
Homework #2

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Problem 1

A function $f \in L^p(\mathbb{R}^n)$ is said to be L^p -continuous if $\tau_h f \rightarrow f$ in $L^p(\mathbb{R}^n)$ as $h \rightarrow 0$ in \mathbb{R}^n , where $\tau_h f(x) = f(x-h)$ is the translation of f by h . Prove that, if $1 \leq p < \infty$, every $f \in L^p(\mathbb{R}^n)$ is L^p -continuous. Give a counter-example to show that this result is not true when $p = \infty$. [Hint: Approximate an L^p function by a C_c function.]

Proof. We begin with the counter-example for L^∞ . Define $f \in L^\infty(\mathbb{R})$ as $f(x) = \mathcal{X}_{[0,1]}$ where \mathcal{X} is the characteristic function. Note that $f(1-\varepsilon) = 1$ for all $\varepsilon > 0$. Let h be a small perturbation, i.e. $0 < |h| \ll 1$, and choose $\varepsilon = \frac{h}{2}$. Then $\forall x \in (0, \varepsilon)$, $\tau_h f(x) = 0$ but $f(x) = 1$, and thus $|\tau_h f(x) - f(x)| = 1$. This shows that $\forall h > 0$, \exists an interval I_h (of positive measure, $\mu(I_h) > 0$) such that $|\tau_h f(x) - f(x)| = 1$ for all $x \in I_h$. Thus $\tau_h f \not\rightarrow f$ in $L^\infty(\mathbb{R})$.

Now let $p \in [1, \infty)$ and let $f \in L^p(\mathbb{R}^n)$. Then by the density of simple functions in L^p , there exist simple functions $f_\ell \in L^p$ such that $\|f_\ell - f\|_p \rightarrow 0$. By the definition of simple functions, there are disjoint, finite-measure sets $E_{\ell,k}$, where $k = 1, \dots, N_\ell$ and constants $a_{\ell,k}$ such that

$$f_\ell = \sum_{k=1}^{N_\ell} a_{\ell,k} \mathcal{X}_{E_{\ell,k}}$$

Next we show that for each $h > 0$, $\tau_h f_\ell \rightarrow f_\ell$ in L^p .

$$\|\tau_h f_\ell - f_\ell\|_p^p = \int_{\mathbb{R}^n} |f_\ell(x+h) - f_\ell(x)|^p dx = \int_{\mathbb{R}^n} |f_\ell(x) - f_\ell(x)|^p dx = \|f_\ell - f_\ell\|_p^p \rightarrow 0$$

by a change of variables. Next we show that for each n , $\tau_h f_\ell \rightarrow f_\ell$ as $h \rightarrow 0$. For each $E_{\ell,k}$, define the set $E_{\ell,k,h}$ as the shifted set $E_{\ell,k}$ by h :

$$E_{\ell,k,h} = \{x+h : x \in E_{\ell,k}\}.$$

Then

$$\begin{aligned} \|\tau_h f_\ell - f_\ell\|_p^p &= \int_{\mathbb{R}^n} \left| \sum_{k=1}^{N_\ell} a_{\ell,k} \mathcal{X}_{E_{\ell,k,h}} - \sum_{k=1}^{N_\ell} a_{\ell,k} \mathcal{X}_{E_{\ell,k}} \right|^p dx \\ &= \int_{\mathbb{R}^n} \left| \sum_{k=1}^{N_\ell} a_{\ell,k} (\mathcal{X}_{E_{\ell,k,h} \setminus E_{\ell,k}} - \mathcal{X}_{E_{\ell,k} \setminus E_{\ell,k,h}}) \right|^p dx \\ &= \int_{\mathbb{R}^n} G_{\ell,h} dx \end{aligned}$$

where

$$G_{\ell,h} = \left| \sum_{k=1}^{N_\ell} a_{\ell,k} (\mathcal{X}_{E_{\ell,k,h} \setminus E_{\ell,k}} - \mathcal{X}_{E_{\ell,k} \setminus E_{\ell,k,h}}) \right|^p.$$

