UC DAVIS, APPLIED MATHEMATICS (MAT207C), SPRING 2016

Homework #3

Sam Fleischer

April 22, 2016

Problem 1

In class we constructed the leading order composite expansion to the initial value problem

$$\varepsilon \ddot{u} + \dot{u} + u = 0,$$
$$u(0) = 0, \qquad \varepsilon \dot{u}(0) = 1.$$

(a) Find the terms at order ε for the inner and outer expansions, perform matching at this order using the intermediate scale, and give the composite expansion.

(b) Compute the exact solution to this problem. Use it to assess the accuracy of the leading order composite expansion and the expansion from part (a) for different values of ε.

Proof. First, we compute the outer solution. Since the layer is located at 0, none of the boundary conditions apply. Let $u = u_0 + \varepsilon u_1 + \varepsilon u_2^2 + \ldots$ and let $\varepsilon \to 0$. Then combine terms of similar order:

$$\begin{aligned}
\dot{u}_0 + u_0 &= 0 \\
\dot{u}_1 + u_1 &= -\ddot{u}_0 \\
\dot{u}_2 + u_2 &= -\ddot{u}_1 \\
&\vdots
\end{aligned}$$

The general forms of the solutions are

$$u_0 = Ae^{-t} \qquad \text{and} \qquad u_1 = Be^{-t} - Ate^{-t}.$$

Next, denote $\tau = \frac{t}{\varepsilon}$ and $U(\tau) = u(t)$. This yields the following inner layer problem (where the boundary conditions apply):

$$\begin{cases} \ddot{U} + \dot{U} + \varepsilon U = 0 \\ U(0) = 0 \\ \dot{U}(0) = 1 \end{cases}$$

Then let $U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots$. Combining terms of similar order gives

$$\begin{cases} \ddot{U}_0 + \dot{U}_0 = 0 \\ U_0(0) = 0 \\ \dot{U}_0(0) = 1 \end{cases}$$
$$\begin{cases} \ddot{U}_1 + \dot{U}_1 = -U_0 \\ U_1(0) = 0 \\ \dot{U}_1(0) = 0 \end{cases}$$
$$\vdots$$

Thus, $U_0(\tau) = 1 - e^{-\tau}$, which subsequentally yields $U_1(\tau) = -\tau(1 + e^{-\tau}) + 2(1 - e^{-\tau})$. Thus,

$$\begin{aligned}
u_{\text{out}}(t) &= Ae^{-t} + \varepsilon\left(Be^{-t} - Ate^{-t}\right) \\
u_{\text{in}}(t) &= \left(1 - e^{-\frac{t}{\varepsilon}}\right) + \varepsilon\left(-\frac{t}{\varepsilon}\left(1 + e^{-\frac{t}{\varepsilon}}\right) + 2\left(1 - e^{-\frac{t}{\varepsilon}}\right)\right) \\
&= \left[1 - e^{-\frac{t}{\varepsilon}} - t\left(1 + e^{-\frac{t}{\varepsilon}}\right)\right] + 2\varepsilon\left[1 - e^{-\frac{t}{\varepsilon}}\right]
\end{aligned}$$

Obviously these don't simply match up, i.e. A and B are not easily determined. So we need to create an intermediate time scale $t_\eta = \frac{t}{\eta}$, such that $\varepsilon \prec \eta \prec 1$. We want to match terms such that u_{in} and u_{out} match up to order ε. So, we want to consider $0 = \lim_{\varepsilon\to 0}\left[\dfrac{u_{\text{in}}(\eta t_\eta) - u_{\text{out}}(\eta t_\eta)}{\varepsilon}\right]$.

