201C, Spring '16, Thomases
Homework 2 due 4/12/16

1. A function $f \in L^p(\mathbb{R}^n)$ is said to be $L^p - $*continuous* if $\tau_h f \to f$ in $L^p(\mathbb{R}^n)$ as $h \to 0$ in $\mathbb{R}^n$, where $\tau_h f(x) = f(x-h)$ is the translation of f by h. Prove that, if $1 \le p < \infty$, every $f \in L^p(\mathbb{R}^n)$ is L^p-continuous. Give a counter-example to show that this result is not true when $p = \infty$. [Hint: Approximate an L^p function by a C_c-function.]

2. Show that $L^\infty(\mathbb{R})$ is not separable. [Hint: There is an uncountable set $\mathcal{F} \subset L^\infty$ such that $\|f-g\|_\infty \ge 1$ for all $f, g \in \mathcal{F}$ with $f \ne g$.]

3. Prove Chebyshev's Inequality: If $f \in L^p$ $(1 \le p < \infty)$, then for any $\alpha > 0$,

$$\mu(\{x : |f(x)| > \alpha\}) \le \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

   [Note that you can find the proof of this simple fact in many texts but you should see if you can figure it out yourself. Also, note that this inequality holds for all $0 < p < \infty$.]

4. Assume that $f, g \in L^1(\mathbb{R}^n)$. Prove that the convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

   is measurable and in $L^1(\mathbb{R}^n)$.

5. Let $f_n = \sqrt{n}\mathbf{1}_{(0,1/n)}$. Prove that f_n converges weakly to 0 in $L^2(0,1)$ and $f_n \to 0$ in $L^1(0,1)$ but f_n does not converge strongly in $L^2(0,1)$.

6. Find a sequence of functions with the property that f^j converges to 0 in $L^2(\Omega)$ weakly, to 0 in $L^{3/2}(\Omega)$ strongly, but it does not converge to 0 strongly in $L^2(\Omega)$.

7. Let f_n and g_n denote two sequences in $L^p(\Omega)$ with $1 \le p \le \infty$ such that $f_n \to f$ in $L^p(\Omega)$, and $g_n \to g$ in $L^p(\Omega)$. Set $h_n = \max\{f_n, g_n\}$ and prove that $h_n \to h$ in $L^p(\Omega)$.

8. Let f_n be a sequence in $L^p(\Omega)$ with $1 \le p < \infty$, and let g_n be a bounded sequence in $L^\infty(\Omega)$. Suppose that $f_n \to f$ in $L^p(\Omega)$ and that $g_n \to g$ pointwise a.e. Prove that $f_n g_n \to fg$ in $L^p(\Omega)$.

9. Prove the the space of continuous functions with compact support $(\mathcal{C}_c^0(\mathbb{R}^n))$ is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.