HW #2

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Lieb and Loss Exercise 1.9

Verify the linearity of the integral as given in 1.5(7) by completing the steps outlined below. In what follows, f and g are nonnegative summable functions.

a)

Show that $f+g$ is also summable. In fact, by a simple argument $\int(f+g) \leq 2\left(\int f + \int g\right)$.

To show $\int(f+g) \leq 2\left(\int f + \int g\right)$, first note that

$$S_{f+g}(t) = \{x \ : \ (f+g)(x) > t\} \subset \left\{x \ : \ f(x) > \frac{t}{2}\right\} \cup \left\{x \ : \ g(x) > \frac{t}{2}\right\} = S_f\left(\frac{t}{2}\right) \cup S_g\left(\frac{t}{2}\right)$$

Since $f(x) \leq \frac{t}{2}$ and $g(x) \leq \frac{t}{2}$ implies $(f+g)(x) = f(x) + g(x) \leq t$. By properties of measures,

$$\mu(S_{f+g}(t)) \leq \mu\left(S_f\left(\frac{t}{2}\right) \cup S_g\left(\frac{t}{2}\right)\right) \leq \mu\left(S_f\left(\frac{t}{2}\right)\right) + \mu\left(S_g\left(\frac{t}{2}\right)\right)$$
$$\implies \int_0^\infty \mu(S_{f+g}(t))\mathrm{d}t \leq \int_0^\infty \mu\left(S_f\left(\frac{t}{2}\right)\right)\mathrm{d}t + \int_0^\infty \mu\left(S_g\left(\frac{t}{2}\right)\right)\mathrm{d}t$$

Note the integral on the right hand side can split linearly because it is a Riemann integral. By u-substitution with $u = \frac{t}{2}$, we get

$$\int_0^\infty \mu(S_{f+g}(t))\mathrm{d}t \leq 2\int_0^\infty S_f(t)\mathrm{d}t + 2\int_0^\infty S_g(t)\mathrm{d}t$$

Note the constant 2 can be factored of each integral on the right hand side linearly because they are Riemann integrals. Thus, by definition,

$$\int(f+g) \leq 2\left(\int f + \int g\right)$$

and since f and g are summable, $\int f$ and $\int g$ are finite, which proves $\int(f+g)$ is finite. Next we confirm $S_{f+g}(t) \in \Sigma$. Construct a function $A \ : \ \Omega \to \mathbb{R}^2$ by

$$A(x) = (f(x), g(x))$$

and a function $B : \mathbb{R}^2 \to \mathbb{R}$ by

$$B(x, y) = x + y$$

Since A and B are measurable, then $B \circ A$ is measurable (since the composition of measurable functions is measurable). Thus $\{x : (f+g)(x) > t\} = \{x : B(A(x)) > t\}$ is measurable, and hence $S_{f+g}(t) \in \Sigma$. Thus $f + g$ is summable. $\square$

b)

For any integer N find two functions f_N and g_N that take only finitely many values, such that $|\int f - \inf f_N| \leq \frac{C}{N}$, $|\int g - \int g_N| \leq \frac{C}{N}$, $|\int (f+g) - \int (f_N - g_N)| \leq \frac{C}{N}$ for some constant C independent of N.

c)

Show that for f_N and g_N as above $\int (f_N + g_N) = \int f_N + \int g_N$, thus proving the addivitivity of te integral for nonnegative functions.

Since f_N and g_N are simple functionsm they take on finitely many values, i.e.

$$f_N = \sum_{i=1}^{M} c_i \mu(E_i) \quad \text{and}$$
$$g_N = \sum_{j=1}^{M} d_j \mu(D_j)$$

Note both summations can be written with the same limit since we can always add finitely many terms where either c_i or d_j are zero.

$$\int_\Omega f_N \mathrm{d}\mu = \int_0^\infty F_{f_N} \mathrm{d}t = \sum_{i=1}^{M} c_i \mu(E_i) \quad \text{and}$$
$$\int_\Omega g_N \mathrm{d}\mu = \int_0^\infty F_{g_N} \mathrm{d}t = \sum_{j=1}^{M} d_j \mu(D_j)$$

Then

$$\int_\Omega (f_N + g_N) = \int_\Omega \sum_{i=1}^{M} (c_i \mu(E_i) + d_i \mu(D_i))$$

d)

In a similar fashion, show that for $f, g \geq 0$, $\int (f - g) = \int f - \int g$.

e)

Now use c) and d) to prove the linearity of the integral.

Lieb and Loss Exercise 1.10

Prove that when we add and subtract the subsets of sets of zero measure to the sets of a sgma-algebra then the result is again a sigma-algebra and the extended measure is again a measure.

Consider a measure space (Ω, Σ, μ) and let $\mathcal{A}$ be the collection of measurable sets of measure zero:

$$\mathcal{A} = \{A \in \Sigma \ : \ \mu(A) = 0\}$$

For each $A \in \mathcal{A}$, let $\mathbb{P}(A)$ be the power set of A, i.e.

$$\mathbb{P}(A) = \{a \ : \ a \subset A\}$$

Next, let $\overline{\Sigma}$ be a superset of Σ, consisting of the "addition" and "subtraction" of the subsets of sets of measure to each set:

$$\overline{\Sigma} = \Sigma \cup \Sigma^+ \cup \Sigma^-$$

where

$$\begin{aligned} \Sigma^+ &= \{\sigma \cup a \ : \ \sigma \in \Sigma \text{ and } a \in \mathbb{P}(A) \text{ for some } A \in \mathcal{A}\} \\ \Sigma^- &= \{\sigma \setminus a \ : \ \sigma \in \Sigma \text{ and } a \in \mathbb{P}(A) \text{ for some } A \in \mathcal{A}\} \end{aligned} \tag{1}$$

