201C, Spring '16, Thomases
Homework 3 due 4/19/16

1. If $u \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, and $u^\varepsilon = \eta_\varepsilon * u$, for η_ε the standard mollifier. Show that
$$u^\varepsilon \to u$$
in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$.

2. Let Ω denote an open and smooth subset of $\mathbb{R}^n$. Prove that $\mathcal{C}_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$.

3. Prove that if $u \in L^1_{loc}(\Omega)$ satisfies $\int_\Omega u(x)v(x)dx = 0$ for all $v \in \mathcal{C}_c^\infty(\Omega)$, then $u = 0$ a.e. in Ω.

4. Let $u \in L^\infty(\mathbb{R}^n)$ and let η_ε be a standard mollifier. For $\varepsilon > 0$ consider the sequence $\psi_\varepsilon \in L^\infty(\mathbb{R}^n)$ such that
$$\|\psi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le 1 \ \ \forall \ \ \varepsilon > 0 \ \ \text{and} \ \ \psi_\varepsilon \to \psi \text{ a.e. in } \mathbb{R}^n,$$
define
$$v^\varepsilon = \eta_\varepsilon * (\psi_\varepsilon u) \ \ \text{and} \ \ v = \psi u.$$
   (a) Prove that $v^\varepsilon \overset{*}{\rightharpoonup}$ in $L^\infty(\mathbb{R}^n)$.
   (b) Prove that $v^\varepsilon \to v$ in $L^1(B)$ for every ball $B \subseteq \mathbb{R}^n$.

5. For $u \in \mathcal{C}^0(\mathbb{R}^n;\mathbb{R})$, spt$(u)$ is the closure of the set $\{x \in \mathbb{R}^n \ : \ u(x) \neq 0\}$, but this definition may not make sense for functions $u \in L^p(\Omega)$. For example what is the support of $\mathbf{1}_\mathbb{Q}$, the indicator over the rationals?

   Let $u : \mathbb{R}^n \to \mathbb{R}$, and let $\{\Omega_\alpha\}_{\alpha \in A}$ denote the collection of all open sets on $\mathbb{R}^n$ such that for each $\alpha \in A$, $u = 0$ a.e. on Ω_α. Define $\Omega = \bigcup_{\alpha \in A} \Omega_\alpha$. Prove that $u = 0$ a.e. on Ω.

   The support of u, spt(u) is Ω^c, the complement of Ω. Notice that if $v = w$ a.e. on $\mathbb{R}^n$, then spt(v)=spt(w); furthermore, if $u \in \mathcal{C}^0(\mathbb{R}^n)$, then $\Omega^c = \overline{\{x \in \mathbb{R}^n | u(x) \neq 0\}}$. (Hint: Since A is not necessarily countable, it is not clear that $f = 0$ a.e. on Ω, so find a countable family U_n of open sets in $\mathbb{R}^n$ such that every open set on $\mathbb{R}^n$ is the union of some of the sets from $\{U_n\}$.)

6. Prove that if $f \in L^1(\mathbb{R}^n)$ and $v \in L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$, then
$$\text{spt}(u * v) \subseteq \overline{\text{spt}(u) + \text{spt}(v)}.$$

7. Suppose that $1 < p < \infty$. If $\tau_y f(x) = f(x - y)$, show that f belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $\tau_y f$ is a Lipschitz function of y with values in $L^p(\mathbb{R}^n)$; that is,
$$\|\tau_y f - \tau_z f\|_{L^p(\mathbb{R}^n)} \le C|y - z|.$$
What happens in the case $p = 1$?

8. If $u \in W^{1,p}(\mathbb{R}^n)$ for some $p \in [1, \infty)$ and $\frac{\partial u}{\partial x_j} = 0$, $j = 1, ..., n$, on a connected open set $\Omega \subseteq \mathbb{R}^n$, show that u is equal a.e. to a constant on Ω. (Hint: approximate u using that $\eta_\varepsilon * u \to u$ in $W^{1,p}(\mathbb{R}^n)$, where η_ε is a sequence of standard mollifiers. Show that $\frac{\partial}{\partial x_j}(\eta_\varepsilon * u) = 0$ on $\Omega_\varepsilon \subset\subset \Omega$ where $\Omega_\varepsilon \nearrow \Omega$ as $\varepsilon \to 0$.)

   More generally, if $\frac{\partial u}{\partial x_j} = f_j \in C(\Omega)$, $1 \leq j \leq n$, show that u is equal a.e. to a function in $\mathcal{C}^1(\Omega)$.