UC DAVIS, APPLIED MATHEMATICS (MAT207B), WINTER 2016

Homework #5

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Problem 1

Suppose that $p : [a,b] \to \mathbb{R}$ is a continuously differentiable function such that $p > 0$ and $q, r : [a,b]$ are continuous functions such that $r > 0$, $q \geq 0$. Define a weighted inner product on $L^2(a,b)$ by

$$\langle u, v\rangle_r = \int_a^b r(x)\overline{u(x)}v(x)\mathrm{d}x.$$

Let $A : D(A) \subset L^2(a,b) \to L^2(a,b)$ by

$$A = \frac{1}{r(x)}\left[-\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right]$$

with Dirichlet boundary conditions and domain

$$D(A) = \left\{u \in H^2(a,b) : u(a) = 0 = u(b)\right\}.$$

(a) *Show that*

$$\langle u, Av\rangle_r = \langle Au, v\rangle_r \qquad \textit{for all } u, v \in D(A),$$

meaning that A is self-adjoint with respect to $\langle\cdot,\cdot\rangle_r$.

Denote the real and complex parts of a function u by u_r and u_i, respectively. Then

$$\begin{aligned}\langle u, Av\rangle_r &= \int_a^b r\overline{u}\frac{1}{r}\left[-(pv')' + qv\right]\mathrm{d}x \\ &= \int_a^b r u_r \frac{1}{r}\left[-(pv')' + qv\right]\mathrm{d}x - i\int_a^b r u_i \frac{1}{r}\left[-(pv')' + qv\right]\mathrm{d}x \\ &= \int_a^b u_r\left[-(pv')' + qv\right]\mathrm{d}x - i\int_a^b u_i\left[-(pv')' + qv\right]\mathrm{d}x\end{aligned}$$

By Homework 4 number 3,

$$\langle u, Av\rangle_r = \left[p\big((u_r'v - u_rv') - i(u_i'v - u_iv')\big)\right]_a^b + \langle rAu_r, v\rangle - i\langle rAu_i, v\rangle$$

where $\langle u, v\rangle$ is the unweighted innerproduct of u and v. The Dirichlet boundary condition $u(a) = u(b) = 0$ implies $u_r(a) = u_r(b) = u_i(a) = u_i(b) = 0$. If we assume the adjoint boundary condition on v,

$$v_r(a) = v_r(b) = v_i(a) = v_i(b) = 0$$

then

$$\langle u, Av\rangle_r = \langle rAu_r, v\rangle - i\langle rAu_i, v\rangle = \langle rAu, v\rangle$$

since inner products are conjugate-linear in the first term. However, since $r > 0$, then

$$\langle ru, v\rangle = \int_a^b \overline{ru}v\mathrm{d}x = \int_a^b r\overline{u}v\mathrm{d}x = \langle u, v\rangle_r$$

which proves

$$\langle u, Av\rangle_r = \langle rAu, v\rangle = \langle Au, v\rangle_r$$

Thus, A is self-adjoint with respect to $\langle\cdot,\cdot\rangle_r$.

(b) *Show that the eigenvalues λ of the weighted Sturm-Liouville eigenvalue problem*

$$-\left(pu'\right)' + qu = \lambda r u, \qquad u(a) = 0 = u(b)$$

are real and positive and eigenfunctions associated with different eigenvalues are orthogonal with respect to $\langle\cdot,\cdot\rangle_r$.

Eigenvalues λ of $-\left(pu'\right)' + qu = \lambda r u$; $u(a) = 0 = u(b)$ are eigenvalues of

$$Au = \lambda u, \qquad u(a) = 0 = u(b)$$

where A is defined above. We showed A is self-adjoint with respect to $\langle\cdot,\cdot\rangle_r$ in part **(a)**. Thus if $Au = \lambda u$,

$$\langle Au, u\rangle_r = \langle \lambda u, u\rangle_r = \overline{\lambda}\langle u, u\rangle_r \qquad \text{and} \qquad \langle u, Au\rangle_r = \langle u, \lambda u\rangle_r = \lambda\langle u, u\rangle_r$$

Thus $\lambda = \overline{\lambda}$ or $u = 0$, i.e. $\lambda \in \mathbb{R}$ if λ is an eigenvalue. Note

$$\lambda = \frac{\langle u, Au\rangle_r}{\langle u, u\rangle_r}$$

implies $\lambda > 0$ since $u \neq 0$ and inner-products are positive-definite.

