UC DAVIS, APPLIED MATHEMATICS (MAT207C), SPRING 2016

Homework #7

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Problem 1

Consider a one-dimensional layered medium with a periodic substructure of alternate layers of material one with thickness $\varepsilon\phi_1$ and diffusion coefficient D_1 and material two with thickness $\varepsilon\phi_2$ and diffusion coefficient D_2. WIthout loss of generality take $\phi_2 = 1-\phi_1$ so that ϕ_i represents the volume fraction of material i and the length of the periodic cell is ε.
The steady-state diffusion equation is

$$\partial_x(D(x)\partial_x u) = f(x),$$

where $D(x) = D_i$ in material i. Let x_* represent a point on the interface between two layers. At such points we require

$$\lim_{x\to x_*^-} u(x) = \lim_{x\to x_*^+} u(x)$$
$$\lim_{x\to x_*^-} D_{u_x} = \lim_{x\to x_*^+} D_{u_x},$$

which enforce continuity of the solution and continuity of the flux. Derive a homogenized steady-state diffusion equation.

Proof. First let $y = \frac{x}{\varepsilon}$. Without loss of generality, rescale our equations such that the periodic structure has boundaries at each integer. Then

$$D(y) = \begin{cases} D_1, & \text{if } y \in (0,\phi) \\ D_2, & \text{if } y \in (\phi,1). \end{cases}$$

Next let $u(x) = v(x,y) = v_0(x,y) + v_1(x,y) + \dots$. Then

$$\partial_x u = \left(\partial_x + \frac{1}{\varepsilon}\partial_y\right)v,$$

which shows

$$\left(\partial_x + \frac{1}{\varepsilon}\partial_y\right)\left(D(y)\left(\partial_x + \frac{1}{\varepsilon}\partial_y\right)(v_0 + \varepsilon v_1 + \dots)\right) = f(x).$$

The highest order is $\frac{1}{\varepsilon^2}$, and the $\mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$ equation is

$$\begin{aligned} &\partial_y\big(D(y)\partial_y v_0\big) = 0 \\ \implies &\begin{cases} \partial_y\big(D_1\partial_y v_0\big) = 0, & \text{if } y \in (0,\phi) \\ \partial_y\big(D_2\partial_y v_0\big) = 0, & \text{if } y \in (\phi,1) \end{cases} \\ \implies v_0(x,y) = &\begin{cases} a_1(x) + b_1(x)y, & \text{if } y \in (0,\phi) \\ a_2(x) + b_2(x)y, & \text{if } y \in (\phi,1). \end{cases} \end{aligned}$$

Continuity of the solution and of the flux imply

$$\underbrace{\begin{cases} a_1(x) + b_1(x)\phi = a_2(x) + b_2(x)\phi \\ a_1(x) = a_2(x) + b_2(x) \end{cases}}_{\text{continuity of the solution}} \quad \text{and} \quad \underbrace{\begin{cases} D_1 b_1(x) = D_2 b_2(x). \end{cases}}_{\text{continuity of the flux}}$$

Continuity of the solution shows us

$$b_1(x) = \frac{\phi-1}{\phi}b_2(x),$$

and thus

$$\left[D_1\frac{\phi-1}{\phi}-D_2\right]b_2(x)=0 \qquad\Longrightarrow\qquad b_2(x)=0 \qquad\Longrightarrow\qquad b_1(x)=0 \qquad\Longrightarrow\qquad a_1(x)=a_2(x)$$

since $D_1, D_2 > 0$ and $\phi < 1$. After defining $a(x) := a_1(x) = a_2(x)$, we get $v_0(x,y) = a(x)$, which shows $\partial_y v_0 = 0$, $\partial_x v_0 = a'(x)$, and $\partial_{xx} v_0 = a''(x)$. The $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ equation is

