UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #5

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Hunter and Nachtergaele 8.2

If $\mathscr{H} = \mathscr{M} \oplus \mathscr{N}$ is an orthogonal direct sum, show that $\mathscr{M}^\perp = \mathscr{N}$ and $\mathscr{N}^\perp = \mathscr{M}$.

Proof. Suppose $x \in \mathscr{M}^\perp$. Then $x \in \mathscr{H} \implies \exists! y, z$ such that $x = y + z$ where $y \in \mathscr{M}$ and $z \in \mathscr{N}$. Then

$$\langle x, x\rangle = \cancelto{0}{\langle x, y\rangle} + \langle x, z\rangle = \langle x, z\rangle \implies \langle x, x - z\rangle = 0 \implies x = z$$

which shows $x \in \mathscr{N}$, i.e. $\mathscr{M}^\perp \subset \mathscr{N}$.
Now suppose $x \notin \mathscr{M}^\perp$. Then $x \in \mathscr{M}$ and $x \neq 0$. Thus $x \notin \mathscr{N}$ since a direct sum implies $\mathscr{N} \cap \mathscr{M} = \{0\}$. Thus $\mathscr{N} \subset \mathscr{M}^\perp \subset \mathscr{N} \implies \mathscr{N} = \mathscr{M}^\perp$.
Switching $\mathscr{N}$ and $\mathscr{M}$ shows $\mathscr{M} = \mathscr{N}^\perp$. □

Hunter and Nachtergaele 8.3

Let $\mathscr{M}$, $\mathscr{N}$ be closed subspaces of a Hilbert space $\mathscr{H}$ and P, Q the orthogonal projections with ran $P = \mathscr{M}$, ran $Q = \mathscr{N}$. *Prove that the following conditions are equivalent: (a) $\mathscr{M} \subset \mathscr{N}$; (b) $QP = P$; (c) $PQ = P$; (d) $\|Px\| \le \|Qx\|$ for all $x \in \mathscr{H}$; (e) $\langle x, Px\rangle \le \langle x, Qx\rangle$ for all $x \in \mathscr{H}$.*

Proof. We will show *(a)* → *(b)* → *(c)* → *(d)* → *(e)* → *(a)*, which proves the statements' equivalence.

(a) → (b). Let $\mathscr{M} \subset \mathscr{N}$ and let $x \in \mathscr{H}$. Then $Px \in \mathscr{M} \subset \mathscr{N} \implies Q(Px) = Px \implies QP = P$.

(b) → (c). Let $QP = P$. Then

$$\left\langle x, Py\right\rangle = \left\langle Px, y\right\rangle = \left\langle QPx, y\right\rangle = \left\langle Px, Qy\right\rangle = \left\langle x, PQy\right\rangle \implies \left\langle x, Py - PQy\right\rangle = 0 \qquad \forall x, y \in \mathscr{H}.$$

Thus $Py - PQy = 0$ for all $y \in \mathscr{H}$, i.e. $PQ = P$.

(c) → (d). Let $PQ = P$. First note $\|Px\| \le \|x\|$ for all $x \in \mathscr{H}$ because

$$\|Px\| = \frac{\|Px\|^2}{\|Px\|} = \frac{\langle Px, Px\rangle}{\|Px\|} = \frac{\left\langle x, P^2x\right\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|} \le \|x\|$$

by the Cauchy-Schwarz inequality. Thus,

$$\|Px\| = \|PQx\| = \|P(Qx)\| \le \|Qx\| \qquad \forall x \in \mathscr{H}$$

(d) → (e). Let $\|Px\| \le \|Qx\|$.

$$\|Px\|^2 \le \|Qx\|^2 \implies \langle Px, Px\rangle \le \langle Qx, Qx\rangle \implies \left\langle x, P^2x\right\rangle \le \left\langle x, Q^2x\right\rangle \implies \langle x, Px\rangle \le \langle x, Qx\rangle \qquad \forall x \in \mathscr{H}$$

(e) → (a). Let $\langle x, Px\rangle \le \langle x, Qx\rangle$ and suppose $x \in M$. Then $Px = x$. Then $\|x\|^2 = \langle x, x\rangle = \langle x, Px\rangle \le \langle x, Qx\rangle$. However, since $\|Qx\| \le \|x\|$ for all x, then $\langle x, Qx\rangle \le \|x\|\|Qx\| \le \|x\|^2$, which shows $\langle x, Qx\rangle = \|x\|^2$. Thus $Qx = x$, which shows $x \in \mathscr{N}$, proving $\mathscr{M} \subset \mathscr{N}$.

