UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #3

Sam Fleischer

April 19, 2016

Problem 1

Let $f \in L^1(\mathbb{R})$, and set

$$g(x) = \int_{-\infty}^{x} f(y)\mathrm{d}y.$$

Prove that g is continuous, and show that $\frac{\mathrm{d}g}{\mathrm{d}x} = f$, where $\frac{\mathrm{d}g}{\mathrm{d}x}$ denotes the weak derivative.
Hint: given $\phi \in C_C^\infty(\mathbb{R})$, use the definition of g to obtain

$$\int_{\mathbb{R}} \phi'(x)g(x)\mathrm{d}x = \int_{\mathbb{R}}\int_{-\infty}^{x} \phi'(x)f(y)\mathrm{d}y\mathrm{d}x.$$

Then write this integral as

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\left[\phi(x+h)-\phi(x)\right]g(x)\mathrm{d}x = -\lim_{h\to 0}\int_{\mathbb{R}}\int_{x}^{x+h} f(y)\phi(x)\mathrm{d}y\mathrm{d}x.$$

Proof. First we show g is continuous. Let $x_n \to x$. Then

$$\begin{aligned}
\lim_{x_n\to x}\left|g(x_n)-g(x)\right| &= \lim_{x_n\to x}\left|\int_{-\infty}^{x_n} f(y)\mathrm{d}y - \int_{-\infty}^{x} f(y)\mathrm{d}y\right| \\
&= \lim_{x_n\to x}\left|\int_{x}^{x_n} f(y)\mathrm{d}y\right| \\
&\le \lim_{x_n\to x}\int_{x}^{x_n}\left|f(y)\right|\mathrm{d}y \\
&= \begin{cases} \lim_{x_n\to x}\left\|f\mathscr{X}_{[x,x_n]}\right\|_1 & \text{, if } x_n > x \\ \lim_{x_n\to x}\left\|f\mathscr{X}_{[x_n,x]}\right\|_1 & \text{, else} \end{cases} \\
&\le \lim_{x_n\to x}\left\|f\right\|_1\left\|\mathscr{X}_{[x,x_n]}\right\|_\infty \qquad \text{without loss of generality} \\
&= \left\|f\right\|_1 \lim_{x_n\to x}\left\|\mathscr{X}_{x_n,x}\right\|_\infty \\
&= 0
\end{aligned}$$

Thus g is continuous. Next we show the weak derivative of g is f. Let ϕ be any test function. Then

$$\begin{aligned}
\int_{\mathbb{R}}\phi'(x)g(x)\mathrm{d}x &= \int_{\mathbb{R}}\phi'(x)\int_{-\infty}^{x} f(y)\mathrm{d}y\mathrm{d}y \\
&= \int_{\mathbb{R}}\int_{-\infty}^{x}\phi'(x)f(y)\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{y}^{\infty}\phi'(x)f(y)\mathrm{d}x\mathrm{d}y \qquad \text{by Fubini's Theorem} \\
&= \int_{\mathbb{R}}\left[\int_{y}^{\infty}\phi'(x)\mathrm{d}x\right]f(y)\mathrm{d}y \\
&= \int_{\mathbb{R}}\phi(y)f(y)\mathrm{d}y \\
&= \int_{\mathbb{R}}\phi(x)f(x)\mathrm{d}x
\end{aligned}$$

Thus f is the weak derivative of g. □

Problem 2

Show that $W^{n,1}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
Hint: $u(x) = \displaystyle\int_{-\infty}^0 \dots \int_{-\infty}^0 \frac{\partial^n}{\partial x_1 \dots \partial x_n} u(x+y)\mathrm{d}y_1 \dots \mathrm{d}y_n$.

Proof. For ease, let $\alpha_1 = (1,0,0,\dots,0)$, $\alpha_2 = (1,1,0,0,\dots,0)$, $\dots$, $\alpha_n = (1,1,1,\dots,1)$. By the hint.

