Functional Analysis Facts and Notes

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April 28, 2016

1 A Short Introduction to L^p Spaces

1.1 Three Pillars of Analysis

Three Pillars of Analysis	Notes
Monotone Convergence Theorem - If a sequence of non-negative functions is increasing, we can pull the limit through an integral. $$\lim_k \int f_k = \int \lim_k f_k$$	$f_k = k\mathcal{X}_{\left[0,\frac{1}{k}\right]}$ converges pointwise to 0, but they are not increasing, and so the Monoton Convergence Theorem does not hold. $$1 = \lim_k \int f_k \neq \int \lim_k f_k = 0$$
Fatou's Lemma - For a sequence of non-negative functions, the integral of the lim inf is less than or equal to the lim inf of the integral. $$\int \liminf_k f_k \leq \liminf_k \int f_k$$	• To remember which direction the inequality goes, draw a bunch of functions and trace out the infimum at each point. Visually, the integral over the pointwise lim inf is clearly less than the lim inf of the integrals of the functions. • $f_k = k\mathcal{X}_{\left[0,\frac{1}{k}\right]}$ is a clear example of this. $$0 = \int \liminf_k f_k \leq \liminf \int f_k = 1$$ • *Prove this thing: $t = 3d$ etc.* Well here is a proof.
Dominated Convergence Theorem - If a sequence converges pointwise almost everywhere and is dominated, then it converges in norm to its pointwise limit. $$\lim_k \int f_k = \int \lim_k f_k = \int f$$ $$\lim_k \|f_k - f\|_1 = 0$$	$f_k = k\mathcal{X}_{\left[0,\frac{1}{k}\right]}$ is not dominated by an L_1 function. At best, it is dominated by $\frac{1}{x}$, and so the Dominated Convergence Theorem does not hold.

1.2 Integrals over Product Spaces

Integrals over Product Spaces	Notes
Fubini's Theorem - If a function is integrable on a product space, then the integral over the product space is equal to both iterated integrals.	
Semi-converse of Fubini's Theorem - If an iterated integral exists of the *absolute value* of a function on a prodct space, then the integral of the product space is equal to both iterated integrals.	This is the most useful theorem regarding integrals over product spaces. You only need a single iterated integral to get the other iterated integral and the integral over the product space. Useful when finding norms of convolved functions since you always get a double integral.
Tonelli's Theorem - If a function is non-negative and measurable on a product space, then the integral over the product space is equal to both iterated integrals.	

1.3 L^p Spaces

L^p **Spaces**	Notes
Convexity is a thing - $x^\lambda \leq (1-\lambda) + \lambda x \qquad \forall \lambda \in (0,1)$ $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b \;\; \forall \lambda \in (0,1), \;\; \forall a, b \geq 0$ $\lvert f+g \rvert^p \leq 2^{p-1}(\lvert f \rvert^p + \lvert g \rvert^p)$	$\mathbb{R}$ is convex. L^p is convex for $1 \leq p \leq \infty$.
Hölder's Inequality - For conjugate exponents p and q, the 1-norm of a product of L^p and L^q functions is finite, and the 1-norm of the product is less than or equal to the product of the norms of the original functions. $\lVert fg \rVert_1 \leq \lVert f \rVert_p \lVert g \rVert_q$	Turn limits of integrals with wonky limits to limits of integrals of functions applied to characteristic functions. $\int_A f = \int f \mathcal{X}_A \leq \lVert f \rVert_p \lVert \mathcal{X}_A \rVert_q = \lVert f \rVert_p \mu(A)^{\frac{1}{q}} < \infty$ This trick is only useful when A has finite measure.
Interpolation Inequality - For $1 \leq r \leq s \leq t \leq \infty$, if u is in L^r and L^t, then u is in L^s and the s-norm is less than of equal to the product of the r- and t-norms raised to the appropriate power. $\lVert u \rVert_s \leq \lVert u \rVert_r^a \lVert u \rVert_t^{1-a} \qquad \text{where } \frac{1}{s} = \frac{a}{r} + \frac{1-a}{t}$ $L^r \cap L^t \subset L^s$	This is helpful in determining ranges of integrability.
Minkowski's Inequality - For functions in L^p, the norm of their sum is less than or equal to the sum of their norms. $\lVert f+g \rVert_p \leq \lVert f \rVert_p + \lVert g \rVert_p$	This is just the triangle inequality.
L^p is a normed linear space.	This is true fact that is true.
L^p is a Banach Space.	Cauchy sequences in L^p converge.

