UC DAVIS, APPLIED MATHEMATICS (MAT207B), WINTER 2016

Homework #6

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Problem 1

Suppose that $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ are two solutions of the homogeneous Sturm-Liouville equation

$$-(pu')' + qu = 0$$

where $p, q : \mathbb{R} \to \mathbb{R}$ are smooth functions and $p > 0$. If $W = u_1 u_2' - u_2 u_1'$ is the Wronskian of u_1, u_2, show that $pW =$ constant.

If u_1, u_2 are solutions of $-(pu')' + qu = 0$, then

$$\begin{aligned}
-(pu_1')' + qu_1 = 0 &\implies -\big(p' u_2 u_1' + p u_2 u_1''\big) + q u_1 u_2 = 0, \qquad \text{and} \\
-(pu_2')' + qu_2 = 0 &\implies -\big(p' u_1 u_2' + p u_1 u_2''\big) + q u_2 u_1 = 0, \\
&\implies p' u_1 u_2' - p' u_2 u_1' + p u_1 u_2'' - p u_2 u_1'' = 0 \\
&\implies \frac{\mathrm{d}}{\mathrm{d}x}\big(pW\big) = 0 \\
&\implies pW = \text{constant}
\end{aligned}$$

Problem 2

Compute the Green's function for the BVP

$$\begin{aligned}
-u'' + u = f(x) \qquad & 0 < x < 1 \\
u(0) = 0, \qquad & u(1) = 0.
\end{aligned}$$

Write down the integral representation of the solution u in terms of f.

Assume the Green's function $G(x;\xi)$ is continuous, and solves $AG(x,\xi) = \delta(x-\xi)$ where $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{Id}$. Then the following four conditions must hold:

1. Initial Value Problem: $-G_{xx} + G = 0$; $G(0,\xi) = 0$ for $x \in [0,\xi)$
2. Final Value Problem: $-G_{xx} + G = 0$; $G(1,\xi) = 0$ for $x \in (\xi, 1]$
3. Continuity: $G(\xi^- \xi) = G(\xi^+, \xi)$
4. Jump Condition: $-[G_x]_{\xi^-}^{\xi^+} = 1$

The solution to $-u'' + u = 0$ is $u(x) = Ae^x + Be^{-x}$. If $u(0) = 0$, then $A = -B$, or $u(x) = A(e^x - e^{-x})$. On the other hand, if $u(1) = 0$ (and $u(x) = Ce^x + De^{-x}$), then $D = -Ce^2$, or $u(x) = C\big(e^x - e^{2-x}\big)$. Thus, the first two conditions imply

$$G(x;\xi) = \begin{cases} A(\xi)(e^x - e^{-x}) & \text{if } x \in [0,\xi) \\ C(\xi)\big(e^x - e^{2-x}\big) & \text{if } x \in (\xi, 1] \end{cases}$$

$$\implies G_x(x;\xi) = \begin{cases} A(\xi)(e^x + e^{-x}) & \text{if } x \in [0,\xi) \\ C(\xi)\left(e^x + e^{2-x}\right) & \text{if } x \in (\xi,1] \end{cases}$$

Continuity of G implies

$$A(\xi)\left(e^\xi - e^{-\xi}\right) = C(\xi)\left(e^\xi - e^{2-\xi}\right)$$
$$\implies A(\xi) = C(\xi)\left[\frac{e^\xi - e^{2-\xi}}{e^\xi - e^{-\xi}}\right]$$

The jump condition implies

$$C(\xi)\left[\frac{2(1-e^2)}{2^\xi - e^{-\xi}}\right] = -1 \qquad \implies \qquad C(\xi) = \frac{e^\xi - e^{-\xi}}{2(e^2-1)} \qquad \implies \qquad A(\xi) = \frac{e^\xi - e^{2-\xi}}{2(e^2-1)}$$

This shows

$$G(x;\xi) = \begin{cases} \dfrac{e^\xi - e^{2-\xi}}{2(e^2-1)}(e^x - e^{-x}) & \text{if } x \in [0,\xi) \\ \dfrac{e^\xi - e^{-\xi}}{2(e^2-1)}\left(e^x - e^{2-x}\right) & \text{if } x \in (\xi,1] \end{cases}$$

Note $G(x;\xi) = G(\xi;x)$. Then the general solution to $-u''(x) + u(x) = f(x)$ is

$$u(x) = \int_0^1 G(x;\xi) f(\xi) \mathrm{d}\xi$$

Problem 3

Compute the Green's function for the BVP

$$\begin{aligned} &-u'' = f(x) \qquad 0 < x < 1 \\ &u(0) + u(1) = 0, \qquad u'(0) + u'(1) = 0. \end{aligned}$$

Write down the integral representation of the solution u in terms of f.

