Homework #4

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Hunder and Nachtergaele 7.9

Suppose that $u(t,x)$ is a smooth solution of the one-dimensional wave equation,

$$u_{tt} - c^2 u_{xx} = 0.$$

Prove that

$$\left(u_t^2 + c^2 u_x^2\right)_t - \left(2c^2 u_t u_x\right)_x = 0.$$

If $u(0,t) = u(1,t) = 0$ for all t, deduce that

$$\int_0^1 |u_t(x,t)|^2 + c^2 |u_x(x,t)|^2 \mathrm{d}x = \textit{constant.}$$

Proof.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ \iff 2u_t u_{tt} &= 2c^2 u_t u_{xx} \\ \iff 2u_t u_{tt} + 2c^2 u_x u_{tx} &= 2c^2 u_t u_{xx} + 2c^2 u_x u_{tx} \\ \iff \left(u_t^2 + c^2 u_x^2\right)_t &= \left(2c^2 u_t u_x\right)_x \end{aligned}$$

Since $u(0,t) = u(1,t) = 0$ for all t, then $u(0,t)_t = u(1,t) = 0$ for all t. Thus

$$\begin{aligned} 0 = 2c^2(u_t(1,t)u_x(1,t) - u_t(0,t)u_x(0,t)) &= (2c^2 u_t u_x)\Big|_{x=0}^1 \\ &= \int_0^1 \left(2c^2 u_t u_x\right)_x \mathrm{d}x \\ &= \int_0^1 \left(u_t^2 + c^2 u_x^2\right)_t \mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(u_t^2 + c^2 u_x^2\right) \mathrm{d}x \\ \iff \int_0^1 (u_t^2 + c^2 u_x^2)\mathrm{d}x &= \text{constant.} \end{aligned}$$

□

Hunder and Nachtergaele 7.10

Show that

$$u(x,t) = f(x+ct) + g(x-ct)$$

is a solution of the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

for arbitrary functions f and g. This solution is called d'Alembert's solution.

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ \implies u_t(x,t) &= c(f'(x+ct) - g'(x-ct)) \\ \implies u_{tt}(x,t) &= c^2(f''(x+ct) + g''(x-ct)) \end{aligned}$$

Also,

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ \implies u_x(x,t) &= f'(x+ct) + g'(x-ct) \\ \implies u_{xx}(x,t) &= f''(x+ct) + g''(x-ct) \end{aligned}$$

Thus,

$$u_{tt}(x,t) = c^2(f''(x+ct) + g''(x-ct)) = c^2 u_{xx}(x,t)$$

Hunder and Nachtergaele 7.14

Consider the logistic map

$$x_{n+1} = 4\mu x_n(1-x_n),$$

where $x_n \in [0,1]$, and $\mu = 1$. Show that the solutions may be written as $x_n = \sin^2\theta_n$ where $\theta_n \in \mathbb{T}$, and

$$\theta_{n+1} = 2\theta_n.$$

What can you say about the orbits of the logistic map, the exitence of an invariant measure, and the validity of an ergodic theorem?

Let $x_n = \sin^2\theta_n$ and $\theta_{n+1} = 2\theta_n$. Then

$$\begin{aligned} \theta_{n+1} &= 2\theta_n \\ \implies \sin^2(\theta_{n+1}) &= \sin^2(2\theta_n) \\ \implies x_{n+1} &= 4\sin^2\theta_n\cos^2\theta_n \\ \implies x_{n+1} &= 4\sin^2\theta_n(1-\sin^2\theta_n) \\ \implies x_{n+1} &= 4x_n(1-x_n) \end{aligned}$$

Thus $x_n = \sin^2\theta_n$, where $\theta_{n+1} = 2\theta_n$, satisfies the logistic map.
FIrst, we show the map $T : [0,1] \to [0,1]$ by $T\theta = 2\theta \mod 1$ preserves the Lebesgue measure $\mathscr{L}$ for all Borel sets on $[0,1]$. Consider the σ-algebra

$$\mathscr{F} = \left\{A : \mathscr{L}\left(T^{-1}A\right) = \mathscr{L}(A)\right\}$$

