UC Davis, Analysis (MAT201C), Spring 2016

Homework #2

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Problem 1

A function $f \in L^p(\mathbb{R}^n)$ is said to be L_p*-continuous* if $\tau_h f \to f$ in $L^p(\mathbb{R}^n)$ as $h \to 0$ in $\mathbb{R}^n$, where $\tau_h f(x) = f(x-h)$ is the translation of f by h. Prove that, if $1 \leq p < \infty$, every $f \in L^p(\mathbb{R}^n)$ is L^p-continuous. Give a counter-example to show that this result is not true when $p = \infty$. [Hint: Approximate an L^p function by a C_c function.]

Proof. We begin with the counter-example for L^∞. Define $f \in L^\infty(\mathbb{R})$ as $f(x) = \mathscr{X}_{[0,1]^n}$ where $\mathscr{X}$ is the characteristic function. Note that $f(1-\varepsilon) = 1$ for all $\varepsilon > 0$. Let h be a small perturbation, i.e. $0 < |h| \ll 1$, and choose $\varepsilon = \frac{h}{2}$. Then $\forall x \in (0, \varepsilon)$, $\tau_h f(x) = 0$ but $f(x) = 1$, and thus $\left|\tau_h f(x) - f(x)\right| = 1$. This shows that $\forall h > 0$, $\exists$ an interval I_h (of positive measure, $\mu(I_n) > 0$) such that $\left|\tau_h f(x) - f(x)\right| = 1$ for all $x \in I_h$. Thus $\tau_h f \not\to f$ in $L^\infty(\mathbb{R}^n)$.

Now let $p \in [1, \infty)$ and let $f \in L^p(\mathbb{R}^n)$. Then by the density of simple functions in L^p, there exist simple functions $f_\ell \in L^p$ such that $\left\|f_\ell - f\right\|_p \to 0$. By the definition of simple functions, there are disjoint, finite-measure sets $E_{\ell,k}$, where $k = 1, \ldots, N_\ell$ and constants $a_{\ell,k}$ such that

$$f_\ell = \sum_{k=1}^{N_\ell} a_{\ell,k} \mathscr{X}_{E_{\ell,k}}$$

Next we show that for each $h > 0$, $\tau_h f_\ell \to \tau_h f$ in L^p.

$$\left\|\tau_h f_\ell - \tau_h f\right\|_p^p = \int_{\mathbb{R}^n} \left|f_\ell(x+h) - f(x+h)\right|^p \mathrm{d}x = \int_{\mathbb{R}^n} \left|f_\ell(x) - f(x)\right|^p \mathrm{d}x = \left\|f_\ell - f\right\|_p^p \to 0$$

by a change of variables. Next we show that for each n, $\tau_h f_\ell \to f_\ell$ as $h \to 0$. For each $E_{\ell,k}$, define the set $E_{\ell,k,h}$ as the shifted set $E_{\ell,k}$ by h:

$$E_{\ell,k,h} = \left\{x + h \,:\, x \in E_{\ell,k}\right\}.$$

Then

$$\begin{aligned}
\left\|\tau_h f_\ell - f_\ell\right\|_p^p &= \int_{\mathbb{R}^n} \left|\sum_{k=1}^{N_\ell} a_{\ell,k} \mathscr{X}_{E_{\ell,k,h}} - \sum_{k=1}^{N_\ell} a_{\ell,k} \mathscr{X}_{E_{\ell,k}}\right|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left|\sum_{k=1}^{N_\ell} a_{\ell,k} \left(\mathscr{X}_{E_{\ell,k,h} \setminus E_{\ell,k}} - \mathscr{X}_{E_{\ell,k} \setminus E_{\ell,k,h}}\right)\right|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^n} G_{\ell,h} \mathrm{d}x
\end{aligned}$$

where

$$G_{\ell,h} = \left|\sum_{k=1}^{N_\ell} a_{\ell,k} \left(\mathscr{X}_{E_{\ell,k,h} \setminus E_{\ell,k}} - \mathscr{X}_{E_{\ell,k} \setminus E_{\ell,k,h}}\right)\right|^p.$$

