

HW #2

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Hunter and Nachtergaele 7.1

Let ϕ_n be the functions defined in (7.7)

$$\phi_n(x) = c_n(1 + \cos x)^n$$

where c_n is chosen such that

$$\int_{\mathbb{T}} \phi_n(x) dx = 1$$

for all n .

(a) Prove (7.5).

$$\lim_{n \rightarrow \infty} \int_{\delta \leq |x| \leq \pi} \phi_n(x) dx = 0$$

for every $\delta > 0$.

Let $\delta > 0$ and for ease, define $\mathbb{D} = [-\pi, -\delta] \cup [\delta, \pi]$.

$$\int_{\mathbb{D}} \phi_n(x) dx = \frac{\int_{\mathbb{D}} (1 + \cos x)^n dx}{\int_{\mathbb{T}} (1 + \cos x)^n dx}$$

since

$$c_n = \frac{1}{\int_{\mathbb{T}} (1 + \cos x)^n dx}$$

Note that

$$\phi'_n(x) = -nc_n(1 + \cos x)^{n-1} \sin x$$

which is positive on $[-\pi, 0)$ and negative on $(0, \pi]$, and thus

$$\max_{x \in \mathbb{D}} \phi_n(x) = \phi_n(\delta)$$

So,

$$\int_{\mathbb{D}} \phi_n(x) dx = \frac{\int_{\mathbb{D}} (1 + \cos x)^n dx}{\int_{\mathbb{T}} (1 + \cos x)^n dx} \leq \frac{2\pi(1 + \cos \delta)^n}{\int_{\mathbb{T}} (1 + \cos x)^n dx}$$

where $\mathbb{E} = [-\frac{\delta}{2}, \frac{\delta}{2}]$. Again, since ϕ_n is decreasing on $(0, \frac{\pi}{2}]$ and ϕ is an even function,

$$\min_{x \in \mathbb{E}} \phi_n(x) = \phi_n\left(\frac{\delta}{2}\right)$$

Thus,

$$\int_{\mathbb{D}} \phi_n(x) dx \leq \frac{2\pi(1 + \cos \delta)^n}{\int_{\mathbb{E}} (1 + \cos x)^n dx} \leq \frac{2\pi}{\delta} \left(\frac{1 + \cos \delta}{1 + \cos \frac{\delta}{2}} \right)^n$$

but

$$\frac{1 + \cos \delta}{1 + \cos \frac{\delta}{2}} < 1$$

since \cos is a decreasing function on $[0, \pi]$. Thus,

$$\lim_{n \rightarrow \infty} \frac{2\pi}{\delta} \left(\frac{1 + \cos \delta}{1 + \cos \frac{\delta}{2}} \right)^n = 0$$

and by the comparison test,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{D}} \phi_n(x) dx = 0$$

- (b) *Prove that if the set \mathcal{P} of trigonometric polynomials is dense in the space of periodic continuous functions on \mathbb{T} with the uniform norm, then \mathcal{P} is dense in the space of all continuous functions on \mathbb{T} with the L^2 -norm.*
- (c) *Is \mathcal{P} dense in the space of all continuous functions on $[0, 2\pi]$ with the uniform norm?*

Hunter and Nachtergaele 7.2

Suppose that $f : \mathbb{T} \rightarrow \mathbb{C}$ is a continuous function, and

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^N \hat{f}_n e^{inx}$$

is the N^{th} partial sum of its Fourier series.

- (a) *Show that $S_N = D_N * f$, where D_N is the Dirichlet kernel*

$$D_N(x) = \frac{1}{2\pi} \frac{\sin \left[\left(N + \frac{1}{2} \right) x \right]}{\sin \left(\frac{x}{2} \right)}.$$

For ease, let $\omega = e^{ix}$. Then note

$$\sum_{n=0}^N \omega^n = \frac{1 - \omega^{N+1}}{1 - \omega}, \quad \text{and} \quad \sum_{n=-N}^{-1} \omega^n = \frac{\omega^{-N} - 1}{1 - \omega}$$