It suffices to show all intervals are contained in $\mathscr{F}$ since the Borel σ-algebra is the smallest σ-algebra to contain the intervals. However, for any interval $[a,b] \subset [0,1]$, $T\left[\frac{a}{2}, \frac{b}{2}\right] = [a,b]$ and $T\left[\frac{3a}{2}, \frac{3b}{2}\right] = [a,b]$ and no other intervals are mapped to $[a,b]$. Also,

$$\mathscr{L}\left(T^{-1}[a,b]\right) = \mathscr{L}\left(\left[\frac{a}{2}, \frac{b}{2}\right] \cup \left[\frac{3a}{2}, \frac{3b}{2}\right]\right) = \mathscr{L}\left(\left[\frac{a}{2}, \frac{b}{2}\right]\right) + \mathscr{L}\left(\left[\frac{3a}{2}, \frac{3b}{2}\right]\right) = \frac{b-a}{2} + \frac{b-a}{2} = b-a = \mathscr{L}([a,b])$$

Thus all intervals are contained in $\mathscr{F}$ and thus T preserves the Lebesgue measure for any Borel set on $[0,1]$.

Hunder and Nachtergaele 7.15

Consider the dynamical system on $\mathbb{T}$*,*

$$x_{n+1} = \alpha x_n \mod 1,$$

where $\alpha = (1+\sqrt{5})/2$ *is the golden ration. Show that the orbit with initial value* $x_0 = 1$ *is not equidistributed on the circle, meaning that it does not satisy (7.39).*
HINT. Show that

$$u_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

satisfies the difference equation

$$u_{n+1} = u_n + u_{n-1}$$

and hence s an integer for every $n \in \mathbb{N}$*. Then use the fact that*

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0 \quad as\ n \to \infty.$$

Let $\phi^+ = \frac{1+\sqrt{5}}{2}$ and $\phi^- = \frac{1-\sqrt{5}}{2}$. Clearly the dynamical system is not equidistributed on $[0,1]$ since if $x_0 = 1$, then $x_1 = \phi^+ \mod 1 = -\phi^-$ and $x_2 = -\phi^- \cdot \phi^+ = 1$. Thus the system has orbit of length 2 and any finite orbit cannot be equidistributed in an interval.

However, the hint is implying the writer intended to ask us to show that the following sequence is not equidistributed on $[0,1]$.

$$\left\{\left(\phi^+\right)^n \mod 1\right\}_{n=0}^{\infty}$$

Note that this is not a dynamical system since it is not recursive. Let $(u_n)_n$ be a sequence defined by

$$u_n = \left(\phi^+\right)^n + \left(\phi^-\right)^n$$

Note that this sequence satisfies the recursion relation

$$u_{n+1} = u_n + u_{n-1}$$

since

$$\begin{aligned} u_n + u_{n-1} &= \left(\phi^+\right)^n + \left(\phi^-\right)^n + \left(\phi^+\right)^{n-1} + \left(\phi^-\right)^{n-1} \\ &= \left(\phi^+\right)^{n-1}\left[1+\phi^+\right] + \left(\phi^-\right)^{n-1}\left[1+\phi^-\right] \end{aligned}$$

$$\begin{aligned}&= \left(\phi^+\right)^{n-1}\left[\frac{3+\sqrt{5}}{2}\right] + \left(\phi^-\right)^{n-1}\left[\frac{3-\sqrt{5}}{2}\right]\\ &= \left(\phi^+\right)^{n+1} + \left(\phi^-\right)^{n+1}\\ &= u_{n+1}\end{aligned}$$

Since $u_0 = 2$ and $u_1 = 1$, then $u_n \in \mathbb{N}$ for all n. Then note that since $\left|\phi^-\right| < 1$, then $\left(\phi^-\right)^n \to 0$. Thus for any ε, there exists N_ε such that, $\left(\phi^+\right)^n \mod 1 \in (0,\varepsilon) \cup (1-\varepsilon, 1]$ for all $n \geq N_\varepsilon$. Thus $\#\{u_n \mid u_n \in [\varepsilon, 1-\varepsilon]\} \leq N_\varepsilon$ and so the sequence is not equidistributed in $[0,1]$.

