HW #1

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Exercise 1.1

Complete the proof of the the Monotone Class Theorem.

Lemma 1. *The arbitrary intersection of monotone classes in a monotone class.*

Proof. Let $\mathcal{S}$ be the arbitrary intersection of monotone classes M_j for $j \in J$, where J is an index set. Then let $S_1 \subset S_2 \subset S_3 \subset \dots$ and $S_i \in \mathcal{S}$ $\forall i = 1, 2, \dots$. Then since each $S_i \in M_j$ for each M_j and each M_j is a monotone class, then $\bigcup_{i=1}^{\infty} S_i \in M_j$ for each M_j. Thus $\bigcup_{i=1}^{\infty} S_i \in \mathcal{S}$. Now let $S_1 \supset S_2 \supset S_3 \supset \dots$ and $S_i \in \mathcal{S}$ $\forall i = 1, 2, \dots$. Then since $S_i in M_j$ for each M_j and each M_j is a monotone class, then $\bigcap_{i=1}^{\infty} S_i \in M_j$ for each M_j. Thus $\bigcap_{i=1}^{\infty} S_i \in \mathcal{S}$. Thus $\mathcal{S}$ is a monotone class. □

Theorem 1 (Monotone Class Theorem)**.** *Let Ω be a set a let $\mathcal{A}$ be an algebra of subsets of Ω such that $\Omega, \emptyset \in \mathcal{A}$. Then there exists a smallest monotone class $\mathcal{S}$ that contain $\mathcal{A}$. That class, $\mathcal{S}$, is also the smallest sigma-algebra that contains $\mathcal{A}$.*

Proof. Let S be the intersection of all monotone classes M_i that contain $\mathcal{A}$. By Lemma 1, S is a monotone class, and thus the smallest monotone class containing $\mathcal{A}$.

Pick $A \in \mathcal{A}$ and construct $C(A) = \{B \in \mathcal{S} \mid B \cup A \in \mathcal{S}\}$. By construction, $C(A) \subset \mathcal{S}$. Since $\mathcal{A}$ is an algebra, $\mathcal{A}$ is closed under finite unions, and thus $\mathcal{A} \subset C(A)$. Now we show $C(A)$ is a monotone class, which would show $\mathcal{S} \subset C(A)$, implying $C(A) = \mathcal{S}$. Take $B_1 \subset B_2 \subset B_3 \subset \dots$ and $B_i \in C(A)$ $\forall i = 1, 2, \dots$. Then $B_i \cup A \in \mathcal{S}$ $\forall i = 1, 2, \dots$ and $(B_1 \cup A) \subset (B_2 \cup A) \subset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcup_{i=1}^{\infty}(B_i \cup A) \in \mathcal{S}$, but $\bigcup_{i=1}^{\infty}(B_i \cup A) = (\bigcup_{i=1}^{\infty} B_i) \cup A$. Thus $\bigcup_{i=1}^{\infty} B_i \in C(A)$. Similarly, take $D_1 \supset D_2 \supset \dots$ and $D_i \in C(A)$ $\forall i = 1, 2, \dots$. Then $D_i \cup A \in \mathcal{S}$ $\forall i = 1, 2, \dots$ and $(D_1 \cup A) \supset (D_2 \cup A) \supset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcap_{i=1}^{\infty}(D_i \cup A) \in \mathcal{S}$, but $\bigcap_{i=1}^{\infty}(D_i \cup A) = (\bigcap_{i=1}^{\infty} D_i) \cup A$. Thus $\bigcap_{i=1}^{\infty} D_i \in C(A)$. This proves that $C(A)$ is a monotone class, and thus $C(A) = \mathcal{S}$.

