Functional Analysis Theorems, Examples, and Counter Exmaples

Sam Fleischer

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1 A Short Introduction to L^p Spaces

1.1 Three Pillars of Analysis

Three Pillars of Analysis	**Examples**
Monotone Convergence Theorem - If a sequence of non-negative functions is increasing, we can pull the limit through an integral. $\lim_k \int f_k = \int \lim_k f_k$	This is an example.
Fatou's Lemma - For a sequence of non-negative functions, the integral of the lim inf is less than or equal to the lim inf of the integral. $\int \liminf_k f_k \leq \liminf_k \int f_k$	This is an example.
Dominated Convergence Theorem - If a sequence converges pointwise almost everywhere and is dominated, then it converges in norm to its pointwise limit. $\lim_k \int f_k = \int \lim_k f_k = \int f$ $\lim_k \lVert f_k - f \rVert_1 = 0$	This is an example.

1.2 Integrals over Product Spaces

Integrals over Product Spaces	**Examples**
Fubini's Theorem - If a function is integrable on a product space, then the integral over the product space is equal to both iterated integrals.	This is an example.
Semi-converse of Fubini's Theorem - If an iterated integral exists of the *absolute value* of a function on a prodct space, then the integral of the product space is equal to both iterated integrals.	This is an example.

Integrals over Product Spaces	**Examples**
Tonelli's Theorem - If a function is non-negative and measurable on a product space, then the integral over the product space is equal to both iterated integrals.	This is an example.

1.3 L^p Spaces

L^p **Spaces**	**Examples**
Convexity is a thing. $x^\lambda \leq (1-\lambda) + \lambda x \qquad \forall \lambda \in (0,1)$ $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b \;\; \forall \lambda \in (0,1), \;\; \forall a, b \geq 0$	This is an example.
Hölder's Inequality - For conjugate exponents p and q, the 1-norm of a product of L^p and L^q functions is finite, and the 1-norm of the product is less than or equal to the product of the norms of the original functions. $\lVert fg \rVert_1 \leq \lVert f \rVert_p \lVert g \rVert_q$	This is an example.
Interpolation Inequality - For $1 \leq r \leq s \leq t \leq \infty$, if u is in L^r and L^t, then u is in L^s and the s-norm is less than of equal to the product of the r- and t-norms raised to the appropriate power. $\lVert u \rVert_s \leq \lVert u \rVert_r^a \lVert u \rVert_t^{1-a} \qquad \text{where } \frac{1}{s} = \frac{a}{r} + \frac{1-a}{t}$ $L^r \cap L^t \subset L^s$	This is an example.
Minkowski's Inequality - For functions in L^p, the norm of their sum is less than or equal to the sum of their norms. $\lVert f+g \rVert_p \leq \lVert f \rVert_p + \lVert g \rVert_p$	This is an example.
L^p is a normed linear space.	This is an example.
L^p is a Banach Space.	This is an example.
Pointwise convergence implies a double implication - If a sequence of functions converge pointwise, then their norms converge if and only if they converge in norm. $f_k \to f \text{ pointwise} \implies$ $\left[\lVert f_k - f \rVert_p \to 0 \iff \lVert f_k \rVert_p \to \lVert f \rVert_p \right]$	This is an example.
L^p **Comparisons** - For $1 \leq r \leq s \leq t \leq \infty$, if a function in L^s can be written as the sum of functions in L^r and L^t. $L^s \subset L^r + L^t$	This is an example.

L^p **Spaces**	**Examples**
L^p **Comparison for Finite Spaces** - For finite measure spaces, a function in L^q is also in L^p for all $q > p$. $L^q \subset L^p$	This is an example.
Approximation of L^p $(p < \infty)$ **by Simple Functions** - The set of Simple Functions are dense in L^p.	This is an example.
Approximation of L^p $(p < \infty)$ **by Continuous Functions** - For bounded measure spaces, the set of continuous functions is dense in L^p.	This is an example.
Approximation of $L^p_{\mathbf{loc}}$ **by Smooth Functions** - For a function f in L^p_{loc}, its mollified functions: 1. are infinitely differentiable, 2. converge pointwise to f, 3. converge uniformly to f on compact subsets of the space (given f is continuous), and 4. converge to f in L^p_{loc}.	This is an example.

