HW #3

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February 05, 2015

Problem 1

Suppose that $u(x)$ is a non-zero solution of the eigenvalue problem

$$\begin{aligned} -u'' &= \lambda u, & 0 < x < 1, \\ u(0) &= 0, & u(1) = 0. \end{aligned}$$

Show that

$$\lambda = \frac{\int_0^1 (u')^2 \mathrm{d}x}{\int_0^1 u^2 \mathrm{d}x}.$$

Deduce that every eigenvalue λ is strictly positive.

Problem 2

Heat flows in a rod of length L with a heat source $(a > 0)$ or sink $(a < 0)$ whose density au is proportional to the temperature u. Suppose that $u(x,t)$ satisfies the IBVP

$$\begin{aligned} u_t &= Du_{xx} + au, & 0 < x < L, && t > 0, \\ u(0,t) &= 0, & u(L,t) = 0, \\ u(x,0) &= f(x). \end{aligned}$$

(a) *Nondimensionalize the problem, and show that the IBVP can be written in nondimensional form as*

$$\begin{aligned} u_t &= u_{xx} + \alpha u, & 0 < x < 1, && t > 0, \\ u(0,t) &= 0, & u(1,t) = 0, && t > 0, \\ u(x,0) &= f(x). \end{aligned}$$

where α is a suitable nondimensional parameter. Give a physical interpretation of α.

(b) *Solve the IBVP in **(a)** be the method of separation of variables.*

(c) *How does your solution behave as $t \to \infty$? For what values of α does $u(x,t) \to 0$ as $t \to \infty$? What happens for larger values of α? Give a physical explanation of this behavior in terms of the thermal energy.*

Problem 3

Solve the following eigenvalue problem for the linear operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *with Neumann BCs:*

$$\begin{aligned} -u'' &= \lambda u, \qquad & 0 < x < 1, \\ u'(0) &= 0, \qquad & u'(1) = 0. \end{aligned}$$

(a) *Find the eigenvalues* $\lambda = \lambda_n$*, where* $n = 0, 1, 2, \ldots,$ *and the corresponding eigenfunctions* $u_n(x)$.

(b) *Show that the eigenfunctions can be normalized so that*

$$\int_0^1 u_m(x)u_n(x)\mathrm{d}x = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

where δ *is the Kronecker Delta.*

(c) *Does your argument in* ***Problem 1*** *that* $\lambda \neq 0$ *work in this case?*

Problem 4

(a) *Solve the following IBVP by the method of separation of variables*

$$\begin{aligned} u_t &= u_{xx}, \qquad & 0 < x < 1, \qquad & t > 0, \\ u_x(0,t) &= 0, \qquad & u_x(1,t) = 0 \qquad & t > 0 \\ u(x,0) &= f(x) \qquad & 0 < x < 1 & \end{aligned}$$

(b) *How does your solution behave as* $t \to \infty$*?*

(c) *Show directly from the IBVP in* ***(a)*** *that*

$$\int_0^1 u(x,t)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x \qquad \text{for all } t \geq 0.$$

Is this result consistent with your answer in ***(b)****? Give a physical explanation of the long-term behavior of* $u(x,t)$.