UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #7

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April 5, 2016

Problem 1

If f and g are measurable functions on Ω, then $\|fg\|_1 \le \|f\|_1\|g\|_\infty$. If $f \in L^1$ and $g \in L^\infty$, then $\|fg\|_1 = \|f\|_1\|g\|_\infty$ if and only if $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \ne 0$.

Proof. Let f and g be measurable functions on Ω. Then

$$\begin{aligned}
\|fg\|_1 &= \int_\Omega |(fg)(x)|\mathrm{d}\mu \\
&= \int_\Omega |f(x)|\,|g(x)|\mathrm{d}\mu \\
&\le \int_\Omega |f(x)| \operatorname*{ess\,sup}_{x\in\Omega} |g(x)|\mathrm{d}\mu \\
&= \operatorname*{ess\,sup}_{x\in\Omega} |g(x)| \int_\Omega |f(x)|\mathrm{d}\mu \\
&= \|f\|_1\|g\|_\infty
\end{aligned}$$

Now let $f \in L^1$ and $g \in L^\infty$. First, suppose $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \ne 0$. In other words, define $A \subset \Omega$ by

$$A = \{x \in \Omega \,:\, f(x) \neq 0\}$$

and assume $|g(x)| = \|g\|_\infty$ for almost all $x \in A$. Again, in other words, define $B \subset A$ by

$$B = \{x \in A \,:\, |g(x)| < \|g\|_\infty\}$$

and assume $\mu(B) = 0$. Then

$$\begin{aligned}
\|fg\|_1 &= \int_\Omega |(fg)(x)|\mathrm{d}\mu \\
&= \int_A |(fg)(x)|\mathrm{d}\mu + \cancelto{0}{\int_{\Omega\setminus A} |(fg)(x)|\mathrm{d}\mu}
\end{aligned}$$

since $f(x) = 0$ for $x \in \Omega \setminus A$ by definition of A. Thus

$$\begin{aligned}
\|fg\|_1 &= \int_A |(fg)(x)|\mathrm{d}\mu \\
&= \cancelto{0}{\int_B |(fg)(x)|\mathrm{d}\mu} + \int_{A\setminus B} |(fg)(x)|\mathrm{d}\mu
\end{aligned}$$

since $\mu(B) = 0$. For $x \in A \setminus B$, $|g(x)| = \|g\|_\infty$. Thus,

$$\begin{aligned}
\|fg\|_1 &= \int_{A\setminus B} |(fg)(x)|\mathrm{d}\mu \\
&= \int_{A\setminus B} |f(x)||g(x)|\mathrm{d}\mu \\
&= \int_{A\setminus B} |f(x)|\|g\|_\infty\mathrm{d}\mu \\
&= \|g\|_\infty \int_{A\setminus B} |f(x)|\mathrm{d}\mu \\
&= \|g\|_\infty \left[\int_{A\setminus B} |f(x)|\mathrm{d}\mu + \int_B |f(x)|\mathrm{d}\mu + \int_{\Omega\setminus A} |f(x)|\mathrm{d}\mu\right]
\end{aligned}$$

since $\mu(B) = 0$ and $f(x) = 0$ for $x \in \Omega \setminus A$ implies

$$\int_B |f(x)| \mathrm{d}\mu = 0 \qquad \text{and} \qquad \int_{\Omega \setminus A} |f(x)| \mathrm{d}\mu = 0$$

Thus,

$$\begin{aligned}
\|fg\|_1 &= \|g\|_\infty \left[\int_{A \setminus B} |f(x)| \mathrm{d}\mu + \int_B |f(x)| \mathrm{d}\mu + \int_{\Omega \setminus A} |f(x)| \mathrm{d}\mu \right] \\
&= \|g\|_\infty \int_\Omega |f(x)| \mathrm{d}\mu \\
&= \|f\|_1 \|g\|_\infty
\end{aligned}$$

Second, suppose $B \subset A$ (as defined above) has positive measure. Then

$$\int_B |(fg)(x)| \mathrm{d}\mu = \int_B |f(x)||g(x)| \mathrm{d}\mu < \int_B |f(x)| \|g\|_\infty \mathrm{d}\mu$$

