HW #1

Writer: Sam Fleischer
Carter Johnson
Xiaoyun Niu

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Problem 1

A damped linear oscillator is a classical mechanical system. One typically analyzes it to death in math, physics, and engineering courses. Its importance lies in the fact that, near equilibrium, many systems behave like a damped linear osciallator. Here, you'll see how this works. Here are three differential equations that govern non-linear osciallors of one sort or another:

1. A mass on a wire (like you saw in homework 3, but here it is not over damped so it obeys a second order differential equation).

$$m\ddot{x} = -b\dot{x} - k\left(\sqrt{x^2+h^2} - \ell_0\right)\frac{x}{\sqrt{x^2+h^2}}$$

A non-dimensional form of this eqaution is

$$\frac{\mathrm{d}^2X}{\mathrm{d}T^2} = \frac{X}{\sqrt{X^2+\alpha^2}} - X - \beta\frac{\mathrm{d}X}{\mathrm{d}T} \tag{1}$$

2. A pendulim on a torsional spring (like you saw on midterm 1, but not over-damped).

$$-m\ell^2\ddot{\theta} = \zeta\dot{\theta} + \kappa\theta - mg\ell\sin\theta$$

A non-dimensional form of this equation is

$$\frac{\mathrm{d}^2X}{\mathrm{d}T^2} = \beta\frac{\mathrm{d}X}{\mathrm{d}T} - \alpha X + \sin X \tag{2}$$

3. The Duffing's oscillator (a model for a slender metal beam interacting with two magnets, which we will likely revisit), in non-dimensional form

$$\frac{\mathrm{d}^2X}{\mathrm{d}T^2} = -\frac{\mathrm{d}X}{\mathrm{d}T} + \beta X - \alpha X^3 \tag{3}$$

a)

Find the fixed points of each oscillator and classify them (i.e. stable node, unstable node, saddle, stable spiral, etc.). In all cases, $\beta > 0$ and $\alpha > 0$.

Consider system (1). Let $x_1 = X$, and $x_2 = \dot{X}$. Then

$$\begin{aligned} f_1(x_1, x_2) &= \dot{x_1} = x_2 \\ f_2(x_1, x_2) &= \dot{x_2} = \frac{x_1}{\sqrt{x_1^2 + \alpha^2}} - x_1 - \beta x_2 \end{aligned}$$

To find fixed points, set $\dot{x_1} = 0$ and $\dot{x_2} = 0$. Thus, $x_2 = 0$, and $x_1 = 0$, and if $|\alpha| < 1$, then x_1 may also equal $\pm\sqrt{1-\alpha^2}$. Thus the fixed points are

$$\vec{x_A}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x_B}^* = \begin{pmatrix} \sqrt{1-\alpha^2} \\ 0 \end{pmatrix} \quad \vec{x_C}^* = \begin{pmatrix} -\sqrt{1-\alpha^2} \\ 0 \end{pmatrix}$$

The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ \dfrac{\alpha^2}{(x_1^2+\alpha^2)^{3/2}} - 1 & -\beta \end{pmatrix}$$

Which yields the following characteristic equation

$$\lambda^2 + \beta\lambda + \left(1 - \frac{\alpha^2}{(x_1^2+\alpha^2)^{3/2}}\right) = 0$$

and the following eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left[-\beta \pm \sqrt{\beta^2 - 4\left(1 - \frac{\alpha^2}{(x_1^2+\alpha^2)^{3/2}}\right)}\right]$$

For the fixed point $\vec{x_A}^*$,

$$\lambda_{1,2} = \frac{1}{2}\left[-\beta \pm \sqrt{\beta^2 - 4\left(1 - \frac{1}{\alpha}\right)}\right]$$

If $\alpha > 1$, then $1 - \frac{1}{\alpha} > 0$, which implies $\mathrm{Re}[\lambda_{1,2}] < 0$. If $\alpha > 1$ and $\beta^2 > 4\big(1 - \frac{1}{\alpha}\big)$, then $\lambda_{1,2} \in \mathbb{R}$ and thus $\vec{x_A}^*$ is a stable node. If $\alpha > 1$ and $\beta^2 < 4\big(1 - \frac{1}{\alpha}\big)$, then $\lambda_{1,2} \notin \mathbb{R}$ and thus $\vec{x_A}^*$ is a stable spiral. If $\alpha < 1$, then $1 - \frac{1}{\alpha} < 0$, which implies $\lambda_2 < 0 < \lambda_1$ and thus $\vec{x_A}^*$ is a saddle.

For the fixed points $\vec{x_B}^*$ and $\vec{x_C}^*$,

$$\lambda_{1,2} = \frac{1}{2}\left[-\beta \pm \sqrt{\beta^2 - 4(1-\alpha^2)}\right]$$

Since the fixed points $\vec{x_B}^*$ and $\vec{x_C}^*$ only exist if $|\alpha| < 1$, we only consider $\alpha < 1$. Thus $\mathrm{Re}[\lambda_{1,2}] < 0$. If $\beta^2 > 4(1-\alpha^2)$, then $\lambda_{1,2} \in \mathbb{R}$ and thus $\vec{x_B}^*$ and $\vec{x_C}^*$ are stable nodes. If

$\beta^2 < 4(1-\alpha^2)$, then $\lambda_{1,2} \notin \mathbb{R}$ and thus $\vec{x_B}^*$ and $\vec{x_C}^*$ are stable spirals.

