous functions on $\mathbb{T}$ with the L^2-norm.*

Let the set of trigonometric polynomials $\mathscr{P}$ be dense in the space of periodic continuous functions on $\mathbb{T}$ with the uniform norm. Then choose $f \in (C(\mathbb{T}), \|\cdot\|_\infty)$. Then

$$\exists (p_n)_n \in \mathscr{P} \text{ such that } \lim_{n\to\infty}\|p_n - f\|_\infty = 0$$

Choose $\varepsilon > 0$ and note $\exists N_\varepsilon$ such that $\|p_n - f\|_\infty < \varepsilon$ whenever $n > N_\varepsilon$. Then if $n \geq N_\varepsilon$,

$$\|p_n - f\|_{L^2}^2 = \int_{\mathbb{T}}|p_n(x) - f(x)|^2\mathrm{d}x \leq \int_{\mathbb{T}}\|p_n - f\|_\infty^2\mathrm{d}x = 2\pi\|p_n - f\|_\infty^2 < 2\pi\varepsilon^2$$

Thus for $n \geq N_\varepsilon$,

$$\|p_n - f\|_{L^2} < \sqrt{2\pi}\varepsilon$$

Since ε was arbitrary, this proves there is a sequence in $\mathscr{P}$ that converges with respect to the L^2-norm to an arbitrary continuous function on $\mathbb{T}$. Thus $\mathscr{P}$ is dense in $\left(C(\mathbb{T}), \|\cdot\|_{L^2}\right)$.

(c) *Is $\mathscr{P}$ dense in the space of all continuous functions on $[0,2\pi]$ with the uniform norm?*

No. Consider a continuous function f in which $f(0) \neq f(2\pi)$. Since any functions $p_n \in \mathscr{P}$ are 2π-periodic, then in order to approximate f either $p_n(0) = p_n(2\pi) = f(0)$ or $p_n(0) = p_n(2\pi) = f(2\pi)$. In either case,

$$\|p_n - f\|_\infty \geq |f(0) - f(2\pi)|$$

This cannot become arbitrarily small since $f(0) \neq f(2\pi)$.

Hunter and Nachtergaele 7.2

Suppose that $f : \mathbb{T} \to \mathbb{C}$ is a continuous function, and

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

is the N^{th} partial sum of its Fourier seriers.

(a) *Show that $S_N = D_N * f$, where D_N is the* Dirichlet kernel

$$D_N(x) = \frac{1}{2\pi} \frac{\sin\left[(N+\frac{1}{2})x\right]}{\sin\left(\frac{x}{2}\right)}.$$

For ease, let $\omega = e^{ix}$. Then note

$$\sum_{n=0}^{N} \omega^n = \frac{1-\omega^{N+1}}{1-\omega}, \quad \text{and} \quad \sum_{n=-N}^{-1} \omega^n = \frac{\omega^{-N}-1}{1-\omega}$$

Then

$$\begin{aligned}\frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} &= \frac{1}{2\pi}\sum_{n=-N}^{N}\omega^n = \frac{1}{2\pi}\frac{\omega^{-N}-\omega^{N+1}}{1-\omega} = \frac{1}{2\pi}\frac{\omega^{-N-\frac{1}{2}}-\omega^{N+\frac{1}{2}}}{\omega^{-\frac{1}{2}}-\omega^{\frac{1}{2}}}\\ &= \frac{1}{2\pi}\frac{\exp\left[ix[N+\frac{1}{2}]\right]-\exp\left[-ix[N+\frac{1}{2}]\right]}{\exp\left[ix[\frac{1}{2}]\right]-\exp\left[-ix[\frac{1}{2}]\right]} = \frac{1}{2\pi}\frac{\sin\left[[N+\frac{1}{2}]x\right]}{\sin\left[\frac{x}{2}\right]} = D_N(x)\end{aligned}$$

