UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #3

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Problem 1

If $u \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, and $u^\varepsilon = \eta_\varepsilon * u$, for η_ε the standard mollifier. Show that

$$u^\varepsilon \to u$$

in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Proof. First we show the following containment:

$$\text{spt}\,(f+g) \subset \text{spt}\,(f) \cup \text{spt}\,(g)$$

Let $x \in \{x \in \Omega \,:\, (f+g)(x) \neq 0\}$. Then $(f+g)(x) = f(x) + g(x) \neq 0$. Then either $f(x) = 0$ or $g(x) = 0$, i.e. $x \in \text{spt}\,(f) \cup \text{spt}\,(g)$. But $\text{spt}\,(f) \cup \text{spt}\,(g)$ is closed, which implies

$$\text{spt}\,(f+g) = \overline{\{x \in \Omega \,:\, (f+g)(x) \neq 0\}} \subset \text{spt}\,(f) \cup \text{spt}\,(g).$$

Now let $\varepsilon > 0$ and approximate u by $\tilde{u} \in C_C^0(\mathbb{R}^n)$ such that $\|u - \tilde{u}\|_p < \dfrac{\varepsilon}{3}$. Since $\tilde{u}$ is continuous on a compact set, it is uniformly continuous, and thus $\exists \delta_\varepsilon > 0$ such that $\left|x - y\right| < \delta_\varepsilon \implies \left|\tilde{u}(x) - \tilde{u}(y)\right| < \mu(K)^{-\frac{1}{p}}\frac{\varepsilon}{3}$ where K is defined below. Define $\delta = \min\left\{\delta_\varepsilon, \frac{1}{2}\right\}$. Define the set $K = \left(\overline{\text{spt}\,(\eta_1) + \text{spt}\,(\tilde{u})}\right) \cup \text{spt}\,(\tilde{u})$. Then since $\mu(\text{spt}\,(\tilde{u})) < \infty$ and $\mu(\text{spt}\,(\eta_1)) < \infty$, then $\mu(K) < \infty$. Then

$$\left\|\eta_\delta * u - \eta_\delta * \tilde{u}\right\|_p = \left\|\eta_\delta * (u - \tilde{u}\right\|_p \le \left\|\eta_\delta\right\|_1 \|u - \tilde{u}\|_p = 1 \cdot \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

Let $J = \text{spt}\,(\eta_\delta * \tilde{u} - \tilde{u})$. Then $J \subset K$ by the first containment shown. Finally,

$$\begin{aligned}
\left\|\eta_\delta * \tilde{u} - \tilde{u}\right\|_p &\le \left[\int_J \left|\int_{B_\delta(x)} \eta_\delta(x-y)\tilde{u}(y)\mathrm{d}y\right|^p \mathrm{d}x - \tilde{u}(x)\right]^{\frac{1}{p}} \\
&\le \left[\int_J \left(\int_{B_\delta(x)} \eta_\delta(x-y)|\tilde{u}(u) - \tilde{u}(x)|\mathrm{d}y\right)^p \mathrm{d}x\right]^{\frac{1}{p}} \\
&< \mu(K)^{-\frac{1}{p}}\frac{\varepsilon}{3}\left[\int_J \mathrm{d}x\right]^{\frac{1}{p}} \\
&= \mu(K)^{-\frac{1}{p}}\frac{\varepsilon}{3}\mu(J)^{\frac{1}{p}} \\
&\le \mu(K)^{-\frac{1}{p}}\frac{\varepsilon}{3}\mu(K)^{\frac{1}{p}} \\
&= \frac{\varepsilon}{3}
\end{aligned}$$

Thus,

$$\left\|\eta_\delta * u - u\right\|_p \le \left\|\eta_\delta * u - \eta_\delta * \tilde{u}\right\|_p + \left\|\eta_\delta * \tilde{u} - \tilde{u}\right\|_p + \|\tilde{u} - u\|_p < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon,$$

which shows, since ε is arbitrarily small, that

$$\eta_\delta * u \to u.$$

□

Problem 2

Let Ω denote an open and smooth subset of $\mathbb{R}^n$. Prove that $\mathscr{C}_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$.

