UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #2

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Problem 1

A function $f \in L^p(\mathbb{R}^n)$ is said to be L_p-*continuous* if $\tau_h f \to f$ in $L^p(\mathbb{R}^n)$ as $h \to 0$ in $\mathbb{R}^n$, where $\tau_h f(x) = f(x-h)$ is the translation of f by h. Prove that, if $1 \le p < \infty$, every $f \in L^p(\mathbb{R}^n)$ is L^p-continuous. Give a counter-example to show that this result is not true when $p = \infty$. [Hint: Approximate an L^p function by a C_c function.]

Proof. Define $f \in L^\infty(\mathbb{R})$ as $f(x) = \mathscr{X}_{[0,1]^n}$ where $\mathscr{X}$ is the characteristic function. Note that $f(1-\varepsilon) = 1$ for all $\varepsilon > 0$. Let h be a small perturbation, i.e. $0 < |h| \ll 1$, and choose $\varepsilon = \frac{h}{2}$. Then $\forall x \in (0, \varepsilon)$, $\tau_h f(x) = 0$ but $f(x) = 1$, and thus $\left|\tau_h f(x) - f(x)\right| = 1$. This shows that $\forall h > 0$, $\exists$ an interval I_h (of positive measure, $\mu(I_n) > 0$) such that $\left|\tau_h f(x) - f(x)\right| = 1$ for all $x \in I_h$. Thus $\tau_h f \not\to f$ in $L^\infty(\mathbb{R}^n)$. □

Problem 2

Show that $L^\infty(\mathbb{R})$ is not separable. [Hint: There is an uncountable set $\mathscr{F} \subset L^\infty$ such that $\left\|f - g\right\|_\infty \ge 1$ for all $f, g \in \mathscr{F}$ with $f \ne g$.]

Proof. Let $\mathscr{F} = \{\mathscr{X}_{[0,\alpha]} \,:\, 0 < \alpha \in \mathbb{R}\}$. $\mathscr{F}$ is clearly uncountable. Consider any two $f, g \in \mathscr{F}$. Then, without loss of generality, $f = \mathscr{X}_{[0,\alpha]}$ and $g = \mathscr{X}_{[0,\beta]}$ where $\alpha < \beta$. Also,

$$\left\|f - g\right\|_\infty = \text{ess sup}\{\mathscr{X}_{(\alpha,\beta]}\} = 1$$

Thus the ball around any $f \in \mathscr{F}$ of radius $\frac{1}{2}$, i.e. $B\left(f, \frac{1}{2}\right)$, contains no other elements of $\mathscr{F}$. Thus $L^\infty(\mathbb{R})$ is not separable since $\mathscr{F}$ is uncountable and not dense. □

Problem 3

Prove Chebyshev's Inequality: If $f \in L^p$ $(1 \le p < \infty)$, then for any $\alpha > 0$,

$$\mu\left(\left\{x \,:\, \left|f(x)\right| > \alpha\right\}\right) \le \left(\frac{\left\|f\right\|_p}{\alpha}\right)^p.$$

[Note that you can find the proof of this simple fact in many texts but you should see if you can figure it out yourself. Also, note that this inequality holds for all $0 < p < \infty$.]

Proof. Let $A_\alpha = \left\{x \,:\, \left|f(x)\right| > \alpha\right\} = \left\{x \,:\, \left|\frac{f(x)}{\alpha}\right| > 1\right\} = \left\{x \,:\, \left|\frac{f(x)}{\alpha}\right|^p > 1\right\}$ for all $p \ge 1$. Then

$$\left(\frac{\left\|f\right\|_p}{\alpha}\right)^p = \int_\Omega \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu = \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu + \int_{\Omega\setminus A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu \ge \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu \ge \int_{A_\alpha} 1 \mathrm{d}\mu = \mu(A_\alpha)$$

which proves the result. □

Problem 4

Assume that $f, g \in L^1(\mathbb{R}^n)$. Prove that the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \mathrm{d}y$$

is measurable and in $L^1(\mathbb{R}^n)$.