L^p **Spaces**	**Notes**
Pointwise convergence implies a double implication - If a sequence of functions converge pointwise, then their norms converge if and only if they converge in norm. $f_k \to f$ pointwise $\implies$ $$\left[\|f_k - f\|_p \to 0 \quad \iff \quad \|f_k\|_p \to \|f\|_p\right]$$	We don't need pointwise convergence to ensure that convergence in norm implies norm convergence. This follows from the triangle inequality. However, we require pointwise convergence and employ convexity to ensure norm convergence implies convergence in norm.
L^p **Comparisons** - For $1 \leq r \leq s \leq t \leq \infty$, if a function in L^s can be written as the sum of functions in L^r and L^t. $L^s \subset L^r + L^t$	Prove this by writing $f = f\mathcal{X}_A + f\mathcal{X}_B$ where $A = \{x \ : \ f(x) \geq 1\}$ and $B = \{x \ : \ f(x) < 1\}$.
L^p **Comparison for Finite Spaces** - For finite measure spaces, a function in L^q is also in L^p for all $q > p$. $L^q \subset L^p$	• Let $\Omega = (0,1)$. Then $\frac{1}{x^\alpha} \in L^p(\Omega)$ if $\alpha p < 1$. • Let $\Omega = (1,\infty)$. Then $\frac{1}{x^\alpha} \in L^p(\Omega)$ if $\alpha p > 1$.
Approximation of L^p $(p < \infty)$ **by Simple Functions** - The set of Simple Functions are dense in L^p.	This is a truthful statement that can be proved using true facts.
Approximation of L^p $(p < \infty)$ **by Continuous Functions** - For bounded measure spaces, the set of continuous functions is dense in L^p.	Dear L-rd please help me understand density. Amen.
Approximation of L^p_{loc} **by Smooth Functions** - For a function f in L^p_{loc}, its mollified functions: 1. are infinitely differentiable, 2. converge pointwise to f, 3. converge uniformly to f on compact subsets of the space (given f is continuous), and 4. converge to f in L^p_{loc}.	First note that $L^p \subset L^p_{\text{loc}}$. If a function is integrable on the whole domain it is certainly integrable on subsets. Functions can fail to be in L^p if they don't taper off fast enough at infinity, or they blow up at a singularity. Functions that don't taper off fast enough can still be in L^p_{loc}. Functions that blow up at a singularity can also still by in L^p_{loc} provided the singularity is at the boundary. • $\frac{1}{x} \notin L^1_{\text{loc}}((-1,1))$. • $\frac{1}{x} \notin L^1((0,1))$, but $\frac{1}{x} \in L^1_{\text{loc}}((0,1))$. • $\mathcal{X}_{\mathbb{R}} \notin L^1(\mathbb{R})$, but $\mathcal{X}_{\mathbb{R}} \in L^1_{\text{loc}}(\mathbb{R})$.