Let $\overline{\mu}$ map sets in $\overline{\Sigma}$ to the nonnegative reals, including infinity, i.e. $\mu \ : \ \overline{\Sigma} \to \mathbb{R}_0^+ \cup \{\infty\}$, by extending the measure μ.

$$\overline{\mu}(\overline{\sigma}) = \begin{cases} \mu(\overline{\sigma}) & \text{if } \overline{\sigma} \in \Sigma \\ \mu(\sigma) & \text{if } \overline{\sigma} \in \Sigma^+ \cup \Sigma^- \text{ where } \sigma \text{ is defined in (1)} \end{cases}$$

We want to show $(\Omega, \overline{\Sigma}, \overline{\mu})$ is a measure space. To do this, we must show $\overline{\Sigma}$ is a σ-algebra and $\overline{\mu}$ is a measure on $\overline{\Sigma}$. First we will show $\overline{\Sigma} = \Sigma^+$ by showing (i) $\Sigma \subset \Sigma^+$ and (ii) $\Sigma^- \subset \Sigma^+$.

(i) Since $\emptyset$ is a subset of all sets, then for $\sigma \in \Sigma$, $\sigma = \sigma \cup \emptyset \in \Sigma^+$. Thus $\Sigma \subset \Sigma^+$.

(ii) Let $\sigma \setminus a \in \Sigma^-$. Then $a \subset A$ for some $A \in \mathcal{A}$. Also, $A \setminus a \subset A$, and

$$\begin{aligned} \sigma \setminus a &= \sigma \cap (a^C) \\ &= \sigma \cap (A^C \cup (A \setminus a)) \\ &= (\sigma \cap A^C) \cup (\sigma \cap (A \setminus a)) \end{aligned}$$

But Σ is a σ-algebra, which implies it is closed under finite intersections, and thus σ, $A^C \in \Sigma$ implies $\hat{\sigma} = \sigma \cap A^C \in \Sigma$. Also, $\hat{a} = \sigma \cap (A \setminus a) \subset A \setminus a \subset A$. Thus

$$\sigma \setminus a = \hat{\sigma} \cup \hat{a} \in \Sigma^+$$

which proves $\Sigma^- \subset \Sigma^+$.

This shows

$$\overline{\Sigma} = \Sigma^+$$

Next we will show $\overline{\Sigma}$ is a σ-algebra. To do this we must show (i) it is closed under complementation, (ii) it is closed under countable unions, and (iii) $\Omega \in \overline{\Sigma}$.

(i) Let $x \in \overline{\Sigma} = \Sigma^+$. Then $x = \sigma \cup a$ for some $\sigma \in \Sigma$ and $a \in \mathbb{P}(A)$. Since $\sigma^C \in \Sigma$, then

$$x^C = (\sigma \cup a)^C = \sigma^C \cap a^C = \sigma^C \setminus a \in \Sigma^- \subset \overline{\Sigma}$$

(ii) Let $x = \bigcup_{n=1}^\infty \overline{\sigma}_n$ where $\overline{\sigma}_n \in \overline{\Sigma} = \Sigma^+$ for $n = 1, 2, \dots$. Each $\overline{\sigma}_n$ can be written as

$$\overline{\sigma}_n = \sigma_n \cup a_n$$

for some $\sigma_n \in \Sigma$ and $a_n \subset A_n$ for some $A_n \in \mathcal{A}$. This means that by the commutativity of unions, we can write x as

$$x = \bigcup_{n=1}^\infty \overline{\sigma}_n = \bigcup_{n=1}^\infty (\sigma_n \cup a_n) = \bigcup_{n=1}^\infty \sigma_n \cup \bigcup_{n=1}^\infty a_n = \hat{\sigma} \cup \bigcup_{n=1}^\infty a_n$$

where $\hat{\sigma} = \bigcup_{n=1}^\infty \sigma_n \in \Sigma$ since Σ is closed under countable unions. Since μ is a measure, it has the property of countable additivity, which means

$$\mu\Big(\bigcup \mathcal{A}\Big) = \mu\left(\bigcup_{A \in \mathcal{A}} A\right) = \sum_{A \in \mathcal{A}} \mu(A) = \sum_{A \in \mathcal{A}} 0 = 0$$

This means $\bigcup \mathcal{A} \in \mathcal{A}$ since it has measure zero. Thus

$$\bigcup_{n=1}^\infty a_n \subset \bigcup_{n=1}^\infty A_n \subset \bigcup \mathcal{A} \in \mathcal{A}$$

Defining $b = \bigcup_{n=1}^\infty a_n \in \mathbb{P}(\bigcup \mathcal{A})$, we can write

$$\bigcup_{n=1}^\infty (\sigma_n \cup a_n) = \hat{\sigma} \cup \bigcup_{n=1}^\infty a_n = \hat{\sigma} \cup b \in \Sigma^+$$

Thus $\overline{\Sigma} = \Sigma^+$ is closed under countable unions.

(iii) Since Σ is a σ-algebra, $\Omega \in \Sigma \subset \overline{\Sigma}$.

Next we will show $\overline{\mu}$ is a measure on $\overline{\Sigma} = \Sigma^+$. We have to show (i) $\overline{\mu}(\emptyset) = 0$, (ii) $\overline{\mu}$ has the property of countable additivity, and (iii) $\overline{\mu}$ is well-defined.

(i) Since $\emptyset \in \Sigma$, then $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$ since μ is a measure on Σ.

