201C, Spring '16, Thomases
Homework 1

1. If f and g are measurable functions on Ω, then $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$. If $f \in L^1$ and $g \in L^\infty$, then $\|fg\|_1 = \|f\|_1 \|g\|_\infty$ if and only if $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$.

2. $\|f_n - f\|_\infty \to 0$ iff there exists a measurable set E such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.

We say $\{f_n\}$ **converges in measure** to f if for every $\epsilon > 0$,

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \text{ as } n \to \infty.$$

3. If $\|f_n - f\|_p \to 0$ $(p < \infty)$ then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n $(p < \infty)$ then $\|f_n - f\|_p \to 0$.

4. If $f_n, f \in L^p$ $(p < \infty)$ and $f_n \to f$ point-wise a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$.

5. Suppose $0 < p < q \leq \infty$. Then $L^p \not\subset L^q$ iff Ω contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff Ω contains sets of arbitrarily large finite measure. [Hint: for the "if" inplication: in the first case there is a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) \leq 2^{-n}$, and in the second case there is a disjoint sequence $\{E_n\}$ with $1 \leq \mu(E_n) < \infty$. Consider $f = \sum a_n \chi_{E_n}$ for suitable constants a_n.]

6. If $f \in L^\infty(\Omega) \cap L^q(\Omega)$ for some q then $f \in L^p(\Omega)$ for all $p > q$ and

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p.$$

7. Prove that when $\infty \geq r \geq q \geq 1$, $f \in L^r(\Omega) \cap L^q(\Omega) \Rightarrow f \in L^p(\Omega)$ for all $r \geq p \geq q$.

8. Prove that a strongly convergent sequence in $L^p(\mathbb{R}^n)$ is also a Cauchy sequence.

9. Give three different examples of ways for a sequence $f_k \in L^p(\mathbb{R}^n)$ to converge weakly to zero, but not converge strongly to anything. Verify your claims for these examples.