Then

$$\begin{aligned}\frac{1}{2\pi} \sum_{n=-N}^N e^{inx} &= \frac{1}{2\pi} \sum_{n=-N}^N \omega^n = \frac{1}{2\pi} \frac{\omega^{-N} - \omega^{N+1}}{1 - \omega} = \frac{1}{2\pi} \frac{\omega^{-N-\frac{1}{2}} - \omega^{N+\frac{1}{2}}}{\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}}} \\ &= \frac{1}{2\pi} \frac{\exp[ix[N + \frac{1}{2}]] - \exp[-ix[N + \frac{1}{2}]]}{\exp[ix[\frac{1}{2}]] - \exp[-ix[\frac{1}{2}]]} = \frac{1}{2\pi} \frac{\sin[(N + \frac{1}{2})x]}{\sin[\frac{x}{2}]} = D_N(x)\end{aligned}$$

Then note

$$\begin{aligned}S_N &= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^N \hat{f}_n e^{inx} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^N \left[\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \right] e^{inx} \\ &= \int_{-\pi}^{\pi} f(y) \left(\frac{1}{2\pi} \sum_{n=-N}^N e^{in(x-y)} \right) dy \\ &= D_N * f\end{aligned}$$

(b) Let T_N be the mean of the first $N + 1$ partial sums,

$$T_N = \frac{1}{N+1} (S_0 + S_1 + \cdots + S_N) = \frac{1}{N+1} \sum_{i=0}^N S_i(x).$$

Show that $T_N = F_N * f$, where F_N is the Fejér kernel

$$F_N(x) = \frac{1}{2\pi(N+1)} \left(\frac{\sin[(N+1)\frac{x}{2}]}{\sin(\frac{x}{2})} \right)^2.$$

First note the following identity:

$$\begin{aligned}\frac{\sin^2[(N+1)\frac{x}{2}]}{\sin[\frac{x}{2}]} &= \frac{1 - \cos[(N+1)x]}{2 \sin[\frac{x}{2}]} \quad \text{by the power-reducing formulas} \\ &= \frac{1}{2 \sin[\frac{x}{2}]} \left([\cos(0x) - \cos(1x)] + [\cos(1x) - \cos(2x)] + \cdots \right. \\ &\quad \left. \cdots + [\cos((N-1)x) - \cos(Nx)] + [\cos(Nx) - \cos((N+1)x)] \right) \\ &\quad \text{using a telescoping series} \\ &= \frac{1}{2 \sin[\frac{x}{2}]} 2 \sin[\frac{x}{2}] \sum_{i=0}^{\infty} \sin\left[\frac{2i+1}{2}x\right] \\ &= \sum_{i=0}^{\infty} \sin\left[\frac{2i+1}{2}x\right]\end{aligned}$$

Then note that

$$F_N(x) = \frac{1}{2\pi(N+1)} \left(\frac{\sin[(N+1)\frac{x}{2}]}{\sin[\frac{x}{2}]} \right)^2$$

$$\begin{aligned}
&= \frac{1}{2\pi(N+1) \sin \left[\frac{x}{2} \right]} \sum_{i=0}^{\infty} \sin \left[\frac{2i+1}{2} x \right] \\
&= \frac{1}{N+1} \sum_{i=0}^N \frac{1}{2\pi} \frac{\sin \left[(i + \frac{1}{2})x \right]}{\sin \left[\frac{x}{2} \right]} \\
&= \frac{1}{N+1} \sum_{i=0}^N D_i(x)
\end{aligned}$$

Lastly,

$$\begin{aligned}
T_N(x) &= \frac{1}{N+1} \sum_{i=0}^N S_i(x) \\
&= \frac{1}{N+1} \sum_{i=0}^N (D_i * f)(x) \quad \text{by part (a)} \\
&= \frac{1}{N+1} \sum_{i=0}^N \int_{\mathbb{T}} f(y) D_i(x-y) dy \\
&= \int_{\mathbb{T}} f(y) \left[\frac{1}{N+1} \sum_{i=0}^N D_i(x-y) \right] dy \\
&= \int_{\mathbb{T}} f(y) F_N(x-y) dy \\
&= (F_N * f)(x)
\end{aligned}$$

- (c) Which of the families (D_N) and (F_N) are approximate identities as $N \rightarrow \infty$? What can you say about the uniform convergence of the partial sums S_N and the averaged partial sums T_N to f ?

