Homework #7

Sam Fleischer

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Problem 1

(a) *Use separation of variables to find the eigenvalues λ and eigenfunctions $u(x,y)$ of the Dirichlet Laplacian on the unit square that satisfy*

$$-\left(u_{xx} + u_{yy}\right) = \lambda u \qquad 0 < x < 1,\ 0 < y < 1$$
$$u(x,0) = 0, \qquad u(x,1) = 0 \qquad 0 \le x \le 1$$
$$u(0,y) = 0, \qquad u(1,y) = 0 \qquad 0 \le y \le 1.$$

Suppose $u(x,y) = F(x)G(y)$. Then

$$-F''G - FG'' = \lambda FG \qquad \Longrightarrow \qquad -F''G = F\left(G'' + \lambda G\right)$$
$$\Longrightarrow \quad -\frac{F''}{F} = \frac{G''}{G} + \lambda$$

Since the left hand side is a function of x and the right hand side is a function of y, then they can only be equal if

$$-\frac{F''}{F} = \frac{G''}{G} + \lambda = \mu$$

where μ is a constant. Note the boundary conditions imply

$$F(0) = F(1) = G(0) = G(1) = 0$$

thus $F'' + \mu F = 0$ and the homogeneous Dirichlet boundary conditions imply $\mu > 0$ and

$$F(x) = A\sin\left(\sqrt{\mu}x\right)$$

The condition $F(1) = 0$ implies

$$0 = A\sin\left(\sqrt{\mu}\right) \qquad \Longrightarrow \qquad \mu = \pi^2 n^2$$

for $n \ge 1$. Then $G'' + (\lambda - \mu)G = 0$ and the homogeneous Dirichlet boundary conditions imply $\lambda - \mu > 0$ and

$$G(y) = B\sin\left(\sqrt{\lambda - \mu}y\right)$$

The condition $G(1) = 0$ implies

$$0 = B\sin\left(\sqrt{\lambda - \mu}\right) \qquad \Longrightarrow \qquad \lambda - \mu = \pi^2 m^2 \qquad \Longrightarrow \qquad \lambda = \pi^2(n^2 + m^2)$$

for $n, m \ge 1$. Thus the solution to $-(u_{xx} + u_{yy}) = \lambda u$ is

$$\sum_{n,m \ge 1} A_{n,m} \sin(n\pi x)\sin\left(m\pi y\right)$$

for some constants $A_{n,m}$. Then the eigenvalues of

$$-\nabla^2 u = \lambda u; \qquad u(x,0) = u(x,1) = u(0,y) = u(1,y)0$$

are $\lambda = \pi^2(m^2 + n^2)$ for all $n, m \ge 1$.

(b) What is the smallest eigenvalue that is not a simple eigenvalue?

The smallest eigenvalue that is not a simple eigenvalue is $5\pi^2$ since this can be acheived when $n = 1$, $m = 2$ or when $n = 2$, $m = 1$. The only smaller eigenvalue is $2\pi^2$, but that is simple since it can only be acheived when $n = m = 1$.

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Problem 2

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(a) Let $\vec{x} = (x, y)$, $\vec{\xi} = (\xi, \eta)$, and $\vec{\xi}^* = (\xi, -\eta)$ where $\eta > 0$. Show that

$$G\left(\vec{x}, \vec{\xi}\right) = -\frac{1}{2\pi} \log\left(\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi}^*\right|}\right)$$

is the solution of

$$-\left(G_{xx} + G_{yy}\right) = \delta\left(\vec{x} - \vec{\xi}\right) \qquad \text{in } -\infty < x < \infty, \qquad y > 0$$

$$G\left(\vec{x}, \vec{\xi}\right) = 0 \qquad \text{on } y = 0.$$

$$-\nabla^2 G = -\nabla^2 \left[-\frac{1}{2\pi} \log\left(\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi}^*\right|}\right) \right]$$

$$= -\nabla^2 \left[-\frac{1}{2\pi} \log\left|\vec{x} - \vec{\xi}\right| - \left(-\frac{1}{2\pi} \log\left|\vec{x} - \vec{\xi}^*\right| \right) \right]$$

$$= -\nabla^2 \left[G_F\left(\left|\vec{x} - \vec{\xi}\right|\right) - G_F\left(\left|\vec{x} - \vec{\xi}^*\right|\right) \right]$$

where

$$G_F(r) = -\frac{1}{2\pi} \log r$$

is the free-space Green's function in two dimensions. By definition,

$$-\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi}\right|\right) = \delta\left(\vec{x} - \vec{\xi}\right) \qquad \text{and} \qquad -\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi}^*\right|\right) = \delta\left(\vec{x} - \vec{\xi}^*\right).$$

But $\delta\left(\vec{x} - \vec{\xi}^*\right) = 0$ for $y > 0$ since $\xi^* = (\xi, -\eta)$ where $\eta > 0$. Thus,

$$-\nabla^2 G = -\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi}\right|\right) + \underbrace{\nabla^2 G_F\left(\left|\vec{x} - \vec{\xi}^*\right|\right)}_{0} = \delta\left(\vec{x} - \vec{\xi}\right) \qquad \text{for } (x, y) \in \{(x, y) \, : \, y > 0\}$$

When $y = 0$, $\left|\vec{x} - \vec{\xi}\right| = \left|\vec{x} - \vec{\xi}^*\right|$, thus

$$\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi}^*\right|} = 1 \qquad \Longrightarrow \qquad G\left(\vec{x}, \vec{\xi}\right) = -\frac{1}{2\pi} \log(1) = 0$$

(b) *Write down the Green's function representation for the solution $u(x, y)$ of the Dirichlet problem for the Laplacian in the upper half plane*

$$u_{xx} + u_{yy} = 0 \qquad \text{in } -\infty < x < \infty, \qquad y > 0$$
$$u(x, 0) = f(x).$$

You can assume that $u(x, y) \to 0$ sufficiently rapidly as $|(x, y)| \to \infty$.

