HW #1

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Problem 1

Consider the following chemical reaction, where one chemical (A) turns into a different chemical (B) and vice versa. Suppose that the total amount of chemical is constant, that is $A(t)+B(t) = C$, where C is a positice constant. This reaction can be represented schematically in the following way:

$$\mathrm{A} \underset{k^-}{\overset{k^+}{\rightleftharpoons}} \mathrm{B}$$

where the two positive constants K^+ and k^- are called rate constants.

The following differential equation describes how A changes with time

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -k^+A + k^-B \tag{1}$$

Recall that, in addition to this differential equation, we have the conservation constraint $A(t) + B(t) = C$.

a)

Solve for $A(t)$, given $A(0) = A_0$, with A_0 being a positive constant such that $A_0 < C$.

Eq. (1) can be simplified by using the conservation constraint as

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= -k^+A + k^-(C - A)\\ &= -\big(k^+ + k^-\big)A + k^-C\end{aligned}$$

Let $u(t) = -(k^+ + k^-)A(t) + k^-C$. Then $\dot{u} = -(k^+ + k^-)\dot{A}$. Thus,

$$\dot{u} = -(k^+ + k^-)u \tag{2}$$

The solution to Eq. (2) is exponential, i.e.

$$\begin{aligned} u(t) &= Z\exp\big(-(k^+ + k^-)t\big) \quad \text{ for some arbitrary constant } Z \\ \implies A(t) &= Z\exp\big(-(k^+ + k^-)t\big) + \frac{k^- C}{k^+ + k^-} \end{aligned}$$

The initial condition $A(0) = A_0$ implies

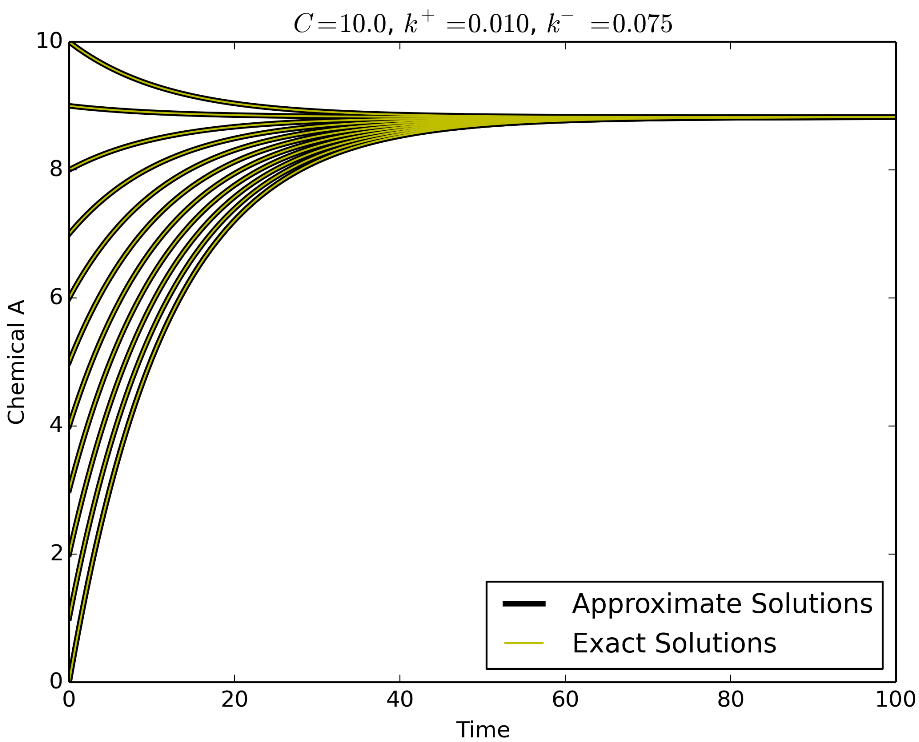
$$Z = A_0 - \frac{k^- C}{k^+ + k^-}$$

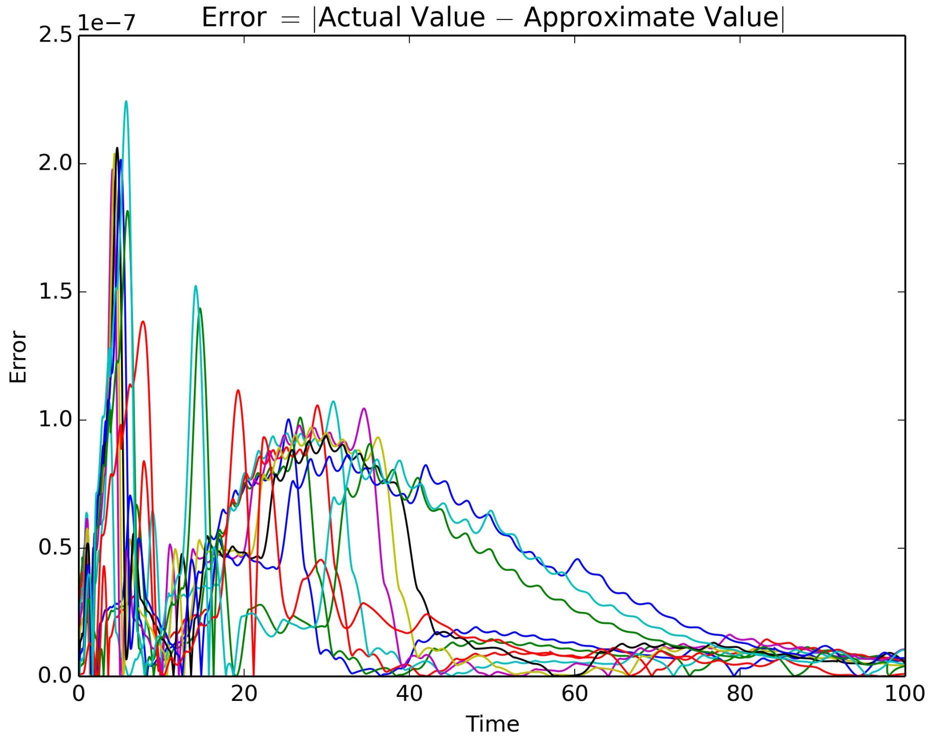
and so

$$A(t) = \left(A_0 - \frac{k^- C}{k^+ + k^-}\right)\exp\big(-(k^+ + k^-)t\big) + \frac{k^- C}{k^+ + k^-} \tag{3}$$

b)

Use Matlab to check your answer for a few choices of A_0, C, k^+, and k^- (I have provided code that will assist you).





