UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #3

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Problem 1

If $u \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, and $u^\varepsilon = \eta_\varepsilon * u$, for η_ε the standard mollifier. Show that

$$u^\varepsilon \to u$$

in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Proof.

$$\begin{aligned}
\|\eta_\varepsilon * u - u\|_p^p &= \int_{\mathbb{R}^n} |\eta_\varepsilon * u - u|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_E(y) u(x-y) \mathrm{d}y - u(x) \right|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_\varepsilon(y)(u(x-y) - u(x)) \mathrm{d}y \right|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left| \int \eta_\varepsilon(x-y)(u(y) - u(x)) \right|^p \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} \frac{C}{\varepsilon^n} \left| \int |u(y) - u(x)| \mathrm{d}y \right|^p \mathrm{d}x
\end{aligned}$$

$\tilde{u} \in C_C(\mathbb{R}^n) \implies \tilde{u}$ is uniformly continuous.

$$\|u - \tilde{u}\|_p \leq \frac{\tilde{\varepsilon}}{\varepsilon}$$

$$\implies \|\eta_\varepsilon * u - u\|_p \leq \|\eta_\varepsilon * u - \eta_\varepsilon * \tilde{u}\|_p + \|\eta_\varepsilon * \tilde{u} - \tilde{u}\|_p + \|\tilde{u} - u\|_p$$

□

Problem 2

Let Ω denote an open and smooth subset of $\mathbb{R}^n$. Prove that $\mathscr{C}_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

Proof. Ω open $\implies$ smooth Urysohn's Lemma: Ω open $\subset \mathbb{R}^n$, and C_0, $C_1 \subset \Omega$ disjoint nonempty, then $\exists f : \Omega \to [0,1]$, smooth, $f(C_0) = \{0\}$, $f(C_1) = \{1\}$. Let $\varepsilon > 0$. Pick $A \subset \Omega$. By inner and outer regularity of Lebesgue measure, there is a compact subset K of Ω and $\omega \subset \Omega$ such that $K \subset A \subset \omega$ with $\mu(\omega \setminus A) < \varepsilon$, $\mu(A \setminus K) < \varepsilon$.
$\Omega \subset \mathbb{R}^n$ implies Ω is locally compact and Hausdorff, which implies $\exists$ precompact $O, U \subset \Omega$ such that $K \subset O \subset \overline{O} \subset U \subset \overline{U} \subset W$. Apply smooth Urysohn's Lemma to $K = C_1$ and $\overline{U} \setminus O = C_0$. $f_k : \Omega \to [0,1]$, $f(K) = \{1\}$, $f(\Omega \setminus W) = \{0\}$.

$$\int_\Omega |\mathscr{X}_A - f_k|^p \mathrm{d}\mu = \int_{A \setminus K} |\mathscr{X}_A - f_k|^p \mathrm{d}\mu + \int_{W \setminus A} |\mathscr{X}_A - f_k|^p \mathrm{d}\mu \leq M2\varepsilon$$

which implies $C_C^\infty(\Omega)$ dense in ISF (Integral Simple Functions) dense in $L^p(\Omega)$.

The integral is split by $\Omega = (\Omega \setminus W) \cup (W \setminus A) \cup (A \setminus K) \cup K$. But integral over $\Omega \setminus W$ and over K are 0 for various reasons.. □

Problem 3

Prove that if $u \in L^1_{\text{loc}}(\Omega)$ satisfies $\int_\Omega u(x)v(x)\mathrm{d}x = 0$ for all $v \in \mathscr{C}_c^\infty(\Omega)$, then $u = 0$ a.e. in Ω.

Proof. Suppose $u \not\equiv 0$. Then $\exists E \subset \Omega$ with $\mu(E) > 0$ and $u(x) \neq 0$ for all $x \in E$. Let $K \subset E$ be compact and set $v = \mathscr{X}_K \operatorname{sgn}(u)$. Then

$$\int_\Omega u(x)v(x)\mathrm{d}x = \int_K |u(x)|\mathrm{d}x > 0$$

This is a contradiction.
If $f \in L^p_{\text{loc}}$ and η_ε is the standard mollifier, then $\eta_\varepsilon * f \to f$ pointwise a.e.

$$\int \eta_\varepsilon(x-y)\mu(y)\mathrm{d}y = 0 \forall \varepsilon > 0$$

$\Omega_\varepsilon = \left\{x \in \Omega \,:\, d\left(x, \Omega^C\right) \geq \varepsilon\right\}$. □

Problem 4

Let $u \in L^\infty(\mathbb{R}^n)$ and let η_ε be a standard mollifier. For $\varepsilon > 0$ consider the sequence $\psi_\varepsilon \in L^\infty(\mathbb{R}^n)$ such that

$$\left\|\psi_\varepsilon\right\|_\infty \leq 1 \ \forall \varepsilon > 0 \ \text{ and } \ \psi_\varepsilon \to \psi \text{ a.e. in } \mathbb{R}^n,$$

define

$$v^\varepsilon = \eta_\varepsilon * (\psi_\varepsilon u) \ \text{ and } \ v = \psi u.$$

(a) Prove that $v^\varepsilon \overset{*}{\rightharpoonup} v$ in $L^\infty(\mathbb{R}^n)$.

(b) Prove that $v^\varepsilon \to v$ in $L^1(B)$ for every ball $B \subset \mathbb{R}^n$.