Now consider eigenfunctions of A, ϕ_n, ϕ_m with eigenvalues λ_n and λ_m, respectively ($\lambda_n \neq \lambda_m$). Then

$$A\phi_n = \lambda_n\phi_n \qquad \text{and} \qquad A\phi_m = \lambda_m\phi_m$$

By multiplying the left equation by ϕ_m and the right equation by ϕ_n, and subtracting the two equations, we see

$$\begin{aligned} \phi_m A\phi_n - \phi_n A\phi_m &= (\lambda_n - \lambda_m)\phi_n\phi_m \\ \implies \left\langle \phi_m, A\phi_n\right\rangle_r - \left\langle A\phi_m, \phi_n\right\rangle_r &= \int_a^b (\lambda_n - \lambda_m)\phi_n\phi_m \mathrm{d}x \end{aligned}$$

Then since A is self-adjoint,

$$\begin{aligned} 0 = \frac{1}{r}0 = (\lambda_n - \lambda_m)\int_a^b \phi_n\phi_m \mathrm{d}x \\ \implies 0 &= \int_a^b \phi_n\phi_m r\,\mathrm{d}x \\ &= \left\langle \phi_n, \phi_m\right\rangle_r \end{aligned}$$

and thus ϕ_n and ϕ_m are orthogonal.

Problem 2

A nonuniform string of length one with wave speed $c_0(x) = \sqrt{\dfrac{T}{\rho_0(x)}} > 0$ is fixed at each end, with zero initial *displacement and nonzero initial velocity. The transverse displaceent $y = u(x,t)$ of the string satisfies the IBVP*

$$\begin{array}{lll} u_{tt} = c_0^2(x)u_{xx}, & 0 < x < 1, & t > 0, \\ u(0,t) = 0, & u(1,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 < x < 1, & \\ u_t(x,0) = g(x), & 0 < x < 1, & \end{array}$$

Find the solution in terms of the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the weighted Sturm-Liouville problem

$$-c_0^2\phi_n'' = \lambda_n\phi_n, \qquad \phi_n(0) = 0, \qquad \phi_n(1) = 0, \qquad n = 1,2,3,\dots$$

Suppose $u(x,t) = F(x)G(t)$. Then

$$F(x)G''(t) = c_0^2(x)F''(x)G(t)$$
$$\left(\frac{G''}{G}\right)(t) = \left(\frac{F''c_0^2}{F}\right)(x) = -\lambda \in \mathbb{R}$$

since the left hand side is a function of t and the right hand side is a function of x. Then

$$-c_0^2(x)F''(x) = \lambda F(x)$$

By Problem 1 ($p \equiv 1$, $q \equiv 0$, and $r \equiv \frac{1}{c_0^2}$) this implies the eigenfunctions $\phi_n(x)$ are orthogonal and its eigenvalues λ_n are real and $\lambda_n > 0$ for all n. Then

