UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #7

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Hunter and Nachtergaele 9.1

Prove that $\rho(A^*) = \overline{\rho(A)}$, *where* $\overline{\rho(A)}$ *is the set* $\{\lambda \in \mathbb{C} \mid \overline{\lambda} \in \rho(A)\}$.

Proof. First note

$$(A^* - \lambda I) = (A^* - (\overline{\lambda} I)^*) = (A - \overline{\lambda} I)^*,$$

and since $(A - \overline{\lambda} I) \in \mathscr{B}(\mathscr{H})$, then $(A - \overline{\lambda} I)$ is invertible if and only if $(A - \overline{\lambda} I)^*$ is invertible. Thus

$$\begin{aligned} \lambda \in \rho(A^*) &\iff (A^* - \lambda I) \text{ invertible} \\ &\iff (A - \overline{\lambda} I)^* \text{ invertible} \\ &\iff (A - \overline{\lambda} I) \text{ invertible} \\ &\iff \overline{\lambda} \in \rho(A) \\ &\iff \lambda \in \overline{\rho(A)} \end{aligned}$$

Thus, $\rho(A^*) = \overline{\rho(A)}$. □

Hunter and Nachtergaele 9.3

Suppose that A is a bounded linear operator on a Hilbert space and $\lambda, \mu \in \rho(A)$. Prove that the resolvent R_λ of A satisfies the resolvent equation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu.$$

Proof. If the resolvent R_λ is defined as $R_\lambda = (\lambda I - A)^{-1}$, then

$$\begin{aligned} \big(\mu I - A\big) - (\lambda I - A) &= (\mu - \lambda) I \\ \implies (\lambda I - A)^{-1}\big[\big(\mu I - A\big) - (\lambda I - A)\big]\big(\mu I - A\big)^{-1} &= (\lambda I - A)^{-1}\big[(\mu - \lambda) I\big]\big(\mu I - A\big)^{-1} \\ \implies (\lambda I - A)^{-1} - \big(\mu I - A\big)^{-1} &= \big(\mu - \lambda\big)(\lambda I - A)^{-1}\big(\mu I - A\big)^{-1} \\ \implies R_\lambda - R_\mu &= (\mu - \lambda) R_\lambda R_\mu \end{aligned}$$

□

Hunter and Nachtergaele 9.4

Prove that the spectrum of an orthogonal projection P is either $\{0\}$, *in which case* $P = 0$, *or* $\{1\}$, *in which case* $P = I$, *or else* $\{0, 1\}$.

Proof. Let λ be an eigenvalue. Then $Px = \lambda x$ for some nonzero vector x. Clearly if $P \equiv 0$, then $0 = Px = \lambda x$ for some $x \neq 0$, which implies $\lambda = 0$. Also, if $P \equiv I$, then $x = Px = \lambda x \implies (1-\lambda)x = 0$ for some $x \neq 0$. Thus $\lambda = 1$. In general, suppose $P \not\equiv 0$ and $P \not\equiv 1$, then for $x \in \text{ran } P$, $x = Px = \lambda x \implies \lambda = 1$. For $x \notin \text{ran } P$, then since P is an orthogonal projection, $x = y + z$ for some $y \in \text{ran } P$ (i.e. $Py = y$) and $z \in \ker P$ (i.e. $Pz = 0$). Thus $Py + Pz = y = \lambda x$. Since $x \notin \text{ran } P$, $\lambda x \in \text{ran } P$ only if $\lambda = 0$. Thus the only eigenvalues of P are 0 and 1 (i.e. the point spectrum of P is contained in $\{0,1\}$).
Since orthogonal projections are bounded and self adjoint, then the residual specturm of P is empty.
Let $a \in \text{ran } P$. Then $(1-\lambda)a \in \text{ran } (P-\lambda I)$ (since $(P-\lambda I)a = Pa - \lambda a = (1-\lambda)a$). If $\lambda \neq 1$ then $a \in \text{ran } (P-\lambda I)$ since ran $(P-\lambda I)$ is closed under scalar multiplication. Let $b \in \ker P$. Then $-\lambda b \in \text{ran } (P-\lambda I)$ (since $(P-\lambda I)b = Pb - \lambda b = -\lambda b$). If $\lambda \neq 0$, then $b \in \text{ran } (P-\lambda I)$ since ran $(P-\lambda I)$ is closed under scalar multiplication. Thus for $\lambda \in \mathbb{C} \setminus \{0,1\}$,

$$\text{ran } P \cup \ker P \subset \text{ran } (P-\lambda I)$$

Since P is an orthogonal projection,

$$\mathscr{H} \subset \text{ran } (P-\lambda I) \subset \mathscr{H}$$

and thus ran $(P-\lambda I)$ is closed. Thus the continuous specturm of P is empty.
Thus, $\sigma(P) = \{0,1\}$. □

Hunter and Nachtergaele 9.5

Let A be a bounded, nonnegative operator on a complex Hilbert space. Prove that $\sigma(A) \subset [0,\infty)$.