Consider system (2). Let $x_1 = X$, and $x_2 = \dot{X}$. Then

$$\begin{aligned} f_1(x_1, x_2) &= \dot{x_1} = x_2 \\ f_2(x_1, x_2) &= \dot{x_2} = -\beta x_2 - \alpha x_1 + \sin x_1 \end{aligned}$$

To find fixed points, set $\dot{x_1} = 0$ and $\dot{x_2} = 0$. Thus, $x_2 = 0$, and $\alpha x_1 = \sin x_1$. If $\alpha \geq 1$, then $x_1 = 0$ is the only solution. However, $\alpha < 1$ yields more solutions. Thus the fixed points are

$$\vec{x_A}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x_{B,1}}^* = \begin{pmatrix} x^*_{B,1} \\ 0 \end{pmatrix} \quad \ldots \quad \vec{x_{B,n}}^* = \begin{pmatrix} x^*_{B,n} \\ 0 \end{pmatrix}$$

where n is dependent on α, and $\alpha x^*_{B,i} = \sin x^*_{B,i}$ for all i. The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ -\alpha + \cos x_1 & -\beta \end{pmatrix}$$

Which yields the following characteristic equation

$$\lambda^2 + \beta\lambda + (\alpha - \cos x_1) = 0$$

and the following eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\Big[-\beta \pm \sqrt{\beta^2 - 4(\alpha - \cos x_1)}\Big]$$

For the fixed point $\vec{x_A}^*$,

$$\lambda_{1,2} = \frac{1}{2}\Big[-\beta \pm \sqrt{\beta^2 - 4(\alpha - 1)}\Big]$$

If $\alpha > 1$, then $\alpha - 1 > 0$, which implies $\mathrm{Re}[\lambda_{1,2}] < 0$. If $\alpha > 1$ and $\beta^2 > 4(\alpha - 1)$, then $\lambda_{1,2} \in \mathbb{R}$ and thus $\vec{x_A}^*$ is a stable node. If $\alpha > 1$ and $\beta^2 < 4(\alpha - 1)$, then $\lambda_{1,2} \notin \mathbb{R}$ and thus $\vec{x_A}^*$ is a stable spiral. If $\alpha < 1$, then $\alpha - 1 < 0$, which implies $\lambda_2 < 0 < \lambda_1$ and thus $\vec{x_A}^*$ is a saddle.

For the fixed points $\vec{x_{B,i}}^*$,

$$\lambda_{1,2} = \frac{1}{2}\Big[-\beta \pm \sqrt{\beta^2 - 4\big(\alpha - \cos x^*_{B,i}\big)}\Big]$$

Since each of $\vec{x_{B,i}}^*$ only exist if $\alpha < 1$, then we only have to consider that case. If $1 > \alpha > \cos x^*_{B,i}$ and $\beta^2 > 4\big(\alpha - \cos x^*_{B,i}\big)$, then $\lambda_{1,2} \in \mathbb{R}$ and $\vec{x_{B,i}}^*$ are stable nodes. If $1 > \alpha > \cos x^*_{B,i}$ and $\beta^2 < 4\big(\alpha - \cos x^*_{B,i}\big)$, then $\lambda_{1,2} \notin \mathbb{R}$ and $\vec{x_{B,i}}^*$ are stable spirals. If $\alpha < \cos x^*_{B,i}$, then $\lambda_2 < 0 < \lambda_1$ and thus $\vec{x_{B,i}}^*$ are saddles.

Consider system (3). Let $x_1 = X$, and $x_2 = \dot{X}$. Then

$$\begin{aligned} f_1(x_1, x_2) &= \dot{x_1} = x_2 \\ f_2(x_1, x_2) &= \dot{x_2} = -x_2 + \beta x_1 - \alpha x_1^3 \end{aligned}$$

To find fixed points, set $\dot{x_1} = 0$ and $\dot{x_2} = 0$. Thus, $x_2 = 0$, and $\alpha x_1 = 0, \pm\sqrt{\frac{\beta}{\alpha}}$. Thus the fixed points are

$$\vec{x_A}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x_B}^* = \begin{pmatrix} \sqrt{\frac{\beta}{\alpha}} \\ 0 \end{pmatrix} \quad \vec{x_C}^* = \begin{pmatrix} -\sqrt{\frac{\beta}{\alpha}} \\ 0 \end{pmatrix}$$

The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ \beta - 3\alpha x_1^2 & -1 \end{pmatrix}$$

Which yields the following characteristic equation

$$\lambda^2 + \lambda + \left(3\alpha x_1^2 - \beta\right) = 0$$

and the following eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left[-1 \pm \sqrt{1 - 4(3\alpha x_1^2 - \beta)}\right]$$

For the fixed point $\vec{x_A}^*$,

$$\lambda_{1,2} = \frac{1}{2}\left[-1 \pm \sqrt{1 + 4\beta}\right]$$

which is a saddle since $\beta > 0$.

For the fixed points $\vec{x_B}^*$ and $\vec{x_C}^*$,

$$\lambda_{1,2} = \frac{1}{2}\left[-1 \pm \sqrt{1 - 8\beta}\right]$$

Since $\beta > 0$, $\mathrm{Re}[\lambda_{1,2}] < 0$. If $8\beta < 1$, then $\lambda_{1,2} \in \mathbb{R}$ and thus $\vec{x_B}^*$ and $\vec{x_C}^*$ are stable nodes. If $8\beta > 1$, then $\lambda_{1,2} \notin \mathbb{R}$ and thus $\vec{x_B}^*$ and $\vec{x_C}^*$ are stable spirals.

b)

For each oscillator, choose a fixed point that is stable in some parameter regime and write the linearized equations.

