UC DAVIS, ANALYSIS (MAT201B), WINTER 2016

Homework #6

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Hunter and Nachtergaele 8.1

If M is a linear subspace of a linear space X, then the quotient space *X/M is the set $\{x+M \mid x+y \in M\}$ of affine spaces*

$$x+M=\{x+y \mid y \in M\}$$

parallel to M.

(a) Show that X/M is a linear space with respect to the operations

$$\lambda(x+M)=\lambda x+M, \qquad (x+M)+(y+M)=(x+y)+M.$$

Proof. Since X is a linear space, then $\alpha x+\beta y \in X$ for every $x, y \in X$, $\alpha, \beta \in \mathbb{F}$. Then

$$\alpha(x+M)+\beta(x+M)=(\alpha x+\beta y)+M \in X/M$$

Define the "zero" vector in X/M by $0+M$ where 0 is the "zero" vector in X. Then

$$(0+M)+(x+M)=(0+x)+M=x+M=(x+0)+M=(x+M)+(0+M)$$

Also, the "one" in $\mathbb{F}$ (1) is the "one" in X/M since

$$1(x+M)=1x+M=x+M$$

Thus X/M is a vector space. □

(b) Suppose that $X=M \oplus N$. Show that N is linearly isomorphic to X/M.

Proof. Define $T: N \to X/M$ by

$$Tn=n+M$$

For any $x, y \in N$, then if $Tx=Ty$, then $x+M=y+M \implies (x-y)+M=0+M \implies x-y \in M$. But since N is a vector space, then $x-y \in N$. Since $X=M \oplus N$, then $M \cap N=\{0\}$, which means $x=y$. Thus T is injective. Now choose $x+M \in X/M$. Then note $P_N x \in N$ and

$$T(P_N x)=P_N x+M=(P_N x+M)+(P_M x+M)=(P_N x+P_M x)+M=x+M$$

Thus T is surjective. Thus T is a bijection. Also, T is a linear map since

$$T(\alpha x+\beta y)=(\alpha x+\beta y)+M=\alpha(x+M)+\beta(y+M)=\alpha Tx+\beta Ty$$

Thus N is linearly isomorphic to X/M. □

(c) The codimension *of M in X is the dimension of X/M. Is a subspace of a Banach space with finite codimension necessarily closed?*

Proof. Let ϕ be an unbounded linear functional. Then let $M = \ker\phi$ and consider $X/(\ker\phi)$. Define the bijection $T : X/(\ker\phi) \to \mathbb{C}$ by

$$T(x + \ker\phi) = \phi(x)$$

Injectivity: If $T(x+\ker\phi) = T(y+\ker\phi)$, then $\phi(x) = \phi(y)$, then since ϕ is linear, $\phi(x-y) = 0$, i.e. $x - y \in \ker\phi$. Thus $x - y + \ker\phi = 0 + \ker\phi$, and so $x + \ker\phi = (y + \ker\phi) + (0 + \ker\phi) = y + \ker\phi$. Thus T is injective. Sujectivity: If $\lambda \in \mathbb{C}$, then choose $y \in X$ with $y \notin \ker\phi$ and note

$$T\left(\lambda\frac{y}{\phi(y)} + \ker\phi\right) = \phi\left(\lambda\frac{y}{\phi(y)}\right) = \frac{\lambda}{\phi(y)}\phi(y) = \lambda$$

Thus T is surjective, which shows T is bijective, and so $\dim(X/(\ker\phi)) = \dim(\mathbb{C}) = 1 < \infty$. Also, $\ker\phi \neq \overline{\ker\phi}$ ($\ker\phi$ is not closed) since ϕ is unbounded (proven below). Thus the codimension of M is finite ($\dim(X/\ker\phi) = 1$) and M is not closed.

Since ϕ is unbounded, $\exists x_n$ such that $\|x_n\| = 1$ and $\phi(x_n) \to \infty$. Then consider $y_n = \frac{x_n}{\phi(x_n)}$. Then $y_n \to 0$ but $\phi(y_n) = \frac{1}{\phi(x_n)}\phi(x_n) = 1$ for $n = 1, 2, \ldots$. Choose any $\tilde{z} \in X$ and define $z = \frac{\tilde{z}}{\phi(\tilde{z})}$. Then $\phi(z) = 1$. Then $\phi(z - y_n) = \phi(z) - \phi(y_n) = 0$ for $n = 1, 2, \ldots$. However, $z - y_n \to z$ since $y_n \to 0$. Thus $(z - y_n) \in \ker\phi$ for all n but $\lim_n(z - y_n) = z \notin \ker\phi$ since $\phi(z) \neq 0$. Thus $\ker\phi$ is not closed. $\square$

Hunter and Nachtergaele 8.10

Let $\{u_\alpha\}$ be an orthonormal basis of $\mathscr{H}$. Prove that $\{\phi_{u_\alpha}\}$ is an orthnormal basis of $\mathscr{H}^*$.