Proof. Since A is nonnegative, then $\langle x, Ax \rangle \geq 0$ for all $x \in \mathscr{H}$ and $A = A^*$. Since A is self-adjoint, its eigenvalues are real and $\sigma(A) \subset [-\|A\|, \|A\|]$. Let λ be an eigenvalue. Then for some $x \neq 0$,

$$\begin{aligned} 0 \leq \langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle &= \lambda \|x\|^2 \\ &\implies 0 \leq \lambda \end{aligned}$$

Thus all eigenvalues are positive (i.e. the point spectrum is contained in $[0,\infty)$). Let $\lambda \in$ continuous spectrum. Then ran $(A-\lambda I)$ is dense $\in \mathscr{H}$. Then there is a sequence $(y_n) \in \text{ran } (A-\lambda I)$ such that $y_n \to 0$. Then there is a sequence $(x_n) \in \mathscr{H}$ such that $(A-\lambda I)x_n = y_n$. Then

$$\lim_{n\to\infty} \langle x_n, (A-\lambda I)x_n \rangle = \lim_{n\to\infty} \langle x_n, y_n \rangle = \left\langle \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n \right\rangle = \left\langle \lim_{n\to\infty}, 0 \right\rangle = 0.$$

Also,

$$\langle x_n, (A-\lambda I)x_n \rangle = \langle x_n, Ax_n \rangle - \lambda \|x_n\|^2 = b_n - \lambda \|x_n\|^2.$$

where $b_n = \langle x_n, Ax_n \rangle \geq 0$ for all n. But

$$b_n - \lambda \|x_n\|^2 \to 0$$

and $\|x_n\| \geq 0$ for all n, which implies $\lambda \geq 0$. Thus the continuous spectrum is contained in $[0,\infty)$. Thus,

$$\sigma(A) \subset [0,\infty).$$

□

Hunter and Nachtergaele 9.6

Let G be a multiplication operator on $L^2(\mathbb{R})$ defined by

$$Gf(x) = g(x)f(x),$$

where g is continuous and bounded. Prove that G is a bounded linear operator on $L^2(\mathbb{R})$ and that its spectrum is given by

$$\sigma(G) = \overline{\{g(x) \,:\, x \in \mathbb{R}\}}.$$

Can an operator of this form have eigenvalues?

Proof. Let $\lambda \notin \overline{\{g(x) \,:\, x \in \mathbb{R}\}}$. Then $\exists \varepsilon$ such that $|g(x) - \lambda| > \varepsilon$ for all $x \in \mathbb{R}$. Thus $\frac{1}{g(x)-\lambda}$ is well-defined and we can define the inverse of $(G - \lambda I)$ as

$$\left((G-\lambda I)^{-1} f\right)(x) = \frac{1}{g(x)-\lambda} f(x)$$

because $\forall f \in L^2(\mathbb{R})$,

$$\left((G-\lambda I)(G-\lambda I)^{-1}\right) f = \left((G-\lambda I)^{-1}(G-\lambda I)\right) f = f$$

Thus $\lambda \in \rho(G)$, i.e. $\lambda \notin \sigma(G)$, which shows

$$\sigma(G) \subset \overline{\{g(x) \,:\, x \in \mathbb{R}\}}$$

Next, consider $\lambda \in \{g(x) \,:\, x \in \mathbb{R}\}$. Then $\exists x_0 \in \mathbb{R}$ such that $g(x_0) = \lambda$. Then consider the characteristic function on $\{x \,:\, |x - x_0| < 1\}$,

$$\mathscr{X}(x) = \begin{cases} 1 & \text{if } |x - x_0| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathscr{X} \notin \operatorname{ran}(G - \lambda I)$ since the only candidate function $\mathscr{C}$ to map to $\mathscr{X}$ is