We know (D_N) can not be an approximate identity since

$$D_3(\pi) = \frac{1}{2\pi} \cdot \frac{\sin \left[\frac{7}{2}\pi \right]}{\sin \left[\frac{\pi}{2} \right]} = -\frac{1}{2\pi} < 0$$

and each function in an approximate identity must be nonnegative on $[-\pi, \pi]$. We claim, however, that (F_N) is an approximate identity. First,

$$F_N(x) = \frac{1}{2\pi(N+1)} \left(\frac{\sin \left[(N+1)\frac{x}{2} \right]}{\sin \left[\frac{x}{2} \right]} \right)^2 \geq \frac{1}{2\pi(N+1)} > 0, \quad \forall N \geq 0, \forall x \in \mathbb{T}$$

Next we show

$$\int_{\mathbb{T}} F_N(x) dx = 1$$

for all $N \geq 0$.

$$\int_{\mathbb{T}} F_N(x) dx = \frac{1}{N+1} \int_{\mathbb{T}} \sum_{j=0}^N D_j(x) dx$$

$$\begin{aligned}
&= \frac{1}{N+1} \int_{\mathbb{T}} \sum_{j=0}^N \left[\frac{1}{2\pi} \sum_{n=-j}^j e^{inx} \right] dx \\
&= \frac{1}{2\pi(N+1)} \sum_{j=0}^N \sum_{n=-j}^j \int_{\mathbb{T}} e^{inx} dx \quad \text{since the sums are finite} \\
&= \frac{1}{2\pi(N+1)} \sum_{j=0}^N \left[2\pi + \sum_{\substack{n=-j \\ n \neq 0}}^j \left[\frac{1}{in} (\cos(nx) + i \sin(nx)) \right]_{-\pi}^{\pi} \right] \\
&= \frac{1}{2\pi(N+1)} \sum_{j=0}^N 2\pi \\
&= \frac{2\pi(N+1)}{2\pi(N+1)} \\
&= 1
\end{aligned}$$

Lastly we show

$$\lim_{N \rightarrow \infty} \int_{\mathbb{D}} F_N(x) dx = 0$$

where $\mathbb{D} = [-\pi, -\delta] \cup [\delta, \pi]$.

Hunter and Nachtergaele 7.3

Prove that the sets $\{e_n \mid n \geq 1\}$ defined by

$$e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx,$$

and $\{f_n : n \geq 1\}$ defined by

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \quad f_n(x) = \sqrt{\frac{2}{\pi}} \cos nx \quad \text{for } n \geq 1,$$

are both orthonormal bases of $L^2([0, \pi])$.

First we show $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$ are orthonormal. Suppose $n \neq m$. Then

$$\begin{aligned}
\langle e_n, e_m \rangle &= \int_0^{\pi} e_n(x) e_m(x) dx \\
&= \frac{2}{\pi} \int_0^{\pi} \sin(nx) \sin(mx) dx \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [\cos(nx - mx) - \cos(nx + mx)] dx \\
&= \frac{1}{\pi} \int_0^{\pi} \cos((n - m)x) dx - \frac{1}{\pi} \int_0^{\pi} \cos((n + m)x) dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \left[\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m} \right]_0^\pi \\
&= 0
\end{aligned}$$

Also,

$$\begin{aligned}
\langle e_n, e_n \rangle &= \frac{2}{\pi} \int_0^\pi \sin^2(nx) dx \\
&= \frac{1}{\pi} \int_0^\pi 1 - \cos(2nx) dx \\
&= \frac{1}{\pi} \left[\pi - \frac{1}{2n} \sin(2n\pi) \right] \\
&= \frac{1}{\pi} \pi \\
&= 1
\end{aligned}$$

Thus $\{e_n\}_{n=1}^\infty$ is orthonormal. Let $n \geq 1$.