(ii) Let $\overline{\sigma}_1, \overline{\sigma}_2, \dots$ be a sequence of disjoint sets in $\overline{\Sigma}$. Then for each n,

$$\overline{\sigma}_n = \sigma_n \cup a_n$$

for some $\sigma_n \in \Sigma$ and $a_n \subset A_n$ for some $A_n \in \mathcal{A}$. This means that $\{\sigma_n\}_n$ is a sequence of disjoint sets and $\{a_n\}_n$ is a sequence of disjoint sets. We showed earlier that

$$\bigcup_{n=1}^\infty a_n \subset \bigcup \mathcal{A}$$

so denote $b = \bigcup_{n=1}^{\infty} a_n$ and note

$$\bigcup_{n=1}^{\infty} \sigma_n \in \Sigma$$

Then

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} \overline{\sigma}_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} (\sigma_n \cup a_n)\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} \sigma_n \cup \bigcup_{n=1}^{\infty} a_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} \sigma_n \cup b\right) = \mu\left(\bigcup_{n=1}^{\infty} \sigma_n\right)$$

by the definition of $\overline{\mu}$. Thus by the countable additivity of μ,

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} \overline{\sigma}_n\right) = \mu\left(\bigcup_{n=1}^{\infty} \sigma_n\right) = \sum_{n=1}^{\infty} \mu(\sigma_n) = \sum_{n=1}^{\infty} \overline{\mu}(\sigma_n \cup a_n) = \sum_{n=1}^{\infty} \overline{\mu}(\overline{\sigma}_n)$$

This shows $\overline{\mu}$ has countable additivity.

(iii) Let $\overline{\sigma} \in \overline{\Sigma}$ be represented in two arbitrary ways:

$$\overline{\sigma} = \sigma_1 \cup a_1 = \sigma_2 \cup a_2$$

for $a_1 \subset A_1$ and $a_2 \subset A_2$ for some $A_1, A_2 \in \mathcal{A}$. Then let $A = A_1 \cup A_2$ and note that $\sigma_1 \subset A$ and $\sigma_2 \subset A$. Note

$$A = [(A \setminus A_1) \cup (A \setminus A_2)] \cup (A_1 \cap A_2)$$

Then

$$\sigma_1 \cup [(A_1 \setminus A_2) \cup (A_1 \cup A_2)] = \sigma_1 \cup A_1 = \sigma_2 \cup A_2 = \sigma_2 \cup [(A_2 \setminus A_1) \cup (A_2 \cup A_1)]$$

but

$$\begin{aligned} A_1 \setminus A_2 &\subset (A \setminus A_1) \cup (A \setminus A_2) \quad \text{and} \\ A_2 \setminus A_1 &\subset (A \setminus A_1) \cup (A \setminus A_2) \end{aligned}$$

and thus

$$\sigma_1 \cup A = \sigma_1 \cup [(A \setminus A_1) \cup (A \setminus A_2)] \cup (A_1 \cap A_2) = \sigma_2 \cup [(A \setminus A_1) \cup (A \setminus A_2)] \cup (A_1 \cap A_2) = \sigma_2 \cup A$$

This implies

$$\mu(\sigma_1) = \mu(\sigma_1) + \mu(A) = \mu(\sigma_1 \cup A) = \mu(\sigma_2 \sup A) = \mu(\sigma_2) + \mu(A) = \mu(\sigma_2)$$

This shows that $\overline{\mu}(\overline{\sigma})$ well-defined regardless of how it is represented.

Thus $\overline{\mu}$ is a measure on $\overline{\Sigma}$.

Lieb and Loss Exercise 1.12

Find a simple condition for $f_n(x)$ so that

$$\sum_{n=0}^{\infty} \int_{\Omega} f_n(x)\mu(\mathrm{d}x) = \int_{\Omega} \left[\sum_{n=0}^{\infty} f_n(x)\right]\mu(\mathrm{d}x)$$

Let f_{n_n} be a sequence of positive functions. Then let

$$g_n = \sum_{i=0}^{n} f_i$$

be the nth partial sum of $\sum_{n=0}^{\infty} f_n$. Then g_n is an increasing sequence of functions that converges pointwise to $\sum_{n=0}^{\infty} f_n$ in Ω. Then by the monotone convergence theorem,

$$\int_{\Omega} \sum_{n=0}^{\infty} f_n \mathrm{d}\mu = \lim_{n\to\infty} \int_{\Omega} g_n \mathrm{d}\mu$$

and thus

$$\int_{\Omega} \left[\sum_{n=0}^{\infty} f_n\right] \mathrm{d}\mu = \lim_{n\to\infty} \int_{\Omega} \left[\sum_{i=0}^{n} f_i\right] \mathrm{d}\mu = \lim_{n\to\infty} \sum_{i=0}^{n} \int_{\Omega} f_i \mathrm{d}\mu = \sum_{n=0}^{\infty} \int_{\Omega} f_n \mathrm{d}\mu$$

which proves the result.

Lieb and Loss Exercise 1.13

Let f be the function on $\mathbb{R}^n$ defined by $f(x) = |x|^{-p}\mathcal{X}_{\{|x|<1\}}(x)$. Compute $\int f\mathrm{d}\mathcal{L}^n$ in two ways: (i) Use polar coordinates and compute the integral by the standard calculus method. (ii) Compute $\mathcal{L}^n(\{x \ : \ f(x) > a\})$ and then use Lebesgue's definition.