Then note

$$\begin{aligned}S_N &= \frac{1}{\sqrt{2\pi}}\sum_{n=-N}^{N}\hat{f}_n e^{inx}\\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=-N}^{N}\left[\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(y)e^{-iny}\mathrm{d}y\right]e^{inx}\\ &= \int_{-\pi}^{\pi} f(y)\left(\frac{1}{2\pi}\sum_{n=-N}^{N} e^{in(x-y)}\right)\mathrm{d}y\\ &= D_N * f\end{aligned}$$

(b) *Let T_N be the mean of the first $N+1$ partial sums,*

$$T_N = \frac{1}{N+1}(S_0 + S_1 + \cdots + S_N) = \frac{1}{N+1}\sum_{i=0}^{N} S_i(x).$$

Show that $T_N = F_N * f$, where F_N is the Fejér kernel

$$F_N(x) = \frac{1}{2\pi(N+1)}\left(\frac{\sin\left[(N+1)\frac{x}{2}\right]}{\sin\left(\frac{x}{2}\right)}\right)^2.$$

First note the following identity:

$$\begin{aligned}\frac{\sin^2\left[(N+1)\frac{x}{2}\right]}{\sin\left[\frac{x}{2}\right]} &= \frac{1-\cos\left[(N+1)x\right]}{2\sin\left[\frac{x}{2}\right]} \quad \text{by the power-reducing formulas}\\ &= \frac{1}{2\sin\left[\frac{x}{2}\right]}\Big([\cos(0x)-\cos(1x)] + [\cos(1x)-\cos(2x)] + \dots\end{aligned}$$

$$\begin{aligned}
&\cdots + [\cos((N-1)x) - \cos(Nx)] + [\cos(Nx) - \cos((N+1)x)]\Big) \\
&\qquad \text{using a telescoping series} \\
&= \frac{1}{2\sin\left[\frac{x}{2}\right]} 2\sin\left[\frac{x}{2}\right] \sum_{i=0}^{\infty} \sin\left[\frac{2i+1}{2}x\right] \\
&= \sum_{i=0}^{\infty} \sin\left[\frac{2i+1}{2}x\right]
\end{aligned}$$

Then note that

$$\begin{aligned}
F_N(x) &= \frac{1}{2\pi(N+1)}\left(\frac{\sin\left[(N+1)\frac{x}{2}\right]}{\sin\left[\frac{x}{2}\right]}\right)^2 \\
&= \frac{1}{2\pi(N+1)\sin\left[\frac{x}{2}\right]} \sum_{i=0}^{\infty} \sin\left[\frac{2i+1}{2}x\right] \\
&= \frac{1}{N+1}\sum_{i=0}^{N} \frac{1}{2\pi} \frac{\sin\left[(i+\frac{1}{2})x\right]}{\sin\left[\frac{x}{2}\right]} \\
&= \frac{1}{N+1}\sum_{i=0}^{N} D_i(x)
\end{aligned}$$

Lastly,

$$\begin{aligned}
T_N(x) &= \frac{1}{N+1}\sum_{i=0}^{N} S_i(x) \\
&= \frac{1}{N+1}\sum_{i=0}^{N} (D_i * f)(x) \quad \text{by part } \textbf{(a)} \\
&= \frac{1}{N+1}\sum_{i=0}^{N} \int_{\mathbb{T}} f(y) D_i(x-y)\mathrm{d}y \\
&= \int_{\mathbb{T}} f(y)\left[\frac{1}{N+1}\sum_{i=0}^{N} D_i(x-y)\right]\mathrm{d}y \\
&= \int_{\mathbb{T}} f(y) F_N(x-y)\mathrm{d}y \\
&= (F_N * f)(x)
\end{aligned}$$

(c) *Which of the families (D_N) and (F_N) are approximate identities as $N \to \infty$? What can you say about the uniform convergence of the partial sums S_N and the averaged partial sums T_N to f?*

