UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #4

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Hunder and Nachtergaele 7.9

Suppose that $u(t,x)$ is a smooth solution of the one-dimensional wave equation,

$$u_{tt} - c^2 u_{xx} = 0.$$

Prove that

$$\left(u_t^2 + c^2 u_x^2\right)_t - \left(2c^2 u_t u_x\right)_x = 0.$$

If $u(0,t) = u(1,t) = 0$ for all t, deduce that

$$\int_0^1 |u_t(x,t)|^2 + c^2 |u_x(x,t)|^2 \mathrm{d}x = \textit{constant}.$$

Proof.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ \iff 2u_t u_{tt} &= 2c^2 u_t u_{xx} \\ \iff 2u_t u_{tt} + 2c^2 u_x u_{tx} &= 2c^2 u_t u_{xx} + 2c^2 u_x u_{tx} \\ \iff \left(u_t^2 + c^2 u_x^2\right)_t &= \left(2c^2 u_t u_x\right)_x \end{aligned}$$

Since $u(0,t) = u(1,t) = 0$ for all t, then $u(0,t)_t = u(1,t) = 0$ for all t. Thus

$$\begin{aligned} 0 = 2c^2(u_t(1,t)u_x(1,t) - u_t(0,t)u_x(0,t)) &= (2c^2 u_t u_x)\Big|_{x=0}^1 \\ &= \int_0^1 \left(2c^2 u_t u_x\right)_x \mathrm{d}x \\ &= \int_0^1 \left(u_t^2 + c^2 u_x^2\right)_t \mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(u_t^2 + c^2 u_x^2\right) \mathrm{d}x \\ \iff \int_0^1 (u_t^2 + c^2 u_x^2) \mathrm{d}x &= \text{constant}. \end{aligned}$$

□

Hunder and Nachtergaele 7.10

Show that

$$u(x,t) = f(x+ct) + g(x-ct)$$

is a solution of the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

for arbitrary functions f and g. This solution is called d'Alembert's solution.

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ \implies u_t(x,t) &= c(f'(x+ct) - g'(x-ct)) \\ \implies u_{tt}(x,t) &= c^2(f''(x+ct) + g''(x-ct)) \end{aligned}$$

Also,

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ \implies u_x(x,t) &= f'(x+ct) + g'(x-ct) \\ \implies u_{xx}(x,t) &= f''(x+ct) + g''(x-ct) \end{aligned}$$

Thus,

$$u_{tt}(x,t) = c^2(f''(x+ct) + g''(x-ct)) = c^2 u_{xx}(x,t)$$

Hunder and Nachtergaele 7.14

Consider the logistic map

$$x_{n+1} = 4\mu x_n(1-x_n),$$

where $x_n \in [0,1]$, and $\mu = 1$. Show that the solutions may be written as $x_n = \sin^2\theta_n$ where $\theta_n \in \mathbb{T}$, and

$$\theta_{n+1} = 2\theta_n.$$

What can you say about the orbits of the logistic map, the exitence of an invariant measure, and the validity of an ergodic theorem?

Let $x_n = \sin^2\theta_n$ and $\theta_{n+1} = 2\theta_n$. Then

$$\begin{aligned} \theta_{n+1} &= 2\theta_n \\ \implies \sin^2(\theta_{n+1}) &= \sin^2(2\theta_n) \\ \implies x_{n+1} &= 4\sin^2\theta_n\cos^2\theta_n \\ \implies x_{n+1} &= 4\sin^2\theta_n(1-\sin^2\theta_n) \\ \implies x_{n+1} &= 4x_n(1-x_n) \end{aligned}$$

Thus $x_n = \sin^2\theta_n$, where $\theta_{n+1} = 2\theta_n$ satisfies the logistic map.