$$\begin{aligned} & G'' = -\lambda G \qquad \lambda > 0 \\ \implies & G(t) = \sum_{n=1}^{\infty}\left[a_n\cos\left(\sqrt{\lambda_n}t\right) + b_n\sin\left(\sqrt{\lambda_n}t\right)\right] \\ \implies & u(x,t) = \sum_{n=1}^{\infty}\left\{\left[a_n\cos\left(\sqrt{\lambda_n}t\right) + b_n\sin\left(\sqrt{\lambda_n}t\right)\right]\phi_n(x)\right\} \\ \implies & 0 = u(x,0) = \sum_{n=1}^{\infty}\left[a_n\phi_n(x)\right] \\ \implies & a_n = 0 \text{ for } n = 1,2,\dots \qquad \text{because } \{\phi_n\}_n \text{ is a linearly independent set} \\ \implies & u(x,t) = \sum_{n=1}^{\infty}\left\{\left[b_n\sin\left(\sqrt{\lambda_n}t\right)\right]\phi_n(x)\right\} \\ \implies & u_t(x,t) = \sum_{n=1}^{\infty}\left\{\sqrt{\lambda_n}b_n\cos\left(\sqrt{\lambda_n}t\right)\phi_n(x)\right\} \\ \implies & g(x) = u_t(x,0) = \sum_{n=1}^{\infty}\left[\sqrt{\lambda_n}b_n\phi_n(x)\right] \end{aligned}$$

Thus the coefficients $\sqrt{\lambda_n}b_n$ are Fourier coefficients with respect to the weighted norm $\langle\cdot,\cdot\rangle_{c_0^2}$, i.e.

$$\sqrt{\lambda_n}b_n = \left\langle g(x), \phi_n(x)\right\rangle_{c_0^2(x)}$$

So,

$$u_t(x,t) = \sum_{n=1}^{\infty}\left\{\left\langle g(x),\phi_n(x)\right\rangle_{c_0^2(x)}\cos\left(\sqrt{\lambda_n}t\right)\phi_n(x)\right\}$$

Problem 3

The Fourier solution of the initial value problem

$$\begin{aligned} &u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ &u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \\ &u(x,0) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} < x < 1 \end{cases}, \\ &= u_t(x,0) = 0, \qquad 0 \leq x \leq 1 \end{aligned}$$

is given by

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right]$$

(a) *Show that the Fourier seires converges to a continuous function. How many spacial (weak) L^2-derivatives does $u(x,t)$ have?*

For a fixed t, it suffices to show the partial sums u_N converge to a function in H^k for $k < \frac{3}{2}$ (and $u \notin H^k$ for $k \geq \frac{3}{2}$) where H^k is the Sobolev space of order k. Then by the Sobolev Embedding Theorem, u converges to a continuous function. Thus, that continuous function has a weak spacial L^2 derivative $u' \notin H^{\frac{1}{2}}$, and so u does not have two weak spacial derivatives.

To show this is true, fix t and denote $c_t = \cos\left[(2n-1)\pi t\right] \in \mathbb{R}$. Then the Fourier coefficients $\hat{u}_n$ are proportional to the following (we do not say equal since we want to consider the standard Fourier basis $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ as opposed to the basis $\left\{1, \sin nx, \cos nx\right\}_{n=1}^{\infty}$).

$$|\hat{u}_n| \propto \frac{1}{(2n-1)^2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \hat{u}_n^2 \propto \sum_{n=1}^{\infty} \frac{1}{n^4}$$

which converges. Then the Fourier coefficients of the weak derivative are proportional to the following:

$$\left|\hat{u'}_n\right| = |n\hat{u}_n| \propto \frac{n}{(2n-1)^2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \hat{u'}_n^2 \propto \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which also conveges. However,

$$\left|\hat{u''}_n\right| = \left|n^2 \hat{u}_n\right| \propto \frac{n^2}{(2n-1)^2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \hat{u''}_n^2 \propto \sum_{n=1}^{\infty} 1$$

which diverges.

