

HW #1

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Problem 1

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(0, 0) = 0$ and

$$f(x, y) = \frac{xy^3}{x^2 + y^6} \quad \text{if } f(x, y) \neq (0, 0)$$

(a) Show that the directional derivatives of f at $(0, 0)$ exist in every direction. What is its Gâteaux derivatives at $(0, 0)$?

(b) Show that f is not Fréchet differentiable at $(0, 0)$. (HINT. A Fréchet differentiable function must be continuous.)

Let $u = \langle u_1, u_2 \rangle$ and consider $D_u f(0, 0)$

$$\begin{aligned} D_u f(0, 0) &= \lim_{\varepsilon \rightarrow 0} \frac{f(0 + \varepsilon u_1, 0 + \varepsilon u_2) - f(0, 0)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon} \frac{(\varepsilon u_1)(\varepsilon u_2)^3}{(\varepsilon u_1)^2 + (\varepsilon u_2)^6} \right] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\varepsilon \frac{u_1 u_2^3}{u_1^2 + \varepsilon^4 u_2^6} \right] \\ &= 0 \end{aligned}$$

provided $u_1 \neq 0$. If $u_1 = 0$,

$$\begin{aligned} D_u f(0, 0) &= \lim_{\varepsilon \rightarrow 0} \frac{f(0, 0 + \varepsilon u_2) - f(0, 0)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon} \frac{(0)(\varepsilon u_2)^3}{0 + (\varepsilon u_2)^6} \right] \\ &= \lim_{\varepsilon \rightarrow 0} [0] \\ &= 0 \end{aligned}$$

Thus all directional derivatives exist and the Gâteaux derivative $\nabla f = f_x \mathbf{i} + f_y \mathbf{j} = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$.

If f is Fréchet differentiable, f must be continuous. However, f is not continuous because consider approaching $(0, 0)$ along the curve $y = x$. Then

$$\lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} f(x, x) = \lim_{x \rightarrow 0} \frac{x^4}{x^2 + x^6} = \lim_{x \rightarrow 0} \frac{x^2}{1 + x^4} = 0$$

Now consider approaching $(0, 0)$ along the curve $y = \sqrt[3]{x}$. Then

$$\lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} f(x, \sqrt[3]{x}) = \lim_{x \rightarrow 0} \frac{x^2}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

Since $0 \neq \frac{1}{2}$, f is not continuous, and thus f is not Fréchet differentiable at $(0, 0)$.

Problem 2

Define $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = x^2 + y^2, \quad g(x, y) = (y - 1)^3 - x^2$$

Find the minimum value of $f(x, y)$ subject to the constraint $g(x, y) = 0$. Show that there does not exist any constant λ such that $\nabla f = \lambda \nabla g$ at some point $(x, y) \in \mathbb{R}^2$. Why does the method of Lagrange multipliers fail in this example?

Note that $\nabla g = \langle -2x, 3(y - 1)^2 \rangle$ and $\nabla f = \langle 2x, 2y \rangle$. Also note that the method of Lagrange multipliers cannot be used if $\nabla g = \mathbf{0}$.

Now assume that $\nabla f = \lambda \nabla g$. Then

$$\begin{aligned} f_x &= \lambda g_x & \text{and} \\ f_y &= \lambda g_y \end{aligned}$$

Thus,

$$\begin{aligned} 2x &= \lambda(-2x) & \text{and} \\ 2y &= \lambda(3(y - 1)^2) \end{aligned}$$

The first equation gives either $\lambda = -1$ or $x = 0$. If $x = 0$, then $g = 0 \implies y = 1$. However, $(x, y) = (0, 1) \implies \nabla g = \mathbf{0}$. Thus $\lambda = -1$. The second equation gives

$$\begin{aligned} 2y &= -3(y^2 - 2y + 1) \\ \implies 3y^2 - 4y + 3 &= 0 \\ \implies y &= \frac{4 \pm \sqrt{-20}}{6} \notin \mathbb{R} \end{aligned}$$

which is not possible since $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$. Thus the Lagrange multiplier method does not work in this case.

Problem 3

Derive the Euler-Lagrange equation for a functional of the form

$$J(u) = \int_a^b F(x, u, u', u'') dx$$

What are the natural boundary conditions for this functional?