Proof. It suffices to show that any convolved function is in $\mathscr{C}_C^\infty$. Let $u \in L^p$ and choose $\varepsilon > 0$. Then since

$$\frac{\partial}{\partial x}\eta_\varepsilon(x-y)u(y) = u(y)\frac{\partial}{\partial x}\eta_\varepsilon(x-y) \leq u(y)M(y)$$

since $\eta_\varepsilon(x-y)$ is an arbitrarily smooth compactly supported function, and thus its derivatives are arbitrarily smooth compactly supported functions. Specifically,

$$M(y) = \max_{x\in B_\varepsilon(y)}\left\{\eta_\varepsilon(x-y)\right\}$$

Next we show M is continuous. Choose $\tilde{\varepsilon} > 0$. Then choose δ such that $|x_1 - x_2| < \delta \implies \left|\eta_\varepsilon(x_1) - \eta_\varepsilon(x_2)\right| < \tilde{\varepsilon}$. Then

$$M(y+\delta) = \max_{x\in B_\varepsilon(y+\delta)}\left\{\eta_\varepsilon(x-y-\delta)\right\} \leq \max_{x\in B_\varepsilon(y+\delta)}\left\{\eta_\varepsilon(x-y)+\tilde{\varepsilon}\right\} = M(y)+\tilde{\varepsilon}$$

since

$$\left|\eta_\varepsilon(x-y-\delta) - \eta_\varepsilon(x-y)\right| < \tilde{\varepsilon}$$

Similarly, $-\tilde{\varepsilon} < M(y+\delta) - M(y)$. Thus,

$$\left|M(y+\delta) - M(y)\right| < \tilde{\varepsilon}$$

and thus M is continuous in M. Since $u \in L^p$ and M is continuous, $u(y)M(y)$ is integrable (and it is also a bounding function of $\frac{\partial}{\partial x}\eta_\varepsilon(x-y)u(y)$). Thus Shkoller Lemma 1.39 applies, and

$$\frac{\mathrm{d}}{\mathrm{d}x}u^\varepsilon = \frac{\mathrm{d}}{\mathrm{d}x}\int_\Omega \eta_\varepsilon(x-y)u(y)\mathrm{d}y = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}x}\eta_\varepsilon(x-y)u(y)\mathrm{d}y = \int_\Omega u(y)\frac{\mathrm{d}}{\mathrm{d}x}\eta_\varepsilon(x-y)\mathrm{d}y = u * \frac{\mathrm{d}}{\mathrm{d}x}\eta_\varepsilon \in \mathscr{C}_C^0(\Omega)$$

since the convolution of an L^p function with a continuous function is continuous. This shows $u^\varepsilon \in \mathscr{C}_C^1(\Omega)$. Now suppose $u^\varepsilon \in \mathscr{C}_C^k(\Omega)$. Then

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\int_\Omega \eta_\varepsilon(x-y)u(y)\mathrm{d}y \in \mathscr{C}_C^0(\Omega)$$

$$\implies \int_\Omega \frac{\mathrm{d}^k}{\mathrm{d}x^k}\eta_\varepsilon(x-y)u(y)\mathrm{d}y = \int_\Omega u(y)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\eta_\varepsilon(x-y)\mathrm{d}y$$

Thus,

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\int_\Omega \eta_\varepsilon(x-y)u(y)\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}\int_\Omega u(y)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\eta_\varepsilon(x-y)\mathrm{d}y$$

By similar arguments as above (which apply since all derivatives of η_ε are arbitrarily smooth),

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\int_\Omega \eta_\varepsilon(x-y)u(y)\mathrm{d}y = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}x}u(y)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\eta_\varepsilon(x-y)\mathrm{d}y = \int_\Omega u(y)\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\eta_\varepsilon(x-y)\mathrm{d}y = u * \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\eta_\varepsilon \in \mathscr{C}_C^0(\Omega)$$

since u is L^p and all derivatives of η_ε are continuous. Thus, by induction, $u^\varepsilon \in \mathscr{C}_C^k(\Omega)$ for all $k = 1, 2, \ldots$, i.e. $u^\varepsilon \in \mathscr{C}_C^\infty(\Omega)$. By problem one, convolutions are dense in $L^p(\Omega)$, and thus $\mathscr{C}_C^\infty$ is dense in $L^p(\Omega)$. $\square$

Problem 3

Prove that if $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfies $\int_\Omega u(x)v(x)\mathrm{d}x = 0$ for all $v \in \mathscr{C}_c^\infty(\Omega)$, then $u = 0$ a.e. in Ω.

