201C, Spring '16, Thomases
Homework 6 due 5/17/16

1. Given $f(x) = \frac{1}{(1+x^2)^2}$ find $\hat{f}(\xi)$. Prove that $\hat{f} \in C^2$. You can use the following fact that follows from complex integration

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}dx = \frac{\pi}{b}e^{-ab}, \quad a, b > 0.$$

2. (a) Prove that if $f, g \in \mathscr{S}(\mathbb{R}^n)$ (the Schwartz class of functions) then $f * g \in \mathscr{S}(\mathbb{R}^n)$.
   (b) Find explicitly $\Psi = \widehat{|x|^2} \in \mathscr{S}'(\mathbb{R}^n)$.

3. Let $0 < \alpha < n/2$.
   (a) Prove that $|x|^{-n+\alpha}$ defines a tempered distribution.
   (b) Prove that
   $$\widehat{|x|^{-n+\alpha}}(\xi) = c_{n,\alpha}|\xi|^{-\alpha}.$$
   Observe that $|x|^{-n+\alpha}\chi_{\{|x|\le 1\}} \in L^1(\mathbb{R})$ and $|x|^{-n+\alpha}\chi_{\{|x|>1\}} \in L^2(\mathbb{R})$. Thus $\widehat{|x|^{-n+\alpha}}(\xi)$ is a function. Show that $\widehat{|x|^{-n+\alpha}}(\xi)$ is radial and homogeneous of order $-\alpha$.

Define the *Hilbert transform* $\mathcal{H}(\varphi)$ of a function $\varphi \in \mathscr{S}(\mathbb{R})$ by

$$\mathcal{H}(\varphi) = \frac{1}{\pi}p.v.\frac{1}{x} * \varphi,$$

where

$$p.v.\frac{1}{x}(\varphi) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<\frac{1}{\varepsilon}} \frac{\varphi(x)}{x}dx.$$

4. If $\varphi \in \mathscr{S}(\mathbb{R})$, prove that $\mathcal{H}(\varphi) \in L^1(\mathbb{R})$ if and only if $\hat{\varphi}(0) = 0$.

5. Prove the following identities:
   (a) $\mathcal{H}(fg) = \mathcal{H}(f)g + f\mathcal{H}(g) + \mathcal{H}(\mathcal{H}(f)\mathcal{H}(g))$.
   (b) $\mathcal{H}(\chi_{(-1,1)}) = \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right|$.