Thus,

$$\begin{aligned}
\|fg\|_1 &= \int_\Omega |(fg)(x)| \mathrm{d}\mu \\
&= \int_B |(fg)(x)| \mathrm{d}\mu + \int_{A \setminus B} |(fg)(x)| \mathrm{d}\mu + \cancelto{0}{\int_{\Omega \setminus A} |(fg)(x)| \mathrm{d}\mu} \\
&< \int_B |f(x)| \|g\|_\infty \mathrm{d}\mu + \int_{A \setminus B} |f(x)| \|g\|_\infty \mathrm{d}\mu \\
&= \|g\|_\infty \int_A |f(x)| \mathrm{d}\mu \\
&= \|g\|_\infty \int_\Omega |f(x)| \mathrm{d}\mu \\
&= \|f\|_1 \|g\|_\infty
\end{aligned}$$

□

Problem 2

$\|f_n - f\|_\infty \to 0$ if and only if there exists a measurable set E such that $\mu(E^C) = 0$ and $f_n \to f$ uniformly on E.

Proof. Assume $\|f_n - f\|_\infty \to 0$. For each n, define K_n by

$$K_n = \inf_K \left\{ |f_n(x) - f(x)| \le K \text{ for almost all } x \in \Omega \right\}$$

Then define E^C by

$$E^C = \left\{ x \in \Omega \, : \, |f_n(x) - f(x)| > K_n \right\}$$

Then $\mu(E^C) = 0$. Also,

$$\|f_n - f\|_{\sup} = \sup_{x \in E} |f_n(x) - f(x)| = K_n \to 0$$

Now assume $f_n \to f$ uniformly on E and $\mu(E^C) = 0$. Then

$$\|f_n - f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f_n(x) - f(x)| = \sup_{x \in E} |f_n(x) - f(x)| \to 0$$

□

Problem 3

We say $\{f_n\}$ *converges in measure* to f if for every $\varepsilon > 0$,

$$\mu\big(\{x : \left|f_n(x) - f(x)\right| \geq \varepsilon\}\big) \to 0 \text{ as } n \to \infty.$$

If $\left\|f_n - f\right\|_p \to 0$ ($p < \infty$) then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand if $f_n \to f$ in measure and $\left|f_n\right| \leq g \in L^p$ for all n ($p < \infty$) then $\left\|f_n - f\right\|_p \to 0$.

Proof. □

Problem 4

If $f_n, f \in L^p$ ($p < \infty$) and $f_n \to f$ point-wise a.e., then $\left\|f_n - f\right\|_p \to 0$ if and only if $\left\|f_n\right\|_p \to \left\|f\right\|_p$.

Proof. Theorem 1.28 is going left □

Problem 5

Suppose $0 < p < q \leq \infty$. Then $L^p \not\subset L^q$ if and only if Ω contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ if and only if Ω contains sets of arbitrarily large finite measure. [Hint: for the "if" implication: in the first case there is a disjoint sequence $\{E_n\}$ with $o < \mu(E_n) \leq 2^{-n}$, and in the second case there is a disjoint sequence $\{E_n\}$ with $1 \leq \mu(E_n) < \infty$. Consider $f = \sum a_n \mathscr{X}_{E_n}$ for suitable constants a_n.]

Proof. □

Problem 6

If $f \in L^\infty(\Omega) \cap L^q(\Omega)$ for some q then $f \in L^p(\Omega)$ for all $p > q$ and

$$\left\|f\right\|_\infty = \lim_{p \to \infty} \left\|f\right\|_p.$$

Proof. Let $p > q$. Then

$$\begin{aligned}
\left\|f\right\|_p^p &= \int_\Omega |f|^p \mathrm{d}\mu \\
&= \int_\Omega |f|^{p-q}|f|^q \mathrm{d}\mu \\
&\leq \int_\Omega \left\|f\right\|_\infty^{p-q}|f|^q \mathrm{d}\mu \\
&= \left\|f\right\|_\infty^{p-q} \int_\Omega |f|^q \mathrm{d}\mu \\
&= \left\|f\right\|_\infty^{p-q}\left\|f\right\|_q^q \\
&< \infty
\end{aligned}$$

since $p-q>0$, $\|f\|_\infty<\infty$, and $\|f\|_q<\infty$. Thus $f\in L^p(\Omega)$. Next we show $\|f\|_\infty=\lim_{p\to\infty}\|f\|_p$. By the above calculation,

$$\begin{aligned}\lim_{p\to\infty}\|f\|_p &\le \lim_{p\to\infty}\left[\|f\|_\infty^{\frac{p-q}{p}}\|f\|_q^{\frac{q}{p}}\right]\\ &= \lim_{p\to\infty}\|f\|_\infty^{\frac{p-q}{p}}\cdot\lim_{p\to\infty}\|f\|_q^{\frac{q}{p}}\\ &= \|f\|_\infty\end{aligned}$$