Problem 2

The position of a moving object in 1-D ($x(t)$) on a damped linear spring obeys the following differential equation

$$m\ddot{x} = -b\dot{x} - kx \tag{4}$$

where m, b, and k are positive constants representing the mass of the object, the damping coefficient and the stiffness of the spring, respectively.

a)

Solve for $x(t)$, given $x(0) = x_0$ and $\dot{x}(0) = v_0$.

Since Eq. (4) is a linear ODE, we can solve for the roots the characteristic polynomial, which is

$$P(\lambda) = m\lambda^2 + b\lambda + k = 0$$

$$\implies \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

and thus the solution to Eq. (4) is

$$x(t) = \begin{cases} C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) & \text{if } \lambda_1 \neq \lambda_2 \text{ and } \lambda_1, \lambda_2 \in \mathbb{R} \\ C_1 \exp(\lambda t) + C_2 t \exp(\lambda t) & \text{if } \lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \\ C_1 \exp(\alpha t)\sin(\beta t) + C_2 \exp(\alpha t)\cos(\beta t) & \text{if } \lambda_{1,2} = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

The conditions simplify based on the formulation of $\lambda_{1,2}$ to

$$x(t) = \begin{cases} C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) & \text{if } b^2 - 4mk > 0 \\ C_1 \exp(\lambda t) + C_2 t \exp(\lambda t) & \text{if } b^2 - 4mk = 0 \\ C_1 \exp(\alpha t) \sin(\beta t) + C_2 \exp(\alpha t) \cos(\beta t) & \text{if } b^2 - 4mk < 0 \end{cases} \tag{5}$$

Case 1: $b^2 - 4mk > 0$

$$\begin{aligned} x(t) &= C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) \\ \dot{x}(t) &= C_1 \lambda_1 \exp(\lambda_1 t) + C_2 \lambda_2 \exp(\lambda_2 t) \end{aligned}$$

Since $x(0) = x_0$ and $\dot{x}(0) = v_0$,

$$\begin{aligned} x_0 = C_1 + C_2 \quad &\text{and} \quad v_0 = C_1 \lambda_1 + C_2 \lambda_2 \\ \Longrightarrow C_1 = \frac{x_0 \lambda_2 - v_0}{\lambda_2 - \lambda_1} \quad &\text{and} \quad C_2 = \frac{x_0 \lambda_1 - v_0}{\lambda_1 - \lambda_2} \\ \Longrightarrow &\boxed{x(t) = \frac{x_0 \lambda_2 - v_0}{\lambda_2 - \lambda_1} \exp(\lambda_1 t) + \frac{x_0 \lambda_1 - v_0}{\lambda_1 - \lambda_2} \exp(\lambda_2 t)} \end{aligned}$$

Case 2: $b^2 - 4mk = 0$

$$\begin{aligned} x(t) &= C_1 \exp(\lambda t) + C_2 t \exp(\lambda t) \\ \dot{x}(t) &= C_1 \lambda \exp(\lambda t) + C_2 \lambda t \exp(\lambda t) + C_2 \exp(\lambda t) \end{aligned}$$

Since $x(0) = x_0$ and $\dot{x}(0) = v_0$,

$$\begin{aligned} x_0 = C_1 \quad &\text{and} \quad v_0 = C_1 \lambda + C_2 \\ \Longrightarrow C_1 = x_0 \quad &\text{and} \quad C_2 = v_0 - x_0 \lambda \\ \Longrightarrow &\boxed{x(t) = x_0 \exp(\lambda t) + (v_0 - x_0 \lambda) t \exp(\lambda t)} \end{aligned}$$

Case 3: $b^2 - 4mk < 0$

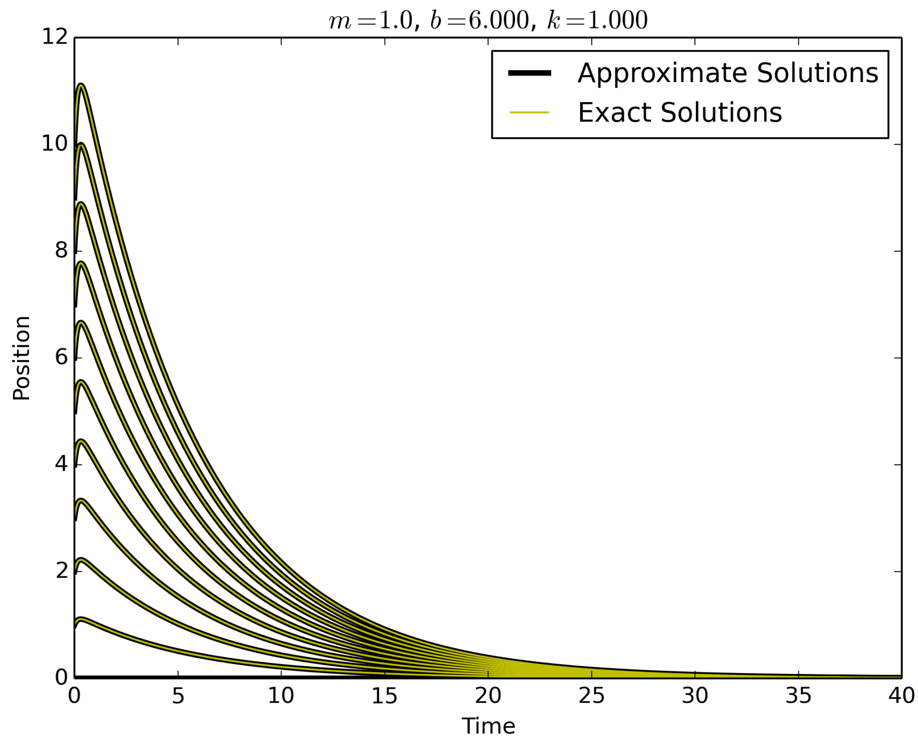
$$\begin{aligned} x(t) &= C_1 \exp(\alpha t) \sin(\beta t) + C_2 \exp(\alpha t) \cos(\beta t) \\ \dot{x}(t) &= C_1 (a \exp(\alpha t) \sin(\beta t) + b \exp(\alpha t) \cos(\beta t)) + C_2 (a \exp(\alpha t) \cos(\beta t) - b \exp(\alpha t) \sin(\beta t)) \end{aligned}$$

