

# HW #1

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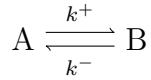
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## Problem 1

*Consider the following chemical reaction, where one chemical ( $A$ ) turns into a different chemical ( $B$ ) and vice versa. Suppose that the total amount of chemical is constant, that is  $A(t) + B(t) = C$ , where  $C$  is a positive constant. This reaction can be represented schematically in the following way:*



*where the two positive constants  $k^+$  and  $k^-$  are called rate constants.*

*The following differential equation describes how  $A$  changes with time*

$$\frac{dA}{dt} = -k^+ A + k^- B \tag{1}$$

*Recall that, in addition to this differential equation, we have the conservation constraint  $A(t) + B(t) = C$ .*

**a)**

*Solve for  $A(t)$ , given  $A(0) = A_0$ , with  $A_0$  being a positive constant such that  $A_0 < C$ .*

Eq. (1) can be simplified by using the conservation constraint as

$$\begin{aligned} \frac{dA}{dt} &= -k^+ A + k^-(C - A) \\ &= -(k^+ + k^-)A + k^-C \end{aligned}$$

Let  $u(t) = -(k^+ + k^-)A(t) + k^-C$ . Then  $\dot{u} = -(k^+ + k^-)\dot{A}$ . Thus,

$$\dot{u} = -(k^+ + k^-)u \tag{2}$$

The solution to Eq. (2) is exponential, i.e.

$$u(t) = Z \exp(-(k^+ + k^-)t) \quad \text{for some arbitrary constant } Z$$

$$\implies A(t) = Z \exp(-(k^+ + k^-)t) + \frac{k^- C}{k^+ + k^-}$$

The initial condition  $A(0) = A_0$  implies

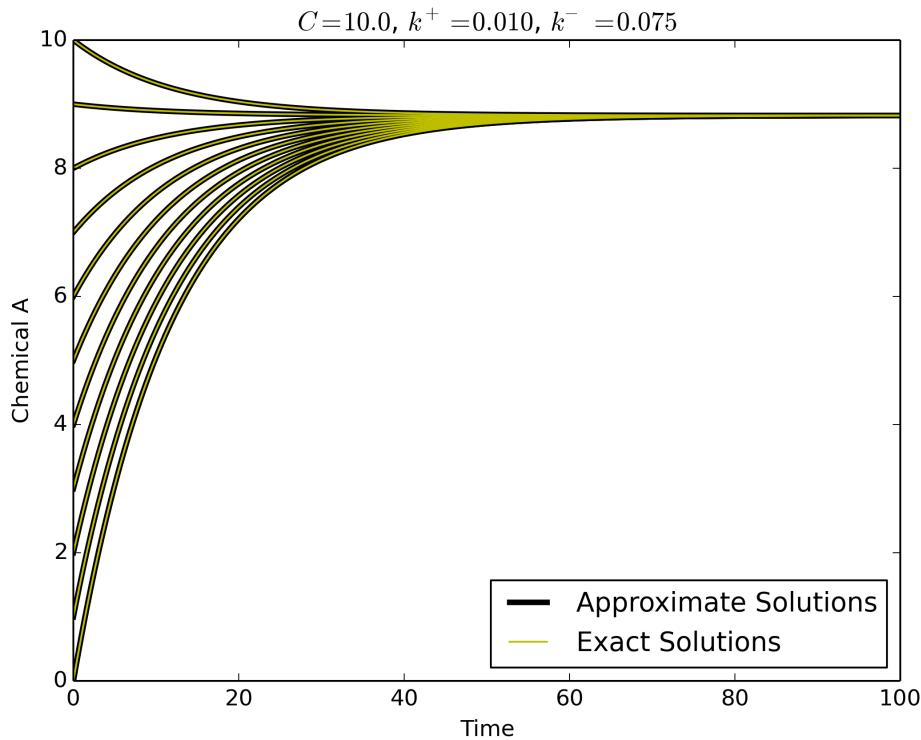
$$Z = A_0 - \frac{k^- C}{k^+ + k^-}$$

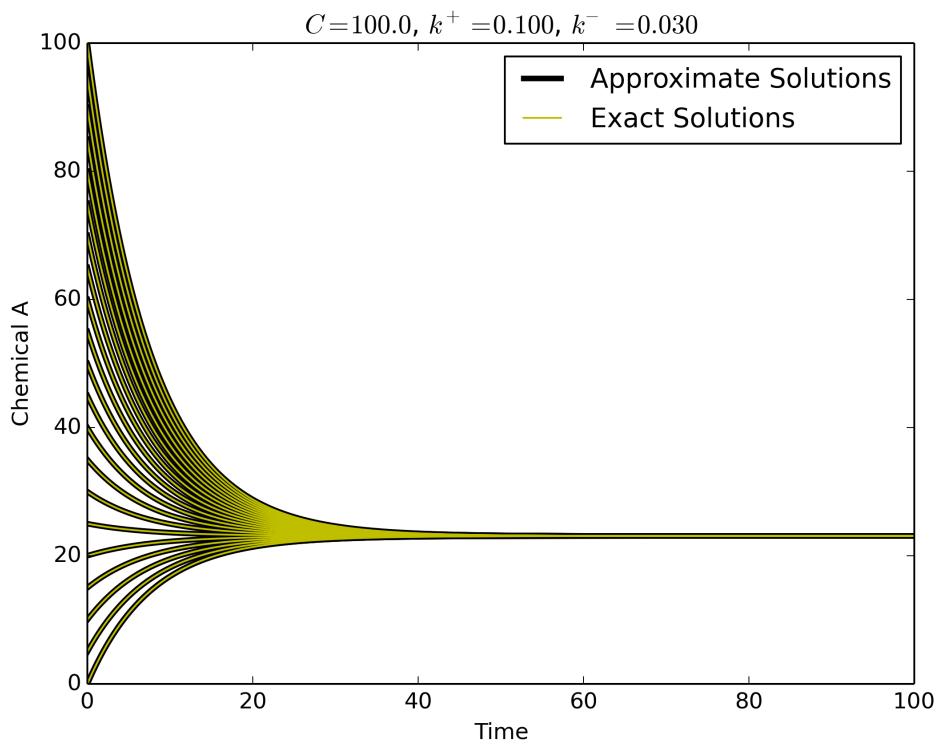
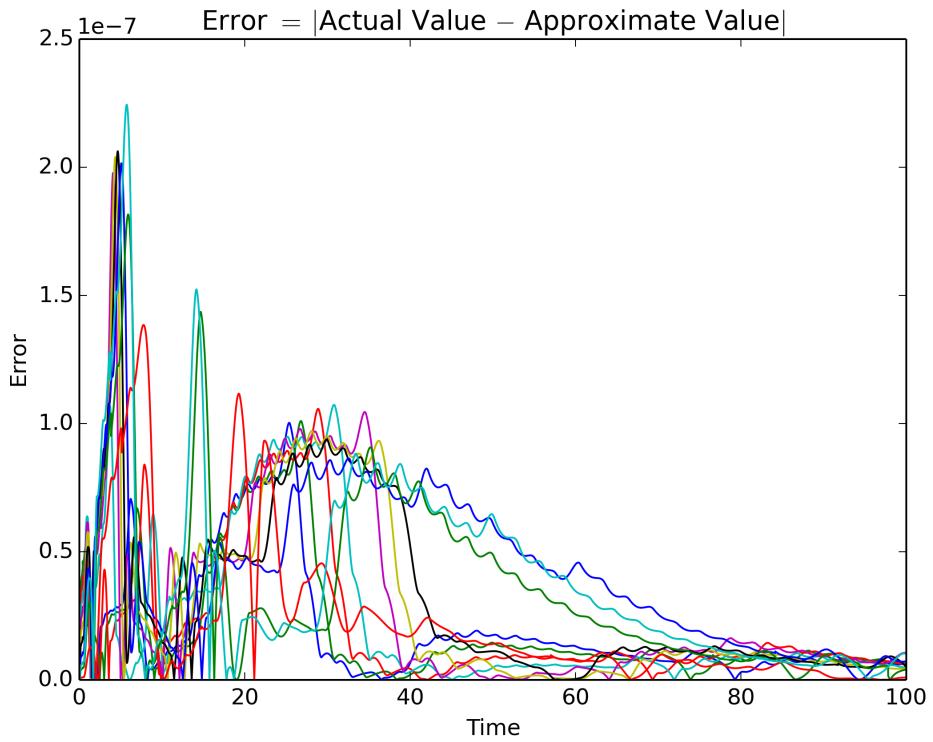
and so