Consider system (1) and the fixed point $\vec{x_A}^* = [0, 0]^T$. The linearized system around $\vec{x_A}^*$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = J|_{\vec{x_A}^*} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1-\alpha}{\alpha} & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $J|_{\vec{x_A}^*}$ is the Jacobian evaluated at $\vec{x_A}^*$. If $\alpha > 1$ and $\beta^2 > 4(1 - \frac{1}{\alpha})$, then by part **a)**, $\vec{x_A}^*$ is a stable node. The linearized system is

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= \frac{1-\alpha}{\alpha} x_1 - \beta x_2 \end{aligned}$$

Consider system (2) and the fixed point $\vec{x_A}^* = [0,0]^T$. The linearized system around $\vec{x_A}^*$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = J|_{\vec{x_A}^*} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha+1 & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $J|_{\vec{x_A}^*}$ is the Jacobian evaluated at $\vec{x_A}^*$. If $\alpha > 1$ and $\beta^2 < 4(\alpha - 1)$, then by part **a)**, $\vec{x_A}^*$ is a stable spiral. The linearized system is

$$\dot{x_1} = x_2$$
$$\dot{x_2} = (-\alpha + 1)x_1 - \beta x_2$$

Consider system (3) and the fixed point $\vec{x_B}^* = \left[\sqrt{\frac{\beta}{\alpha}}, 0\right]^T$. The linearized system around $\vec{x_B}^*$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = J|_{\vec{x_B}^*} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\beta & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $J|_{\vec{x_B}^*}$ is the Jacobian evaluated at $\vec{x_B}^*$. If $8\beta > 1$, then by part **a)**, $\vec{x_B}^*$ is a stable spiral. The linearized system is

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -2\beta x_1 - x_2$$

c)

Compare your linearization to that of a linear oscialltor $\ddot{x} = -(k/m)x - (b/m)\dot{x}$ and determine the effective spring constant k/m and effective damping b/m for each system.

For the linear oscillator, let $x_1 = x$ and $x_2 = \dot{x}$. Thus

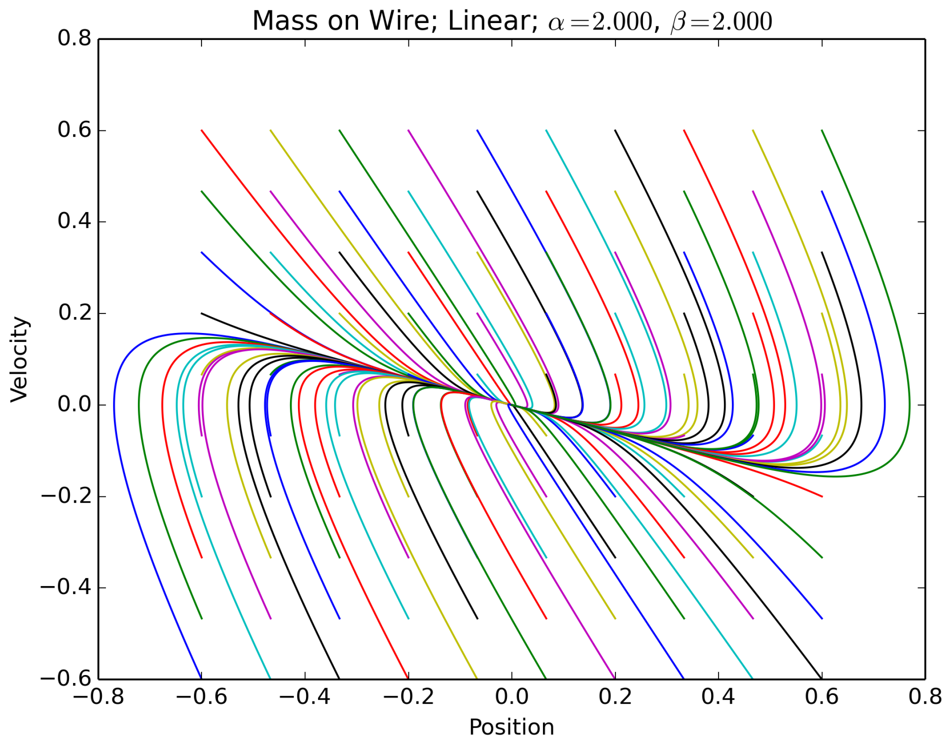
$$\dot{x_1} = x_2$$
$$\dot{x_2} = -\frac{k}{m}x_1 - \frac{b}{m}x_2$$

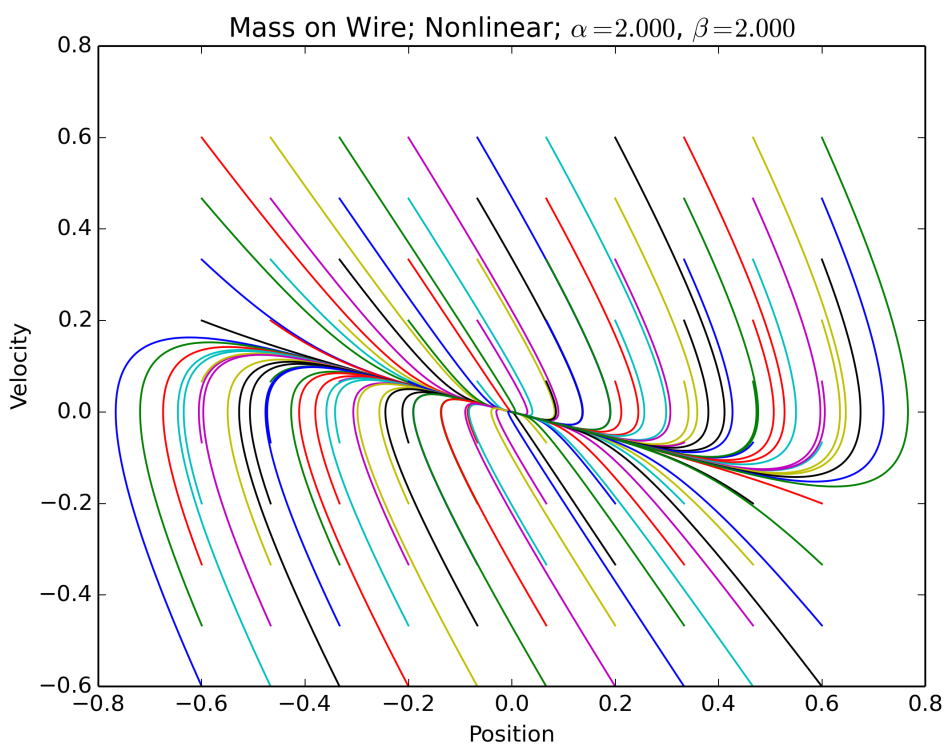
Clearly, for system (1), the effective spring constant is $\frac{\alpha-1}{\alpha}$ and the effective damping is β. For system (2), the effective spring constant is $\alpha - 1$ and the effective damping is β. For system (3), the effective spring constant is -2β and the effective damping is -1.