1.4 Convolutions and (in general) Integral Operators

Convolutions and Integral Operators	Notes
Boundedness of Integral Operators - An integral operator has bounded norm (and is hence continuous) if both of the absolute iterated integrals of its kernel are bounded (say by C_1 and C_2). $$\|K\|_{\mathcal{B}(L^p(\mathbb{R}^n))} \leq C_1^{\frac{1}{p}} C_2^{\frac{1}{q}}$$	
Cauchy-Young Inequality - If p and q are conjugate exponents, then for all nonnegative a and b, $$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$	
Cauchy-Young Inequality with δ - If p and q are conjugate exponents, the for all nonnegative a and b, $$ab \leq \delta a^p + C_\delta b^q, \quad \delta > 0, \quad C_\delta = (\delta p)^{-\frac{q}{p}} q^{-1}$$	
Simple Version of Young's Inequality - For L^1 function k and L^p function f, the p-norm of their convolution is less than or equal to the product of their respective norms. $$\|k * f\|_p \leq \|k\|_1 \|f\|_p$$	
(More general) Young's Inequality for Convolution - For L^p function k and L^q function f, the r-norm of their convolution is bounded by the product of their respective norms, given $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. $$\|k * f\|_r \leq \|k\|_p \|f\|_q, \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$	

1.5 The Dual Space and Weak Topology

The Dual Space and Weak Topology	Notes
Norm of an Integral Operator is the Norm of its Kernel - For conjugate exponents p and q, integration of an L^p function against an L^q function is a continuous linear functional on L^p and the operator norm is equal to the norm of the L^q function. $$F_g(f) = \int fg \qquad \text{and} \qquad \|F_g\|_{\text{op}} = \|g\|_q$$	

The Dual Space and Weak Topology	Notes
Riesz Representation Theorem $(1 < p < \infty)$ - For conjugate exponents p and q, every bounded (continuous) linear functional on L^p can be represented as an integral operator whose kernel is in L^q. $$\phi \in (L^p)^* \implies \exists g \in L^q$$ such that $$\phi(f) = \int fg \ \ \forall f \in L^p$$	
Reflexivity of L^p $(1 < p < \infty)$ - The dual space of the dual space of L^p is isomorphic to L^p.	$L^p \cong ((L^p)^*)^*$ for $1 < p < \infty$. That is, L^p is reflexive. On the other hand, $(L^1)^* \cong L^\infty$ but $(L^\infty)^* \ncong L^1$. In fact, $(L^\infty)^* \cong A \supset L^1$.
Radon-Nikodym Theorem - If μ and ν are two finite measures on a measure space where ν is absolutely continuous with respect to μ, then there exists an L^1 function h to change the measure of integration as follows: $$\int F \mathrm{d}\nu = \int F h \mathrm{d}\mu$$ for every positive measurable function F.	Is this literally just u-substitution?
Converse to Hölder's Inequality - For finite measure spaces, if a product of a measurable function and any simple function is L^1, and if the supremum of the L^1-norm of the product (for simple functions of L^p-norm 1) is finite, then the measurable function is in L^q and its L^q-norm is equal to that supremum. $$M(g) = \sup_{\substack{\Vert f \Vert_p = 1 \\ f \text{ is simple}}} \left\{ \left\vert \int_\Omega fg \mathrm{d}\mu \right\vert \right\} < \infty$$ $$\implies$$ $$g \in L^q(\Omega) \ \text{ and } \ \Vert g \Vert_q = M(g)$$	Ok lets try this again. Hölder's inequality states $$f \in L^p, \ g \in L^{p^*} \implies fg \in L^1.$$ The converse states that if $fg \in L^1$ for every $f \in L^p$ such that $\Vert f \Vert_p = 1$ and f is simple, then $g \in L^{p^*}$. In addition, $$\Vert g \Vert_{p^*} = \sup_{\substack{\Vert f \Vert_p = 1 \\ f \text{ is simple}}} \left\{ \left\vert \int_\Omega fg \mathrm{d}\mu \right\vert \right\}$$
Alaoglu's Lemma - The closed unit ball in the dual of a Banach space is compact in the weak-* topology.	Another way to say this is that the closed unit ball is compact in the weak topology. These are equivalent for reflexive spaces (in particular L^p for $1 < p < \infty$).
Weak Compactness for $L^p(\Omega)$ **for** $1 < p < \infty$ - Every bounded sequence in L^p has a weakly convergent subsequence.	
Weak-* compactness for L^∞ - Every bounded sequence in L^∞ has a weak* convergent subsequence.	
Convergence implies weak convergence - Convergent sequences in L^p are weakly convergent.	