Proof. Suppose u satisfies the hypothetical conditions, and also that $u \not\equiv 0$. Then $\exists E \subset \Omega$ with $\mu(E) > 0$ and $u(x) \neq 0$ for all $x \in E$. Without loss of generality, suppose $u(x) > 0$ for all $x \in E$. Next let $K \subset L \subset E$, with K compact and L open. By Urysohn's Lemma for smooth functions, construct the test function v such that $v(x) = 1$ for all $x \in K$ and $v(x) = 0$ for all $x \in \overline{E^C}$ and $v(x) \in [0,1]$ for all $x \in \overline{E_C} \setminus K$. Then

$$\int_\Omega u(x)v(x)\mathrm{d}x \geq \int_K |u(x)|\mathrm{d}x > 0$$

This is a contradiction. Thus if $u \in L^1_{\text{loc}}(\Omega)$ satisfies $\int_\Omega u(x)v(x)\mathrm{d}x = 0$ for all test functions v, then $u = 0$ a.e. in Ω.

□

Problem 4

Let $u \in L^\infty(\mathbb{R}^n)$ and let η_ε be a standard mollifier. For $\varepsilon > 0$ consider the sequence $\psi_\varepsilon \in L^\infty(\mathbb{R}^n)$ such that

$$\|\psi_\varepsilon\|_\infty \leq 1 \ \forall \varepsilon > 0 \ \text{ and } \ \psi_\varepsilon \to \psi \text{ a.e. in } \mathbb{R}^n,$$

define

$$v^\varepsilon = \eta_\varepsilon * (\psi_\varepsilon u) \ \text{ and } \ v = \psi u.$$

(a) Prove that $v^\varepsilon \overset{*}{\rightharpoonup} v$ in $L^\infty(\mathbb{R}^n)$.

(b) Prove that $v^\varepsilon \to v$ in $L^1(B)$ for every ball $B \subset \mathbb{R}^n$.

Proof. (a) We want to show $\phi_{v^\varepsilon}(f) \to \phi_v(f)$ for all $f \in L^1(\mathbb{R})$, where ϕ_v and ϕ_{v^ε} are the continuous linear functionals associated with v and v^ε, respectively.

□

Problem 5

For $u \in \mathscr{C}^0(\mathbb{R}^n;\mathbb{R})$, spt (u) is the closure of the set $\{x \in \mathbb{R}^n : u(x) \neq 0\}$, but this definition may not make sense for functions $u \in L^p(\Omega)$. For example what is the support of $\mathscr{X}_\mathbb{Q}$, the indicator over the rationals?

Let $u : \mathbb{R}^n \to \mathbb{R}$, and let $\{\Omega_\alpha\}_{\alpha \in A}$ denote the collection of all open sets on $\mathbb{R}^n$ such that for each $\alpha \in A$, $u = 0$ a.e. on Ω_α. Define $\Omega = \bigcup_{\alpha \in A} \Omega_\alpha$. Prove that $u = 0$ a.e. on Ω.

The support of u, spt (u), is Ω^C, the complement of Ω. Notice that if $v = w$ a.e. on $\mathbb{R}^n$, then spt (v) = spt (w); furthermore, if $u \in \mathscr{C}^0(\mathbb{R}^n)$, then $\Omega^C = \overline{\{x \in \mathbb{R}^n : u(x) \neq 0\}}$. (Hint: Since A is not necessarily countable, it is not clear that $f = 0$ a.e. on Ω, so find a countable family U_n of open sets in $\mathbb{R}^n$ such that every open set on $\mathbb{R}^n$ is the union of some of the sets from $\{U_n\}$.)