(b) *Verify from the Fourier solution that*

$$\int_0^1 \left[u_t^2(x,t) + u_x^2(x,t)\right] \mathrm{d}x = \textit{constant} \qquad \textit{for } -\infty < t < \infty.$$

First note

$$u_x(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}(2n-1)\pi\cos((2n-1)\pi x)\cos((2n-1)\pi t), \qquad \text{and}$$
$$u_t(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2n-1)^2}(2n-1)\pi\sin((2n-1)\pi x)\sin((2n-1)\pi t)$$

and since

$$\int_0^1 \cos((2n-1)\pi x)\cos((2m-1)\pi x)\mathrm{d}x = \int_0^1 \sin((2n-1)\pi x)\sin((2m-1)\pi x)\mathrm{d}x = \frac{1}{2}\delta_{n,m} = \begin{cases}\frac{1}{2} & \text{if } n = m\\ 0 & \text{if } n \neq m\end{cases}$$

then

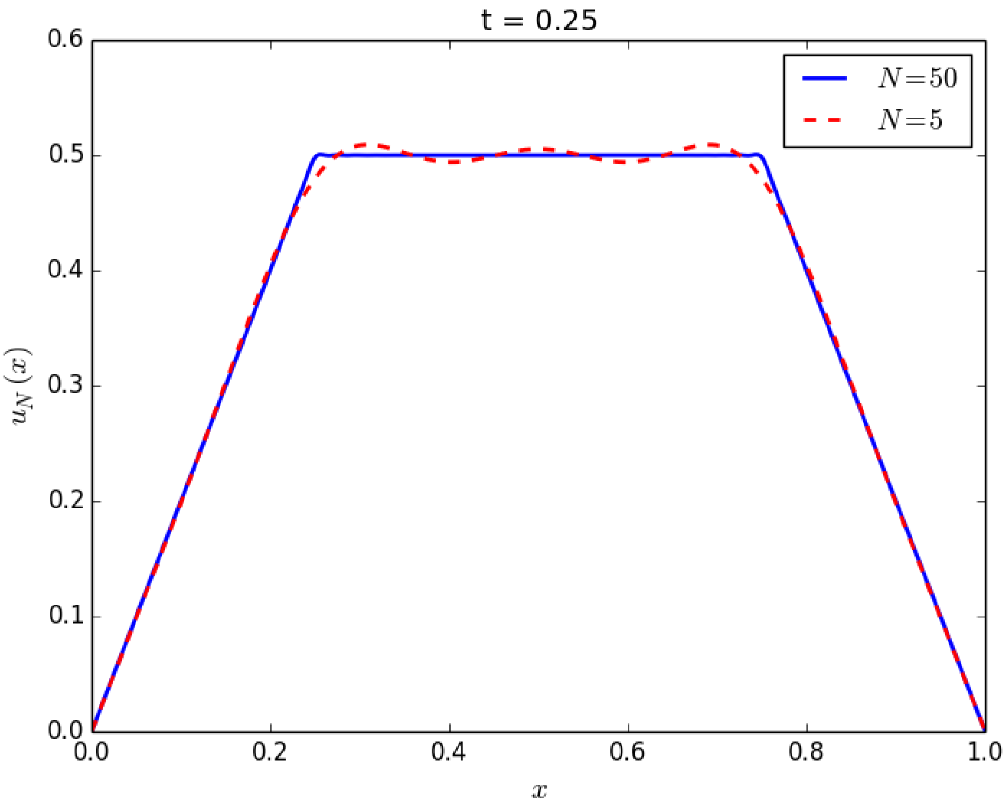
$$\int_0^1 u_t^2\mathrm{d}x = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin^2((2n-1)\pi t)\int_0^1\sin^2((2n-1)\pi x)\mathrm{d}x, \qquad \text{and}$$
$$\int_0^1 u_x^2\mathrm{d}x = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos^2((2n-1)\pi t)\int_0^1\cos^2((2n-1)\pi x)\mathrm{d}x$$

$$\begin{aligned}
\implies &\int_0^1 [u_t^2 + u_x^2]\mathrm{d}x\\
&= \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[\sin^2((2n-1)\pi t)\int_0^1\sin^2((2n-1)\pi x)\mathrm{d}x + \cos^2((2n-1)\pi t)\int_0^1\cos^2((2n-1)\pi x)\mathrm{d}x\right]\\
&= \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[\sin^2((2n-1)\pi t) + \cos^2((2n-1)\pi t)\right]\\
&= \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\\
&= \frac{32}{\pi^2}\cdot\frac{\pi^2}{8}\\
&= 4
\end{aligned}$$

(c) *Use* `MATLAB` *(or another program) to compute the partial sum*

$$u_N(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{(2n-1)^2}\sin[(2n-1)\pi x]\cos[(2n-1)\pi t]$$

at $t = 0.25$ *for* $N = 5$ *and* $N = 50$.