We know (D_N) can not be an approximate identity since

$$D_3(\pi) = \frac{1}{2\pi} \cdot \frac{\sin\left[\frac{7}{2}\pi\right]}{\sin\left[\frac{\pi}{2}\right]} = -\frac{1}{2\pi} < 0$$

and each function in an approximate identity must be nonnegative on $[-\pi, \pi]$. We claim, however, that (F_N) is an approximate identity. First,

$$F_N(x) = \frac{1}{2\pi(N+1)}\left(\frac{\sin\left[(N+1)\frac{x}{2}\right]}{\sin\left[\frac{x}{2}\right]}\right)^2 \geq \frac{1}{2\pi(N+1)} > 0, \qquad \forall N \geq 0, \forall x \in \mathbb{T}$$

Next we show

$$\int_{\mathbb{T}} F_N(x)\mathrm{d}x = 1$$

for all $N \geq 0$.

$$\begin{aligned}
\int_{\mathbb{T}} F_N(x)\mathrm{d}x &= \frac{1}{N+1}\int_{\mathbb{T}} \sum_{j=0}^{N} D_j(x)\mathrm{d}x \\
&= \frac{1}{N+1}\int_{\mathbb{T}} \sum_{j=0}^{N}\left[\frac{1}{2\pi}\sum_{n=-j}^{j} e^{inx}\right]\mathrm{d}x \\
&= \frac{1}{2\pi(N+1)}\sum_{j=0}^{N}\sum_{n=-j}^{j}\int_{\mathbb{T}} e^{inx}\mathrm{d}x \qquad \text{since the sums are finite} \\
&= \frac{1}{2\pi(N+1)}\sum_{j=0}^{N}\left[2\pi + \sum_{\substack{n=-j \\ n\neq 0}}^{j}\left[\frac{1}{in}(\cos(nx)+i\sin(nx))\right]_{-\pi}^{\pi}\right] \\
&= \frac{1}{2\pi(N+1)}\sum_{j=0}^{N} 2\pi \\
&= \frac{2\pi(N+1)}{2\pi(N+1)} \\
&= 1
\end{aligned}$$

Lastly we show

$$\lim_{N\to\infty}\int_{\mathbb{D}} F_N(x)\mathrm{d}x = 0$$

where $\mathbb{D} = [-\pi,-\delta]\cup[\delta,\pi]$. However,

$$\begin{aligned}
\int_{\mathbb{D}} F_N(x)\mathrm{d}x &= \frac{1}{2\pi(N+1)}\int_{\mathbb{D}}\left(\frac{\sin\left[(N+1)\frac{x}{2}\right]}{\sin\left[\frac{x}{2}\right]}\right)^2\mathrm{d}x \\
&\leq \frac{1}{2\pi(N+1)}\int_{\mathbb{D}}\left(\frac{1}{\sin\left[\frac{\delta}{2}\right]}\right)^2\mathrm{d}x \\
&= \frac{\pi-\delta}{\pi(N+1)\sin^2\left[\frac{\delta}{2}\right]}
\end{aligned}$$

since $\sin\left[\frac{x}{2}\right]$ is a symmetric, increasing function on $[\delta,\pi]$. But the sequence

$$\frac{\pi-\delta}{\pi(N+1)\sin^2\left[\frac{\delta}{2}\right]} \to 0$$

as $N\to\infty$. Thus, by the comparison test,

$$\lim_{N\to\infty}\int_{\mathbb{D}} F_N(x)\mathrm{d}x = 0$$

This shows (F_N) is an approximate identity.

Hunter and Nachtergaele 7.3

Prove that the sets $\{e_n \mid n \geq 1\}$ *defined by*

$$e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx,$$

and $\{f_n \,:\, n \geq 1\}$ *defined by*

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \quad f_n(x) = \sqrt{\frac{2}{\pi}} \cos nx \quad \textit{for } n \geq 1,$$

are both orthonormal bases of $L^2([0,\pi])$.