Proof. First note that $\|f\|_1\|g\|_1 < \infty$ since they are in $L^1(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f\|_1\|g\|_1 &= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \|f\|_1 |g(y)|\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|\mathrm{d}x|g(y)|\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)||g(y)|\mathrm{d}x\mathrm{d}y \\
\implies \int_{\mathbb{R}^{2n}} |f(x-y)g(y)|\mathrm{d}x\mathrm{d}y < \infty
\end{aligned}$$

Thus, by Fubini's Theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)g(y)|\mathrm{d}y\mathrm{d}x < \infty \\
\implies \|f*g\|_1 = \int_{\mathbb{R}^n} |(f*g)(x)|\mathrm{d}x = \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y\right|\mathrm{d}x \leq &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)g(y)|\mathrm{d}y\mathrm{d}x < \infty
\end{aligned}$$

which shows $(f*g) \in L^1(\mathbb{R}^n)$. Since all L^1 functions are measurable, $(f*g)$ is measurable. □

Problem 5

Let $f_n = \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}$. Prove that f_n converges weakly to 0 in $L^2(0,1)$ and $f_n \to 0$ in $L^1(0,1)$ but f_n does not converge strongly in $L^2(0,1)$.

Proof.

$$\|f_n\|_2^2 = \int_0^1 n\mathscr{X}_{(0,\frac{1}{n})}\mathrm{d}x = \int_0^{\frac{1}{n}} n\mathrm{d}x = 1$$

Thus $\|f_n\|_2 = 1$ for all n, and thus does not converge strongly to 0 in $L^2(0,1)$.

$$\|f_n\|_1 = \int_0^1 \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}\mathrm{d}x = \int_0^{\frac{1}{n}} \sqrt{n}\mathrm{d}x = \frac{1}{\sqrt{n}}$$

Thus $\|f_n\|_1 \to 0$ as $n \to 0$, which shows $f_n \to 0$ strongly in $L^1(0,1)$. Let $L \in L^2(0,1)^* \cong L^2(0,1)$. Thus $\exists \ell \in L^2(0,1)$ such that

$$L(f) = \int_0^1 \ell(x)f(x)\mathrm{d}x$$

for all $f \in L^2$. Then

$$L(f_n) = \int_0^1 \ell(x)\sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}\mathrm{d}x = \int_0^{\frac{1}{n}} \ell(x)\sqrt{n}\mathrm{d}x \leq \left(\int_0^{\frac{1}{n}} |\ell(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^{\frac{1}{n}} n\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_0^{\frac{1}{n}} |\ell(x)|^2\mathrm{d}x\right) \to 0$$

since ℓ is fixed and $\mu\big((0,\frac{1}{n})\big) \to 0$ as $n \to 0$. Thus $f_n \rightharpoonup 0$ in $L^2(0,1)$. □

Problem 6

Find a sequence of functions with the property that f_j converges to 0 in $L^2(\Omega)$ weakly, to 0 in $L^{\frac{3}{2}}(\Omega)$ strongly, but it does not converge to 0 strongly in $L^2(\Omega)$.

Proof. Let $f_n = \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}$. Then by number 5, $f_n \not\to 0$ in $L^2(0,1)$ but $f_n \rightharpoonup 0$ in $L^2(0,1)$. Also,

$$\|f_n\|_{\frac{3}{2}}^{\frac{3}{2}} = \int_0^1 \left|n^{\frac{1}{2}}\mathscr{X}_{(0,\frac{1}{n})}\right|^{\frac{3}{2}} \mathrm{d}x = \int_0^{\frac{1}{n}} n^{\frac{3}{4}} \mathrm{d}x = n^{-\frac{1}{4}} \to 0 \text{ as } n \to \infty$$

Thus $f_n \to 0$ in $L^{\frac{3}{2}}(0,1)$. □

Problem 7

Let f_n and g_n denote two sequences in $L^p(\Omega)$ with $1 \le p \le \infty$ such that $f_n \to f$ in $L^p(\Omega)$, and $g_n \to g$ in $L^p(\Omega)$. Set $h_n = \max\{f_n, g_n\}$ and prove that $h_n \to h$ in $L^p(\Omega)$.

Proof. First note that

$$\begin{aligned} h_n(x) &= \frac{1}{2}\big(f_n(x)+g_n(x)\big) + \frac{1}{2}\big|f_n(x)-g_n(x)\big|, \qquad \text{and} \\ h(x) &= \frac{1}{2}\big(f(x)+g(x)\big) + \frac{1}{2}\big|f(x)-g(x)\big|. \end{aligned}$$