Then since $\mu(E_{\ell,k,h} \setminus E_{\ell,k}) \rightarrow 0$ and $\mu(E_{\ell,k} \setminus E_{\ell,k,h}) \rightarrow 0$ as $h \rightarrow 0$, then $G_{\ell,h} \rightarrow 0$ pointwise as $h \rightarrow 0$. Also,

$$|G_{\ell,h}(x)| \leq \left| \sum_{k=1}^{N_\ell} a_{\ell,k} 2 \mathcal{X}_{E_{\ell,k}}(x) \right|^p = 2^p \left| \sum_{k=1}^{N_\ell} a_{\ell,k} \mathcal{X}_{E_{\ell,k}}(x) \right|^p = 2^p |f_\ell(x)|^p$$

That is, for a given n , $G_{\ell,k}$ is uniformly bounded by $2^p |f_\ell(x)|^p$. Then by the Dominated Convergence Theorem, the integral of $G_{\ell,k}$ converges to the integral of its pointwise limit, which is 0. Thus, as $h \rightarrow 0$,

$$\|\tau_h f_\ell - f_\ell\|_p^p \rightarrow \int_{\mathbb{R}^n} 0 dx = 0.$$

Thus,

$$\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h f_\ell\|_p + \|\tau_h f_\ell - f_\ell\|_p + \|f_\ell - f\|_p \rightarrow 0$$

since each of the three norms on the right hand side converge to 0. \square

Problem 2

Show that $L^\infty(\mathbb{R})$ is not separable. [Hint: There is an uncountable set $\mathcal{F} \subset L^\infty$ such that $\|f - g\|_\infty \geq 1$ for all $f, g \in \mathcal{F}$ with $f \neq g$.]

Proof. Let $\mathcal{F} = \{\mathcal{X}_{[0, \alpha]} : 0 < \alpha \in \mathbb{R}\}$. \mathcal{F} is clearly uncountable. Consider any two $f, g \in \mathcal{F}$. Then, without loss of generality, $f = \mathcal{X}_{[0, \alpha]}$ and $g = \mathcal{X}_{[0, \beta]}$ where $\alpha < \beta$. Also,

$$\|f - g\|_\infty = \text{ess sup}\{\mathcal{X}_{(\alpha, \beta]}\} = 1$$

Thus the ball around any $f \in \mathcal{F}$ of radius $\frac{1}{2}$, i.e. $B(f, \frac{1}{2})$, contains no other elements of \mathcal{F} . Thus $L^\infty(\mathbb{R})$ is not separable since \mathcal{F} is uncountable and not dense. \square

Problem 3

Prove Chebyshev's Inequality: If $f \in L^p$ ($1 \leq p < \infty$), then for any $\alpha > 0$,

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

[Note that you can find the proof of this simple fact in many texts but you should see if you can figure it out yourself. Also, note that this inequality holds for all $0 < p < \infty$.]

Proof. Let $A_\alpha = \{x : |f(x)| > \alpha\} = \{x : \left|\frac{f(x)}{\alpha}\right| > 1\} = \{x : \left|\frac{f(x)}{\alpha}\right|^p > 1\}$ for all $p \geq 1$. Then

$$\left(\frac{\|f\|_p}{\alpha}\right)^p = \int_\Omega \left|\frac{f(x)}{\alpha}\right|^p d\mu = \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p d\mu + \int_{\Omega \setminus A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p d\mu \geq \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p d\mu \geq \int_{A_\alpha} 1 d\mu = \mu(A_\alpha)$$

which proves the result. \square

Problem 4

Assume that $f, g \in L^1(\mathbb{R}^n)$. Prove that the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy$$

is measurable and in $L^1(\mathbb{R}^n)$.

Proof. First note that $\|f\|_1 \|g\|_1 < \infty$ since they are in $L^1(\mathbb{R}^n)$. Then

$$\|f\|_1 \|g\|_1 = \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dy$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n} \|f\|_1 |g(y)| dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)| dx |g(y)| dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)| |g(y)| dx dy \\
&\Rightarrow \int_{\mathbb{R}^{2n}} |f(x-y)g(y)| dx dy < \infty
\end{aligned}$$

Thus, by Fubini's Theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| dy dx < \infty \\
\Rightarrow \|f * g\|_1 &= \int_{\mathbb{R}^n} |(f * g)(x)| dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y) dy \right| dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| dy dx < \infty
\end{aligned}$$

which shows $(f * g) \in L^1(\mathbb{R}^n)$. Since all L^1 functions are measurable, $(f * g)$ is measurable. \square

Problem 5

Let $f_n = \sqrt{n}\mathcal{X}_{(0, \frac{1}{n})}$. Prove that f_n converges weakly to 0 in $L^2(0, 1)$ and $f_n \rightarrow 0$ in $L^1(0, 1)$ but f_n does not converge strongly in $L^2(0, 1)$.

