
Homework #6

Sam Fleischer

March 4, 2016

Problem 1	2
Problem 2	2
Problem 3	3
Problem 4	4
Problem 5	6

Problem 1

Suppose that $u_1, u_2 : \mathbb{R} \rightarrow \mathbb{R}$ are two solutions of the homogeneous Sturm-Liouville equation

$$-(pu')' + qu = 0$$

where $p, q : \mathbb{R} \rightarrow \mathbb{R}$ are smooth functions and $p > 0$. If $W = u_1 u_2' - u_2 u_1'$ is the Wronskian of u_1, u_2 , show that $pW = \text{constant}$.

If u_1, u_2 are solutions of $-(pu')' + qu = 0$, then

$$\begin{aligned} -(pu_1')' + qu_1 &= 0 \implies -(p'u_2 u_1' + pu_2 u_1'') + qu_1 u_2 = 0, & \text{and} \\ -(pu_2')' + qu_2 &= 0 \implies -(p'u_1 u_2' + pu_1 u_2'') + qu_2 u_1 = 0, \\ &\implies p'u_1 u_2' - p'u_2 u_1' + pu_1 u_2'' - pu_2 u_1'' = 0 \\ &\implies \frac{d}{dx}(pW) = 0 \\ &\implies pW = \text{constant} \end{aligned}$$

Problem 2

Compute the Green's function for the BVP

$$\begin{aligned} -u'' + u &= f(x) & 0 < x < 1 \\ u(0) &= 0, & u(1) = 0. \end{aligned}$$

Write down the integral representation of the solution u in terms of f .

Assume the Green's function $G(x; \xi)$ is continuous, and solves $AG(x, \xi) = \delta(x - \xi)$ where $A = -\frac{d^2}{dx^2} + \text{Id}$. Then the following four conditions must hold:

1. Initial Value Problem: $-G_{xx} + G = 0$; $G(0, \xi) = 0$ for $x \in [0, \xi)$
2. Final Value Problem: $-G_{xx} + G = 0$; $G(1, \xi) = 0$ for $x \in (\xi, 1]$
3. Continuity: $G(\xi^-, \xi) = G(\xi^+, \xi)$
4. Jump Condition: $-[G_x]_{\xi^-}^{\xi^+} = 1$

The solution to $-u'' + u = 0$ is $u(x) = Ae^x + Be^{-x}$. If $u(0) = 0$, then $A = -B$, or $u(x) = A(e^x - e^{-x})$. On the other hand, if $u(1) = 0$ (and $u(x) = Ce^x + De^{-x}$), then $D = -Ce^2$, or $u(x) = C(e^x - e^{2-x})$. Thus, the first two conditions imply

$$G(x; \xi) = \begin{cases} A(\xi)(e^x - e^{-x}) & \text{if } x \in [0, \xi) \\ C(\xi)(e^x - e^{2-x}) & \text{if } x \in (\xi, 1] \end{cases}$$

$$\Rightarrow G_x(x; \xi) = \begin{cases} A(\xi)(e^x + e^{-x}) & \text{if } x \in [0, \xi) \\ C(\xi)(e^x + e^{2-x}) & \text{if } x \in (\xi, 1] \end{cases}$$

Continuity of G implies

$$\begin{aligned} A(\xi)(e^\xi - e^{-\xi}) &= C(\xi)(e^\xi - e^{2-\xi}) \\ \Rightarrow A(\xi) &= C(\xi) \left[\frac{e^\xi - e^{2-\xi}}{e^\xi - e^{-\xi}} \right] \end{aligned}$$

The jump condition implies

$$C(\xi) \left[\frac{2(1 - e^2)}{2^\xi - e^{-\xi}} \right] = -1 \quad \Rightarrow \quad C(\xi) = \frac{e^\xi - e^{-\xi}}{2(e^2 - 1)} \quad \Rightarrow \quad A(\xi) = \frac{e^\xi - e^{2-\xi}}{2(e^2 - 1)}$$

This shows

$$G(x; \xi) = \begin{cases} \frac{e^\xi - e^{2-\xi}}{2(e^2 - 1)}(e^x - e^{-x}) & \text{if } x \in [0, \xi) \\ \frac{e^\xi - e^{-\xi}}{2(e^2 - 1)}(e^x - e^{2-x}) & \text{if } x \in (\xi, 1] \end{cases}$$

Note $G(x; \xi) = G(\xi; x)$. Then the general solution to $-u''(x) + u(x) = f(x)$ is

$$u(x) = \int_0^1 G(x; \xi) f(\xi) d\xi$$

Problem 3

Compute the Green's function for the BVP

$$\begin{aligned} -u'' &= f(x) & 0 < x < 1 \\ u(0) + u(1) &= 0, & u'(0) + u'(1) &= 0. \end{aligned}$$

Write down the integral representation of the solution u in terms of f .