$$\mathscr{C}(x) = \frac{\mathscr{X}(x)}{g(x) - \lambda}$$

but this function is not square-integrable, i.e. $\mathscr{C} \notin L^2(\mathbb{R})$. Thus $(G - \lambda I)$ is not surjective, which shows $\lambda \in \sigma(G)$. Thus

$$\{g(x) \,:\, x \in \mathbb{R}\} \subset \sigma(G)$$

However, since $\sigma(G)$ is closed (all spectrums are closed), then any closure of a subset of $\sigma(G)$ is also a subset of $\sigma(G)$. Thus

$$\overline{\{g(x) \,:\, x \in \mathbb{R}\}} \subset \sigma(G)$$

which shows

$$\overline{\{g(x) \,:\, x \in \mathbb{R}\}} = \sigma(G)$$

It is possible for G to have eigenvalues. Consider a function g and $\lambda \in \mathbb{R}$ such that $\mu\big(\{x : g(x) = \lambda\}\big) > 0$. Then λ is an eigenvalue of G and any function f such that $\operatorname{supp} f \subset \{x : g(x) = \lambda\}$ is an eigenvector with respect to λ since

$$(Gf)(x) = g(x)f(x) = \mathscr{X}_{\operatorname{supp} f}\, g(x)f(x) + \cancelto{0}{\mathscr{X}_{\mathbb{R}\setminus\operatorname{supp} f}\, g(x)f(x)} = \lambda f(x)$$

$\square$

Hunter and Nachtergaele 9.7

Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\,\mathrm{d}y.$$

a) *Find the adjoint operator K^*.*

Proof.

$$\begin{aligned}
\langle f, Kg\rangle &= \int_0^1 \overline{f}(x)\int_0^x g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \overline{f}(x)\int_0^1 g(y)\mathscr{X}_{0<y<x<1}\,\mathrm{d}y\,\mathrm{d}x, \qquad \text{where } \mathscr{X} \text{ is the characteristic function} \\
&= \int_0^1\int_0^1 \overline{f}(x)g(y)\mathscr{X}_{0<y<x<1}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 g(y)\int_0^1 \overline{f}(x)\mathscr{X}_{0<y<x<1}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left[\int_y^1 \overline{f}(x)\,\mathrm{d}x\right] g(y)\,\mathrm{d}y \\
&= \int_0^1 \overline{\left[\int_y^1 f(x)\,\mathrm{d}x\right]}\, g(y)\,\mathrm{d}y \\
&= \langle K^* f, g\rangle
\end{aligned}$$

Thus,

$$K^* f(x) = \int_x^1 f(y)\,\mathrm{d}y$$

$\square$

b) *Show that $\|K\| = \frac{2}{\pi}$.*

Proof. Define $f \in L^2([0,1])$ by

$$f(x) = \sqrt{2}\cos\frac{\pi}{2}x.$$

Note $\|f\|_{L^2} = 1$ since

$$\int_0^1 \cos^2 \frac{\pi}{2} x = \int_0^1 \frac{1+\cos \pi x}{2} \mathrm{d}x = \frac{1}{2}$$

Then

$$\|Kf\|^2 = \int_0^1 \left(\int_0^x \sqrt{2} \cos \frac{\pi}{2} y \mathrm{d}y \right)^2 \mathrm{d}x = 2 \int_0^1 \sin^2 \frac{\pi}{2} x \mathrm{d}x = \frac{4}{\pi^2}$$

and thus $\|K\| \geq \dfrac{2}{\pi}$.

Let $\|f\| = 1$. Then

$$\begin{aligned}
\|Kf\|^2 &= \int_0^1 (Kf)^2 \mathrm{d}x = \int_0^1 \left(\int_0^x |f(y)| \mathrm{d}y \right)^2 \mathrm{d}x = \int_0^1 \left(\int_0^x \sqrt{\cos \frac{\pi}{2} y} \frac{|f(y)|}{\sqrt{\cos \frac{\pi}{2}}} \mathrm{d}y \right)^2 \mathrm{d}x \\
&\leq \int_0^1 \left(\int_0^1 \cos \frac{\pi}{2} y \mathrm{d}y \right) \left(\int_0^x \frac{|f(y)|^2}{\cos \frac{\pi}{2} y} \right) \mathrm{d}x = \frac{2}{\pi} \int_0^1 \sin\left(\frac{\pi}{2} x\right) \left(\int_0^x \frac{|f(y)|^2}{\cos \frac{\pi}{2} y} \mathrm{d}y \right) \mathrm{d}x \\
&= \frac{2}{\pi} \int_0^1 \frac{|f(y)|^2}{\cos \frac{\pi}{2} y} \left(\int_y^1 \sin\left(\frac{\pi}{2} x\right) \mathrm{d}x \right) \mathrm{d}y = \frac{4}{\pi^2} \int_0^1 |f(y)|^2 \mathrm{d}y \\
&= \frac{4}{\pi^2} \|f\|^2 = \frac{4}{\pi^2}
\end{aligned}$$