First note the homogeneous problem is not singular since a linear function $u(x) = a + bx$ would solve $-u'' = 0$, but $u(0)+u(1) = 0 = u'(0)+u'(1)$ implies $a = b = 0$. Since $\{1, x\}$ form a fundamental set of solutions for the homogeneous problem on $[0,\xi)$ and $(\xi,1]$, then let G be the Green's function that solves $-G(x;\xi) = \delta(x-\xi)$:

$$G(x;\xi) = \begin{cases} A_1(\xi) + A_2(\xi)x & \text{if } x \in [0,\xi) \\ B_1(\xi) + B_2(\xi)x & \text{if } x \in (\xi,1] \end{cases}$$
$$\implies G_x(x;\xi) = \begin{cases} A_2(\xi) & \text{if } x \in [0,\xi) \\ B_2(\xi) & \text{if } x \in (\xi,1] \end{cases}$$

The boundary condition $u'(0) = -u'(1)$ implies $G_x(0,\xi) = -G_x(1,\xi)$, or $A_2(\xi) = -B_2(\xi)$, and thus

$$G(x;\xi) = \begin{cases} A_1(\xi) + A_2(\xi)x & \text{if } x \in [0,\xi) \\ B_1(\xi) - A_2(\xi)x & \text{if } x \in (\xi,1] \end{cases}$$

$$\implies G_x(x;\xi) = \begin{cases} A_2(\xi) & \text{if } x \in [0,\xi) \\ -A_2(\xi) & \text{if } x \in (\xi,1] \end{cases}$$

Continuity of G implies

$$A_1(\xi) + A_2(\xi)\xi = B_1(\xi) - A_2(\xi)\xi$$

and the jump condition $-[G_x]_{\xi^-}^{\xi^+} = 1$ implies

$$-\Big[-A_2(\xi) - A_2(\xi)\Big] = 1 \qquad \implies \qquad A_2(\xi) = \frac{1}{2}$$

and thus $A_1(\xi) = B_1(\xi) - \xi$, which shows

$$G(x;\xi) = \begin{cases} B_1(\xi) - \xi + \frac{x}{2} & \text{if } x \in [0,\xi) \\ B_1(\xi) - \frac{x}{2} & \text{if } x \in (\xi,1] \end{cases}$$

Then the boundary condition $u(0) = -u(1)$ implies $G(0,\xi) = -G(1,\xi)$, or $B_1(\xi) - \xi = -B_1(\xi) + \frac{1}{2}$, or $B_1(\xi) = \frac{\xi}{2} + \frac{1}{4}$. This shows

$$G(x;\xi) = -\frac{1}{4} + \frac{1}{2}\begin{cases} x - \xi & \text{if } x \in [0,\xi) \\ \xi - x & \text{if } x \in (\xi,1] \end{cases} = -\frac{1}{4} + \frac{1}{2}\big(x_< - x_>\big)$$

Problem 4

Compute the generalized Green's function $G(x;\xi)$ for the BVP

$$\begin{aligned} -u'' &= \pi^2 u + f(x) \qquad 0 < x < 1 \\ u(0) &= 0, \qquad u(1) = 0. \end{aligned}$$

State the equations that are satisfied by the function

$$u(x) = \int_0^1 G(x;\xi) f(\xi)\, d\xi.$$

Define the differential operator $A = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} - \pi^2$. Then $A(\sqrt{2}\sin\pi x) = 0$, which shows A is a singular operator for the homogeneous problem. *We use* $\sqrt{2}\sin\pi x$ *since* $\left\|\sqrt{2}\sin\pi x\right\|_{L^2} = 1$. This means we must orthogonally project the onto the kernel:

$$\Big\langle \sqrt{2}\sin\pi x, f(x)\Big\rangle = \int_0^1 \sqrt{2}\sin\pi x f(x)\mathrm{d}x = 0$$