$\square$

Hunter and Nachtergaele 8.4

Suppose that (P_n) is a sequence of orthogonal projections on a Hilbert space $\mathscr{H}$ such that

$$\operatorname{ran} P_{n+1} \supset \operatorname{ran} P_n, \qquad \bigcup_{n=1}^{\infty} \operatorname{ran} P_n = \mathscr{H}.$$

Prove that (P_n) converges strongly to the identity operator I as $n \to \infty$. Show that (P_n) does not converge to the identity operator with respect to the operator norm unless $P_n = I$ for all sufficiently large n.

Proof. Let $x \in \mathscr{H}$. Then $\exists N$ such that $x \in \operatorname{ran} P_N$. Thus $x \in \operatorname{ran} P_n$ for all $n \geq N$. Since each P_n is an orthogonal projection, then $x = P_n x$ for all $n \geq N$. Thus

$$\lim_{n\to\infty} P_n x = \lim_{n\to\infty} x = x = Ix$$

where I is the identity operator. Thus (P_n) converges strongly to I.
If $P_n = I$ for all sufficiently large n, then obviously

$$\lim_{n\to\infty} \|P_n - I\| = \lim_{n\to\infty} \|I - I\| = \|0\| = 0$$

Thus P_n converges to I with respect to the operator norm. If it is not true that $P_n = I$ for all sufficiently large n, then $\operatorname{ran} P_n \subset P_{n+1}\ \forall n \implies P_n \not\equiv I$ for any n. Then $\forall n$, $\ker P_n \neq \{0\}$, i.e. $\dim \ker P_n > 0$, and so $\exists e_n \in \ker P_n$ with $\|e_n\| = 1$. Then $P_n e_n = 0$ and $\forall n$,

$$\|P_n - I\| \geq \|(P_n - I)e_n\| = \|e_n\| = 1 \implies \lim_{n\to\infty} \|P_n - I\| \geq 1$$

which shows P_n does not converge to the identity operator with respect to the operator norm. $\square$

Hunter and Nachtergaele 8.6

Show that a linear operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ is unitary if and only if it is an isometric isomorphism of normed linear spaces. Show that an invertible linear map is unitary if and only if its inverse is.

Proof. Let $U : \mathscr{H}_1 \to \mathscr{H}_2$ be unitary. Then U is invertible and $\left\langle Ux, Uy \right\rangle_{\mathscr{H}_2} = \left\langle x, y \right\rangle_{\mathscr{H}_1}$ for all $x, y \in \mathscr{H}_1$. Thus

$$\|Ux\|^2_{\mathscr{H}_2} = \langle Ux, Ux \rangle_{\mathscr{H}_2} = \langle x, x \rangle_{\mathscr{H}_1} = \|x\|^2_{\mathscr{H}_1}$$

Thus U preserves norms and is thus an isometric isomorphism.

Now suppose $U : \mathscr{H}_1 \to \mathscr{H}_2$ is an isometric isomorphism. By the definition of isomorphism, U^{-1} exists and $\|Ux\|_{\mathscr{H}_2} = \|x\|_{\mathscr{H}_1}$ (or $\langle Ux, Ux\rangle = \langle x, x\rangle$) for all $x \in \mathscr{H}_1$. Also,

$$\begin{aligned}
\langle x, y\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right) \\
&= \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle - i\langle x+iy, x+iy\rangle + i\langle x-iy, x-iy\rangle\right) \\
&= \frac{1}{4}\left(\langle Ux+Uy, Ux+Uy\rangle - \langle Ux-Uy, Ux-Uy\rangle - i\langle Ux+iUy, Ux+iUy\rangle + i\langle Ux-iUy, Ux-iUy\rangle\right) \\
&= \frac{1}{4}\Bigg(\left[\|Ux\|^2 + \langle Ux, Uy\rangle + \underline{\langle Uy, Ux\rangle} + \|Uy\|^2\right] - \left[\|Ux\|^2 - \langle Ux, Uy\rangle - \underline{\langle Uy, Ux\rangle} + \|Uy\|^2\right] \\
&\qquad - i\left[\|Ux\|^2 + i\langle Ux, Uy\rangle - i\underline{\langle Uy, Ux\rangle} - \|Uy\|^2\right] + i\left[\|Ux\|^2 - i\langle Ux, Uy\rangle + i\underline{\langle Uy, Ux\rangle} - \|Uy\|^2\right]\Bigg) \\
&= \frac{1}{4}\left(4\langle Ux, Uy\rangle\right) \\
&= \langle Ux, Uy\rangle
\end{aligned}$$