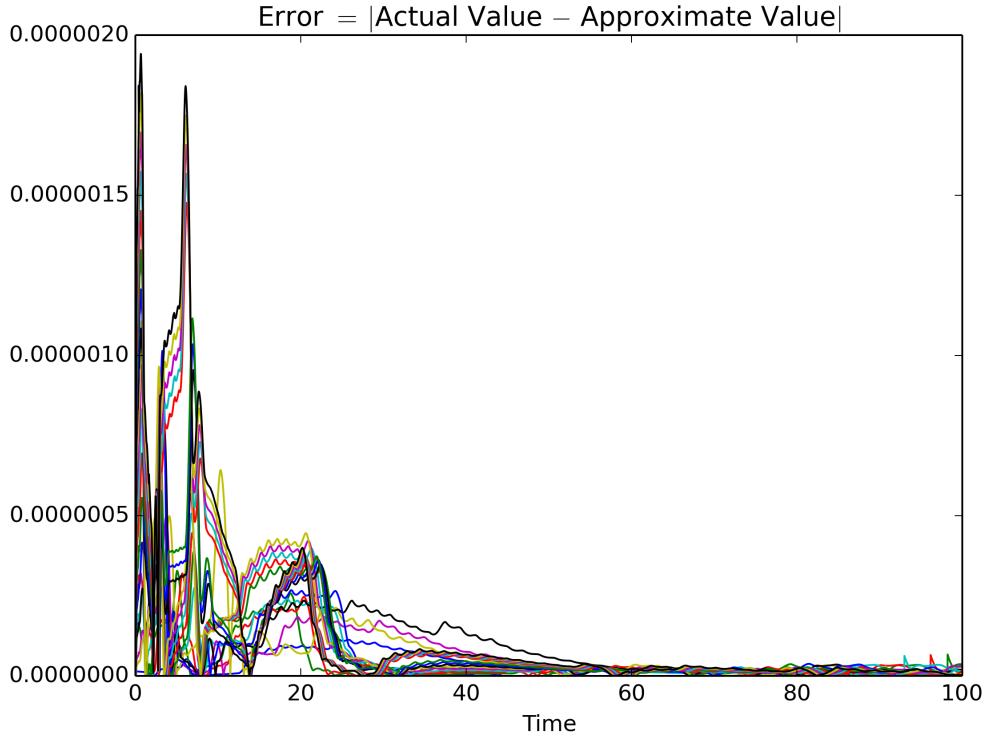
$$A(t) = \left( A_0 - \frac{k^- C}{k^+ + k^-} \right) \exp(-(k^+ + k^-)t) + \frac{k^- C}{k^+ + k^-} \quad (3)$$

**b)**

Use Matlab to check your answer for a few choices of  $A_0$ ,  $C$ ,  $k^+$ , and  $k^-$  (I have provided code that will assist you).







## Problem 2

The position of a moving object in 1-D ( $x(t)$ ) on a damped linear spring obeys the following differential equation

$$m\ddot{x} = -b\dot{x} - kx \quad (4)$$

where  $m$ ,  $b$ , and  $k$  are positive constants representing the mass of the object, the damping coefficient and the stiffness of the spring, respectively.

a)

Solve for  $x(t)$ , given  $x(0) = x_0$  and  $\dot{x}(0) = v_0$ .

Since Eq. (4) is a linear ODE, we can solve for the roots the characteristic polynomial, which is

$$\begin{aligned} P(\lambda) &= m\lambda^2 + b\lambda + k = 0 \\ \implies \lambda_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \end{aligned}$$

and thus the solution to Eq. (4) is

$$x(t) = \begin{cases} C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) & \text{if } \lambda_1 \neq \lambda_2 \text{ and } \lambda_1, \lambda_2 \in \mathbb{R} \\ C_1 \exp(\lambda t) + C_2 t \exp(\lambda t) & \text{if } \lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \\ C_1 \exp(\alpha t) \sin(\beta t) + C_2 \exp(\alpha t) \cos(\beta t) & \text{if } \lambda_{1,2} = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

The conditions simplify based on the formulation of  $\lambda_{1,2}$  to

$$x(t) = \begin{cases} C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) & \text{if } b^2 - 4mk > 0 \\ C_1 \exp(\lambda t) + C_2 t \exp(\lambda t) & \text{if } b^2 - 4mk = 0 \\ C_1 \exp(\alpha t) \sin(\beta t) + C_2 \exp(\alpha t) \cos(\beta t) & \text{if } b^2 - 4mk < 0 \end{cases} \quad (5)$$

**Case 1:**  $b^2 - 4mk > 0$

$$\begin{aligned} x(t) &= C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) \\ \dot{x}(t) &= C_1 \lambda_1 \exp(\lambda_1 t) + C_2 \lambda_2 \exp(\lambda_2 t) \end{aligned}$$