In other words, the solvability condition for the boundary value problem is $f \perp \sqrt{2}\sin\pi x$. In this case, if $Av = f$ (v is a solution to the nonhomogeneous problem), then $u = v + c\sqrt{2}\sin\pi x$ is a solution to the nonhomogeneous problem since

$$Au = Av + Ac\sqrt{2}\sin\pi x = Av + 0 = Av = f$$

Thus, consider $u \perp \sqrt{2}\sin\pi x$ and solve the nonsingular problem

$$Au = f - 2\left\langle \sin\pi x, f\right\rangle \sin\pi x, \qquad u \perp \sqrt{2}\sin\pi x, \qquad u(0) = 0 = u(1) \tag{1}$$

Suppose the Green's function $G(x;\xi)$ is the solution to the above boundary value problem for $f = \delta(x-\xi)$. Then note that

$$f(x) = 0 \ \text{ for} x \neq \xi \qquad \text{and} \qquad \langle \sin\pi x, \delta(x-\xi)\rangle = \int_0^1 \sin\pi x \delta(x-\xi)\mathrm{d}x = \sin\pi\xi$$

Then G satisfies the following conditions:

1. Initial Value Problem: $G_{xx} + \pi^2 G = 2\sin\pi\xi\sin\pi x$ for $x \in [0,\xi)$, $G(0,\xi) = 0$, $G_x(0,\xi) = h_0(\xi)$
2. Final Value Problem: $G_{xx} + \pi^2 G = 2\sin\pi\xi\sin\pi x$ for $x \in (\xi,\xi]$, $G(1,\xi) = 0$, $G_x(1,\xi) = h_1(\xi)$
3. Continuity: $G(\xi^-,\xi) = G(\xi^+,\xi)$
4. Orthogonality: $G \perp \sin\pi x$

The homogeneous solution u_h to $u'' + \pi^2 u = 0$ is given by

$$u_h(x) = a\cos\pi x + b\sin\pi x$$

and guess the particular solution $Y(x) = x[c\sin\pi x + d\cos\pi x]$ to $u'' + \pi^2 u = \sin\pi x$. Then

$$\begin{aligned}
Y'(x) &= x\pi[c\cos\pi x - d\sin\pi x] + [c\sin\pi x + d\cos\pi x] \\
Y''(x) &= -x\pi^2[c\sin\pi x + d\cos\pi x] + 2\pi[c\cos\pi x - d\sin\pi x] \\
\implies Y'' + \pi^2 Y &= 2\pi c\cos\pi x - 2\pi d\sin\pi x = \sin\pi x \\
&\implies c = 0 \implies d = -\frac{1}{2\pi}
\end{aligned}$$

Thus, the particular solution $Y(x) = -\frac{x}{2\pi}\cos\pi x$ and the complete solution is

$$u(x) = a\cos\pi x + b\sin\pi x - \frac{x}{2\pi}\cos\pi x$$

The initial condition $u(0) = 0$ implies $a = 0$, and thus for $x \in [0,\xi)$,

$$\begin{aligned}
u(x) &= b\sin\pi x - \frac{x}{2\pi}\cos\pi x \\
\implies u'(x) &= b\pi\cos\pi x - \frac{1}{2\pi}\cos\pi x + \frac{x}{2}\sin\pi x
\end{aligned}$$

The initial condition $u'(0) = h_0$ implies

$$h_0 = b\pi - \frac{1}{2\pi} \qquad \implies \qquad b = \frac{2\pi h_0 - 1}{2\pi^2}$$

which shows, for $x \in [0,\xi)$,

$$u(x) = \frac{2\pi h_0 - 1}{2\pi^2}\sin\pi x - \frac{x}{2\pi}\cos\pi x$$

For $x \in (\xi, 1]$, the final condition $u(1) = 0$ implies $a = \frac{1}{2\pi}$, and thus

$$u(x) = \frac{1}{2\pi}\cos\pi x + b\sin\pi x - \frac{x}{2\pi}\cos\pi x$$

$$\implies u'(x) = -\frac{1}{2}\sin\pi x + b\pi\cos\pi x - \frac{1}{2\pi}\cos\pi x + \frac{x}{2}\sin\pi x$$

