UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #5

Sam Fleischer

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Hunter and Nachtergaele 8.2

If $\mathscr{H} = \mathscr{M} \oplus \mathscr{N}$ is an orthogonal direct sum, show that $\mathscr{M}^\perp = \mathscr{N}$ and $\mathscr{N}^\perp = \mathscr{M}$.

Proof. Suppose $x \in \mathscr{M}^\perp$. Then $x \in \mathscr{H} \implies \exists! y, z$ such that $x = y + z$ where $y \in \mathscr{M}$ and $z \in \mathscr{N}$. Then

$$\langle x, x\rangle = \cancelto{0}{\langle x, y\rangle} + \langle x, z\rangle = \langle x, z\rangle \implies \langle x, x - z\rangle = 0 \implies x = z$$

which shows $x \in \mathscr{N}$, i.e. $\mathscr{M}^\perp \subset \mathscr{N}$.
Now suppose $x \notin \mathscr{M}^\perp$. Then $x \in \mathscr{M}$ and $x \neq 0$. Thus $x \notin \mathscr{N}$ since a direct sum implies $\mathscr{N} \cap \mathscr{M} = \{0\}$. Thus $\mathscr{N} \subset \mathscr{M}^\perp \subset \mathscr{N} \implies \mathscr{N} = \mathscr{M}^\perp$.
Switching $\mathscr{N}$ and $\mathscr{M}$ in the above paragraphs shows $\mathscr{M} = \mathscr{N}^\perp$. $\square$

Hunter and Nachtergaele 8.3

Let $\mathscr{M}$, $\mathscr{N}$ be closed subspaces of a Hilbert space $\mathscr{H}$ and P, Q the orthogonal projections with ran $P = \mathscr{M}$, ran $Q = \mathscr{N}$. *Prove that the following conditions are equivalent: (a) $\mathscr{M} \subset \mathscr{N}$; (b) $QP = P$; (c) $PQ = P$; (d) $\|Px\| \le \|Qx\|$ for all $x \in \mathscr{H}$; (e) $\langle x, Px\rangle \le \langle x, Qx\rangle$ for all $x \in \mathscr{H}$.*

Proof. We will show *(a) → (b) → (c) → (d) → (e) → (a)*, which proves the statements' equivalence.

(a) → (b). Let $\mathscr{M} \subset \mathscr{N}$ and let $x \in \mathscr{H}$. Then $Px \in \mathscr{M} \subset \mathscr{N} \implies Q(Px) = Px \implies QP = P$.

(b) → (c). Let $QP = P$. Then

$$\langle x, Py\rangle = \langle Px, y\rangle = \langle QPx, y\rangle = \langle Px, Qy\rangle = \langle x, PQy\rangle \implies \langle x, Py - PQy\rangle = 0 \qquad \forall x, y \in \mathscr{H}.$$

Thus $Py - PQy = 0$ for all $y \in \mathscr{H}$, i.e. $PQ = P$.

(c) → (d). Let $PQ = P$. First note $\|Px\| \le \|x\|$ for all $x \in \mathscr{H}$ because

$$\|Px\| = \frac{\|Px\|^2}{\|Px\|} = \frac{\langle Px, Px\rangle}{\|Px\|} = \frac{\langle x, P^2x\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|} \le \|x\|$$

by the Cauchy-Schwarz inequality. Thus,

$$\|Px\| = \|PQx\| = \|P(Qx)\| \le \|Qx\| \qquad \forall x \in \mathscr{H}$$

(d) → (e). Let $\|Px\| \le \|Qx\|$.

$$\|Px\|^2 \le \|Qx\|^2 \implies \langle Px, Px\rangle \le \langle Qx, Qx\rangle \implies \langle x, P^2x\rangle \le \langle x, Q^2x\rangle \implies \langle x, Px\rangle \le \langle x, Qx\rangle \qquad \forall x \in \mathscr{H}$$

(e) → (a). Let $\langle x, Px\rangle \le \langle x, Qx\rangle$ and suppose $x \in M$. Then $Px = x$. Then $\|x\|^2 = \langle x, x\rangle = \langle x, Px\rangle \le \langle x, Qx\rangle$. However, since $\|Qx\| \le \|x\|$ for all x, then $\langle x, Qx\rangle \le \|x\|\|Qx\| \le \|x\|^2$, which shows $\langle x, Qx\rangle = \|x\|^2$. Thus $Qx = x$, which shows $x \in \mathscr{N}$, proving $\mathscr{M} \subset \mathscr{N}$.

□

Hunter and Nachtergaele 8.4

Suppose that (P_n) *is a sequence of orthogonal projections on a Hilbert space* $\mathscr{H}$ *such that*

$$\text{ran } P_{n+1} \supset \text{ran } P_n, \qquad \bigcup_{n=1}^{\infty} \text{ran } P_n = \mathscr{H}.$$

Prove that (P_n) *converges strongly to the identity operator* I *as* $n \to \infty$. *Show that* (P_n) *does not converge to the identity operator with respect to the operator norm unless* $P_n = I$ *for all sufficiently large* n.