Thus U is unitary.
Suppose U is an invertible, unitary map. Then $\langle Ux, Uy\rangle = \langle x, y\rangle\ \forall x, y \in \mathscr{H}$. The invertibility of U implies $\langle x, y\rangle = \langle U(U^{-1}(x)), U(U^{-1}(y))\rangle = \langle U^{-1}(x), U^{-1}(y)\rangle\ \forall x, y \in \mathscr{H}$. Thus U^{-1} is unitary. Similarly, suppose U is invertible and U^{-1} is unitary. Then $\langle x, y\rangle = \langle U^{-1}(U(x)), U^{-1}(U(y))\rangle = \langle Ux, Uy\rangle$. Then U is unitary. Thus, an invertible linear map is unitary if and only if its inverse is. □

Hunter and Nachtergaele 8.7

If ϕ_y is the bounded linear functional defined in (8.5),

$$\phi_y(x) = \langle y, x\rangle \tag{8.5}$$

prove that $\|\phi_y\| = \|y\|$.

Proof. First we prove $\|\phi_y\|$ is bounded above by $\|y\|$.

$$\|\phi_y\| = \sup_{\|x\|=1}\|\phi_y(x)\| = \sup_{\|x\|=1}|\langle y, x\rangle| \le \sup_{\|x\|=1}\|y\|\|x\| = \|y\|$$

Next, consider $x = \frac{y}{\|y\|}$ (note $\|x\| = 1$):

$$\|\phi_y(x)\| = \|\langle y, x\rangle\| = \left\|\frac{\langle y, y\rangle}{\|y\|}\right\| = \left\|\frac{\|y\|^2}{\|y\|}\right\| = \|y\|$$

and thus $\|\phi_y\| \ge \|y\|$, which proves $\|\phi_y\| = \|y\|$. □

Hunter and Nachtergaele 8.8

Prove that $\mathcal{H}^*$ is a Hilbert space with the inner product defined by

$$\langle \phi_x, \phi_y \rangle_{\mathcal{H}^*} = \langle y, x \rangle_{\mathcal{H}}.$$

Proof. First note that for $\phi_{y_1}, \phi_{y_2} \in \mathcal{H}^*$, $\|\phi_{y_1} + \phi_{y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$ where y_1 and y_2 are the associated vectors in $\mathcal{H}$ guaranteed in the Riesz Representation Theorem. This is true because $\phi_{y_1}(x) = \langle y_1, x \rangle_{\mathcal{H}}$ and $\phi_{y_2}(x) = \langle y_2, x \rangle_{\mathcal{H}}$ imply

$$\phi_{y_1+y_2}(x) = \langle y_1 + y_2, x \rangle_{\mathcal{H}} = \langle y_1, x \rangle_{\mathcal{H}} + \langle y_2, x \rangle_{\mathcal{H}} = \phi_{y_1}(x) + \phi_{y_2}(x)$$

and since $\|\phi_{y_1+y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$ by (Hunter and Nachtergaele 8.7), then $\|\phi_{y_1} + \phi_{y_2}\|_{\mathcal{H}^*} = \|y_1 + y_2\|_{\mathcal{H}}$. Let (ϕ_n) be a Cauchy sequence in $\mathcal{H}^*$. Then $\forall \varepsilon > 0$, $\exists N$ such that $\|\phi_m - \phi_n\|_{\mathcal{H}^*} < \varepsilon$ for $m, n \geq N$. By the Riesz Representation Theorem, $\exists (y_n)_n \in \mathcal{H}$ such that for every n, $\phi_n(x) = \langle y_n, x \rangle$ $\forall x \in \mathcal{H}$. $(y_n)_n$ is Cauchy since given $\varepsilon > 0$ we can find N such that $\|y_n - y_m\|_{\mathcal{H}} = \|\phi_n - \phi_m\|_{\mathcal{H}^*} < \varepsilon$ for $m, n \geq N$. Since $\mathcal{H}$ is a Hilbert space, then $(y_n)_n$ is convergent to some $y \in \mathcal{H}$. By the Riesz Representation Theorem, $\exists \phi \in \mathcal{H}^*$ such that $\phi(x) = \langle y, x \rangle$ $\forall x \in \mathcal{H}$. Then $(\phi_n)_n$ converges to ϕ because $\|\phi_n - \phi\|_{\mathcal{H}^*} = \|y_n - y\|_{\mathcal{H}}$, which can be made arbitrary small by the definition of convergence. Thus $\mathcal{H}^*$ is complete. Also, $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$ is a well-defined inner product since $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is a well-defined inner product (i.e. the properties of inner product hold). Thus $\mathcal{H}^*$ is a Hilbert space. □

