UC DAVIS, ANALYSIS (MAT201C), SPRING 2016

Homework #1

Sam Fleischer

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Problem 1

If f and g are measurable functions on Ω, then $\|fg\|_1 \le \|f\|_1\|g\|_\infty$. If $f \in L^1$ and $g \in L^\infty$, then $\|fg\|_1 = \|f\|_1\|g\|_\infty$ if and only if $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$.

Proof. Let f and g be measurable functions on Ω. Then

$$\begin{aligned}\|fg\|_1 &= \int_\Omega |(fg)(x)|\mathrm{d}\mu \\ &= \int_\Omega |f(x)|\,|g(x)|\mathrm{d}\mu \\ &\le \int_\Omega |f(x)| \operatorname*{ess\,sup}_{x\in\Omega}|g(x)|\mathrm{d}\mu \\ &= \operatorname*{ess\,sup}_{x\in\Omega}|g(x)|\int_\Omega |f(x)|\mathrm{d}\mu \\ &= \|f\|_1\|g\|_\infty\end{aligned}$$

Now let $f \in L^1$ and $g \in L^\infty$. First, suppose $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$. In other words, define $A \subset \Omega$ by

$$A = \{x \in \Omega \,:\, f(x) \neq 0\}$$

and assume $|g(x)| = \|g\|_\infty$ for almost all $x \in A$. Again, in other words, define $B \subset A$ by

$$B = \{x \in A \,:\, |g(x)| < \|g\|_\infty\}$$

and assume $\mu(B) = 0$. Then

$$\begin{aligned}\|fg\|_1 &= \int_\Omega |(fg)(x)|\mathrm{d}\mu \\ &= \int_A |(fg)(x)|\mathrm{d}\mu + \cancelto{0}{\int_{\Omega\setminus A}|(fg)(x)|\mathrm{d}\mu}\end{aligned}$$

since $f(x) = 0$ for $x \in \Omega \setminus A$ by definition of A. Thus

$$\begin{aligned}\|fg\|_1 &= \int_A |(fg)(x)|\mathrm{d}\mu \\ &= \cancelto{0}{\int_B |(fg)(x)|\mathrm{d}\mu} + \int_{A\setminus B}|(fg)(x)|\mathrm{d}\mu\end{aligned}$$

since $\mu(B) = 0$. For $x \in A \setminus B$, $|g(x)| = \|g\|_\infty$. Thus,

$$\begin{aligned}\|fg\|_1 &= \int_{A\setminus B}|(fg)(x)|\mathrm{d}\mu \\ &= \int_{A\setminus B}|f(x)||g(x)|\mathrm{d}\mu \\ &= \int_{A\setminus B}|f(x)|\|g\|_\infty\mathrm{d}\mu \\ &= \|g\|_\infty\int_{A\setminus B}|f(x)|\mathrm{d}\mu \\ &= \|g\|_\infty\left[\int_{A\setminus B}|f(x)|\mathrm{d}\mu + \int_B|f(x)|\mathrm{d}\mu + \int_{\Omega\setminus A}|f(x)|\mathrm{d}\mu\right]\end{aligned}$$

since $\mu(B) = 0$ and $f(x) = 0$ for $x \in \Omega \setminus A$ implies

$$\int_B |f(x)| \mathrm{d}\mu = 0 \qquad \text{and} \qquad \int_{\Omega \setminus A} |f(x)| \mathrm{d}\mu = 0$$

Thus,

$$\begin{aligned}
\|fg\|_1 &= \|g\|_\infty \left[\int_{A \setminus B} |f(x)| \mathrm{d}\mu + \int_B |f(x)| \mathrm{d}\mu + \int_{\Omega \setminus A} |f(x)| \mathrm{d}\mu \right] \\
&= \|g\|_\infty \int_\Omega |f(x)| \mathrm{d}\mu \\
&= \|f\|_1 \|g\|_\infty
\end{aligned}$$

Second, suppose $B \subset A$ (as defined above) has positive measure. Then

$$\int_B |(fg)(x)| \mathrm{d}\mu = \int_B |f(x)| |g(x)| \mathrm{d}\mu < \int_B |f(x)| \|g\|_\infty \mathrm{d}\mu$$