Now we extend the definition of $C(A)$ to be defined for any $A \in \mathcal{S}$. Pick $A' \in \mathcal{S}$. Then since $A' \in C(A)$ $\forall A \in \mathcal{A}$, then $A \in C(A')$ $\forall A' \in \mathcal{A}$. Thus $\mathcal{A} \subset C(A')$. Now we show $C(A')$ is a monotone class, which would show $\mathcal{S} \subset C(A')$, implying $C(A') = \mathcal{S}$. Take $B_1 \subset B_2 \subset B_3 \subset \dots$ and $B_i \in C(A')$ $\forall i = 1, 2, \dots$. Then $B_i \cup A' \in \mathcal{S}$ $\forall i = 1, 2, \dots$ and $(B_1 \cup A') \subset (B_2 \cup A') \subset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcup_{i=1}^{\infty}(B_i \cup A') \in \mathcal{S}$, but $\bigcup_{i=1}^{\infty}(B_i \cup A') = (\bigcup_{i=1}^{\infty} B_i) \cup A'$. Thus $\bigcup_{i=1}^{\infty} B_i \in C(A')$. Similarly, take $D_1 \supset D_2 \supset \dots$ and $D_i \in C(A')$ $\forall i = 1, 2, \dots$. Then $D_i \cup A' \in \mathcal{S}$ $\forall i = 1, 2, \dots$ and $(D_1 \cup A') \supset (D_2 \cup A') \supset \dots$. Then since $\mathcal{S}$ is a monotone class, $\bigcap_{i=1}^{\infty}(D_i \cup A') \in \mathcal{S}$, but $\bigcap_{i=1}^{\infty}(D_i \cup A') = (\bigcap_{i=1}^{\infty} D_i) \cup A'$. Thus $\bigcap_{i=1}^{\infty} D_i \in C(A')$. This proves that $C(A')$ is a monotone class, and thus $C(A') = \mathcal{S}$. Thus $\mathcal{S}$ is closed under finite unions.

Now define $C = \{B \in \mathcal{S} \mid B^C \in \mathcal{S}\}$. Since $\mathcal{A}$ is an algebra, $\mathcal{A}$ is closed under complimentation, and thus $\mathcal{A} \subset C$. Now take $B_1 \subset B_2 \subset \dots$ and $B_i \in C\ \forall i = 1, 2, \dots$. Then since $B_1^C \supset B_2^C \supset \dots$ and $B_i^C \in \mathcal{S}\ \forall i = 1, 2, \dots$, and since $\mathcal{S}$ is a monotone class, then $\bigcap_{i=1}^{\infty} \left(B_i^C\right) \in S$. However, $\bigcap_{i=1}^{\infty} \left(B_i^C\right) = \left(\bigcup_{i=1}^{\infty} B_i\right)^C$, and thus $\bigcup_{i=1}^{\infty} B_i \in C$. Then take $D_1 \supset D_2 \supset \dots$ and $D_i \in C\ \forall i = 1, 2, \dots$. Then since $D_1^C \subset D_2^C \subset \dots$ and $D_i^C \in \mathcal{S}$, and since $\mathcal{S}$ is a monotone class, then $\bigcup_{i=1}^{\infty} \left(D_1^C\right) \in S$. However, $\bigcup_{i=1}^{\infty} \left(D_i^C\right) = \left(\bigcap_{i=1}^{\infty} D_C\right)^C$, and thus $\bigcap_{i=1}^{\infty} D_i \in C$. Thus C is a monotone class containing $\mathcal{A}$, and thus $\mathcal{S} \subset C$, proving $C = \mathcal{S}$. Thus $\mathcal{S}$ is closed under complementation.

Now we show $\mathcal{S}$ is closed under countable unions and intersections. Consider a sequene of sets $\{A_i\}_{i=1}^{\infty} \in \mathcal{S}$. Then form $B_n = \bigcup_{i=1}^{n}$. Since each B_n is a finite union of elements in $\mathcal{S}$, then each $B_n \in \mathcal{S}$. Also, $B_1 \subset B_2 \subset \dots$. Since $\mathcal{S}$ is a monotone class, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{S}$, but $\bigcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty} B_n$, and thus $\mathcal{S}$ is closed under countable unions. Similarly, form $D_n = \bigcup_{i=1}^{n} A_i^C$. Since each D_i is a finite union of elements in $\mathcal{S}$ ($\mathcal{S}$ is closed under complementation), then each $D_i \in \mathcal{S}$. Also, $D_1 \subset D_2 \subset \dots$. Since $\mathcal{S}$ is a monotone class, $\bigcup_{n=1}^{\infty} D_n \in S$, but $\bigcup_{n=1}^{\infty} D_n = \bigcup_{i=1}^{\infty} \left(A_i^C\right) = \left(\bigcap_{i=1}^{\infty} A_i\right)^C$. Again, since $\mathcal{S}$ is closed under complementation, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{S}$. Thus $\mathcal{S}$ is closed under countable unions and intersections.