$$\begin{aligned}
u(x) &= \int_{-\infty}^0 \dots \int_{-\infty}^0 \frac{\partial^n}{\partial x_1 \dots \partial x_n} u(x+y)\mathrm{d}y_1 \dots \mathrm{d}y_n \\
&= \int_{-\infty}^0 \int_{-\infty}^0 D^{\alpha_n} u(x+y)\mathrm{d}y_1 \dots \mathrm{d}y_n \\
&= \int_{-\infty}^{x_n} \dots \int_{-\infty}^{x_1} D^{\alpha_n} u(t)\mathrm{d}t_1 \dots \mathrm{d}t_n
\end{aligned}$$

by some change of variables. Thus,

$$\begin{aligned}
\sup|u(x)| &= \sup\left|\int_{-\infty}^{x_n} \dots \int_{-\infty}^{x_1} D^{\alpha_n} u(t)\mathrm{d}t_1 \dots \mathrm{d}t_n\right| \\
&= \sup \int_{-\infty}^{x_n} \dots \int_{-\infty}^{x_1} \left|D^{\alpha_n} u(t)\right|\mathrm{d}t_1 \dots \mathrm{d}t_n \\
&\leq \sup \int_{\mathbb{R}^n} \left|D^{\alpha_n} u(t)\right|\mathrm{d}t_1 \dots \mathrm{d}t_n \\
&= \left\|D^{\alpha_n} u(t)\right\|_\infty \\
&< \infty
\end{aligned}$$

since $|\alpha| = n$ and hence $D^{\alpha_n} u(t) \in L^1(\Omega)$. Thus u is bounded, i.e. $u \in L^\infty(\mathbb{R}^n)$. Next we show u is continuous. For ease, denote $g_i = D^{\alpha_i} u \in L^1$. Then

$$g_{i-1} = \int_\infty^{x_i} g_i(t)\mathrm{d}x_i$$

for $i = 2, \dots, n$ and $u = \int_\infty^{x_1} g_1(t)\mathrm{d}t_1$. Then by problem 1, g_{n-1} is continuous. But since $g_{n-1} \in L^1$, then g_{n-2} is continuous. Since n is finite, we can do this $n-2$ times to show g_1 is continuous. Again, since $g_1 \in L^1$, then u is continuous. Thus $u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. □

Problem 3

If $u \in L^1_{\text{loc}}(\mathbb{R})$ and if $\dfrac{\mathrm{d}u}{\mathrm{d}x} = f \in L^1(\mathbb{R})$, then

$$u(x) = C + \int_{-\infty}^x f(y)\mathrm{d}y, \qquad a.e.\ x \in \mathbb{R}$$

for some constant C.

Proof. First let $v(x) := C + \int_\infty^x f(y)\mathrm{d}y$. Then by problem 1, $\frac{\mathrm{d}v}{\mathrm{d}x} = f$. Then for all test functions ϕ,

$$\int_\mathbb{R} u(x)\phi'(x)\mathrm{d}x = -\int_\mathbb{R} f(x)\phi(x)\mathrm{d}x = \int_\mathbb{R} v(x)\phi'(x)\mathrm{d}x$$

Every test function is the derivative of some other test function, and so we can say that for all test functions ψ,

$$\int_{\mathbb{R}} u(x)\psi(x)\mathrm{d}x = \int_{\mathbb{R}} v(x)\phi(x)\mathrm{d}x$$

Since this holds for all test functions, it holds in particular for $\psi = \eta_\varepsilon$ for any $\varepsilon > 0$. Then

$$\int_{\mathbb{R}} u(x)\eta_\varepsilon(x-y)\mathrm{d}x = \int_{\mathbb{R}} v(x)\eta_\varepsilon(x-y)\mathrm{d}x \qquad \iff \qquad u^\varepsilon = v^\varepsilon$$

Since $u^\varepsilon \to u$ and $v^\varepsilon \to v$, and $\lim u^\varepsilon = \lim v^\varepsilon$, then $u = v$, i.e.

$$u(x) = C + \int_{-\infty}^{x} f(y)\mathrm{d}y, \qquad a.e.\ x \in \mathbb{R}$$

$\square$

Problem 4

Let $\Omega := B\left(0, \frac{1}{2}\right) \subset \mathbb{R}^2$ denote the open ball of radius $\frac{1}{2}$. For $x = (x_1, x_2) \in \Omega$, let

$$u(x_1, x_2) = x_1 x_2 \log\big(\big|\log(|x|)\big|\big) \text{ where } |x| = \sqrt{x_1^2 + x_2^2}.$$

(a) Show that $u \in C^1(\bar{\Omega})$.

(b) Show that $\dfrac{\partial^2 u}{\partial x_j^2} \in C(\bar{\Omega})$ for $j = 1, 2$ but $u \notin C^2(\bar{\Omega})$.

(c) Show that $u \in H^2(\Omega)$.