Thus,

$$\begin{aligned}
\|fg\|_1 &= \int_\Omega |(fg)(x)| \mathrm{d}\mu \\
&= \int_B |(fg)(x)| \mathrm{d}\mu + \int_{A \setminus B} |(fg)(x)| \mathrm{d}\mu + \cancelto{0}{\int_{\Omega \setminus A} |(fg)(x)| \mathrm{d}\mu} \\
&< \int_B |f(x)| \|g\|_\infty \mathrm{d}\mu + \int_{A \setminus B} |f(x)| \|g\|_\infty \mathrm{d}\mu \\
&= \|g\|_\infty \int_A |f(x)| \mathrm{d}\mu \\
&= \|g\|_\infty \int_\Omega |f(x)| \mathrm{d}\mu \\
&= \|f\|_1 \|g\|_\infty
\end{aligned}$$

□

Problem 2

$\|f_n - f\|_\infty \to 0$ if and only if there exists a measurable set E such that $\mu(E^C) = 0$ and $f_n \to f$ uniformly on E.

Proof. Assume $\|f_n - f\|_\infty \to 0$. For each n, define K_n by

$$K_n = \inf_K \left\{ |f_n(x) - f(x)| \leq K \text{ for almost all } x \in \Omega \right\}$$

Then define E^C by

$$E^C = \left\{ x \in \Omega \,:\, |f_n(x) - f(x)| > K_n \right\}$$

Then $\mu(E^C) = 0$. Also,

$$\|f_n - f\|_{\sup} = \sup_{x \in E} |f_n(x) - f(x)| = K_n \to 0$$

Now assume $f_n \to f$ uniformly on E and $\mu(E^C) = 0$. Then

$$\|f_n - f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f_n(x) - f(x)| = \sup_{x \in E} |f_n(x) - f(x)| \to 0$$

□

Problem 3

We say $\{f_n\}$ *converges in measure* to f if for every $\varepsilon > 0$,

$$\mu\big(\{x : |f_n(x) - f(x)| \geq \varepsilon\}\big) \to 0 \text{ as } n \to \infty.$$

If $\|f_n - f\|_p \to 0$ ($p < \infty$) then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n ($p < \infty$) then $\|f_n - f\|_p \to 0$.

Proof. Suppose $\|f_n - f\|_p \to 0$. Then $\int_\Omega |f_n - f|^p \mathrm{d}\mu \to 0$. Choose $\varepsilon > 0$ and define $A_{n,\varepsilon}$ as

$$A_{n,\varepsilon} = \left\{x : |f_n(x) - f(x)| \geq \varepsilon\right\}.$$

Then

$$0 \leftarrow \int_\Omega |f_n - f| \mathrm{d}\mu = \int_{A_{n,\varepsilon}} |f_n - f| \mathrm{d}\mu + \int_{\Omega \setminus A_{n,\varepsilon}} |f_n - f| \mathrm{d}\mu$$

Since each integrand is positive, each integral is positive, and thus

$$\int_{A_{n,\varepsilon}} |f_n - f| \mathrm{d}\mu \to 0 \qquad \text{and} \qquad \int_{\Omega \setminus A_{n\varepsilon}} |f_n - f| \mathrm{d}\mu \to 0$$

But since $|f_n(x) - f(x)| \geq \varepsilon$ for all $x \in A_{n,\varepsilon}$, then the only way for $\int_{A_{n,\varepsilon}} |f_n - f| \mathrm{d}\mu$ to converge to 0 is for $\mu(A_{n,\varepsilon}) \to 0$ as $n \to \infty$. Thus f_n converges to f in measure.
Next we show a subsequence of $\{f_n\}$ converges to f pointwise a.e. Consider $\varepsilon_k \to 0$. Then $\exists n_k$ such that $\forall n \geq n_k$, $\mu(A_{n,\varepsilon_k}) < 2^{-k}$. Define $A_k = A_{n_k,\varepsilon_k}$ and note $\mu(A_k) < 2^{-k}$. Then define B_m by

$$B_m = \bigcup_{k=m}^{\infty} A_k \qquad \text{and note} \qquad \mu(B_m) \leq \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}.$$

