UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #6

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Problem 1

Given $f(x) = \frac{1}{(1+x^2)^2}$ find $\widehat{f}(\xi)$. Prove that $\widehat{f} \in C^2$. You can use the following fact that follows from complex integration

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\mathrm{d}x = \frac{\pi}{b}e^{-ab}, \qquad a, b > 0.$$

Proof. □

Problem 2

(a) Prove that if $f, g \in \mathscr{S}(\mathbb{R}^n)$ (the Schwartz class of functions) then $f * g \in \mathscr{S}(\mathbb{R}^n)$.

(b) Find explicitly $\Psi = \widehat{|x|^2} \in \mathscr{S}'(\mathbb{R}^n)$.

(a) *Proof.* First note that the Fourier transform is an isomorphism from $\mathscr{S}(\mathbb{R}^n)$ onto itself. Thus it suffices to show that for $f, g \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{f * g} \in \mathscr{S}(\mathbb{R}^n)$. However, $\widehat{f * g} = \widehat{f}\widehat{g} \in \mathscr{S}(\mathbb{R}^n)$ since $\widehat{f}$ and $\widehat{g}$ are Schwartz functions and the product of Schwartz functions is a Schwartz function. Thus $\widehat{f * g} \in \mathscr{S}(\mathbb{R}^n)$, which shows $f * g \in \mathscr{S}(\mathbb{R}^n)$. □

(b) *Proof.* something □

Problem 3

Let $0 < \alpha < \frac{n}{2}$.

(a) Prove that $|x|^{-n+\alpha}$ defines a tempered distribution.

(b) Prove that

$$\widehat{|x|^{-n+\alpha}}(\xi) = c_{n,\alpha}|\xi|^{-\alpha}.$$

Observe that $|x|^{-n+\alpha}\mathscr{X}_{\{|x|\le 1\}} \in L^1(\mathbb{R})$ and $|x|^{-n+\alpha}\mathscr{X}_{\{|x|>1\}} \in L^2(\mathbb{R})$. Thus $\widehat{|x|^{-n+\alpha}}(\xi)$ is a function. Show that $\widehat{|x|^{-n+\alpha}}(\xi)$ is radial and homogeneous of order $-\alpha$.

Define the *Hilbert transform* $\mathscr{H}(\phi)$ of a function $\phi \in \mathscr{S}(\mathbb{R})$ by

$$\mathscr{H}(\phi) = \frac{1}{\pi}\text{p.v.}\left(\frac{1}{x}\right) * \phi,$$

where

$$\text{p.v.}\left(\frac{1}{x}\right)(\phi) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<\frac{1}{\varepsilon}} \frac{\phi(x)}{x}\mathrm{d}x.$$

(a) *Proof.*

$$\int_{\mathbb{R}} |x|^{-n+\alpha}$$

□

(b) *Proof.* □

Problem 4

If $\phi \in \mathscr{S}(\mathbb{R})$, prove that $\mathscr{H}(\phi) \in L^1(\mathbb{R})$ if and only if $\widehat{\phi}(0) = 0$.

Proof. $\square$

Problem 5

Prove the following identities:

(a) $\mathscr{H}(fg) = \mathscr{H}(f)g + f\mathscr{H}(g) + \mathscr{H}\big(\mathscr{H}(f)\mathscr{H}(g)\big)$.

(b) $\mathscr{H}\big(\mathscr{X}_{(-1,1)}\big) = \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right|$.

Proof. (a) First note that since

$$\widehat{\text{p.v.}\left(\frac{1}{x}\right)} = -i\pi\text{sgn}\,(\xi),$$

then the Fourier transform of the Hilbert transform is

$$\widehat{\mathscr{H}(\phi)} = \frac{1}{\pi}\widehat{\text{p.v.}\left(\frac{1}{x}\right)}\widehat{\phi} = -i\text{sgn}\,(\xi)\widehat{\phi}.$$

Also note that

$$\text{sgn}\,(x-y)\text{sgn}\,(y) = \text{sgn}\,(x)\text{sgn}\,(y) + \text{sgn}\,(x-y)\text{sgn}\,(x) - 1$$