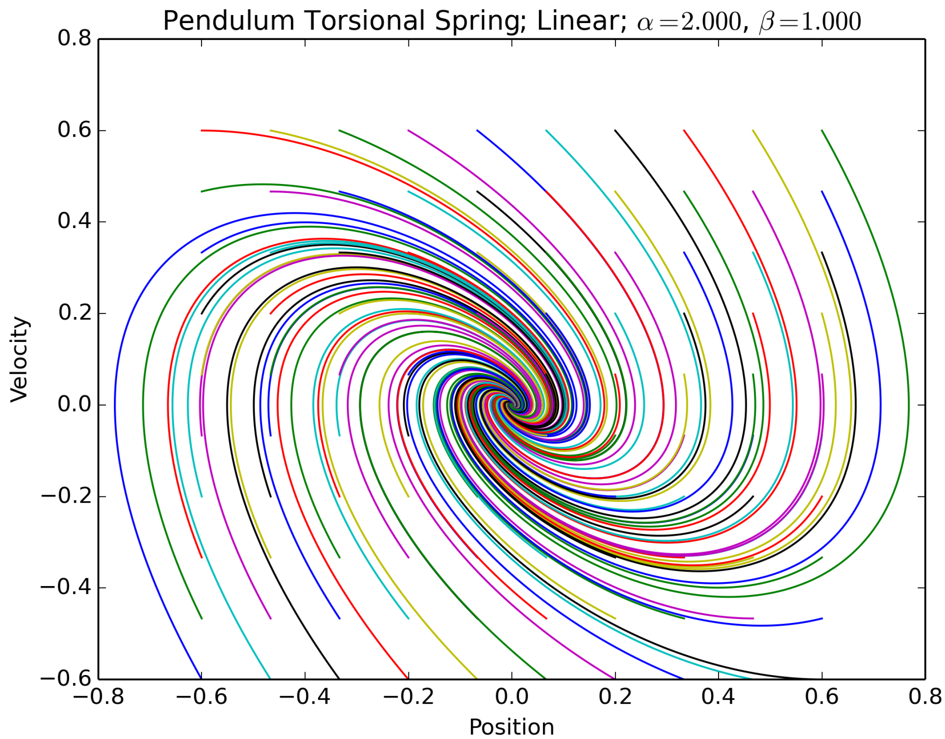
d)

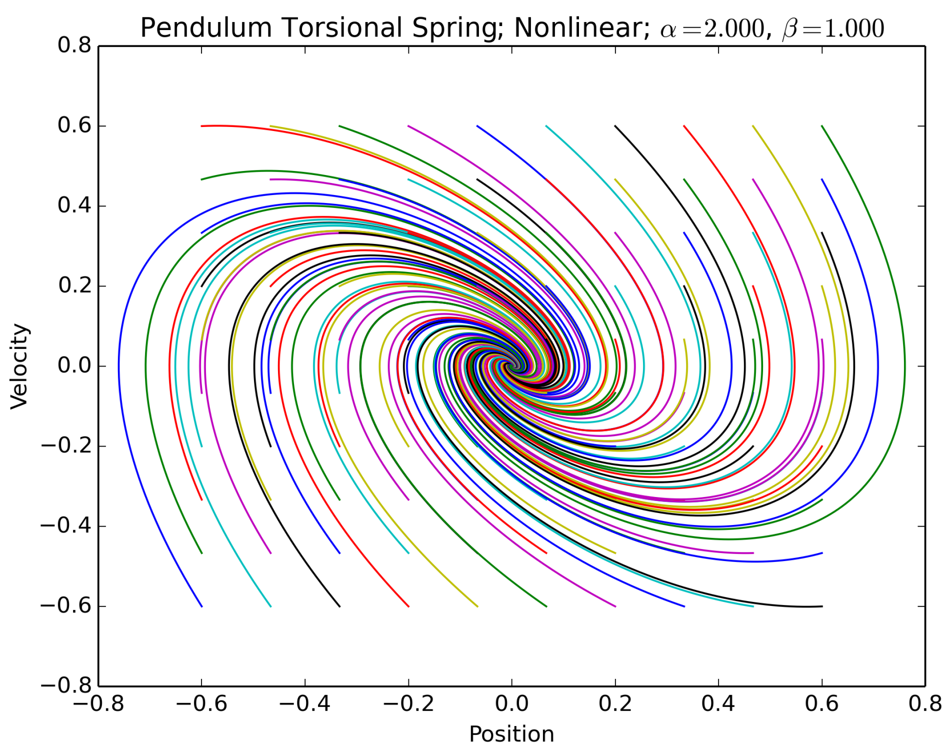
Use Matlab to check your work. Pick values of α and β and run some simulations of the three non-linear oscillators. Compare these to the predictions of the linear system you found in part c, which can be solved analytically (as you did on HW 1).