Hunder and Nachtergaele 7.17

Let B_n and V_n be defined in (7.46) and (7.47). Prove that $\bigcup_{n=0}^N B_n$ is an orthonormal basis of V_N.
HINT. Prove that the set is orthonormal and count its elements.

Let V_n be the finite dimensional subspace of $L^2[0,1]$

$$V_n = \left\{ f \,:\, f \text{ is constant on } \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \text{ for } k = 0, \ldots, 2^n - 1\right\}$$

and define subsets B_n of V_n by

$$B_0 = \{\phi_{0,0}\}, \qquad B_{n+1} = \{\psi_{n,k} \mid k = 0, 1, \ldots, 2^n - 1\}$$

where

$$\phi_{n,k}(x) = 2^{\frac{n}{2}}\phi(2^n - k), \qquad \psi_{n,k}(x) = 2^{\frac{n}{2}}\psi(2^n x - k)$$

and

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad \psi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise}\end{cases}$$

First define e_k, $k = 0, 1, \ldots, 2^N - 1$ by

$$e_k(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \\ 0 & \text{otherwise}\end{cases}$$

and note $E = \{e_k\}_{k=0}^{2^N-1}$ is a basis since if $f \in V_N$ then f is constant on each interval and hence can be written as a linear combination of elements of E. Thus $\dim(V_N) = 2^N$.
Next define $B = \bigcup_{n=0}^N B_N$ and note $\#B = 2^N$. Next we show that $\#B$ is an orthonormal set. Fix n and $k \in \mathbb{Z} \cap [0, 2^n - 1]$. Then

$$\left\langle \psi_{n,k}, \psi_{n,k}\right\rangle_{L^2} = \int_0^1 \psi_{n,k}^2 \mathrm{d}x = \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \left(2^{\frac{n}{2}}\right)^2 = 2^n\left(\frac{1}{2^n}\right) = 1$$

Also, the support of ψ_{n,k_1} is disjoint from the support of ψ_{n,k_2}, thus

$$\left\langle \psi_{n,k_1}, \psi_{n,k_2}\right\rangle = \int_0^1 \psi_{n,k_1}\psi_{n,k_2}\mathrm{d}x = 0$$

For $n_2 > n_1$, then the support of ψ_{n_2,k_2} is either (i) disjoint from the support of ψ_{n_1,k_1} or (ii) contained in either the left or right halves of the support of ψ_{n_1,k_1}. If (i), then their inner product is zero since their supports are disjoint. If (ii), then without loss of generality suppose the support of ψ_{n_2,k_2} is contained in the right half of the support of ψ_{n_1,k_1}. Then

$$\left\langle \psi_{n_1,k_1}, \psi_{n_2,k_2} \right\rangle = \int_0^1 \psi_{n_1,k_1}\psi_{n_2,k_2}\,\mathrm{d}x = \int_{\frac{k_2}{2^{n_2}}}^{\frac{k_2+1}{2^{n_2}}} 2^{\frac{n_1}{2}}\psi_{n_2,k_2}\,\mathrm{d}x = 2^{\frac{n_1}{2}}\int_{\frac{2k_2}{2^{n_2+1}}}^{\frac{2k_2+1}{2^{n_2+1}}} 2^{\frac{n_2}{2}}\,\mathrm{d}x + 2^{\frac{n_1}{2}}\int_{\frac{2k_2+1}{2^{n_2+1}}}^{\frac{2k_2+2}{2^{n_2+1}}} \left(-2^{\frac{n_2}{2}}\right)\mathrm{d}x = 0$$

Thus B is an orthonormal set. Also, $B \subset V_N$ since each $b \in B$ is constant on each of the required intervals. Thus B is an orthonormal basis of V_N.

