201C, Spring '16, Thomases
Homework 5 due 5/10/16

1. (a) For $f \in L^1(\mathbb{R})$, set $S_R f(x) = (2\pi)^{-1/2} \int_{-R}^{R} \widehat{f}(\xi) e^{ix\xi} d\xi$. Show that

$$S_R f(x) = K_R * f(x) = \int_{-\infty}^{\infty} K_R(x-y) f(y) dy$$

   where

$$K_R(x) = (2\pi)^{-1} \int_{-R}^{R} e^{ix\xi} d\xi = \frac{\sin Rx}{\pi x}.$$

   (b) Show that if $f \in L^2(\mathbb{R})$, then $S_R f \to f$ in $L^2(\mathbb{R})$ as $R \to \infty$.

2. Show that for any $R \in (0, \infty)$, there exists $f \in L^1(\mathbb{R})$ such that $S_R f \notin L^1(\mathbb{R})$. Note that $K_R \notin L^1(\mathbb{R})$.

3. Assume $w \in \mathscr{S}'(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$ and $w(x) \geq 0$. Show that if $\widehat{w} \in L^\infty(\mathbb{R}^n)$ then $w \in L^1(\mathbb{R}^n)$ and

$$\|\widehat{w}\|_{L^\infty(\mathbb{R}^n)} = (2\pi)^{-n/2} \|w\|_{L^1(\mathbb{R}^n)}.$$

   Hint: Consider $w_j(x) = \psi\left(\frac{x}{j}\right) w(x)$ with $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and $\psi(0) = 1$. Use the fact that $w_j \to w$ in $\mathscr{S}'(\mathbb{R}^n)$.

4. Consider the Poisson equation on $\mathbb{R}: u_{xx} = f$.

   (a) Show that $\varphi(x) = \frac{x+|x|}{2}$ and $\phi(x) = \frac{|x|}{2}$ are both distributional solutions to $u_{xx} = \delta_0$.

   (b) Let f be continuous with compact support in $\mathbb{R}$. Show that

$$u(x) = \int_{\mathbb{R}} \varphi(x-y) f(y) dy$$

   and

$$v(x) = \int_{\mathbb{R}} \phi(x-y) f(y) dy$$

   both solve the Poisson equation $w_{xx}(x) = f(x)$ without relying upon distribution theory.

5. Let $T \in \mathscr{S}'(\mathbb{R}^n)$ and $f \in \mathscr{S}(\mathbb{R}^n)$. Show that the Leibniz rule for distributional derivatives holds:

$$\frac{\partial}{\partial x_i}(fT) = f\frac{\partial T}{\partial x_i} + \frac{\partial f}{\partial x_i} T$$

   in the sense of distributions.

6. Show that a function $f \in L^2(\mathbb{R}^n)$ is real if and only if

$$\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}.$$