(i) First note that

$$f(x) = \begin{cases} |x|^{-p} & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}$$

Then note that polar coordinates on $\mathbb{R}^n$ are $(r, \phi, \theta_1, \theta_2, \ldots, \theta_{n-2})$ where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $\theta_i \in [0, \pi)$ for $i = 1, 2, \ldots, n-2$. When transforming rectangular coordinates to polar coordinates in n dimensions, we multiply by the determinant of the Jacobian matrix, and so

$$\int f\mathrm{d}\mathcal{L}^n = \int_0^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} r^{-p}\left[\frac{1}{r^{1-n}}\prod_{k=1}^{n-2}\sin^{n-k-1}(\theta_k)\right] \mathrm{d}r\ \mathrm{d}\phi\ \mathrm{d}\theta_1 \ldots\ \mathrm{d}\theta_{n-3}\ \mathrm{d}\theta_{n-2}$$

The integrand is separable, and thus

$$\int f\mathrm{d}\mathcal{L}^n = \int_0^{\infty} \frac{1}{r^{p-n+1}}\mathrm{d}r \left[\prod_{k=1}^{n-2}\int_0^{\pi} \sin^{n-k-1}(\theta_k)\mathrm{d}\theta_k\right]\int_0^{2\pi} \mathrm{d}\phi$$

$$= 2\pi \left[\prod_{k=1}^{n-2} \int_0^{\pi} \sin^{n-k-1}(\theta_k) \mathrm{d}\theta_k \right] \int_0^{\infty} \frac{1}{r^{p-n+1}} \mathrm{d}r$$

But

$$\left| \mathbb{S}^{n-1} \right| = 2\pi \prod_{k=1}^{n-2} \int_0^{\pi} \sin^{n-k-1}(\theta_k) \mathrm{d}\theta_k$$

where $|\mathbb{S}^{n-1}|$ is the surface area of an n-dimensional sphere of radius 1. Thus,

$$\int f \mathrm{d}\mathcal{L}^n = |\mathbb{S}^{n-1}| \int_0^{\infty} \frac{1}{r^{p-n+1}} \mathrm{d}r = |\mathbb{S}^{n-1}| \int_0^{1} \frac{1}{r^{p-n+1}} \mathrm{d}r = \begin{cases} |\mathbb{S}^{n-1}| \frac{1}{n-p} & , n > p \\ +\infty & , n \leq p \end{cases}$$

(ii) First note the Lebesgue measure of $f^{-1}(t, \infty)$ for a fixed t.

$$\begin{aligned} \mathcal{L}^n(f^{-1}(t,\infty)) &= \mathcal{L}^n(\{x \ : \ f(x) > t\}) \\ &= \mathcal{L}^n(\{x \in B_1(0) \ : \ |x| < t^{-1/p}\}) \\ &= \begin{cases} \mathcal{L}^n(B_1(0)) & \text{if } t^{-1/p} \geq 1 \\ \mathcal{L}^n(B_{t^{-1/p}}(0)) & \text{if } t^{-1/p} < 1 \end{cases} \\ &= \begin{cases} \frac{1}{n}|\mathbb{S}^{n-1}| & \text{if } t^{-1/p} \geq 1 \\ \frac{1}{n}|\mathbb{S}^{n-1}| t^{-n/p} & \text{if } t^{-1/p} < 1 \end{cases} \end{aligned}$$

Now integrate over $t \in [0, \infty]$:

$$\begin{aligned} \int_0^{\infty} \mathcal{L}^{n-1}(f^{-1}(t,\infty)) &= \int_0^1 \left[\frac{1}{n} |\mathbb{S}^{n-1}| \right] \mathrm{d}t + \int_1^{\infty} \left[\frac{1}{n} |\mathbb{S}^{n-1}| t^{-n/p} \right] \mathrm{d}t \\ &= \frac{1}{n} |\mathbb{S}^{n-1}| \left(1 + \left[\frac{t^{-\frac{n}{p}+1}}{-\frac{n}{p}+1} \right]_1^{\infty} \right) \end{aligned}$$

If $p \geq n$,

$$\left(1 + \left[\frac{t^{-\frac{n}{p}+1}}{-\frac{n}{p}+1} \right]_1^{\infty} \right) = \infty$$

but if $p < n$,

$$\left(1 + \left[\frac{t^{-\frac{n}{p}+1}}{-\frac{n}{p}+1} \right]_1^{\infty} \right) = \frac{p}{p-n}$$

Thus,

$$\int_0^{\infty} \mathcal{L}^{n-1}(f^{-1}(t,\infty)) = \begin{cases} \infty & \text{if } p \geq n \\ \frac{1}{n-p} & \text{if } p < n \end{cases}$$

which matches with our answer in part (i).

Lieb and Loss Exercise 1.17

Show that the infimum of a family of continuous functions is upper semi-continuous.

Let $\mathcal{F} = \{f_i \in [\Omega \to \mathbb{R}] \ : \ f_i \text{ is continuous, and } i \in I\}$ where I is some index set. Then define $f \in [\Omega \to \mathbb{R}]$ by

$$f(x) = \inf_{i \in I} f_i(x)$$

Assume f is not upper semi-continuous at x. Then there is a sequence $\{x_n\}_n \to x$ such that

$$\limsup_{x_n \to x} f(x_n) > f(x)$$

So there is some ε such that $\limsup_{x_n \to x} f(x_n) = f(x) + \varepsilon$. By definition of the infimum, there is a function f_i such that $f_i(x) < f(x) + \frac{\varepsilon}{2}$. The continuity of f_i implies $\exists \delta$ such that $|x - x_0| < \delta \implies |f_i(x) - f_i(x_0)| < \frac{\varepsilon}{2}$. Then for $|x_n - x| < \delta$, $|f_i(x_n) - f_i(x)| < \frac{\varepsilon}{2}$. But $f_i(x) < f(x) + \frac{\varepsilon}{2}$, and thus

$$\begin{aligned} f_i(x_n) - \left(f(x) + \frac{\varepsilon}{2}\right) &< f_i(x_n) - f_i(x) < \frac{\varepsilon}{2} \\ \implies f_i(x_n) - f(x) &< \varepsilon \end{aligned}$$

However, by the definition of f,

$$f(x_n) \leq f_i(x_n) < f(x) + \varepsilon$$

which implies

$$\limsup_{x_n \to x} f(x_n) < f(x) + \varepsilon$$

which is a contradiction since $\limsup_{x_n \to x} f(x_n) = f(x) + \varepsilon$. Thus f is upper semi-continuous at x. Since x was arbitrary, f is upper semi-continuous on its domain.