Thus $\|K\| \leq \dfrac{2}{\pi}$. Thus,

$$\|K\| = \frac{2}{\pi}.$$

□

c) *Show that the spectral radius of K is equal to zero.*

Proof. Let $|\lambda| > 0$ and note $L^2([0,1]) = \overline{\text{ran }(K - \lambda I)} \oplus \ker(K - \lambda I)^*$. Note $(K - \lambda I)^* = (K^* - \overline{\lambda} I)$ where K^* is defined above. If $(K^* - \overline{\lambda} I) f_1 = (K^* - \overline{\lambda} I) f_2$ then $\forall x \in [0,1]$,

$$\begin{aligned}
\int_x^1 f_1(y) \mathrm{d}y &= \int_x^1 f_2(y) \mathrm{d}y \\
\implies F_1(1) - F_1(x) &= F_2(1) - F_2(x)
\end{aligned}$$

where $F_1' = f_1$ and $F_2' = f_2$. Thus $f_1 \equiv f_2$. This shows $(K^* - \overline{\lambda} I)$ is injective, which shows $\ker(K^* - \overline{\lambda} I) = \{0\}$. Thus,

$$L^2([0,1]) = \overline{\text{ran }(K - \lambda I)}$$

i.e. ran $(K - \lambda I)$ is dense in $L^2([0,1])$. Thus showing ran $(K - \lambda I)$ is closed will imply it is surjective.

Let $(g_n)_n \in$ ran $(K - \lambda I)$ be such that $g_n \to g$. Then $\exists f_n$ such that $(K - \lambda I) f_n = g_n$. Since $g_n \to g$, then by continuity (boundedness) of $(K - \lambda I)$, $\exists f$ such that $f_n \to f$. Then

$$\begin{aligned}
(K - \lambda I) f &= \int_0^x f(y) \mathrm{d}y - \lambda f(x) \\
&= \int_0^x \lim_{n\to\infty} f_n(y) \mathrm{d}y - \lambda \lim_{n\to\infty} \lambda f_n(x)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty}\left[\int_0^x f_n(y)\,\mathrm{d}y - \lambda f_n(x)\right], \qquad \text{by Lebesgue's Dominated Convergence Theorem}\\
&= \lim_{n\to\infty}(K-\lambda I)f_n\\
&= \lim_{n\to\infty} g_n\\
&= g
\end{aligned}$$

Thus $g \in \operatorname{ran}(K-\lambda I)$, i.e. $\operatorname{ran}(K-\lambda I)$ is closed, and since $\operatorname{ran}(K-\lambda I)$ is dense in $L^2([0,1])$, then $(K-\lambda I)$ is surjective. Thus $(K-\lambda I)$ is bijective, proving $\lambda \in \rho(K)$. Thus the spectral radius of K, $r(K)$, is equal to 0. □

d) *Show that* 0 *belongs to the continuous spectrum of* K.

Proof. K is not onto since K is the integral operator, and thus the range of K is equal to the set of differentiable functions. However, not all functions in L^2 are differentiable. Thus K is not onto. However, differentiable functions are dense in L^2, and thus 0 is in the continuous spectrum of K. □