Since  $x(0) = x_0$  and  $\dot{x}(0) = v_0$ ,

$$\begin{aligned} x_0 &= C_1 + C_2 \quad \text{and} \quad v_0 = C_1 \lambda_1 + C_2 \lambda_2 \\ \implies C_1 &= \frac{x_0 \lambda_2 - v_0}{\lambda_2 - \lambda_1} \quad \text{and} \quad C_2 = \frac{x_0 \lambda_1 - v_0}{\lambda_1 - \lambda_2} \\ \implies x(t) &= \boxed{\frac{x_0 \lambda_2 - v_0}{\lambda_2 - \lambda_1} \exp(\lambda_1 t) + \frac{x_0 \lambda_1 - v_0}{\lambda_1 - \lambda_2} \exp(\lambda_2 t)} \end{aligned}$$

**Case 2:**  $b^2 - 4mk = 0$

$$\begin{aligned} x(t) &= C_1 \exp(\lambda t) + C_2 t \exp(\lambda t) \\ \dot{x}(t) &= C_1 \lambda \exp(\lambda t) + C_2 \lambda t \exp(\lambda t) + C_2 \exp(\lambda t) \end{aligned}$$

Since  $x(0) = x_0$  and  $\dot{x}(0) = v_0$ ,

$$\begin{aligned} x_0 &= C_1 \quad \text{and} \quad v_0 = C_1 \lambda + C_2 \\ \implies C_1 &= x_0 \quad \text{and} \quad C_2 = v_0 - x_0 \lambda \\ \implies x(t) &= \boxed{x_0 \exp(\lambda t) + (v_0 - x_0 \lambda) t \exp(\lambda t)} \end{aligned}$$

**Case 3:**  $b^2 - 4mk < 0$

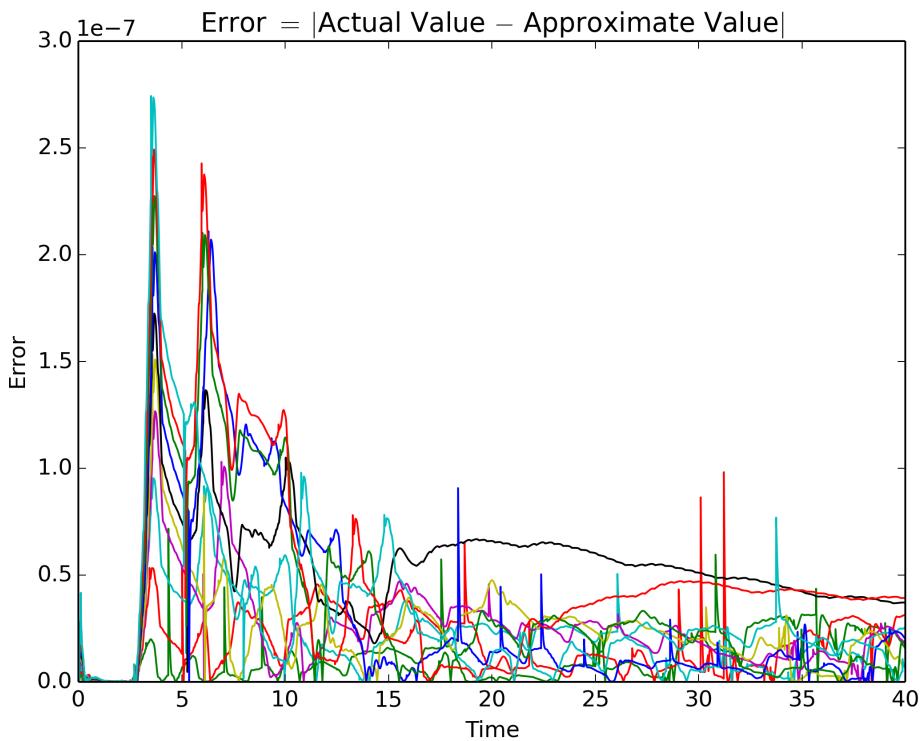
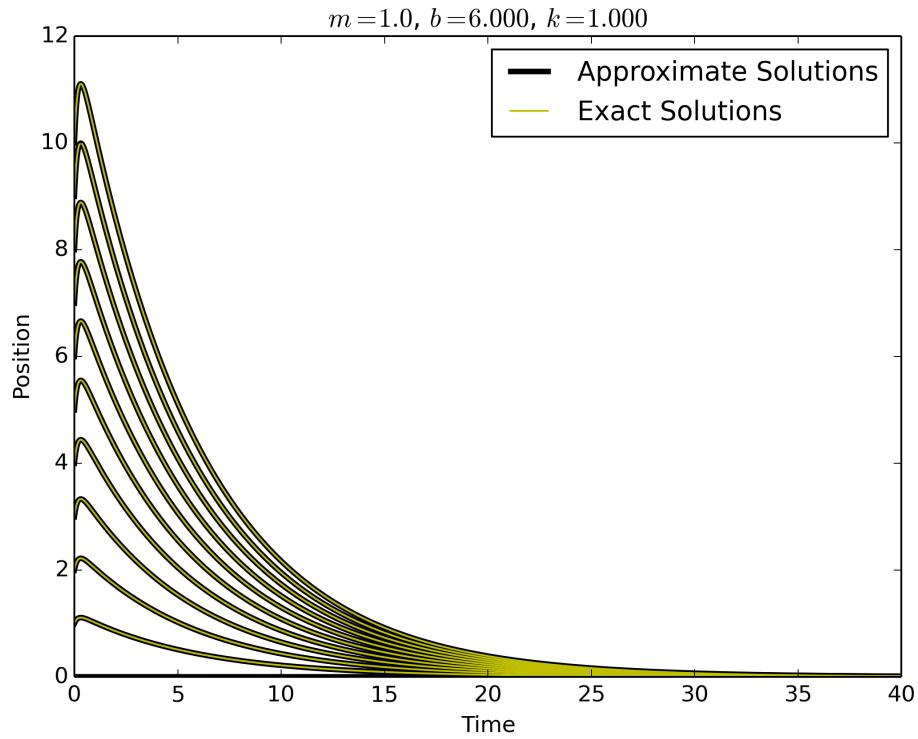
$$\begin{aligned} x(t) &= C_1 \exp(\alpha t) \sin(\beta t) + C_2 \exp(\alpha t) \cos(\beta t) \\ \dot{x}(t) &= C_1 (\alpha \exp(\alpha t) \sin(\beta t) + b \exp(\alpha t) \cos(\beta t)) + C_2 (\alpha \exp(\alpha t) \cos(\beta t) - b \exp(\alpha t) \sin(\beta t)) \end{aligned}$$

