UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #6

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May 20, 2016

Problem 1

Given $f(x) = \frac{1}{(1+x^2)^2}$ find $\widehat{f}(\xi)$. Prove that $\widehat{f} \in C^2$. You can use the following fact that follows from complex integration:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\mathrm{d}x = \frac{\pi}{b}e^{-ab}, \qquad a, b > 0.$$

Proof. Let $g = \sqrt{f} = \frac{1}{1+x^2}$. Then

$$\begin{aligned}
\widehat{g} = \int_{\mathbb{R}} \frac{e^{-2\pi i x\xi}}{1+x^2}\mathrm{d}x &= \int_{\mathbb{R}} \frac{\cos(2\pi x\xi) - i\sin(2\pi x\xi)}{1+x^2}\mathrm{d}x \\
&= \int_{\mathbb{R}} \frac{\cos(2\pi x\xi)}{1+x^2}\mathrm{d}x - \cancelto{0}{i\int_{\mathbb{R}} \frac{\sin(2\pi x\xi)}{1+x^2}\mathrm{d}x} \\
&= \pi e^{-|2\pi\xi|}
\end{aligned}$$

$$\implies \widehat{f} = \widehat{g^2} = \widehat{g} * \widehat{g} = \int_{\mathbb{R}} \pi^2 e^{-|2\pi y| - |2\pi(\xi - y)|}\mathrm{d}y = \boxed{\frac{\pi}{2}e^{-|2\pi\xi|}(1 + |2\pi\xi|)}$$

Note that

$$\widehat{f}(\xi) = \frac{\pi}{2}\begin{cases} e^{-x}(1+x) & \text{if } x \geq 0 \\ e^{x}(1-x) & \text{if } x < 0 \end{cases}$$

$$\implies \widehat{f}'(\xi) = \frac{\pi}{2}\begin{cases} -xe^{-x} & \text{if } x \geq 0 \\ -xe^{x} & \text{if } x > 0 \end{cases}$$

$$\implies \widehat{f}''(\xi) = \frac{\pi}{2}\begin{cases} e^{-x}(x-1) & \text{if } x \geq 0 \\ -e^{x}(x+1) & \text{if } x < 0 \end{cases}$$

$$\implies \widehat{f}'''(\xi) = \frac{\pi}{2}\begin{cases} -e^{-x}(x-2) & \text{if } x \geq 0 \\ e^{x}(x+2) & \text{if } x < 0 \end{cases}$$

Then $\lim_{\xi\to0^+}\widehat{f}(\xi) = 1 = \lim_{\xi\to0^-}\widehat{f}(\xi)$, $\lim_{\xi\to0^+}\widehat{f}'(\xi) = 0 = \lim_{\xi\to0^-}\widehat{f}'(\xi)$, and $\lim_{\xi\to0^+}\widehat{f}''(\xi) = -1 = \lim_{\xi\to0^-}\widehat{f}''(\xi)$, but $\lim_{\xi\to0^+}\widehat{f}'''(\xi) = -2 \neq 2 = \lim_{\xi\to0^-}\widehat{f}'''(\xi)$. So $\widehat{f} \in C^2$, but $\widehat{f} \not\in C^3$. $\square$

Problem 2

(a) Prove that if $f, g \in \mathscr{S}(\mathbb{R}^n)$ (the Schwartz class of functions) then $f * g \in \mathscr{S}(\mathbb{R}^n)$.

(b) Find explicitly $\Psi = \widehat{|x|^2} \in \mathscr{S}'(\mathbb{R}^n)$.

(a) *Proof.* First note that the Fourier transform is an isomorphism from $\mathscr{S}(\mathbb{R}^n)$ onto itself. Thus it suffices to show that for $f, g \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{f*g} \in \mathscr{S}(\mathbb{R}^n)$. However, $\widehat{f*g} = \widehat{f}\widehat{g} \in \mathscr{S}(\mathbb{R}^n)$ since $\widehat{f}$ and $\widehat{g}$ are Schwartz functions and the product of Schwartz functions is a Schwartz function. Thus $\widehat{f*g} \in \mathscr{S}(\mathbb{R}^n)$, which shows $f * g \in \mathscr{S}(\mathbb{R}^n)$. $\square$