$$\begin{aligned}
\langle f_0, f_n \rangle &= \frac{\sqrt{2}}{\pi} \int_0^\pi \cos(nx) dx \\
&= \frac{\sqrt{2}}{\pi} \frac{1}{n} \sin(nx) \Big|_0^\pi \\
&= 0
\end{aligned}$$

Let $1 \leq n < m$. Then

$$\begin{aligned}
\langle f_n, f_m \rangle &= \frac{2}{\pi} \int_0^\pi \cos(nx) \cos(mx) dx \\
&= \frac{1}{\pi} \int_0^\pi [\cos((n-m)x) + \cos((n+m)x)] dx \\
&= \frac{1}{\pi} \left(\frac{\sin((n-m)x)}{n-m} + \frac{\sin((n+m)x)}{n+m} \right) \Big|_0^\pi \\
&= 0
\end{aligned}$$

Also,

$$\langle f_0, f_0 \rangle = \frac{1}{\pi} \int_0^\pi dx = \frac{\pi}{\pi} = 1$$

and for $n \geq 1$,

$$\begin{aligned}
\langle f_n, f_n \rangle &= \frac{2}{\pi} \int_0^\pi \cos^2(nx) dx \\
&= \frac{1}{\pi} \int_0^\pi (1 + \cos(2nx)) dx \\
&= \frac{1}{\pi} \left[\pi + \left(\frac{1}{2} \sin(2nx) \right) \Big|_0^\pi \right]
\end{aligned}$$

$$= 1$$

Thue $\{f_n\}_{n=0}^\infty$ is orthonormal. Next we show $\{f_n\}_{n=0}^\infty$ and $\{e_n\}_{n=1}^\infty$ are each bases of $L^2[0, \pi]$.

Let $f \in L^2([0, \pi])$. Then extend f to its odd extension $f_{\text{odd}} \in L^2([-\pi, \pi])$ by

$$f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x \in (0, \pi] \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x \in [-\pi, 0) \end{cases}$$

We know $\{e_n\}_{n=1}^\infty \cup \{f_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2[-\pi, \pi]$ and thus f_{odd} can be written as a Fourier series like so

$$f_{\text{odd}}(x) = \frac{1}{2}f_0 + \sum_{n=1}^{\infty} (a_n f_n + b_n e_n)$$

But since f_{odd} is constructed to be odd,

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n e_n$$

Thus on $[0, \pi]$,

$$f(x) = \sum_{n=1}^{\infty} e_n \sin(nx)$$

Thus $\{e_n\}_{n=1}^\infty$ is a basis of $L^2[0, \pi]$. Now extend f to its even extension $f_{\text{even}} \in L^2[-\pi, \pi]$ be

$$f_{\text{even}}(x) = \begin{cases} f(x) & \text{if } x \in [0, \pi] \\ f(-x) & \text{if } x \in [-\pi, 0) \end{cases}$$

Again, we know $\{e_n\}_{n=1}^\infty \cup \{f_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2[-\pi, \pi]$ and thus f_{even} can be written as a Fourier series like so

$$f_{\text{even}}(x) = \frac{1}{2}f_0 + \sum_{n=1}^{\infty} (a_n f_n + b_n e_n)$$

But since f_{even} is constructed to be even,

$$f_{\text{even}}(x) = \frac{1}{2}f_0 + \sum_{n=1}^{\infty} a_n f_n$$

Thus $\{f_n\}_{n=0}^\infty$ is a basis of $L^2[0, \pi]$.

Hunter and Nachtergaele 7.4

Let $T, S \in L^2(\mathbb{T})$ be the triangular and square wave, respectively, defined by

$$T(x) = |x|, \quad \text{if } |x| \leq \pi, \quad S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \end{cases}$$

(a) Compute the Fourier series of T and S .