The final condition $u'(1) = h_1$ implies

$$h_1 = -b\pi + \frac{1}{2\pi} \qquad \implies \qquad b = \frac{1-2\pi h_1}{2\pi^2}$$

which shows, for $x \in (\xi, 1]$,

$$u(x) = \frac{1-2\pi h_1}{2\pi^2}\sin\pi x + \frac{1-x}{2\pi}\cos\pi x$$

Thus G is defined as

$$G(x;\xi) = \frac{\sin\pi\xi}{\pi^2}\begin{cases}(2\pi h_0 - 1)\sin\pi x - \pi x\cos\pi x & \text{if } x \in [0,\xi) \\ (1-2\pi h_1)\sin\pi x + \pi(1-x)\cos\pi x & \text{if } x \in (\xi, x]\end{cases}$$

Note the extra factor of $2\sin\pi\xi$, which is multiplied to the right hand side of the initial and final value problem. To solve for h_0 and h_1 we impose continuity

$$\begin{aligned}(2\pi h_0 - 1)\sin\pi\xi - \pi\xi\cos\pi\xi &= (1-2\pi h_1)\sin\pi\xi + \pi(1-\xi)\cos\pi\xi \\ \implies (2\pi h_0 - 1)\sin\pi\xi &= (1-2\pi h_1)\sin\pi\xi + \pi\cos\pi\xi\end{aligned}$$

and orthogonality $\langle G, \sin\pi x\rangle = 0$

$$\begin{aligned}\int_0^\xi (2\pi h_0 - 1)\sin^2\pi x \mathrm{d}x - \pi\int_0^\xi x\cos\pi x\sin\pi x\mathrm{d}x + \int_\xi^1 (1-2\pi h_1)\sin^2\pi x\mathrm{d}x + \pi\int_\xi^1 (1-x)\cos\pi x\sin\pi x\mathrm{d}x &= 0 \\ (2\pi h_0 - 1)\int_0^\xi \sin^2\pi x\mathrm{d}x + (1-2\pi h_1)\int_\xi^1 \sin^2\pi x\mathrm{d}x - \cancelto{0}{\pi\int_0^1 x\cos\pi x\sin\pi x\mathrm{d}x} + \pi\int_\xi^1 \cos\pi x\sin\pi x\mathrm{d}x &= 0 \\ (2\pi h_0 - 1)\left(\frac{\xi}{2} - \frac{\sin 2\pi\xi}{4\pi}\right) + (1-2\pi h_1)\left(\frac{1}{2} - \frac{\xi}{2} + \frac{\sin 2\pi\xi}{4\pi}\right) - \frac{1}{4}\Big[1 - \cos 2\pi\xi\Big] &= 0 \\ 2\pi(h_0 + h_1)\left(\frac{\xi}{2} - \frac{\sin 2\pi\xi}{4\pi}\right) - \pi h_1 + \frac{1}{4} - \xi + \frac{\sin 2\pi\xi}{2\pi} + \frac{\cos 2\pi\xi}{4} &= 0\end{aligned}$$

Continuity implies

$$h_0 = \frac{1}{\pi} - h_1 + \frac{\cot\pi\xi}{2}$$

Substituting this in to the orthogonality condition gives

$$2\pi\xi\cot\pi\xi - 4\pi h_1 = 0 \qquad \implies \qquad h_1 = \frac{\xi\cot\pi\xi}{2} \qquad \implies \qquad h_0 = \frac{(1-\xi)\cot\pi\xi}{2}$$

Thus,

$$\begin{aligned}G(x;\xi) &= \frac{\sin\pi\xi}{\pi^2}\begin{cases}\left(2\pi\frac{(1-\xi)\cot\pi\xi}{2} - 1\right)\sin\pi x - \pi x\cos\pi x & \text{if } x \in [0,\xi) \\ \left(1 - 2\pi\frac{\xi\cot\pi\xi}{2}\right)\sin\pi x + \pi(1-x)\cos\pi x & \text{if } x \in (\xi, x]\end{cases} \\ &= \frac{1}{\pi^2}\begin{cases}\pi(1-\xi)\cos\pi\xi\sin\pi x - \sin\pi\xi\sin\pi x - \pi x\sin\pi\xi\cos\pi x & \text{if } x \in [0,\xi) \\ \pi(1-x)\cos\pi x\sin\pi\xi - \sin\pi x\sin\pi\xi - \pi\xi\sin\pi x\cos\pi\xi & \text{if } x \in (\xi, 1]\end{cases}\end{aligned}$$