since as $p\to\infty$, $\frac{p-q}{p}\to 1$ and $\frac{q}{p}\to 0$. Also, the definition of $\|\cdot\|_\infty$ implies that for any ε, $\mu(E_\varepsilon)>0$ where

$$E_\varepsilon=\left\{x\,:\,|f(x)|\ge\|f\|_\infty-\varepsilon\right\}.$$

but $\mu(E_\varepsilon)\to 0$ and $\varepsilon\to 0$. Thus,

$$\begin{aligned}\|f\|_p^p &= \int_\Omega |f|^p\mathrm{d}\mu\\ &= \int_{\Omega\setminus E_\varepsilon}|f|^p\mathrm{d}\mu+\int_{E_\varepsilon}|f|^p\mathrm{d}\mu\\ &\ge \int_{E_\varepsilon}|f|^p\mathrm{d}\mu\\ &\ge \int_{E_\varepsilon}\big|\|f\|_\infty-\varepsilon\big|^p\mathrm{d}\mu\\ &= \mu(E_\varepsilon)\big|\|f\|_\infty-\varepsilon\big|^p\\ \implies \lim_{p\to\infty}\|f\|_p &= \lim_{p\to\infty}\left[\mu(E_\varepsilon)^{\frac{1}{p}}\big|\|f\|_\infty-\varepsilon\big|\right]\\ &= \big|\|f\|_\infty-\varepsilon\big|\end{aligned}$$

Since ε is arbitrarily small, we find $\|f\|_\infty\le\lim_{p\to\infty}\|f\|_p$. Thus,

$$\|f\|_\infty=\lim_{p\to\infty}\|f\|_p$$

□

Problem 7

Prove that when $\infty\ge r\ge q\ge 1$, $f\in L^r(\Omega)\cap L^q(\Omega)\implies f\in L^p(\Omega)$ for all $r\ge p\ge q$.

Proof. Let $f\in L^r(\Omega)\cap L^q(\Omega)$. For $p\in[r,q]$, by convexity of $\mathbb{R}$, $\exists a\in[0,1]$ such that

$$\frac{1}{p}=\frac{a}{r}+\frac{1-a}{q}$$

Then

$$\begin{aligned}\|f\|_p^p &= \int_\Omega|f|^p\mathrm{d}\mu\\ &= \int_\Omega|f|^{pa}|f|^{p(1-a)}\mathrm{d}\mu\\ &\le \left(\int_\Omega|f|^{(pa)\left(\frac{r}{pa}\right)}\mathrm{d}\mu\right)^{\frac{pa}{r}}\left(\int_\Omega|f|^{(p(1-a))\left(\frac{q}{p(1-a)}\right)}\mathrm{d}\mu\right)^{\frac{p(1-a)}{q}} \qquad \text{by Hölder's Inequality}\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_\Omega |f|^r\right)^{\frac{pa}{r}} \left(\int_\Omega |f|^q\right)^{\frac{p(1-a)}{q}} \\
&= \|f\|_r^{pa} \|f\|_q^{p(1-a)} \\
\implies \|f\|_p &\le \|f\|_r^a \|f\|_q^{1-a} < \infty \\
\implies f &\in L^p(\Omega)
\end{aligned}
$$

□

Problem 8

Prove that a strongly convergent sequence in $L^p(\mathbb{R}^n)$ is also a Cauchy sequence.

Proof. Let $\{f_n\}_n$ be a strongly convergent sequence in $L^p(\mathbb{R}^n)$ and let $\epsilon > 0$. Then there is some N such that $\|f_N - f\| < \frac{\varepsilon}{2}^{\frac{1}{p}}$. Then for all $m, n \ge N$,

$$\|f_n - f_m\|_p^p \le \|f_n - f\|_p^p + \|f_m - f\|_p^p$$

since $|a+b|^p \le |a|^p + |b|^p$ for all $a, b \in \mathbb{C}$ and $p \in (0, \infty]$. Then

$$\|f_n - f_m\|_p^p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $\{f_n\}_n$ is Cauchy. □

Problem 9

Give three different examples of ways for a sequence $f_k \in L^p(\mathbb{R}^n)$ to converge weakly to zero, but not strongly to anything. Verify your claims for these exmples.

Proof. □