UC DAVIS, APPLIED MATHEMATICS (MAT207B), WINTER 2016

Homework #7

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Problem 1

(a) *Use separation of variables to find the eigenvalues λ and eigenfunctions $u(x,y)$ of the Dirichlet Laplacian on the unit square that satisfy*

$$\begin{gathered} -\left(u_{xx}+u_{yy}\right)=\lambda u \qquad 0<x<1,\ 0<y<1 \\ u(x,0)=0, \qquad u(x,1)=0 \qquad 0\le x\le 1 \\ u(0,y)=0, \qquad u(1,y)=0 \qquad 0\le y\le 1. \end{gathered}$$

Suppose $u(x,y)=F(x)G(y)$. Then

$$\begin{aligned} -F''G-FG''=\lambda FG \qquad &\implies \qquad -F''G=F\left(G''+\lambda G\right) \\ &\implies -\frac{F''}{F}=\frac{G''}{G}+\lambda \end{aligned}$$

Since the left hand side is a function of x and the right hand side is a function of y, then they can only be equal if

$$-\frac{F''}{F}=\frac{G''}{G}+\lambda=\mu$$

where μ is a constant. Note the boundary conditions imply

$$F(0)=F(1)=G(0)=G(1)=0$$

thus $F''+\mu F=0$ and the homogeneous Dirichlet boundary conditions imply $\mu>0$ and

$$F(x)=A\sin\left(\sqrt{\mu}x\right)$$

The condition $F(1)=0$ implies

$$0=A\sin\left(\sqrt{\mu}\right) \qquad \implies \qquad \mu=\pi^2n^2$$

for $n\ge 1$. Then $G''+(\lambda-\mu)G=0$ and the homogeneous Dirichlet boundary conditions imply $\lambda-\mu>0$ and

$$G(y)=B\sin\left(\sqrt{\lambda-\mu}y\right)$$

The condition $G(1)=0$ implies

$$0=B\sin\left(\sqrt{\lambda-\mu}\right) \qquad \implies \qquad \lambda-\mu=\pi^2m^2 \qquad \implies \qquad \lambda=\pi^2(n^2+m^2)$$

for $n,m\ge 1$. Thus the solution to $-(u_{xx}+u_{yy})=\lambda u$ is

$$\sum_{n,m\ge 1} A_{n,m}\sin(n\pi x)\sin\left(m\pi y\right)$$

for some constants $A_{n,m}$. Then the eigenvalues of

$$-\nabla^2u=\lambda u; \qquad u(x,0)=u(x,1)=u(0,y)=u(1,y)0$$

are $\lambda=\pi^2(m^2+n^2)$ for all $n,m\ge 1$.

(b) What is the smallest eigenvalue that is not a simple eigenvalue?

The smallest eigenvalue that is not a simple eigenvalue is $5\pi^2$ since this can be acheived when $n = 1$, $m = 2$ or when $n = 2$, $m = 1$. The only smaller eigenvalue is $2\pi^2$, but that is simple since it can only be acheived when $n = m = 1$.

Problem 2

(a) Let $\vec{x} = (x, y)$, $\vec{\xi} = (\xi, \eta)$, and $\vec{\xi^*} = (\xi, -\eta)$ where $\eta > 0$. Show that

$$G\left(\vec{x}, \vec{\xi}\right) = -\frac{1}{2\pi}\log\left(\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi^*}\right|}\right)$$

is the solution of

$$-\left(G_{xx} + G_{yy}\right) = \delta\left(\vec{x} - \vec{\xi}\right) \qquad \text{in } -\infty < x < \infty, \qquad y > 0$$
$$G\left(\vec{x}, \vec{\xi}\right) = 0 \qquad \text{on } y = 0.$$

$$\begin{aligned}
-\nabla^2 G &= -\nabla^2\left[-\frac{1}{2\pi}\log\left(\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi^*}\right|}\right)\right] \\
&= -\nabla^2\left[-\frac{1}{2\pi}\log\left|\vec{x} - \vec{\xi}\right| - \left(-\frac{1}{2\pi}\log\left|\vec{x} - \vec{\xi^*}\right|\right)\right] \\
&= -\nabla^2\left[G_F\left(\left|\vec{x} - \vec{\xi}\right|\right) - G_F\left(\left|\vec{x} - \vec{\xi^*}\right|\right)\right]
\end{aligned}$$

where

$$G_F(r) = -\frac{1}{2\pi}\log r$$

is the free-space Green's function in two dimensions. By definition,

$$-\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi}\right|\right) = \delta\left(\vec{x} - \vec{\xi}\right) \qquad \text{and} \qquad -\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi^*}\right|\right) = \delta\left(\vec{x} - \vec{\xi^*}\right).$$

But $\delta\left(\vec{x} - \vec{\xi^*}\right) = 0$ for $y > 0$ since $\xi^* = (\xi, -\eta)$ where $\eta > 0$. Thus,

$$-\nabla^2 G = -\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi}\right|\right) + \cancelto{0}{\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi^*}\right|\right)} = \delta\left(\vec{x} - \vec{\xi}\right) \qquad \text{for } (x, y) \in \{(x, y) \,:\, y > 0\}$$

When $y = 0$, $\left|\vec{x} - \vec{\xi}\right| = \left|\vec{x} - \vec{\xi^*}\right|$, thus

$$\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi^*}\right|} = 1 \qquad \Longrightarrow \qquad G\left(\vec{x}, \vec{\xi}\right) = -\frac{1}{2\pi}\log(1) = 0$$

(b) *Write down the Green's function representation for the solution $u(x,y)$ of the Dirichlet problem for the Laplacian in the upper half plane*

$$u_{xx} + u_{yy} = 0 \qquad \text{in } -\infty < x < \infty, \qquad y > 0$$
$$u(x,0) = f(x).$$

You can assume that $u(x,y) \to 0$ sufficiently rapidly as $|(x,y)| \to \infty$.