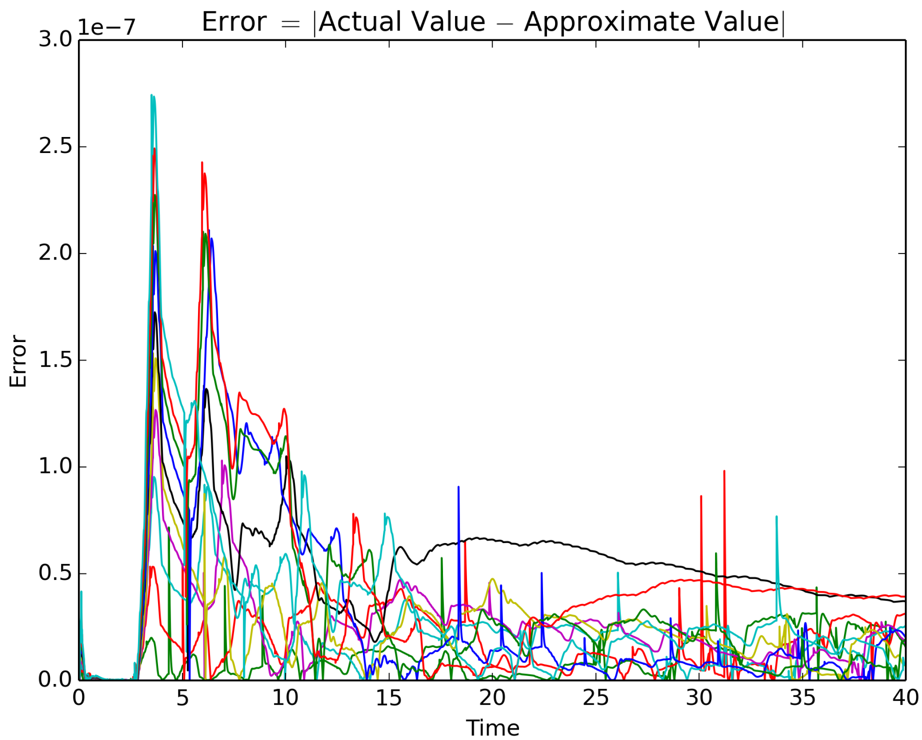
Since $x(0) = x_0$ and $\dot{x}(0) = v_0$,

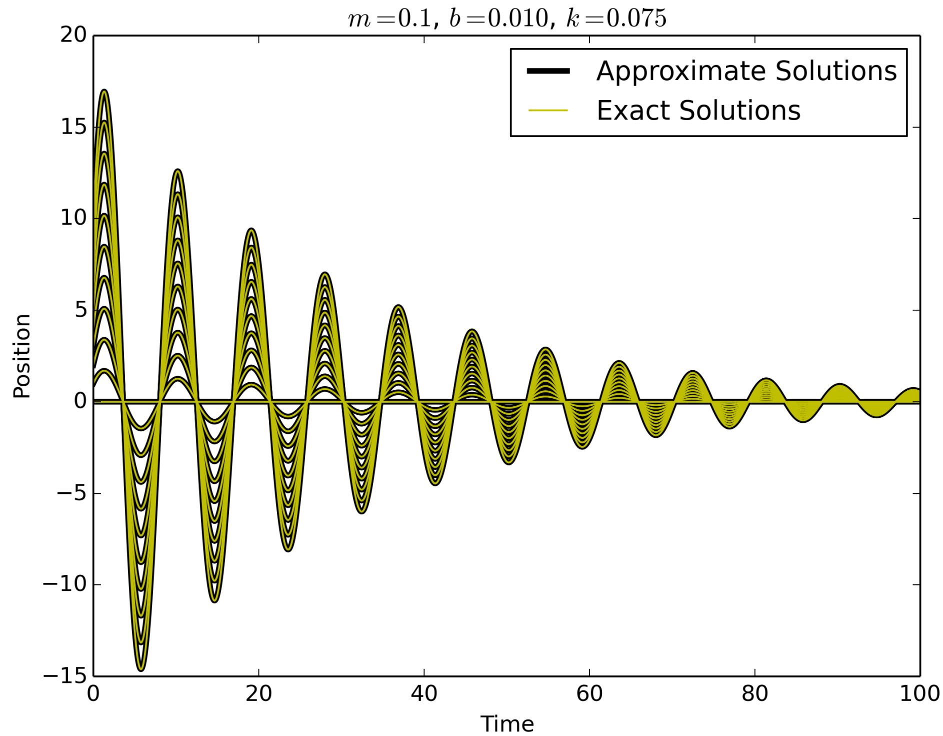
$$\begin{aligned} x_0 = C_2 \quad &\text{and} \quad v_0 = C_1 \beta + C_2 \alpha \\ \Longrightarrow C_1 = \frac{v_0 - x_0 \alpha}{\beta} \quad &\text{and} \quad C_2 = x_0 \\ \Longrightarrow &\boxed{x(t) = \frac{v_0 - x_0 \alpha}{\beta} \exp(\alpha t) \sin(\beta t) + x_0 \exp(\alpha t) \cos(\beta t)} \end{aligned}$$

