UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #3

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Problem 1

If $u \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, and $u^\varepsilon = \eta_\varepsilon * u$, for η_ε the standard mollifier. Show that

$$u^\varepsilon \to u$$

in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Proof. □

Problem 2

Let Ω denote an open and smooth subset of $\mathbb{R}^n$. Prove that $\mathscr{C}_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

Proof. □

Problem 3

Prove that if $u \in L^1_{\text{loc}}(\Omega)$ satisfies $\int_\Omega u(x)v(x)\mathrm{d}x = 0$ for all $v \in \mathscr{C}_c^\infty(\Omega)$, then $u = 0$ a.e. in Ω.

Proof. □

Problem 4

Let $u \in L^\infty(\mathbb{R}^n)$ and let η_ε be a standard mollifier. For $\varepsilon > 0$ consider the sequence $\psi_\varepsilon \in L^\infty(\mathbb{R}^n)$ such that

$$\left\|\psi_\varepsilon\right\|_\infty \leq 1 \ \forall \varepsilon > 0 \ \text{ and } \ \psi_\varepsilon \to \psi \text{ a.e. in } \mathbb{R}^n,$$

define

$$v^\varepsilon = \eta_\varepsilon * (\psi_\varepsilon u) \ \text{ and } \ v = \psi u.$$

(a) Prove that $v^\varepsilon \overset{*}{\rightharpoonup} v$ in $L^\infty(\mathbb{R}^n)$.

(b) Prove that $v^\varepsilon \to v$ in $L^1(B)$ for every ball $B \subset \mathbb{R}^n$.

Proof. (a) We want to show $\phi_{v^\varepsilon}(f) \to \phi_v(f)$ for all $f \in L^1(\mathbb{R})$, where ϕ_v and ϕ_{v^ε} are the continuous linear functionals associated with v and v^ε, respectively.

□

Problem 5

For $u \in \mathscr{C}^0(\mathbb{R}^n;\mathbb{R})$, spt (u) is the closure of the set $\{x \in \mathbb{R}^n : u(x) \neq 0\}$, but this definition may not make sense for functions $u \in L^p(\Omega)$. For example what is the support of $\mathscr{X}_{\mathbb{Q}}$, the indicator over the rationals?

Let $u : \mathbb{R}^n \to \mathbb{R}$, and let $\{\Omega_\alpha\}_{\alpha \in A}$ denote the collection of all open sets on $\mathbb{R}^n$ such that for each $\alpha \in A$, $u = 0$ a.e. on Ω_α. Define $\Omega = \bigcup_{\alpha \in A} \Omega_\alpha$. Prove that $u = 0$ a.e. on Ω.

The support of u, spt (u), is Ω^C, the complement of Ω. Notice that if $v = w$ a.e. on $\mathbb{R}^n$, then spt $(v) =$ spt (w); furthermore, if $u \in \mathscr{C}^0(\mathbb{R}^n)$, then $\Omega^C = \overline{\{x \in \mathbb{R}^n : u(x) \neq 0\}}$. (Hint: Since A is not necessarily countable, it is not clear that $f = 0$ a.e. on Ω, so find a countable family U_n of open sets in $\mathbb{R}^n$ such that every open set on $\mathbb{R}^n$ is the union of some of the sets from $\{U_n\}$.)

Proof. Since $\mathscr{X}_{\mathbb{Q}}$ is nonzero on $\mathbb{R} \setminus \mathbb{Q}$, which is a dense subset of $\mathbb{R}$, then spt $(\mathscr{X}_{\mathbb{Q}}) = \mathbb{R}$. This is nonsence, however, since $\mathscr{X}_{\mathbb{Q}}$ is equivalent to 0 in $L^p(\mathbb{R})$. □

Problem 6

Prove that if $u \in L^1(\mathbb{R}^n)$ and $v \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, then

$$\text{spt}\,(u * v) \subset \overline{\text{spt}\,(u) + \text{spt}\,(v)}.$$

Proof. Suppose $x \notin \overline{\text{spt}\,(u) + \text{spt}\,(v)}$ and define the set $[x - \text{spt}\,(u)]$ as the shift of the support of u by the vector x:

$$[x - \text{spt}\,(u)] = \{y : x - y \in \text{spt}\,(u)\}$$

Then

$$(u * v)(x) = \int_{\mathbb{R}^n} u(x-y)v(y)\mathrm{d}y = \int_{[x-\text{spt}\,(u)]\cap\text{spt}\,(v)} u(x-y)v(y)\mathrm{d}y$$

If $x \in \text{spt}\,(v) \cap [x - \text{spt}\,(u)]$, then $x \in \text{spt}\,(v)$ and $x - x = 0 \in \text{spt}\,(u)$. Then since $x = 0 + x$, then $x \in \text{spt}\,(u) + \text{spt}\,(v)$, which is a contradiction since $x \notin \overline{\text{spt}\,(u) + \text{spt}\,(v)}$. Thus $[x - \text{spt}\,(u)] \cap \text{spt}\,(v) = \emptyset$, and therefore

$$(u * v)(x) = \int_{[x-\text{spt}\,(u)]\cap\text{spt}\,(v)} u(x-y)v(y)\mathrm{d}y = \int_{\emptyset} u(x-y)v(y)\mathrm{d}y = 0$$

and thus $x \notin \text{spt}\,(u * v)$. This shows

$$\text{spt}\,(u * v) \subset \overline{\text{spt}\,(u) + \text{spt}\,(v)}.$$

□

Problem 7

Suppose that $1 < p < \infty$. If $\tau_y f(x) = f(x-y)$, show that f belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $\tau_y f$ is a Lipschitz function of y with values in $L^p(\mathbb{R}^n)$; that is,

$$\left\|\tau_y f - \tau_z f\right\|_p \leq C|y - z|.$$

What happens in the case $p = 1$?

Proof. □

Problem 8

If $u \in W^{1,p}(\mathbb{R}^n)$ for some $p \in [1,\infty)$ and $\frac{\partial u}{\partial x_j} = 0$, $j = 1,\ldots,n$, on a connected open set $\Omega \subset \mathbb{R}^n$, show that u is equal a.e. to a constant on Ω. (Hint: approximate u using that $\eta_\varepsilon * u \to u$ in $W^{1,p}(\mathbb{R}^n)$, where η_ε is a sequence of standard mollifiers. Show that $\frac{\partial}{\partial x_j}(\eta_\varepsilon * u) = 0$ on $\Omega_\varepsilon \subset\subset \Omega$ where $\Omega_\varepsilon \nearrow \Omega$ as $\varepsilon \to 0$.)

More generally, if $\frac{\partial u}{\partial x_j} - f_j \in C(\Omega)$, $1 \le j \le n$, show that u is equal a.e. to a funtion in $\mathscr{C}^1(\Omega)$.

Proof. □