Since T is an even function, we can represent T with a cosine series

$$T(x) = \frac{1}{2}\hat{T}_0 + \sum_{n=1}^{\infty} \hat{T}_n \cos(nx)$$

where

$$\begin{aligned} \hat{T}_0 &= \frac{1}{\pi} \int_{\mathbb{T}} T(x) dx \quad \text{and} \\ \hat{T}_n &= \frac{1}{\pi} \int_{\mathbb{T}} T(x) \cos(nx) dx, \quad n = 1, 2, \dots \end{aligned}$$

Because \cos is even and T is even, $T \sin$ is odd, and so

$$\hat{T}_0 = \frac{1}{\pi} \int_{\mathbb{T}} T(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \frac{\pi^2}{2} = \pi$$

and for $n = 1, 2, \dots$,

$$\hat{T}_n = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx$$

Utilizing integration by parts, we find

$$\begin{aligned} \hat{T}_n &= \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2}{\pi} \left[\left(\frac{x}{n} \sin(nx) \right) \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin(nx) dx \right] \\ &= \frac{2}{\pi} \left[\frac{1}{n^2} \cos(nx) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right] \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Thus,

$$T(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} \cos((2n-1)x) \right]$$

Since S is an odd function, we can represent S with a sin series

$$S(x) = \sum_{n=1}^{\infty} \hat{S}_n \sin(nx)$$

where

$$\hat{S}_n = \frac{1}{\pi} \int_{\mathbb{T}} S(x) \sin(nx) dx, \quad n = 1, 2, \dots$$

Because \sin is odd and S is odd, $\sin S$ is even, and thus

$$\begin{aligned} \hat{S}_n &= \frac{1}{\pi} \int_{\mathbb{T}} S(x) \sin(nx) dx \\ &= \frac{2}{\pi} \int_0^\pi \sin(nx) dx \\ &= \frac{2}{\pi} \left[-\frac{1}{n} \cos(nx) \right]_0^\pi \\ &= -\frac{2}{\pi n} ((-1)^n - 1) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Thus,

$$S(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)} \sin((2n-1)x) \right]$$

(b) Show that $T \in H^1(\mathbb{T})$ and $T' = S$.

First we turn $T(x)$ into a Fourier series with $\{e^{inx}\}_{n \in \mathbb{Z}}$ as the basis using

$$\cos x = \frac{1}{2} [e^{ix} + e^{-ix}]$$

Thus,

$$\begin{aligned} T(x) &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} \cos((2n-1)x) \right] \\ &= \frac{\pi}{2} - \frac{2}{\pi} \sum_{n \in \mathbb{Z}} \frac{\exp[i(2n-1)x]}{(2n-1)^2} \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{\pi^2}{\sqrt{2\pi}} - \frac{4}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \frac{\exp[i(2n-1)x]}{(2n-1)^2} \right] \end{aligned}$$

To show $T \in H^1(\mathbb{T})$, we show

$$\sum_{n \in \mathbb{Z}} n^2 |\hat{T}_n|^2 < \infty$$

but this is true because

$$\sum_{n \in \mathbb{Z}} n^2 |\hat{T}_n|^2 = \frac{8}{\pi} \sum_{n \in \mathbb{Z}} \frac{(2n-1)^2}{(2n-1)^4} < \infty$$

by the comparison test. Thus $T \in H^1(\mathbb{T})$.

Next note that $S(x)$ can be turned into a Fourier series with $\{e^{inx}\}_{n \in \mathbb{Z}}$ as a basis by using the following:

$$\sin x = \frac{1}{2i} [e^{ix} - e^{-ix}]$$

Thus,

$$\begin{aligned} S(x) &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)} \sin((2n-1)x) \right] \\ &= -\frac{2i}{\pi} \sum_{n \in \mathbb{Z}} \frac{\exp[i(2n-1)x]}{2n-1} \end{aligned}$$

We can explicitly calculate $in\hat{T}_n$ for each n :

$$T' = \frac{1}{\sqrt{2\pi}} \left[\frac{\pi^2}{\sqrt{2\pi}}(0i) - \frac{4}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} ((2n-1)i) \frac{\exp[i(2n-1)x]}{(2n-1)^2} \right] = -\frac{2i}{\pi} \sum_{n \in \mathbb{Z}} \frac{\exp[i(2n-1)x]}{2n-1} = S$$

(c) Show that $S \notin H^1(\mathbb{T})$.