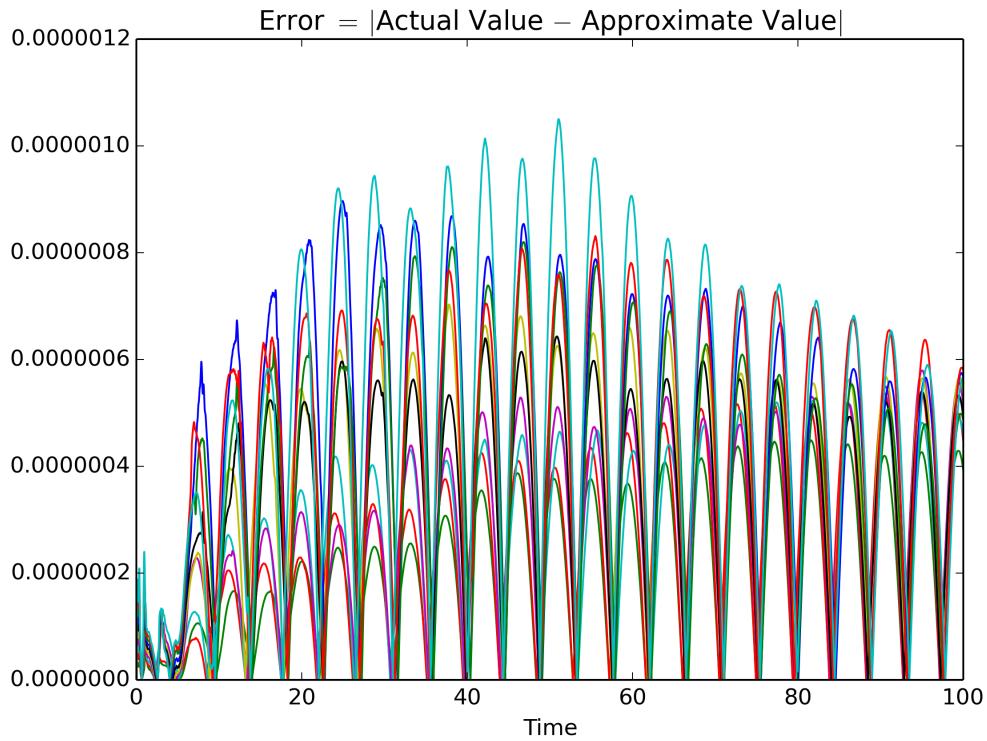
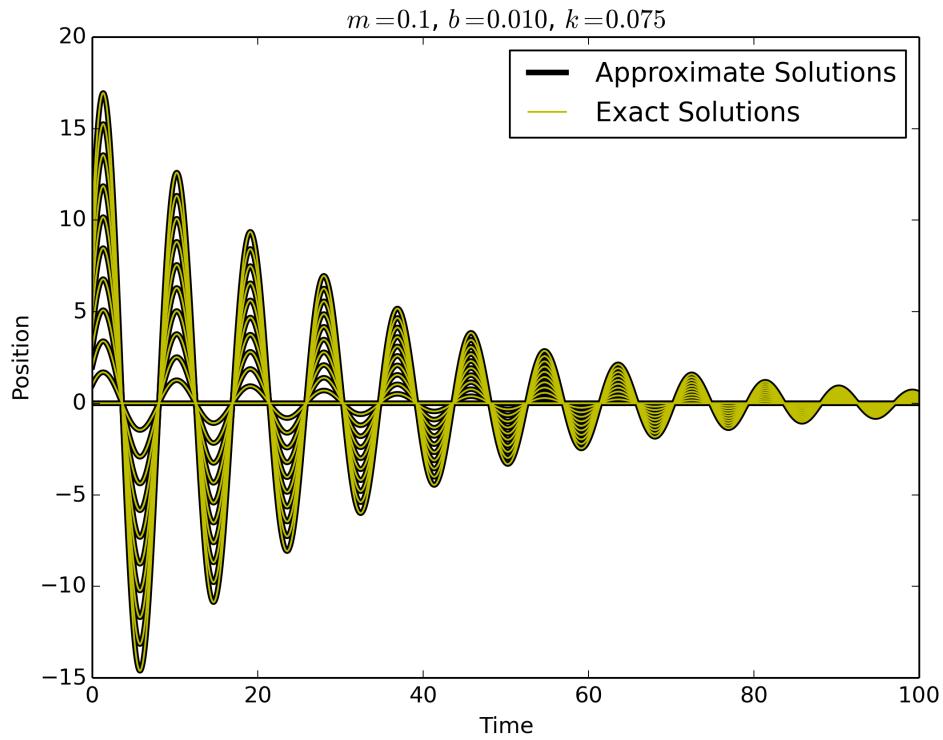
Since  $x(0) = x_0$  and  $\dot{x}(0) = v_0$ ,

$$\begin{aligned} x_0 &= C_2 \quad \text{and} \quad v_0 = C_1 \beta + C_2 \alpha \\ \implies C_1 &= \frac{v_0 - x_0 \alpha}{\beta} \quad \text{and} \quad C_2 = x_0 \\ \implies x(t) &= \boxed{\frac{v_0 - x_0 \alpha}{\beta} \exp(\alpha t) \sin(\beta t) + x_0 \exp(\alpha t) \cos(\beta t)} \end{aligned}$$

b)

Use Matlab to check your answer for a few choices of  $x_0$ ,  $v_0$ ,  $b$ ,  $m$ , and  $k$  (I have provided code that will assist you).





c)

Assuming that both  $x_0$  and  $v_0$  are not zero, what inequality must  $m$ ,  $b$ , and  $k$  satisfy in order to give an oscillatory solution (i.e. a solution  $x(t)$  that contains sine/cosine terms)?

From part a), the solution is only oscillatory if  $b^2 - 4mk < 0$ . The period of the oscillations depend on  $\text{Im}[\lambda] = \beta$ . The magnitude of the oscillations depend on  $\text{Re}[\lambda] = \alpha$ . If  $\alpha > 0$  then the magnitude of the oscillations will grow exponentially, if  $\alpha < 0$  then the magnitude of the oscillations will decay exponentially, and if  $\alpha = 0$ , then the magnitude of the oscillations will stay constant. However, we know  $\alpha = \frac{-b}{2m}$ , where  $b$  and  $m$  are positive constants, and so  $\alpha < 0$ , resulting in decaying oscillations.