The Dual Space and Weak Topology	Notes
Weak Limits have Bounded Norms - The L^p norm of a weak limit is bounded by the lim inf of the L^p norms of its sequence.	
Weakly convergent Sequences are bounded - Weakly convergent L^p sequences have bounded L^p norms.	
Egoroff's Theorem - For pointwise convergent sequences on finite domains, there exist arbitrarily small (positive measure) subsets such that the sequence converges uniformly on its complement. $\forall \varepsilon < 0,\ \exists E \subset \Omega$ with $\lvert E\rvert < \varepsilon$ such that $f_k \to f$ uniformly on $\Omega \setminus E$	
Almost everywhere convergence of a bounded (in L^p) sequence in a bounded domain implies weak convergence for $1 < p < \infty$. $\left\{\begin{array}{l}\Omega \subset \mathbb{R}^n \text{ bounded,}\\ \sup_k \lVert f_k\rVert_p \leq M < \infty, \text{and}\\ f_k \to f \quad \text{a.e.}\end{array}\right\} \implies f_k \rightharpoonup f$	
Weak and Strong Convergence Imply Strong Integral Convergence - If $u_k \rightharpoonup u$ and $v_k \to v$ in $L^p(\Omega)$, then $\int_\Omega u_k v_k \mathrm{d}x \to \int_\Omega uv \mathrm{d}x$	
Weak Convergence Sometimes Implies Strong Convergence - Suppose $u_k \rightharpoonup u$ in $L^p(\Omega)$. If $\lVert u\rVert_p = \lim \lVert u_k\rVert_p$, the $u_k \to u$ in $L^p(\Omega)$.	

2 Sobolev Spaces and the Fourier Transform

2.1 Sobolev Spaces $W^{k,p}$ for Integers $k \geq 0$

Sobolev Spaces	Notes
Divergence Theorem - Let $w : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$. If $\partial\Omega$ is the graph of a Lipschitz function, then $\int_\Omega \boldsymbol{\nabla} \cdot w \mathrm{d}x = \int_{\partial\Omega} w \cdot N \mathrm{d}S$ where N is the outward-facing normal vector.	
Multi-Dimensional Version of Integration by Parts - Suppose $g, h : \Omega \to \mathbb{R}$. Then $\int_\Omega g h_{x_i} \mathrm{d}x = \int_{\partial\Omega} g h N^i \mathrm{d}S - \int_\Omega g_{x_i} h \mathrm{d}x$ where g_{x_i} and h_{x_i} are the i^{th} partial derivatives of g and h, respectively, and N^i is the i^{th} component of the outward-facing normal vector.	