First note the homogeneous problem is not singular since a linear function $u(x) = a + bx$ would solve $-u'' = 0$, but $u(0) + u(1) = 0 = u'(0) + u'(1)$ implies $a = b = 0$. Since $\{1, x\}$ form a fundamental set of solutions for the homogeneous problem on $[0, \xi)$ and $(\xi, 1]$, then let G be the Green's function that solves $-G(x; \xi) = \delta(x - \xi)$:

$$\begin{aligned} G(x; \xi) &= \begin{cases} A_1(\xi) + A_2(\xi)x & \text{if } x \in [0, \xi) \\ B_1(\xi) + B_2(\xi)x & \text{if } x \in (\xi, 1] \end{cases} \\ \Rightarrow G_x(x; \xi) &= \begin{cases} A_2(\xi) & \text{if } x \in [0, \xi) \\ B_2(\xi) & \text{if } x \in (\xi, 1] \end{cases} \end{aligned}$$

The boundary condition $u'(0) = -u'(1)$ implies $G_x(0, \xi) = -G_x(1, \xi)$, or $A_2(\xi) = -B_2(\xi)$, and thus

$$G(x; \xi) = \begin{cases} A_1(\xi) + A_2(\xi)x & \text{if } x \in [0, \xi) \\ B_1(\xi) - A_2(\xi)x & \text{if } x \in (\xi, 1] \end{cases}$$

$$\Rightarrow G_x(x; \xi) = \begin{cases} A_2(\xi) & \text{if } x \in [0, \xi) \\ -A_2(\xi) & \text{if } x \in (\xi, 1] \end{cases}$$

Continuity of G implies

$$A_1(\xi) + A_2(\xi)\xi = B_1(\xi) - A_2(\xi)\xi$$

and the jump condition $-[G_x]_{\xi^-}^{\xi^+} = 1$ implies

$$-[-A_2(\xi) - A_2(\xi)] = 1 \quad \Rightarrow \quad A_2(\xi) = \frac{1}{2}$$

and thus $A_1(\xi) = B_1(\xi) - \xi$, which shows

$$G(x; \xi) = \begin{cases} B_1(\xi) - \xi + \frac{x}{2} & \text{if } x \in [0, \xi) \\ B_1(\xi) - \frac{x}{2} & \text{if } x \in (\xi, 1] \end{cases}$$

Then the boundary condition $u(0) = -u(1)$ implies $G(0, \xi) = -G(1, \xi)$, or $B_1(\xi) - \xi = -B_1(\xi) + \frac{1}{2}$, or $B_1(\xi) = \frac{\xi}{2} + \frac{1}{4}$. This shows

$$G(x; \xi) = -\frac{1}{4} + \frac{1}{2} \begin{cases} x - \xi & \text{if } x \in [0, \xi) \\ \xi - x & \text{if } x \in (\xi, 1] \end{cases} = -\frac{1}{4} + \frac{1}{2}(x_{<} - x_{>})$$

Problem 4

Compute the generalized Green's function $G(x; \xi)$ for the BVP

$$\begin{aligned} -u'' &= \pi^2 u + f(x) & 0 < x < 1 \\ u(0) &= 0, & u(1) &= 0. \end{aligned}$$

State the equations that are satisfied by the function

$$u(x) = \int_0^1 G(x; \xi) f(\xi) d\xi.$$

Define the differential operator $A = -\frac{d^2}{dx^2} - \pi^2$. Then $A(\sqrt{2} \sin \pi x) = 0$, which shows A is a singular operator for the homogeneous problem. We use $\sqrt{2} \sin \pi x$ since $\|\sqrt{2} \sin \pi x\|_{L^2} = 1$. This means we must orthogonally project the onto the kernel:

$$\langle \sqrt{2} \sin \pi x, f(x) \rangle = \int_0^1 \sqrt{2} \sin \pi x f(x) dx = 0$$

