HW #1

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Problem 1

Define $f \mid \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x,y) = \frac{xy^3}{x^2+y^6} \qquad \text{if } f(x,y) \neq (0,0)$$

(a) Show that the directional derivatives of f at $(0,0)$ exist in every direction. What is its Gâteaux derivatives at $(0,0)$?
(b) Show that f is not Fréchet differentiable at $(0,0)$. (HINT. A Fréchet differentiable function must be continuous.)

Let $u = \langle u_1, u_2 \rangle$ and consider $D_u f(0,0)$

$$\begin{aligned} D_u f(0,0) &= \lim_{\varepsilon \to 0} \frac{f(0+\varepsilon u_1, 0+\varepsilon u_2) - f(0,0)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} \frac{(\varepsilon u_1)(\varepsilon u_2)^3}{(\varepsilon u_1)^2 + (\varepsilon u_2)^6} \right] \\ &= \lim_{\varepsilon \to 0} \left[\varepsilon \frac{u_1 u_2^3}{u_1^2 + \varepsilon^4 u_2^6} \right] \\ &= 0 \end{aligned}$$

provided $u_1 \neq 0$. If $u_1 = 0$,

$$\begin{aligned} D_u f(0,0) &= \lim_{\varepsilon \to 0} \frac{f(0, 0+\varepsilon u_2) - f(0,0)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} \frac{(0)(\varepsilon u_2)^3}{0 + (\varepsilon u_2)^6} \right] \\ &= \lim_{\varepsilon \to 0} [0] \\ &= 0 \end{aligned}$$

Thus all directional derivatives exist and the Gâteaux derivative $\boldsymbol{\nabla} f = f_x \mathbf{i} + f_y \mathbf{j} = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$.

If f is Fréchet differentiable, f must be continuous. However, f is not continuous because consider approaching $(0,0)$ along the curve $y = x$. Then

$$\lim_{x \to 0} f(x,y) = \lim_{x \to 0} f(x,x) = \lim_{x \to 0} \frac{x^4}{x^2+x^6} = \lim_{x \to 0} \frac{x^2}{1+x^4} = 0$$

Now consider approaching $(0,0)$ along the curve $y = \sqrt[3]{x}$. Then

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} f(x, \sqrt[3]{x}) = \lim_{x\to 0} \frac{x^2}{x^2 + x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Since $0 \neq \frac{1}{2}$, f is not continuous, and thus f is not Fréchet differentiable at $(0,0)$.

Problem 2

Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = x^2 + y^2, \qquad g(x,y) = (y-1)^3 - x^2$$

Find the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = 0$. Show that there does not exist any constant λ such that $\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g$ at some point $(x,y) \in \mathbb{R}^2$. Why does the method of Lagrange multipliers fail in this example?

Note that $\boldsymbol{\nabla} g = \langle -2x, 3(y-1)^2 \rangle$ and $\boldsymbol{\nabla} f = \langle 2x, 2y \rangle$. Also note that the method of Lagrange multipliers cannot be used if $\boldsymbol{\nabla} g = \mathbf{0}$.

Now assume that $\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g$. Then

$$\begin{aligned} f_x &= \lambda g_x \quad \text{and} \\ f_y &= \lambda g_y \end{aligned}$$

Thus,

$$\begin{aligned} 2x &= \lambda(-2x) \quad \text{and} \\ 2y &= \lambda(3(y-1)^2) \end{aligned}$$

The first equation gives either $\lambda = -1$ or $x = 0$. If $x = 0$, then $g = 0 \implies y = 1$. However, $(x,y) = (0,1) \implies \boldsymbol{\nabla} g = \mathbf{0}$. Thus $\lambda = -1$. The second equation gives

$$\begin{aligned} 2y &= -3(y^2 - 2y + 1) \\ \implies 3y^2 - 4y + 3 &= 0 \\ \implies y &= \frac{4 \pm \sqrt{-20}}{6} \notin \mathbb{R} \end{aligned}$$

which is not possible since $f, g : \mathbb{R}^2 \to \mathbb{R}$. Thus the Lagrange multiplier method does not work in this case.

Problem 3

Derive the Euler-Lagrange equation for a functional of the form

$$J(u) = \int_a^b F(x, u, u', u'')\mathrm{d}x$$

What are the natural boundary conditions for this functional?

Assume u minimizes the functional. Then

$$\mathrm{d}J(u,\phi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_a^b F(x, u+\varepsilon\phi, u'+\varepsilon\phi', u''+\varepsilon\phi'')\Big|_{\varepsilon=0} = 0$$

$$\implies \int_a^b [F_u(x,u,u',u'')\phi + F_{u'}(x,u,u',u'')\phi' + F_{u''}(x,u,u',u'')\phi'']\mathrm{d}x = 0$$

By integrating by parts, we get

$$\int_a^b F_{u'}(x,u,u',u'')\phi'\mathrm{d}x = [F_{u'}(x,u,u',u'')\phi]_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}F_{u'}(x,u,u',u'')\right)\phi\mathrm{d}x$$

and

$$\int_a^b F_{u''}(x,u,u',u'')\phi''\mathrm{d}x = [F_{u''}(x,u,u',u'')\phi']_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}F_{u''}(x,u,u',u'')\right)\phi'\mathrm{d}x$$