Clearly, $G(x;\xi) = G(\xi;x)$, so we can define $x_> = \max\{x,\xi\}$ and $x_< = \min\{x,\xi\}$ and write

$$G(x;\xi) = \frac{1}{\pi^2}\Big[\pi(1-x_>)\cos\pi x_> \sin\pi x_< - \sin\pi x_> \sin\pi x_< - \pi x_< \sin\pi x_> \cos\pi x_<\Big]$$

Then the solution to the projected problem (1) is

$$u(x) = \int_0^1 G(x;\xi)f(\xi)\mathrm{d}\xi$$

Problem 5

Consider the Sturm-Liouville equation

$$-(pu')' + qu = \lambda r u, \qquad a < x < b$$

where $p, q, r : [a,b] \to \mathbb{R}$ are smooth functions and $p(x), r(x) > 0$ for $a \le x \le b$. Show that the change of variables

$$t = \int_a^x \sqrt{\frac{r(s)}{p(s)}}\, ds, \qquad v(t) = \big[r(x)p(x)\big]^{1/4} u(x)$$

transforms this equation into a Sturm-Liouville equation with $p = r = 1$ of the form

$$-v'' + Qv = \lambda v, \qquad 0 < t < c.$$

What are c and $Q : [0,c] \to \mathbb{R}$?

First note that $u = v(rp)^{-1/4}$ and that

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \sqrt{\frac{r(x)}{p(x)}}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}v = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}v = \sqrt{\frac{r}{p}}v' \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}v' = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}v' = \sqrt{\frac{r}{p}}v''$$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}u = \frac{\mathrm{d}}{\mathrm{d}x}\big[v(rp)^{-1/4}\big] &= \sqrt{\frac{r}{p}}v'(rp)^{-1/4} + v\left(-\frac{1}{4}(rp)^{-5/4}(rp)'\right) = v'r^{1/4}p^{-3/4} - \frac{1}{4}v(rp)'(rp)^{-5/4} \\
\Longrightarrow p\frac{\mathrm{d}}{\mathrm{d}x}u &= v'(rp)^{1/4} - \frac{1}{4}v(rp)'r^{-5/4}p^{-1/4} \\
\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}}{\mathrm{d}x}u\right] &= \sqrt{\frac{r}{p}}v''(rp)^{1/4} + \frac{1}{4}v'(rp)^{-3/4}(rp)' - \frac{1}{4}\Bigg\{v(rp)'r^{5/4}\frac{-1}{4}p^{-5/4}p' + \Bigg[v(rp)'\frac{-5}{4}r^{-9/4}r' \\
&\qquad + \left(v(rp)'' + \sqrt{\frac{r}{p}}v'(rp)'\right)r^{-5/4}\Bigg]p^{-1/4}\Bigg\}
\end{aligned}$$

$$= v''\left[r^{3/4}p^{-1/4}\right] + v\left[\frac{1}{16}(rp)'p'r^{5/4}p^{-5/4} + \frac{5}{16}(rp)'r'r^{-9/4}p^{-1/4} - \frac{1}{4}(rp)''r^{-5/4}p^{-1/4}\right]$$

$$\implies -\left(r^{-3/4}p^{1/4}\right)(pu')' = -v'' - v\left[\frac{1}{16}(rp)'p'r^{1/2}p^{-1} + \frac{5}{16}(rp)'r'r^{-3} - \frac{1}{4}(rp)''r^{-2}\right]$$

Also,

$$\left(r^{-3/4}p^{1/4}\right)qu = \left(r^{-3/4}p^{1/4}\right)qvr^{-1/4}p^{-1/4} = qvr^{-1}$$

and

$$\left(r^{-3/4}p^{1/4}\right)\lambda ru = \left(r^{1/4}p^{1/4}\right)\lambda vr^{-1/4}p^{-1/4} = \lambda v$$

Thus,

$$-v'' + Qv = \lambda v$$

where

$$\boxed{Q = -\left[\frac{1}{16}(rp)'p'r^{1/2}p^{-1} + \frac{5}{16}(rp)'r'r^{-3} - \frac{1}{4}(rp)''r^{-2} + qr^{-1}\right]}$$

Finally, when $x = b$,

$$\boxed{c = \int_a^b \sqrt{\frac{r(s)}{p(s)}}\,\mathrm{d}s}$$