Then

$$\begin{aligned} \|h_n - h\|_p &= \left\|\frac{1}{2}(f_n+g_n) + \frac{1}{2}|f_n-g_n| - \left[\frac{1}{2}(f+g)+\frac{1}{2}|f-g|\right]\right\| \\ &\le \frac{1}{2}\left[\|f_n-f\|_p + \|g_n-g\|_p + \big\||f_n-g_n|-|f-g|\big\|_p\right] \\ &\le \frac{1}{2}\left[\|f_n-f\|_p + \|g_n-g\|_p + \|(f_n-g_n)-(f-g)\|_p\right] \\ &\le \frac{1}{2}\left[\|f_n-f\|_p + \|g_n-g\|_p + \|f_n-f\|_p + \|g_n-g\|_p\right] \\ &= \|f_n-f\|_p + \|g_n+g\|_p \end{aligned}$$

But since $f_n \to f$ and $g_n \to g$ in L^p, then there is an N such that $\|f_n - f\|_p < \frac{\varepsilon}{2}$ and $\|g_n - g\|_p < \frac{\varepsilon}{2}$ for all $n \ge N$. Thus if $n \ge N$,

$$\|h_n - h\|_p \le \|f_n - f\|_p + \|g_n + g\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since ε is arbitrary, $h_n \to h$ in L^p. □

Problem 8

Let f_n be a sequence in $L^p(\Omega)$ with $1 \le p < \infty$, and let g_n be a bounded sequence in $L^\infty(\Omega)$. Suppose that $f_n \to f$ in $L^p(\Omega)$ and that $g_n \to g$ pointwise a.e. Prove that $f_n g_n \to fg$ in $L^p(\Omega)$.

Proof.

$$\begin{aligned}\left\|f_n g_n - fg\right\|_p &\leq \left\|f_n g_n - f g_n\right\|_p + \left\|f g_n - fg\right\|_p \\ &= \left[\int_\Omega \left|g_n\right|^p \left|f_n - f\right|^p\right]^{\frac{1}{p}} + \left[\int_\Omega \left|f g_n - fg\right|^p\right]^{\frac{1}{p}}\end{aligned}$$

Since g_n is a bounded sequence in L^∞, $\exists M$ such that $\left|g_n(x)\right| \leq \left\|g_n\right\| \leq M$ almost everywhere. Thus

$$\left[\int_\Omega \left|g_n\right|^p \left|f_n - f\right|^p\right]^{\frac{1}{p}} \leq \left[\int_\Omega M^p \left|f_n - f\right|^p\right]^{\frac{1}{p}} = M\left\|f_n - f\right\|_p \to 0$$

since $f_n \to f$ in L^p. Next note $g \in L^\infty$ since

$$\left\|g\right\|_\infty = \text{ess sup}\{g(x)\} = \text{ess sup}\left\{\lim_n g_n(x)\right\} \leq M$$

Since $g_n \to g$ pointwise, then $f g_n \to fg$ pointwise and thus $\left|f g_n - fg\right|^p \to 0$ pointwise. Define h_n as

$$h_n = \left|f g_n - fg\right|^p.$$

Then $h_n \in L^1(\Omega)$ since

$$\|h_n\|_1^p = \left\|f g_n - fg\right\|_p \leq \left\|f g_n\right\|_p + \left\|fg\right\|_p \leq \left\|f\right\|_p \left\|g_n\right\|_\infty + \left\|f\right\|_p \left\|g\right\|_\infty \leq 2\left\|f\right\|_p M$$

Also, $|h_n(x)|^{\frac{1}{p}} = \left|(f g_n - fg)(x)\right| \leq \left|f(x) g_n(x) - f(x) g(x)\right| \leq \left|f(x)\right|\left|g_n(x)\right| + \left|f(x)\right|\left|g(x)\right| \leq 2M\left|f(x)\right|$, which implies h is dominated:

$$|h(x)| \leq 2^p M^p \left|f(x)\right|^p$$

Thus, by the dominated convergence theorem,

$$\begin{aligned}\lim_n \int_\Omega h_n = \int_\Omega \lim_n h_n = \int_\Omega 0 = 0 \\ \implies \lim_n \int_\Omega \left|f g_n - fg\right|^p = 0\end{aligned}$$

Thus,

$$\lim_n \left\|f_n g_n - fg\right\|_p = \lim_n \left[\int_\Omega \left|g_n\right|^p \left|f_n - f\right|^p\right]^{\frac{1}{p}} + \lim_n \left[\int_\Omega \left|f g_n - fg\right|^p\right]^{\frac{1}{p}} = 0$$

which shows $f_n g_n \to fg$ in $L^p(\Omega)$. □

Problem 9

Prove that the space of continuous functions with compat support $\mathscr{C}_c^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Proof. □