Proof. (a) First, we calculate the first partial derivatives:

$$\frac{\partial u}{\partial x_i} = \frac{x_i^2 x_j}{|x|^2 \big|\log(|x|)\big|} + x_j \log\big(\big|\log(|x|)\big|\big)$$

Note that as $|x| \to 0$, then by L'Hospital, each of the above terms $\to 0$. Thus $u \in C1(\bar{\Omega})$.

(b) Next, we calculate each non-mixed second partial derivative:

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} &= \frac{2x_i x_j |x|^2 \big|\log(|x|)\big|}{|x|^2 \big|\log(|x|)\big|^2} - \frac{x_i^2}{|x|^2 \big|\log(|x|)\big|^2} + \frac{2x_i^3 x_j \big|\log(|x|)\big|}{|x|^2 \big|\log(|x|)\big|^2} + x_i x_j \\ &= \frac{2x_i x_j}{\big|\log(|x|)\big|} - \frac{x_i^2}{|x|^2 \big|\log(|x|)\big|^2} + \frac{2x_i^3 x_j}{|x| \big|\log(|x|)\big|} + x_i x_j\end{aligned}$$

Similar to the first partials, each term $\to 0$ as $|x| \to 0$. Thus $\dfrac{\partial^2 u}{\partial x_i^2} \in C(\bar{\Omega})$ for $i = 1, 2$. However,

$$\frac{\partial^2 u}{\partial x_j \partial x_i} = \cancelto{0}{\frac{x_i^2}{\cancel{|x|}\big|\log(|x|)\big|}} + \cancelto{0}{\frac{x_i^2 x_j^2}{\cancel{|x|^4}\big|\log(|x|)\big|^2}} + \cancelto{0}{\frac{2x_i^2 x_j^2}{\cancel{|x|^4}\big|\log(|x|)\big|}} + \cancelto{0}{\frac{x_j^2}{\cancel{|x|}\big|\log(|x|)\big|}} + \cancelto{\infty}{\big|\log\big(\big|\log(|x|)\big|\big)\big|}$$

which diverges to ∞ as $|x| \to 0$. Thus $u \notin C^2(\bar{\Omega})$.

(c) Although $\dfrac{\partial^2 u}{\partial x_j \partial x_i}$ is not continuous, it is integrable, and thus there is a v such that for all test functions ϕ,

$$\int_\Omega u \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \mathrm{d}x = (-1)^2 \int_\Omega v\phi \mathrm{d}x + \cancelto{0}{\int_{\partial B_{\frac{1}{2}}(x)} \left[\frac{\partial u}{\partial x_1}\frac{\partial \phi}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial \phi}{\partial x_2} \right] \mathrm{d}s},$$

which shows the second weak derivatives of u exist for any α, which shows $u \in H^2(\Omega)$.

□

Problem 5

Prove that $C_C^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for integers $k \geq 0$ and $1 \leq p < \infty$.

Proof. First note that $C_C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$. Let $u \in W^{k,p}(\mathbb{R}^n)$. Then in particular, $\eta_\varepsilon * u \in C_C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and

$$\left\| \eta_\varepsilon * u - u \right\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \left\| D^\alpha(\eta_\varepsilon * u) - D^\alpha u \right\|_{L^p} \right)^{\frac{1}{p}} = \left(\sum_{|\alpha| \leq k} \left\| \eta_\varepsilon * D^\alpha u - D^\alpha u \right\|_{L^p} \right)^{\frac{1}{p}} \to 0$$

since each $D^\alpha u \in L^p$ and convolutions approximate functions. Thus $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Next we show $C_C^\infty(\mathbb{R}^n)$ is dense in $C_C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$. Suppose $f \in C_C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$. Define $\phi \in C_C^\infty(\mathbb{R}^n)$ as follows:

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Then for $R = 1, 2, \dots$, define ϕ_R as the dilation of ϕ.

$$\phi_R(x) = \phi\left(\frac{x}{R}\right).$$

Also define $f_R = \phi_R f$. Note $f_R \in C_C^\infty(\mathbb{R}^n)$ since $\phi_R \in C_C^\infty(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n)$. Then by the Liebnitz rule,

$$\partial^\alpha f_R = \partial^\alpha[\phi_R f] = \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \partial^\beta \phi_R \partial^{\alpha-\beta} f = \phi_R \partial^\alpha f + \sum_{\substack{|\beta| \leq |\alpha| \\ \beta \neq 0}} \binom{\alpha}{\beta} \partial^\beta \phi_R \partial^{\alpha - \beta} f$$