We utilize Green's identity

$$\int_\Omega \left(u\nabla^2 v - v\nabla^2 u\right)\mathrm{d}\vec{\xi} = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n\left(\vec{\xi}\right)} - v\frac{\partial u}{\partial n\left(\vec{\xi}\right)}\right)\mathrm{d}s\left(\vec{\xi}\right)$$

by setting $\Omega = \{(x,y) \ : \ y > 0\}$ and $v = G$ where G is the free-space Green's function which solves $-\nabla^2 G = \delta(x)$. Thus,

$$\int_\Omega \left(u\nabla^2 G - G\nabla^2 u\right)\mathrm{d}\vec{\xi} = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n\left(\vec{\xi}\right)} - G\frac{\partial u}{\partial n\left(\vec{\xi}\right)}\right)\mathrm{d}s\left(\vec{\xi}\right)$$

However, u is harmonic in Ω, i.e. $\nabla^2 u \equiv 0$ for $(x,y) \in \Omega$, and $-\nabla^2 G = \delta\left(\vec{x} - \vec{\xi}\right)$, i.e.

$$\int_\Omega \left(u\nabla^2 G - \cancelto{0}{G\nabla^2 u}\right)\mathrm{d}\vec{\xi} = -\int_\Omega u\left(\vec{\xi}\right)\delta\left(\vec{x} - \vec{\xi}\right)\mathrm{d}\vec{\xi} = -u(\vec{x})$$

Also, the Dirichlet condition $u(\vec{x}) = f(x)$ on $\partial\Omega$ give us $G \equiv 0$ for $y = 0$ (i.e. for $(x,y) \in \partial\Omega$). Thus,

$$-u(\vec{x}) = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n\left(\vec{\xi}\right)} - \cancelto{0}{G\frac{\partial u}{\partial n\left(\vec{\xi}\right)}}\right)\mathrm{d}s\left(\vec{\xi}\right) = \int_{\partial\Omega} u\left(\vec{\xi}\right)\frac{\partial G\left(x,y;\xi,\eta\right)}{\partial n\left(\vec{\xi}\right)}\mathrm{d}s\left(\vec{\xi}\right)$$

Finally, the unit normal vector on the boundary of Ω is $-\eta$, and thus

$$\boxed{u(\vec{x}) = \int_{\mathbb{R}} u\left(\vec{\xi}\right)\frac{\partial G\left(x,y;\xi,\eta\right)}{\partial\eta}\mathrm{d}\xi}$$

(c) *Use the Green's function representation to show that*

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(x-t)^2 + y^2}\mathrm{d}t.$$

Rewrite $G\left(\vec{x};\vec{\xi}\right) = G\left(x,y;\xi,\eta\right)$. Then

$$G\left(\vec{x},\vec{\xi}\right) = \frac{1}{2\pi}\log\left(\frac{\left|\vec{x}-\vec{\xi}\right|}{\left|\vec{x}-\vec{\xi}^*\right|}\right) = -\frac{1}{4\pi}\log\left(\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}\right)$$

$$\implies \left.\frac{\partial G}{\partial\eta}\right|_{\eta=0} = \left[-\frac{1}{4\pi}\cdot\frac{\left[(x-\xi)^2 + (y+\eta)^2\right](-2)(y-\eta) - \left[(x-\xi)^2 + (y-\eta)^2\right](2)(y+\eta)}{\left((x-\xi)^2 + (y+\eta)^2\right)^2}\right]_{\eta=0}$$

$$= -\frac{1}{4\pi}\cdot\frac{\left[(x-\xi)^2 + y^2\right](-2y) - \left[(x-\xi)^2 + y^2\right](2y)}{\left((x-\xi)^2 + y^2\right)^2}$$

$$= -\frac{1}{4\pi} \cdot \frac{1}{(x-\xi)^2 + y^2} \cdot (-4y)$$
$$= \frac{y}{\pi} \frac{1}{(x-\xi)^2 + y^2}$$

Thus,

$$u(x,y) = u(\vec{x}) = \int_{\mathbb{R}} u\left(\vec{\xi}\right) \frac{\partial G(x,y;\xi,\eta)}{\partial \eta} \mathrm{d}\xi = \frac{y}{\pi} \int_{\mathbb{R}} \frac{u\left(\vec{\xi}\right)}{(x-\xi)^2 + y^2} \mathrm{d}\xi$$

Define $t := \xi$ and note $u\left(\vec{\xi}\right) = f(\xi)$ when $\eta = 0$. Thus,

$$\boxed{u(x,y) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(x-t)^2 + y^2} \mathrm{d}t}$$