UC Davis, Analysis (MAT201C), Spring 2016

Homework #2

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Problem 1

A function $f \in L^p(\mathbb{R}^n)$ is said to be L_p-*continuous* if $\tau_h f \to f$ in $L^p(\mathbb{R}^n)$ as $h \to 0$ in $\mathbb{R}^n$, where $\tau_h f(x) = f(x-h)$ is the translation of f by h. Prove that, if $1 \le p < \infty$, every $f \in L^p(\mathbb{R}^n)$ is L^p-continuous. Give a counter-example to show that this result is not true when $p = \infty$. [Hint: Approximate an L^p function by a C_c function.]

Proof. Define $f \in L^\infty(\mathbb{R})$ as $f(x) = \mathscr{X}_{[0,1]^n}$ where $\mathscr{X}$ is the characteristic function. Note that $f(1-\varepsilon) = 1$ for all $\varepsilon > 0$. Let h be a small perturbation, i.e. $0 < |h| \ll 1$, and choose $\varepsilon = \frac{h}{2}$. Then $\forall x \in (0, \varepsilon)$, $\tau_h f(x) = 0$ but $f(x) = 1$, and thus $\left|\tau_h f(x) - f(x)\right| = 1$. This shows that $\forall h > 0$, $\exists$ an interval I_h (of positive measure, $\mu(I_n) > 0$) such that $\left|\tau_h f(x) - f(x)\right| = 1$ for all $x \in I_h$. Thus $\tau_h f \not\to f$ in $L^\infty(\mathbb{R}^n)$. $\square$

Problem 2

Show that $L^\infty(\mathbb{R})$ is not separable. [Hint: There is an uncountable set $\mathscr{F} \subset L^\infty$ such that $\left\|f - g\right\|_\infty \ge 1$ for all $f, g \in \mathscr{F}$ with $f \ne g$.]

Proof. Let $\mathscr{F} = \{\mathscr{X}_{[0,\alpha]} \,:\, 0 < \alpha \in \mathbb{R}\}$. $\mathscr{F}$ is clearly uncountable. Consider any two $f, g \in \mathscr{F}$. Then, without loss of generality, $f = \mathscr{X}_{[0,\alpha]}$ and $g = \mathscr{X}_{[0,\beta]}$ where $\alpha < \beta$. Also,

$$\left\|f - g\right\|_\infty = \text{ess sup}\{\mathscr{X}_{(\alpha,\beta]}\} = 1$$

Thus the ball around any $f \in \mathscr{F}$ of radius $\frac{1}{2}$, i.e. $B\left(f, \frac{1}{2}\right)$, contains no other elements of $\mathscr{F}$. Thus $L^\infty(\mathbb{R})$ is not separable since $\mathscr{F}$ is uncountable and not dense. $\square$

Problem 3

Prove Chebyshev's Inequality: If $f \in L^p$ $(1 \le p < \infty)$, then for any $\alpha > 0$,

$$\mu\left(\left\{x \,:\, \left|f(x)\right| > \alpha\right\}\right) \le \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

[Note that you can find the proof of this simple fact in many texts but you should see if you can figure it out yourself. Also, note that this inequality holds for all $0 < p < \infty$.]

Proof. Let $A_\alpha = \left\{x \,:\, \left|f(x)\right| > \alpha\right\} = \left\{x \,:\, \left|\frac{f(x)}{\alpha}\right| > 1\right\} = \left\{x \,:\, \left|\frac{f(x)}{\alpha}\right|^p > 1\right\}$ for all $p \ge 1$. Then

$$\left(\frac{\|f\|_p}{\alpha}\right)^p = \int_\Omega \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu = \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu + \int_{\Omega \setminus A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu \ge \int_{A_\alpha} \left|\frac{f(x)}{\alpha}\right|^p \mathrm{d}\mu \ge \int_{A_\alpha} 1 \mathrm{d}\mu = \mu(A_\alpha)$$

which proves the result. $\square$

Problem 4

Assume that $f, g \in L^1(\mathbb{R}^n)$. Prove that the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \mathrm{d}y$$

is measurable and in $L^1(\mathbb{R}^n)$.