Again, by integrating by parts,

$$\int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}F_{u''}(x,u,u',u'')\right)\phi'\mathrm{d}x = \left[\left(\frac{\mathrm{d}}{\mathrm{d}x}F_{u''}(x,u,u',u'')\right)\phi\right]_a^b - \int_a^b \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{u''}(x,u,u',u'')\right)\phi\mathrm{d}x$$

Thus,

$$\int_a^b \left[F_u\phi - \left(\frac{\mathrm{d}}{\mathrm{d}x}F_{u'}\right)\phi + \left(\frac{\mathrm{d}}{\mathrm{d}x^2}F_{u''}\right)\phi\right]\mathrm{d}x + \left[F_{u'}\phi + F_{u''}\phi' - \left(\frac{\mathrm{d}}{\mathrm{d}x}F_{u''}\right)\phi\right]_a^b = 0$$

Thus, given natural boundary conditions, the Euler-Lagrange equation is

$$F_u - \frac{\mathrm{d}}{\mathrm{d}x}F_{u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{u''} = 0$$

The required boundary conditions are

$$\left[\left(F_{u'} - \frac{\mathrm{d}}{\mathrm{d}x}F_{u''}\right)\phi\right]_a^b + [F_{u''}\phi'] = 0$$

The most "natural" boundary conditions are

$$\left[F_{u'} - \frac{\mathrm{d}}{\mathrm{d}x}F_{u''}\right]\Bigg|_{x=a} = \left[F_{u'} - \frac{\mathrm{d}}{\mathrm{d}x}F_{u''}\right]\Bigg|_{x=b} = (F_{u''})\Big|_{x=a} = (F_{u''})\Big|_{x=b} = 0$$

Problem 4

A curve $y = u(x)$ with $a \le x \le b$, $u(x) > 0$, and $u(a) = u_0$, $u(b) = u_1$ is rotated about the x-axis. Find the curve that minimizes the area of the surface of revolution,

$$J(u) = \int_a^b u\sqrt{1+(u')^2}\mathrm{d}x$$

Let $F(x,u,u') = F(u,u') = u\sqrt{1+(u')^2}$. Since $J(u) = \int_a^b F(u,u')\mathrm{d}x$, and since $F(u,u')$ is explicitly independent of x, we can guarantee

$$F - u'F_{u'} = C \in \mathbb{R}$$

Note

$$F_{u'} = \frac{uu'}{\sqrt{1+(u')^2}}$$

Thus,

$$\begin{aligned}
u\sqrt{1+(u')^2} - u'\frac{uu'}{\sqrt{1+(u')^2}} &= C \\
\implies \frac{u(1+(u')^2)}{\sqrt{1+(u')^2}} - \frac{u(u')^2}{\sqrt{1+(u')^2}} &= C \\
\implies u &= C\sqrt{1+(u')^2} \\
\implies \frac{\mathrm{d}u}{\mathrm{d}x} &= \sqrt{\left(\frac{u}{C}\right)^2 - 1}
\end{aligned}$$

This is a separable differential equation, so

$$\begin{aligned}
C\int \frac{1}{\sqrt{u^2 - C^2}}\mathrm{d}u &= \int \mathrm{d}x \\
\implies C\ln\left(\sqrt{u^2-C^2} + u\right) &= x + K
\end{aligned}$$

where $K \in \mathbb{R}$. We obtain a system of equations by noting the Dirichlet boundary conditions $u(a) = u_0$, $u(b) = u_1$:

$$\begin{aligned}
C\ln\left(\sqrt{u_0^2 - C^2} + u_0\right) &= a + K \\
C\ln\left(\sqrt{u_1^2 - C^2} + u_1\right) &= b + K
\end{aligned}$$

We can solve this system for $C = C(u_0,u_1,a,b)$ and $K = K(u_0,u_1,a,b)$ in terms of u_0, u_1, a, and b to obtain the implicit curve

$$C\ln\left(\sqrt{u^2-C^2} + u\right) = x + K$$

Problem 5

Let X be the space of smooth functions $u : [0,1] \to \mathbb{R}$ such that $u(0) = 0$, $u(1) = 0$. Define functionals $J, K : X \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_0^1 (u')^2\mathrm{d}x, \qquad K(u) = \frac{1}{2}\int_0^1 u^2\mathrm{d}x$$

(a) Introduce a Lagrange multiplier and write down the Euler-Lagrange equation for extremals in X of the functional $J(u)$ subject to the constraint $K(u) = 1$.

(b) Solve the eigenvalue problem in (a) and find all of the extremals. Which one minimizes $J(u)$?