Lieb and Loss Exercise 1.18

Simple facts about measure:

a)

Show that the condition $\{x \ : \ f(x) > a\}$ is measureiable for all $a \in \mathbb{R}$ holds if and only if it holds for all rational a.

Suppose $\{x \ : \ f(x) > a\} \in \Sigma$ for all $a \in \mathbb{Q}$. Then for $a \in \mathbb{R} \setminus \mathbb{Q}$, let $\{a_i\}_i$ be an increasing sequence in $\mathbb{Q}$ such that $\{a_i\} \to a$. Then

$$\{x \ : \ f(x) > a\} = \bigcap_{n=1}^{\infty} \{x \ : \ f(x) > a_n\} \in \Sigma$$

because Σ is closed under countable intersections.

b)

For rational a, show that

$$\{x\ :\ f(x)+g(x)>a\}=\bigcup_{b\in\mathbb{Q}}\left(\{x\ :\ f(x)>b\}\cap\{x\ :\ g(x)>a-b\}\right)$$

For ease, define $A=\{x\ :\ f(x)+g(x)>a\}$ and $B=\bigcup_{b\in\mathbb{Q}}\left(\{x\ :\ f(x)>b\}\cap\{x\ :\ g(x)>a-b\}\right)$.

Suppose $x\in A$. Then $f(x)+g(x)>a$. Then $\exists\varepsilon>0$ such that $f(x)+g(x)=a+\varepsilon$. Now choose $b\in\mathbb{Q}\cap(f(x)-\varepsilon,f(x))$. Then $f(x)-\varepsilon<b<f(x)$, i.e. $f(x)<b+\varepsilon<f(x)+\varepsilon$. If $g(x)\leq a-b$, then $f(x)+g(x)\leq f(x)+a-b<b+\varepsilon+a-b=a+\varepsilon$, which is a contradiction since $f(x)+g(x)=a+\varepsilon$. Thus $x\in B$, showing $A\subset B$.

Suppose $x\in B$. Then $\exists b\in\mathbb{Q}$ such that $f(x)>b$ and $g(x)>a-b$. Then $f(x)+g(x)>b+a-b=a$, and thus $x\in A$, showing $B\subset A$.

Thus,

$$\{x\ :\ f(x)+g(x)>a\}=\bigcup_{b\in\mathbb{Q}}\left(\{x\ :\ f(x)>b\}\cap\{x\ :\ g(x)>a-b\}\right)$$

c)

In a similar way, prove that fg is measurable if f and g are measurable.

We want to show if $f^{-1}(t,\infty)\in\Sigma$ and $g^{-1}(t,\infty)\in\Sigma$, then $(fg)^{-1}(t,\infty)\in\Sigma$. We will show this for $t\in\mathbb{Q}$, but by part **b)**, this is equivalent to showing it for $t\in\mathbb{R}$.

To show $(fg)^{-1}(t,\infty)\in\Sigma$, we will show

$$(fg)^{-1}(t,\infty)=\bigcup_{b\in\mathbb{Q}}\left(f^{-1}(b,\infty)\cap g^{-1}\left(\frac{a}{b},\infty\right)\right)$$

For ease, define $A=(fg)^{-1}(t,\infty)$ and $B=\bigcup_{b\in\mathbb{Q}}\left(f^{-1}(b,\infty)\cap g^{-1}\left(\frac{a}{b},\infty\right)\right)$.

Suppose $x\in A$. Then $f(x)+g(x)>a$. Then $\exists\varepsilon>0$ such that $fg(x)=a(1+\varepsilon)$. Now choose $b\in\mathbb{Q}\cap\left(\dfrac{f(x)}{1+\varepsilon},f(x)\right)$. Then $\dfrac{f(x)}{1+\varepsilon}<b<f(x)$, i.e. $f(x)<b(1+\varepsilon)<f(x)(1+\varepsilon)$. If $g(x)\leq\dfrac{a}{b}$, then $(fg)(x)\leq\dfrac{af(x)}{b}<\dfrac{ab(1+\varepsilon)}{b}=a(1+\varepsilon)$, which is a contradiction since $(fg)(x)=a(1+\varepsilon)$. Thus $x\in B$, showing $A\subset B$.

Suppose $x\in B$. Then $\exists b\in\mathbb{Q}$ such that $f(x)>b$ and $g(x)>\dfrac{a}{b}$. Then $(fg)(x)>\dfrac{ba}{b}=a$, and thus $x\in A$, showing $B\subset A$.

Thus, for $t\in\mathbb{Q}$,

$$(fg)^{-1}(t,\infty)=\bigcup_{b\in\mathbb{Q}}\left(f^{-1}(b,\infty)\cap g^{-1}\left(\frac{a}{b},\infty\right)\right)$$

Then since $\mathbb{Q}$ is countable and $(fg)^{-1}(t,\infty)$ is a countable union and intersection of elements in Σ, then $(fg)^{-1}(t,\infty)\in\Sigma$. By part **a)**, this shows the above holds for $t\in\mathbb{R}$ and thus f,g measurable imply fg is measurable.

Hunter and Nachtergaele Exercise 6.1

Prove that a closed, convex subset of a Hilbert space has a unique point of minimum norm.