$$\begin{aligned}
\cancelto{0}{\partial_x\big(D(y)\partial_y v_0\big)} + \partial_y\big(D(y)\partial_x v_0\big) + \partial_y(D(y)\partial_y v_1) = 0 \\
\begin{cases} D_1\Big(\cancelto{0}{\partial_y a'(x)} + \partial_{yy}v_1\Big) = 0, & \text{if } y\in(0,\phi) \\ D_2\Big(\cancelto{0}{\partial_y a'(x)} + \partial_{yy}v_1\Big) = 0, & \text{if } y\in(\phi,1) \end{cases} \\
\Longrightarrow\ v_1(x,y) = \begin{cases} c_1(x)+d_1(x)y, & \text{if } y\in(0,\phi) \\ c_2(x)+d_2(x)y, & \text{if } y\in(\phi,1). \end{cases}
\end{aligned}$$

Continuity of the solution and of the flux imply

$$\underbrace{\begin{cases} c_1(x)+d_1(x)\phi = c_2(x)+d_2(x)\phi \\ c_1(x) = c_2(x)+d_2(x) \end{cases}}_{\text{continuity of the solution}} \qquad \text{and} \qquad \underbrace{\Big\{D_1[a'(x)+d_1(x)] = D_2[a'(x)+d_2(x)].}_{\text{continuity of the flux}}$$

Continuity of the solution shows us

$$d_1(x) = \frac{\phi-1}{\phi}d_2(x),$$

and thus

$$\begin{aligned}
D_1\left[a'(x)+\frac{\phi-1}{\phi}d_2(x)\right] = D_2\big[a'(x)+d_2(x)\big] \\
\Longrightarrow \qquad d_2(x) = \frac{\phi(D_2-D_1)a'(x)}{(D_1-D_2)\phi-D_1} \qquad \Longrightarrow \qquad d_1(x) = \frac{\big(\phi-1\big)(D_2-D_1)a'(x)}{(D_1-D_2)\phi-D_1}.
\end{aligned}$$

Finally, the $\mathcal{O}(1)$ equation is

$$\begin{aligned}
\partial_x\big(D(y)\partial_x v_0\big)+\partial_x\big(D(y)\partial_y v_1\big)+\partial_y\big(D(y)\partial_x v_1\big)+\partial_y\big(D(y)\partial_y v_2\big) = f(x) \\
\Longrightarrow \int_0^1 D(y)\big(a''(x)+\partial_{xy}v_1(x,y)\big)\mathrm{d}y + \underbrace{\cancelto{0}{D(y)\big[\partial_x v_1(x,y)+\partial_y v_2(x,y)\big]\Big|_{y=0}^{y=1}}}_{\text{cancels due to periodicity in } y} = f(x) \\
\Longrightarrow D_1 a''(x)\int_0^\phi \mathrm{d}y + D_2 a''(x)\int_\phi^1 \mathrm{d}y + D_1 d_1'(x)\int_0^\phi \mathrm{d}y + D_2 d_2'(x)\int_\phi^1 \mathrm{d}y = f(x) \\
\Longrightarrow D_1\big(a''(x)+d_1'(x)\big)\phi + D_2\big(a''(x)+d_2'(x)\big)(1-\phi) = f(x).
\end{aligned}$$

Since

$$d_1''(x) = \frac{(\phi-1)(D_2-D_1)}{(D_1-D_2)\phi-D_1}a''(x) \qquad \text{and} \qquad d_2''(x) = \frac{\phi(D_2-D_1)}{(D_1-D_2)\phi-D_1}a''(x),$$

then

$$D^*(x) := \frac{a''(x)D_1D_2}{D_1(1-\phi)+D_2\phi} = f(x).$$

Finally, the homogenized equation is

$$\boxed{\partial_x\big(D^*(x)\partial_x u\big) = f(x)}$$

□

Problem 2

Express the below equation in the standard in the standard form to apply the averaging theorem $\left(\dot{\mathbf{x}} = \varepsilon\mathbf{f}(\mathbf{x}, t), \mathbf{f} \text{ periodic in time}\right)$, and give the averaged equations.

$$\ddot{u} + 4\varepsilon\left(\cos^2 t\right)\dot{u} + u = 0.$$

Generate a numerical solution to the time-averaged equations and compare it with the solution of

$$\ddot{v} + 2\varepsilon\dot{v} + v = 0,$$

in which the coefficient has been replaced with the time-averaged value without invoking the averaging theorem.