Hunder and Nachtergaele 7.15

Consider the dynamical system on $\mathbb{T}$,

$$x_{n+1} = \alpha x_n \mod 1,$$

where $\alpha = (1+\sqrt{5})/2$ is the golden ration. Show that the orbit with initial value $x_0 = 1$ is not equidistributed on the circle, meaning that it does not satisy (7.39).
HINT. Show that

$$u_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

satisfies the difference equation

$$u_{n+1} = u_n + u_{n-1}$$

and hence s an integer for every $n \in \mathbb{N}$. Then use the fact that

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0 \quad \textit{as } n \to \infty.$$

Let $\phi^+ = \frac{1+\sqrt{5}}{2}$ and $\phi^- = \frac{1-\sqrt{5}}{2}$. Clearly the dynamical system is not equidistributed on $[0,1]$ since if $x_0 = 1$, then $x_1 = \phi^+ \mod 1 = -\phi^-$ and $x_2 = -\phi^- \cdot \phi^+ = 1$. Thus the system has orbit of length 2 and any finite orbit cannot be equidistributed in an interval.

However, the hint is implying the writer intended to ask us to show that the following sequence is not equidistributed on $[0,1]$.

$$\left\{\left(\phi^+\right)^n \mod 1\right\}_{n=0}^{\infty}$$

Note that this is not a dynamical system since it is not recursive. Let $(u_n)_n$ be a sequence defined by

$$u_n = \left(\phi^+\right)^n + \left(\phi^-\right)^n$$

Note that this sequence satisfies the recursion relation

$$u_{n+1} = u_n + u_{n-1}$$

since

$$\begin{aligned}
u_n + u_{n-1} &= \left(\phi^+\right)^n + \left(\phi^-\right)^n + \left(\phi^+\right)^{n-1} + \left(\phi^-\right)^{n-1} \\
&= \left(\phi^+\right)^{n-1}\left[1+\phi^+\right] + \left(\phi^-\right)^{n-1}\left[1+\phi^-\right] \\
&= \left(\phi^+\right)^{n-1}\left[\frac{3+\sqrt{5}}{2}\right] + \left(\phi^-\right)^{n-1}\left[\frac{3-\sqrt{5}}{2}\right] \\
&= \left(\phi^+\right)^{n+1} + \left(\phi^-\right)^{n+1} \\
&= u_{n+1}
\end{aligned}$$

Since $u_0 = 2$ and $u_1 = 1$, then $u_n \in \mathbb{N}$ for all n. Then note that since $\left|\phi^-\right| < 1$, then $\left(\phi^-\right)^n \to 0$. Thus for any ε, there exists N_ε such that, $\left(\phi^+\right)^n \mod 1 \in (0,\varepsilon) \cup (1-\varepsilon, 1]$ for all $n \geq N_\varepsilon$. Thus $\#\{u_n \mid u_n \in [\varepsilon, 1-\varepsilon]\} \leq N_\varepsilon$ and so the sequence is not equidistributed in $[0,1]$.

Hunder and Nachtergaele 7.17

Let B_n and V_n be defined in (7.46) and (7.47). Prove that $\bigcup_{n=0}^{N} B_n$ in an orthonormal basis of V_N.

HINT. Prove that the set is orthonormal and count its elements.

Hunder and Nachtergaele 7.18

Suppose that $B = \{e_n(x)\}_{n=1}^{\infty}$ *is an orthonormal basis for* $L^2([0,1])$. *Prove the following:*

(a) *For any* $a \in \mathbb{R}$, *the set* $B_a = \{e_n(x-a)\}_{n=1}^{\infty}$ *is an orthonormal basis for* $L^2([a,a+1])$.

(b) *For any* $c > 0$, *the set* $B^C = \{\sqrt{c}e_n(cx)\}_{n=1}^{\infty}$ *is an orthonormal basis for* $L^2([0,c^{-1]]})$.

(c) *With the convention that functions are extended to a larger domain than their original domain by setting them equal to* 0, *prove that* $B \cup B_1$ *is an orthonormal basis for* $L^2([0,1])$.

(d) *Prove that* $\bigcup_{k\in\mathbb{Z}} B_k$ *is an orthonormal basis for* $L^2(\mathbb{R})$