Let A be a closed and convex subset of a Hilbert space $\mathcal{H}$. Let d be the distance of $\vec{0}$ from A,

$$d = \inf_{x \in A} \{\|x\|\}$$

First we prove that there is a closest point $z \in A$ at which this infimum is attained. From the definition of d, there is a sequence of elements $z_n \in A$ such that

$$\lim_{n \to \infty} \|z_n\| = d$$

Thus $\forall \varepsilon$, $\exists N_\varepsilon$ such that

$$\|z_n\| \leq d + \varepsilon \qquad \text{when } n \geq N_\varepsilon$$

Next we will show $\{z_n\}_n$ is Cauchy. Let $n, m \geq N_\varepsilon$. The parallelogram law implies

$$\|z_m - z_n\|^2 + \|z_m + z_n\|^2 = 2\|z_n\|^2 + 2\|z_m\|^2$$

Since A is convex, $\frac{z_m + z_n}{2} \in A$, and thus

$$\left\| \frac{z_m + z_n}{2} \right\| \leq d$$

by the definition of d. Thus,

$$\|z_m + z_n\|^2 \leq 4d^2$$

which implies

$$\begin{aligned}
\|z_m - z_n\|^2 + 4d^2 &\leq 2\|z_m\|^2 + 2\|z_n\|^2 \\
\implies \|z_m - z_n\|^2 &\leq 2(d+\varepsilon)^2 + 2(d+\varepsilon)^2 - 4d^2 \\
&= 4\varepsilon(2d + \varepsilon)
\end{aligned}$$

which is arbitrarily small as $\varepsilon \to 0$. Thus $\{z_n\}_n$ is Cauchy. The completeness of Hilbert spaces implies $\{z_n\}_n$ converges to a limit, but since A is closed, $\lim_{n \to \infty} z_n = z \in A$. By the continuity of $\|\cdot\|$,

$$\|z\| = \left\| \lim_{n \to \infty} z_n \right\| = \lim_{n \to \infty} \|z_n\| = d$$

Thus there is a point at which A achieves minimum norm. Next, we prove uniqueness. Suppose $\|z_1\| = \|z_2\| = 0$. Then by the parallelogram law,

$$2\|z_1\|^2 + 2\|z_2\|^2 = \|z_1 + z_2\|^2 + \|z_1 - z_2\|^2$$

Again, the convexity of A implies $\frac{z_1 + z_2}{2} \in A$, and thus

$$\|z_1 - z_2\|^2 = 4d^2 - 4\left\| \frac{z_1 + z_2}{2} \right\|^2 \leq 4d^2 - 4d^2 = 0$$

But norm is non-negative, i.e. $\|z_1 - z_2\| = 0$. Thus $z_1 = z_2$. Thus the point of minimum norm is unique.

Hunter and Nachtergaele Exercise 6.3

If A is a subset of a Hilbert space, prove that

$$A^{\perp} = \overline{A}^{\perp},$$

where $\overline{A}$ is the closure of A. If $\mathcal{M}$ is a linear subspace of a Hilbert space, prove that

$$\mathcal{M}^{\perp\perp} = \overline{\mathcal{M}}.$$

Let $x \in A^{\perp}$ and choose any $y \in \overline{A}$. Then $\exists \{y_n\}_n \in A$ such that $y_n \to y$. But since $y_n \in A$, $x \perp y_n$ for all n. Thus, by the continuity of inner products,

$$\langle x, y \rangle = \langle x, \lim_{n\to\infty} y_n \rangle = \lim_{n\to\infty} \langle x, y_n \rangle = \lim_{n\to\infty} 0 = 0$$

and thus $x \perp y$, which shows $x \in \overline{A}^{\perp}$, and hence $A^{\perp} \subset \overline{A}^{\perp}$.

Now let $x \in \overline{A}^{\perp}$. Then $x \perp y \ \forall y \in \overline{A}$. But $A \subset \overline{A}$, and thus trivially, $x \perp y \ \forall y \in A$, i.e. $x \in A^{\perp}$. Hence $\overline{A}^{\perp} \subset A^{\perp}$.

Thus, $A^{\perp} = \overline{A}^{\perp}$.

Let $\mathcal{M}$ be a linear subspace of $\mathcal{H}$. Assume $x \in \overline{\mathcal{M}}$. Then there is a sequence $x_n \in \mathcal{M}$ such that $x_n \to x$. Then $\langle x_n, y \rangle = 0 \ \forall y \in \mathcal{M}^{\perp}$. Then by continuity of inner products,

$$\langle x, y \rangle = \langle \lim_{n\to\infty} x_n, y \rangle = \lim_{n\to\infty} \langle x_n, y \rangle = \lim_{n\to\infty} 0 = 0 \qquad \forall y \in \mathcal{M}^{\perp}$$

Then $x \in \mathcal{M}^{\perp\perp}$, which shows $\overline{\mathcal{M}} \subset \overline{M}^{\perp\perp}$.

Now assume $x \notin \overline{\mathcal{M}}$. Since $\overline{\mathcal{M}}$ is closed, the by the Projection Theorem, $\exists y \in \overline{\mathcal{M}}$ such that $(x-y) \perp \overline{\mathcal{M}}$. Since $y \in \overline{\mathcal{M}}$, $\langle x-y, y \rangle = 0$. Since $x \neq y$ ($x \notin \overline{\mathcal{M}}$ and $y \in \overline{\mathcal{M}}$), then $\langle x-y, x-y \rangle \neq 0$. However, $\langle x-y, x-y \rangle = \langle x-y, x \rangle - \langle x-y, y \rangle = \langle x-y, x \rangle$. Since $x - y \perp \overline{\mathcal{M}}$, then $x - y \perp \mathcal{M}$, i.e. $x - y \in \mathcal{M}^{\perp}$. Then since $\langle x-y, x \rangle \neq 0$, then $x \notin \overline{\mathcal{M}}^{\perp\perp} = \mathcal{M}^{\perp\perp}$, which shows $\mathcal{M}^{\perp\perp} \subset \overline{\mathcal{M}}$.