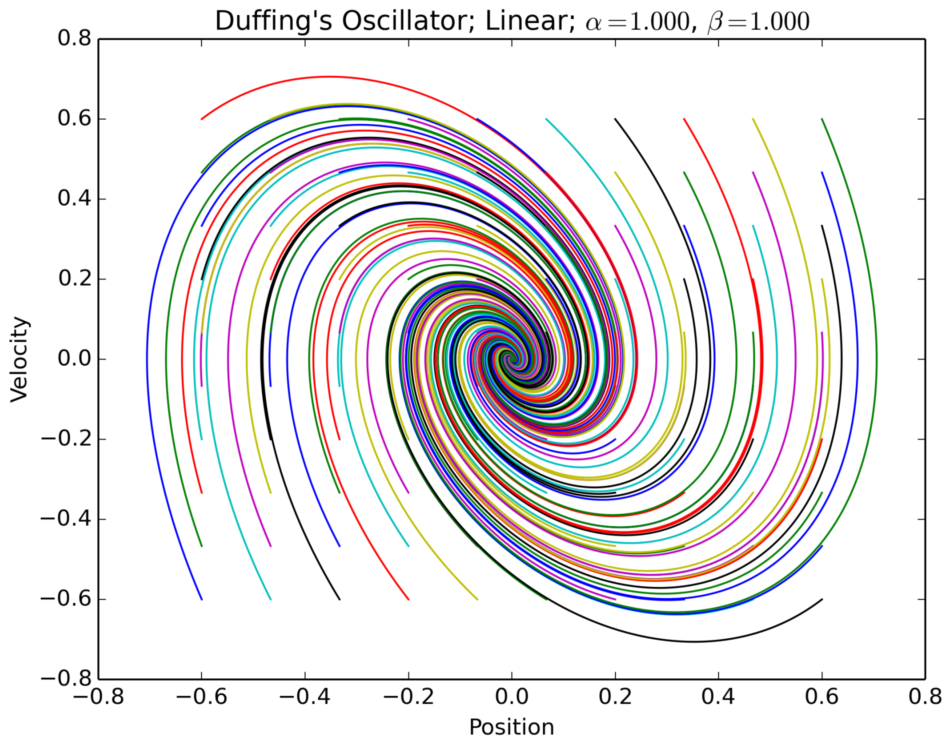
In the following simluations of each system, we chose α and β such that the fixed points chosen in part **b)** are stable. Then we generated the phase trajectories of the linear and nonlinear systems near the stable fixed points.

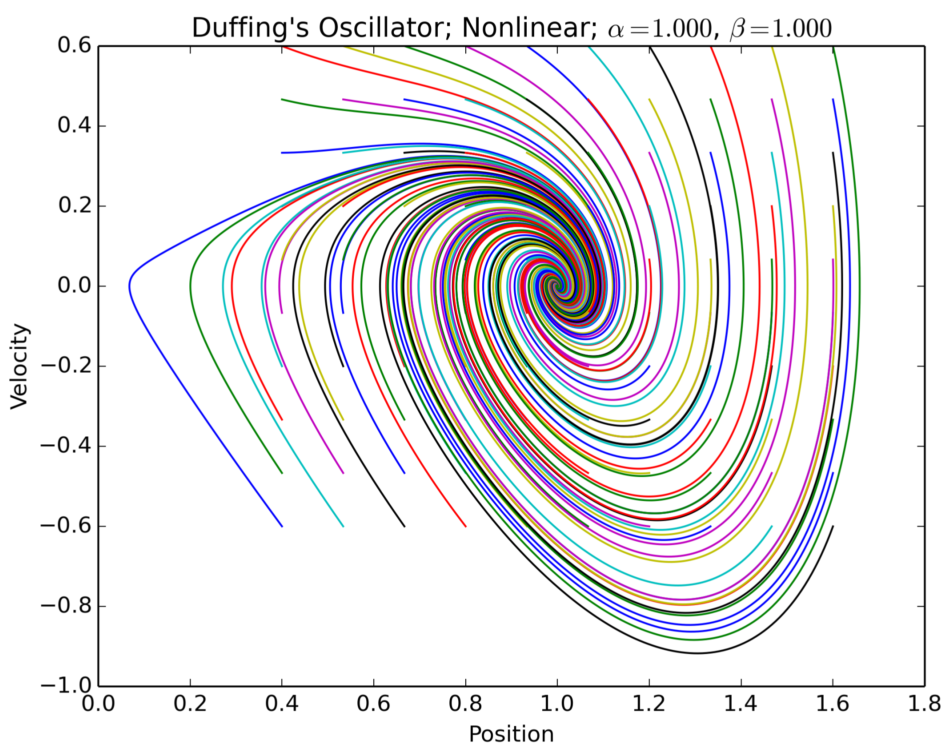

Mass on Wire; Linear; $\alpha = 2.000$, $\beta = 2.000$
Velocity
Position
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8


Mass on Wire; Nonlinear; $\alpha = 2.000$, $\beta = 2.000$
Velocity
Position
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8


Pendulum Torsional Spring; Linear; $\alpha = 2.000$, $\beta = 1.000$
Velocity
Position
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8


Pendulum Torsional Spring; Nonlinear; $\alpha = 2.000$, $\beta = 1.000$
Velocity
Position
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8


Duffing's Oscillator; Linear; $\alpha = 1.000$, $\beta = 1.000$
Velocity
Position
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
0.0
0.2
0.4
0.6
0.8


Duffing's Oscillator; Nonlinear; $\alpha = 1.000$, $\beta = 1.000$
Velocity
Position
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8

Problem 2

Problem 6.3.3

a)

Find all fixed points, classify them and fill in the rest of your phase portrait for the following system of equations.

$$\dot{x} = 1 + y - e^{-x}$$
$$\dot{y} = x^3 - y$$

To solve for equilibrium points, set $\dot{x} = \dot{y} = 0$. Then solve both for y to yield

