UC DAVIS, APPLIED MATHEMATICS (MAT207B), WINTER 2016

Homework #4

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Problem 1

The following nonhomogeneous IBVP describes heat flow in a rod whose ends are held at temperatures u_0, u_1:

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = u_0, \qquad u(1,t) = u_1 \\ &u(x,0) = f(x) \end{aligned}$$

(a) *Find te steady state temperature $U(x)$ taht satisfies*

$$\begin{aligned} &U_{xx} = 0 \quad 0 < x < 1 \\ &U(0) = u_0, \qquad U(1) = u_1 \end{aligned}$$

(b) *Write $u(x,t) = U(x) + v(x,t)$ and find the corresponding IBVP for v. Use separation of variables to solve for v and hence u.*

(c) *How does $u(x,t)$ behave as $t \to \infty$?*

Problem 2

Define a first-order differential operator with complex coefficients acting $L^2(0,2\pi)$ by

$$A = -i\frac{\mathrm{d}}{\mathrm{d}x}.$$

(a) *Show that A is formally self-adjoint.*

(b) *Show that A with periodic boundary conditions $u(0) = u(2\pi)$ is self-adjoint, and find the eigenvalues and eigenfunctions of the corresponding eigenvalue problem*

$$-iu' = \lambda u, \qquad u(0) = u(2\pi).$$

(c) *What are the adjoint boundary conditions to the Dirichlet condition $u(0) = 0$ at $x = 0$? Is A with this Dirichlet boundary condition self-adjoint? Find all eigenvalues and eigenfunctions of the corresponding eigenvalue problem*

$$-iu' = \lambda u, \qquad u(0) = 0.$$

How does your result compare with the properties of finite-dimensional eigenvalue problems for matrices?

Problem 3

Let A be a Sturm-Liouville operator, given by

$$Au = -(pu')' + qu,$$

acting in $L^2(a,b)$. Verify that A with the Robin boundary conditions

$$\alpha u'(a) + u(a) = 0, \qquad u;(b) + \beta u(b) = 0$$

is self adjoint.

Problem 4

Show that the eigenvalues of the Sturm-Lioville problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1 \\ u(0) = 0, &\qquad u'(1) + \beta u(1) = 0 \end{aligned}$$

are given by $\lambda = k^2$ where $k > 0$ satisfies the equation

$$\beta \tan k + k = 0.$$

Show graphically that there is an infinite sequence of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. What is the asymptotic behavior of λ_n as $n \to \infty$?

Problem 5

THe following IBVP describes heat flow in a rod whose left end is held at temperature 0 and whose right end loses heat to the surroundings according to Newton's law of cooling (heat flux is proportional to the temperature difference):

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \qquad u'(1,t) = -\beta u(1,t) \\ &u(x,0) = f(x) \end{aligned}$$

Solve this IBVP by the method of separation of variables.