Define $L : X \to \mathbb{R}$ by

$$L(u) = K(u) - 1 = \frac{1}{2}\int_0^1 u^2 \mathrm{d}x - \int_0^1 \mathrm{d}x = \int_0^1 \left(\frac{1}{2}u^2 - 1\right)\mathrm{d}x$$

Then the constraint $K(u) = 1$ is equivalent to $L(u) = 0$. Thus we consider

$$\frac{\delta J}{\delta u} = \lambda \frac{\delta L}{\delta u}$$

Define $F(x, u, u') = F(u') = \frac{1}{2}(u')^2$ and $G(x, u, u') = G(u) = \frac{1}{2}u^2 - 1$. Then

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}F_{u'} + F_u\right] = \lambda\left[-\frac{\mathrm{d}}{\mathrm{d}x}G_{u'} + G_u\right]$$

Note that $F_{u'} = u'$, $F_u = 0$, $G_{u'} = 0$ and $G_u = u$. Thus,

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}u' &= \lambda u \\ \implies u'' + \lambda u &= 0 \\ \implies u(x) &= a\exp\left(\sqrt{-\lambda}x\right) + b\exp\left(-\sqrt{-\lambda}x\right) \end{aligned}$$

If $\lambda < 0$, then $u(0) = 0$ implies $0 = a + b$ and $u(1) = 0$ implies

$$0 = a\sinh\left(\sqrt{\lambda}\right)$$

Thus either $a = 0$ or $\lambda = 0$. But $\lambda < 0 \implies \lambda \neq 0$. Thus $a = 0$, and thus $b = 0$. So

$$u \equiv \mathbf{0}$$

But $L(\mathbf{0}) = \int_0^1 -1\mathrm{d}x = -1 \neq 0$, which contradicts our constrain requirement. Thus $\lambda > 0$. This gives

$$u(x) = a\sin\left(\sqrt{\lambda}x\right) + b\cos\left(\sqrt{\lambda}x\right)$$

Using the boundary requirements,

$$\begin{aligned} 0 &= b \qquad \text{and} \\ 0 &= a\sin\left(\sqrt{\lambda}\right) \end{aligned}$$

Thus $\lambda = \pi^2$ or $a = 0$. However, if $a = 0$ then $u \equiv \mathbf{0}$, which is a contradiction as showed above. Thus $\lambda = \pi^2$. So,

$$u(x) = a\sin(\pi x)$$

Since $L(u) = 0$,

$$\frac{1}{2}\int_0^1 (a\sin(\pi x))^2 \mathrm{d}x - 1 = 0$$

$$
\begin{aligned}
\implies \frac{a^2}{2}\int_0^1 (\sin(\pi x))^2 \mathrm{d}x - 1 = 0 \\
\implies \frac{a^2}{2}\frac{1}{2} - 1 = 0 \\
\implies a^2 = 4 \\
\implies a = \pm 2
\end{aligned}
$$

Thus $u_+(x) = 2\sin(\pi x)$ and $u_-(x) = -2\sin(\pi x)$ minimize $J(u)$ subject to $L(u) = 0$. Also note that $(u_+')^2 = (u_-')^2 = 4\pi^2(\cos(\pi x))^2$ and so $J(u_+) = J(u_-) = \pi^2$.

Problem 6

(a) Make a change of variable $x = \phi(t)$, $v(t) = u(\phi(t))$, where $\phi'(t) > 0$, in the functional

$$J(u) = \int_a^b F(x, u, u')\mathrm{d}x$$

Show that $J(u) = K(v)$ where $K(v)$ has the form

$$K(v) = \int_c^d G(t, v, v')\mathrm{d}t$$

and express G in terms of F and ϕ.
(b) Show that the Euler-Lagrange equation for $K(v)$ is the same as what you get by changing variables in the Euler-Lagrange equation for $J(u)$.

Let $x = \phi(t)$ and $v(t) = u(\phi(t))$ where $\phi'(t) > 0$. Then note, by the chain rule, $\mathrm{d}x = \phi'(t)\mathrm{d}t$ and $v'(t) = u_\phi \phi'(t)$. Thus, $x = a \implies t = c = \phi^{-1}(a)$ and $x = b \implies t = d = \phi^{-1}(b)$. Also,

$$F(x, u, u') = F\left(\phi(t), v(t), \frac{v'(t)}{\phi'(t)}\right)$$

So let $G(t, v, v') = \phi'(t)F\left(\phi(t), v(t)\frac{v'(t)}{\phi'(t)}\right)$. Then

$$J(u) = \int_a^b F(x, u, u') = \int_c^d G(t, v, v')\mathrm{d}t = K(v)$$

Also,

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}x}F_{u'} + F_u = 0 \\
\implies -\frac{1}{\phi'}\frac{\mathrm{d}}{\mathrm{d}t}F_{\frac{v'}{\phi'}}\left(\phi, v, \frac{v'}{\phi'}\right) + F_v\left(\phi, v, \frac{v'}{\phi'}\right) = 0 \\
\implies -\frac{1}{\phi'}\frac{\mathrm{d}}{\mathrm{d}t}G_{v'} + \frac{1}{\phi'}G_v = 0 \\
\implies -\frac{\mathrm{d}}{\mathrm{d}t}G_{v'} + G_v = 0
\end{aligned}
$$

since $\phi' > 0$.