UC DAVIS, APPLIED MATHEMATICS (MAT207B), WINTER 2016

Homework #4

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Problem 1

The following nonhomogeneous IBVP describes heat flow in a rod whose ends are held at temperatures u_0, u_1:

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = u_0, \qquad u(1,t) = u_1 \\ &u(x,0) = f(x) \end{aligned} \tag{1}$$

(a) *Find the steady state temperature $U(x)$ that satisfies*

$$\begin{aligned} &U_{xx} = 0 \quad 0 < x < 1 \\ &U(0) = u_0, \quad U(1) = u_1 \end{aligned}$$

$U_{xx} = 0$ implies $U(x)$ is a linear function.

$$U(x) = a + bx$$

The boundary conditions imply

$$U(x) = u_0 + (u_1 - u_0)x$$

(b) *Write $u(x,t) = U(x) + v(x,t)$ and find the corresponding IBVP for v. Use separation of variables to solve for v and hence u.*

If $u(x,t) = U(x) + v(x,t)$, then $u(0,t) = u_0 = U(0) + v(0,t)$ implies $v(0,t) = 0$. Similarly, $v(1,t) = 0$. Also, $u(x,0) = f(x) = U(x) + v(x,0)$ implies $v(x,0) = f(x) - U(x)$. Lastly, partial differential equation $u_t = u_{xx}$ implies $v_t = U''(x) + v_{xx}$ but since $U'' = 0$, then $v_t = v_{xx}$. The problem becomes

$$\begin{aligned} &v_t = v_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &v(0,t) = 0, \qquad v(1,t) = 0 \\ &v(x,0) = f(x) - U(x) \end{aligned} \tag{2}$$

By our last homework, the solution of (2) is

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

where

$$c_n = 2\int_0^1 \big(f(x) - U(x)\big) \sin(n\pi x) \mathrm{d}x$$

Thus the solution to (1) is

$$u(x,t) = u_0 + (u_1 - u_0)x + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

(c) *How does $u(x,t)$ behave as $t \to \infty$?*

As $t \to \infty$, the coefficients of the sin series decay hyper-exponentially, and thus the solution rapidly approaches the linear function $U(x)$, i.e.

$$\lim_{t\to\infty} u(x,t) = U(x)$$

Problem 2

Define a first-order differential operator with complex coefficients acting $L^2(0,2\pi)$ by

$$A = -i\frac{\mathrm{d}}{\mathrm{d}x}.$$

(a) *Show that A is formally self-adjoint.*

$$\begin{aligned}
\langle u, Av\rangle &= \int_0^{2\pi} -i\overline{u}v'\mathrm{d}x \\
&= -i\left[\left(\overline{u}v\right)_0^{2\pi} - \int_0^{2\pi} \overline{u'}v\mathrm{d}x\right] \\
&= -i\left(\overline{u(2\pi)}v(2\pi) - \overline{u(0)}v(0)\right) + \int_0^{2\pi} i\overline{u'}v\mathrm{d}x \\
&= -i\left(\overline{u(2\pi)}v(2\pi) - \overline{u(0)}v(0)\right) + \int_0^{2\pi} \overline{-iu'}v\mathrm{d}x \\
&= -i\left(\overline{u(2\pi)}v(2\pi) - \overline{u(0)}v(0)\right) + \langle Au, v\rangle
\end{aligned}$$

Thus, given adequate boundary conditions, A is self-adjoint, i.e. A is formally self-adjoint.

(b) *Show that A with periodic boundary conditions $u(0) = u(2\pi)$ is self-adjoint, and find the eigenvalues and eigenfunctions of the corresponding eigenvalue problem*

$$-iu' = \lambda u, \qquad u(0) = u(2\pi).$$

If $u(0) = u(2\pi)$, then $\overline{0} = \overline{u(2\pi)}$. Thus,

$$\langle u, Av\rangle = -i(u(0)[v(2\pi) - v(0)]) + \langle Au, v\rangle$$

and so the adjoint boundary condition is $v(0) = v(2\pi)$. In this case, A is self-adjoint.

The solution to $-iu' = \lambda u$ is

$$u(x) = c\exp(\lambda i x)$$

for some constant c. The periodicity of u implies

$$c = c\exp(\lambda i 2\pi) \implies \lambda = n, \quad n \in \mathbb{Z}$$

(c) *What are the adjoint boundary conditions to the Dirichlet condition $u(0) = 0$ at $x = 0$? Is A with this Dirichlet boundary condition self-adjoint? Find all eigenvalues and eigenfunctions of the corresponding eigenvalue problem*

$$-iu' = \lambda u, \qquad u(0) = 0.$$

How does your result compare with the properties of finite-dimensional eigenvalue problems for matrices?