Note that any partial derivative of ϕ_R contains a factor of $\frac{1}{R}$ by the chain rule. Thus,

$$\partial^\alpha f_R = \phi_R \partial^\alpha f + \frac{1}{R} h_R$$

where h_R is a bounded function in L^p. It is also uniformly bounded in R, which implies

$$\left\| \frac{1}{R} h_R \right\|_p \leq \frac{1}{R} \|h_R\| \leq \frac{M}{R} \to 0 \text{ as } R \to \infty.$$

Thus, $\partial^\alpha f_R \to \partial^\alpha f$ in L^p for each $|\alpha| \leq k$, which implies $f_R \to f$ in $W^{k,p}(\mathbb{R}^n)$. Since each $f_R \in C_C^\infty(\mathbb{R}^n)$, then $C_C^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$. □

Problem 6

Let η_ε denote the standard mollifier, and for $u \in H^3(\mathbb{R}^3)$, set $u^\varepsilon = \eta_\varepsilon * u$. Prove that

$$\left\| u^\varepsilon - u \right\|_{L^\infty(\mathbb{R}^3)} \leq C\sqrt{\varepsilon}\|u\|_{H^2(\mathbb{R}^3)},$$

and that

$$\left\| u^\varepsilon - u \right\|_{L^\infty(\mathbb{R}^3)} \leq C\varepsilon\|u\|_{H^3(\mathbb{R}^3)}.$$

Proof. Let $u \in H^3(\mathbb{R}^3) = W^{3,2}(\mathbb{R}^3)$. Then since $3 < 2\cdot 3 = 6$, then by Morrey's Inequality for $W^{k,p}(\mathbb{R}^n)$, where $(k,p,n) = (3,2,3)$,

$$\|u\|_{C^{3-1-\left[\frac{3}{2}\right],1+\left[\frac{3}{2}\right]-\frac{3}{2}}(\mathbb{R}^3)} = \|u\|_{C^{1,\frac{1}{2}}(\mathbb{R}^n)} \leq C\|u\|_{W^{3,2}}\left(\mathbb{R}^3\right) = C\|u\|_{H^3(\mathbb{R}^3)}$$

Similary, since $H^3(\mathbb{R}^3) \subset H^2(\mathbb{R}^3)$, then since $3 < 2\cdot 2 = 4$, then

$$\|u\|_{C^{2-1-\left[\frac{3}{2}\right],1+\left[\frac{3}{2}\right]-\frac{3}{2}}(\mathbb{R}^3)} = \|u\|_{C^{0,\frac{1}{2}}(\mathbb{R}^n)} \leq C\|u\|_{W^{2,2}}\left(\mathbb{R}^3\right) = C\|u\|_{H^2(\mathbb{R}^3)}$$

Since $\|u\|_{C^{1,\frac{1}{2}}(\mathbb{R}^n)} < \infty$ and $\|u\|_{C^{0,\frac{1}{2}}(\mathbb{R}^n)} < \infty$, then u is continuous, and thus $\|u\|_{L^\infty} \leq \|u\|_{C^{i,\frac{1}{2}}(\mathbb{R}^3)}$ for $i = 0,1$. Thus $\|u^\varepsilon - u\|_{L^\infty} \leq C\|u^\varepsilon - u\|_{H^j(\mathbb{R}^3)}$ for $j = 2,3$. Thus

$$\left\| u^\varepsilon - u \right\|_{L^\infty(\mathbb{R}^3)} \leq C\sqrt{\varepsilon}\|u\|_{H^2(\mathbb{R}^3)} \qquad \text{and} \qquad \left\| u^\varepsilon - u \right\|_{L^\infty(\mathbb{R}^3)} \leq C\varepsilon\|u\|_{H^3(\mathbb{R}^3)}.$$

□

Problem 7

Let $D := B(0,1) \subset \mathbb{R}^2$ denote the unit disc, and let

$$u(x) = \left[-\log|x|\right]^\alpha.$$

Prove that the *weak derivative* of u exists for all $\alpha \geq 0$.

Proof. For any $0 < \delta < 1$, set $D_\delta = B(0,1) \setminus B(0,\delta)$. Then for $u(x) = \left[-\log|x|\right]^\alpha$, the strong derivative u' exists on D_δ, and thus the weak derivative of u exists on D_δ for all $\alpha \geq 0$. It remains to be shown that the boundary terms of the integral over the remaining δ-ball limit to 0 as $\delta \to 0$. □