Finally, Define $B = \bigcap_{m=1}^\infty B_m$ and note $\mu(B) \leq \mu(B_m) \leq 2^{-m+1}$ for any integer m. Since $2^{-m+1} \to 0$ as $m \to \infty$, this shows $\mu(B)$ is arbitrarily small, i.e. $\mu(B) = 0$. Finally, choose $x \notin B$. Then $x \notin B_m$ for some $m \geq 1$, and thus $x \notin A_k$ for all $k \geq m$. This shows $\exists \{n_k\}$ subsequence of $\{f_n\}$ such that

$$|f_{n_k}(x) - f(x)| < \varepsilon_k$$

for all k. Since $\varepsilon_k \to 0$, this shows there is a subsequence $\{f_{n_k}\}$ which converges pointwise for all $x \notin B$, but since $\mu(B) = 0$, this is pointwise a.e. □

Problem 4

If $f_n, f \in L^p$ ($p < \infty$) and $f_n \to f$ point-wise a.e., then $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$.

Proof. Suppose $f_n, f \in L^p(\Omega)$ and $p < \infty$. Also suppose $f_n \to f$ point-wise a.e. Let $\|f_n - f\|_p \to 0$. Then by the reverse triangle inequality,

$$0 \leq \Big| \|f_n\|_p - \|f\|_p \Big| \leq \|f_n - f\|_p \to 0$$

Thus $\|f_n\|_p \to \|f\|_p$. Now let $\|f_n\|_p \to \|f\|_p$. Then by Theorem 1.9 from Lieb and Loss ("Missing term in Fatou's lemma"),

$$\lim_{n\to\infty} \int_\Omega \Big| |f_n(x)|^p - |f_n(x) - f(x)|^p - |f(x)|^p \Big| \mathrm{d}\mu = 0$$

By the triangle inequality,

$$\int_\Omega |f_n|^p \mathrm{d}\mu \le \int_\Omega |f|^p \mathrm{d}\mu + \int_\Omega |f - f_n|^p \mathrm{d}\mu$$
$$\implies \|f_n\|_p^p - \|f\|_p^p \le \|f - f_n\|_p^p$$

□

Problem 5

Suppose $0 < p < q \le \infty$. Then $L^p \not\subset L^q$ if and only if Ω contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ if and only if Ω contains sets of arbitrarily large finite measure. [Hint: for the "if" implication: in the first case there is a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) \le 2^{-n}$, and in the second case there is a disjoint sequence $\{E_n\}$ with $1 \le \mu(E_n) < \infty$. Consider $f = \sum a_n \mathscr{X}_{E_n}$ for suitable constants a_n.]

Proof. Suppose $0 < p < q \le \infty$.

(a) Let Ω contain sets of arbitrarily small positive measure. That is, $\exists$ disjoint sets E_n and integers k_n with $0 < k_1 < k_2 < \ldots$ such that $2^{-k_{n+1}} < \mu(E_n) < 2^{-k_n}$. Note $n \le k_n$ for all integers n. Define f by

$$f = \sum_{n=1}^\infty 2^{\frac{2n}{p+q}} \mathscr{X}_{E_n}$$

The following calculations show $\|f\|_p < \infty$ but $\|f\|_q = \infty$, and thus $L^p \not\subset L^q$.

$$\|f\|_p^p = \int_\Omega |f|^p \mathrm{d}x = \sum_{n=1}^\infty \int_{E_n} 2^{\frac{2np}{p+q}} \mathrm{d}x = \sum_{n=1}^\infty 2^{\frac{2np}{p+q}} \mu(E_n) \le \sum_{n=1}^\infty 2^{\frac{2np}{p+q}} 2^{-k_n} \le \sum_{n=1}^\infty 2^{\frac{2k_np}{p+q}} 2^{-k_n} = \sum_{n=1}^\infty \left(2^{\frac{p-q}{p+q}}\right)^{k_n} < \infty$$

since $p - q < 0$ and thus $2^{\frac{p-q}{p+q}} < 1$.

$$\|f\|_q^q = \int_\Omega |f|^q \mathrm{d}x = \sum_{n=1}^\infty \int_{E_n} 2^{\frac{2nq}{p+q}} \mathrm{d}x = \sum_{n=1}^\infty 2^{\frac{2nq}{p+q}} \mu(E_n) \ge \sum_{n=1}^\infty 2^{\frac{2nq}{p+q}} 2^{-k_{n+1}} \ge \sum_{n=1}^\infty 2^{\frac{2nq}{p+q}} 2^{-(n+1)} = \frac{1}{2} \sum_{n=1}^\infty \left(2^{\frac{q-p}{p+q}}\right)^n = \infty$$

since $q - p > 0$ and thus $2^{\frac{q-p}{p+q}} > 1$.