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\left[1 - e^{-\frac{\eta t_\eta}{\varepsilon}} - \eta t_\eta\left(1 + e^{-\frac{\eta t_\eta}{\varepsilon}}\right)\right] + 2\varepsilon\left[1 - e^{-\frac{\eta t_\eta}{\varepsilon}}\right] - Ae^{-\eta t_\eta} - \varepsilon\left(Be^{-\eta t_\eta} - A\eta t_\eta e^{-\eta t_\eta}\right)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\left[1 - e^{-\frac{\eta t_\eta}{\varepsilon}} - \eta t_\eta\left(1 + e^{-\frac{\eta t_\eta}{\varepsilon}}\right) - Ae^{-\eta t_\eta}\right] + \varepsilon\left[2\left(1 - e^{-\frac{\eta t_\eta}{\varepsilon}}\right) - Be^{-\eta t_\eta} + A\eta t_\eta e^{-\eta t_\eta}\right]\right)\\
&= \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon} - \frac{1}{\varepsilon}\exp\left[-\frac{\eta t_\eta}{\varepsilon}\right] - \frac{\eta}{\varepsilon}t_\eta - \frac{\eta}{\varepsilon}t_\eta\exp\left[-\frac{\eta t_\eta}{\varepsilon}\right] - \frac{A}{\varepsilon}e^{-\eta t_\eta} + 2 - 2\exp\left[-\frac{\eta t_\eta}{\varepsilon}\right] - Be^{-\eta t_\eta} + A\eta t_\eta e^{-\eta t_\eta}\right]
\end{aligned}$$

We neglect the transcendentally small terms (terms containing $\exp\left[-\frac{\eta t_\eta}{\varepsilon}\right]$).

$$0 = \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon} - \frac{\eta}{\varepsilon}t_\eta - \frac{A}{\varepsilon}e^{-\eta t_\eta} + 2 - Be^{-\eta t_\eta} + A\eta t_\eta e^{-\eta t_\eta}\right]$$

Note that since $\varepsilon \to 0$ then $\eta \to 0$ since $\varepsilon \prec \eta \prec 1$. Next we employ Taylor expansions of $e^{-\eta t_\eta}$.

$$\begin{aligned}
0 = \lim_{\varepsilon\to 0}\Bigg[&\frac{1}{\varepsilon}\left(1 - A\left[1 - \eta t_\eta + \frac{1}{2}(\eta t_\eta)^2 + \dots\right]\right) + \left(2 - B\left[1 - \eta t_\eta + \frac{1}{2}(\eta t_\eta)^2 + \dots\right]\right) - \frac{\eta}{\varepsilon}t_\eta\\
&+ A\eta t_\eta\left[1 - \eta t_\eta + \frac{1}{2}(\eta t_\eta)^2 + \dots\right]\Bigg]
\end{aligned}$$

Assuming $\eta^2 \prec \varepsilon$, then $\frac{\eta^2}{\varepsilon} \to 0$, and so

$$\frac{1}{\varepsilon} - \frac{A}{\varepsilon} = 0 \quad\text{and}\quad 2 - B = 0 \quad\Longrightarrow\quad A = 1 \quad\text{and}\quad B = 2.$$

Thus,

$$\begin{aligned}
u_{\text{out}}(t) &= e^{-t} + \varepsilon\left(2e^{-t} - te^{-t}\right)\\
u_{\text{in}}(t) &= \left[1 - e^{-\frac{t}{\varepsilon}} - t\left(1 + e^{-\frac{t}{\varepsilon}}\right)\right] + 2\varepsilon\left[1 - e^{-\frac{t}{\varepsilon}}\right]
\end{aligned}$$

Finally, the composite expansion for this problem is

$$\begin{aligned}
u(t) &= u_{\text{out}}(t) + u_{\text{in}}(t) - u_{\text{match}}(t)\\
&= \underbrace{e^{-t} + \varepsilon\left(2e^{-t} - te^{-t}\right)}_{\text{outer layer expasion}} + \underbrace{\left[1 - e^{-\frac{t}{\varepsilon}} - t\left(1 + e^{-\frac{t}{\varepsilon}}\right)\right] + 2\varepsilon\left[1 - e^{-\frac{t}{\varepsilon}}\right]}_{\text{inner layer expansion}} - \underbrace{[1 - t + 2\varepsilon]}_{\text{matched terms}}\\
&= \left[e^{-t} - (1+t)e^{-\frac{t}{\varepsilon}}\right] + \varepsilon\left[(2-t)e^{-t} - 2e^{-\frac{t}{\varepsilon}}\right].
\end{aligned}$$