Sobolev Spaces	Notes
Green's First Identity - Suppose $u \in \mathcal{C}^2(\overline{\Omega})$ and $v \in \mathcal{C}^1(\overline{\Omega})$. Then $$\int_\Omega \nabla v \cdot \nabla u \mathrm{d}x + \int_\Omega v \nabla^2 u \mathrm{d}x = \int_\Omega \nabla \cdot (v \nabla u) \mathrm{d}x = \int_{\partial\Omega} v \frac{\partial u}{\partial N} \mathrm{d}X.$$ where $\nabla^2 = \frac{\partial^2}{\partial x_i^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.	
Green's Second Identity - Suppose both u and v are in $\mathcal{C}^2(\overline{\Omega})$. Then $$\int_\Omega (v\nabla^2 u - u\nabla^2 v)\mathrm{d}x = \int_{\partial\Omega} \left[v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N} \right] \mathrm{d}S.$$	
Liebnitz Rule (Product Rule) - Suppose $u \in W^{k,p}$ and ϕ is a test function. Then $\phi u \in W^{k,p}$ and $$D^\alpha(\phi u) = \sum_{\lvert\beta\rvert \leq \lvert\alpha\rvert} \binom{\alpha}{\beta} D^\alpha \phi D^{\alpha-\beta} u$$	
Sobolev Spaces are Banach Spaces - $W^{k,p}$ is a Banach Space.	
Sobolev Embedding in 2D - Suppose ϕ is a test function. Then it is absolutely bounded by a constant multiple of its norm in $W^{k,p}(\mathbb{R}^2)$. $$\max_{x\in\mathbb{R}^2} \lvert u(x)\rvert \leq C\lVert u\rVert_{W^{k,p}(\mathbb{R}^2)}$$	
Local Approximation of Sobolev Functions by Smooth Functions - For nonnegative k and finite p, and for $u \in W^{k,p}$, 1. $u^\varepsilon = \eta_\varepsilon * u$ is infinitely continuous (not necessarily compactly supported) on Ω_ε, and 2. $u^\varepsilon \to u$ in $W^{k,p}_{\mathrm{loc}}$	
Global Approximaton of Sobolev Functions by Smooth Functions - For open and bounded Ω and for finite p, infinitely smooth Sobolev functions are dense in Sobolev Space. $$\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega) \quad \text{is dense in} \quad W^{k,p}(\Omega)$$ with respect to the $W^{k,p}$ norm.	
Global Approximation of Sobolev Functions on the Closure of the Domain - For smooth, bounded, open subsets of $\mathbb{R}^n$, Sobolev functions can be approximated by infinitely smooth functions on the closure of the domain. $$\mathcal{C}^\infty(\overline{\Omega}) \quad \text{is dense in} \quad W^{k,p}(\Omega)$$ with respect to the $W^{k,p}$ norm.	

Sobolev Spaces	Notes
Morrey's Inequality - Sobolev functions on a ball have bounded differences. Denote $B_r \subset \mathbb{R}^n$ as a ball of radius r and let $n < p \leq \infty$. $\lvert u(x) - u(y) \rvert \leq C \lvert x - y \rvert^{1-\frac{n}{p}} \lVert Du \rVert_{L^p(B_r)}$	
Sobolev Embedding for $k = 1$ - The $\mathcal{C}^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ norm (Hölder Space norm) of a Sobolev function is bounded by a constant multiple (dependent on p and n) of the $W^{1,p}(\mathbb{R}^n)$ norm. $\lVert u \rVert_{\mathcal{C}^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C \lVert u \rVert_{W^{1,p}(\mathbb{R}^n)}$	
Sobolev Embedding for $kp > n$ - The Hölder Space norm of a Sobolev function is bounded by a constant multiple (dependent on k, p, and n) of the Sobolev norm. $\lVert u \rVert_{\mathcal{C}^{k-\left[\frac{n}{p}\right]-1,\gamma}(\mathbb{R}^n)} \leq C \lVert u \rVert_{W^{k,p}(\mathbb{R}^n)}$ where $\gamma = \begin{cases} \left[\dfrac{n}{p}\right] + 1 - \dfrac{n}{p} & \text{if } \dfrac{n}{p} \notin \mathbb{N}, \\ \text{any } \alpha \in \mathbb{R} \cap (0,1) & \text{if } \dfrac{n}{p} \in \mathbb{N}. \end{cases}$	
Almost-Everywhere Differentiability - For $n < p \leq \infty$, local Sobolev functions are almost-everywhere differentiable and its gradient and weak gradient agree almost everywhere.	
Gagliardo-Nirenberg-Sobolev Inequality - For $1 \leq p < n$, set $p^* = \dfrac{np}{n-p}$. Then the L^{p^*} norm of a Sobolev function is bounded by a constant multiple (dependent on n and p) of the L^p norm of its derivative. $\lVert u \rVert_{L^{p^*}(\mathbb{R}^n)} \leq C \lVert Du \rVert_{L^p(\mathbb{R}^n)}$	