Hunter and Nachtergaele 8.9

Let $A \subset \mathcal{H}$ be such that

$$\mathcal{M} = \{x \in \mathcal{H} \mid x \textit{ is a finite linear combination of elements in } A\}$$

is a dense linear subspace of $\mathcal{H}$. Prove that any bounded linear functional on $\mathcal{H}$ is uniquely determined by its values on A. If $\{u_\alpha\}$ is an orthonormal basis, find a necessary and sufficient condition on a family of complex numbers c_α for there to be a bounded linear functional ϕ such that $\phi(u_\alpha) = c_\alpha$.

Proof. Suppose ϕ_1 and ϕ_2 are two bounded linear functionals such that $\phi_1(a) = \phi_2(a)$ for all $a \in A$. Let $x \in \mathcal{H}$. By denisty of $\mathcal{M}$, $\exists m_i \in \mathcal{M}$ such that $m_i \to x$. By linearity of ϕ_1 and ϕ_2, $\phi_1(m_i) = \phi_2(m_i)$ for all $i = 1, 2, \ldots$. By the Riesz Representation Theorem, $\exists y_1, y_2$ such that $\phi_1(x) = \langle y_1, x \rangle$ and $\phi_2(x) = \langle y_2, x \rangle$ for all $x \in \mathcal{H}$. Thus by continuity of inner products,

$$\phi_1(x) = \langle y_1, x \rangle = \lim_{i\to\infty} \langle y_1, m_i \rangle = \lim_{i\to\infty} \phi_1(m_i) = \lim_{i\to\infty} \phi_2(m_i) = \lim_{i\to\infty} \langle y_2, m_i \rangle = \langle y_2, x \rangle = \phi_2(x) \qquad \forall x \in \mathcal{H}.$$

Thus $\phi_1 \equiv \phi_2$, i.e. bounded linear functionals are uniqueley determined by their values on A.
Let $\{u_\alpha\}$ is an orthonormal basis on $\mathcal{H}$. Then if $\exists \phi \in \mathcal{H}^*$ such that $\phi(u_\alpha) = c_\alpha$ then by the Riesz Representation Theorem, $\exists y \in \mathcal{H}$ such that $\langle y, u_\alpha \rangle = c_\alpha$. Then $y = \sum_\alpha c_\alpha u_\alpha$. Thus the necessary condition $\sum_\alpha |c_\alpha|^2 < \infty$.

Suppose $\sum_\alpha |c_\alpha|^2 < \infty$. Then define $y = \sum_\alpha c_\alpha u_\alpha$. Since $\{u_\alpha\}$ is an orthonormal basis, then $c_\alpha = \langle y, u_\alpha \rangle$. Then by the Riesz Representation Theorem, $\exists \phi \in \mathscr{H}^*$ such that $\phi(x) = \langle y, x \rangle \; \forall x \in \mathscr{H}$. In particular, $\phi(u_\alpha) = c_\alpha$.
Thus, given a family of complex numbers $\{c_\alpha\}$,

$$\sum_\alpha |c_\alpha|^2 < \infty \iff \exists \phi \in \mathscr{H}^* \text{ such that } \phi(u_\alpha) = c_\alpha$$

□

Hunter and Nachtergaele 8.11

Prove that if $A : \mathscr{H} \to \mathscr{H}$ is a linear map and $\dim \mathscr{H} < \infty$, then

$$\dim \ker A + \dim \operatorname{ran} A = \dim \mathscr{H}.$$

Prove that if $\dim \mathscr{H} < \infty$, then $\dim \ker A = \dim \ker A^*$. In particular, $\ker A = \{0\}$ if and only if $\ker A^* = \{0\}$.