We utilize Green's identity

$$\int_\Omega \left(u \nabla^2 v - v \nabla^2 u \right) \mathrm{d}\vec{\xi} = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n(\vec{\xi})} - v \frac{\partial u}{\partial n(\vec{\xi})} \right) \mathrm{d}s(\vec{\xi})$$

by setting $\Omega = \{(x, y) \; : \; y > 0\}$ and $v = G$ where G is the free-space Green's function which solves $-\nabla^2 G = \delta(x)$. Thus,

$$\int_\Omega \left(u \nabla^2 G - G \nabla^2 u \right) \mathrm{d}\vec{\xi} = \int_{\partial\Omega} \left(u \frac{\partial G}{\partial n(\vec{\xi})} - G \frac{\partial u}{\partial n(\vec{\xi})} \right) \mathrm{d}s(\vec{\xi})$$

However, u is harmonic in Ω, i.e. $\nabla^2 u \equiv 0$ for $(x, y) \in \Omega$, and $-\nabla^2 G = \delta\left(\vec{x} - \vec{\xi}\right)$, i.e.

$$\int_\Omega \left(u \nabla^2 G - G \overset{0}{\cancel{\nabla^2 u}} \right) \mathrm{d}\vec{\xi} = -\int_\Omega u\left(\vec{\xi}\right) \delta\left(\vec{x} - \vec{\xi}\right) \mathrm{d}\vec{\xi} = -u(\vec{x})$$

Also, the Dirichlet condition $u(\vec{x}) = f(x)$ on $\partial\Omega$ give us $G \equiv 0$ for $y = 0$ (i.e. for $(x, y) \in \partial\Omega$). Thus,

$$-u(\vec{x}) = \int_{\partial\Omega} \left(u \frac{\partial G}{\partial n(\vec{\xi})} - \overset{0}{\cancel{G \frac{\partial u}{\partial n(\vec{\xi})}}} \right) \mathrm{d}s(\vec{\xi}) = \int_{\partial\Omega} u\left(\vec{\xi}\right) \frac{\partial G(x, y; \xi, \eta)}{\partial n(\vec{\xi})} \mathrm{d}s(\vec{\xi})$$

Finally, the unit normal vector on the boundary of Ω is $-\eta$, and thus

$$\boxed{u(\vec{x}) = -\int_{\mathbb{R}} u\left(\vec{\xi}\right) \frac{\partial G(x, y; \xi, \eta)}{\partial \eta} \mathrm{d}\xi}$$

(c) *Use the Green's function representation to show that*

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x - t)^2 + y^2} \mathrm{d}t.$$

Rewrite $G\left(\vec{x}; \vec{\xi}\right) = G(x, y; \xi, \eta)$. Then

$$G\left(\vec{x}, \vec{\xi}\right) = \frac{1}{2\pi} \log\left(\frac{\left|\vec{x} - \vec{\xi}\right|}{\left|\vec{x} - \vec{\xi}^*\right|} \right) = -\frac{1}{4\pi} \log\left(\frac{(x - \xi)^2 + (y - \eta)^2}{(x - \xi)^2 + (y + \eta)^2} \right)$$

$$\implies \left. \frac{\partial G}{\partial \eta} \right|_{\eta=0} = \left[-\frac{1}{4\pi} \cdot \frac{\left[(x - \xi)^2 + (y + \eta)^2\right](-2)(y - \eta) - \left[(x - \xi)^2 + (y - \eta)^2\right](2)(y + \eta)}{\left((x - \xi)^2 + (y + \eta)^2\right)^2} \right]_{\eta=0}$$

$$= -\frac{1}{4\pi} \cdot \frac{\left[(x - \xi)^2 + y^2\right](-2y) - \left[(x - \xi)^2 + y^2\right](2y)}{\left((x - \xi)^2 + y^2\right)^2}$$

$$= -\frac{1}{4\pi} \cdot \frac{1}{(x-\xi)^2 + y^2} \cdot (-4y)$$

$$= \frac{y}{\pi} \frac{1}{(x-\xi)^2 + y^2}$$

Thus,

$$u(x, y) = u(\vec{x}) = \int_{\mathbb{R}} u\left(\vec{\xi}\right) \frac{\partial G(x, y; \xi, \eta)}{\partial \eta} \,\mathrm{d}\xi = \frac{y}{\pi} \int_{\mathbb{R}} \frac{u\left(\vec{\xi}\right)}{(x-\xi)^2 + y^2} \,\mathrm{d}\xi$$

Define $t := \xi$ and note $u\left(\vec{\xi}\right) = f(\xi)$ when $\eta = 0$. Thus,

$$\boxed{u(x, y) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(x-t)^2 + y^2} \,\mathrm{d}t}$$