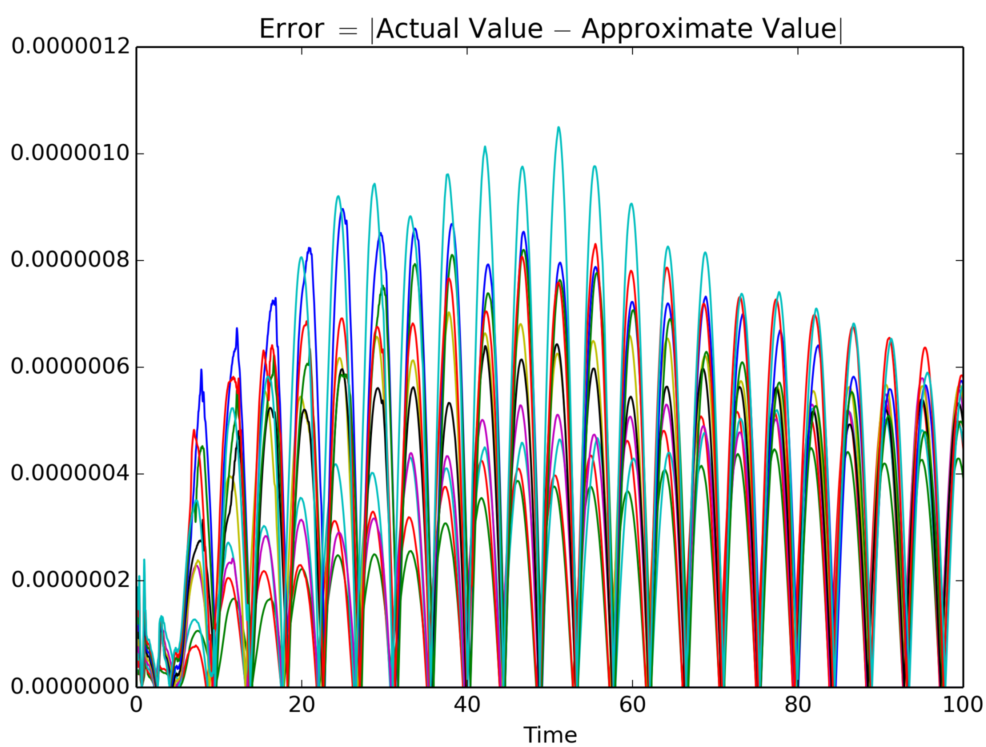
b)

Use Matlab to check your answer for a few choices of x_0, v_0, b, m, and k (I have provided code that will assist you).









c)

Assuming that both x_0 and v_0 are not zero, what inequality must m, b, and k satisfy in order to given an oscillatory solution (i.e. a solution $x(t)$ that contains sine/cosine terms)?

From part a), the solution is only oscillatory if $b^2 - 4mk < 0$. The period of the oscillations depend on $\mathrm{Im}[\lambda] = \beta$. The magnitude of the oscillations depend on $\mathrm{Re}[\lambda] = \alpha$. If $\alpha > 0$ then the magnitude of the oscillations will grow exponentially, if $\alpha < 0$ then the magnitude of the oscillations will decay exponentially, and if $\alpha = 0$, then the magnitude of the oscillations will stay constant. However, we know $\alpha = \frac{-b}{2m}$, where b and m are positive constants, and so $\alpha < 0$, resulting in decaying oscillations.

Problem 3

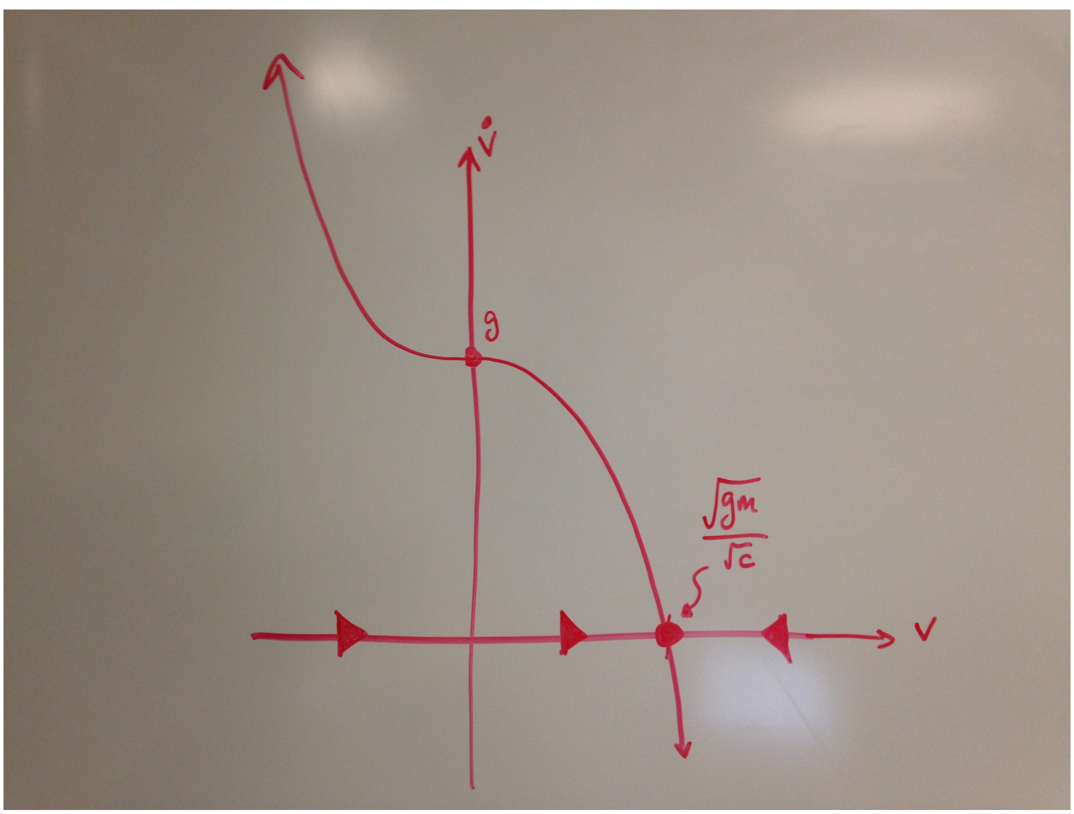
The following equation describes the velocity $v(t)$ of a relatively large object falling through and inviscid medium (e.g. a baseball falling through the air)

$$m\dot{v} = -cv|v| + mg \tag{6}$$

where m, c, and g are positive constants representing the mass of the object, the drag of the fluid and the pull of gravity, respectively.