If $u(0) = 0$ then $\overline{u(0)} = 0$ and

$$\langle u, Av\rangle = -i\overline{u(2\pi)}v(2\pi) + \langle Au, v\rangle$$

Thus the adjoint condition is $v(2\pi) = 0$, and in this case, A is self-adjoint. The solution to $-iu' = \lambda u$ is

$$u(x) = c\exp(\lambda i x)$$

for some constant c. The boundary condition $u(0) = 0$ implies $c = 0$, and thus $u \equiv 0$. Since eigenvalues cannot have 0 eigenfunctions, there are no eigenvalues for this eigenvalue problem. This does not happen in finite-dimensional eigenvalue problems since all matrices (over algebraically closed fields, like $\mathbb{C}$) have at least one eigenvalue.

Problem 3

Let A be a Sturm-Liouville operator, given by

$$Au = -(pu')' + qu,$$

acting in $L^2(a,b)$. Verify that A with the Robin boundary conditions

$$\alpha u'(a) + u(a) = 0, \qquad u'(b) + \beta u(b) = 0$$

is self adjoint.

$$\begin{aligned}\langle u, Av\rangle &= \int_a^b \left[u\big(-(pv')' + qv\big)\right]\mathrm{d}x \\ &= -\int_a^b up'v'\mathrm{d}x - \int_a^b upv''\mathrm{d}x + \int_a^b uqv\mathrm{d}x\end{aligned}$$

We can integrate the middle integral by parts:

$$\begin{aligned}&\int_a^b upv''\mathrm{d}x = \left[upv'\right]_a^b - \int_a^b (up'v' + u'pv')\mathrm{d}x \\ \implies &\langle u, Av\rangle = -\left[upv'\right]_a^b + \int_a^b u'pv'\mathrm{d}x + \int_a^b uqv\mathrm{d}x\end{aligned}$$

Again, we can integrate the leftmost integral by parts:

$$\int_a^b u'pv'\mathrm{d}x = \left[u'pv\right]_a^b - \int_a^b (u''pv - u'p'v)\mathrm{d}x$$

$$\begin{aligned}
\implies \langle u, Av\rangle &= \big[p(u'v - uv')\big]_a^b + \int_a^b (-(u''pv + u'p'v) + uqv)\,\mathrm{d}x \\
&= \big[p(u'v - uv')\big]_a^b + \int_a^b \big[\big(-(pu')' + qu\big)v\big]\,\mathrm{d}x \\
&= \big[p(u'v - uv')\big]_a^b + \langle Au, v\rangle
\end{aligned}$$

The Robin boundary conditions imply $u'(a) = -\frac{u(a)}{\alpha}$ and $u'(b) = -\beta u(b)$. If we impose the adjoint condition

$$\alpha v'(a) + v(a) = 0, \qquad v'(b) + \beta v(b) = 0,$$

then

$$\begin{aligned}
\big[p(u'v - uv')\big]_a^b &= p(b)\big(u'(b)v(b) - u(b)v'(b)\big) - p(a)\big(u'(a)v(a) - u(a)v'(a)\big) \\
&= p(b)\big(-\beta u(b)v(b) + \beta u(b)v(b)\big) - p(a)\left(-\frac{1}{\alpha}u(a)v(a) + \frac{1}{\alpha}u(a)v(a)\right) \\
&= 0
\end{aligned}$$

and thus A is self-adjoint.

Problem 4

Show that the eigenvalues of the Sturm-Liouville problem

$$\begin{gathered}
-u'' = \lambda u \qquad 0 < x < 1 \\
u(0) = 0, \qquad u'(1) + \beta u(1) = 0
\end{gathered}$$

are given by $\lambda = k^2$ where $k > 0$ satisfies the equation

$$\beta \tan k + k = 0.$$

Show graphically that there is an infinite sequence of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$ with $\lambda_n \to \infty$ as $n \to \infty$. What is the asymptotic behavior of λ_n as $n \to \infty$?

If $\lambda = 0$, the solutions to $-u'' = \lambda u$ are

$$u(x) = c_1 + c_2 x$$

Then $u(0) = 0 \implies c_1 = 0$, i.e.

$$\begin{aligned}
u(x) &= c_2 x \\
u'(x) &= c_2
\end{aligned}$$

The Robin boundary condition $u'(1) + \beta u(1) = 0$ implies

$$0 = c_2 + \beta c_2 = c_2(1 + \beta)$$

If $\beta \neq -1$, then $c_2 = 0$ and thus $u \equiv 0$. If $\lambda \neq 0$, the solutions to $-u'' = \lambda u$ are

$$u(x) = c_1 \exp\left(\sqrt{\lambda}x\right) + c_2 \exp\left(-\sqrt{\lambda}x\right)$$

If $\lambda > 0$, then $u(0) = 0$ implies $c_1 = -c_2$, i.e.

$$u(x) = c_1\left[\exp\left(\sqrt{\lambda}x\right) - \exp\left(-\sqrt{\lambda}x\right)\right]$$
$$u'(x) = c_1\sqrt{\lambda}\left[\exp\left(\sqrt{\lambda}x\right) + \exp\left(-\sqrt{\lambda}x\right)\right]$$