Finally,

$$\begin{aligned}
\widehat{\mathscr{H}(f)g + f\mathscr{H}(g) + \mathscr{H}\big(\mathscr{H}(f)\mathscr{H}(g)\big)} &= \left[-i\text{sgn}\,\widehat{f}\right] * \widehat{g} + \left[-i\text{sgn}\,\widehat{g}\right] * \widehat{f} - i\text{sgn}\,\left[\widehat{\mathscr{H}(f)\mathscr{H}(g)}\right] \\
&= \left[-i\text{sgn}\,\widehat{f}\right] * \widehat{g} + \left[-i\text{sgn}\,\widehat{g}\right] * \widehat{f} - i\text{sgn}\,\left[\left(-i\text{sgn}\,\widehat{f}\right) * \left(-i\text{sgn}\,\widehat{g}\right)\right] \\
&= \int_{\mathbb{R}} -i\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y + \int_{\mathbb{R}} -i\text{sgn}\,(y)\widehat{g}(y)\widehat{f}(\xi-y)\mathrm{d}y \\
&\quad - i\text{sgn}\,(\xi)\int_{\mathbb{R}} -\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\text{sgn}\,(y)\widehat{g}(y)\mathrm{d}y \\
&= \int_{\mathbb{R}} -i\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y + \int_{\mathbb{R}} -i\text{sgn}\,(y)\widehat{g}(y)\widehat{f}(\xi-y)\mathrm{d}y \\
&\quad - i\text{sgn}\,(\xi)\int_{\mathbb{R}} \widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&\quad + i\text{sgn}\,(\xi)\int_{\mathbb{R}} \text{sgn}\,(\xi)\text{sgn}\,(y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&\quad + i\text{sgn}\,(\xi)\int_{\mathbb{R}} \text{sgn}\,(\xi-y)\text{sgn}\,(\xi)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&= \underline{\int_{\mathbb{R}} -i\text{sgn}\,(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y} + \underline{\underline{\int_{\mathbb{R}} -i\text{sgn}\,(y)\widehat{g}(y)\widehat{f}(\xi-y)\mathrm{d}y}} \\
&\quad - i\int_{\mathbb{R}} \text{sgn}\,(\xi)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y \\
&\quad \underline{\underline{+ i\int_{\mathbb{R}} \text{sgn}\,(y)\widehat{f}(\xi-y)\widehat{g}(y)\mathrm{d}y}}
\end{aligned}$$

$$
\begin{aligned}
&\underline{+i\int_{\mathbb{R}} \operatorname{sgn}(\xi-y)\widehat{f}(\xi-y)\widehat{g}(y)\,\mathrm{d}y} \\
&= -i\operatorname{sgn}(\xi)\int_{\mathbb{R}} \widehat{f}(\xi-y)\widehat{g}(y)\,\mathrm{d}y \\
&= -i\operatorname{sgn}(\xi)\widehat{f}*\widehat{g} = -i\operatorname{sgn}(\xi)\widehat{fg} = \widehat{\mathscr{H}(fg)}
\end{aligned}
$$

Since the Fourier transform is an isomorphism, the identity holds since we can take the inverse Fourier transform of both sides.

(b)

$$
\begin{aligned}
\mathscr{H}\big(\mathscr{X}_{(-1,1)}\big)(x) &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<y<\frac{1}{\varepsilon}} \frac{\mathscr{X}_{-1,1)}(x-y)}{y}\,\mathrm{d}y \\
&= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|<\frac{1}{\varepsilon}} \frac{\mathscr{X}_{(-1,1)}(y)}{x-y}\,\mathrm{d}y \\
&= \begin{cases}
\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon}\frac{1}{x-y}\,\mathrm{d}y + \int_{x+\varepsilon}^{1}\frac{1}{x-y}\,\mathrm{d}y\right] & \text{if } x\in(-1,1) \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{-1+\varepsilon}^{1}\frac{1}{x-y}\,\mathrm{d}y & \text{if } x=-1 \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{-1}^{1-\varepsilon}\frac{1}{x-y}\,\mathrm{d}y & \text{if } x=1 \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{-1}^{1}\frac{1}{x-y}\,\mathrm{d}y & \text{if } x\notin[-1,1]
\end{cases} \\
&= \begin{cases}
\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[\underline{-\log|\varepsilon|}+\log|x+1|-\log|x-1|\underline{+\log|\varepsilon|}\right] & \text{if } x\in(-1,1) \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[-\log|x-1|+\log|x+1-\varepsilon|\right] & \text{if } x=-1 \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[-\log|x-1+\varepsilon|+\log|x+1|\right] & \text{if } x=1 \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[-\log|x-1|+\log|x+1|\right] & \text{if } x\notin[-1,1]
\end{cases} \\
&= \begin{cases}
\frac{1}{\pi}\lim_{\varepsilon\to 0}\log\left|\frac{x+1}{x-1}\right| & \text{if } x\in(-1,1) \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\log\left|\frac{x+1-\varepsilon}{x-1}\right| & \text{if } x=-1 \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\log\left|\frac{x+1}{x-1+\varepsilon}\right| & \text{if } x=1 \\
\frac{1}{\pi}\lim_{\varepsilon\to 0}\log\left|\frac{x+1}{x-1}\right| & \text{if } x\notin[-1,1]
\end{cases} \\
&= \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right| \qquad \forall x\in\mathbb{R}
\end{aligned}
$$

$\square$