$$y = e^{-x} - 1$$
$$y = x^3$$

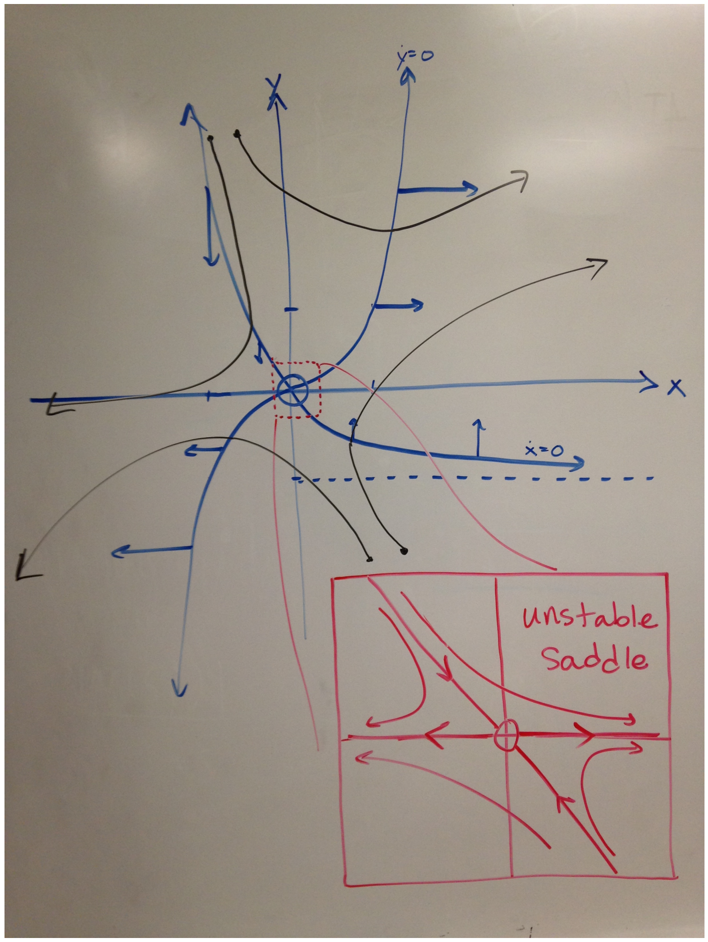
Since $e^{-x} - 1$ is a strictly decreasing function of x, and x^3 is a strictly increasing function of x, this system has at most one solution. Since $(x, y) = (0, 0)$ is a solution, it must be the only solution. Thus $\vec{x}^* = [0, 0]^T$ is the only fixed point. The Jacobian is

$$J = \begin{pmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{pmatrix}$$
$$\implies J_{\vec{x}^*} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

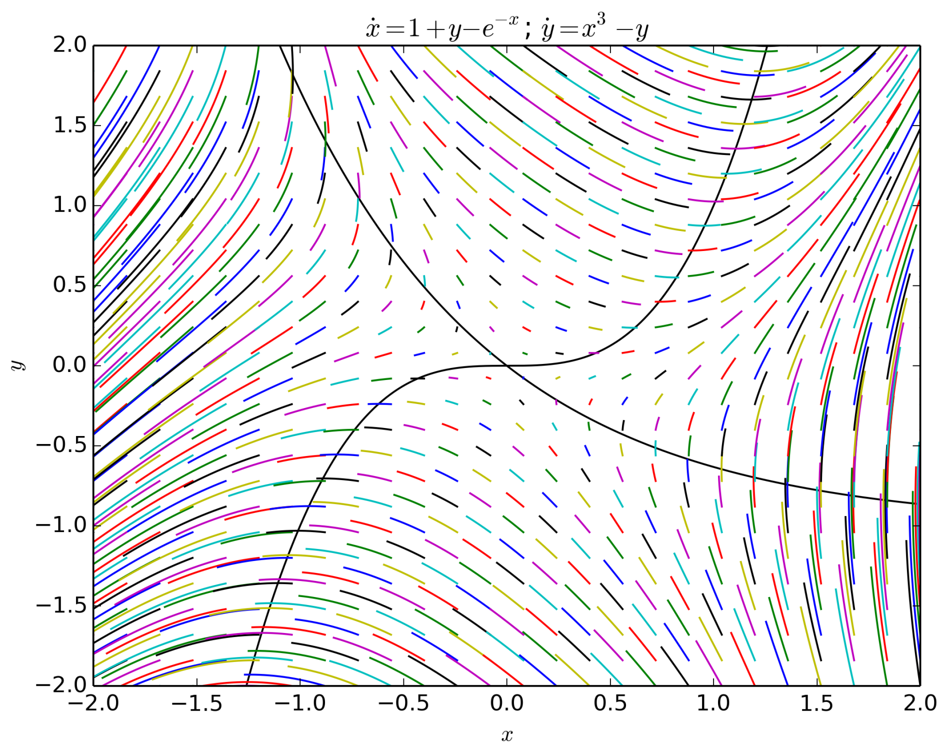
which is a triangular matrix, and so the eigenvalues are $\lambda_{1,2} = \pm 1$, which shows $\vec{x}^*$ is a saddle.

b)

Check your answers by generating a phase portrait with Matlab. (Simulate several initial conditions, and plot x and y)



$\dot{y}=0$
y
x
$\dot{x}=0$
unstable
Saddle



$\dot{x}=1+y-e^{-x}$; $\dot{y}=x^3-y$
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x
y

Problem 3

Problem 6.3.6

a)

Find all fixed points, classify them and fill in the rest of your phase portrait for the following system of equations.