Figures (0.1) and (0.2) depicts the accuracy of the expansion for various values of ε. $\square$

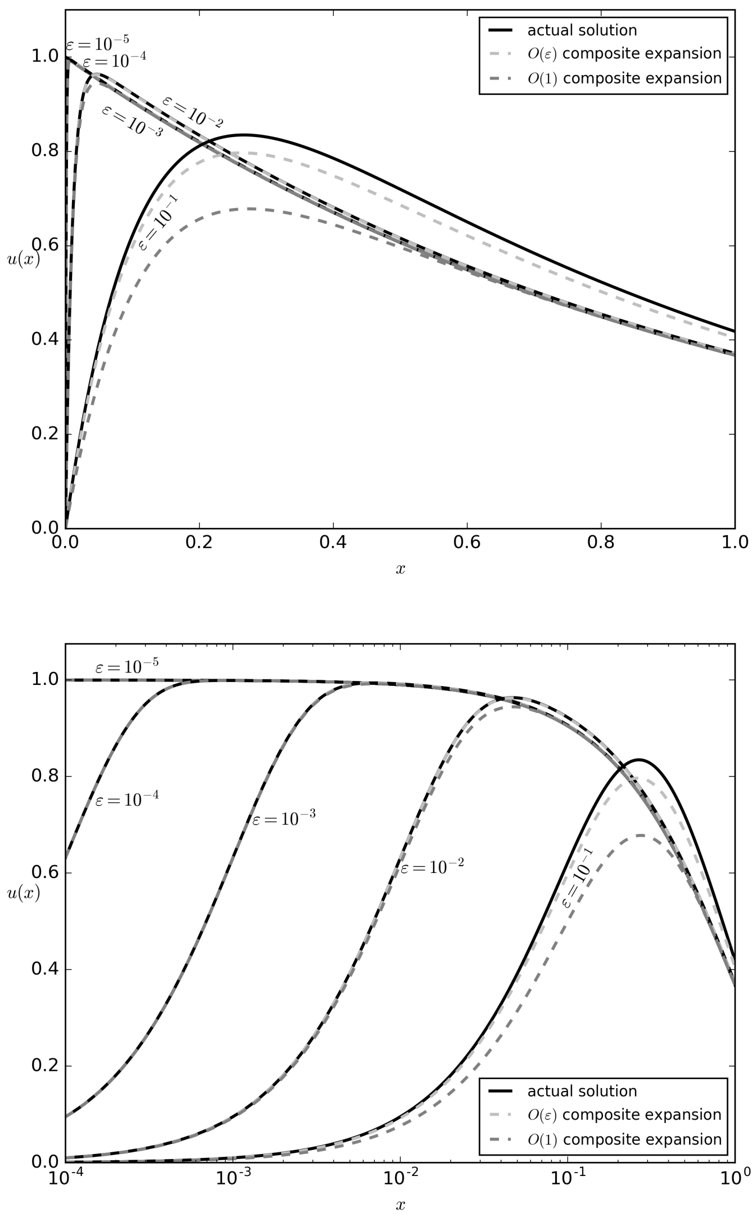
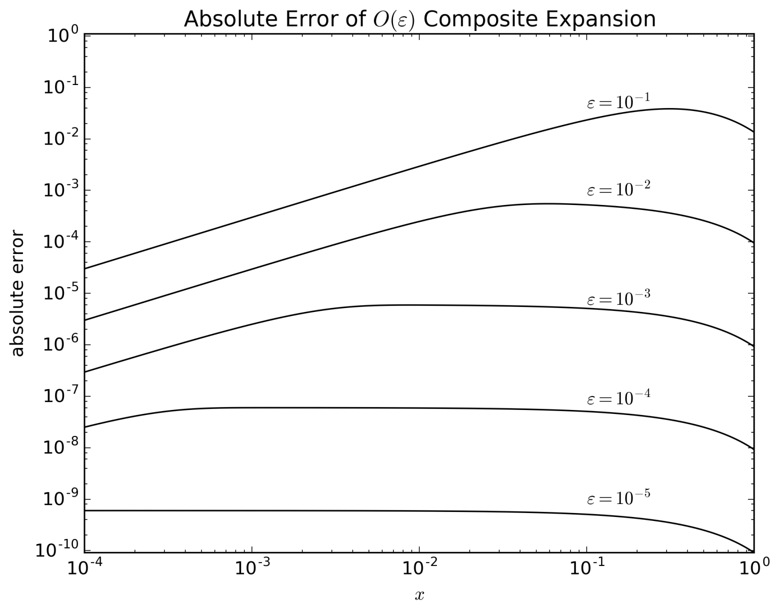