Assume u minimizes the functional. Then

$$\begin{aligned} dJ(u, \phi) &= \frac{d}{d\varepsilon} \int_a^b F(x, u + \varepsilon\phi, u' + \varepsilon\phi', u'' + \varepsilon\phi'') \Big|_{\varepsilon=0} = 0 \\ \implies \int_a^b [F_u(x, u, u', u'')\phi + F_{u'}(x, u, u', u'')\phi' + F_{u''}(x, u, u', u'')\phi''] dx &= 0 \end{aligned}$$

By integrating by parts, we get

$$\int_a^b F_{u'}(x, u, u', u'')\phi' dx = [F_{u'}(x, u, u', u'')\phi]_a^b - \int_a^b \left(\frac{d}{dx} F_{u'}(x, u, u', u'') \right) \phi dx$$

and

$$\int_a^b F_{u''}(x, u, u', u'')\phi'' dx = [F_{u''}(x, u, u', u'')\phi']_a^b - \int_a^b \left(\frac{d}{dx} F_{u''}(x, u, u', u'') \right) \phi' dx$$

Again, by integrating by parts,

$$\int_a^b \left(\frac{d}{dx} F_{u''}(x, u, u', u'') \right) \phi' dx = \left[\left(\frac{d}{dx} F_{u''}(x, u, u', u'') \right) \phi \right]_a^b - \int_a^b \left(\frac{d^2}{dx^2} F_{u''}(x, u, u', u'') \right) \phi dx$$

Thus,

$$\int_a^b \left[F_u \phi - \left(\frac{d}{dx} F_{u'} \right) \phi + \left(\frac{d}{dx^2} F_{u''} \right) \phi \right] dx + \left[F_{u'} \phi + F_{u''} \phi' - \left(\frac{d}{dx} F_{u''} \right) \phi \right]_a^b = 0$$

Thus, given natural boundary conditions, the Euler-Lagrange equation is

$$F_u - \frac{d}{dx} F_{u'} + \frac{d^2}{dx^2} F_{u''} = 0$$

The required boundary conditions are

$$\left[\left(F_{u'} - \frac{d}{dx} F_{u''} \right) \phi \right]_a^b + [F_{u''} \phi'] = 0$$

The most “natural” boundary conditions are

$$\left[F_{u'} - \frac{d}{dx} F_{u''} \right] \Big|_{x=a} = \left[F_{u'} - \frac{d}{dx} F_{u''} \right] \Big|_{x=b} = (F_{u'}) \Big|_{x=a} = (F_{u''}) \Big|_{x=b} = 0$$

Problem 4

A curve $y = u(x)$ with $a \leq x \leq b$, $u(x) > 0$, and $u(a) = u_0$, $u(b) = u_1$ is rotated about the x -axis. Find the curve that minimizes the area of the surface of revolution,

$$J(u) = \int_a^b u \sqrt{1 + (u')^2} dx$$

Let $F(x, u, u') = F(u, u') = u\sqrt{1 + (u')^2}$. Since $J(u) = \int_a^b F(u, u')dx$, and since $F(u, u')$ is explicitly independent of x , we can guarantee

$$F - u'F_{u'} = C \in \mathbb{R}$$

Note

$$F_{u'} = \frac{uu'}{\sqrt{1 + (u')^2}}$$

Thus,

$$\begin{aligned} u\sqrt{1 + (u')^2} - u'\frac{uu'}{\sqrt{1 + (u')^2}} &= C \\ \Rightarrow \frac{u(1 + (u')^2)}{\sqrt{1 + (u')^2}} - \frac{u(u')^2}{\sqrt{1 + (u')^2}} &= C \\ &\Rightarrow u = C\sqrt{1 + (u')^2} \\ &\Rightarrow \frac{du}{dx} = \sqrt{\left(\frac{u}{C}\right)^2 - 1} \end{aligned}$$

This is a separable differential equation, so

$$\begin{aligned} C \int \frac{1}{\sqrt{u^2 - C^2}} du &= \int dx \\ \Rightarrow C \ln \left(\sqrt{u^2 - C^2} + u \right) &= x + K \end{aligned}$$

where $K \in \mathbb{R}$. We obtain a system of equations by noting the Dirichlet boundary conditions $u(a) = u_0$, $u(b) = u_1$:

$$\begin{aligned} C \ln \left(\sqrt{u_0^2 - C^2} + u_0 \right) &= a + K \\ C \ln \left(\sqrt{u_1^2 - C^2} + u_1 \right) &= b + K \end{aligned}$$

We can solve this system for $C = C(u_0, u_1, a, b)$ and $K = K(u_0, u_1, a, b)$ in terms of u_0 , u_1 , a , and b to obtain the implicit curve

$$C \ln \left(\sqrt{u^2 - C^2} + u \right) = x + K$$

Problem 5

Let X be the space of smooth functions $u : [0, 1] \rightarrow \mathbb{R}$ such that $u(0) = 0$, $u(1) = 0$. Define functionals $J, K : X \rightarrow \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_0^1 (u')^2 dx, \quad K(u) = \frac{1}{2} \int_0^1 u^2 dx$$

(a) Introduce a Lagrange multiplier and write down the Euler-Lagrange equation for extremals in X of the functional $J(u)$ subject to the constraint $K(u) = 1$.

