HW #2

Sam Fleischer

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Hunter and NachterGaele 7.1

Let ϕ_n be the functions defined in (7.7)

$$\phi_n(x) = c_n(1 + \cos x)^n$$

where c_n is chosen such that

$$\int_{\mathbb{T}} \phi_n = 1$$

for all n.

(a) *Prove (7.5).*

$$\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} \phi_n(x)\mathrm{d}x = 0$$

for every $\delta > 0$.

(b) *Prove that if the set $\mathcal{P}$ of trigonometric polynomials is dense in the space of periodic continuous functions on $\mathbb{T}$ with the uniform norm, then $\mathcal{P}$ is dense in the space of all continuous functions on $\mathbb{T}$ with the L^2-norm.*

(c) *Is $\mathcal{P}$ dense in the space of all continuous functions on $[0, 2\pi]$ with the uniform norm?*

Hunter and NachterGaele 7.2

Suppose that $f : \mathbb{T} \to \mathbb{C}$ is a continuous function, and

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

is the N^{th} partial sum of its Fourier seriers.

(a) *Show that $S_N = D_N * f$, where D_N is the* Dirichlet kernel

$$D_N(x) = \frac{1}{2\pi}\frac{\sin\left[(N + \frac{1}{2})x\right]}{\sin\left(\frac{x}{2}\right)}.$$

(b) *Let T_N be the mean of the first $N+1$ partial sums,*

$$T_N = \frac{1}{N+1}.$$

Show that $T_N = F_N * f$, where F_N is the Fejér kernel

$$F_N(x) = \frac{1}{2\pi(N+1)}\left(\frac{\sin\left[(N+1)\frac{x}{2}\right]}{\sin\left(\frac{x}{2}\right)}\right)^2.$$

(c) *Which of the families (D_N) and (F_N) are approximate identities as $N \to \infty$? What can you say about the uniform convergence of the partial sums S_N and the averaged partial sums T_N to f?*

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Prove that the sets $\{e_n \mid n \geq 1\}$ defined by

$$e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx,$$

and $\{f_n \ : \ n \geq 1\}$ defined by

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \qquad f_n(x) = \sqrt{\frac{2}{\pi}}\cos nx \qquad \textit{for } n \geq 1,$$

are both orthonormal bases of $L^2([0,1])$.

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Let $T, S \in L^2(\mathbb{T})$ be the triangular and square wave, respectively, defined by

$$T(x) = |x|, \qquad \text{if } |x| \leq \pi, \qquad S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \end{cases}$$

(a) *Compute the Fourier series of T and S.*

(b) *Show that $T \in H^1(\mathbb{T})$ and $T' = S$.*

(c) *Show that $S \notin H^1(\mathbb{T})$.*

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Consider $f \ : \ \mathbb{T}^d \to \mathbb{C}$ defined by

$$f(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{in\cdot x},$$

where $x = (x_1, x_2, \ldots, x_d)$, $n = (n_1, n_2, \ldots, n_d)$, and $n \cdot x = n_1x_1 + n_2x_2 + \cdots + n_dx_d$. Prove that if

$$\sum_{n \in \mathbb{Z}^d} |n|^{2k}|a_n|^2 < \infty$$

for some $k > \frac{d}{2}$, then f is continuous.

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Suppose that $f \in H^1([a,b])$ *and* $f(a) = f(b) = 0$*. Prove the* Poincaré inequality

$$\int_a^b |f(x)|^2 \mathrm{d}x \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \mathrm{d}x.$$

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Solve the following initial-boundary value problem for the hear equation,

$$\begin{array}{ll} u_t = u_{xx}, & \\ u(0,t) = 0, \quad u(L,t) = 0 & \text{for } t > 0 \\ u(x,0) = f(x) & \text{for } 0 \leq x \leq L \end{array}$$