Hunder and Nachtergaele 7.18

Suppose that $B = \{e_n(x)\}_{n=1}^\infty$ is an orthonormal basis for $L^2([0,1])$. Prove the following:

(a) *For any $a \in \mathbb{R}$, the set $B_a = \{e_n(x-a)\}_{n=1}^\infty$ is an orthonormal basis for $L^2([a,a+1])$.*

Since $e_n(x-a)$ is a horizontal shift then

$$\langle e_n(x-a), e_n(x-a)\rangle_{L^2([a,a+1])} = \int_a^{a+1}(e_n(x-a))^2\mathrm{d}x = \int_0^1 (e_n(x))^2\mathrm{d}x = \langle e_n(x), e_n(x)\rangle_{L^2([0,1])} = 1$$

Similarly,

$$\langle e_n(x-a), e_m(x-a)\rangle_{L^2([a,a+1])} = \int_a^{a+1} e_n(x-a)e_m(x-a)\mathrm{d}x = \int_0^1 e_n(x)e_m(x) = \langle e_n(x), e_m(x)\rangle_{L^2([0,1])} = 0$$

Thus $\{e_n(x-a)\}_{n=1}^\infty$ is orthonormal. Suppose $\left\langle f(x-a), e_n(x-a)\right\rangle_{L^2([a,a+1])} = 0$ for all $n = 1,2,\ldots$. Then

$$0 = \int_a^{a+1} f(x-a)e_n(x-a)\mathrm{d}x = \int_0^1 f(x)e_n(x)\mathrm{d}x = \left\langle f(x), e_n(x)\right\rangle_{L^2([0,1])}$$

for all n. Since $\{e_n(x)\}_{n=1}^\infty$ is a basis of $L^2([0,1])$, then $f(x) \equiv 0$. Thus $f(x-a) \equiv 0$, proving $\{e_n(x-a)\}_{n=1}^\infty$ is a basis of $L^2([a,a+1])$.

(b) *For any $c > 0$, the set $B^C = \left\{\sqrt{c}e_n(cx)\right\}_{n=1}^\infty$ is an orthonormal basis for $L^2\left(\left[0,c^{-1}\right]\right)$.*

Since $e_n(cx)$ is a horizontal compression,

$$\left\langle \sqrt{c}e_n(cx), \sqrt{c}e_n(cx)\right\rangle_{L^2([0,c^{-1}])} = c\int_0^{c^{-1}}(e_n(cx))^2\mathrm{d}x = c\cdot c^{-1}\int_0^1 (e_n(x))^2\mathrm{d}x = \langle e_n(x), e_n(x)\rangle_{L^2([0,1])} = 1$$

Similarly,

$$\left\langle \sqrt{c}e_n(cx), \sqrt{c}e_m(cx)\right\rangle_{L^2([0,c^{-1}])} = c\int_0^{c^{-1}} e_n(cx)e_m(cx)\mathrm{d}x = c\cdot c^{-1}\int_0^1 e_n(x)e_m(x)\mathrm{d}x = \langle e_n(x), e_m(x)\rangle_{L^2([0,1])} = 0$$

Thus B^C is orthonormal. Suppose $\left\langle \sqrt{c}f(cx), \sqrt{c}e_n(cx)\right\rangle = 0$ for all $n = 1,2,\ldots$. Then

$$0 = c\int_0^{c^{-1}} f(cx)e_n(cx)\mathrm{d}x = c\cdot c^{-1}\int_0^1 f(x)e_n(x)\mathrm{d}x = \left\langle f(x), e_n(x)\right\rangle_{L^2([0,1])}$$

for all n. Since $\{e_n(x)\}_{n=1}^\infty$ is a basis of $L^2([0,1])$, then $f(x) \equiv 0$. Thus $\sqrt{c}f(cx) \equiv 0$, proving B^C is a basis of $L^2\left(\left[0,c^{-1}\right]\right)$.