$$\dot{x} = xy - 1$$
$$\dot{y} = x - y^3$$

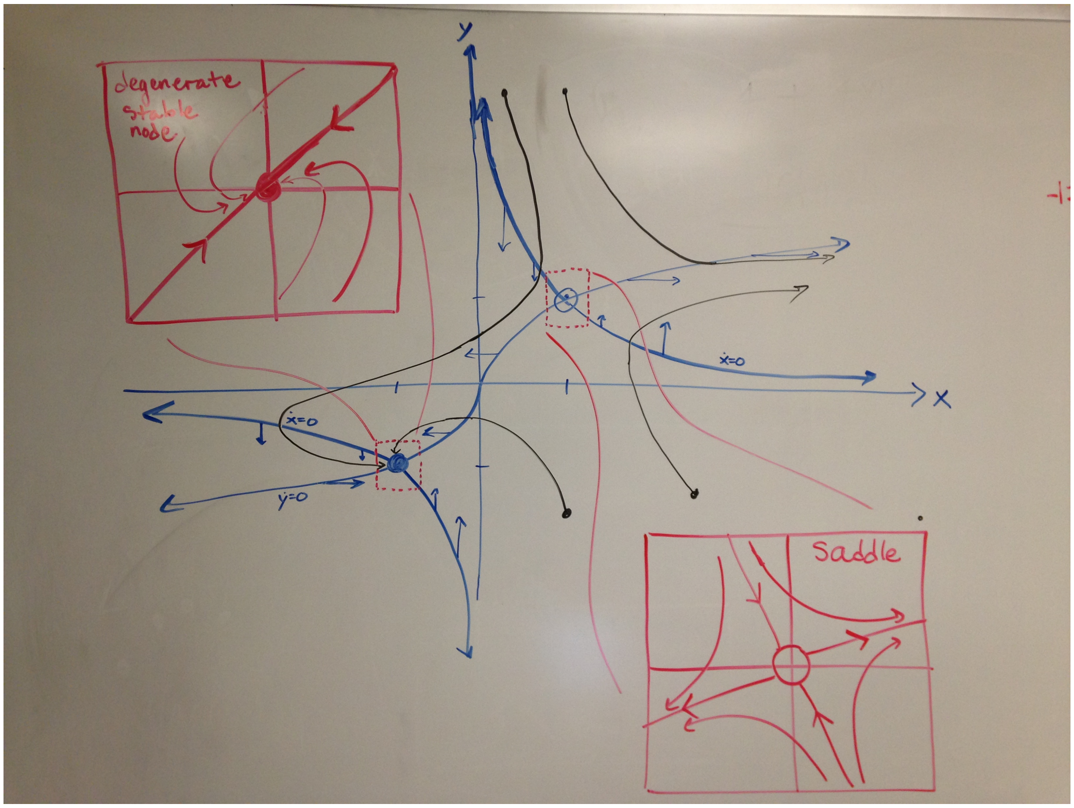
Set $\dot{x} = \dot{y} = 0$, and substitute $x = y^3$ in to $\dot{x} = 0$, yielding $y^4 = 1$, i.e. $y = \pm 1$. $y = 1 \implies x = 1$, and $y = -1 \implies x = -1$. Thus the fixed points are $\vec{x_A}^* = [1, 1]^T$ and $\vec{x_B}^* = [-1, -1]^T$. The Jacobian is

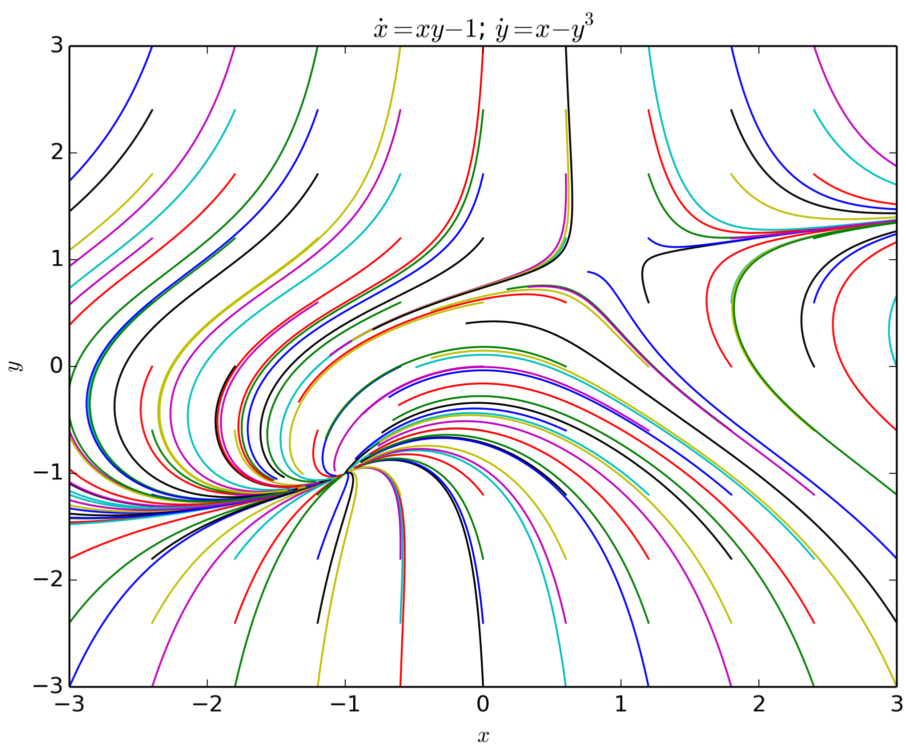
$$J = \begin{pmatrix} y & x \\ 1 & -3y^2 \end{pmatrix}$$
$$\implies J_{\vec{x_A}^*} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \quad \text{and} \quad J_{\vec{x_B}^*} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$$

Thus the eigenvalues for $\vec{x_A}^*$ are $\lambda_{1,2} = -1 \pm \sqrt{5}$, and the eigenvalues for $\vec{x_B}^*$ are $\lambda_{1,2} = -2$, which shows $\vec{x_A}^*$ is a saddle and $\vec{x_B}^*$ is a stable node.

b)

Check your answers by generating a phase portrait with Matlab. (Simulate several initial conditions, and plot x and y)


degenerate
stable
node
Saddle
y
x
ẋ=0
y=0


$\dot{x}=xy-1;\ \dot{y}=x-y^3$
x
y

Problem 4

Problem 6.4.1

a)

The following is a "rabbits vs. sheep" model of two species competing for resources (in this case, rabbits and sheep competing for grass). They are discussed in §6.4 in the book.

$$\dot{x} = x(3 - x - y)$$
$$\dot{y} = y(2 - x - y)$$

Find the fixed points, investigate their stability and draw plausible phase portraits.