Thus $\overline{\mathcal{M}} = \mathcal{M}^{\perp\perp}$.

Hunter and Nachtergaele Exercise 6.5

Suppose that $\{\mathcal{H}_n \ : \ n \in \mathbb{N}\}$ is a set of orthogonal closed subspaces of a Hilbert space $\mathcal{H}$. We define the infinite direct sum

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \left\{ \sum_{n=1}^{\infty} x_n \mid x_n \in \mathcal{H}_n \quad \textit{and} \quad \sum_{n=1}^{\infty} \|x_n\|^2 < +\infty \right\}.$$

Prove that $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a closed linear subspace of $\mathcal{H}$.

First we show $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is linear. Consider $x, y \in \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ where

$$x = \sum_{n=1}^{\infty} x_n \qquad \text{and} \qquad y = \sum_{n=1}^{\infty} y_n$$

Then since each $\mathcal{H}_n$ is linear, then $c_n = ax_n + by_n \in \mathcal{H}_n$ for each n. Thus

$$ax + by = a\sum_{n=1}^{\infty} x_n + b\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} (ax_n + by_n) = \sum_{n=1}^{\infty} c_n$$

Now we need to show $\sum_{n=1}^{\infty} \|c_n\|^2 < \infty$. Consider $x^{(N)}$ and $y^{(N)}$ where

$$x^{(N)} = \sum_{n=1}^{N} x_n \quad \text{and} \quad y^{(N)} = \sum_{n=1}^{N} y_n$$

Then

$$\left\|ax^{(N)} + by^{(N)}\right\|^2 = \left\|\sum_{n=1}^{N} (ax_n + by_n)\right\|^2 = \left\|\sum_{n=1}^{N} c_n\right\|^2 = \sum_{n=1}^{N} \|c_n\|^2$$

by the pythagorean theorem. However, since the norm is continuous,

$$\lim_{N\to\infty} \left\|ax^{(N)} + by^{(N)}\right\|^2 = \|ax + by\|^2 = \sum_{n=1}^{\infty} \|c_n\|^2$$

Since $ax + by \in \mathcal{H}$, then $\|ax + by\| \in \mathbb{R}$ by the definition of norm. Thus $\|ax + by\|^2 \in \mathbb{R}$ and hence $< \infty$. Thus $ax + by \in \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, which shows $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a linear subspace.

Hunter and Nachtergaele Exercise 6.8

Let $\mathcal{X} = \{x_n \ : \ n \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space. Show that the sum $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n}$ converges unconditionally but not absolutely.

Let $y_n = \dfrac{x_n}{n}$ and let $\mathcal{Y} = \{y_n \ : \ n \in \mathbb{N}\}$. Since each y_n is a scalar multiple of x_n for all n, and since $\mathcal{X}$ is an orthonomal set, then $\mathcal{Y}$ is an orthogonal set. Thus by the Pythagorean Theorem, $\sum_{n=1}^{\infty} y_n$ converges unconditionally if and only if $\sum_{n=1}^{\infty} \|y_n\|^2$ converges. But

$$\sum_{n=1}^{\infty} \|y_n\|^2 = \sum_{n=1}^{\infty} \frac{\|x_n\|^2}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

by the p-series test. Thus $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{x_n}{n}$ converges unconditionally. However,

$$\sum_{n=1}^{\infty} \left\|\frac{x_n}{n}\right\| = \sum_{n=1}^{\infty} \frac{\|x_n\|}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$$

And so $\sum_{n=1}^{\infty} \frac{x_n}{n}$ does not converge absolutely.

Hunter and Nachtergaele Exercise 6.12

Define the Legendre polynomials P_n *by*

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

(a) *Compute the first few Legendre polynomials, and compare with what you get by Gram-Schmidt orthogonalization of the monomials* $\{1, x, x^2, \dots\}$ *in* $L^2([-1,1])$.

$$\begin{aligned}
P_0(x) &= \frac{1}{2^0 0!}(x^2-1)^0 = 1\\
P_1(x) &= \frac{1}{2^1 1!}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)^1 = \frac{1}{2}2x = x\\
P_2(x) &= \frac{1}{2^2 2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 = \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^4-2x^2+1) = \frac{1}{8}(12x^2-4) = \frac{3}{2}x^2 - \frac{1}{2}\\
P_3(x) &= \frac{1}{2^3 3!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3 = \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^6-3x^4+3x^2-1) = \frac{1}{48}(120x^3-72x) = \frac{5}{2}x^3 - \frac{3}{2}x
\end{aligned}$$

These polynomials are scalar multiples of the results of the Gram-Schmidt orthogonalization of the monomials $\{1, x, x^2, \dots\}$ in $L^2([-1,1])$.

(b) *Show that the Legendre polynomials are orthogonal in* $L^2([-1,1])$, *and that they are obtained by Gram-Schmidt orthongonalization of the monomials*

Fix n and pick $m < n$. Then

$$\begin{aligned}
\langle x^m, P_n\rangle &= \int_{-1}^{1} x^m P_n \mathrm{d}x\\
&= \int_{-1}^{1} x^m \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \mathrm{d}x\\
\implies 2^n n!\langle x^m, P_n\rangle &= \int_{-1}^{1} x^m \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \mathrm{d}x\\
&= (-1)^m m! \int_{-1}^{1} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}(x^2-1)\mathrm{d}x \quad \text{through integration by parts } m \text{ times}\\
&= (-1)^m m! \left[\frac{\mathrm{d}^{n-m-1}}{\mathrm{d}x^{n-m-1}}(x^2-1)^n\right]_{-1}^{1}\\
&= 0
\end{aligned}$$

because x^2-1 is a factor of $\dfrac{\mathrm{d}^{n-m-1}}{\mathrm{d}x^{n-m-1}}(x^2-1)^n$. Thus $x^m \perp P_n$ for all $m < n$. However P_m is a linear combination of elements from $\{1, x, \dots, x^m\}$, and thus $P_m \perp P_n$. Thus the Legendre polynomials are orthogonal in $L^2([-1,1])$.

