# HW #2

**Sam Fleischer**

**January 29, 2015**

## Problem 1

*A particle of mass $m$ with position $\vec{x}(t)$ at time $t$ has potential energy $V(\vec{x})$ and kinetic energy*

$$T = \frac{1}{2}m|\vec{x}_t|^2.$$

*The action of the particle over times $t_0 \leq t \leq t_1$ is the time-integral of the difference between the kinetic and potential energy:*

$$S(\vec{x}) = \int_{t_0}^{t_1} (T - V)\mathrm{d}t.$$

(a) *Show that an extremal $\vec{x}(t)$ of $S$ satisfies Newton's second law $\vec{F} = m\vec{a}$ for motion in a conservative force field $\vec{F} = -\nabla V$.*

Let $\vec{\phi}$ be a perturbation such that $\vec{\phi}(t_0) = \vec{\phi}(t_1) = 0$. Then consider

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(S(\vec{x} + \varepsilon\vec{\phi}))\bigg|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{t_0}^{t_1}\left\{\frac{1}{2}m|\vec{x}_t + \varepsilon\vec{\phi}_t|^2 - V(\vec{x} + \varepsilon\vec{\phi})\right\}\mathrm{d}t\bigg|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\left\{m|\vec{x}_t + \varepsilon\vec{\phi}_t|\vec{\phi}_t - \vec{\phi}\nabla V(\vec{x} + \varepsilon\vec{\phi})\mathrm{d}t\right\}\mathrm{d}t\bigg|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\vec{\phi}_t m|\vec{x}_t| - \vec{\phi}\nabla V(\vec{x})\mathrm{d}t \\
&= \int_{t_0}^{t_1} -\vec{\phi}\nabla V(\vec{x})\mathrm{d}t + \cancelto{0}{\left[\vec{\phi}m|\vec{x}_y|\right]_{t_0}^{t_1}} - \int_{t_0}^{t_1}\vec{\phi}m|\vec{x}_{tt}|\mathrm{d}t \\
&= \int_{t_0}^{t_1}\vec{\phi}[-\nabla V - m|\vec{a}|]\mathrm{d}t
\end{aligned}$$

Since this holds for any perturbation $\vec{\phi}$, then by the Fundamental Theorem of Calculus of Variations, this shows

$$\begin{aligned}
&-\nabla V - m|\vec{a}| = 0 \\
\implies &F = -\nabla V = m|\vec{a}|
\end{aligned}$$

(b) *Show that the total energy of the particle $E = T + V$ is a constant independent of time.*

We will show the time derivative of $E = T + V$ is zero:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[T+V] &= \frac{\mathrm{d}}{\mathrm{d}t}T + \frac{\mathrm{d}}{\mathrm{d}t}V \\ &= m|\vec{x}_{tt}| + \nabla V \\ &= m|\vec{a}| - F \\ &= 0 \end{aligned}$$

Thus the total energy $E$ is time-independent.

# Problem 2

*Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary (so the divergence theorem holds) and $f : \overline{\Omega} \to \mathbb{R}$ a smooth function. Derive the Euler-Lagrange equation and natural boundary condition that are satisfied by a smooth extremal $u : \overline{\Omega} \to \mathbb{R}$ of the functional*

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right)\mathrm{d}x.$$

Let $\phi$ be a perturbation of $u$. Then

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u+\varepsilon\phi)\bigg|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega \left(\frac{1}{2}|\nabla(u+\varepsilon\phi)|^2 - f(u+\varepsilon\phi)\right)\mathrm{d}x\bigg|_{\varepsilon=0} \\ &= \int_\Omega (\nabla(u+\varepsilon\phi)\nabla\phi - f\phi)\mathrm{d}x\bigg|_{\varepsilon=0} \\ &= \int_\Omega (\nabla u \nabla\phi - f\phi)\mathrm{d}x \\ &= \int_{\partial\Omega} \phi\nabla u \cdot n\mathrm{d}s - \int_\Omega \phi\Delta u\mathrm{d}x - \int_\Omega f\phi\mathrm{d}x \\ &= \int_{\partial\Omega} \phi\frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}s - \int_\Omega \phi(\Delta u + f)\mathrm{d}x \end{aligned}$$

So the natural boundary conditions are the Neumann boundary conditions:

$$\frac{\mathrm{d}u}{\mathrm{d}n} = 0 \qquad \text{for } x \in \partial\Omega$$

and since $\phi$ is an arbitrary perturbation, then

$$f = -\Delta u \qquad \text{for } x \in \Omega$$

# Problem 3

*The transverse displacement at position $x$ and time $t$ of an elastic string vibrating in the $(x, y)$-plane is given by $y = u(x, t)$, where $a \leq x \leq b$ and $t_0 \leq t \leq t_1$. If the density of the string per unit length is $\rho(x)$ and the tension in the string is a constant $T$, then (for small displacements) the motion of the string is an extremum of the action*

$$S(u) = \int_{t_0}^{t_1} \int_a^b \left(\frac{1}{2}\rho u_t^2 - \frac{1}{2}T u_x^2\right) \mathrm{d}x\mathrm{d}t.$$