Then since $\mu\left(E_{\ell,k,h} \setminus E_{\ell,k}\right) \to 0$ and $\mu\left(E_{\ell,k} \setminus E_{\ell,k,h}\right) \to 0$ as $h \to 0$, then $G_{\ell,h} \to 0$ pointwise as $h \to 0$. Also,

$$\left|G_{\ell,h}(x)\right| \leq \left|\sum_{k=1}^{N_\ell} a_{\ell,k} 2 \mathscr{X}_{E_{\ell,k}}(x)\right|^p = 2^p \left|\sum_{k=1}^{N_\ell} a_{\ell,k} \mathscr{X}_{E_{\ell,k}}(x)\right| = 2^p \left|f_\ell(x)\right|^p$$

That is, for a given n, $G_{\ell,k}$ is uniformly bounded by $2^p \left|f_\ell(x)\right|^p$. Then by the Dominated Convergence Theorem, the integral of $G_{\ell,k}$ converges to the integral of its pointwise limit, which is 0. Thus, as $h \to 0$,

$$\left\|\tau_h f_\ell - f_\ell\right\|_p^p \to \int_{\mathbb{R}^n} 0 \mathrm{d}x = 0.$$

Thus,

$$\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h f_\ell\|_p + \|\tau_h f_\ell - f_\ell\|_p + \|f_\ell - f\|_p \to 0$$

since each of the three norms on the right hand side converge to 0. □

Problem 2

Show that $L^\infty(\mathbb{R})$ is not separable. [Hint: There is an uncountable set $\mathscr{F} \subset L^\infty$ such that $\|f-g\|_\infty \geq 1$ for all $f, g \in \mathscr{F}$ with $f \neq g$.]

Proof. Let $\mathscr{F} = \{\mathscr{X}_{[0,\alpha]} \,:\, 0 < \alpha \in \mathbb{R}\}$. $\mathscr{F}$ is clearly uncountable. Consider any two $f, g \in \mathscr{F}$. Then, without loss of generality, $f = \mathscr{X}_{[0,\alpha]}$ and $g = \mathscr{X}_{[0,\beta]}$ where $\alpha < \beta$. Also,

$$\|f - g\|_\infty = \text{ess sup}\{\mathscr{X}_{(\alpha,\beta]}\} = 1$$

Thus the ball around any $f \in \mathscr{F}$ of radius $\frac{1}{2}$, i.e. $B\left(f, \frac{1}{2}\right)$, contains no other elements of $\mathscr{F}$. Thus $L^\infty(\mathbb{R})$ is not separable since $\mathscr{F}$ is uncountable and not dense. □

Problem 3

Prove Chebyshev's Inequality: If $f \in L^p$ $(1 \leq p < \infty)$, then for any $\alpha > 0$,

$$\mu\left(\{x \,:\, |f(x)| > \alpha\}\right) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

[Note that you can find the proof of this simple fact in many texts but you should see if you can figure it out yourself. Also, note that this inequality holds for all $0 < p < \infty$.]

Proof. Let $A_\alpha = \{x \,:\, |f(x)| > \alpha\} = \left\{x \,:\, \left|\frac{f(x)}{\alpha}\right| > 1\right\} = \left\{x \,:\, \left|\frac{f(x)}{\alpha}\right|^p > 1\right\}$ for all $p \geq 1$. Then

$$\left(\frac{\|f\|_p}{\alpha}\right)^p = \int_\Omega \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu = \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu + \int_{\Omega\setminus A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu \geq \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu \geq \int_{A_\alpha} 1 \mathrm{d}\mu = \mu(A_\alpha)$$

which proves the result. □

Problem 4

Assume that $f, g \in L^1(\mathbb{R}^n)$. Prove that the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y$$

is measurable and in $L^1(\mathbb{R}^n)$.