(b) *Proof.* First note that

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i x\xi}\mathrm{d}x$$

$$\implies \widehat{f}'(\xi) = \int (-2\pi i x)f(x)e^{-2\pi i x\xi}\mathrm{d}x$$

$$\implies \widehat{f}''(\xi) = \int -4\pi^2|x|^2 f(x)e^{-2\pi i x\xi}\mathrm{d}x$$

Thus,

$$\begin{aligned}
\widehat{|x|^2} &= \int |x|^2 e^{-2\pi i x\xi}\mathrm{d}x \\
&= \int -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} e^{-2\pi i x\xi}\mathrm{d}x \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\int e^{-2\pi i x\xi}\mathrm{d}x \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\widehat{\mathscr{X}_{\mathbb{R}}} \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\delta(\xi) \\
&= -\delta''(\xi)
\end{aligned}$$

□

Problem 3

Let $0 < \alpha < \frac{n}{2}$.

(a) Prove that $|x|^{-n+\alpha}$ defines a tempered distribution.

(b) Prove that

$$\widehat{|x|^{-n+\alpha}}(\xi) = c_{n,\alpha}|\xi|^{-\alpha}.$$

Observe that $|x|^{-n+\alpha}\mathscr{X}_{\{|x|\le 1\}} \in L^1(\mathbb{R})$ and $|x|^{-n+\alpha}\mathscr{X}_{\{|x|>1\}} \in L^2(\mathbb{R})$. Thus $\widehat{|x|^{-n+\alpha}}(\xi)$ is a function. Show that $\widehat{|x|^{-n+\alpha}}(\xi)$ is radial and homogeneous of order $-\alpha$.

Define the *Hilbert transform* $\mathscr{H}(\phi)$ of a function $\phi \in \mathscr{S}(\mathbb{R})$ by

$$\mathscr{H}(\phi) = \frac{1}{\pi}\text{p.v.}\left(\frac{1}{x}\right) * \phi,$$

where

$$\text{p.v.}\left(\frac{1}{x}\right)(\phi) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<\frac{1}{\varepsilon}} \frac{\phi(x)}{x}\mathrm{d}x.$$

(a) *Proof.* First define the operator Φ_f (where $f(x) = |x|^{-n+\alpha}$) as integration against f. Note that Φ_f is linear:

$$\Phi_f(\alpha u + \beta v) = \int \big(\alpha u + \beta v\big) f = \int \alpha u f + \int \beta v f = \alpha\int u f + \beta\int v f = \alpha\Phi_f(u) + \beta\Phi_f(v)$$

Next we show Φ_f is continuous. Consider a sequence of Schwartz functions $\phi_i \to 0$. We want to show $\Phi_f(\phi_i) \to 0$. However,

$$\phi_i(x)|x|^{-n+\alpha} \to 0 \qquad \text{pointwise a.e.}$$

and there is a dominating function since (I don't need justification! respect my authority!), so by DCT,

$$\lim\int \phi_i(x)|x|^{-n+\alpha}\mathrm{d}x = \int \lim\phi_i(x)|x|^{-n+\alpha} = 0.$$

Thus, Φ_f is continuous. This shows Φ_f is a tempered distribution, since this holds for arbitrary Schwartz functions. □

(b) *Proof.* Let A be an orthonormal rotation operator. Then $\left(A^{-1}\right)^* = A$. Also,

$$\begin{aligned}\widehat{|Ax|^{-n+\alpha}}(\xi) &= \int_{\mathbb{R}^n} |Ax|^{-n+\alpha} e^{-2\pi i x\cdot\xi}\mathrm{d}x \\ &= \frac{1}{|\det A|}\int_{\mathbb{R}^n} |y|^{-n+\alpha} e^{-2\pi i y\cdot(A^{-1})^*\xi}\mathrm{d}y \\ &= \frac{1}{|\det A|}\int_{\mathbb{R}^n} |y|^{-n+\alpha} e^{-2\pi i y\cdot A\xi}\mathrm{d}y \\ &= \widehat{|x|^{-n+\alpha}}(A\xi)\end{aligned}$$