Proof. □

Problem 5

Let $f_n = \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}$. Prove that f_n converges weakly to 0 in $L^2(0,1)$ and $f_n \to 0$ in $L^1(0,1)$ but f_n does not converge strongly in $L^2(0,1)$.

Proof.

$$\|f_n\|_2^2 = \int_0^1 n\mathscr{X}_{(0,\frac{1}{n})}\mathrm{d}x = \int_0^{\frac{1}{n}} n\mathrm{d}x = 1$$

Thus $\|f_n\|_2 = 1$ for all n, and thus does not converge strongly to 0 in $L^2(0,1)$.

$$\|f_n\|_1 = \int_0^1 \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}\mathrm{d}x = \int_0^{\frac{1}{n}} \sqrt{n}\mathrm{d}x = \frac{1}{\sqrt{n}}$$

Thus $\|f_n\|_1 \to 0$ as $n \to 0$, which shows $f_n \to 0$ strongly in $L^1(0,1)$. Let $L \in L^2(0,1)^* \cong L^2(0,1)$. Thus $\exists \ell \in L^2(0,1)$ such that

$$L(f) = \int_0^1 \ell(x)f(x)\mathrm{d}x$$

for all $f \in L^2$. Then

$$L(f_n) = \int_0^1 \ell(x)\sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}\mathrm{d}x = \int_0^{\frac{1}{n}} \ell(x)\sqrt{n}\mathrm{d}x \leq \left(\int_0^{\frac{1}{n}} |\ell(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^{\frac{1}{n}} n\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_0^{\frac{1}{n}} |\ell(x)|^2\mathrm{d}x\right) \to 0$$

since ℓ is fixed and $\mu\big((0,\frac{1}{n})\big) \to 0$ as $n \to 0$. Thus $f_n \rightharpoonup 0$ in $L^2(0,1)$. □

Problem 6

Find a sequence of functions with the property that f_j converges to 0 in $L^2(\Omega)$ weakly, to 0 in $L^{\frac{3}{2}}(\Omega)$ strongly, but it does not converge to 0 strongly in $L^2(\Omega)$.

Proof. Let $f_n = \sqrt{n}\mathscr{X}_{(0,\frac{1}{n})}$. Then by number 5, $f_n \not\to 0$ in $L^2(0,1)$ but $f_n \rightharpoonup 0$ in $L^2(0,1)$. Also,

$$\|f_n\|_{\frac{3}{2}}^{\frac{3}{2}} = \int_0^1 \left|n^{\frac{1}{2}}\mathscr{X}_{(0,\frac{1}{n})}\right|^{\frac{3}{2}}\mathrm{d}x = \int_0^{\frac{1}{n}} n^{\frac{3}{4}}\mathrm{d}x = n^{-\frac{1}{4}} \to 0 \text{ as } n \to \infty$$

Thus $f_n \to 0$ in $L^{\frac{3}{2}}(0,1)$. □

Problem 7

Let f_n and g_n denote two sequences in $L^p(\Omega)$ with $1 \leq p \leq \infty$ such that $f_n \to f$ in $L^p(\Omega)$, and $g_n \to g$ in $L^p(\Omega)$. Set $h_n = \max\{f_n, g_n\}$ and prove that $h_n \to h$ in $L^p(\Omega)$.

Proof. □

Problem 8

Let f_n be a sequence in $L^p(\Omega)$ with $1 \leq p < \infty$, and let g_n be a bounded sequence in $L^\infty(\Omega)$. Suppose that $f_n \to f$ in $L^p(\Omega)$ and that $g_n \to g$ pointwise a.e. Prove that $f_n g_n \to fg$ in $L^p(\Omega)$.

Proof. □

Problem 9

Prove that the space of continuous functions with compat support $\mathscr{C}_c^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Proof. □