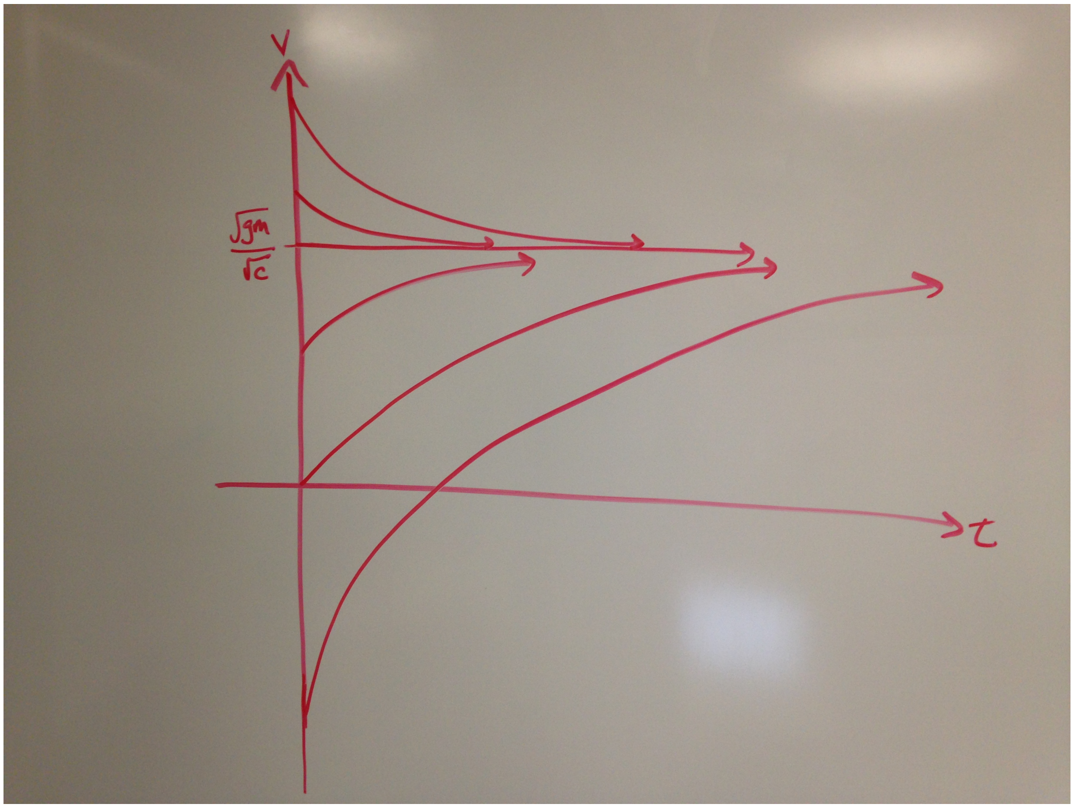
a)

Draw a plot of $\dot{v}$ vs. v. Label any fixed point(s) and indicate the stability of each. On the horizontal axis (v), indicate the flow direction.



b)

Without solving the equation, sketch $v(t)$ as a function of t for several initial conditions.



c)

Solve the equation for $v(t)$ given $v(0) = 0$. (It will simplify your life considerably to assume that $v \geq 0$ to get rid of the absolute value sign. Once you have a solution, you can determine whether this is a reasonable assumption.)

First, assume $v \geq 0$, and thus $m\dot{v} = -cv^2 + mg$. Eq. (6) is a separable equation, i.e.

$$m \int \frac{\mathrm{d}v}{mg - cv^2} = \int \mathrm{d}t$$

The left hand side integral requires partial fractions:

$$\frac{1}{mg - cv^2} = \frac{A}{\sqrt{mg} - \sqrt{c}v} + \frac{B}{\sqrt{mg} + \sqrt{c}v}$$

$$\implies 1 = (A + B)\sqrt{mg} + (B - A)\sqrt{c}v$$

$$\implies A = B = \frac{1}{2\sqrt{mg}}$$

and thus,

$$\frac{\sqrt{m}}{2\sqrt{g}} \left(\int \frac{\mathrm{d}v}{\sqrt{mg} - \sqrt{c}v} + \int \frac{\mathrm{d}v}{\sqrt{mg} - \sqrt{c}v} \right) = \int \mathrm{d}t$$

Using u-substitution ($u = \sqrt{mg} \pm \sqrt{c}v$),

$$\int \frac{\mathrm{d}v}{\sqrt{mg} - \sqrt{c}v} = -\frac{1}{\sqrt{c}} \int \frac{\mathrm{d}u}{u} = -\frac{\ln\left(\sqrt{mg} - \sqrt{c}v\right)}{\sqrt{c}} \quad \text{and}$$

$$\int \frac{\mathrm{d}v}{\sqrt{mg} + \sqrt{c}v} = \frac{1}{\sqrt{c}} \int \frac{\mathrm{d}u}{u} = \frac{\ln\left(\sqrt{mg} + \sqrt{c}v\right)}{\sqrt{c}}$$

Thus,

$$\frac{\sqrt{m}}{2\sqrt{cg}} \ln\left(\frac{\sqrt{mg} + \sqrt{c}v}{\sqrt{mg} - \sqrt{c}v}\right) = t + C \quad \text{for some arbitrary constant } C$$

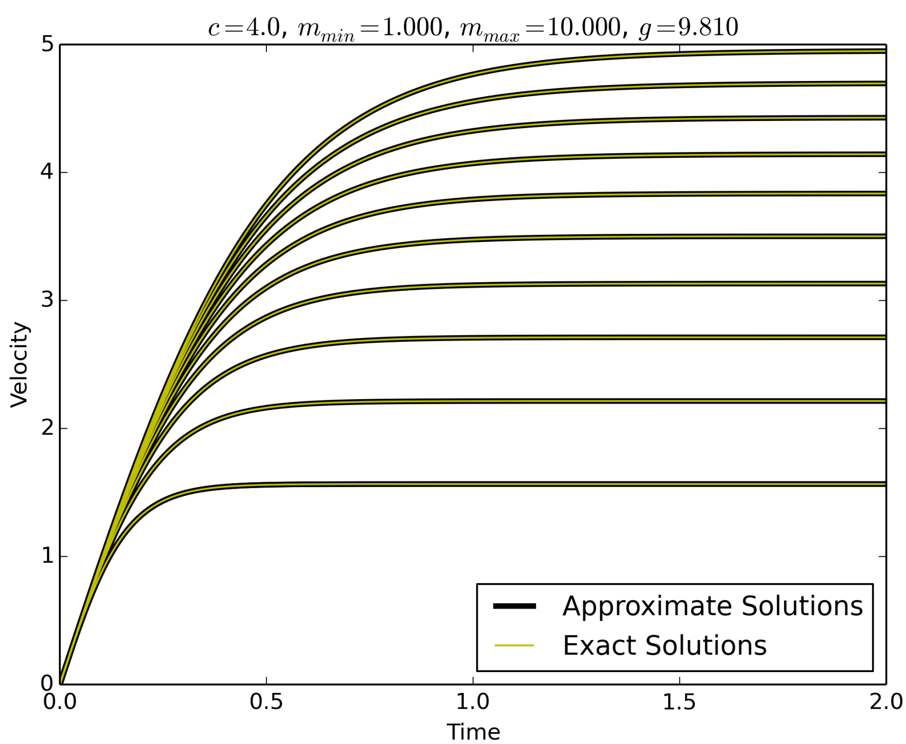
The initial condition $v(0) = 0$ implies $C = 0$. Then the above can be solved for v:

