UC DAVIS, APPLIED MATHEMATICS (MAT207B), WINTER 2016

Homework #7

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Problem 1

(a) *Use separation of variables to find the eigenvalues λ and eigenfunctions $u(x,y)$ of the Dirichlet Laplacian on the unit square that satisfy*

$$\begin{aligned} -\big(u_{xx}+u_{yy}\big) &= \lambda u \qquad 0<x<1,\ 0<y<1 \\ u(x,0)=0, &\qquad u(x,1)=0 \qquad 0\le x\le 1 \\ u(0,y)=0, &\qquad u(1,y)=0 \qquad 0\le y\le 1. \end{aligned}$$

Suppose $u(x,y)=F(x)G(y)$. Then

$$-F''G-FG''=\lambda FG \qquad \implies \qquad -F''G=F\big(G''+\lambda G\big)$$
$$\implies -\frac{F''}{F}=\frac{G''}{G}+\lambda$$

Since the left hand side is a function of x and the right hand side is a function of y, then they can only be equal if

$$-\frac{F''}{F}=\frac{G''}{G}+\lambda=\mu$$

where μ is a constant. Note the boundary conditions imply

$$F(0)=F(1)=G(0)=G(1)=0$$

thus $F''+\mu F=0$ and the homogeneous Dirichlet boundary conditions imply $\mu<0$ and

$$F(x)=A\sin\big(\sqrt{-\mu}x\big)$$

The condition $F(1)=0$ implies

$$0=A\sin\big(\sqrt{-\mu}\big) \qquad \implies \qquad \mu=-\pi^2n^2$$

for $n\ge 1$. Then $G''+(\lambda-\mu)G=0$ and the homogeneous Dirichlet boundary conditions imply $\mu-\lambda>0$ and

$$G(y)=B\sin\Big(\sqrt{\mu-\lambda}\,y\Big)$$

The condition $G(1)=0$ implies

$$0=B\sin\Big(\sqrt{\mu-\lambda}\Big) \qquad \implies \qquad \lambda-\mu=-\pi^2m^2 \qquad \implies \qquad \lambda=-\pi^2(n^2+m^2)$$

for $n,m\ge 1$. Thus the solution to $-(u_{xx}+u_{yy})=\lambda u$ is

$$\sum_{n,m\ge 1}A_{n,m}\sin(n\pi x)\sin\big(m\pi y\big)$$

for some constants $A_{n,m}$. Then the eigenvalues of

$$\nabla^2u=\lambda u; \qquad u(x,0)=u(x,1)=u(0,y)=u(1,y)0$$

are $\lambda=\pi^2(m^2+n^2)$ for all $n,m\ge 1$.

(b) What is the smallest eigenvalue that is not a simple eigenvalue?

The smallest eigenvalue that is not a simple eigenvalue is $5\pi^2$ since this can be acheived when $n = 1$, $m = 2$ or when $n = 2$, $m = 1$. The only smaller eigenvalue is $2\pi^2$, but that is simple since it can only be acheived when $n = m = 1$.

Problem 2

(a) Let $\vec{x} = (x, y)$, $\vec{\xi} = (\xi, \eta)$, and $\vec{\xi^*} = (\xi, -\eta)$ where $\eta > 0$. Show that

$$G\left(\vec{x}, \vec{\xi}\right) = -\frac{1}{2\pi} \log\left(\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi^*}\right|}\right)$$

is the solution of

$$\begin{aligned} -\left(G_{xx} + G_{yy}\right) = \delta\left(\vec{x} - \vec{\xi}\right) \qquad & \text{in } -\infty < x < \infty, \qquad y > 0 \\ G\left(\vec{x}, \vec{\xi}\right) = 0 \qquad & \text{on } y = 0. \end{aligned}$$

(b) Write down the Green's function representation for the solution $u(x, y)$ of the Dirichlet problem for the Laplacian in the upper half plane

$$\begin{aligned} u_{xx} + u_{yy} = 0 \qquad & \text{in } -\infty < x < \infty, \qquad y > 0 \\ & u(x, 0) = f(x). \end{aligned}$$

You can assume that $u(x, y) \to 0$ sufficiently rapidly as $|(x, y)| \to \infty$.

(c) Use the Green's function representation to show that

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x - t)^2 + y^2} \, \mathrm{d}t.$$