201C, Spring '16, Thomases
Homework 4 due 4/26/16

1. Let $f \in L^1(\mathbb{R})$, and set
$$g(x) = \int_{-\infty}^{x} f(y)dy.$$
Prove that f is continuous, and show that $\frac{dg}{dx} = f$, where $\frac{dg}{dx}$ denotes the weak derivative.
Hint: given $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, use the definition of g to obtain
$$\int_{\mathbb{R}} \phi'(x)g(x)dx = \int_{\mathbb{R}} \int_{-\infty}^{x} \phi'(x)f(y)dydx.$$
Then write this integral as
$$\lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}} \left[\phi(x+h) - \phi(x)\right] g(x)dx = -\lim_{h\to 0} \int_{\mathbb{R}} \int_{x}^{x+h} f(y)\phi(x)dydx.$$

2. Show that $W^{n,1}(\mathbb{R}^n) \subseteq C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
Hint: $u(x) = \int_{-\infty}^{0} \dots \int_{-\infty}^{0} \frac{\partial^n}{\partial x_1 \dots \partial x_n} u(x+y)dy_1 \dots dy_n$.

3. If $u \in L^1_{\text{loc}}(\mathbb{R})$ and if $\frac{du}{dx} = f \in L^1(\mathbb{R})$, then
$$u(x) = C + \int_{-\infty}^{x} f(y)dy, \quad \text{a.e. } x \in \mathbb{R},$$
for some constant C.

4. Let $\Omega := B(0, 1/2) \subseteq \mathbb{R}^2$ denote the open ball of radius $1/2$. For $x = (x_1, x_2) \in \Omega$, let
$$u(x_1, x_2) = x_1 x_2 \log(|\log(|x|)|) \quad \text{where} \quad |x| = \sqrt{x_1^2 + x_2^2}.$$
   (a) Show that $u \in \mathcal{C}^1(\bar{\Omega})$.
   (b) Show that $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{C}(\bar{\Omega})$ for $j = 1, 2$ but $u \notin \mathcal{C}^2(\bar{\Omega})$.
   (c) Show that $u \in H^2(\Omega)$.

5. Prove that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for integers $k \geq 0$ and $1 \leq p < \infty$.

6. Let η_ε denote the standard mollifier, and for $u \in H^3(\mathbb{R}^3)$, set $u^\varepsilon = \eta_\varepsilon * u$. Prove that

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^3)} \leq C\sqrt{\varepsilon}\|u\|_{H^2(\mathbb{R}^3)},$$

and that

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^3)} \leq C\varepsilon\|u\|_{H^3(\mathbb{R}^3)}.$$

7. Let $D := B(0,1) \subseteq \mathbb{R}^2$ denote the unit disc, and let

$$u(x) = [-\log|x|]^\alpha .$$

Prove that the *weak derivative* of u exists for all $\alpha \geq 0$.