First we show $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$ are orthonormal. Suppose $n \neq m$. Then

$$\begin{aligned}
\langle e_n, e_m \rangle &= \int_0^{\pi} e_n(x) e_m(x) \mathrm{d}x \\
&= \frac{2}{\pi} \int_0^{\pi} \sin(nx) \sin(mx) \mathrm{d}x \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [\cos(nx - mx) - \cos(nx + mx)] \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^{\pi} \cos((n-m)x) \mathrm{d}x - \frac{1}{\pi} \int_0^{\pi} \cos((n+m)x) \\
&= \frac{1}{\pi} \left[\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m} \right]_0^{\pi} \\
&= 0
\end{aligned}$$

Also,

$$\begin{aligned}
\langle e_n, e_n \rangle &= \frac{2}{\pi} \int_0^{\pi} \sin^2(nx) \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^{\pi} 1 - \cos(2nx) \mathrm{d}x \\
&= \frac{1}{\pi} \left[\pi - \frac{1}{2n} \sin(2n\pi) \right] \\
&= \frac{1}{\pi} \pi \\
&= 1
\end{aligned}$$

Thus $\{e_n\}_{n=1}^{\infty}$ is orthonormal. Let $n \geq 1$.

$$\begin{aligned}
\langle f_0, f_n \rangle &= \frac{\sqrt{2}}{\pi} \int_0^{\pi} \cos(nx) \mathrm{d}x \\
&= \frac{\sqrt{2}}{\pi} \frac{1}{n} \sin(nx) \Big|_0^{\pi} \\
&= 0
\end{aligned}$$

Let $1 \leq n < m$. Then

$$\langle f_n, f_m \rangle = \frac{2}{\pi} \int_0^{\pi} \cos(nx) \cos(mx) \mathrm{d}x$$

$$\begin{aligned}&= \frac{1}{\pi}\int_0^\pi [\cos((n-m)x)+\cos((n+m)x)]\mathrm{d}x\\ &= \frac{1}{\pi}\left(\frac{\sin((n-m)x)}{n-m}+\frac{\sin((n+m)x)}{n+m}\right)\Bigg|_0^\pi\\ &= 0\end{aligned}$$

Also,

$$\langle f_0, f_0\rangle = \frac{1}{\pi}\int_0^\pi \mathrm{d}x = \frac{\pi}{\pi} = 1$$

and for $n \geq 1$,

$$\begin{aligned}\langle f_n, f_n\rangle &= \frac{2}{\pi}\int_0^\pi \cos^2(nx)\mathrm{d}x\\ &= \frac{1}{\pi}\int_0^\pi (1+\cos(2nx))\mathrm{d}x\\ &= \frac{1}{\pi}\left[\pi + \left(\frac{1}{2}\sin(2nx)\right)_0^\pi\right]\\ &= 1\end{aligned}$$

Thue $\{f_n\}_{n=0}^\infty$ is orthonormal. Next we show $\{f_n\}_{n=0}^\infty$ and $\{e_n\}_{n=1}^\infty$ are each bases of $L^2[0,\pi]$.
Let $f \in L^2([0,\pi])$. Then extend f to its odd extension $f_{\text{odd}} \in L^2([-\pi,\pi])$ by

$$f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x \in (0,\pi]\\ 0 & \text{if } x = 0\\ -f(-x) & \text{if } x \in [-\pi, 0)\end{cases}$$

We know $\{e_n\}_{n=1}^\infty \cup \{f_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2[-\pi,\pi]$ and thus f_{odd} can be written as a Fourier series like so

$$f_{\text{odd}}(x) = \frac{1}{2}f_0 + \sum_{n=1}^\infty \left(a_n f_n + b_n e_n\right)$$

But since f_{odd} is constructed to be odd,

$$f_{\text{odd}}(x) = \sum_{n=1}^\infty b_n e_n$$

Thus on $[0,\pi]$,

$$f(x) = \sum_{n=1}^\infty e_n \sin(nx)$$

Thus $\{e_n\}_{n=1}^\infty$ is a basis of $L^2[0,\pi]$. Now extend f to its even extension $f_{\text{even}} \in L^2[-\pi,\pi]$ be

$$f_{\text{even}}(x) = \begin{cases} f(x) & \text{if } x \in [0,\pi]\\ f(-x) & \text{if } x \in [-\pi, 0)\end{cases}$$

Again, we know $\{e_n\}_{n=1}^\infty \cup \{f_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2[-\pi,\pi]$ and thus f_{even} can be written as a Fourier series like so

$$f_{\text{even}}(x) = \frac{1}{2}f_0 + \sum_{n=1}^\infty \left(a_n f_n + b_n e_n\right)$$

But since f_{even} is constructed to be even,

$$f_{\text{even}}(x) = \frac{1}{2}f_0 + \sum_{n=1}^{\infty} a_n f_n$$

Thus $\{f_n\}_{n=0}^{\infty}$ is a basis of $L^2[0,\pi]$.