(d) *Use the addition formula for sines to show that the Fourier solution can be written in the form of the d'Alembert solution as*

$$u(x,t) = F(x-t) + F(x+t)$$

for a suitable function $F : \mathbb{R} \to \mathbb{R}$. What is F?

Define $\tilde{F}$ as half the initial data, i.e.

$$\tilde{F}(x) = \begin{cases} x & \text{if } 0 \le x \le n + \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

Let F be it's odd 2-periodic expansion,

$$F(x) = \begin{cases} x & \text{if } n - \frac{1}{2} \le x \le n + \frac{1}{2} \\ 1 - x & \text{if } n + \frac{1}{2} < x < n + \frac{3}{2} \end{cases} \qquad \frac{n}{2} \in \mathbb{Z}$$

and note its Fourier series representation:

$$F(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)\pi x)$$

Then note $F(x-t) + F(x+t) = u(x,t)$ because

$$F(x-t) + F(x+t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)\pi(x-t)) + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)\pi(x+t))$$

$$\begin{aligned}
&= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\frac{1}{2}[\sin((2n-1)\pi(x-t))+\sin((2n-1)\pi(x+t))] \\
&= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)\pi x)\cos((2n-1)\pi t) \\
&= u(x,t)
\end{aligned}$$

Problem 4

Suppose that $u(x,t)$ is a smooth solution of the wave equation

$$u_{tt} = c_0^2\Delta u,$$

where $x \in \mathbb{R}^n$, the wave speed $c_0 > 0$ is a constant.

(a) *Show that u satisfies the energy equation*

$$\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right)_t - \nabla\cdot\left(c_0^2 u_t\nabla u\right) = 0.$$

$$\begin{aligned}
u_{tt} &= c_0^2\Delta u \\
\iff u_t u_{tt} &= c^2 u_t\Delta u \\
\iff u_t u_{tt} + c^2\nabla u\cdot\nabla u_t &= c^2 u_t\Delta u + c^2\nabla u\cdot\nabla u_t \\
\iff \frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2\right)_t &= \nabla\cdot\left(c^2 u_t\nabla u\right)
\end{aligned}$$

(b) *For $T > 0$, let $\Omega_T \subset \mathbb{R}^{n+1}$ be the space-time cone*

$$\Omega_T = \left\{(x,t)\in\mathbb{R}^{n+1} \,:\, |x| < c_0(T-t),\ 0 < t < T\right\},$$

and for $0 \le t \le T$, let $B(T-t)$ be the spatial cross-section of Ω_T at time t

$$B(T-t) = \left\{x\in\mathbb{R}^n \,:\, |x| < c_0(T-t)\right\}.$$

Define

$$e_T(t) = \frac{1}{2}\int_{B(T-t)}\left(u_t^2 + c_0^2|\nabla u|^2\right)\mathrm{d}x,$$

and show that $e_T(t) \le e_T(0)$.