(b) Let Ω contain sets of arbitrarily large positive measure. That is, $\exists$ disjoint sets E_n and integers k_n with $0 < k_1 < k_2 < \ldots$ such that $2^{k_n} \le \mu(E_n) \le 2^{k_{n+1}}$. Note $n \le k_n$ for all integers n. Define f by

$$f = \sum_{n=1}^\infty 2^{-\frac{2n}{p+q}} \mathscr{X}_{E_n}$$

The following calculations show $\|f\| = \infty$ but $\|f\|_q < \infty$, and thus $L^q \not\subset L^p$.

$$\|f\|_p^p = \int_\Omega |f|^p \mathrm{d}x = \sum_{n=1}^\infty \int_{E_n} 2^{\frac{-2np}{p+q}} \mathrm{d}x = \sum_{n=1}^\infty 2^{\frac{-2np}{p+q}} \mu(E_n) \ge \sum_{n=1}^\infty 2^{\frac{-2np}{p+q}} 2^{k_n} \ge \sum_{n=1}^\infty 2^{\frac{-2np}{p+q}} 2^n = \sum_{n=1}^\infty \left(2^{\frac{q-p}{p+q}}\right)^n = \infty$$

since $q > p$ and thus $2^{\frac{q-p}{p+q}} > 1$.

$$\|f\|_q^q = \int_\Omega |f|^q \mathrm{d}x = \sum_{n=1}^\infty \int_{E_n} 2^{\frac{-2nq}{p+q}} \mathrm{d}x = \sum_{n=1}^\infty 2^{\frac{-2nq}{p+q}} \mu(E_n) \le \sum_{n=1}^\infty 2^{\frac{-2nq}{p+q}} 2^{k_{n+1}} \le \sum_{n=1}^\infty 2^{\frac{-2k_{n+1}p}{p+q}} 2^{k_{n+1}} = \sum_{n=1}^\infty \left(2^{\frac{p-q}{p+q}}\right)^{k_{n+1}} < \infty$$

since $p - q < 0$ and thus $2^{\frac{p-q}{p+q}} < 1$.

□

Problem 6

If $f \in L^\infty(\Omega) \cap L^q(\Omega)$ for some q then $f \in L^p(\Omega)$ for all $p > q$ and

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

Proof. Let $p > q$. Then

$$\begin{aligned}
\|f\|_p^p &= \int_\Omega |f|^p \mathrm{d}\mu \\
&= \int_\Omega |f|^{p-q}|f|^q \mathrm{d}\mu \\
&\leq \int_\Omega \|f\|_\infty^{p-q}|f|^q \mathrm{d}\mu \\
&= \|f\|_\infty^{p-q} \int_\Omega |f|^q \mathrm{d}\mu \\
&= \|f\|_\infty^{p-q}\|f\|_q^q \\
&< \infty
\end{aligned}$$

since $p - q > 0$, $\|f\|_\infty < \infty$, and $\|f\|_q < \infty$. Thus $f \in L^p(\Omega)$. Next we show $\|f\|_\infty = \lim_{p\to\infty}\|f\|_p$. By the above calculation,

$$\begin{aligned}
\lim_{p\to\infty}\|f\|_p &\leq \lim_{p\to\infty}\left[\|f\|_\infty^{\frac{p-q}{p}}\|f\|_q^{\frac{q}{p}}\right] \\
&= \lim_{p\to\infty}\|f\|_\infty^{\frac{p-q}{p}} \cdot \lim_{p\to\infty}\|f\|_q^{\frac{q}{p}} \\
&= \|f\|_\infty
\end{aligned}$$

since as $p \to \infty$, $\frac{p-q}{p} \to 1$ and $\frac{q}{p} \to 0$. Also, the definition of $\|\cdot\|_\infty$ implies that for any ε, $\mu(E_\varepsilon) > 0$ where