The Robin boundary condition $u'(1) + \beta u(1) = 0$ implies

$$\begin{aligned} 0 &= c_1\sqrt{\lambda}\left[\exp\left(\sqrt{\lambda}\right) + \exp\left(-\sqrt{\lambda}\right)\right] + \beta c_1\left[\exp\left(\sqrt{\lambda}\right) - \exp\left(-\sqrt{\lambda}\right)\right] \\ &= \left(\sqrt{\lambda} + \beta\right)\exp\left(\sqrt{\lambda}\right) + \left(\sqrt{\lambda} - \beta\right)\exp\left(-\sqrt{\lambda}\right) \end{aligned}$$

If $\lambda < 0$, then the solution is given by

$$u(x) = c_1\cos\left(\sqrt{-\lambda}x\right) + c_2\sin\left(\sqrt{-\lambda}x\right)$$

The boundary condition $u(0) = 0$ implies $c_1 = 0$, i.e.

$$u(x) = c_2\sin\left(\sqrt{-\lambda}x\right)$$
$$u'(x) = c_2\sqrt{-\lambda}\cos\left(\sqrt{-\lambda}x\right)$$

The Robin boundary condition $u'(1) + \beta u(1) = 0$ implies

$$\begin{aligned} 0 &= c_2\sqrt{-\lambda}\cos\left(\sqrt{-\lambda}\right) + \beta c_2\sin\left(\sqrt{-\lambda}\right) \\ &= \sqrt{-\lambda} + \beta\tan\left(\sqrt{-\lambda}\right) \qquad \text{provided } c_2 \neq 0 \\ &= \beta\tan k + k \end{aligned}$$

where $k^2 = -\lambda$. Let the solutions to $0 = \beta\tan k + k$ be k_n where $\frac{\pi}{2} < |k_{n+1} - k_n| < \pi$ for all n. Also, $k_n \to \infty$, and thus $\lambda_n \to \infty$ as $n \to \infty$.

Problem 5

THe following IBVP describes heat flow in a rod whose left end is held at temperature 0 *and whose right end loses heat to the surroundings according to Newton's law of cooling (heat flux is proportional to the temperature difference):*

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \qquad u_x(1,t) = -\beta u(1,t) \\ &u(x,0) = f(x) \end{aligned}$$

Solve this IBVP by the method of separation of variables.

Assume the solution $u(x,t) = F(x)G(t)$. Then $FG' = F''G$, which implies $\frac{G'}{G} = \frac{F''}{F} = \lambda$, i.e.

$$G' - \lambda G = 0 \qquad \text{and} \qquad F'' - \lambda F = 0$$

Then

$$G(t) = c\exp[\lambda t]$$

and, if $\lambda \neq 0$, then

$$F(x) = c_1 \exp\left[\sqrt{\lambda}x\right] + c_2 \exp\left[-\sqrt{\lambda}x\right]$$

If $\lambda = 0$, then

$$F(x) = c_1 + c_2 x$$

The left boundary condition implies $F(0) = 0$, and so if $\lambda = 0$ then $c_1 = 0$. Then $F(x) = c_2 x$. The right boundary condition implies $F'(1) = -\beta F(1)$. Thus $c_2 = -\beta c_2 \iff c_2(1+\beta) = 0 \iff \beta = -1$ for nontrivial solutions F. If $\lambda > 0$, then the left boundary condition implies $c_1 = -c_2$, and thus

$$\begin{aligned} F(x) &= c_1\left(\exp\left[\sqrt{\lambda}x\right] - \exp\left[-\sqrt{\lambda}x\right]\right) \\ F'(x) &= c_1\sqrt{\lambda}\left(\exp\left[\sqrt{\lambda}x\right] + \exp\left[-\sqrt{\lambda}x\right]\right) \end{aligned}$$

The right boundary condition implies

$$\begin{aligned} -c_1\beta\left(\exp\left[\sqrt{\lambda}\right] - \exp\left[-\sqrt{\lambda}\right]\right) &= c_1\sqrt{\lambda}\left(\exp\left[\sqrt{\lambda}\right] + \exp\left[-\sqrt{\lambda}\right]\right) \\ \left(\sqrt{\lambda} + \beta\right)\exp\left[\sqrt{\lambda}\right] + \left(\sqrt{\lambda} - \beta\right)\exp\left[-\sqrt{\lambda}\right] &= 0 \end{aligned}$$

If $\lambda < 0$, then

$$F(x) = c_1 \sin\left(\sqrt{-\lambda}x\right) + c_2 \cos\left(\sqrt{-\lambda}x\right)$$

By number 4,

$$F(x) = c_1 \sin\left(\sqrt{-\lambda}x\right)$$

Thus the general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp[\lambda_n t] \sin\left(\sqrt{-\lambda_n}x\right)$$

where $\lambda_n < 0$ for all n and $\lambda_n \to -\infty$ as $n \to \infty$.