1.4 Convolutions and (in general) Integral Operators

L^p **Spaces**	**Examples**
Boundedness of Integral Operators - An integral operator has bounded norm (and is hence continuous) if both of the absolute iterated integrals of its kernel are bounded (say by C_1 and C_2). $\Vert K\Vert_{\mathcal{B}(L^p(\mathbb{R}^n))} \leq C_1^{\frac{1}{p}} C_2^{\frac{1}{q}}$	This is an example.
Cauchy-Young Inequality - If p and q are conjugate exponents, then for all nonnegative a and b, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$	This is an example.
Cauchy-Young Inequality with δ - If p and q are conjugate exponents, the for all nonnegative a and b, $ab \leq \delta a^p + C_\delta b^q, \quad \delta > 0, \quad C_\delta = (\delta p)^{-\frac{q}{p}} q^{-1}$	This is an example.

L^p **Spaces**	**Examples**
Simple Version of Young's Inequality - For L^1 function k and L^p function f, the p-norm of their convolution is less than or equal to the product of their respective norms. $\|k * f\|_p \leq \|k\|_1 \|f\|_p$	This is an example.
(More general) Young's Inequality for Convolution - For L^p function k and L^q function f, the r-norm of their convolution is bounded by the product of their respective norms, given $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. $$\|k * f\|_r \leq \|k\|_p \|f\|_q, \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$	This is an example.

1.5 The Dual Space and Weak Topology

L^p **Spaces**	**Examples**
Norm of an Integral Operator is the Norm of its Kernel - For conjugate exponents p and q, integration of an L^p function against an L^q function is a continuous linear functional on L^p and the operator norm is equal to the norm of the L^q function. $$F_g(f) = \int fg \qquad \text{and} \qquad \|F_g\|_{\text{op}} = \|g\|_q$$	This is an example.
Riesz Representation Theorem $(1 < p < \infty)$ - For conjugate exponents p and q, every bounded (continuous) linear functional on L^p can be represented as an integral operator whose kernel is in L^q. $$\phi \in (L^p)^* \quad \Longrightarrow \quad \exists g \in L^q$$ such that $$\phi(f) = \int fg \ \ \forall f \in L^p$$	This is an example.
Reflexivity of L^p $(1 < p < \infty)$ - The dual space of the dual space of L^p is isomorphic to L^p.	This is an example.
Radon-Nikodym Theorem - If μ and ν are two finite measures on a measure space where ν is absolutely continuous with respect to μ, then there exists an L^1 function h to change the measure of integration as follows: $$\int F \mathrm{d}\nu = \int F h \mathrm{d}\mu$$ for every positive measurable function F.	This is an example.