Hunter and Nachtergaele 9.8

Define the right shift operator S *on* $\ell^2(\mathbb{Z})$ *by*

$$S(x)_k = x_{k-1} \qquad \textit{for all } k \in \mathbb{Z},$$

where $x = (x_k)_{k=-\infty}^{\infty}$ *is in* $\ell^2(\mathbb{Z})$. *Prove the following facts.*

a) *The point spectrum of* S *is empty.*

Proof. Suppose λ is in the point specturm of S. The for $Sx = \lambda x$ for some nonzero $x \in \ell^2(\mathbb{Z})$. If $\lambda = 0$, the $x \equiv 0$, which is a contradiction. If $\lambda = 1$, then $x_k = e_j$ for all $k, j \in \mathbb{Z}$, i.e. x is constant. However constant bi-infinite sequences are not in ℓ^2 unless they are uniformly 0. This is a contradiction since eigenvectors are nonzero. If $|\lambda| > 1$ and $0 < |\lambda| < 1$, then for all $kin\mathbb{Z}$, x_k, $Sx_k = \lambda x_{k-1}$, and thus for all $n \in \mathbb{Z}$,

$$x_k = \lambda^{k-n} x_n, \qquad \forall n \in \mathbb{Z}.$$

Thus x_k can be made arbitrarily large, which is a contradiction since this is true for all $k \in \mathbb{Z}$. Thus there are no eigenvalues of S (i.e. the point spectrum is empty). □

b) $\operatorname{ran}(\lambda I - S) = \ell^2(\mathbb{Z})$ *for every* $\lambda \in \mathbb{C}$ *with* $|\lambda| > 1$.

Proof. If $\|(x)_n\| = 1$, then $\|S(x)_n\| = \|(x)_{n+1}\| = \|(x)_n\| = 1$. Thus $\|S\| = 1$. Then any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$ has $\lambda \in \rho(S)$, and thus $\lambda I - S$ is bijective. Thus $\operatorname{ran}(\lambda I - S) = \ell^2(\mathbb{Z})$. □

c) $\operatorname{ran}(\lambda I - S) = \ell^2(\mathbb{Z})$ *for every* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$.

Proof. Let $(y_n) \in \ell^2(\mathbb{Z})$. Then since $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})$, let $\mathscr{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be an isomorphism. So $\exists (a_n)_n$ such that $\mathscr{F}((y_n)) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$. Then

$$\mathscr{F}(S(y)_n) = \tilde{S}\left(\sum_{n\in\mathbb{Z}} a_n e^{inx}\right) = e^{ix}\sum_{n\in\mathbb{Z}} a_n e^{inx} = \sum_{n\in\mathbb{Z}} a_n e^{i(n+1)x}$$

where $\tilde{S}$ is the shift operator in $L^2(\mathbb{T})$ ($\tilde{S} = \mathscr{F}\circ S$). Let $|\lambda| < 1$. Then $\left(\tilde{S} - \lambda I\right)g = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ where g is defined as

$$g = \frac{\sum_{n\in\mathbb{Z}} a_n e^{inx}}{e^{ix} - \lambda}$$

since

$$\left(\tilde{S} - \lambda I\right)\left(\frac{\sum_{n\in\mathbb{Z}} a_n e^{inx}}{e^{ix} - \lambda}\right) = \frac{e^{ix}\sum_{n\in\mathbb{Z}} a_n e^{inx} - \lambda\sum_{n\in\mathbb{Z}} a_n e^{inx}}{e^{ix} - \lambda} = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

Thus $(\tilde{S} - \lambda I)$ is surjective, which shows $(S - \lambda I)$ is surjective. □

d) The spectrum of S consists of the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and is purely continuous.

Proof. If $|\lambda| = 1$, then $\lambda = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then $\left(\tilde{S} - \lambda I\right)$ is a multiplication operator, $\left(\left(\tilde{S} - \lambda I\right)f\right)(x) = \left(e^{ix} - e^{i\theta}\right)f(x)$. Let $f \in L^2(\mathbb{T})$. Define $g_n \in L^2(\mathbb{T})$ by

$$g_n(x) = \begin{cases} \dfrac{f}{e^{ix} - e^{i\theta}} & \text{if } x \in [0, 2\pi) \setminus [\theta - \varepsilon_n, \theta + \varepsilon_n] \\ 0 & \text{otherwise} \end{cases}$$

Then define $f_n = \left(\tilde{S} - \lambda I\right)g_n$. Then

$$f_n = \begin{cases} f & \text{if } x \in [0, 2\pi) \setminus [\theta - \varepsilon_n, \theta + \varepsilon_n] \\ 0 & \text{otherwise} \end{cases}$$

Let $\varepsilon_n \to 0$. Then

$$\left\|f_n - f\right\|^2 = \cancelto{0}{\int_0^{\theta-\varepsilon_n} |f - f|^2 \mathrm{d}x} + \int_{\theta-\varepsilon_n}^{\theta+\varepsilon_n} |0 - f|^2 \mathrm{d}x + \cancelto{0}{\int_{\theta+\varepsilon_n}^{2\pi} |f - f|^2 \mathrm{d}x} = \int_{\theta-\varepsilon_n}^{\theta+\varepsilon_n} f^2 \mathrm{d}x \to 0 \text{ as } \varepsilon_n \to 0$$