In other words, the solvability condition for the boundary value problem is $f \perp \sqrt{2} \sin \pi x$. In this case, if $Av = f$ (v is a solution to the nonhomogeneous problem), then $u = v + c\sqrt{2} \sin \pi x$ is a solution to the nonhomogeneous problem since

$$Au = Av + Ac\sqrt{2} \sin \pi x = Av + 0 = Av = f$$

Thus, consider $u \perp \sqrt{2} \sin \pi x$ and solve the nonsingular problem

$$Au = f - 2 \langle \sin \pi x, f \rangle \sin \pi x, \quad u \perp \sqrt{2} \sin \pi x, \quad u(0) = 0 = u(1)$$

Suppose the Green's function $G(x; \xi)$ is the solution to the above boundary value problem for $f = \delta(x - \xi)$. Then note that

$$f(x) = 0 \text{ for } x \neq \xi \quad \text{and} \quad \langle \sin \pi x, \delta(x - \xi) \rangle = \int_0^1 \sin \pi x \delta(x - \xi) dx = \sin \pi \xi$$

Then G satisfies the following conditions:

1. Initial Value Problem: $G_{xx} + \pi^2 G = 2 \sin \pi \xi \sin \pi x$ for $x \in [0, \xi]$, $G(0, \xi) = 0$, $G_x(0, \xi) = h_0(\xi)$
2. Final Value Problem: $G_{xx} + \pi^2 G = 2 \sin \pi \xi \sin \pi x$ for $x \in (\xi, 1]$, $G(1, \xi) = 0$, $G_x(1, \xi) = h_1(\xi)$
3. Continuity: $G(\xi^-, \xi) = G(\xi^+, \xi)$
4. Orthogonality: $G \perp \sin \pi x$

The homogeneous solution u_h to $u'' + \pi^2 u = 0$ is given by

$$u_h(x) = a \cos \pi x + b \sin \pi x$$

and guess the particular solution $Y(x) = x[c \sin \pi x + d \cos \pi x]$ to $u'' + \pi^2 u = \sin \pi x$. Then

$$\begin{aligned} Y'(x) &= x\pi[c \cos \pi x - d \sin \pi x] + [c \sin \pi x + d \cos \pi x] \\ Y''(x) &= -x\pi^2[c \sin \pi x + d \cos \pi x] + 2\pi[c \cos \pi x - d \sin \pi x] \\ \implies Y'' + \pi^2 Y &= 2\pi c \cos \pi x - 2\pi d \sin \pi x = \sin \pi x \\ \implies c &= 0 \implies d = -\frac{1}{2\pi} \end{aligned}$$

Thus, the particular solution $Y(x) = -\frac{x}{2\pi} \cos \pi x$ and the complete solution is

$$u(x) = a \cos \pi x + b \sin \pi x - \frac{x}{2\pi} \cos \pi x$$

The initial condition $u(0) = 0$ implies $a = 0$, and thus for $x \in [0, \xi]$,

$$\begin{aligned} u(x) &= b \sin \pi x - \frac{x}{2\pi} \cos \pi x \\ \implies u'(x) &= b\pi \cos \pi x - \frac{1}{2\pi} \cos \pi x + \frac{x}{2} \sin \pi x \end{aligned}$$

The initial condition $u'(0) = h_0$ implies

$$h_0 = b\pi - \frac{1}{2\pi} \implies b = \frac{2\pi h_0 - 1}{2\pi^2}$$

which shows, for $x \in [0, \xi]$,

$$u(x) = \frac{2\pi h_0 - 1}{2\pi^2} \sin \pi x - \frac{x}{2\pi} \cos \pi x$$

For $x \in (\xi, 1]$, the final condition $u(1) = 0$ implies $a = \frac{1}{2\pi}$, and thus

$$u(x) = \frac{1}{2\pi} \cos \pi x + b \sin \pi x - \frac{x}{2\pi} \cos \pi x$$

$$\Rightarrow u'(x) = -\frac{1}{2} \sin \pi x + b \cos \pi x - \frac{1}{2\pi} \cos \pi x + \frac{x}{2} \sin \pi x$$