Proof.

$$\|f_n\|_2^2 = \int_0^1 n\mathcal{X}_{(0, \frac{1}{n})} dx = \int_0^{\frac{1}{n}} n dx = 1$$

Thus $\|f_n\|_2 = 1$ for all n , and thus does not converge strongly to 0 in $L^2(0, 1)$.

$$\|f_n\|_1 = \int_0^1 \sqrt{n}\mathcal{X}_{(0, \frac{1}{n})} dx = \int_0^{\frac{1}{n}} \sqrt{n} dx = \frac{1}{\sqrt{n}}$$

Thus $\|f_n\|_1 \rightarrow 0$ as $n \rightarrow \infty$, which shows $f_n \rightarrow 0$ strongly in $L^1(0, 1)$. Let $L \in L^2(0, 1)^* \cong L^2(0, 1)$. Thus $\exists \ell \in L^2(0, 1)$ such that

$$L(f) = \int_0^1 \ell(x)f(x) dx$$

for all $f \in L^2$. Then

$$L(f_n) = \int_0^1 \ell(x)\sqrt{n}\mathcal{X}_{(0, \frac{1}{n})} dx = \int_0^{\frac{1}{n}} \ell(x)\sqrt{n} dx \leq \left(\int_0^{\frac{1}{n}} |\ell(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_0^{\frac{1}{n}} n dx \right)^{\frac{1}{2}} = \left(\int_0^{\frac{1}{n}} |\ell(x)|^2 dx \right)^{\frac{1}{2}} \rightarrow 0$$

since ℓ is fixed and $\mu((0, \frac{1}{n})) \rightarrow 0$ as $n \rightarrow \infty$. Thus $f_n \rightarrow 0$ in $L^2(0, 1)$. \square

Problem 6

Find a sequence of functions with the property that f_j converges to 0 in $L^2(\Omega)$ weakly, to 0 in $L^{\frac{3}{2}}(\Omega)$ strongly, but it does not converge to 0 strongly in $L^2(\Omega)$.

Proof. Let $f_n = \sqrt{n}\mathcal{X}_{(0, \frac{1}{n})}$. Then by number 5, $f_n \not\rightarrow 0$ in $L^2(0, 1)$ but $f_n \rightarrow 0$ in $L^2(0, 1)$. Also,

$$\|f_n\|_{\frac{3}{2}}^{\frac{3}{2}} = \int_0^1 \left| n^{\frac{1}{2}} \mathcal{X}_{(0, \frac{1}{n})} \right|^{\frac{3}{2}} dx = \int_0^{\frac{1}{n}} n^{\frac{3}{4}} dx = n^{-\frac{1}{4}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Thus $f_n \rightarrow 0$ in $L^{\frac{3}{2}}(0, 1)$. □

Problem 7

Let f_n and g_n denote two sequences in $L^p(\Omega)$ with $1 \leq p \leq \infty$ such that $f_n \rightarrow f$ in $L^p(\Omega)$, and $g_n \rightarrow g$ in $L^p(\Omega)$. Set $h_n = \max\{f_n, g_n\}$ and prove that $h_n \rightarrow h$ in $L^p(\Omega)$.

Proof. First note that

$$\begin{aligned} h_n(x) &= \frac{1}{2}(f_n(x) + g_n(x)) + \frac{1}{2}|f_n(x) - g_n(x)|, \quad \text{and} \\ h(x) &= \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|. \end{aligned}$$