$\dot{x} = 0 \implies x = 0$ or $y = -x + 3$. $\dot{y} = 0 \implies y = 0$ or $y = -x + 2$. Thus the fixed points are

$$\vec{x_A}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x_B}^* = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \vec{x_C}^* = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

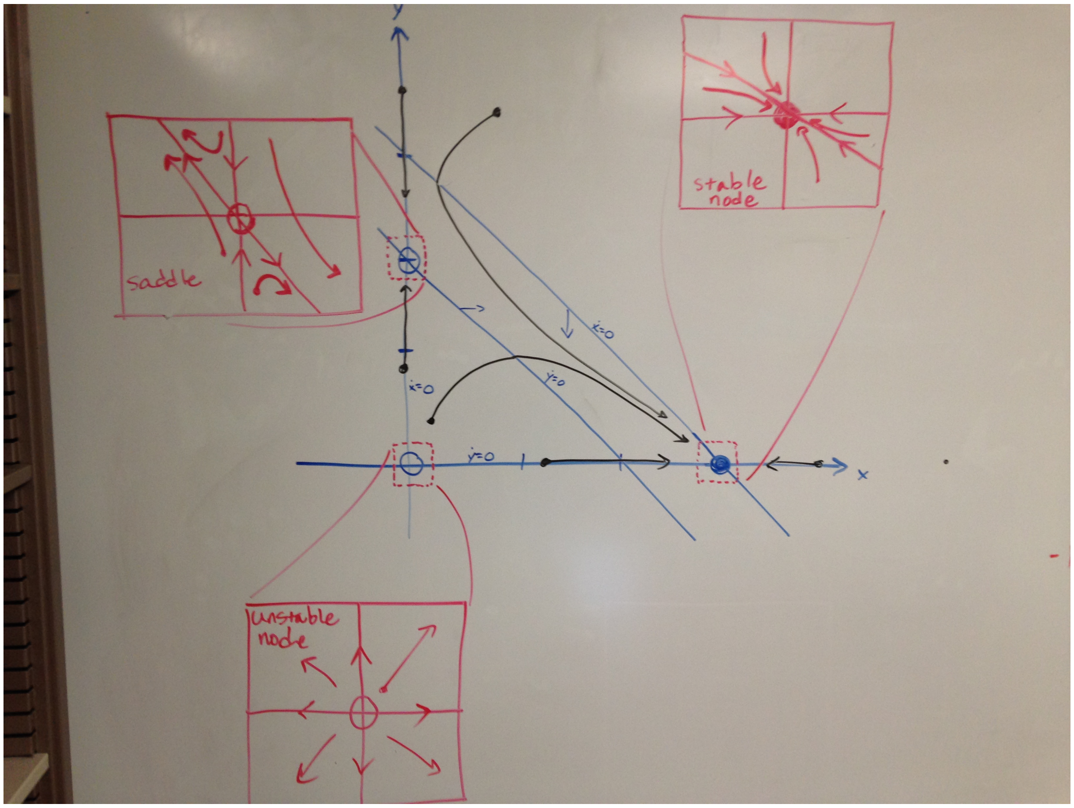
The Jacobian is

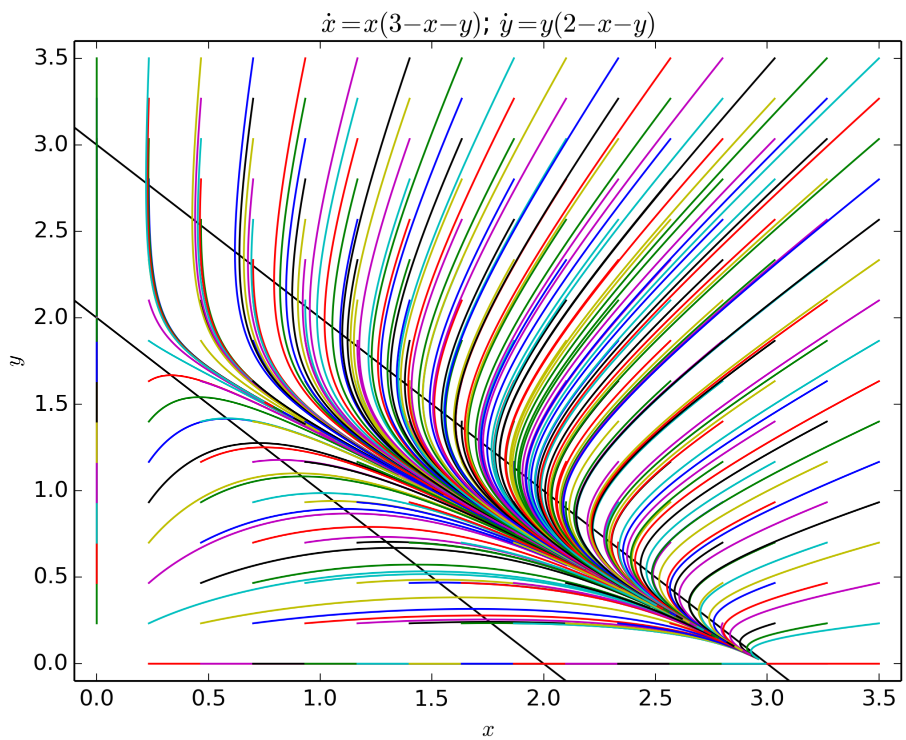
$$J = \begin{pmatrix} 3 - 2x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}$$
$$\implies J_{\vec{x_A}^*} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } J_{\vec{x_B}^*} = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \text{ and } J_{\vec{x_C}^*} = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$$

Thus the eigenvalues for $\vec{x_A}^*$ are 3 and 2, and so it is an unstable node. The eigenvalues for $\vec{x_B}^*$ are -3 and -1, and so it is a stable node. The eigenvalues for $\vec{x_C}^*$ are 1 and -2, and so it is a saddle.

b)

Check your answers by generating a phase portrait with Matlab. (Simulate several initial conditions, and plot x and y)


saddle
stable node
unstable node


$\dot{x} = x(3-x-y)$; $\dot{y} = y(2-x-y)$
x
y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5