L^p **Spaces**	**Examples**
Converse to Hölder's Inequality - For finite measure spaces, if a product of a measurable function and any simple function is L^1, and if the supremum of the L^1-norm of the product (for simple functions of L^p-norm 1) is finite, then the measurable function is in L^q and its L^q-norm is equal to that supremum. $M(g) = \sup_{\|f\|_p=1} \left\{ \left\| \int_\Omega fg \mathrm{d}\mu \right\| : f \text{ is simple} \right\} < \infty$ $\implies$ $g \in L^q(\Omega) \text{ and } \|g\|_q = M(g)$	This is an example.
Alaoglu's Lemma - The closed unit ball in the dual of a Banach space is compact in the weak-* topology.	This is an example.
Weak Compactness for $L^p(\Omega)$ **for** $1 < p < \infty$ - Every bounded sequence in L^p has a weakly convergent subsequence.	This is an example.
Weak-* compactness for L^∞ - Every bounded sequence in L^∞ has a weak* convergent subsequence.	This is an example.
Convergence implies weak convergence - Convergent sequences in L^p are weakly convergent.	This is an example.
Weak Limits have Bounded Norms - The L^p norm of a weak limit is bounded by the lim inf of the L^p norms of its sequence.	This is an example.
Weakly convergent Sequences are bounded - Weakly convergent L^p sequences have bounded L^p norms.	This is an example.
Egoroff's Theorem - For pointwise convergent sequences on finite domains, there exist arbitrarily small (positive measure) subsets such that the sequence converges uniformly on its complement. $\forall \varepsilon < 0,\ \exists E \subset \Omega \text{ with } \|E\| < \varepsilon$ such that $f_k \to f$ uniformly on $\Omega \setminus E$	This is an example.
Almost everywhere convergence of a bounded (in L^p) sequence in a bounded domain implies weak convergence for $1 < p < \infty$. $\left\{ \begin{array}{l} \Omega \subset \mathbb{R}^n \text{ bounded,} \\ \sup_k \|f_k\|_p \leq M < \infty, \text{and} \\ f_k \to f \quad \text{a.e.} \end{array} \right\} \implies f_k \rightharpoonup f$	This is an example.

L^p **Spaces**	**Examples**
Weak and Strong Convergence Imply Strong Integral Convergence - If $u_k \rightharpoonup u$ and $v_k \to v$ in $L^p(\Omega)$, then $$\int_\Omega u_k v_k \mathrm{d}x \to \int_\Omega uv\mathrm{d}x$$	This is an example.
Weak Convergence Sometimes Implies Strong Convergence - Suppose $u_k \rightharpoonup u$ in $L^p(\Omega)$. If $\|u\|_p = \lim \|u_k\|_p$, the $u_k \to u$ in $L^p(\Omega)$.	This is an example.

2 Sobolev Spaces and the Fourier Transform

2.1 Sobolev Spaces $W^{k,p}$ for Integers $k \geq 0$

L^p **Spaces**	**Examples**
Divergence Theorem - Let $w \ : \ \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$. If $\partial\Omega$ is the graph of a Lipschitz function, then $$\int_\Omega \boldsymbol{\nabla} \cdot w \mathrm{d}x = \int_{\partial\Omega} w \cdot N \mathrm{d}S$$ where N is the outward-facing normal vector.	This is an example.
Multi-Dimensional Version of Integration by Parts - Suppose $g, h \ : \ \Omega \to \mathbb{R}$. Then $$\int_\Omega g h_{x_i} \mathrm{d}x = \int_{\partial\Omega} g h N^i \mathrm{d}S - \int_\Omega g_{x_i} h \mathrm{d}x$$ where g_{x_i} and h_{x_i} are the i^{th} partial derivatives of g and h, respectively, and N^i is the i^{th} component of the outward-facing normal vector.	This is an example.
Green's First Identity - Suppose $u \in \mathcal{C}^2(\overline{\Omega})$ and $v \in \mathcal{C}^1(\overline{\Omega})$. Then $$\begin{aligned} \int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u \mathrm{d}x + \int_\Omega v \nabla^2 u \mathrm{d}x &= \int_\Omega \boldsymbol{\nabla} \cdot (v \boldsymbol{\nabla} u) \mathrm{d}x \\ &= \int_{\partial\Omega} v \frac{\partial u}{\partial N} \mathrm{d}X. \end{aligned}$$ where $\nabla^2 = \dfrac{\partial^2}{\partial x_i^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$.	This is an example.
Green's Second Identity - Suppose both u and v are in $\mathcal{C}^2(\overline{\Omega})$. Then $$\int_\Omega (v\nabla^2 u - u \nabla^2 v) \mathrm{d}x = \int_{\partial\Omega} \left[v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right] \mathrm{d}S.$$	This is an example.
Liebnitz Rule (Product Rule) - Suppose $u \in W^{k,p}$ and ϕ is a test function. Then $\phi u \in W^{k,p}$ and $$D^\alpha(\phi u) = \sum_{\lvert\beta\rvert \leq \lvert\alpha\rvert} \binom{\alpha}{\beta} D^\alpha \phi D^{\alpha - \beta} u$$	This is an example.
Sobolev Spaces are Banach Spaces - $W^{k,p}$ is a Banach Space.	This is an example.