Proof. First note that $\|f\|_1 \|g\|_1 < \infty$ since they are in $L^1(\mathbb{R}^n)$. Then

$$\|f\|_1 \|g\|_1 = \|f\|_1 \int_{\mathbb{R}^n} |g(y)| \mathrm{d}y$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \|f\|_1 |g(y)| \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)| \mathrm{d}x |g(y)| \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)| \mathrm{d}x \mathrm{d}y \\
\implies \int_{\mathbb{R}^{2n}} |f(x-y)g(y)| \mathrm{d}x\mathrm{d}y < \infty
\end{aligned}
$$

Thus, by Fubini's Theorem,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| \mathrm{d}y\mathrm{d}x < \infty \\
\implies \|f * g\|_1 = \int_{\mathbb{R}^n} |(f*g)(x)| \mathrm{d}x = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y \right| \mathrm{d}x \leq &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| \mathrm{d}y\mathrm{d}x < \infty
\end{aligned}
$$

which shows $(f * g) \in L^1(\mathbb{R}^n)$. Since all L^1 functions are measurable, $(f * g)$ is measurable. □

Problem 5

Let $f_n = \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}$. Prove that f_n converges weakly to 0 in $L^2(0,1)$ and $f_n \to 0$ in $L^1(0,1)$ but f_n does not converge strongly in $L^2(0,1)$.

Proof.

$$\|f_n\|_2^2 = \int_0^1 n\mathscr{X}_{(0,\frac{1}{n})} \mathrm{d}x = \int_0^{\frac{1}{n}} n \mathrm{d}x = 1$$

Thus $\|f_n\|_2 = 1$ for all n, and thus does not converge strongly to 0 in $L^2(0,1)$.

$$\|f_n\|_1 = \int_0^1 \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})} \mathrm{d}x = \int_0^{\frac{1}{n}} \sqrt{n} \mathrm{d}x = \frac{1}{\sqrt{n}}$$

Thus $\|f_n\|_1 \to 0$ as $n \to 0$, which shows $f_n \to 0$ strongly in $L^1(0,1)$. Let $L \in L^2(0,1)^* \cong L^2(0,1)$. Thus $\exists \ell \in L^2(0,1)$ such that

$$L(f) = \int_0^1 \ell(x) f(x) \mathrm{d}x$$

for all $f \in L^2$. Then

$$L(f_n) = \int_0^1 \ell(x)\sqrt{n}\mathscr{X}_{(0,\frac{1}{n})} \mathrm{d}x = \int_0^{\frac{1}{n}} \ell(x)\sqrt{n}\mathrm{d}x \leq \left(\int_0^{\frac{1}{n}} |\ell(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_0^{\frac{1}{n}} n \mathrm{d}x \right)^{\frac{1}{2}} = \left(\int_0^{\frac{1}{n}} |\ell(x)|^2 \mathrm{d}x \right) \to 0$$

since ℓ is fixed and $\mu\left((0,\frac{1}{n})\right) \to 0$ as $n \to 0$. Thus $f_n \rightharpoonup 0$ in $L^2(0,1)$. □

Problem 6

Find a sequence of functions with the property that f_j converges to 0 in $L^2(\Omega)$ weakly, to 0 in $L^{\frac{3}{2}}(\Omega)$ strongly, but it does not converge to 0 strongly in $L^2(\Omega)$.

Proof. Let $f_n = \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}$. Then by number 5, $f_n \not\rightharpoonup 0$ in $L^2(0,1)$ but $f_n \rightharpoonup 0$ in $L^2(0,1)$. Also,

$$\|f_n\|_{\frac{3}{2}}^{\frac{3}{2}} = \int_0^1 \left| n^{\frac{1}{2}}\mathscr{X}_{(0,\frac{1}{n})}\right|^{\frac{3}{2}} \mathrm{d}x = \int_0^{\frac{1}{n}} n^{\frac{3}{4}}\,\mathrm{d}x = n^{-\frac{1}{4}} \to 0 \text{ as } n \to \infty$$

Thus $f_n \to 0$ in $L^{\frac{3}{2}}(0,1)$. □

Problem 7

Let f_n and g_n denote two sequences in $L^p(\Omega)$ with $1 \le p \le \infty$ such that $f_n \to f$ in $L^p(\Omega)$, and $g_n \to g$ in $L^p(\Omega)$. Set $h_n = \max\{f_n, g_n\}$ and prove that $h_n \to h$ in $L^p(\Omega)$.