## Problem 3

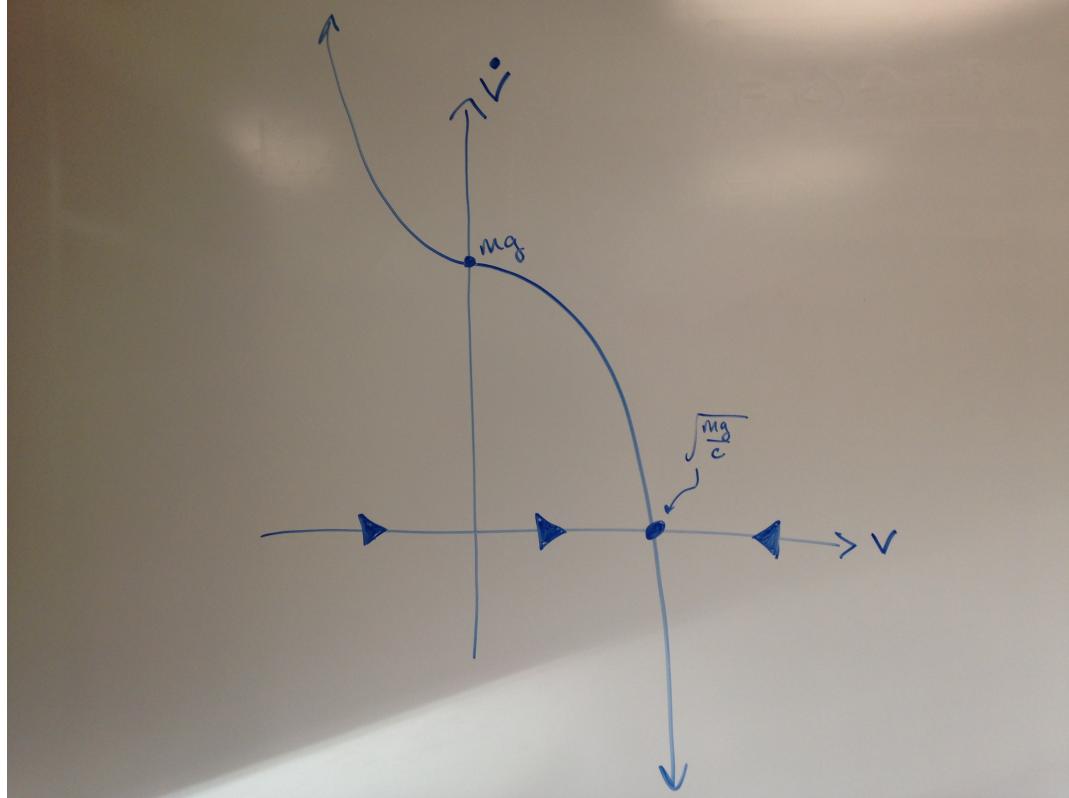
The following equation describes the velocity  $v(t)$  of a relatively large object falling through an inviscid medium (e.g. a baseball falling through the air)

$$m\dot{v} = -cv|v| + mg \quad (6)$$

where  $m$ ,  $c$ , and  $g$  are positive constants representing the mass of the object, the drag of the fluid and the pull of gravity, respectively.

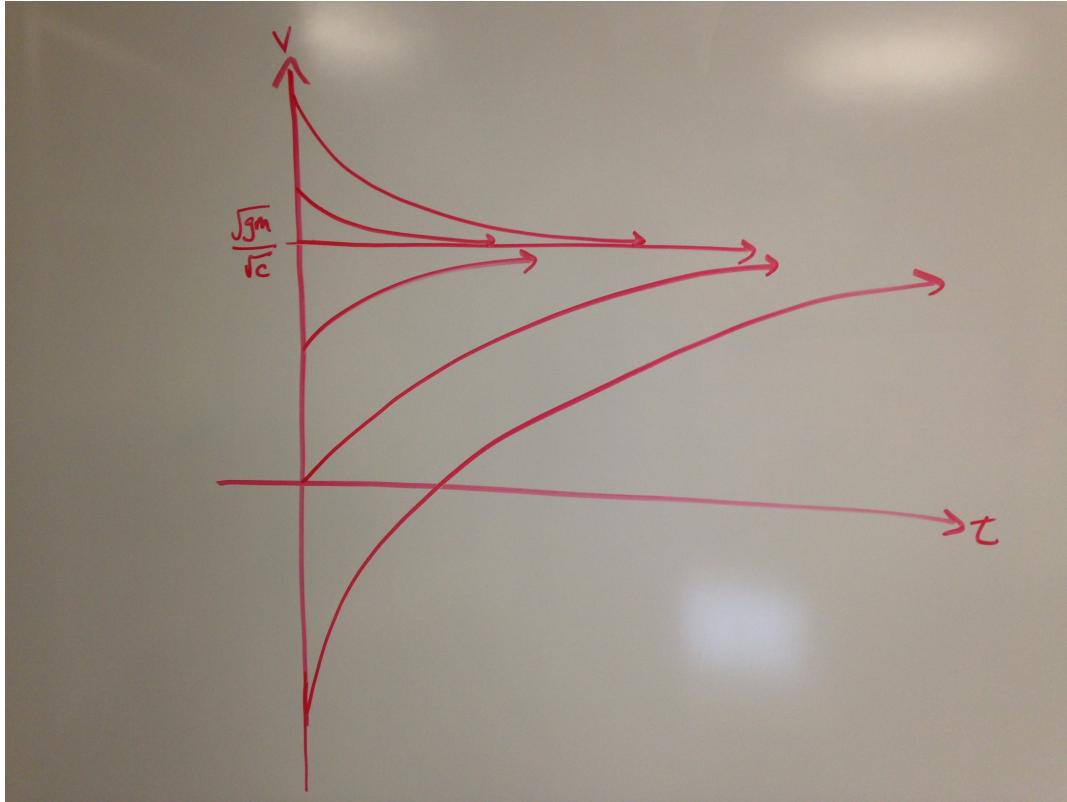
a)

Draw a plot of  $\dot{v}$  vs.  $v$ . Label any fixed point(s) and indicate the stability of each. On the horizontal axis ( $v$ ), indicate the flow direction.



b)

Without solving the equation, sketch  $v(t)$  as a function of  $t$  for several initial conditions.



c)

Solve the equation for  $v(t)$  given  $v(0) = 0$ . (It will simplify your life considerably to assume that  $v \geq 0$  to get rid of the absolute value sign. Once you have a solution, you can determine whether this is a reasonable assumption.)

First, assume  $v \geq 0$ , and thus  $m\dot{v} = -cv^2 + mg$ . Eq. (6) is a separable equation, i.e.

$$m \int \frac{dv}{mg - cv^2} = \int dt$$

The left hand side integral requires partial fractions:

$$\begin{aligned} \frac{1}{mg - cv^2} &= \frac{A}{\sqrt{mg} - \sqrt{cv}} + \frac{B}{\sqrt{mg} + \sqrt{cv}} \\ \implies 1 &= (A + B)\sqrt{mg} + (B - A)\sqrt{cv} \\ \implies A = B &= \frac{1}{2\sqrt{mg}} \end{aligned}$$

and thus,

$$\frac{\sqrt{m}}{2\sqrt{g}} \left( \int \frac{dv}{\sqrt{mg} - \sqrt{cv}} + \int \frac{dv}{\sqrt{mg} + \sqrt{cv}} \right) = \int dt$$

Using  $u$ -substitution ( $u = \sqrt{mg} \pm \sqrt{cv}$ ),

$$\int \frac{dv}{\sqrt{mg} - \sqrt{cv}} = -\frac{1}{\sqrt{c}} \int \frac{du}{u} = -\frac{\ln(\sqrt{mg} - \sqrt{cv})}{\sqrt{c}} \quad \text{and}$$

$$\int \frac{dv}{\sqrt{mg} + \sqrt{cv}} = \frac{1}{\sqrt{c}} \int \frac{du}{u} = \frac{\ln(\sqrt{mg} + \sqrt{cv})}{\sqrt{c}}$$