Proof. Define the basis B of the standard topology on $\mathbb{R}^n$ by

$$B = \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{Q} \ \forall 1 \leq i \leq n\}.$$

B is clearly countable, and is a basis of the standard topology on $\mathbb{R}^n$ because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Since Ω_α is open for each $\alpha \in A$, then Ω_α can be written as a union of open sets in B:

$$\Omega_\alpha = \bigcup_{i=1}^\infty B_{\alpha,i}$$

where $B_{\alpha,i} \in B$. The union above is countable since B is countable. Also, $\bigcup_{\alpha\in A}\Omega_\alpha$ can be re-indexed as

$$\Omega = \bigcup_{\alpha\in A}\Omega_\alpha = \bigcup_{\alpha\in A}\bigcup_{i=1}^{\infty}B_{\alpha,i} = \bigcup_{k=1}^{\infty}B_k$$

where $B_k \in B$. This union is countable since each $\bigcup_{i=1}^{\infty}B_{\alpha,i}$ is countable and all $B_{\alpha,i}\in B$, which is countable. Thus,

$$\mu(\{x\in\Omega\,:\,u(x)\neq 0\}) = \mu\left(\left\{x\in\bigcup_{k=1}^{\infty}B_k\,:\,u(x)\neq 0\right\}\right)\underbrace{\leq}_{\text{countable additivity}}\sum_{k=1}^{\infty}\mu\Big(\big\{x\in\Omega\,:\,u(x)\neq 0\big\}\Big) = 0$$

In other words, $x = 0$ a.e. on Ω. $\square$

Problem 6

Prove that if $u\in L^1(\mathbb{R}^n)$ and $v\in L^p(\mathbb{R}^n)$ for $1\leq p\leq\infty$, then

$$\text{spt}\,(u*v)\subset\overline{\text{spt}\,(u)+\text{spt}\,(v)}.$$

Proof. Suppose $x\notin\overline{\text{spt}\,(u)+\text{spt}\,(v)}$ and define the set $\big[x-\text{spt}\,(u)\big]$ as the shift of the support of u by the vector x:

$$\big[x-\text{spt}\,(u)\big] = \big\{y\,:\,x-y\in\text{spt}\,(u)\big\}$$

Then

$$(u*v)(x) = \int_{\mathbb{R}^n}u(x-y)v(y)\mathrm{d}y = \int_{[x-\text{spt}\,(u)]\cap\text{spt}\,(v)}u(x-y)v(y)\mathrm{d}y$$

If $x_0\in\text{spt}\,(v)\cap\big[x-\text{spt}\,(u)\big]$, then $x_0\in\text{spt}\,(v)$ and $x-x_0 = 0\in\text{spt}\,(u)$. Then since $x = (x-x_0)+(x_0)$, then $x\in\text{spt}\,(u)+\text{spt}\,(v)$, which is a contradiction since $x\notin\overline{\text{spt}\,(u)+\text{spt}\,(v)}$. Thus $\big[x-\text{spt}\,(u)\big]\cap\text{spt}\,(v) = \emptyset$, and therefore

$$(u*v)(x) = \int_{[x-\text{spt}\,(u)]\cap\text{spt}\,(v)}u(x-y)v(y)\mathrm{d}y = \int_{\emptyset}u(x-y)v(y)\mathrm{d}y = 0.$$

Since $\overline{\text{spt}\,(u)+\text{spt}\,(v)}$ is closed, its complement is open. So $\exists\varepsilon$ such that $B_\varepsilon(x)\subset\overline{\text{spt}\,(u)+\text{spt}\,(v)}^C$. Thus $(u*v)(x) = 0$ for all $x\in B_\varepsilon(x)$. Then

$$B_\varepsilon(x)\cap\{x\in\Omega\,:\,(u*v)(x)\neq 0\} = \emptyset.$$

So there is a neighborhood around x which does not intersect $\{x\in\Omega\,:\,(u*v)(x)\neq 0\}$. Thus,

$$x\notin\overline{\{x\in\Omega\,:\,(u*v)(x)\neq 0\}} = \text{spt}\,(u*v).$$

This shows

$$\text{spt}\,(u*v)\subset\overline{\text{spt}\,(u)+\text{spt}\,(v)}.$$

$\square$

Problem 7

Suppose that $1 < p < \infty$. If $\tau_y f(x) = f(x-y)$, show that f belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $\tau_y f$ is a Lipschitz function of y with values in $L^p(\mathbb{R}^n)$; that is,

$$\left\|\tau_y f - \tau_z f\right\|_p \leq C\left|y-z\right|.$$

What happens in the case $p = 1$?