Proof. First note $\{\phi_{u_\alpha}\}$ is an orthonormal set since

$$\left\langle \phi_{u_1}, \phi_{u_2} \right\rangle = \langle u_2, u_1 \rangle = \delta_{u_2,u_1} = \begin{cases} 1 & \text{if } u_1 = u_2 \\ 0 & \text{if } u_1 \neq n_2 \end{cases}$$

Next let $\phi \in \mathscr{H}^*$. By the Riesz Representation Theorem, $\exists u \in \mathscr{H}$ such that $\phi(x) = \langle x, u \rangle$ for all $x \in \mathscr{H}$. Then since $\{u_\alpha\}$ is an orthonormal basis of $\mathscr{H}$, then $\exists\{c_\alpha\}$ such that $\sum_\alpha |c_\alpha|^2 < \infty$ and $u = \sum_\alpha c_\alpha u_\alpha$. Then

$$\phi(x) = \langle x, u \rangle = \left\langle x, \sum_\alpha c_\alpha u_\alpha \right\rangle = \sum_\alpha c_\alpha \langle x, u_\alpha \rangle = \sum_\alpha c_\alpha \phi_{u_\alpha}$$

where ϕ_{u_α} is the functional in $\mathscr{H}^*$ such that $\phi_{u_\alpha}(x) = \langle x, u_\alpha \rangle$ for all $x \in \mathscr{H}$. Thus $\{\phi_{u_\alpha}\}$ spans $\mathscr{H}^*$, and hence $\{\phi_{u_\alpha}\}$ is an orthonormal basis of $\mathscr{H}^*$. $\square$

Hunter and Nachtergaele 8.13

Prove that an orthonormal set of vectors $\{u_\alpha \mid \alpha \in \mathscr{A}\}$ is a Hilbert space $\mathscr{H}$ is an orthonormal basis if and only if

$$\sum_{\alpha \in \mathscr{A}} u_\alpha \otimes u_\alpha = I.$$

Proof. Let $\{u_\alpha\}$ be an orthonormal basis of $\mathscr{H}$. Then $\forall x \in \mathscr{H}$, $x = \sum_\alpha \langle u_\alpha, x\rangle\, u_\alpha$. However, the projection P_{u_α} is defined as

$$P_{u_\alpha} x = \langle u_\alpha, x\rangle\, u_\alpha$$

and hence, for every $x \in \mathscr{H}$, $Ix = x = \sum_\alpha \langle u_\alpha, x\rangle\, u_\alpha = \sum_\alpha P_{u_\alpha} x = \sum_\alpha (u_\alpha \otimes u_\alpha) x$. In other words, $I = \sum_\alpha u_\alpha \otimes u_\alpha$. Now let $\sum_\alpha u_\alpha \otimes u_\alpha = I$. Then $x = \sum_\alpha P_{u_\alpha} x = \sum_\alpha \langle u_\alpha, x\rangle\, u_\alpha$. Thus $\{u_\alpha\}$ is an orthonormal basis of $\mathscr{H}$. □

Hunter and Nachtergaele 8.14

Suppose that $A, B \in \mathscr{B}(\mathscr{H})$ satisfy

$$\left\langle x, Ay\right\rangle = \left\langle x, By\right\rangle \qquad \textit{for all } x, y \in \mathscr{H}.$$

Prove that $A = B$. Use a polarization-type identity to prove that if $\mathscr{H}$ is a complex Hilbert space and

$$\langle x, Ax\rangle = \langle x, Bx\rangle \qquad \textit{for all } x \in \mathscr{H},$$

then $A = B$. What can you say about A and B for real Hilbert spaces?