Proof. Let $\dim \mathscr{H} < \infty$ (say $\dim \mathscr{H} = n$). Then $\dim \ker A < \infty$ and $\dim \operatorname{ran} A < \infty$ since $\ker A$ and $\operatorname{ran} A$ are subspaces of $\mathscr{H}$. Let $\dim \ker A = m \le n$ and let $\{u_1, u_2, \ldots, u_m\}$ be a basis of $\ker A$. Since $\ker A$ is a subspace of $\mathscr{H}$, this basis can be extended to a basis $\mathscr{U}$ of $\mathscr{H}$: $\mathscr{U} = \{u_1, u_2, \ldots, u_m, v_{m+1}, v_{m+2}, \ldots, v_n,\}$. Let $x \in \mathscr{H}$. Then

$$\begin{aligned} x &= a_1 u_1 + a_2 u_2 + \cdots + a_m u_m + a_{m+1} v_{m+1} + a_{m+2} v_{m+2} + \cdots + a_n v_n \\ \implies Ax &= \cancelto{0}{a_1 A u_1} + \cancelto{0}{a_2 A u_2} + \cdots + \cancelto{0}{a_m A u_m} + a_{m+1} A v_{m+1} + a_{m+2} A v_{m+2} + \cdots + a_n A v_n \\ &= a_{m+1} A v_{m+1} + a_{m+2} A v_{m+2} + \cdots + a_n A v_n \end{aligned}$$

since $u_i \in \ker A$ for $i = 1, 2, \ldots, m$. Thus, $\{A v_{m+1}, A v_{m+2}, \ldots, A v_n\}$ spans $\operatorname{ran} A$. However it is also linearly independent since

$$\begin{aligned} & c_{m+1} A v_{m+1} + c_{m+2} A v_{m+2} + \cdots + c_n A v_n = 0 \\ \implies & A(c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \cdots + c_n v_n) = 0 \\ \implies & c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \cdots + c_n v_n \in \ker A \\ \implies & c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \cdots + c_n v_n = d_1 u_1 + d_2 u_2 + \cdots + d_m u_m \qquad \text{for some } d_i \in \mathbb{C} \\ \implies & d_1 = d_2 = \cdots = d_m = c_{m+1} = c_{m+2} = \cdots = c_n = 0 \qquad \text{since } \mathscr{U} \text{ is a basis} \end{aligned}$$

Thus $\{A v_{m+1}, A v_{m+2}, \ldots, A v_n\}$ is linearly independent. Since it also spans $\operatorname{ran} A$ then it is a basis of $\operatorname{ran} A$. Thus $\dim \operatorname{ran} A = n - m$. Thus, since $m + (n - m) = n$, then

$$\dim \ker A + \dim \operatorname{ran} A = \dim \mathscr{H}.$$

Let $x \in \mathscr{H}$. Then

$$\begin{aligned} x \in \ker A^* &\iff A^* x = 0 \\ &\iff \langle y, A^* x \rangle = 0 \qquad \forall y \in \mathscr{H} \\ &\iff \langle Ay, x \rangle = 0 \qquad \forall y \in \mathscr{H} \end{aligned}$$

$$\begin{aligned}
&\iff x \perp Ay \qquad \forall y \in \mathscr{H} \\
&\iff x \perp (\text{ran } A) \\
&\iff x \in (\text{ran } A)^\perp
\end{aligned}$$

Thus $\ker A^* = (\text{ran } A)^\perp$ and $\dim\ker A^* = \dim(\text{ran } A)^\perp$. However since $\dim\ker A + \dim\text{ran } A = \dim\mathscr{H}$ and $\dim\text{ran } A + \dim(\text{ran } A)^\perp = \dim\mathscr{H}$, then $\dim(\text{ran } A)^\perp = \dim\ker A$. Thus,

$$\dim\ker A = \dim\ker A^*.$$

□

Hunter and Nachtergaele 8.12

Suppose that $A : \mathscr{H} \to \mathscr{H}$ is a bounded, self-adjoint linear operator such that there is a constant $c > 0$ with

$$c\|x\| \leq \|Ax\| \qquad \textit{for all } x \in \mathscr{H}.$$

Prove that there is a unique solution x of the equation $Ax = y$ for every $y \in \mathscr{H}$.

Proof. Let $x \neq 0$. Then $\|x\| > 0$. Then $\frac{\|Ax\|}{c} \geq \|x\| > 0 \implies \|Ax\| > 0$, which shows $Ax \neq 0$, and thus $\ker A = \{0\}$. Since A is self-adjoint, then $A = A^*$ and thus $\ker A^* = \{0\}$. Since $\langle y, 0 \rangle = 0\ \forall y \in \mathscr{H}$, then $y \perp \ker A^*$. Then by Theorem 8.18 in Hunter Nachtergaele, $\exists x \in \mathscr{H}$ such that $y = Ax$. Suppose $y = Ax_1 = Ax_2$. Then $A(x_1 - x_2) = 0$, which implies $x_1 - x_2 \in \ker A = \{0\}$, thus $x_1 = x_2$, i.e. the solution to $y = Ax$ is unique. □