Proof. First note that

$$\begin{aligned} h_n(x) &= \frac{1}{2}\big(f_n(x) + g_n(x)\big) + \frac{1}{2}\left|f_n(x) - g_n(x)\right|, \qquad \text{and} \\ h(x) &= \frac{1}{2}\big(f(x) + g(x)\big) + \frac{1}{2}\left|f(x) - g(x)\right|. \end{aligned}$$

Then

$$\begin{aligned} \|h_n - h\|_p &= \left\| \frac{1}{2}(f_n + g_n) + \frac{1}{2}|f_n - g_n| - \left[\frac{1}{2}(f+g) + \frac{1}{2}|f-g|\right]\right\| \\ &\le \frac{1}{2}\left[\|f_n - f\|_p + \|g_n - g\|_p + \big\||f_n - g_n| - |f-g|\big\|_p\right] \\ &\le \frac{1}{2}\left[\|f_n - f\|_p + \|g_n - g\|_p + \|(f_n - g_n) - (f-g)\|_p\right] \\ &\le \frac{1}{2}\left[\|f_n - f\|_p + \|g_n - g\|_p + \|f_n - f\|_p + \|g_n - g\|_p\right] \\ &= \|f_n - f\|_p + \|g_n + g\|_p \end{aligned}$$

But since $f_n \to f$ and $g_n \to g$ in L^p, then there is an N such that $\|f_n - f\|_p < \frac{\varepsilon}{2}$ and $\|g_n - g\|_p < \frac{\varepsilon}{2}$ for all $n \ge N$. Thus if $n \ge N$,

$$\|h_n - h\|_p \le \|f_n - f\|_p + \|g_n + g\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since ε is arbitrary, $h_n \to h$ in L^p. □

Problem 8

Let f_n be a sequence in $L^p(\Omega)$ with $1 \le p < \infty$, and let g_n be a bounded sequence in $L^\infty(\Omega)$. Suppose that $f_n \to f$ in $L^p(\Omega)$ and that $g_n \to g$ pointwise a.e. Prove that $f_n g_n \to fg$ in $L^p(\Omega)$.

Proof.

$$\begin{aligned} \|f_n g_n - fg\|_p &\le \|f_n g_n - f g_n\|_p + \|f g_n - fg\|_p \\ &= \left[\int_\Omega |g_n|^p |f_n - f|^p\right]^{\frac{1}{p}} + \left[\int_\Omega |f g_n - fg|^p\right]^{\frac{1}{p}} \end{aligned}$$

Since g_n is a bounded sequence in L^∞, $\exists M$ such that $|g_n(x)| \le \|g_n\| \le M$ almost everywhere. Thus

$$\left[\int_\Omega |g_n|^p |f_n - f|^p\right]^{\frac{1}{p}} \le \left[\int_\Omega M^p |f_n - f|^p\right]^{\frac{1}{p}} = M\|f_n - f\|_p \to 0$$

since $f_n \to f$ in L^p. Next note $g \in L^\infty$ since

$$\|g\|_\infty = \text{ess sup}\{g(x)\} = \text{ess sup}\left\{\lim_n g_n(x)\right\} \le M$$

Since $g_n \to g$ pointwise, then $fg_n \to fg$ pointwise and thus $|fg_n - fg|^p \to 0$ pointwise. Define h_n as

$$h_n = |fg_n - fg|^p.$$

Then $h_n \in L^1(\Omega)$ since

$$\|h_n\|_1^p = \|fg_n - fg\|_p \le \|fg_n\|_p + \|fg\|_p \le \|f\|_p \|g_n\|_\infty + \|f\|_p \|g\|_\infty \le 2\|f\|_p M$$