Proof. □

Problem 8

If $u \in W^{1,p}(\mathbb{R}^n)$ for some $p \in [1,\infty)$ and $\frac{\partial u}{\partial x_j} = 0$, $j = 1,\ldots,n$, on a connected open set $\Omega \subset \mathbb{R}^n$, show that u is equal a.e. to a constant on Ω. (Hint: approximate u using that $\eta_\varepsilon * u \to u$ in $W^{1,p}(\mathbb{R}^n)$, where η_ε is a sequence of standard mollifiers. Show that $\frac{\partial}{\partial x_j}(\eta_\varepsilon * u) = 0$ on $\Omega_\varepsilon \subset\subset \Omega$ where $\Omega_\varepsilon \nearrow \Omega$ as $\varepsilon \to 0$.)

More generally, if $\frac{\partial u}{\partial x_j} - f_j \in C(\Omega)$, $1 \leq j \leq n$, show that u is equal a.e. to a funtion in $\mathscr{C}^1(\Omega)$.

Proof. We want to show that $\frac{\partial}{\partial x_i} u^\varepsilon = 0$ for any $i = 1,2,\ldots,n$. This would imply $u^\varepsilon = C_\varepsilon$, for C_ε some constant. By Theorem 1.40 in Shkoller's Notes, $u^\varepsilon \to u$ pointwise almost everywhere, and thus would imply $C_\varepsilon \to C \equiv u$.

$$\begin{aligned}
\frac{\partial}{\partial x_i}(u^\varepsilon) &= \frac{\partial}{\partial x_i}\int_{\Omega_\varepsilon} \eta_\varepsilon(x)u(x-y)\mathrm{d}x \\
&= \int_{\Omega_\varepsilon} \frac{\partial}{\partial x_i}\left[\eta_\varepsilon(x)u(x-y)\right]\mathrm{d}x.
\end{aligned}$$

We can interchange the integral and the derivative since the hypotheses of Theorem 1.39 in Shkoller's Notes holds. In particular,

$$\frac{\partial}{\partial x_i}\left[\eta_\varepsilon(x)u(x-y)\right] = \left[\frac{\partial}{\partial x_i}\eta_\varepsilon(x)\right]u(x-y) + \cancelto{0}{\eta_\varepsilon(x)\left[\frac{\partial}{\partial x_i}u(x-y)\right]}$$

since all first partial derivatives of u are assumed to be 0 on Ω. Thus,

$$\frac{\partial}{\partial x_i}\left[\eta_\varepsilon(x)u(x-y)\right] = \left[\frac{\partial}{\partial x_i}\eta_\varepsilon(x)\right]u(x-y) \leq Mu(x-y)$$

where M is the maximum of the i^{th} derivative of η_ε, which is attained since η_ε is continuous on a compact set. Since $u \in W^{1,p}(\mathbb{R}^n)$, $Mu(x-y)$ is integrable, and thus Theorem 1.39 holds. Then

$$\begin{aligned}
\frac{\partial}{\partial x_i}(u^\varepsilon) &= \int_{\Omega_\varepsilon}\left[\frac{\partial}{\partial x_i}\eta_\varepsilon(x)\right]u(x-y)\mathrm{d}x \\
&= \int_{\Omega_\varepsilon}\left[\frac{\partial}{\partial x_i}\eta_\varepsilon(x-y)\right]u(y)\mathrm{d}y \\
&= -\int_{\Omega_\varepsilon}\left[\frac{\partial}{\partial y_i}\eta_\varepsilon(x-y)\right]u(y)\mathrm{d}y \qquad \text{by a suitable change of variables} \\
&= -\int_{\Omega_\varepsilon}\eta_\varepsilon(x-y)\frac{\partial}{\partial y_i}u(y)\mathrm{d}y \qquad \text{by the definition of weak derivative of } u \in W^{1,p}
\end{aligned}$$

$$= 0 \qquad \text{by assumption of all first partial derivatives}$$

Thus, $u^\varepsilon = C_\varepsilon$ is constant, which shows u is constant. since $u^\varepsilon \to u$ pointwise a.e. $\square$