Hunter and Nachtergaele 7.4

Let $T, S \in L^2(\mathbb{T})$ be the triangular and square wave, respectively, defined by

$$T(x) = |x|, \quad \text{if } |x| \leq \pi, \quad S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \end{cases}$$

(a) *Compute the Fourier series of T and S.*

Since T is an even function, we can represent T with a cosine series

$$T(x) = \frac{1}{2}\hat{T}_0 + \sum_{n=1}^{\infty} \hat{T}_n \cos(nx)$$

where

$$\hat{T}_0 = \frac{1}{\pi}\int_{\mathbb{T}} T(x)\mathrm{d}x \quad \text{and}$$
$$\hat{T}_n = \frac{1}{\pi}\int_{\mathbb{T}} T(x)\cos(nx)\mathrm{d}x, \quad n = 1, 2, \ldots$$

Because cos is even and T is even, $T\sin$ is even, and so

$$\hat{T}_0 = \frac{1}{\pi}\int_{\mathbb{T}} T(x)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\mathrm{d}x = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

and for $n = 1, 2, \ldots,$

$$\hat{T}_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x$$

Utilizing integration by parts, we find

$$\begin{aligned}
\hat{T}_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x \\
&= \frac{2}{\pi}\left[\left(\frac{x}{n}\sin(nx)\right)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\mathrm{d}x\right] \\
&= \frac{2}{\pi}\left[\frac{1}{n^2}\cos(nx)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Thus,

$$\boxed{T(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2}\cos((2n-1)x)\right]}$$

Since S is an odd function, we can represent S with a sin series

$$S(x) = \sum_{n=1}^{\infty}\hat{S}_n \sin(nx)$$

where

$$\hat{S}_n = \frac{1}{\pi}\int_{\mathbb{T}} S(x)\sin(nx)\mathrm{d}x, \quad n = 1, 2, \ldots$$

Because sin is odd and S is odd, $\sin S$ is even, and thus

$$\begin{aligned}
\hat{S}_n &= \frac{1}{\pi}\int_{\mathbb{T}} S(x)\sin(nx)\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi}\sin(nx)\mathrm{d}x \\
&= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} \\
&= -\frac{2}{\pi n}\left((-1)^n - 1\right) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Thus,

$$\boxed{S(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)}\sin((2n-1)x)\right]}$$

(b) *Show that $T \in H^1(\mathbb{T})$ and $T' = S$.*

First we turn $T(x)$ into a a Fourier series with $\{e^{inx}\}_{n\in\mathbb{Z}}$ as the basis using

$$\cos x = \frac{1}{2}\left[e^{ix} + e^{-ix}\right]$$

Thus,

$$\begin{aligned}
T(x) &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2}\cos((2n-1)x)\right] \\
&= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n\in\mathbb{Z}}\frac{\exp[i(2n-1)x]}{(2n-1)^2} \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{\pi^2}{\sqrt{2\pi}} - \frac{4}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\frac{\exp[i(2n-1)x]}{(2n-1)^2}\right]
\end{aligned}$$

To show $T \in H^1(\mathbb{T})$, we show

$$\sum_{n\in\mathbb{Z}} n^2|\hat{T}_n|^2 < \infty$$

but this is true because

$$\sum_{n\in\mathbb{Z}} n^2|\hat{T}_n|^2 = \frac{8}{\pi}\sum_{n\in\mathbb{Z}}\frac{(2n-1)^2}{(2n-1)^4} < \infty$$

by the comparison test. Thus $T \in H^1(\mathbb{T})$.