*Derive the Euler-Lagrange equation for $u(x, t)$.*

Let $\phi$ be a perturbation of $u$. Then

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} S(u + \varepsilon\phi)\bigg|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_{t_0}^{t_1} \int_a^b \left(\frac{1}{2}\rho(u + \varepsilon\phi)_t^2 - \frac{1}{2}T(u + \varepsilon\phi)_x^2\right) \mathrm{d}x\mathrm{d}t\bigg|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1} \int_a^b (\rho(u + \varepsilon\phi)_t\phi_t - T(u + \varepsilon\phi)_x\phi_x) \mathrm{d}x\mathrm{d}t\bigg|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1} \int_a^b (\rho u_t\phi_t - T u_x\phi_x) \mathrm{d}x\mathrm{d}t \\
&= \int_{t_0}^{t_1} \int_a^b \rho u_t\phi_t \mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1} \int_a^b T u_x\phi_x \mathrm{d}x\mathrm{d}t \\
&= \int_a^b \rho \int_{t_0}^{t_1} u_t\phi_t \mathrm{d}t\mathrm{d}x - T \int_{t_0}^{t_1} \int_a^b u_x\phi_x \mathrm{d}x\mathrm{d}t
\end{aligned}$$

We can use integration by parts for each inner-integral to get

$$\begin{aligned}
0 &= \int_a^b \rho \left[u_t\phi\bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \phi u_{tt} \mathrm{d}t\right] \mathrm{d}x - T \int_{t_0}^{t_1} \left[u_x\phi\bigg|_a^b - \int_a^b \phi u_{xx} \mathrm{d}x\right] \mathrm{d}t \\
&= \int_{t_0}^{t_1} \int_a^b \phi[\rho u_{tt} + T u_{xx}] \mathrm{d}x\mathrm{d}t - \int_a^b \rho[-u_t(t_0, x)\phi(t_0, x) - u_t(t_1, x)\phi(t_1, x)] \mathrm{d}x \\
&\qquad - T \int_{t_0}^{t_1} [u_x(t, a)\phi(t, a) - u_x(t, b)\phi(t, b)] \mathrm{d}t
\end{aligned}$$

Since there is constant tension, let us assume the endpoints of the string are held constant. In other words, the perturbation is nill at the endpoints:

$$\phi(t, a) = \phi(t, b) = 0 \quad \forall t \in [t_0, t_1]$$

Let us also assume the perturbation only lasts in the time interval $[t_0, t_1]$. In other words, before $t_0$ and after $t_1$ there is no perturbation:

$$\phi(t_0, x) = \phi(t_1, x) = 0 \quad \forall x \in [a, b]$$

These are the Dirichlet boundary conditions on the rectangle $[t_0, t_1] \times [a, b] \subset \mathbb{R}^2$. Thus,

$$0 = \int_{t_0}^{t_1} \int_a^b \phi[-\rho u_{tt} + T u_{xx}] \mathrm{d}x\mathrm{d}t$$

Since this holds for all perturbations $\phi$, then by the Fundamental Theorem of Calculus of Variations,

$$\rho u_{tt} = T u_{xx}$$

## Problem 4

*The (n-dimensional) area of a surface $y = u(x)$ over a region $\Omega \subset \mathbb{R}^n$ is given by*

$$J(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} \mathrm{d}x.$$

*Find the Euler-Lagrange equation (called the minimal surface equation) that is satisfied by a smooth extremum of this functional.*

Let $\phi$ be a perturbation. Then

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u + \varepsilon\phi)\bigg|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_\Omega \sqrt{1 + |\nabla(u + \varepsilon\phi)|^2} \mathrm{d}x \bigg|_{\varepsilon=0} \\
&= \int_\Omega \frac{1}{2} \left(1 + |\nabla(u + \varepsilon\phi)|^2\right)^{-\frac{1}{2}} 2\nabla(u + \varepsilon\phi)\nabla\phi \mathrm{d}x \bigg|_{\varepsilon=0} \\
&= \int_\Omega \frac{\nabla u \cdot \nabla\phi}{\sqrt{1 + |\nabla u|^2}} \mathrm{d}x \\
&= \int_\Omega \nabla f \cdot \nabla\phi \mathrm{d}x
\end{aligned}$$

where

$$f = \frac{u}{\sqrt{1 + |\nabla u|^2}}$$

However, by Green's first identity,

$$\nabla f \cdot \nabla\phi = \nabla \cdot (\phi\nabla f) - \phi\Delta f$$

and thus

$$\begin{aligned}
0 &= \int_\Omega \nabla f \nabla\phi \mathrm{d}x \\
&= \int_\Omega [\nabla \cdot (\phi\nabla f) - \phi\Delta f] \mathrm{d}x \\
&= \int_{\partial\Omega} (\phi\nabla f) \cdot n \ \mathrm{d}s - \int_\Omega \phi\Delta f \mathrm{d}x
\end{aligned}$$

by the divergence theorem. Either natural or Dirichlet boundary conditions will give

$$\int_{\partial\Omega} (\phi\nabla f) \cdot n \ \mathrm{d}s = 0$$

and thus

$$\int_\Omega \phi\Delta f \mathrm{d}x = 0$$

for all perturbations $\phi$. Thus the Euler-Lagrange equation for $J(u)$ is

$$\begin{aligned}
\Delta f &= 0 \\
\implies \Delta\left(\frac{u}{\sqrt{1 + |\nabla u|^2}}\right) &= 0
\end{aligned}$$