Thus $f_n \to f$ in $L^2(\mathbb{T})$. Thus $\operatorname{ran}\left(\tilde{S} - \lambda I\right)$ is dense in $L^2(\mathbb{T})$. However $\left(\tilde{S} - \lambda I\right)$ is not surjective since the only candidate function g that would be mapped by $\left(\tilde{S} - \lambda I\right)$ to f is

$$g(x) = \frac{f}{e^{ix} - e^{i\theta}}$$

but $g \notin L^2(\mathbb{T})$. Thus for $|\lambda| = 1$, $\lambda \in \sigma(\tilde{S})$ and the spectrum is purely continuous. □

Hunter and Nachtergaele 9.18

Let $P_1, \ldots, P_N$ be orthogonal projections with orthogonal ranges. Let

$$A = \sum_{n=1}^{N} \lambda_n P_n$$

be a finite linear combination of these projections. Let $\tilde{f} : \sigma(A) \to \mathbb{C}$ be a continuous function and define $f : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ by

$$f(A) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_n) P_n. \tag{9.23}$$

Suppose that A is a compact self-adjoint operator. Let $f \in C(\sigma(A))$ and consider $f(A)$ defined by (9.23). Prove that

$$\|f(A)\| = \sup\{|\tilde{f}(\lambda_n) \mid n \in \mathbb{N}\}.$$

Let $(\tilde{q}_N)$ be a sequence of polynomials of degree N, converging uniformly to $\tilde{f}$ on $\sigma(A)$. The existence of such a sequence is a consequence of the Weierstrass approximation theorem. Prove that $(q_N(A))$ converges in norm, and that its limit equals $f(A)$ as defined in (9.23).

Proof. The definition of operator norm guarantees that $\forall \varepsilon, \exists u$ with $\|u\| = 1$ such that $\|f(A)\|_{\text{op}} \leq \|f(A)u\| + \varepsilon$. But by the spectral theorem,

$$u = \sum_{n=1}^{\infty} \langle \phi_n, u \rangle \phi_n$$

where $\{\phi_n\}$ is an orthonormal basis of eigenvectors of $f(A)$ with eigenvalues λ_n, respectively. Thus, for any $\varepsilon > 0$,

$$\begin{aligned}
\|f(A)\|_{\text{op}} &\leq \|f(A)u\| + \varepsilon \\
&= \left\| f(A) \sum_{n=1}^{\infty} \langle \phi_n, u \rangle \phi_n \right\| + \varepsilon
\end{aligned}$$

However, since $f(A) \in \mathscr{B}(\mathscr{H})$, then

$$\begin{aligned}
\left\| f(A) \sum_{n\in\mathbb{Z}} \langle \phi_n, u \rangle \phi_n \right\|^2 = \left\| \sum_{n\in\mathbb{Z}} \langle \phi_n, u \rangle f(A)\phi_n \right\|^2 &= \left\| \sum_{n\in\mathbb{Z}} \langle \phi_n, u \rangle \lambda_n \phi_n \right\|^2 \\
&\leq \sup_{n\in\mathbb{Z}} \{|\lambda_n|\}^2 \left\| \sum_{n\in\mathbb{Z}} \langle \phi_n, u \rangle \phi_n \right\|^2 \\
&= \sup_{n\in\mathbb{Z}} \{|\lambda_n|\}^2 \|u\|^2, \qquad \text{by Parseval's Identity} \\
&= \sup_{n\in\mathbb{Z}} \{|\lambda_n|\}^2
\end{aligned}$$

Since ε is arbitrarily small,

$$\|f(A)\|_{\text{op}} \leq \sup_{n\in\mathbb{Z}} \{|\lambda_n|\}$$

Also note, by the Spectral Mapping Theorem, that $\sigma(f(A)) = \tilde{f}(\sigma(A))$, and thus

$$\sup_{n\in\mathbb{Z}} \{|\lambda_n|\} = r(f(A)) \leq \|f(A)\|_{\text{op}},$$

which proves

$$\|f(A)\|_{\text{op}} = \sup_{n\in\mathbb{Z}} \{|\lambda_n|\}$$

$\square$