$$E_\varepsilon = \left\{x : |f(x)| \geq \|f\|_\infty - \varepsilon\right\}.$$

but $\mu(E_\varepsilon) \to 0$ and $\varepsilon \to 0$. Thus,

$$\begin{aligned}
\|f\|_p^p &= \int_\Omega |f|^p \mathrm{d}\mu \\
&= \int_{\Omega\setminus E_\varepsilon} |f|^p \mathrm{d}\mu + \int_{E_\varepsilon} |f|^p \mathrm{d}\mu \\
&\geq \int_{E_\varepsilon} |f|^p \mathrm{d}\mu \\
&\geq \int_{E_\varepsilon} \left|\|f\|_\infty - \varepsilon\right|^p \mathrm{d}\mu \\
&= \mu(E_\varepsilon)\left|\|f\|_\infty - \varepsilon\right|^p \\
\implies \lim_{p\to\infty}\|f\|_p &= \lim_{p\to\infty}\left[\mu(E_\varepsilon)^{\frac{1}{p}}\left|\|f\|_\infty - \varepsilon\right|\right] \\
&= \left|\|f\|_\infty - \varepsilon\right|
\end{aligned}$$

Since ε is arbitrarily small, we find $\|f\|_\infty \le \lim_{p\to\infty}\|f\|_p$. Thus,

$$\|f\|_\infty = \lim_{p\to\infty}\|f\|_p$$

$\square$

Problem 7

Prove that when $\infty \ge r \ge q \ge 1$, $f \in L^r(\Omega)\cap L^q(\Omega) \implies f \in L^p(\Omega)$ for all $r \ge p \ge q$.

Proof. Let $f \in L^q(\Omega)\cap L^r(\Omega)$. For $p \in [q,r]$ where $r < \infty$, by convexity of $\mathbb{R}$, $\exists a \in [0,1]$ such that

$$\frac{1}{p} = \frac{a}{r} + \frac{1-a}{q}$$

Then

$$\begin{aligned}
\|f\|_p^p &= \int_\Omega |f|^p \mathrm{d}\mu \\
&= \int_\Omega |f|^{pa}|f|^{p(1-a)}\mathrm{d}\mu \\
&\le \left(\int_\Omega |f|^{(pa)\left(\frac{r}{pa}\right)}\mathrm{d}\mu\right)^{\frac{pa}{r}}\left(\int_\Omega |f|^{(p(1-a))\left(\frac{q}{p(1-a)}\right)}\mathrm{d}\mu\right)^{\frac{p(1-a)}{q}} && \text{by Hölder's Inequality} \\
&= \left(\int_\Omega |f|^r\right)^{\frac{pa}{r}}\left(\int_\Omega |f|^q\right)^{\frac{p(1-a)}{q}} \\
&= \|f\|_r^{pa}\|f\|_q^{p(1-a)} \\
\implies \|f\|_p &\le \|f\|_r^a\|f\|_q^{1-a} < \infty \\
\implies f &\in L^p(\Omega)
\end{aligned}$$

For $r = \infty$, problem 6 implies $f \in L^p(\Omega)$. $\square$

Problem 8

Prove that a strongly convergent sequence in $L^p(\mathbb{R}^n)$ is also a Cauchy sequence.

Proof. Let $\{f_n\}_n$ be a strongly convergent sequence in $L^p(\mathbb{R}^n)$ and let $\epsilon > 0$. Then there is some N such that $\|f_N - f\| < \frac{\varepsilon}{2}^{\frac{1}{p}}$. Then for all $m, n \ge N$,

$$\|f_n - f_m\|_p^p \le \|f_n - f\|_p^p + \|f_m - f\|_p^p$$

since $|a+b|^p \le |a|^p + |b|^p$ for all $a, b \in \mathbb{C}$ and $p \in (0,\infty]$. Then

$$\|f_n - f_m\|_p^p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $\{f_n\}_n$ is Cauchy. $\square$

Problem 9

Give three different examples of ways for a sequence $f_k \in L^p(\mathbb{R}^n)$ to converge weakly to zero, but not strongly to anything. Verify your claims for these exmples.