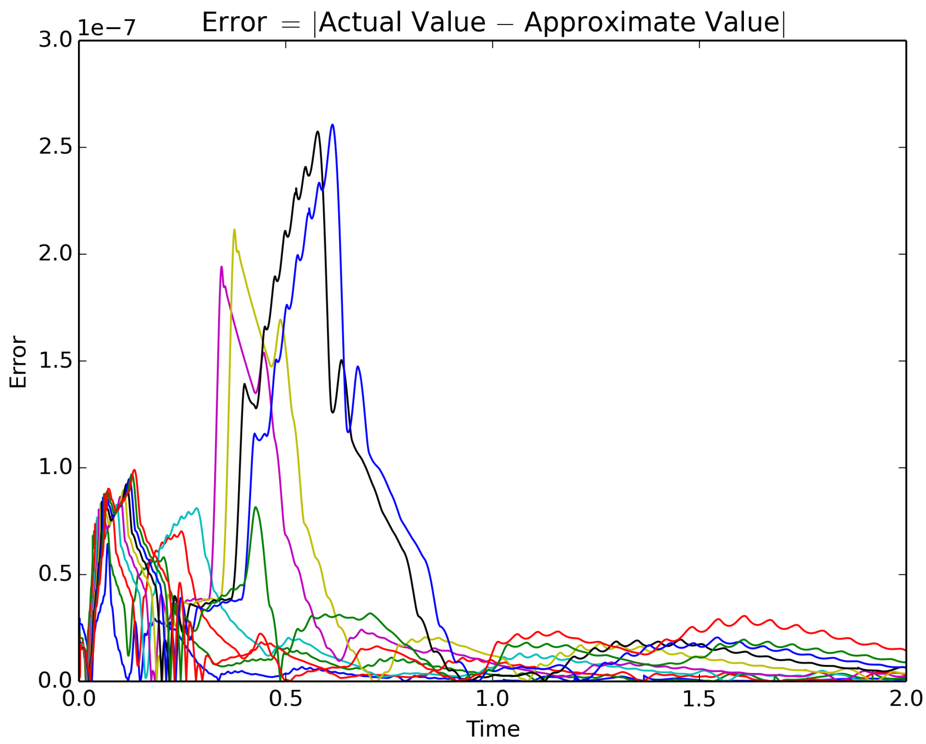
$$v(t) = \frac{\sqrt{mg}\left(\exp\left(\frac{2\sqrt{cg}}{\sqrt{m}}t\right) - 1\right)}{\sqrt{c}\left(\exp\left(\frac{2\sqrt{cg}}{\sqrt{m}}t\right) + 1\right)} \tag{7}$$

Since $\frac{2\sqrt{cg}}{\sqrt{m}} > 0$ and $t \geq 0$, then $\exp\left(\frac{2\sqrt{cg}}{\sqrt{m}}t\right) \geq 1$, which implies $v(t) \geq 0$ for all $t > 0$. Thus our assumption ($v \geq 0$) is still reasonable.

d)

Use Matlab to check your solution. I have not provided code, but you should be able to modify the code for Problem 1.





Problem 4

The DNA in your body codes for specific proteins. Proteins affect how cells in your body behave - even whether they live or die. In some cells, you might want a high concentration of a particular protein; in other cells, you might want only a small concentration of that protein.

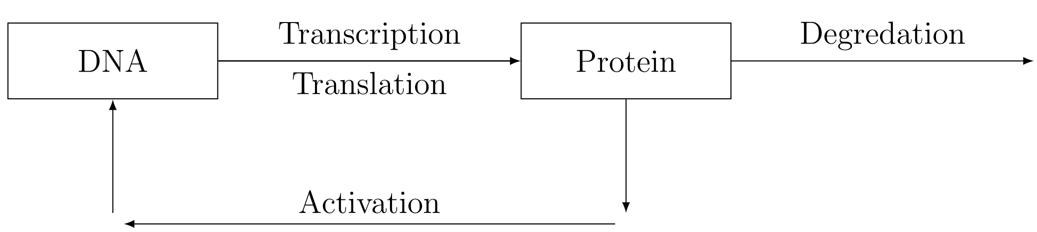


Figure 1: DNA/Protein Interface

Cells in your body typically destroy proteins; they have only a finite lifetime. Thus, the amount of a particular protein in one of your cells at a given time, $p(t)$, *is determined by a balance between the rate of protein formation and destruction. Sometimes, the amount of protein affects how fast protein is made. Here we consider such a situation, as shown schematically in* Figure 1 *above.*