Proof. Let $x \in \mathscr{H}$. Then $\exists N$ such that $x \in \text{ran } P_N$. Thus $x \in \text{ran } P_n$ for all $n \geq N$. Since each P_n is an orthogonal projection, then $x = P_n x$ for all $n \geq N$. Thus

$$\lim_{n\to\infty} P_n x = \lim_{n\to\infty} x = x = Ix$$

where I is the identity operator. Thus (P_n) converges strongly to I.
If $P_n = I$ for all sufficiently large n, then obviously

$$\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \|I\| = \|I\|$$

□

Hunter and Nachtergaele 8.6

Show that a linear operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ *is unitary if and only if it is an isometric isomorphism of normed linear spaces. Show that an invertible linear map is unitary if and only if its inverse is.*

Hunter and Nachtergaele 8.7

If ϕ_y *is the bounded linear functional defined in (8.5)*

$$\phi_y(x) = \langle y, x \rangle \tag{8.5}$$

Prove that $\|\phi_y\| = \|y\|$.

Proof. First we prove $\|\phi_y\|$ is bounded above by $\|y\|$.

$$\|\phi_y\| = \sup_{\|x\|=1} \|\phi_y(x)\| = \sup_{\|x\|=1} \langle y, x\rangle \leq \sup_{\|x\|=1} \|y\|\|x\| = \|y\|$$

Next, consider $x = \frac{y}{\|y\|}$ (note $\|x\| = 1$):

$$\|\phi_y(x)\| = \|\langle y, x\rangle\| = \left\|\frac{\langle y, y\rangle}{\|y\|}\right\| = \left\|\frac{\|y\|^2}{\|y\|}\right\| = \|y\|$$

and thus $\|\phi_y\| \geq \|y\|$, which proves $\|\phi_y\| = \|y\|$. □

Hunter and Nachtergaele 8.8

Prove that $\mathscr{H}^*$ is a Hilbert space with the inner product defined by

$$\langle \phi_x, \phi_y\rangle_{\mathscr{H}^*} = \langle y, x\rangle_{\mathscr{H}}.$$

Proof. First note $\langle\cdot,\cdot\rangle_{\mathscr{H}^*}$ is a well-defined inner product since $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ is a well-defined inner product (i.e. the properties of inner product hold). It suffices to show $\mathscr{H}^*$ is closed. Let (ϕ_n) be a convergent sequence of linear functionals in $\mathscr{H}^*$. By the Riesz Representation Theorem, for all n, $\exists y_n$ such that $\phi_n(x) = \langle y_n, x\rangle_{\mathscr{H}}$ for all $x \in \mathscr{H}$. Since (ϕ_n) converges, then $\forall x \in \mathscr{H}$, $\langle y_n, x\rangle_{\mathscr{H}}$ converges. By continuity of inner product, then y_n converges to some y. By completeness of $\mathscr{H}$, $y \in \mathscr{H}$. Again by the Riesz Representation Theorem, $\exists \phi \in \mathscr{H}^*$ such that $\phi(x) = \langle y, x\rangle_{\mathscr{H}}$ for all $x \in \mathscr{H}$. Moreover,

$$\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \langle y_n, \cdot\rangle_{\mathscr{H}} = \langle y, \cdot\rangle_{\mathscr{H}} = \phi_y$$

Thus $\mathscr{H}^*$ is closed, which proves $\mathscr{H}^*$ is a Hilbert space. □

Hunter and Nachtergaele 8.9

Let $A \subset \mathscr{H}$ be such that

$$\mathscr{M} = \left\{x \in \mathscr{H} \mid x \text{ is a finite linear combination of elements in } A\right\}$$

is a dense linear subspace of $\mathscr{H}$. Prove that any bounded linear functional on $\mathscr{H}$ is uniquely determined by its values on A. If $\{u_\alpha\}$ is an orthonormal basis, find a necessary and sufficient condition on a family of complex numbers c_α for there to be a bounded linear functional ϕ such that $\phi_{u_\alpha} = c_\alpha$.

Hunter and Nachtergaele 8.11

Prove that if $A : \mathscr{H} \to \mathscr{H}$ *is a linear map and* $\dim \mathscr{H} < \infty$, *then*

$$\dim \ker A + \dim \operatorname{ran} A = \dim \mathscr{H}.$$

Prove that, if $\dim \mathscr{H} < \infty$, *then* $\dim \ker A = \dim \ker A^*$. *In particular,* $\ker A = \{0\}$ *if and only if* $\ker A^* = \{0\}$.

Hunter and Nachtergaele 8.12

Suppose that $A : \mathscr{H} \to \mathscr{H}$ *is a bounded, self-adjoint linear operator such that there is a constant* $c > 0$ *with*

$$c\|x\| \le \|Ax\| \qquad \text{for all } x \in \mathscr{H}.$$

Prove that there is a unique solution x *of the equation* $Ax = y$ *for every* $y \in \mathscr{H}$.