To show $S \notin H^1(\mathbb{T})$, we show

$$\sum_{n \in \mathbb{Z}} n^2 |\hat{S}_n|^2 < \infty$$

but this is true because

$$\sum_{n \in \mathbb{Z}} n^2 |\hat{S}_n|^2 = \frac{4}{\pi^2} \sum_{n \in \mathbb{Z}} \frac{(2n-1)^2}{(2n-1)^2} = \infty$$

by the n^{th} term test. Thus $S \notin H^1(\mathbb{T})$.

Hunter and Nachtergaele 7.5

Consider $f : \mathbb{T}^d \rightarrow \mathbb{C}$ defined by

$$f(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{in \cdot x},$$

where $x = (x_1, x_2, \dots, x_d)$, $n = (n_1, n_2, \dots, n_d)$, and $n \cdot x = n_1 x_1 + n_2 x_2 + \dots + n_d x_d$. Prove that if

$$\sum_{n \in \mathbb{Z}^d} |n|^{2k} |a_n|^2 < \infty$$

for some $k > \frac{d}{2}$, then f is continuous.

Let $f \in H^k(\mathbb{T}^d)$ with $k > \frac{1}{2}$. Define the partial sums S_N of the Fourier series of f by

$$S_N(x) = \sum_{([-N, N] \cap \mathbb{Z})^d} \hat{f}_n e^{in \cdot x}$$

and define the norm of the k^{th} weak derivative of f as

$$\|f^k\|^2 = \sum_{n \in \mathbb{Z}^d} |n|^{2k} |\hat{f}_n|^2$$

We will show the sequence $S_N \rightarrow f$ uniformly by showing $(S_N)_N$ is a Cauchy sequence and since $C(\mathbb{T}^d)$ is complete with respect to the supremum norm, this implies the limit of $(S_N)_N$ is contained in $C(\mathbb{T}^d)$.

$$\begin{aligned} \|S_N - S_M\|_\infty &= \left\| \sum_{n \in ((\pm N, \pm M] \cap \mathbb{Z})^d} \hat{f}_n e^{in \cdot x} \right\|_\infty \\ &\leq \sum_{n \in ((\pm N, \pm M] \cap \mathbb{Z})^d} |\hat{f}_n| |e^{in \cdot x}| \\ &\quad \text{by the Triangle Inequality} \\ &= \sum_{n \in ((\pm N, \pm M] \cap \mathbb{Z})^d} |\hat{f}_n| \\ &= \sum_{n \in ((\pm N, \pm M] \cap \mathbb{Z})^d} |n|^k |\hat{f}_n| \frac{1}{|n|^k} \\ &\leq \sqrt{\sum_{n \in ((\pm N, \pm M] \cap \mathbb{Z})^d} |n|^{2k} |\hat{f}_n|^2} \cdot \sqrt{\sum_{n \in ((\pm N, \pm M] \cap \mathbb{Z})^d} \frac{1}{|n|^{2k}}} \\ &\quad \text{by the Cauchy-Schwarz Inequality} \\ &\leq \|f^{(k)}\| \sqrt{\sum_{n \in ((\pm N, \pm M] \cap \mathbb{Z})^d} \frac{1}{|n|^{2k}}} \\ &\quad \text{since the Fourier transform is an isomorphism and thus preserves norm} \\ &\leq \|f^{(k)}\|_\infty \sqrt{|\mathbb{S}^{d-1}| \int_N^\infty \frac{r^{d-1}}{r^{2k}} dr} \\ &\quad \text{where } |\mathbb{S}^{d-1}| \text{ is the area of the unit sphere in } d \text{ dimensions} \\ &= \|f^{(k)}\|_\infty \sqrt{|\mathbb{S}^{d-1}|} \left[\frac{r^{d-2k}}{d-2k} \right]_N^\infty \\ &= \begin{cases} \infty & \text{if } \frac{d}{2} \geq k \\ \|f^{(k)}\|_\infty \sqrt{|\mathbb{S}^{d-1}|} ((2k-d)N^{2k-d})^{-\frac{1}{2}} & \text{if } \frac{d}{2} < k \end{cases} \end{aligned}$$

