UC DAVIS, APPLIED MATHEMATICS (MAT207B), WINTER 2016

Homework #7

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Problem 1

(a) Use separation of variables to find the eigenvalues λ and eigenfunctions $u(x,y)$ of the Dirichlet Laplacian on the unit square that satisfy

$$\begin{aligned} -\left(u_{xx}+u_{yy}\right) &= \lambda u \qquad 0<x<1,\ 0<y<1 \\ u(x,0)=0, \qquad u(x,1)&=0 \qquad 0\le x\le 1 \\ u(0,y)=0, \qquad u(1,y)&=0 \qquad 0\le y\le 1. \end{aligned}$$

(b) What is the smallest eigenvalue that is not a simple eigenvale?

Problem 2

(a) Let $\vec{x}=(x,y)$, $\vec{\xi}=(\xi,\eta)$, and $\vec{\xi}^*=(\xi,-\eta)$ where $\eta>0$. Show that

$$G\left(\vec{x},\vec{\xi}\right) = -\frac{1}{2\pi}\log\left(\frac{\left|\vec{x}-\vec{\xi}\right|}{\left|\vec{x}-\vec{\xi}^*\right|}\right)$$

is the solution of

$$\begin{aligned} -\left(G_{xx}+G_{yy}\right) = \delta\left(\vec{x}-\vec{\xi}\right) \qquad &\text{in } -\infty<x<\infty, \qquad y>0 \\ G\left(\vec{x},\vec{\xi}\right)=0 \qquad &\text{on } y=0. \end{aligned}$$

(b) Write down the Green's function representation for the solution $u(x,y)$ of the Dirichlet problem for the Laplacian in the upper half plane

$$\begin{aligned} u_{xx}+u_{yy}=0 \qquad \text{in } -\infty<x<\infty, \qquad y>0 \\ u(x,0)=f(x). \end{aligned}$$

You can assume that $u(x,y)\to 0$ sufficiently rapidly as $|(x,y)|\to\infty$.

(c) Use the Green's function representation to show that

$$u)x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(x-t)^2+y^2}\,\mathrm{d}t.$$