Next note that $S(x)$ can be turned into a Fourier series with $\{e^{inx}\}_{n\in\mathbb{Z}}$ asa basis by using the following:

$$\sin x = \frac{1}{2i}\left[e^{ix} - e^{-ix}\right]$$

Thus,

$$\begin{aligned} S(x) &= \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)}\sin((2n-1)x)\right] \\ &= -\frac{2i}{\pi}\sum_{n\in\mathbb{Z}}\frac{\exp[i(2n-1)x]}{2n-1} \end{aligned}$$

We can explicitly calculuate $in\hat{T}_n$ for each n:

$$T' = \frac{1}{\sqrt{2\pi}}\left[\frac{\pi^2}{\sqrt{2\pi}}(0i) - \frac{4}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}((2n-1)i)\frac{\exp[i(2n-1)x]}{(2n-1)^2}\right] = -\frac{2i}{\pi}\sum_{n\in\mathbb{Z}}\frac{\exp[i(2n-1)x]}{2n-1} = S$$

(c) *Show that $S \notin H^1(\mathbb{T})$.*

To show $S \notin H^1(\mathbb{T})$, we show

$$\sum_{n\in\mathbb{Z}} n^2|\hat{S}_n|^2 < \infty$$

but this is true because

$$\sum_{n\in\mathbb{Z}} n^2|\hat{S}_n|^2 = \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{(2n-1)^2}{(2n-1)^2} = \infty$$

by the n^{th} term test. Thus $S \notin H^1(\mathbb{T})$.

Hunter and Nachtergaele 7.5

Consider $f : \mathbb{T}^d \to \mathbb{C}$ defined by

$$f(x) = \sum_{n\in\mathbb{Z}^d} a_n e^{in\cdot x},$$

where $x = (x_1, x_2, \ldots, x_d)$, $n = (n_1, n_2, \ldots, n_d)$, and $n \cdot x = n_1x_1 + n_2x_2 + \cdots + n_dx_d$. Prove that if

$$\sum_{n\in\mathbb{Z}^d} |n|^{2k}|a_n|^2 < \infty$$

for some $k > \frac{d}{2}$, then f is continuous.

Let $f \in H^k(\mathbb{T}^d)$ with $k > \frac{1}{2}$. Define the partial sums S_N of the Fourier series of f by

$$S_N(x) = \sum_{([-N,N]\cap\mathbb{Z})^d} \hat{f}_n e^{in\cdot x}$$

and define the norm of the k^{th} weak derivative of f as

$$\left\| f^k \right\|^2 = \sum_{n\in\mathbb{Z}^d} |n|^{2k} |\hat{f}_n|^2$$

We will show the sequence $S_N \to f$ uniformly by showing $(S_N)_N$ is a Cauchy sequence and since $C(\mathbb{T}^d)$ is complete with respect to the supremum norm, this implies the limit of $(S_N)_N$ is contained in $C(\mathbb{T}^d)$.

$$\begin{aligned}
\|S_N - S_M\|_\infty &= \left\| \sum_{n\in((\pm N,\pm M]\cap\mathbb{Z})^d} \hat{f}_n e^{in\cdot x} \right\|_\infty \\
&\leq \sum_{n\in((\pm N,\pm M]\cap\mathbb{Z})^d} |\hat{f}_n||e^{in\cdot x}| \\
&\qquad \text{by the Triangle Inequality} \\
&= \sum_{n\in((\pm N,\pm M]\cap\mathbb{Z})^d} |\hat{f}_n| \\
&= \sum_{n\in((\pm N,\pm M]\cap\mathbb{Z})^d} |n|^k |\hat{f}_n| \frac{1}{|n|^k} \\
&\leq \sqrt{\sum_{n\in((\pm N,\pm M]\cap\mathbb{Z})^d} |n|^{2k} |\hat{f}_n^2|} \cdot \sqrt{\sum_{n\in((\pm N,\pm M]\cap\mathbb{Z})^d} \frac{1}{|n|^{2k}}} \\
&\qquad \text{by the Cauchy-Schwarz Inequality} \\
&\leq \left\| f^{(k)} \right\| \sqrt{\sum_{n\in((\pm N,\pm M]\cap\mathbb{Z})^d} \frac{1}{|n|^{2k}}} \\
&\qquad \text{since the Fourier transform is an isomorphism and thus preserves norm} \\
&\leq \left\| f^{(k)} \right\|_\infty \sqrt{|\mathbb{S}^{d-1}| \int_N^\infty \frac{r^{d-1}}{r^{2k}} \mathrm{d}r} \\
&\qquad \text{where } |\mathbb{S}^{d-1}| \text{ is the area of the unit sphere in } d \text{ dimensions} \\
&= \left\| f^{(k)} \right\|_\infty \sqrt{|\mathbb{S}^{d-1}|} \sqrt{\left. \frac{r^{d-2k}}{d-2k} \right|_N^\infty} \\
&= \begin{cases} \infty & \text{if } \frac{d}{2} \geq k \\ \|f^{(k)}\|_\infty \sqrt{|\mathbb{S}^{d-1}|} \left((2k-d)N^{2k-d}\right)^{-\frac{1}{2}} & \text{if } \frac{d}{2} < k \end{cases}
\end{aligned}$$

