UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #5

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Problem 1

(a) For $f \in L^1(\mathbb{R})$, set $S_R f(x) = (2\pi)^{-\frac{1}{2}} \int_{-R}^{R} \widehat{f}(\xi) e^{ix\xi} \mathrm{d}\xi$. Show that

$$S_R f(x) = K_R * f(x) = \int_{-\infty}^{\infty} K_R(x-y) f(y) \mathrm{d}y$$

where

$$K_R(x) = (2\pi)^{-1} \int_{-R}^{R} e^{ix\xi} \mathrm{d}\xi = \frac{\sin Rx}{\pi x}.$$

(b) Show that if $f \in L^2(\mathbb{R})$, then $S_R f \to f$ in $L^2(\mathbb{R})$ as $R \to \infty$.

Proof. (a) This proof is simply calculation:

$$\begin{aligned}
(K_R * f)(x) &= \int_{-\infty}^{\infty} K_R(x-y) f(y) \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-R}^{R} e^{i(x-y)\xi} \mathrm{d}\xi f(y) \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-R}^{R} e^{ix\xi} e^{-iy\xi} f(y) \mathrm{d}\xi \mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} \mathrm{d}y \right] e^{ix\xi} \mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \widehat{f}(\xi) e^{ix\xi} \mathrm{d}\xi \\
&= S_R f(x)
\end{aligned}$$

(b) Next up, some analysis!

□

Problem 2

Show that for any $R \in (0, \infty)$, there exists $f \in L^1(\mathbb{R})$ such that $S_R f \notin L^1(\mathbb{R})$. Note that $K_R \notin L^1(\mathbb{R})$.

Proof. □

Problem 3

Assume $w \in \mathscr{S}'(\mathbb{R}^n)$ and $w(x) \geq 0$. Show that if $\widehat{w} \in L^\infty(\mathbb{R}^n)$ then $w \in L^1(\mathbb{R}^n)$ and

$$\|\widehat{x}\|_{L^\infty(\mathbb{R}^n)} = (2\pi)^{-\frac{n}{2}} \|x\|_{L^1(\mathbb{R}^n)}.$$

Hint: Consider $w_j(x) = \psi\left(\frac{x}{j}\right) w(x)$ with $\psi \in \mathscr{C}_C^\infty(\mathbb{R}^n)$ and $\psi(0) = 1$. Use the fact that $w_j \to w$ in $\mathscr{S}'(\mathbb{R}^n)$.

Proof. □

Problem 4

Consider the Poisson equation on $\mathbb{R}$: $u_{xx} = f$.

(a) Show that $\varphi(x) = \frac{x+|x|}{2}$ and $\phi(x) = \frac{|x|}{2}$ are both distributional solutions to $u_{xx} = \delta_0$.

(b) Let f be continuous with compact support in $\mathbb{R}$. Show that

$$u(x) = \int_{\mathbb{R}} \varphi(x-y) f(y) \mathrm{d}y$$

and

$$v(x) = \int_{\mathbb{R}} \phi(x-y) f(y) \mathrm{d}y$$

both solve the Poisson equation $w_{xx}(x) = f(x)$ without relying upon distribution theory.

Proof. □

Problem 5

Let $T \in \mathscr{S}'(\mathbb{R}^n)$ and $f \in \mathscr{S}(\mathbb{R}^n)$. Show that the Liebniz rule for distributional derivatives holds:

$$\frac{\partial}{\partial x_i}(fT) = f\frac{\partial T}{\partial x_i} + \frac{\partial f}{\partial x_i} T$$

in the sense of distributions.

Proof.

$$\begin{aligned}
\langle fT, \phi' \rangle &= \langle T, f\phi' \rangle && \text{by the definition of multiplication of } \mathscr{S} \text{ and } \mathscr{S}' \\
&= \langle T, f\phi' \rangle + \langle T, f'\phi \rangle - \langle T, f'\phi \rangle && \text{by adding and subtracting } \langle T, f'\phi \rangle \\
&= \langle T, f\phi' + f'\phi \rangle - \langle T, f'\phi \rangle && \text{by linearity of dual pairings} \\
&= \left\langle T, (f\phi)' \right\rangle - \langle T, f'\phi \rangle && \text{by the product rule of functions in } S \\
&= -\langle T', f\phi \rangle - \langle T, f'\phi \rangle && \text{by the definition of the distributional derivative of } T \in \mathscr{S}^* \\
&= -\langle T', f\phi \rangle - \langle f'T, \phi \rangle && \text{by the definition of multiplication of } \mathscr{S} \text{ and } \mathscr{S}' \\
&= -\langle fT', \phi \rangle - \langle f'T, \phi \rangle && \text{by the definition of multiplication of } \mathscr{S} \text{ and } \mathscr{S}' \\
&= -\langle fT' + f'T, \phi \rangle && \text{by linearity of dual pairings}
\end{aligned}$$

Thus, by the definition of distributional derivative, $(fT)' = fT' + f'T$. □

Problem 6

Show that a function $f \in L^2(\mathbb{R}^n)$ is real if and only if

$$\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}.$$

Proof. First note the following equality:

$$\overline{\widehat{f}(\xi)} = \overline{\int_{\mathbb{R}} f(x) e^{-ix\xi} \mathrm{d}x} = \int_{\mathbb{R}} \overline{f(x)} e^{ix\xi} \mathrm{d}x = \int_{\mathbb{R}} \overline{f(x)} e^{-ix(-\xi)} \mathrm{d}x = \widehat{\overline{f}}(-\xi)$$

If f is real, then $\overline{f} = f$, and thus $\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$. On the other hand, if $\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$, then $\widehat{f}(-\xi) = \widehat{\overline{f}}(-\xi)$. Since $f \in L^2$, then the Fourier transform is a bijection, and thus $f = \overline{f}$, which proves f is a real-valued function. $\square$