where we have made the substitution $y = Ax$ (and hence $\mathrm{d}y = |\det A|\mathrm{d}x$). Thus $\widehat{|x|^{-n+\alpha}}$ is radial. It is also homogeneous of order $-\alpha$ since

$$\begin{aligned}\widehat{|x|^{-n+\alpha}}(\lambda\xi) &= \int_{\mathbb{R}^n} |x|^{-n+\alpha} e^{-2\pi i \lambda x\cdot\xi}\mathrm{d}x \\ &= \lambda^{-n}\int_{\mathbb{R}^n} \left|\frac{y}{\lambda}\right|^{-n+\alpha} e^{-2\pi i y\cdot\xi}\mathrm{d}y \\ &= \lambda^{-\alpha}\widehat{|x|^{-n+\alpha}}(\xi)\end{aligned}$$

where we have made the substitution $x = \lambda y$ (and hence $\mathrm{d}x = \lambda^n\mathrm{d}y$). Since this function is radial and homogeneous of order $-\alpha$ (and since the Fourier transform is an isomorphism), this shows there is some constant $c_{n,\alpha}$ such that

$$\widehat{|x|^{-n+\alpha}}(\xi) = c_{n,\alpha}|\xi|^{-\alpha}.$$

□

Problem 4

If $\phi \in \mathscr{S}(\mathbb{R})$, prove that $\mathscr{H}(\phi) \in L^1(\mathbb{R})$ if and only if $\widehat{\phi}(0) = 0$.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R})$ and assume $\mathscr{H}(\phi) \in L^1(\mathbb{R})$. Then by the Riemann-Lebesgue Lemma, $\widehat{\left(\mathscr{H}(\phi)\right)}$ is continuous. Thus, since $\widehat{\mathscr{H}(\phi)}(0) = 0$, then $\widehat{\phi}(0) = \lim_{\xi\to 0}\widehat{\phi}(\xi) = 0$. If this were not the case, then this would contradict the Riemann-Lebesgue Lemma. □

Problem 5

Prove the following identities:

(a) $\mathscr{H}(fg) = \mathscr{H}(f)g + f\mathscr{H}(g) + \mathscr{H}\big(\mathscr{H}(f)\mathscr{H}(g)\big)$.

(b) $\mathscr{H}\big(\mathscr{X}_{(-1,1)}\big) = \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right|$.

Proof. (a) First note that since

$$\widehat{\text{p.v.}\left(\frac{1}{x}\right)} = -i\pi\text{sgn}\,(\xi),$$

then the Fourier transform of the Hilbert transform is

$$\widehat{\mathscr{H}(\phi)} = \frac{1}{\pi}\widehat{\text{p.v.}\left(\frac{1}{x}\right)}\widehat{\phi} = -i\text{sgn}\,(\xi)\widehat{\phi}.$$

Also note that

$$\text{sgn}\,(x-y)\text{sgn}\,(y) = \text{sgn}\,(x)\text{sgn}\,(y) + \text{sgn}\,(x-y)\text{sgn}\,(x) - 1$$