Supposing $\frac{d}{2} < k$,

$$\|S_N - S_M\|_\infty \leq \frac{\|f^{(k)}\|_\infty \sqrt{|\mathbb{S}^{d-1}|}}{\sqrt{(2k-d)N^{2k-d}}}$$

which goes to zero as $N \rightarrow \infty$. Thus $(S_N)_N$ is a Cauchy sequence and thus converges to a limit in $C(\mathbb{T}^d)$. But S_N are the partial sums of the Fourier series of f , and thus $S_N \rightarrow f$. Thus $f \in C(\mathbb{T}^d)$, i.e. f is continuous.

Hunter and Nachtergaele 7.6

Suppose that $f \in H^1([a, b])$ and $f(a) = f(b) = 0$. Prove the Poincaré inequality

$$\int_a^b |f(x)|^2 dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 dx.$$

Hunter and Nachtergaele 7.7

Solve the following initial-boundary value problem for the heat equation,

$$\begin{aligned} u_t &= u_{xx}, \\ u(0, t) &= 0, \quad u(L, t) = 0 \quad \text{for } t > 0 \\ u(x, 0) &= f(x) \quad \text{for } 0 \leq x \leq L \end{aligned}$$

Suppose $u(x, t) = F(x)G(t)$ is a solution. Then

$$\begin{aligned} u_t &= u_{xx} \\ \implies F(x)G'(t) &= F''(x)G(t) \\ \implies \frac{F''(x)}{F(x)} &= \frac{G'(t)}{G(t)} \end{aligned}$$

Since the left hand side is a function of x and the right hand side is a function of t , they can only be equal if they are both constant, i.e.

$$\frac{F''(x)}{F(x)} = C = \frac{G'(t)}{G(t)}$$

for some $C \in \mathbb{R}$. Thus,

$$G'(t) - CG(t) = 0, \quad \text{and} \tag{1}$$

$$F''(x) - CF(x) = 0 \tag{2}$$

The solutions of (1) are

$$G(t) = c_1 e^{Ct}$$

Let $\lambda = \sqrt{C}$. If $C \neq 0$, the solutions of (2) are

$$F(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

The initial condition

$$u(0, t) = 0 \implies F(0)G(t) = 0 \implies F(0) = 0$$

provided u is not the trivial solution. Similarly,

$$F(L) = 0$$

If $C > 0$,

$$F(0) = 0 \implies 0 = c_1 + c_2 \implies F(x) = c_1(e^{\lambda x} - e^{-\lambda x})$$

Also,

$$F(L) = 0 \implies 0 = c_1(e^{\lambda L} - e^{-\lambda L}) \implies c_1 = 0$$

Thus u is the trivial solution. If $C = 0$, then either $F'' = 0$ or $F \equiv 0$, but regardless, if $F'' = 0$, the initial conditions imply that $F \equiv 0$. So let $C < 0$ and define $\lambda = \sqrt{-C}$. Then

$$F(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x)$$

Then

$$F(0) = 0 \implies 0 = c_2 \implies F(x) = c_1 \sin(\lambda x)$$

Also,

$$F(L) = 0 \implies 0 = c_1 \sin(\lambda L) \implies \lambda L = \pi n$$

for integer values n . Thus $\lambda = \frac{n\pi}{L}$ for $n = \pm 1, \pm 2, \dots$. Note $n \neq 0$ since that would imply $\lambda^2 = 0 = C$. Thus,

$$u(t, x) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2 \pi^2}{L^2} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

The initial condition $u(0, x) = f(x)$ implies

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$$

This is a Fourier series, and thus the coefficients c_n are given by

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

Thus the full solution is

$$u(t, x) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L \left[f(x) \sin\left(\frac{n\pi}{L} x\right) \right] dx \cdot \exp\left(-\frac{n^2 \pi^2}{L^2} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right) \right)$$