Figure 0.1: The first two graphs show the actual solutions and composite expansions between 0 and 1 for $\varepsilon = 10^{-k}$ for $k = 1, 2, 3, 4, 5$ (*calculated and plotted using Python 2.7*).



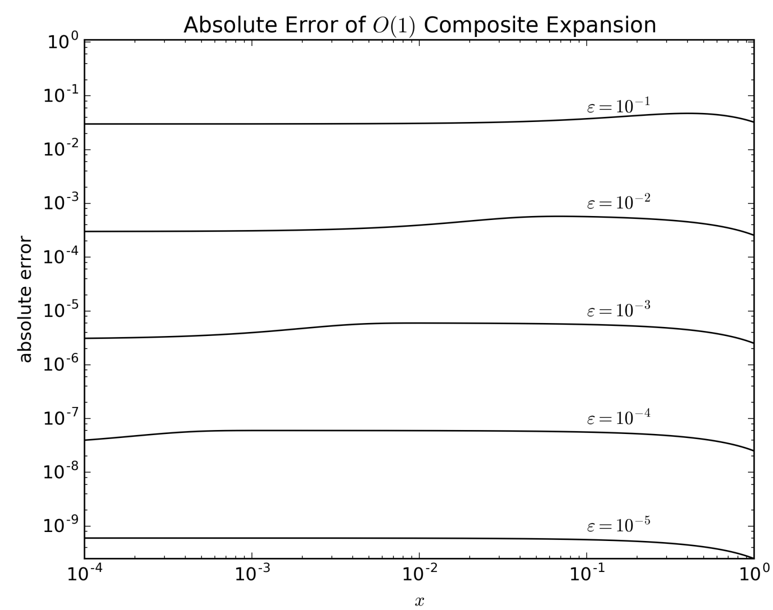


Figure 0.2: The two graphs show the aboslute errors between the actual solutions and composite expansions for $\varepsilon = 10^{-k}$ for $k = 1, 2, 3, 4, 5$ (*calculated and plotted using Python 2.7*).

Problem 2

Compute the leading order composite expansion to the problem

$$\varepsilon u'' + \sqrt{x}u' - u = 0,$$
$$u(0) = 0, \qquad u(1) = e^2.$$

Proof. First let $X = \frac{x-x_0}{\varepsilon^\alpha}$ and $U(X) = u(x)$. Then $x = x_0 + \varepsilon^\alpha X$ and

$$\varepsilon \ddot{U} + \sqrt{x_0 + \varepsilon^\alpha X}\varepsilon^\alpha \dot{U} - \varepsilon^{2\alpha} U = 0$$

There are three possibilities in matching two of these three terms in ε order: $\alpha = 1, 0, \frac{1}{2}$. $\alpha = \frac{1}{2}$ does not work since the term not matched is lower order than the others. $\alpha = 0$ is the original problem, and so the only choice is $\alpha = 1$. Then

$$\ddot{U} + \sqrt{x_0 + \varepsilon X}\dot{U} - \varepsilon U = 0$$

Assuming $U = U_0 + U_1\varepsilon + U_2\varepsilon^2 + \dots$, then the $O(1)$ expansion (U_0) gives

$$\ddot{U}_0 + \sqrt{x_0}\dot{U}_0 = 0$$
$$\implies U_0 = A + Be^{-\sqrt{x_0}X}.$$