Thus,

$$\frac{\sqrt{m}}{2\sqrt{cg}} \ln \left( \frac{\sqrt{mg} + \sqrt{cv}}{\sqrt{mg} - \sqrt{cv}} \right) = t + C \quad \text{for some arbitrary constant } C$$

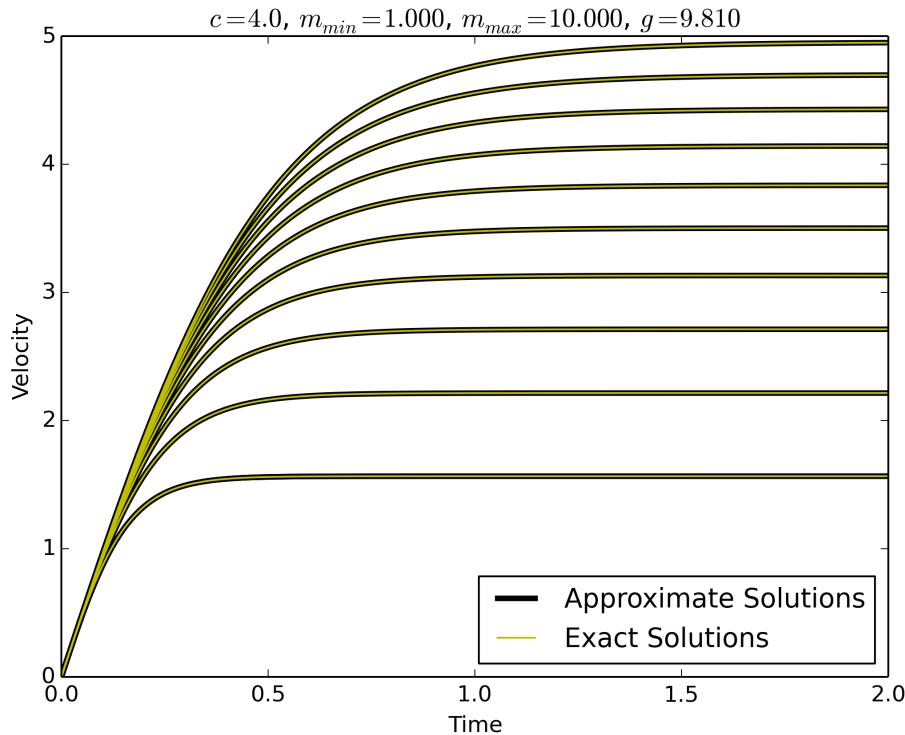
The initial condition  $v(0) = 0$  implies  $C = 0$ . Then the above can be solved for  $v$ :

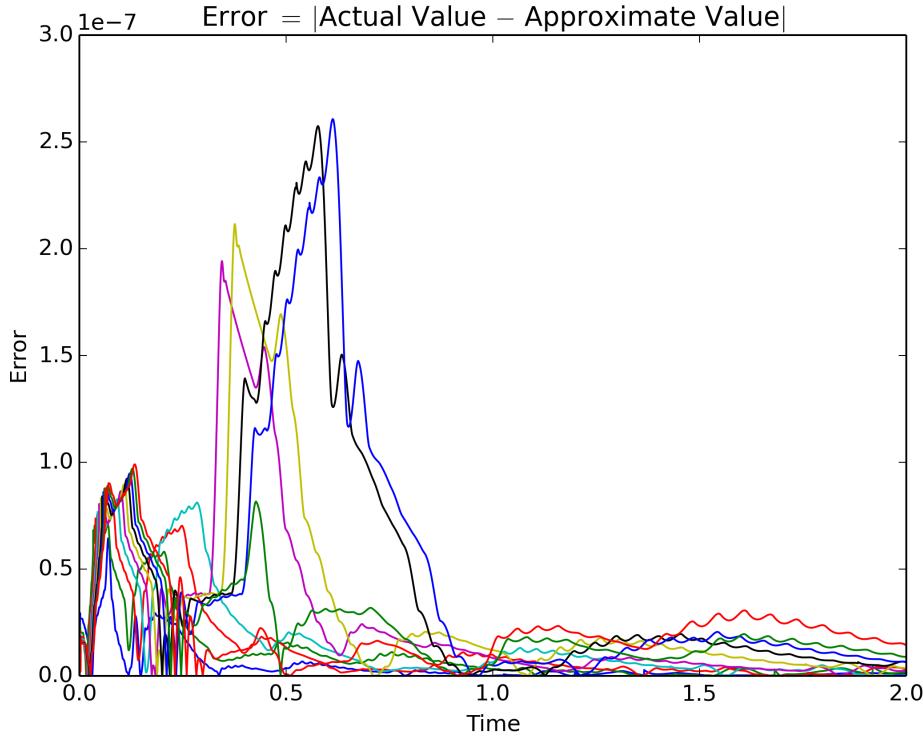
$$v(t) = \frac{\sqrt{mg} \left( \exp \left( \frac{2\sqrt{cg}}{\sqrt{m}} t \right) - 1 \right)}{\sqrt{c} \left( \exp \left( \frac{2\sqrt{cg}}{\sqrt{m}} t \right) + 1 \right)} \quad (7)$$

Since  $\frac{2\sqrt{cg}}{\sqrt{m}} > 0$  and  $t \geq 0$ , then  $\exp \left( \frac{2\sqrt{cg}}{\sqrt{m}} t \right) \geq 1$ , which implies  $v(t) \geq 0$  for all  $t > 0$ . Thus our assumption ( $v \geq 0$ ) is still reasonable.

**d)**

Use Matlab to check your solution. I have not provided code, but you should be able to modify the code for Problem 1.





## Problem 4

The DNA in your body codes for specific proteins. Proteins affect how cells in your body behave - even whether they live or die. In some cells, you might want a high concentration of a particular protein; in other cells, you might want only a small concentration of that protein.

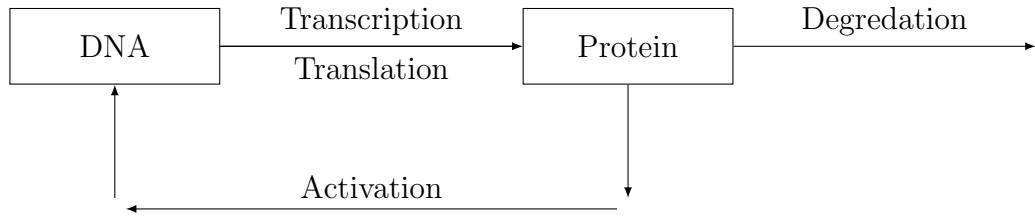


Figure 1: DNA/Protein Interface

Cells in your body typically destroy proteins; they have only a finite lifetime. Thus, the amount of a particular protein in one of your cells at a given time,  $p(t)$ , is determined by a balance between the rate of protein formation and destruction. Sometimes, the amount of protein affects how fast protein is made. Here we consider such a situation, as shown schematically in Figure 1 above.