Proof. If $\left\langle x, Ay\right\rangle = \left\langle x, By\right\rangle$, then $\left\langle x, (A-B)y\right\rangle = 0$ for all $x, y \in \mathscr{H}$. Then $A - B = 0$, i.e. $A = B$.
Let $\langle x, Ax\rangle = \langle x, Bx\rangle$ for all $x \in \mathscr{H}$. Then $\langle x, (A-B)x\rangle = 0$ for all $x \in \mathscr{H}$. Thus,

$$\begin{aligned}
0 &= \left\langle x+y, (A-B)(x+y)\right\rangle = \langle x, (A-B)x\rangle + \left\langle y, (A-B)x\right\rangle + \left\langle x, (A-B)y\right\rangle + \left\langle y, (A-B)y\right\rangle \\
&= \left\langle y, (A-B)x\right\rangle + \left\langle x, (A-B)y\right\rangle \\
\implies & \left\langle y, (A-B)x\right\rangle = -\left\langle x, (A-B)y\right\rangle
\end{aligned}$$

Also,

$$\begin{aligned}
0 &= \left\langle x+iy, (A-B)(x+iy)\right\rangle = \langle x, (A-B)x\rangle + \left\langle iy, (A-B)x\right\rangle + \left\langle x, (A-B)(iy)\right\rangle + \left\langle iy, (A-B)(iy)\right\rangle \\
&= -i\left\langle y, (A-B)x\right\rangle + i\left\langle x, (A-B)(y)\right\rangle \\
\implies & \left\langle y, (A-B)x\right\rangle = \left\langle x, (A-B)y\right\rangle = -\left\langle y, (A-B)x\right\rangle \\
\implies & \left\langle y, (A-B)x\right\rangle = 0 \qquad \forall x, y \in \mathscr{H}
\end{aligned}$$

Thus, $A - B \equiv 0$, or $A \equiv B$. For real Hilbert spaces, it is possible for $\langle x, Ax\rangle = \langle x, Bx\rangle$ for all x but $A \not\equiv B$. Consider $A, B \in \mathscr{B}(\mathbb{R}^n)$ by

$$A(x_1, \ldots, x_n) = (-x_2, x_1, x_3, x_4, \ldots, x_n) \qquad \text{and} \qquad B(x_1, \ldots, x_n) = (x_2, -x_1, x_3, x_4, \ldots, x_n)$$

Then

$$\langle x, Ax\rangle = \sum_{i=3}^{n} x_i^2 = \langle x, Bx\rangle \qquad \forall x \in \mathbb{R}^n$$

but clearly $A \not\equiv B$. □

Hunter and Nachtergaele 8.15

Prove that for all $A, B \in \mathscr{B}(\mathscr{H})$, and $\lambda \in \mathbb{C}$, we have (a) $A^{**} = A$; (b) $(AB)^* = B^*A^*$; (c) $(\lambda A)^* = \overline{\lambda}A^*$; (d) $(A+B)^* = A^* + B^*$; (e) $\|A^*\| = \|A\|$.

Proof. *(a)* For all $x, y \in \mathscr{H}$, $\langle x, Ay\rangle = \langle A^*x, y\rangle = \langle x, (A^*)^*y\rangle$. Thus $\langle x, (A - A^{**})y\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $A = A^{**}$.

(b) For all $x, y \in \mathscr{H}$, $\langle x, (AB)^*y\rangle = \langle ABx, y\rangle = \langle Bx, A^*y\rangle = \langle x, B^*A^*y\rangle \implies \langle x, ((AB)^* - B^*A^*)y\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $(AB)^* = B^*A^*$.

(c) For all $x, y \in \mathscr{H}$, $\langle x, (\lambda A)^*y\rangle = \langle \lambda Ax, y\rangle = \overline{\lambda}\langle Ax, y\rangle = \overline{\lambda}\langle x, A^*y\rangle \implies \left\langle x, ((\lambda A)^* - \overline{\lambda}A^*)y\right\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $(\lambda A)^* = \overline{\lambda}A^*$.

(d) For all $x, y \in \mathscr{H}$, $\langle x, (A+B)^*y\rangle = \langle (A+B)x, y\rangle = \langle Ax, y\rangle + \langle Bx, y\rangle = \langle x, A^*y\rangle + \langle x, B^*y\rangle = \langle x, (A^* + B^*)y\rangle \implies \langle x, ((A+B)^* - (A^* + B^*))y\rangle = 0$ for all $x, y \in \mathscr{H}$. Thus $(A+B)^* = A^* + B^*$.