# Problem 5

*Let* $X = \{u \in C^1([-1,1]) \ : \ u(-1) = -1, \ u(1) = 1\}$*, where* $C^1([a,b])$ *denotes the space of continuously differentiable functions on* $[a,b]$*. Define* $J : X \to \mathbb{R}$ *by*

$$J(u) = \int_{-1}^{1} x^4 (u')^2 \mathrm{d}x.$$

(a) *Show that*

$$\inf_{u \in X} J(u) = 0,$$

*but* $J(u) > 0$ *for every* $u \in X$ *(so* $J$ *does not attain its infimum on* $X$*).*

Let $\varepsilon_n = 2^{-n}$ and consider the following family of functions:

$$f_n(x) = \begin{cases} -1 & x \in [-1, -\varepsilon_n] \\ \dfrac{-1}{2\varepsilon^3} x(x^2 - 3\varepsilon^2) & x \in [-\varepsilon_n, \varepsilon_n] \\ 1 & x \in [\varepsilon_n, 1] \end{cases}$$

Note $u' = 0$ when $\varepsilon \leq |x| < 1$, and the maximum of $x$ when $u' \neq 0$ is $\varepsilon_n$. Also, the maximal derivative of $f_n$ is $\frac{3}{2\varepsilon_n}$, which occurs at $x = 0$. Thus,

$$\begin{aligned} J(f_n) &= \int_{-1}^{1} x^4 (u')^2 \mathrm{d}x \\ &= \cancelto{0}{\int_{-1}^{-\varepsilon_n} x^4 (u')^2 \mathrm{d}x} + \int_{-\varepsilon_n}^{\varepsilon_n} x^4 (u')^2 \mathrm{d}x + \cancelto{0}{\int_{\varepsilon_n}^{1} x^4 (u')^2 \mathrm{d}x} \\ &\leq \int_{-\varepsilon_n}^{\varepsilon_n} (\varepsilon_n)^4 \left( \frac{3}{2\varepsilon_n} \right)^2 \mathrm{d}x \\ &= \frac{9\varepsilon_n^2}{4} (2\varepsilon_n) \\ &= \frac{9\varepsilon_n^3}{2} \end{aligned}$$

Since $\varepsilon_n \to 0$, this means $J(f_n) \to 0$, and thus $\inf_{u \in X} J(u) \leq 0$. But since $x^4 (u')^2 \geq 0 \ \forall u \in X$ then $J(u) \geq 0 \ \forall u \in X$ proving $\inf_{u \in X} J(u) \geq 0$, and thus

$$\inf_{u \in X} J(u) = 0$$

However, if $J(u) = 0$ for some function $u$, then $u' = 0$ for $x \in [-1, 1]$, i.e. $u = c$ where $c$ is a constant. This is a contradiction since $u(-1) = -1$, $u(1) = 1$, and $u$ is continuous. Thus $J$ never attains its infimum on $X$.

(b) *What happens when you try to solve the Euler-Lagrange equation for extremals of* $J$*?*

Since $J(u) = \int_{-1}^{1} x^4 (u')^2 \mathrm{d}x$, then the Euler-Lagrange equation is

$$-\frac{\mathrm{d}}{\mathrm{d}x} F_{u'} + F_u = 0$$

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}x}\left[2x^4u'\right] = 0 \\
\implies &-2x^4u' = c
\end{aligned}
$$

where $c$ is a constant. Notice this is only possible for all $x \in [-1, 1]$ if $c = 0$ since when $x = 0$ the equation reduces to

$$
0 = c
$$

So

$$
x^4u' = 0
$$

Assuming $x \neq 0$, this implies $u' = 0$ or $u = c_1$ where $c_1$ is arbitrary. When $x = 0$, $u'$ has no restriction, and thus $u(0) = c_2$ where $c_2$ is arbitrary. However, when we restrict the solutions with $u(-1) = -1$ and $u(1) = 1$, we see that we cannot form a continuous function that satisfies all conditions. The best we can do is

$$
u(x) = \begin{cases} -1 & x \in [-1, 0) \\ 0 & x = 0 \\ 1 & x \in (0, 1] \end{cases}
$$

Note this "ideal" function is approximated by $\lim_{n\to\infty} f_n$, but cannot be considered since it is not continuous.