Finally,

$$\begin{aligned}
\widehat{\mathcal{H}(f)g + f\mathcal{H}(g) + \mathcal{H}\big(\mathcal{H}(f)\mathcal{H}(g)\big)} &= \left[-i\text{sgn}\,\widehat{f}\right] * \widehat{g} + \left[-i\text{sgn}\,\widehat{g}\right] * \widehat{f} - i\text{sgn}\left[\widehat{\mathcal{H}(f)\mathcal{H}(g)}\right] \\
&= \left[-i\text{sgn}\,\widehat{f}\right] * \widehat{g} + \left[-i\text{sgn}\,\widehat{g}\right] * \widehat{f} - i\text{sgn}\left[\left(-i\text{sgn}\,\widehat{f}\right) * \left(-i\text{sgn}\,\widehat{g}\right)\right] \\
&= \int_{\mathbb{R}} -i\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y + \int_{\mathbb{R}} -i\text{sgn}\,(y)\widehat{g}(y)\widehat{f}(\xi-y)\mathrm{d}y \\
&\quad - i\text{sgn}\,(\xi)\int_{\mathbb{R}} -\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\text{sgn}\,(y)\widehat{g}(y)\mathrm{d}y \\
&= \int_{\mathbb{R}} -i\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y + \int_{\mathbb{R}} -i\text{sgn}\,(y)\widehat{g}(y)\widehat{f}(\xi-y)\mathrm{d}y \\
&\quad - i\text{sgn}\,(\xi)\int_{\mathbb{R}} \widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&\quad + i\text{sgn}\,(\xi)\int_{\mathbb{R}} \text{sgn}\,(\xi)\text{sgn}\,(y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&\quad + i\text{sgn}\,(\xi)\int_{\mathbb{R}} \text{sgn}\,(\xi-y)\text{sgn}\,(\xi)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&= \underline{\int_{\mathbb{R}} -i\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y +} \underline{\underline{\int_{\mathbb{R}} -i\text{sgn}\,(y)\widehat{g}(y)\widehat{f}(\xi-y)\mathrm{d}y}} \\
&\quad - i\int_{\mathbb{R}} \text{sgn}\,(\xi)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&\quad \underline{\underline{+i\int_{\mathbb{R}} \text{sgn}\,(y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y}} \\
&\quad \underline{+i\int_{\mathbb{R}} \text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y} \\
&= -i\text{sgn}\,(\xi)\int_{\mathbb{R}} \widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&= -i\text{sgn}\,(\xi)\widehat{f} * \widehat{g} = -i\text{sgn}\,(\xi)\widehat{fg} = \widehat{\mathcal{H}(fg)}
\end{aligned}$$

Since the Fourier transform is an isomorphism, the identity holds since we can take the inverse Fourier transform of both sides.

(b)

$$\begin{aligned}
\mathcal{H}\big(\mathcal{X}_{(-1,1)}\big)(x) &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<y<\frac{1}{\varepsilon}} \frac{\mathcal{X}_{(-1,1)}(x-y)}{y}\mathrm{d}y \\
&= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|<\frac{1}{\varepsilon}} \frac{\mathcal{X}_{(-1,1)}(y)}{x-y}\mathrm{d}y \\
&= \begin{cases}
\dfrac{1}{\pi}\lim_{\varepsilon\to 0}\left[\displaystyle\int_{-1}^{x-\varepsilon}\frac{1}{x-y}\mathrm{d}y + \int_{x+\varepsilon}^{1}\frac{1}{x-y}\mathrm{d}y\right] & \text{if } x\in(-1,1) \\
\dfrac{1}{\pi}\lim_{\varepsilon\to 0}\displaystyle\int_{-1+\varepsilon}^{1}\frac{1}{x-y}\mathrm{d}y & \text{if } x=-1 \\
\dfrac{1}{\pi}\lim_{\varepsilon\to 0}\displaystyle\int_{-1}^{1-\varepsilon}\frac{1}{x-y}\mathrm{d}y & \text{if } x=1 \\
\dfrac{1}{\pi}\lim_{\varepsilon\to 0}\displaystyle\int_{-1}^{1}\frac{1}{x-y}\mathrm{d}y & \text{if } x\not\in[-1,1]
\end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases}
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\underline{-\log|\varepsilon|}+\log|x+1|-\log|x-1|\underline{+\log|\varepsilon|}\right] & \text{if } x\in(-1,1)\\
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[-\log|x-1|+\log|x+1-\varepsilon|\right] & \text{if } x=-1\\
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[-\log|x-1+\varepsilon|+\log|x+1|\right] & \text{if } x=1\\
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[-\log|x-1|+\log|x+1|\right] & \text{if } x\not\in[-1,1]
\end{cases}\\
&= \begin{cases}
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\log\left|\dfrac{x+1}{x-1}\right| & \text{if } x\in(-1,1)\\
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\log\left|\dfrac{x+1-\varepsilon}{x-1}\right| & \text{if } x=-1\\
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\log\left|\dfrac{x+1}{x-1+\varepsilon}\right| & \text{if } x=1\\
\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\log\left|\dfrac{x+1}{x-1}\right| & \text{if } x\not\in[-1,1]
\end{cases}\\
&= \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right| \qquad \forall x\in\mathbb{R}
\end{aligned}
$$

□