Supposing $\frac{d}{2} < k$,

$$\|S_N - S_M\|_\infty \leq \frac{\|f^{(k)}\|_\infty \sqrt{|\mathbb{S}^{d-1}|}}{\sqrt{(2k-d)N^{2k-d}}}$$

which goes to zero as $N \to \infty$. Thus $(S_N)_N$ is a Cauchy sequence and thus converges to a limit in $C(\mathbb{T}^d)$. But S_N are the partial sums of the Fourier series of f, and thus $S_N \to f$. Thus $f \in C(\mathbb{T}^d)$, i.e. f is continuous.

Hunter and Nachtergaele 7.6

Suppose that $f \in H^1([a,b])$ *and* $f(a) = f(b) = 0$. *Prove the* Poincaré inequality

$$\int_a^b |f(x)|^2 \mathrm{d}x \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \mathrm{d}x.$$

Let $f_{\text{odd}} \in H^1([a-(b-a),a])$ by

$$f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x \in [a,b] \\ f(a+(a-x)) & \text{if } x \in [a-(b-a),a] \end{cases}$$

We know $f_{\text{odd}} \in H^1([a,b])$ because of the boundary condition $f(a) = 0$. We can see that

$$\frac{1}{2}\int_{a-(b-a)}^b |f_{\text{odd}}(x)|^2 \mathrm{d}x = \int_a^b |f_{\text{odd}}(x)|^2 \mathrm{d}x = \int_a^b |f(x)|^2 \mathrm{d}x$$

and since $f, f_{\text{odd}} \in H^1$, their derivatives exist, and moreover,

$$\frac{1}{2}\int_{a-(b-a)}^b |f'_{\text{odd}}(x)|^2 \mathrm{d}x = \int_a^b |f'_{\text{odd}}(x)|^2 \mathrm{d}x = \int_a^b |f'(x)|^2 \mathrm{d}x$$

For ease, we define a linear transformation $L : [a-(b-a)] \to [-\pi,\pi]$ by

$$L(x) = \left(\frac{a\pi}{a+b}\right) + \left(\frac{\pi}{a+b}\right)x$$

and note that the Fourier coefficients of the odd extension f_{odd} are

$$\begin{aligned} \hat{f'_{\text{odd},n}} &= \int_{a-(b-a)}^b \exp[-inL(x)] f'_{\text{odd}}(x) \mathrm{d}x \\ &= \frac{\pi}{b-a} in \hat{f_{\text{odd},n}} \end{aligned}$$