Proof. Three *types* of examples are given in Lieb and Loss section 2.9:

(a) "Oscillates to Death" Let $x \in \mathbb{R}^n$ be denoted $x = (x_1, x_2, \dots, x_n)$. Define $f_k \in L^p$ as

$$f_k(x) = \left[\sum_{i=1}^{n} \sin(k\pi x_i)\right]\mathscr{X}_{[0,1]^n}$$

(b) "Goes Up the Spout" Let $p \geq 1$ and define $f(x) = \mathscr{X}_{[-1,1]^n}$ where $\mathscr{X}$ is the characteristic function. Define $f_k \in L^p(\mathbb{R}^n)$ as

$$f_k(x) = k^{\frac{k}{p}} f(kx) = k^{\frac{n}{p}} \mathscr{X}_{[-\frac{1}{k},\frac{1}{k}]^n}$$

Then for all k,

$$\|f_k\|_p^p = \int_{[-\frac{1}{k},\frac{1}{k}]^n} \left(k^{\frac{n}{p}}\right)^p \mathrm{d}x = k^n\left(\frac{2^n}{k^n}\right) = 2^n$$

So clearly $f_k \not\to 0$ in $\|\cdot\|_p$. However, for a fixed functional $L \in L^p(\mathbb{R}^n)^*$, there is a function $\ell \in L^q(\mathbb{R}^n)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) such that

$$L(f) = \int_{\mathbb{R}^n} \ell(x) f(x) \mathrm{d}x$$

for all $f \in L^p(\mathbb{R}^n)$. Note

$$\begin{aligned}
L(f_k) &= \int_{\mathbb{R}^n} \ell(x) f_k(x) \mathrm{d}x \\
&= \int_{[-\frac{1}{k},\frac{1}{k}]^n} \ell(x) k^{\frac{n}{p}} \mathrm{d}x \\
&= k^{\frac{n}{p}} \int_{[-\frac{1}{k},\frac{1}{k}]^n} \ell(x) \mathrm{d}x \\
&\leq k^{\frac{n}{p}} \left(\int_{[-\frac{1}{k},\frac{1}{k}]^n} 1 \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{[-\frac{1}{k},\frac{1}{k}]^n} |\ell(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} \quad \text{by Hölder's Inequality} \\
&= 2^{\frac{n}{p}} \left(\int_{[-\frac{1}{k},\frac{1}{k}]^n} |\ell(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} \xrightarrow[k\to\infty]{} 0
\end{aligned}$$

since $\left[-\frac{1}{k},\frac{1}{k}\right]^n \to \{0\}$. Thus $f_k \rightharpoonup 0$ but $f_k \not\to 0$. Since f_k does not converge strongly to 0, it does not strongly to anything, since if it did, it would also weakly converge there (a contradiction). The only candidate function for f_k to converge strongly to is a delta function, but $\delta(x) \not\in L^p(\mathbb{R}^n)$.

(c) "Wanders Off to Infinity" Let $f(x) = \mathscr{X}_{[0,1]^n}$. Define $f_k \in L^p(\mathbb{R}^n)$ as

$$f_k(x) = f(x - (k, 0, \dots, 0))$$

Then $\|f_k\|_p = 1$ for all k, and thus $f_k \not\to 0$. However, for any $\ell \in L^q(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_{[k,k+1]\times[0,1]^{n-1}} |\ell(x)|^q \mathrm{d}x \to 0$$

since

$$\int_{\mathbb{R}^n} |\ell(x)|^q < \infty$$

Thus, for any $L \in L^p(\mathbb{R}^n)^*$, $\exists \ell \in L^q(\mathbb{R}^n)$ such that

$$L(f) = \int_{\mathbb{R}^n} \ell(x) f(x) \mathrm{d}x$$

for all $f \in L^p(\mathbb{R}^n)$, and thus

$$L(f_k) = \int_{\mathbb{R}^n} \ell(x) \mathscr{X}_{[k,k+1]\times[0,1]^{n-1}} \mathrm{d}x = \int_{[k,k+1]\times[0,1]^{n-1}} \ell(x) \mathrm{d}x \to 0$$

which shows $f_k \rightharpoonup 0$. Since $f_k \not\to 0$, then f_k does not strongly converge at all.

$\square$