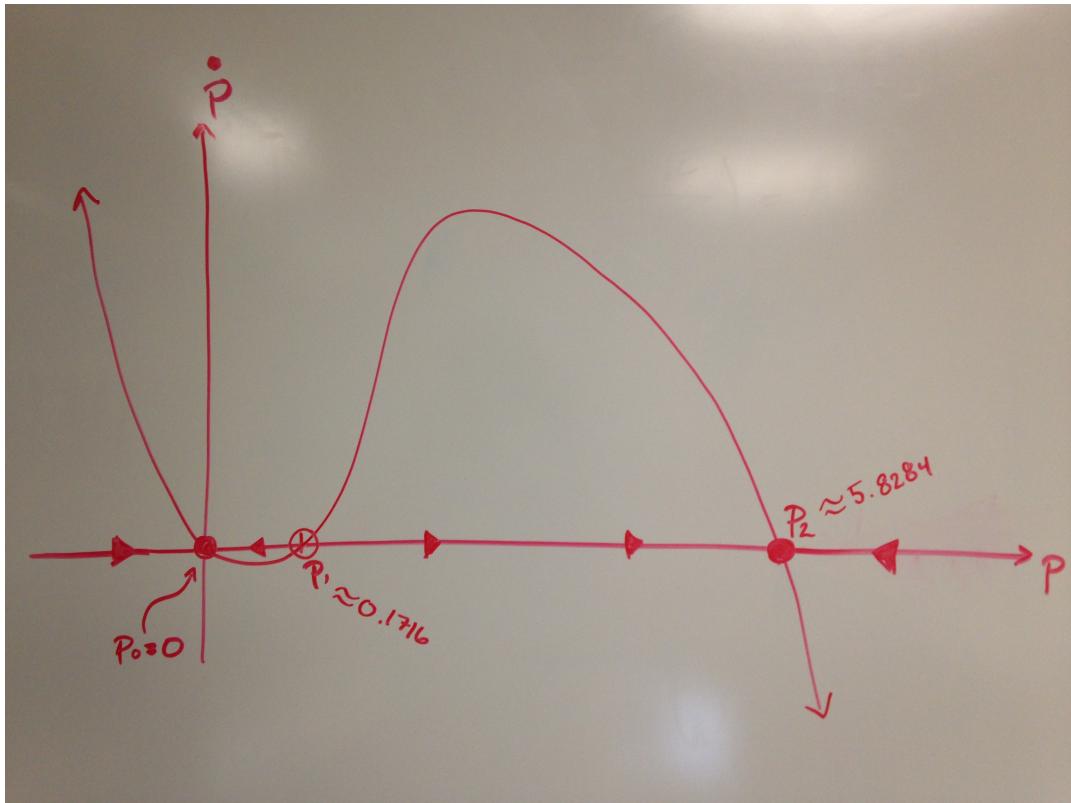
Mathematically, the system can be expressed with the following equation:

$$\dot{p} = -kp + A \frac{p^2}{p^2 + B^2}$$

where  $k$ ,  $A$ , and  $B$  are positive constants that determine how quickly the protein is destroyed, how quickly the protein is formed, and how strongly the protein affects its formation, respectively. For simplicity, assume that  $k = B = 1$ .

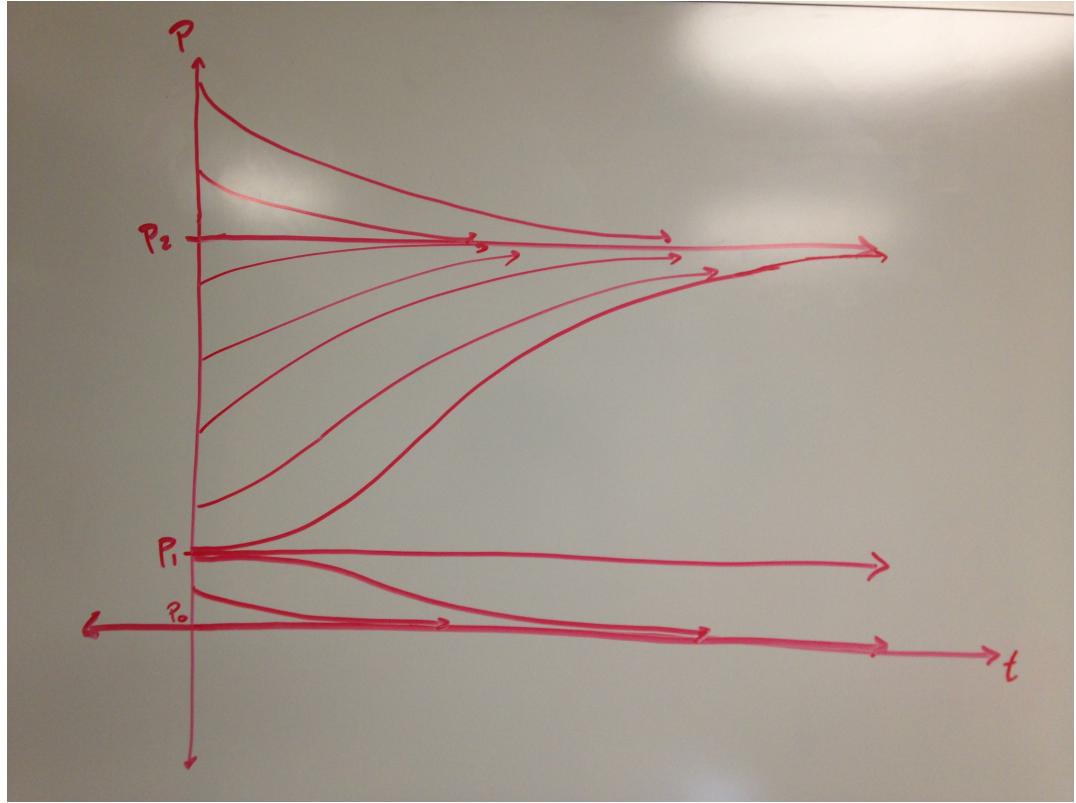
a)

Assuming that  $k = B = 1$ , draw a phase portrait for the case where  $A = 6$  (i.e. plot  $\dot{p}$  vs.  $p$ , indicate any fixed point(s) and indicate the stability of each. On the horizontal axis ( $p$ ) indicate the flow direction of phase points).