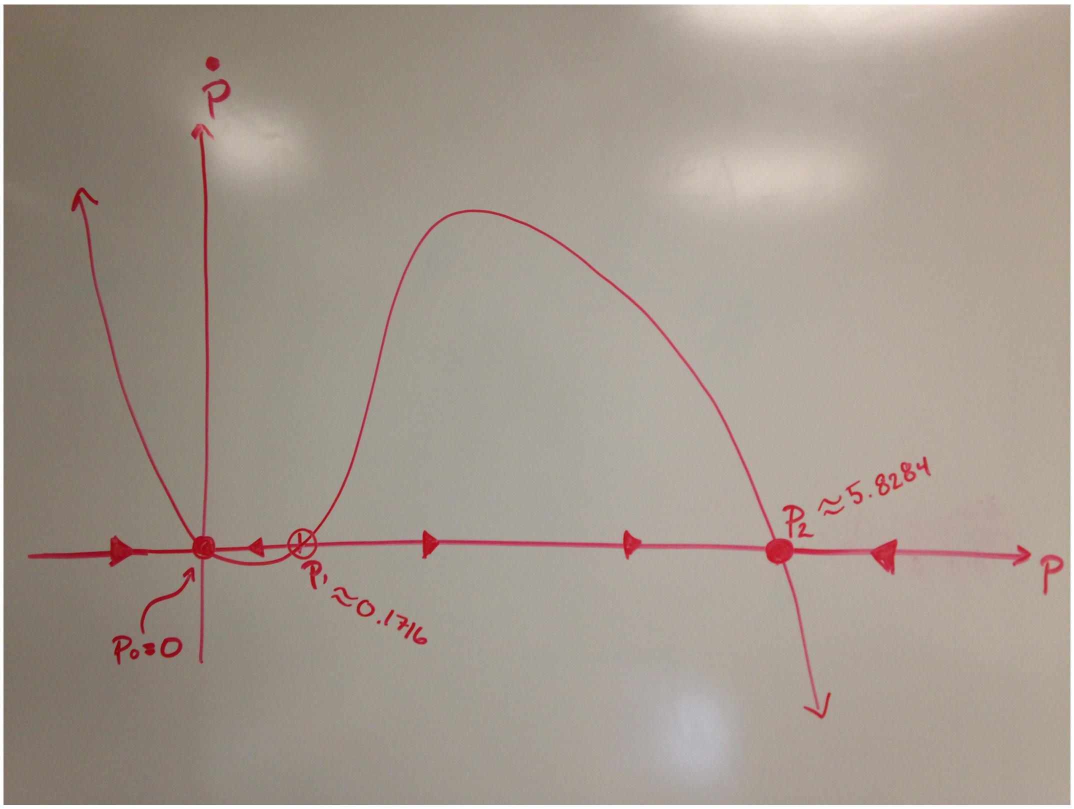
Mathematically, the system can be expressed with the following equation:

$$\dot{p} = -kp + A\frac{p^2}{p^2 + B^2}$$

where k, A, and B are positive constants that determine how quickly the protein is destroyed, how quickly the protein is formed, and how strongly the protein affects its formation, respectively. For simplicity, assume that $k = B = 1$.

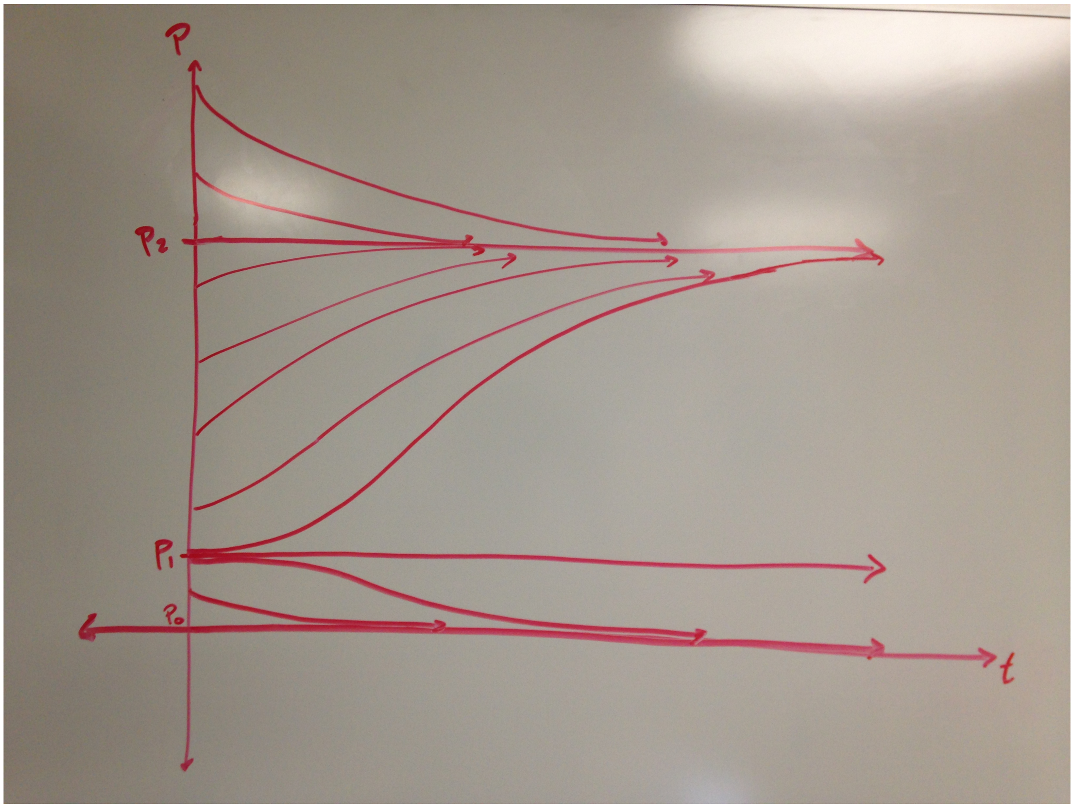
a)

Assuming that $k = B = 1$, draw a phase protrait for the case where $A = 6$ (i.e. plot $\dot{p}$ vs. p, indicate any fixed point(s) and indicate the stability of each. On the horizontal axis (p) indicate the flow direction of phase points).