This proves $\mathcal{S}$ is a σ-algebra. However, every σ-algebra is a monotone class, and thus $\mathcal{S}$ must be the smallest σ-algebra containing $\mathcal{A}$ since it is defined as the smallest monotone class containing $\mathcal{A}$. $\square$

Exercise 1.2

With regard to the remark about continuous functions in Section 1.5, show that f is continuous (in the sense of the usual ε, δ definition) if and only if f is both upper and lower semicontinuous. Show that f is upper semicontinuous at x if and only if, for every sequence $x_1, x_2, \dots$ converging to x, we have $f(x) \geq \overline{\lim}_{n\to\infty} f(x_n)$.

Definition Consider $f : \Omega \to \mathbb{R}$, and define $L_f(t) = \{x \in \Omega \mid f(x) > t\}$ and $U_f(t) = \{x \in \Omega \mid f(x) < t\}$. Then f is *lower semicontinuous* if $L_f(t)$ is open $\forall t \in \mathbb{R}$ and f is *upper semicontinous* if $U_f(t)$ is open $\forall t \in \mathbb{R}$.

Theorem 2. *Let $f : \Omega \to \mathbb{R}$. Then f is continuous if and only if f is both upper and lower semincontinuous.*

Proof. Let f a continuous function. Then $\forall x \in \Omega$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Fix $t \in \mathbb{R}$ and let $x_L \in L_f(t)$. Then $f(x_L) = t + \ell$ for some $\ell > 0$. Now take $\varepsilon = \ell$. Then by the continuity of f, $\exists \delta_\ell$ such that $f(B_{\delta_\ell}(x_L)) \subset B_\ell(t + \ell)$. But since $t_0 \in B_\ell(t + \ell) \implies t_0 > t$, then $B_{\delta_\ell}(x_L) \subset L_f(t)$. Thus $L_f(t)$ is open. Now let $x_U \in U_f(t)$. Then $f(x_U) = t - u$ for some $u > 0$. Again, take $\varepsilon = u$, and again by the continuity of f, $\exists \delta_u$ such that $f(B_{\delta_u}(x_U)) \subset B_u(f(x_U))$. But since $t_0 \in B_u(t - u) \implies t_0 < t$, then $B_{\delta_u}(x_U) \subset U_f(t)$. Thus $U_f(t)$ is open. Thus f is both upper and lower semicontinuous.

Now let f be both upper and lower semicontinuous. Thus $\forall t \in \mathbb{R}$, $L_f(t)$ and $U_f(t)$ are open. Then pick $x \in \Omega$ and let $t = f(x)$. Choose $\varepsilon > 0$ and let $t_1 = t - \varepsilon$ and $t_2 = t + \varepsilon$. Then $x \in L_f(t_2)$ and $x \in U_f(t_1)$. Since $L_f(t_2)$ and $U_f(t_1)$ are open, then $\exists \delta_L$ and δ_U such that $f(B_{\delta_L}(x)) \subset B_\varepsilon(t_2)$ and $f(B_{\delta_U}(x)) \subset B_\varepsilon(t_1)$. Choose $\delta = \min(\delta_L, \delta_U)$ and let $x_0 \in B_\delta(x)$. Then $t_1 < f(x_0) < t_2$, which shows $f(x_0) \in B_\varepsilon(t)$, and thus f is continuous.