Then

$$\begin{aligned} \|h_n - h\|_p &= \left\| \frac{1}{2}(f_n + g_n) + \frac{1}{2}|f_n - g_n| - \left[\frac{1}{2}(f + g) + \frac{1}{2}|f - g| \right] \right\|_p \\ &\leq \frac{1}{2} \left[\|f_n - f\|_p + \|g_n - g\|_p + \left| |f_n - g_n| - |f - g| \right|_p \right] \\ &\leq \frac{1}{2} \left[\|f_n - f\|_p + \|g_n - g\|_p + \|(f_n - g_n) - (f - g)\|_p \right] \\ &\leq \frac{1}{2} \left[\|f_n - f\|_p + \|g_n - g\|_p + \|f_n - f\|_p + \|g_n - g\|_p \right] \\ &= \|f_n - f\|_p + \|g_n - g\|_p \end{aligned}$$

But since $f_n \rightarrow f$ and $g_n \rightarrow g$ in L^p , then there is an N such that $\|f_n - f\|_p < \frac{\varepsilon}{2}$ and $\|g_n - g\|_p < \frac{\varepsilon}{2}$ for all $n \geq N$. Thus if $n \geq N$,

$$\|h_n - h\|_p \leq \|f_n - f\|_p + \|g_n - g\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since ε is arbitrary, $h_n \rightarrow h$ in L^p . □

Problem 8

Let f_n be a sequence in $L^p(\Omega)$ with $1 \leq p < \infty$, and let g_n be a bounded sequence in $L^\infty(\Omega)$. Suppose that $f_n \rightarrow f$ in $L^p(\Omega)$ and that $g_n \rightarrow g$ pointwise a.e. Prove that $f_n g_n \rightarrow f g$ in $L^p(\Omega)$.

Proof.

$$\begin{aligned} \|f_n g_n - f g\|_p &\leq \|f_n g_n - f g_n\|_p + \|f g_n - f g\|_p \\ &= \left[\int_\Omega |g_n|^p |f_n - f|^p \right]^{\frac{1}{p}} + \left[\int_\Omega |f g_n - f g|^p \right]^{\frac{1}{p}} \end{aligned}$$

Since g_n is a bounded sequence in L^∞ , $\exists M$ such that $|g_n(x)| \leq \|g_n\| \leq M$ almost everywhere. Thus

$$\left[\int_{\Omega} |g_n|^p |f_n - f|^p \right]^{\frac{1}{p}} \leq \left[\int_{\Omega} M^p |f_n - f|^p \right]^{\frac{1}{p}} = M \|f_n - f\|_p \rightarrow 0$$

since $f_n \rightarrow f$ in L^p . Next note $g \in L^\infty$ since

$$\|g\|_\infty = \text{ess sup}\{g(x)\} = \text{ess sup}\left\{\lim_n g_n(x)\right\} \leq M$$

Since $g_n \rightarrow g$ pointwise, then $f g_n \rightarrow f g$ pointwise and thus $|f g_n - f g|^p \rightarrow 0$ pointwise. Define h_n as

$$h_n = |f g_n - f g|^p.$$

Then $h_n \in L^1(\Omega)$ since

$$\|h_n\|_1^p = \|f g_n - f g\|_p^p \leq \|f g_n\|_p^p + \|f g\|_p^p \leq \|f\|_p^p \|g_n\|_\infty^p + \|f\|_p^p \|g\|_\infty^p \leq 2 \|f\|_p^p M^p$$

Also, $|h_n(x)|^{\frac{1}{p}} = |(f g_n - f g)(x)| \leq |f(x) g_n(x) - f(x) g(x)| \leq |f(x)| |g_n(x)| + |f(x)| |g(x)| \leq 2M |f(x)|$, which implies h is dominated:

$$|h(x)| \leq 2^p M^p |f(x)|^p$$

Thus, by the dominated convergence theorem,

$$\begin{aligned} \lim_n \int_{\Omega} h_n &= \int_{\Omega} \lim_n h_n = \int_{\Omega} 0 = 0 \\ \implies \lim_n \int_{\Omega} |f g_n - f g|^p &= 0 \end{aligned}$$

Thus,

$$\lim_n \|f_n g_n - f g\|_p = \lim_n \left[\int_{\Omega} |g_n|^p |f_n - f|^p \right]^{\frac{1}{p}} + \lim_n \left[\int_{\Omega} |f g_n - f g|^p \right]^{\frac{1}{p}} = 0$$

which shows $f_n g_n \rightarrow f g$ in $L^p(\Omega)$. □

Problem 9

Prove that the space of continuous functions with compact support $\mathcal{C}_c^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Proof. □