(c) *With the convention that functions are extended to a larger domain than their original domain by setting them equal to* 0*, prove that* $B \cup B_1$ *is an orthonormal basis for* $L^2([0,2])$*.*

We know B is an orthonormal basis of $L^2([0,1])$ and by part (a), B_1 is an orthonormal basis of $L^2([1,2])$. Then let $e_n \in B$ and $e_m \in B_1$. Then $e_n(x) = 0$ for $x \in (1,2)$ and $e_m(x) = 0$ for $x \in (0,1)$. Thus

$$\langle e_n, e_m \rangle_{L^2([0,2])} = \int_0^2 e_n e_m \mathrm{d}x = \int_0^1 e_n e_m \mathrm{d}x + \int_1^2 e_n e_m \mathrm{d}x = 0$$

Thus $B \cup B_1$ is an orthonormal set. Let $f \in L^2([0,2])$. Then $f = f_1 + f_2$ where

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in [0,1) \\ 0 & \text{if } x \in [1,2] \end{cases} \qquad \text{and} \qquad f_2(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ f(x) & \text{if } x \in [1,2] \end{cases}$$

Thus $f_1 \in L^2([0,1])$ and $f_2 \in L^2([1,2])$, and so

$$f_1(x) = \sum_{i=1}^{\infty} \alpha_i e_{n_i}(x) \qquad \text{and} \qquad f_2(x) = \sum_{j=1}^{\infty} \beta_j e_{n_j}(x-1)$$

which implies

$$f = f_1 + f_2 = \sum_{i=1}^{\infty} \alpha_i e_{n_i}(x) + \sum_{j=1}^{\infty} \beta_j e_{n_j}(x-1)$$

This shows f can be written as a linear combination of elements of $B \cup B_1$ and thus $B \cup B_1$ is a basis of $L^2([0,2])$.

(d) *Prove that* $\bigcup_{k\in\mathbb{Z}} B_k$ *is an orthonormal basis for* $L^2(\mathbb{R})$*.*

By part (a), each B_k is an orthonormal basis of $L^2([k,k+1])$. Also, for $k_1 < k_2$, let $e_n(x-k_1) \in B_{k_1}$ and $e_m(x-k_2) \in B_{k_2}$. Then

$$\begin{aligned}
\langle e_n(x-k_1), e_m(x-k_2) \rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} e_n(x-k_1) e_m(x-k_2) \mathrm{d}x \\
&= \cancelto{0}{\int_{-\infty}^{k_1} e_n(x-k_1) e_m(x-k_2) \mathrm{d}x} + \int_{k_1}^{k_1+1} e_n(x-k_1) e_m(x-k_2) \mathrm{d}x \\
&\quad + \cancelto{0}{\int_{k_1+1}^{k_2} e_n(x-k_1) e_m(x-k_2) \mathrm{d}x} + \int_{k_2}^{k_2+1} e_n(x-k_1) e_m(x-k_2) \mathrm{d}x \\
&\quad + \cancelto{0}{\int_{k_2+1}^{\infty} e_n(x-k_1) e_m(x-k_2) \mathrm{d}x} \\
&= \int_{k_1}^{k_1+1} e_n(x-k_1) \cdot 0 \mathrm{d}x + \int_{k_2}^{k_2+1} 0 \cdot e_m(x-k_2) \mathrm{d}x \\
&= 0
\end{aligned}$$

Thus $\bigcup_{k\in\mathbb{Z}}$ is orthonormal. Then let $f \in L^2(\mathbb{R})$. Then

$$f = \sum_{k\in\mathbb{Z}} f_k$$

where $f_k \in B_k$ and for each k,

$$f_k(x) = \begin{cases} f(x) & \text{if } x \in [k, k+1) \\ 0 & \text{otherwise} \end{cases}$$

Then since $f_k \in B_k$,

$$f_k = \sum_{i_k=1}^{\infty} \alpha_{i_k} e_{n_{i_k}}(x-k)$$

since $\{e_n(x-k)\}_{n=1}^{\infty}$ is a basis for B_k. Thus,

$$f = \sum_{k \in \mathbb{Z}} f_k = \sum_{k \in \mathbb{Z}} \left[\sum_{i_k=1}^{\infty} \left(\alpha_{i_k} e_{n_{i_k}}(x-k) \right) \right]$$

and thus f can be written as a linear combination of elements of $\bigcup_{k \in \mathbb{Z}} B_k$. Thus $\bigcup_{k \in \mathbb{Z}}$ is a basis of $L^2(\mathbb{R})$.