(e) First define $M \in \mathscr{H}^*$ by $Mx = \langle y, Ax\rangle$. Then M is a bounded linear functional since

$$M(ax_1 + bx_2) = \langle y, A(ax_1 + bx_2)\rangle = \langle y, aAx_1\rangle + \langle y, bAx_2\rangle = a\langle y, Ax_1\rangle + b\langle y, Ax_2\rangle = aMx_1 + bMx_2$$

and

$$\|M\| = \sup_{\|x\|=1} \langle y, Ax\rangle \leq \sup_{\|x\|=1} \|y\|\|Ax\| \leq \|y\|\|A\|$$

and since $A \in \mathscr{B}(\mathscr{H})$, then $\|M\| < \infty$. Then since $M \in \mathscr{H}^*$, The Riesz Representation Theorem guarantees a unique vector $v \in \mathscr{H}$ such that

$$Mx = \langle v, x\rangle = \langle y, Ax\rangle = \langle A^*y, x\rangle$$

Thus $v = A^*y$. Finally,

$$\|A^*y\| = \sup_{\|x\|=1} |\langle v, x\rangle| = \sup_{\|x\|=1} |\langle y, Ax\rangle| \leq \sup_{\|x\|=1} \|y\|\|Ax\| \leq \sup_{\|x\|=1} \|y\|\|A\|\|x\| = \|y\|\|A\|$$

Thus $\|A^*\| \leq \|A\|$. This also implies $\|A\| = \|A^{**}\| = \|(A^*)^*\| \leq \|A^*\|$. Thus, $\|A\| = \|A^*\|$.

□

Hunter and Nachtergaele 8.16

Let $U : L^2(\Omega, P) \to L^2(\Omega, P)$ by

$$Uf = f \circ T \tag{8.16}$$

where $T : (\Omega, P) \to (\Omega, P)$ is measure preserving, i.e. $P(A) = P(T^{-1}A)\ \forall$ measurable $A \subset \Omega$. Prove that the operator U defined in (8.16) is unitary.

Proof. Since T is measure-preserving, then T is bijective (by definition) and for any $f \in L^2(\Omega, P)$, we have $\mathscr{X} f = \mathscr{X} f \circ T$ (where $\mathscr{X}$ is the characteristic function), or

$$\int_\Omega f \mathrm{d}P = \int_\Omega f \circ T \mathrm{d}P$$

Then since $\overline{f} g \in L^2(\Omega, P)$, then

$$\int_\Omega \overline{f} g \mathrm{d}P = \int_\Omega (\overline{f} g) \circ T \mathrm{d}P$$

Thus,

$$\langle Uf, Ug \rangle = \int_\Omega \overline{f(T(\omega))} g(T(\omega)) \mathrm{d}P(\omega) = \int_\Omega \left(\left(\overline{f} g \right) \circ T \right)(\omega) \mathrm{d}P(\omega) = \int_\Omega \left(\overline{f} g \right)(\omega) \mathrm{d}P(\omega) = \langle f, g \rangle$$

Also, since T is bijective, T^{-1} exists and $U^{-1} f$ can be defined as

$$U^{-1} f = f \circ T^{-1}$$

Clearly

$$U^{-1}(Uf) = U^{-1}(f \circ T) = (f \circ T) \circ T^{-1} = f \circ (T \circ T^{-1}) = f \circ \mathbb{1} = f$$

and

$$U(U^{-1} f) = U(f \circ T^{-1}) = (f \circ T^{-1}) \circ T = f \circ (T^{-1} \circ T) = f \circ \mathbb{1} = f$$

$\square$

Hunter and Nachtergaele 8.17

Prove that strong convergence implies weak convergence. Also prove that strong and weak convergence are equivalent in a finite-dimensional Hilbert space.

Proof. Let $x_n \to x$ strongly, i.e. $\|x_n - x\| \to 0$. Then

$$\langle x_n, y \rangle - \langle x, y \rangle = \langle x_n - x, y \rangle \leq \|x_n - x\| y \to 0 \qquad \forall y \in \mathscr{H}$$

Then $x_n \to x$ weakly. Suppose $\dim \mathscr{H} = n < \infty$ and $x_n \to x$ weakly. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathscr{H}$. Then $x = \sum_{i=1}^n c_i e_i$ where $c_i = \langle e_i, x \rangle$. Next, define the ℓ^1 norm by

$$\|x\|_1 = \sum_{i=1}^n |c_i|$$

Since $x_n \to x$ weakly, then $\langle x_n, y \rangle \to \langle x, y \rangle$ for all $y \in \mathscr{H}$. This implies $\langle e_i, x_n \rangle \to \langle e_i, x \rangle$ for each $i = 1, \ldots, n$. Also, $x_n - x = \sum_{i=1}^n \langle e_i, x_n - x \rangle e_i$, and thus

$$\|x_n - x\|_1 = \sum_{i=1}^n |\langle e_i, x_n - x \rangle| = \sum_{i=1}^n |\langle e_i, x_n \rangle - \langle e_i, x \rangle| \to 0$$