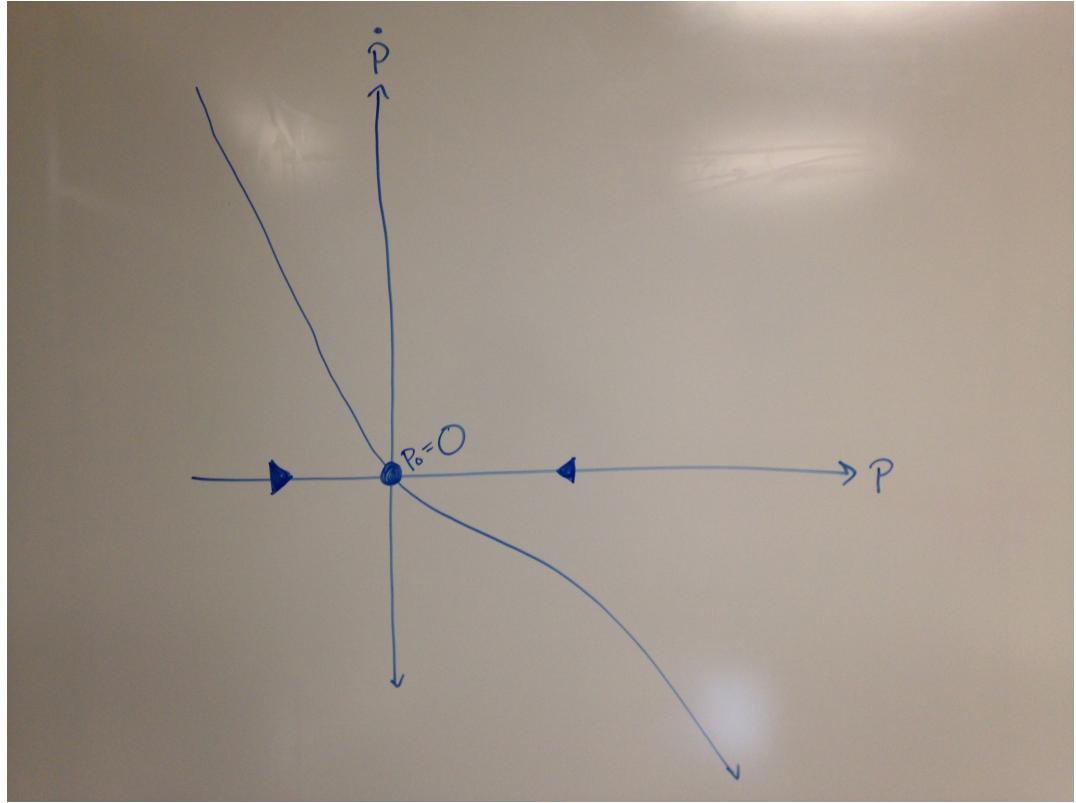
b)

Without solving the equation, sketch  $p(t)$  as a function of  $t$  for several initial conditions for  $A = 6$ .



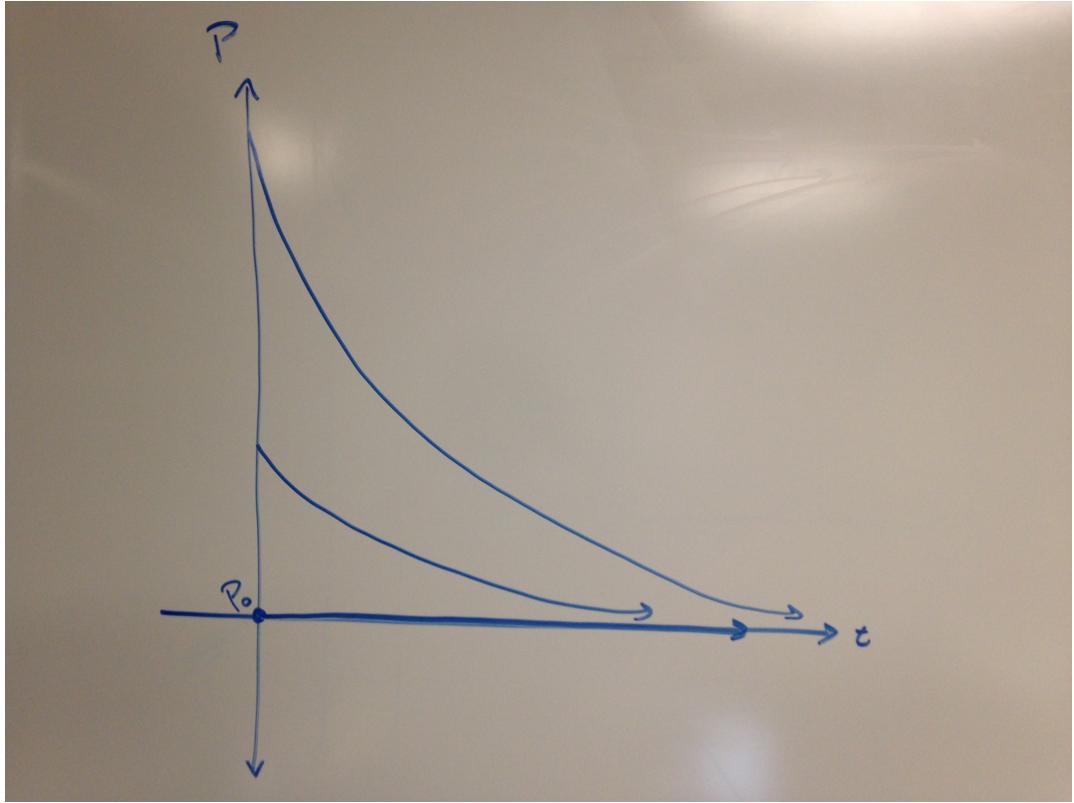
c)

Assuming that  $k = B = 1$ , draw a phase portrait for the case where  $A = 1$  (i.e. plot  $\dot{p}$  vs.  $p$ , indicate any fixed point(s) and indicate the stability of each. On the horizontal axis ( $p$ ) indicate the flow direction of phase points).



d)

Without solving the equation, sketch  $p(t)$  as a function of  $t$  for several initial conditions for  $A = 1$ .



e)

Suppose you wanted to maintain a non-zero steady-state concentration of protein in one of your cells. Would you choose  $A = 1$  or  $A = 6$ ? What else would you have to do to ensure a non-zero steady state?

Clearly, if  $A = 1$ , there is only one stable fixed point: the trivial fixed point  $p_0 = 0$ . However, if  $A = 6$  there are three fixed points: the trivial fixed point  $p_0 = 0$ , which is stable, an unstable non-zero fixed point  $p_1 \approx 0.1716$ , and a stable non-zero fixed point  $p_2 \approx 5.8284$ . Thus, if we want to maintain a non-zero steady-state concentration of protein in one of our cells, we should choose  $A = 6$ . We also need to ensure the initial condition  $p(0) > p_1$ . This way, all solutions send the protein concentration to  $p_2$ . If  $p(0) = p_1$ , a negative perturbation would send the protein concentration to 0. If  $p(0) < p_1$ , all solutions send the protein concentration to 0.