Proof. (a) We want to show $\phi_{v^\varepsilon}(f) \to \phi_v(f)$ for all $f \in L^1(\mathbb{R})$, where ϕ_v and ϕ_{v^ε} are the continuous linear functionals associated with v and v^ε, respectively.

□

Problem 5

For $u \in \mathscr{C}^0(\mathbb{R}^n;\mathbb{R})$, $\operatorname{spt}(u)$ is the closure of the set $\left\{x \in \mathbb{R}^n \,:\, u(x) \neq 0\right\}$, but this definition may not make sense for functions $u \in L^p(\Omega)$. For example what is the support of $\mathscr{X}_\mathbb{Q}$, the indicator over the rationals?

Let $u : \mathbb{R}^n \to \mathbb{R}$, and let $\{\Omega_\alpha\}_{\alpha \in A}$ denote the collection of all open sets on $\mathbb{R}^n$ such that for each $\alpha \in A$, $u = 0$ a.e. on Ω_α. Define $\Omega = \bigcup_{\alpha \in A} \Omega_\alpha$. Prove that $u = 0$ a.e. on Ω.

The support of u, $\operatorname{spt}(u)$, is Ω^C, the complement of Ω. Notice that if $v = w$ a.e. on $\mathbb{R}^n$, then $\operatorname{spt}(v) = \operatorname{spt}(w)$; furthermore, if $u \in \mathscr{C}^0(\mathbb{R}^n)$, then $\Omega^C = \overline{\{x \in \mathbb{R}^n \,:\, u(x) \neq 0\}}$. (Hint: Since A is not necessarily countable, it is not clear that $f = 0$ a.e. on Ω, so find a countable family U_n of open sets in $\mathbb{R}^n$ such that every open set on $\mathbb{R}^n$ is the union of some of the sets from $\{U_n\}$.)

Proof. Since $\mathscr{X}_\mathbb{Q}$ is nonzero on $\mathbb{R} \setminus \mathbb{Q}$, which is a dense subset of $\mathbb{R}$, then $\operatorname{spt}\left(\mathscr{X}_\mathbb{Q}\right) = \mathbb{R}$. This is nonsence, however, since $\mathscr{X}_\mathbb{Q}$ is equivalent to 0 in $L^p(\mathbb{R})$. □

Problem 6

Prove that if $u \in L^1(\mathbb{R}^n)$ and $v \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, then

$$\operatorname{spt}(u * v) \subset \overline{\operatorname{spt}(u) + \operatorname{spt}(v)}.$$

Proof. Suppose $x \notin \overline{\text{spt}(u) + \text{spt}(v)}$ and define the set $\left[x - \text{spt}(u)\right]$ as the shift of the support of u by the vector x:

$$\left[x - \text{spt}(u)\right] = \left\{y \,:\, x - y \in \text{spt}(u)\right\}$$

Then

$$(u * v)(x) = \int_{\mathbb{R}^n} u(x-y)v(y)\mathrm{d}y = \int_{[x - \text{spt}(u)] \cap \text{spt}(v)} u(x-y)v(y)\mathrm{d}y$$

If $x_0 \in \text{spt}(v) \cap \left[x - \text{spt}(u)\right]$, then $x_0 \in \text{spt}(v)$ and $x - x_0 = 0 \in \text{spt}(u)$. Then since $x = (x - x_0) + (x_0)$, then $x \in \text{spt}(u) + \text{spt}(v)$, which is a contradiction since $x \notin \overline{\text{spt}(u) + \text{spt}(v)}$. Thus $\left[x - \text{spt}(u)\right] \cap \text{spt}(v) = \emptyset$, and therefore

$$(u * v)(x) = \int_{[x - \text{spt}(u)] \cap \text{spt}(v)} u(x-y)v(y)\mathrm{d}y = \int_{\emptyset} u(x-y)v(y)\mathrm{d}y = 0$$

and thus $x \notin \text{spt}(u * v)$. This shows

$$\text{spt}(u * v) \subset \overline{\text{spt}(u) + \text{spt}(v)}.$$

$\square$

Problem 7

Suppose that $1 < p < \infty$. If $\tau_y f(x) = f(x-y)$, show that f belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $\tau_y f$ is a Lipschitz function of y with values in $L^p(\mathbb{R}^n)$; that is,

$$\left\|\tau_y f - \tau_z f\right\|_p \leq C\left|y - z\right|.$$

What happens in the case $p = 1$?

Proof. $\square$

Problem 8

If $u \in W^{1,p}(\mathbb{R}^n)$ for some $p \in [1, \infty)$ and $\frac{\partial u}{\partial x_j} = 0$, $j = 1, \dots, n$, on a connected open set $\Omega \subset \mathbb{R}^n$, show that u is equal a.e. to a constant on Ω. (Hint: approximate u using that $\eta_\varepsilon * u \to u$ in $W^{1,p}(\mathbb{R}^n)$, where η_ε is a sequence of standard mollifiers. Show that $\frac{\partial}{\partial x_j}(\eta_\varepsilon * u) = 0$ on $\Omega_\varepsilon \subset\subset \Omega$ where $\Omega_\varepsilon \nearrow \Omega$ as $\varepsilon \to 0$.)

More generally, if $\frac{\partial u}{\partial x_j} - f_j \in C(\Omega)$, $1 \leq j \leq n$, show that u is equal a.e. to a funtion in $\mathscr{C}^1(\Omega)$.

Proof. $\square$