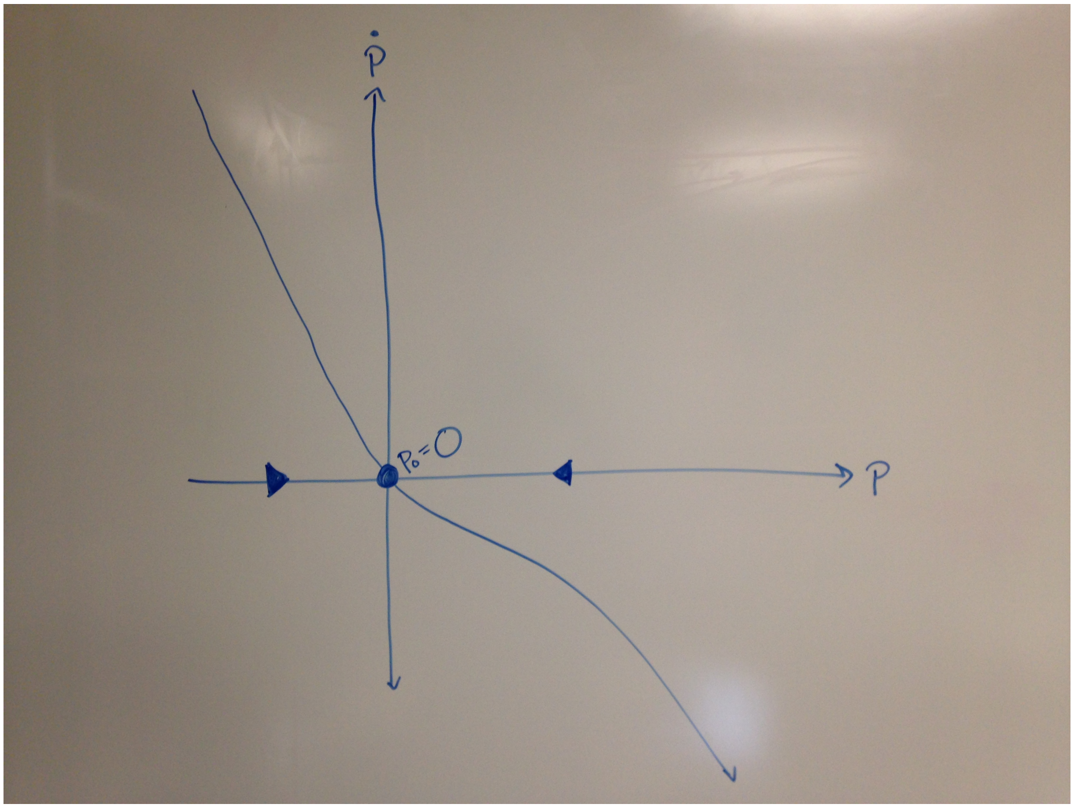
b)

Without solving the equation, sketch $p(t)$ as a function of t for several initial conditions for $A = 6$.



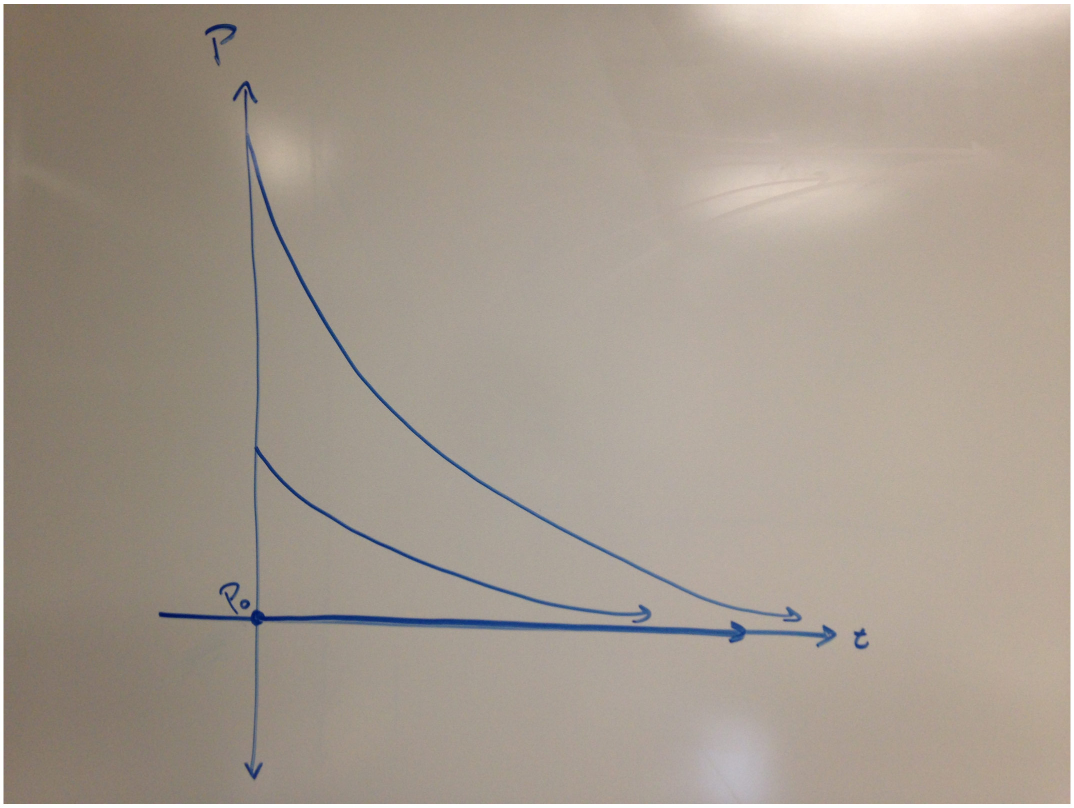
c)

Assuming that $k = B = 1$, draw a phase protrait for the case where $A = 1$ (i.e. plot $\dot{p}$ vs. p, indicate any fixed point(s) and indicate the stability of each. On the horizontal axis (p) indicate the flow direction of phase points).



d)

Without solving the equation, sketch $p(t)$ as a function of t for several initial conditions for $A = 1$.



e)

Suppose you wanted to maintain a non-zero steady-state concentration of protein in one of your cells. Would you choose $A = 1$ or $A = 6$? What else would you have to do to ensure a non-zero steady state?

Clearly, if $A = 1$, there is only one stable fixed point: the trivial fixed point $p_0 = 0$. However, if $A = 6$ there are three fixed points: the trivial fixed point $p_0 = 0$, which is stable, an unstable non-zero fixed point $p_1 \approx 0.1716$, and a stable non-zero fixed point $p_2 \approx 5.8284$. Thus, if we want to maintain a non-zero steady-state concentration of protein in one of our cells, we should choose $A = 6$. We also need to ensure the initial condition $p(0) > p_1$. This way, all solutions send the protein concentration to p_2. If $p(0) = p_1$, a negative perturbation would send the protein concentration to 0. If $p(0) < p_1$, all solutions send the protein concentration to 0.