Then by Parseval's Theorem,

$$\begin{aligned} \int_a^b |f(x)|^2 \mathrm{d}x &= \frac{1}{2}\int_{a-(b-a)}^b |f_{\text{odd}}(x)|^2 \mathrm{d}x \\ &= \frac{1}{2}\cdot 2(b-a) \sum_{n\in\mathbb{Z}} \left|\hat{f_{\text{odd},n}}\right|^2 \\ &\leq \frac{1}{2}\cdot 2(b-a) \sum_{n\in\mathbb{Z}} |-in\hat{f_{\text{odd},n}}|^2 \\ &= \frac{1}{2}\cdot 2(b-a) \sum_{n\in\mathbb{Z}} \left[\left(\frac{\pi}{b-a}\right)^2 \left|-in\hat{f_{\text{odd},n}}\right|^2 \left(\frac{b-a}{\pi}\right)^2\right] \\ &= \left(\frac{b-a}{\pi}\right)^2 \cdot \frac{1}{2}\cdot 2(b-a) \sum_{n\in\mathbb{Z}} \left|\hat{f'_{\text{odd},n}}\right|^2 \\ &= \left(\frac{b-a}{\pi}\right)^2 \cdot \frac{1}{2}\int_{a-(b-a)}^b |f'_{\text{odd}}(x)|^2 \mathrm{d}x \\ &= \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 \mathrm{d}x \end{aligned}$$

which proves the result.

Hunter and Nachtergaele 7.7

Solve the following initial-boundary value problem for the heat equation,

$$\begin{aligned} &u_t = u_{xx}, \\ &u(0,t) = 0, \quad u(L,t) = 0 \qquad \text{for } t > 0 \\ &u(x,0) = f(x) \qquad\qquad\qquad \text{for } 0 \le x \le L \end{aligned}$$

Suppose $u(x,t) = F(x)G(t)$ is a solution. Then

$$\begin{aligned} u_t &= u_{xx} \\ \implies F(x)G'(t) &= F''(x)G(t) \\ \implies \frac{F''(x)}{F(x)} &= \frac{G'(t)}{G(t)} \end{aligned}$$

Since the left hand side is a function of x and the right hand side is a function of t, they can only be equal if they are both constant, i.e.

$$\frac{F''(x)}{F(x)} = C = \frac{G'(t)}{G(t)}$$

for some $C \in \mathbb{R}$. Thus,

$$G'(t) - CG(t) = 0, \quad \text{and} \tag{0.1}$$
$$F''(x) - CF(x) = 0 \tag{0.2}$$

The solutions of (1) are

$$G(t) = c_1 e^{Ct}$$

Let $\lambda = \sqrt{C}$. If $C \neq 0$, the solutions of (2) are

$$F(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

The initial condition

$$u(0,t) = 0 \implies F(0)G(t) = 0 \implies F(0) = 0$$

provided u is not the trivial solution. Similarly,

$$F(L) = 0$$

If $C > 0$,

$$F(0) = 0 \implies 0 = c_1 + c_2 \implies F(x) = c_1\left(e^{\lambda x} - e^{-\lambda x}\right)$$

Also,

$$F(L) = 0 \implies 0 = c_1\left(e^{\lambda L} - e^{-\lambda L}\right) \implies c_1 = 0$$

Thus u is the trivial solution. If $C = 0$, then either $F'' = 0$ or $F \equiv 0$, but regardless, if $F'' = 0$, the initial conditions imply that $F \equiv 0$. So let $C < 0$ and define $\lambda = \sqrt{-C}$. Then

$$F(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x)$$

Then

$$F(0) = 0 \implies 0 = c_2 \implies F(x) = c_1 \sin(\lambda x)$$

Also,

$$F(L) = 0 \implies 0 = c_1 \sin(\lambda L) \implies \lambda L = \pi n$$

for integer values n. Thus $\lambda = \frac{n\pi}{L}$ for $n = \pm 1, \pm 2, \ldots$. Note $n \neq 0$ since that would imply $\lambda^2 = 0 = C$. Thus,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2}{L^2} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

The initial condition $u(0,x) = f(x)$ implies

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$$

This is a Fourier series, and thus the coefficients c_n are given by

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) \mathrm{d}x$$

Thus the full solution is

$$u(t,x) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L \left[f(x) \sin\left(\frac{n\pi}{L} x\right)\right] \mathrm{d}x \cdot \exp\left(-\frac{n^2\pi^2}{L^2} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right)\right)$$