HINT. Apply the divergence theorem in space-time to the equation in ***(a)*** *over the truncated cone $\left\{(x,t')\in\Omega_T \,:\, 0 < t' < t\right\}$, and note that the space-time normal to the side of the cone Ω_T is $N = \dfrac{(\hat{x},c_0)}{\sqrt{1+c_0^2}}$ where $\hat{x} = \dfrac{x}{|x|}$.*

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$F(\vec{x},t) = \left\langle -c_0^2 u_t\nabla u, \frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right)\right\rangle$$

$$\begin{aligned}\implies \nabla\cdot F &= \left(-c_0^2 u_t\nabla u\right)_{\vec{x}} + \frac{1}{2}\left(u_t + c_0^2 + |\nabla u|^2\right)_t\\ &= \nabla\left(-c_0^2 u_t \nabla u\right) + \frac{1}{2}\left(u_t + c_0^2 + |\nabla u|^2\right)_t\end{aligned}$$

For ease, denote $\Omega_{T,t}$ to be the truncated cone, i.e.

$$\Omega_{T,t} = \left\{(x,t') \in \Omega_T \,:\, 0 < t' < t\right\}$$

and denote the curved "side" part of the boundary of the cone as

$$\partial\Omega_{T,t,\text{side}} = \left\{x \in \partial\Omega_{T,t} \,:\, \vec{n} \neq (0,1) \text{ or } \vec{n} \neq (0,-1)\right\}$$

Thus by the divergence theorem,

$$\begin{aligned}0 &= \int_{\Omega_{T,t}} \nabla\cdot F \mathrm{d}V\\ &= \int_{\partial\Omega_{T,t}} F\cdot\vec{n}\mathrm{d}s\\ &= \int_{B(T)} F\cdot\langle 0,-1\rangle\,\mathrm{d}s + \int_{B(T-t)} F\cdot\langle 0,1\rangle\,\mathrm{d}s + \int_{\partial\Omega_{T,t,\text{side}}} F\cdot\frac{\langle \hat{x}, c_0\rangle}{\sqrt{1+c_0^2}}\mathrm{d}s\\ &= -e_T(0) + e_T(t) + \int_{\partial\Omega_{T,t,\text{side}}}\left[\cancelto{0}{-\frac{c_0^2 u_t \nabla u x}{\sqrt{1+c_0^2}}} + \frac{\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right)c_0}{\sqrt{1+c_0^2}}\right]\mathrm{d}s\end{aligned}$$

Since the integrand is always positive, then

$$0 \leq e_T(t) \leq e_T(0)$$

(c) *Suppose that u_1, u_2 are smooth solutions of the wave equation such that*

$$u_i(x,0) = f_i(x), \qquad u_{i\,t}(x,0) = g_i(x), \qquad i = 1,2$$

where $f_i = g_i$ in $|x| \leq c_0 T$. Show that $u_1 = u_2 \in \Omega_T$.
HINT. Consider $u = u_1 - u_2$.

Let $u = u_1 - u_2$. Then since u is a linear combination of solutions of the wave equation then u is a solution of the wave equation. Then note $u(x,0) = u_1(x,0) - u_2(x,0) = 0$ and $u_t(x,0) = u_{1\,t}(x,0) - u_{2\,t}(x,0) = 0$. Also, since u is sufficiently smooth, then

$$(\nabla u)(x,0) = \nabla(u(x,0)) = \nabla 0 = 0$$

for all x. Then fix $t \in [0,T]$ and note that by part **(b)**,

$$0 \leq e_T(t) \leq e_T(0) = \int_{B(T)}\left[u_t^2(x,0) + c_0^2|\nabla u(x,0)|^2\right]\mathrm{d}x$$

However, initial conditions $u_t(x,0) = \nabla u(x,0) = 0$, and so

$$\begin{aligned}0 \leq e_T(t) &\leq \int_B (T)\left[\cancelto{0}{u_t^2(x,0)} + c_0^2\cancelto{0}{|\nabla u(x,0)|^2}\right] = 0\\ &\implies e_T(t) = 0 \qquad \forall t \in [0,T].\end{aligned}$$

Since $u_t^2 \geq 0$ and $|\nabla u|^2 \geq 0$, then $u_t = 0$ and $\nabla u = 0$ for all $t \in [0, T]$. This shows u is constant, i.e.

$$u(x, t) = K \in \mathbb{R}$$

but $u(x, 0) = 0 \implies K = 0$, i.e. $u \equiv 0$, or

$$u_1 = u_2.$$