(b) Solve the eigenvalue problem in (a) and find all of the extremals. Which one minimizes $J(u)$?

Define $L : X \rightarrow \mathbb{R}$ by

$$L(u) = K(u) - 1 = \frac{1}{2} \int_0^1 u^2 dx - \int_0^1 dx = \int_0^1 \left(\frac{1}{2} u^2 - 1 \right) dx$$

Then the constraint $K(u) = 1$ is equivalent to $L(u) = 0$. Thus we consider

$$\frac{\delta J}{\delta u} = \lambda \frac{\delta L}{\delta u}$$

Define $F(x, u, u') = F(u') = \frac{1}{2}(u')^2$ and $G(x, u, u') = G(u) = \frac{1}{2}u^2 - 1$. Then

$$\left[-\frac{d}{dx} F_{u'} + F_u \right] = \lambda \left[-\frac{d}{dx} G_{u'} + G_u \right]$$

Note that $F_{u'} = u'$, $F_u = 0$, $G_{u'} = 0$ and $G_u = u$. Thus,

$$\begin{aligned} -\frac{d}{dx} u' &= \lambda u \\ \implies u'' + \lambda u &= 0 \\ \implies u(x) &= a \exp(\sqrt{-\lambda}x) + b \exp(-\sqrt{-\lambda}x) \end{aligned}$$

If $\lambda < 0$, then $u(0) = 0$ implies $0 = a + b$ and $u(1) = 0$ implies

$$0 = a \sinh(\sqrt{\lambda})$$

Thus either $a = 0$ or $\lambda = 0$. But $\lambda < 0 \implies \lambda \neq 0$. Thus $a = 0$, and thus $b = 0$. So

$$u \equiv \mathbf{0}$$

But $L(\mathbf{0}) = \int_0^1 -1 dx = -1 \neq 0$, which contradicts our constrain requirement. Thus $\lambda > 0$. This gives

$$u(x) = a \sin(\sqrt{\lambda}x) + b \cos(\sqrt{\lambda}x)$$

Using the boundary requirements,

$$\begin{aligned} 0 &= b \quad \text{and} \\ 0 &= a \sin(\sqrt{\lambda}) \end{aligned}$$

Thus $\lambda = \pi^2$ or $a = 0$. However, if $a = 0$ then $u \equiv \mathbf{0}$, which is a contradiction as showed above. Thus $\lambda = \pi^2$. So,

$$u(x) = a \sin(\pi x)$$

Since $L(u) = 0$,

$$\frac{1}{2} \int_0^1 (a \sin(\pi x))^2 dx - 1 = 0$$

$$\begin{aligned}
\implies \frac{a^2}{2} \int_0^1 (\sin(\pi x))^2 dx - 1 &= 0 \\
\implies \frac{a^2}{2} \frac{1}{2} - 1 &= 0 \\
\implies a^2 &= 4 \\
\implies a &= \pm 2
\end{aligned}$$

Thus $u_+(x) = 2 \sin(\pi x)$ and $u_-(x) = -2 \sin(\pi x)$ minimize $J(u)$ subject to $L(u) = 0$. Also note that $(u'_+)^2 = (u'_-)^2 = 4\pi^2(\cos(\pi x))^2$ and so $J(u_+) = J(u_-) = \pi^2$.

Problem 6

(a) Make a change of variable $x = \phi(t)$, $v(t) = u(\phi(t))$, where $\phi'(t) > 0$, in the functional

$$J(u) = \int_a^b F(x, u, u') dx$$

Show that $J(u) = K(v)$ where $K(v)$ has the form

$$K(v) = \int_c^d G(t, v, v') dt$$

and express G in terms of F and ϕ .

(b) Show that the Euler-Lagrange equation for $K(v)$ is the same as what you get by changing variables in the Euler-Lagrange equation for $J(u)$.