UC DAVIS, APPLIED MATHEMATICS (MAT207C), SPRING 2016

Homework #1

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Problem 1

Find the two-term asymptotic expansion for small ε for all real roots x of the below equations.

(a) $\varepsilon^2 x^3 - x + \varepsilon = 0$

(b) $\varepsilon \exp\left(x^2\right) = 1 + \dfrac{\varepsilon}{1+x^2}$

Problem 2

The Exponential integral function is defined as

$$\mathrm{Ei}(x) = \int_x^\infty \frac{\exp(-s)}{s}\,\mathrm{d}s.$$

Derive an asymptotic expansion for $\mathrm{Ei}(x)$ for large x. Use a computer (e.g. expint(x) in MATLAB) to check the accuracy of your expansion for different values of x and for different numbers of terms. Discuss your results.

By integration by parts,

$$\begin{aligned}
\int_x^\infty \frac{\exp(-s)}{s}\,\mathrm{d}s &= \left.\frac{-\exp(-s)}{s}\right|_x^\infty - \int_x^\infty \frac{\exp(-s)}{s^2}\,\mathrm{d}s \\
&= \frac{\exp(-x)}{x} - \left[\left.\frac{-\exp(-s)}{s^2}\right|_x^\infty - 2\int_x^\infty \frac{\exp(-s)}{s^3}\,\mathrm{d}s\right] \\
&\vdots \\
&= \exp(-x)\left[\frac{(-1)^0 0!}{x} + \frac{(-1)^1 1!}{x^2} + \frac{(-1)^2 2!}{x^3} + \cdots + \frac{(-1)^n n!}{x^{n+1}}\right] + R_n(x)
\end{aligned}$$

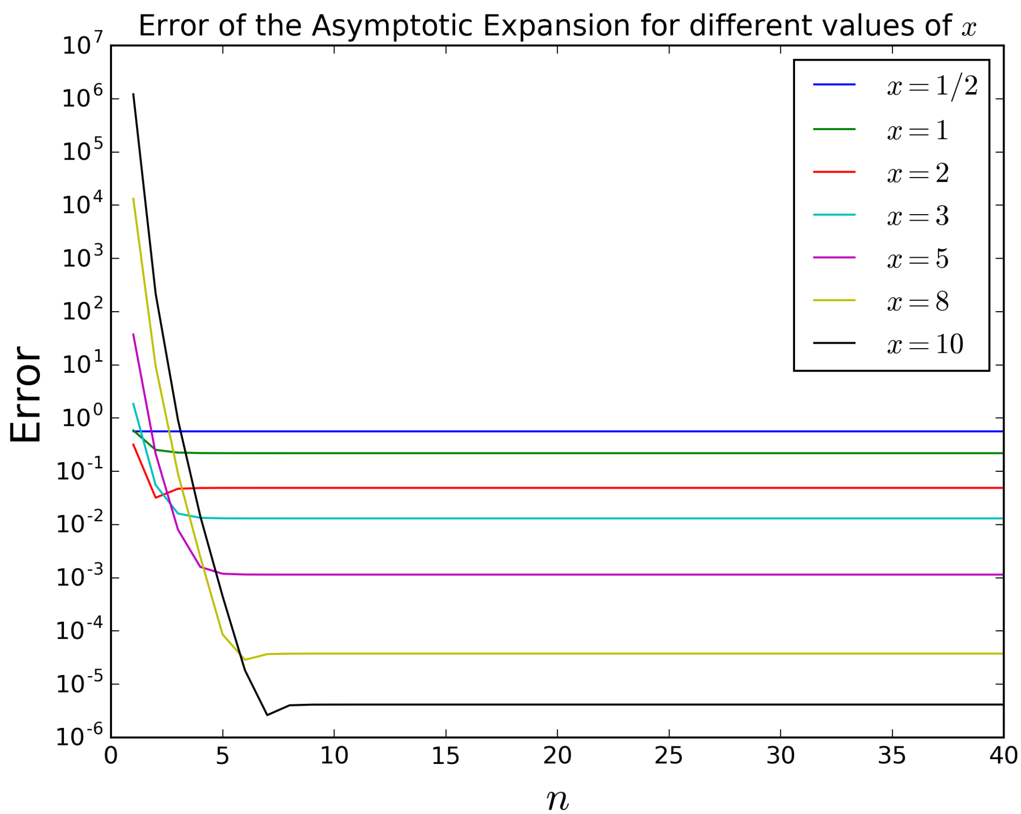
where

$$R_n(x) = (-1)^{n+1}(n+1)!\int_x^\infty \frac{\exp(-s)}{s^{n+2}}\,\mathrm{d}s$$

Thus the asymptotic expansion for $\mathrm{Ei}(x)$ for large x is

$$\mathrm{Ei}(x) \sim \exp(-x)\sum_{n=0}^\infty \frac{(-1)^n n!}{x^{n+1}}.$$

This is a divergent series in n, but converges to 0 as $x \to \infty$ when n is fixed. The following graph shows the error between the asymptotic expansion and the exponential integral function, as calculated by MATLAB and graphed in Python.


Error of the Asymptotic Expansion for different values of x
$x = 1/2$
$x = 1$
$x = 2$
$x = 3$
$x = 5$
$x = 8$
$x = 10$
Error
n
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
40