(c) *Show that*

$$\int_{-1}^{1} P_n(x)^2 \mathrm{d}x = \frac{2}{2n+1}.$$

$$
\begin{aligned}
\int_{-1}^{1} P_n(x)^2 \mathrm{d}x &= \int_{-1}^{1} \left(\frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \right)^2 \mathrm{d}x \\
&= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \right)^2 \mathrm{d}x \\
&= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(x^2-1)^n \mathrm{d}x \qquad \text{through integration by parts } n \text{ times} \\
&= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \mathrm{d}x \qquad \text{through integration by parts } 2n \text{ times}
\end{aligned}
$$

Now just consider the integral

$$
\begin{aligned}
\int_{-1}^{1} (x^2-1)^n \mathrm{d}x &= \int_{-1}^{1} (x-1)^n (x+1)^n \mathrm{d}x \\
&= \frac{(n!)^2(-1)^n}{(2n)!} \int_{-1}^{1} (x+1)^{2n} \mathrm{d}x \qquad \text{through integration by parts } n \text{ times} \\
&= \frac{(n!)^2(-1)^n}{(2n)!} \left[\frac{(x+1)^{2n+1}}{2n+1} \right]_{-1}^{1} \\
&= \frac{(n!)^2(-1)^n 2^{2n+1}}{(2n)!(2n+1)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_{-1}^{1} P_n(x)^2 \mathrm{d}x &= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \\
&= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \cdot \frac{(n!)^2(-1)^n 2^{2n+1}}{(2n)!(2n+1)} \\
&= \frac{2}{2n+1}
\end{aligned}
$$

(d) *Prove that the Legendre polynomials form an orthogonal basis of $L^2([-1,1])$. Suppose that $f \in L^2([-1,1])$ is given by*

$$
f(x) = \sum_{n=0}^{\infty} c_n P_n(x).
$$

Compute c_n and say explicitly in what sense the series converges.

Since $\{P_n\}_n$ can be obtained using the Gram-Schmidt from an orthogonal basis (namely the monomials $\{1, x, x^2, \dots\}$), the $\{P_n\}_n$ is an orthogonal basis of $L^2([-1,1])$.

Bessel's inequality says that since $\{P_n\}_n$ is an orthogonal basis, then

$$
c_n = \left\langle \frac{P_n}{\|P_n\|}, f \right\rangle
$$

(e) *Prove that the Legendre polynomial P_n is an eigenfunction of the differential operator*

$$L = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}$$

with eigenvalue $\lambda_n = n(n+1)$, meaning that

$$LP_n = \lambda_n P_n.$$

Let $u(x) = (x^2-1)^n$ and let D be the differential operator. Then note that

$$(x^2-1)Du = (x^2-1)n(x^2-1)^{n-1} \cdot 2x = 2nxu$$

Apply D^{n+1} to both sides and use Liebnitz's Rule for $(fg)^{(n)}$ to acheive

$$\begin{aligned}\frac{(n+1)n}{2} \cdot 2 \cdot D^{n-1}Du + (n+1)2xD^nDu + (x^2-1)D^{n+1}Du &= 2n(n+1)D^nu + 2nxD^{n+1}u \\ \implies 2xD^{n+1}u + (x^2-1)D^{n+2}u &= n(n+1)D^nu \\ \implies LD^nu &= n(n+1)D^nu\end{aligned}$$

which shows D^n is an eigenfunction of L with eigenvalue $\lambda_n = n(n-1)$. Since $2^n n! P_n = D^n u$ (i.e. P_n is linearly dependent on D^n), then P_n is an eigenfunction of L with eigenvalue $\lambda_n = n(n+1)$.

Extra Problem: Convolution is Continuous

Prove that the convolution of two continuous functions on the unit circle in continuous.

Choose $x \in [0, 2\pi]$ and let $\varepsilon > 0$. The continuity of g implies $\exists \delta$ such that

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \varepsilon$$

Then let $|x - x_0| < \delta$ (which also means $|(x-y) - (x_0 - y)| < \delta$). Then

$$\begin{aligned}|(f*g)(x) - (f*g)(x_0)| &= \left| \int_0^{2\pi} f(y)g(x-y) - f(y)g(x_0-y)\mathrm{d}y \right| \\ &= \left| \int_0^{2\pi} f(y)[g(x-y) - g(x_0-y)]\mathrm{d}y \right| \\ &\leq \int_0^{2\pi} |f(y)|\ |g(x-y) - g(x_0-y)|\mathrm{d}y \\ &< \int_0^{2\pi} |f(y)|\varepsilon \mathrm{d}x\end{aligned}$$

But by the continuity of f, f is bounded on $[0, 2\pi]$ since $[0, 2\pi]$ is compact. Thus $|f(y)| \leq C$ for some $C \in \mathbb{R}^+$. Thus

$$\begin{aligned}|(f*g)(x) - (f*g)(x_0)| &< C\varepsilon \int_0^{2\pi} \mathrm{d}x \\ &= 2\pi C\varepsilon\end{aligned}$$

Since ε was arbitrary, this shows that $f*g$ is continuous.