Also, $|h_n(x)|^{\frac{1}{p}} = |(fg_n - fg)(x)| \le |f(x)g_n(x) - f(x)g(x)| \le |f(x)||g_n(x)| + |f(x)||g(x)| \le 2M|f(x)|$, which implies h is dominated:

$$|h(x)| \le 2^p M^p |f(x)|^p$$

Thus, by the dominated convergence theorem,

$$\begin{aligned} \lim_n \int_\Omega h_n = \int_\Omega \lim_n h_n = \int_\Omega 0 = 0 \\ \implies \lim_n \int_\Omega |fg_n - fg|^p = 0 \end{aligned}$$

Thus,

$$\lim_n \|f_n g_n - fg\|_p = \lim_n \left[\int_\Omega |g_n|^p |f_n - f|^p\right]^{\frac{1}{p}} + \lim_n \left[\int_\Omega |fg_n - fg|^p\right]^{\frac{1}{p}} = 0$$

which shows $f_n g_n \to fg$ in $L^p(\Omega)$. □

Problem 9

Prove that the space of continuous functions with compat support $\mathscr{C}_c^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.

Proof. First we will show that $\mathscr{C}_c^0(\mathbb{R}^n)$ is dense in the space of characteristic functions. Then since simple functions are dense in $L^p(\mathbb{R}^n)$ and simple functions are finite linear combinations of simple functions, this will show $\mathscr{C}_c^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Let $\mathscr{X}_A$ be the characteristic function on a bounded, measurable subset $A \subset \mathbb{R}^n$. By the alternate definition of the Lebesgue measure, for any $\varepsilon > 0$, there is an open set $G \supset A$ and compact set $K \subset A$ such that $\mu(G \setminus K) < \varepsilon$. By Urysohn's Lemma, there is some continuous function g such that

$$g(x) \in \begin{cases} \{1\} & \text{if } x \in K \\ \{0\} & \text{if } x \in G^C \\ [0,1] & \text{if } x \in G \setminus K \end{cases}$$

Then

$$\|g - \mathscr{X}_A\|_p^p = \int_{G\setminus K} \left|g(x) - \mathscr{X}_A(x)\right|^p \mathrm{d}x \le \int_{G\setminus K} 1^p \mathrm{d}x = \mu(G\setminus K) < \varepsilon$$

Thus there are continuous functions that are arbitrarily close in L^p-norm to any characteristic function on bounded measurable sets. Now let $f \in L^p(\mathbb{R}^n)$. Then define f_n as

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| \le n \\ 0 & \text{if } |x| > n \end{cases}.$$

Then by definition, each f_n is compactly supported and converges to f in L^p. Since simple functions are dense in the space of compactly supported functions, so define g_n as

$$h_n = \sum_{k=1}^{N_n} a_{n,k}\mathscr{X}_{n,k}$$

where $a_{n,k}$ are constants and $\mathscr{X}_{n,k}$ are bounded measurable sets, and $\|f_n - h_n\|_p < \varepsilon$. This is possible since for each $\mathscr{X}_{n,k}$, we can define $g_{n,k}$ such that

$$\|g_{n,k} - a_{n,k}\mathscr{X}_{n,k}\| < \frac{\varepsilon}{\max\limits_{k=1,\ldots,N_n}\{a_{n,k}\}N_n}.$$

Then define the continuous function $\tilde{g} = \sum_{k=1}^{N_n} g_{n,k}$. Then

$$\begin{aligned}\|f-g\|_p &\le \|f-f_n\|_p + \|f_n - h_n\|_p + \|h_n - g\|_p \\ &\le \|f-f_n\|_p + \|f_n - h_n\|_p + \sum_{k=1}^{N_n}\|a_{n,k}\mathscr{X}_{n,k} - g_{n,k}\|_p \to 0\end{aligned}$$

Thus there is a continuous function g that arbitrarily approximates any L^p function f. $\square$