L^p **Spaces**	**Examples**
Sobolev Embedding in 2D - Suppose ϕ is a test function. Then it is absolutely bounded by a constant multiple of its norm in $W^{k,p}(\mathbb{R}^2)$. $\max_{x\in\mathbb{R}^2} \lvert u(x)\rvert \leq C\lVert u\rVert_{W^{k,p}(\mathbb{R}^2)}$	This is an example.
Local Approximation of Sobolev Functions by Smooth Functions - For nonnegative k and finite p, and for $u \in W^{k,p}$, 1. $u^\varepsilon = \eta_\varepsilon * u$ is infinitely continuous (not necessarily compactly supported) on Ω_ε, and 2. $u^\varepsilon \to u$ in $W^{k,p}_{\text{loc}}$	This is an example.
Global Approximaton of Sobolev Functions by Smooth Functions - For open and bounded Ω and for finite p, infinitely smooth Sobolev functions are dense in Sobolev Space. $\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega) \quad \text{is dense in} \quad W^{k,p}(\Omega)$ with respect to the $W^{k,p}$ norm.	This is an example.
Global Approximation of Sobolev Functions on the Closure of the Domain - For smooth, bounded, open subsets of $\mathbb{R}^n$, Sobolev functions can be approximated by infinitely smooth functions on the closure of the domain. $\mathcal{C}^\infty(\overline{\Omega}) \quad \text{is dense in} \quad W^{k,p}(\Omega)$ with respect to the $W^{k,p}$ norm.	This is an example.
Morrey's Inequality - Sobolev functions on a ball have bounded differences. Denote $B_r \subset \mathbb{R}^n$ as a ball of radius r and let $n < p \leq \infty$. $\lvert u(x) - u(y)\rvert \leq C\lvert x-y\rvert^{1-\frac{n}{p}}\lVert Du\rVert_{L^p(B_r)}$	This is an example.
Sobolev Embedding for $k = 1$ - The $\mathcal{C}^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ norm (Hölder Space norm) of a Sobolev function is bounded by a constant multiple (dependent on p and n) of the $W^{1,p}(\mathbb{R}^n)$ norm. $\lVert u\rVert_{\mathcal{C}^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\lVert u\rVert_{W^{1,p}(\mathbb{R}^n)}$	This is an example.
Sobolev Embedding for $kp > n$ - The Hölder Space norm of a Sobolev function is bounded by a constant multiple (dependent on k, p, and n) of the Sobolev norm. $\lVert u\rVert_{\mathcal{C}^{k-[\frac{n}{p}]-1,\gamma}(\mathbb{R}^n)} \leq C\lVert u\rVert_{W^{k,p}(\mathbb{R}^n)}$ where $\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N}, \\ \text{any } \alpha \in \mathbb{R} \cap (0,1) & \text{if } \frac{n}{p} \in \mathbb{N}. \end{cases}$	This is an example.

L^p **Spaces**	**Examples**
Almost-Everywhere Differentiability - For $n < p \leq \infty$, local Sobolev functions are almost-everywhere differentiable and its gradient and weak gradient agree almost everywhere.	This is an example.
Gagliardo-Nirenberg-Sobolev Inequality - For $1 \leq p < n$, set $p^* = \dfrac{np}{n-p}$. Then the L^{p^*} norm of a Sobolev function is bounded by a constant multiple (dependent on n and p) of the L^p norm of its derivative. $\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$	This is an example.