The final condition $u'(1) = h_1$ implies

$$h_1 = -b\pi + \frac{1}{2\pi} \quad \Rightarrow \quad b = \frac{1-2\pi h_1}{2\pi^2}$$

which shows, for $x \in (\xi, 1]$,

$$u(x) = \frac{1-2\pi h_1}{2\pi^2} \sin \pi x + \frac{1-x}{2\pi} \cos \pi x$$

Thus G is defined as

$$G(x; \xi) = \frac{\sin \pi \xi}{\pi^2} \begin{cases} (2\pi h_0 - 1) \sin \pi x - \pi x \cos \pi x & \text{if } x \in [0, \xi) \\ (1 - 2\pi h_1) \sin \pi x + \pi(1-x) \cos \pi x & \text{if } x \in (\xi, 1] \end{cases}$$

Note the extra factor of $2 \sin \pi \xi$, which is multiplied to the right hand side of the initial and final value problem. To solve for h_0 and h_1 we impose continuity

$$\begin{aligned} (2\pi h_0 - 1) \sin \pi \xi - \pi \xi \cos \pi \xi &= (1 - 2\pi h_1) \sin \pi \xi + \pi(1 - \xi) \cos \pi \xi \\ \Rightarrow (2\pi h_0 - 1) \sin \pi \xi &= (1 - 2\pi h_1) \sin \pi \xi + \pi \cos \pi \xi \end{aligned}$$

and orthogonality $\langle G, \sin \pi x \rangle = 0$

$$\begin{aligned} \int_0^\xi (2\pi h_0 - 1) \sin^2 \pi x dx - \pi \int_0^\xi x \cos \pi x \sin \pi x dx + \int_\xi^1 (1 - 2\pi h_1) \sin^2 \pi x dx + \pi \int_\xi^1 (1-x) \cos \pi x \sin \pi x dx &= 0 \\ (2\pi h_0 - 1) \int_0^\xi \sin^2 \pi x dx + (1 - 2\pi h_1) \int_\xi^1 \sin^2 \pi x dx - \pi \int_0^1 x \cos \pi x \sin \pi x dx + \pi \int_\xi^1 \cos \pi x \sin \pi x dx &= 0 \\ (2\pi h_0 - 1) \left(\frac{\xi}{2} - \frac{\sin 2\pi \xi}{4\pi} \right) + (1 - 2\pi h_1) \left(\frac{1}{2} - \frac{\xi}{2} + \frac{\sin 2\pi \xi}{4\pi} \right) - \frac{1}{4} [1 - \cos 2\pi \xi] &= 0 \\ 2\pi(h_0 + h_1) \left(\frac{\xi}{2} - \frac{\sin 2\pi \xi}{4\pi} \right) - \pi h_1 + \frac{1}{4} - \xi + \frac{\sin 2\pi \xi}{2\pi} + \frac{\cos 2\pi \xi}{4} &= 0 \end{aligned}$$

Continuity implies

$$h_0 = \frac{1}{\pi} - h_1 + \frac{\cot \pi \xi}{2}$$

Substituting this in to the orthogonality condition gives

$$2\pi \xi \cot \pi \xi - 4\pi h_1 = 0 \quad \Rightarrow \quad h_1 = \frac{\xi \cot \pi \xi}{2} \quad \Rightarrow \quad h_0 = \frac{(1 - \xi) \cot \pi \xi}{2}$$

Problem 5

Consider the Sturm-Liouville equation

$$-(pu')' + qu = \lambda ru, \quad a < x < b$$

where $p, q, r : [a, b] \rightarrow \mathbb{R}$ are smooth functions and $p(x), r(x) > 0$ for $a \leq x \leq b$. Show that the change of variables

$$t = \int_a^x \sqrt{\frac{r(s)}{p(s)}} ds, \quad v(t) = [r(x)p(x)]^{1/4} u(x)$$

transforms this equation into a Sturm-Liouville equation with $p = r = 1$ of the form

$$-v'' + Qv = \lambda v, \quad 0 < t < c.$$

What are c and $Q : [0, c] \rightarrow \mathbb{R}$?