However, $\|\cdot\|_1 \equiv \|\cdot\|_{\mathscr{H}}$ since all norms are equivalent in finite-dimensional spaces, and thus $x_n \to x$ strongly.

$\square$

Hunter and Nachtergaele 8.18

Let (u_n) be a sequence of orthonormal vectors in a Hilbert space. Prove that $u_n \rightharpoonup 0$ weakly.

Proof. Let $y \in \mathscr{H}$. Then by Bessel's inequality, $\|y\| \geq \|\sum_{n=0}^{\infty} c_n u_n\|$ where $\sum_{n=0}^{\infty} |c_n|^2 < \infty$ and $c_n = \langle u_n, y\rangle$ for $n = 0, 1, \ldots$. Thus, $|c_n| \to 0 \implies \langle c_n, y\rangle \to 0$, and thus

$$\forall\, y \in \mathscr{H}, \qquad \langle u_n, y\rangle \to 0 \qquad \implies \qquad \langle u_n, y\rangle \to \langle 0, y\rangle \qquad \implies \qquad u_n \rightharpoonup 0 \text{ weakly.}$$

□

Hunter and Nachtergaele 8.19

Prove that a strongly lower-semicontinuous convex function $f : \mathscr{H} \to \mathbb{R}$ on a Hilbert space $\mathscr{H}$ is weakly lower-semicontinuous.

Proof. Let f be a strongly lower-semicontinuous function on a Hilbert space $\mathscr{H}$, $f : \mathscr{H} \to \mathbb{R}$. Let $u_n \in \mathscr{H}$ such that $u_n \rightharpoonup u$ weakly. Then define $y_n = u_n - u$, $y_n \rightharpoonup 0$ weakly. Assume

$$f(0) > \underline{\lim}_n f(y_n)$$

This assumption will lead to a contradiction, which will prove f is weakly lower-semicontinuous. The assumption implies $\exists$ a subsequence y_{n_k} such that $f(0) - \epsilon > f(y_{n_k})$. Note $y_{n_k} \rightharpoonup 0$ since $y_n \rightharpoonup 0$. By Mazur's Theorem, $\exists$ subsequence of y_{n_k} (call it $y_{n_{k_\ell}}$) and z_ℓ defined by

$$z_\ell = \frac{1}{\ell}\left(y_{n_{k_1}} + y_{n_{k_2}} + \cdots + y_{n_{k_\ell}}\right)$$

and $z_\ell \to 0$ strongly (the limit is 0 since the weak limit of y_{n_k} is 0). Since f is strongly lower-semicontinuous, then

$$f(0) \leq \underline{\lim}_\ell f(z_\ell) = \underline{\lim}_\ell f\left(\frac{1}{\ell}\sum_{i=1}^{\ell} y_{n_{k_i}}\right)$$

Convexity of f implies

$$\underline{\lim}_\ell f\left(\sum_{i=1}^{\ell} y_{n_{k_i}}\right) \leq \underline{\lim}_\ell \sum_{i=1}^{\ell} \frac{1}{\ell} f\left(y_{n_{k_i}}\right)$$

However, since $y_{n_{k_i}}$ is a subsequence of f_{n_k}, then

$$f(0) \leq \underline{\lim}_\ell \sum_{i=1}^{\ell} \frac{1}{\ell} f\left(y_{n_{k_i}}\right) \leq \underline{\lim}_\ell \sum_{i=1}^{\ell} \frac{1}{\ell} f\left(y_{n_k}\right) < \underline{\lim}_\ell \sum_{i=1}^{\ell} \frac{1}{\ell} f(0) - \epsilon$$

which is a contradiction. Thus $f(0) \leq \underline{\lim}_n f(y_n)$, which means f is weakly lower-semicontinuous for the sequence y_n. Thus f is weakly lower-semicontinuous for the sequence u_n, and since u_n was an arbitrary weakly convergent sequence, then f is weakly lower-semicontinuous. □