Thus f is continuous if and only if f is both lower and upper semicontinuous. $\square$

Theorem 3. *Let $f : \Omega \to \mathbb{R}$. Then f is upper semicontinuous at x if and only if for every sequence $\{x_i\} \in \Omega$ such that $x_i \to x$, we have $f(x) \geq \overline{\lim}_{i\to\infty} f(x_i)$.*

Proof. Fix $x \in \Omega$ and let f by upper semicontinuous at x. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_\Omega(x,y) < \delta \implies f(y) - f(x) < \varepsilon$, i.e. $f(y) < f(x) + \varepsilon$. Now consider a sequence $\{x_i\} \in \Omega$ such that $x_i \to x$. Then consider a sequence ε_k such that $\varepsilon_k \to 0$. By the upper semicontinuity of f, $\exists I_k$ such that $\sup_{i \geq I_k}\{f(x_i)\} \leq f(x) + \varepsilon_k$ for each $k = 1, 2, \ldots$. Form a sequence $L_k = \max\{k, I_k\}$ and note $L_k \to \infty$, $L_k \geq I_k$, and $L_k \geq k$ for each $k = 1, 2, \ldots$. Then

$$
\begin{aligned}
\sup_{i \geq L_k}\{f(x_i)\} &\leq \sup_{i \geq I_k}\{f(x_i)\} \leq f(x) + \varepsilon_k \\
\implies \lim_{k\to\infty}\left(\sup_{i \geq L_k}\{f(x_i)\}\right) &\leq \lim_{k\to\infty}\left(f(x) + \varepsilon_k\right) \\
\implies \lim_{k\to\infty}\left(\sup_{i \geq k}\{f(x_i)\}\right) &\leq f(x) + 0 = f(x) \\
\implies \overline{\lim}_{i\to\infty} f(x_i) &\leq f(x)
\end{aligned}
$$

□

Exercise 1.3

Prove the assertion made in Section 1.5 that for any Borel set $A \subset \mathbb{R}$ and any σ-algebra Σ the set $\{x \mid f(x) \in A\} = f^{-1}(A)$ is Σ-measurable whenever the function f is Σ-measurable.

Definition Consider $f : \Omega \to \mathbb{R}$ and let Σ be a σ-algebra on Ω. We say that f is a *measurable function* (with respect to Σ) if the set $L_f(t) = \{x \in \Omega \mid f(x) > t\} = f^{-1}((t, \infty))$ is measurable, i.e. $L_f(t) \in \Sigma$, for every $t \in \mathbb{R}$.

Theorem 4. *Let $f : \Omega \to \mathbb{R}$. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and let $A \in \mathcal{B}$. Then the set $P_A = \{x \mid f(x) \in A\} = f^{-1}(A)$ is Σ-measurable whenever f is Σ-measurable.*

Proof. Consider the interval $(a, b) \subset \mathbb{R}$ $(a < b)$. Note that

$$
\begin{aligned}
(a,b) &= (-\infty, b) \cap (a, \infty) \\
&= [b, \infty)^C \cap (a, \infty) \\
&= \left[\bigcap_{i=1}^{\infty} (b - 2^{-i}, \infty)\right]^C \cap (a, \infty)
\end{aligned}
$$

This shows that any open ball in $\mathbb{R}$ is countably σ-algebraicly generated from sets of the form (t, ∞). Due to the properties of preimages,

$$
\begin{aligned}
f^{-1}((a,b)) &= f^{-1}\left(\left[\bigcap_{i=1}^{\infty} (b - 2^{-i}, \infty)\right]^C \cap (a, \infty)\right) \\
&= \left[\bigcap_{i=1}^{\infty} f^{-1}(b - 2^{-i}, \infty)\right]^C \cap f^{-1}((a, \infty))
\end{aligned}
$$

$$= \left[\bigcap_{i=1}^{\infty} L_f(b - 2^{-i}) \right]^C \cap L_f(a)$$

Thus, since Σ is closed under countable unions, countable intersections, and complements, $f^{-1}((a, b)) \in \Sigma$.

Now consider $A \in \mathcal{B}$. Then since $\mathcal{B}$ is the smallest σ-algebra containing the open balls, then A is countably σ-algebraicly generated from open balls. However, preimages are closed under arbitrary unions, arbitrary intersections, and complements, and thus $f^{-1}(A) = P_A \in \Sigma$. Thus P_A is Σ-measurable whenever f is Σ-measurable. $\square$