This forces $x_0 = 0$, and thus the equation

$$\varepsilon \ddot{U} + \sqrt{x_0 + \varepsilon^\alpha X}\varepsilon^\alpha \dot{U} - \varepsilon^{2\alpha} U = 0$$
$$\varepsilon \ddot{U} + \sqrt{X}\varepsilon^{\frac{3\alpha}{2}} \dot{U} - \varepsilon^{2\alpha} U = 0$$

Now there are three different α possibilities to match two of the three terms: $\alpha = 0, \frac{1}{2}, \frac{2}{3}$. $\alpha = 0$ is the original problem, $\alpha = \frac{1}{2}$ does not work since the term not matched is lower order than the others, and so the only choice is $\alpha = \frac{2}{3}$.

$$\varepsilon \ddot{U} + \varepsilon\sqrt{X}\dot{U} - \varepsilon^{\frac{4}{3}} U = 0$$
$$\implies \ddot{U} + \sqrt{X}\dot{U} - \varepsilon^{\frac{1}{3}} U = 0$$

Assume $U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots$. Then setting $\varepsilon \to 0$ yields

$$\ddot{U}_0 + \sqrt{X}\dot{U}_0 = 0 \qquad \implies \qquad U_0(X) = \int_0^X \left[C\exp\left[-\frac{2}{3}T^{\frac{3}{2}}\right]\right]\mathrm{d}T \qquad \implies \qquad u_{\text{in}}(x) \approx \int_0^{x\varepsilon^{-\frac{2}{3}}} \left[C\exp\left[-\frac{2}{3}T^{\frac{3}{2}}\right]\right]\mathrm{d}T$$

The original outer problem is

$$\varepsilon \ddot{u} + \sqrt{x}\dot{u} - u = 0, \qquad u(0) = 0$$

Assuming $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$,

$$\sqrt{x}\dot{u}_0 - u_0 = 0, \qquad u_0(1) = e^2$$

which has the solution

$$u_0(x) = e^{2\sqrt{x}} \qquad \implies \qquad u_{\text{out}}(x) \approx e^{2\sqrt{x}}$$

To match u_{in} with u_{out}, we want

$$\begin{aligned}
\lim_{x\to 0} u_{\text{out}} &= \lim_{x\to\infty} u_{\text{in}} \\
\lim_{x\to 0} e^{2\sqrt{x}} &= \lim_{x\to\infty} \int_0^{x\varepsilon^{-\frac{2}{3}}} \left[C\exp\left[-\frac{2}{3}T^{\frac{3}{2}}\right]\right] \mathrm{d}T \\
1 &= \int_0^{\infty} \left[C\exp\left[-\frac{2}{3}T^{\frac{3}{2}}\right]\right] \mathrm{d}T \\
\implies C &= \frac{1}{\displaystyle\int_0^{\infty} \left[\exp\left[-\frac{2}{3}T^{\frac{3}{2}}\right]\right] \mathrm{d}T}.
\end{aligned}$$

Also, this says $u_{\text{match}} = 1$ (since $u_{\text{match}} = \lim_{x\to 0} u_{\text{out}} = \lim_{x\to\infty} u_{\text{in}} = 1$). Thus,

$$\begin{aligned}
u(x) &\approx u_{\text{in}}(x) + u_{\text{out}}(x) - u_{\text{match}}(x) \\
&= e^{2\sqrt{x}} + \frac{\displaystyle\int_0^{x\varepsilon^{-\frac{2}{3}}} \left[\exp\left[-\frac{2}{3}T^{\frac{3}{2}}\right]\right] \mathrm{d}T}{\displaystyle\int_0^{\infty} \left[\exp\left[-\frac{2}{3}T^{\frac{3}{2}}\right]\right] \mathrm{d}T} - 1
\end{aligned}$$

□