Proof. Let $\varepsilon \to 0$. Thus $\ddot{u} + u = 0$. This has the solution $u(t) = a(t)\cos\left(t + \phi(t)\right)$, where a and ϕ are parameters based on the small timescale. Define $u_1 = u$ and $u_2 = \dot{u}$ (this is the derivative with respect to the long timescale. Then

$$\begin{aligned}
u_1 &= a(t)\cos\left(t + \phi(t)\right),\\
u_2 &= -a(t)\sin\left(t + \phi(t)\right),
\end{aligned}$$

and thus

$$\begin{aligned}
a'(t)\cos\left(t + \phi(t)\right) - a(t)\sin\left(t + \phi(t)\right)\left(\phi'(t) + 1\right) &= \dot{u}_1 = u_2 = -a(t)\sin\left(t + \phi(t)\right),\\
-a'(t)\sin\left(t + \phi(t)\right) - a(t)\cos\left(t + \phi(t)\right)\left(\phi'(t) + 1\right) &= \dot{u}_2 = -4\varepsilon\left(\cos^2(t)\right)u_1 - u_2 = -4\varepsilon\cos^2(t)\left[-a(t)\sin\left(t + \phi(t)\right)\right] - a(t)\cos\left(t + \phi(t)\right)
\end{aligned}$$

We can solve for $a'(t)$ and $\phi'(t)$:

$$\begin{aligned}
a'(t) &= -4\varepsilon\cos^2(t)\sin^2(t + \phi(t)),\\
\phi'(t) &= -4\varepsilon\cos^2(t)\sin\left(t + \phi(t)\right)\cos\left(t + \phi(t)\right).
\end{aligned}$$

This is of the form $\dot{\mathbf{x}} = \varepsilon\mathbf{f}(\mathbf{x}, t)$ with $\mathbf{f}$ periodic in t, and we can thus employ the averaging theorem:

$$\begin{aligned}
\overline{a}'(t) &= \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t} a'(t)\,\mathrm{d}t\\
\overline{\phi}'(t) &= \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t} \phi'(t)\,\mathrm{d}t.
\end{aligned}$$

We calculate these integrals:

$$\begin{aligned}
\overline{a}'(t) &= \varepsilon\lim_{t\to\infty}\frac{1}{2t}\left[\left(t + \sin(2t) + \frac{1}{4}\sin(4t)\right)\cos\left(2\overline{\phi}(t)\right) - 2t - \sin(2t)\right]\\
&= \varepsilon\lim_{t\to\infty}\left[\left(\frac{1}{2} + \cancelto{0}{\frac{\sin(2t) + \frac{1}{4}\sin(4t)}{2t}}\right)\cos\left(2\overline{\phi}(t)\right) - 1 - \cancelto{0}{\frac{\sin(2t)}{2t}}\right]\\
&= \varepsilon\left[\frac{1}{2}\cos\left(2\overline{\phi}(t)\right) - 1\right]\\
\overline{\phi}'(t) &= \varepsilon\lim_{t\to\infty}\frac{1}{2t}\left[-\frac{1}{4}(4t + 4\sin(2t) + \sin(4t))\sin\left(2\overline{\phi}(t)\right)\right]\\
&= \varepsilon\lim_{t\to\infty}\left[\left(-\frac{1}{2} - \cancelto{0}{\frac{\sin(2t)\frac{1}{4}\sin(4t)}{2t}}\right)\sin\left(2\overline{\phi}(t)\right)\right]
\end{aligned}$$

$$= -\varepsilon \frac{1}{2} \sin\Big(2\overline{\phi}(t)\Big)$$

and thus the averaged system is

$$\boxed{\begin{bmatrix} \overline{a} \\ \overline{\phi} \end{bmatrix}' = \varepsilon \begin{bmatrix} \frac{1}{2}\cos\Big(2\overline{\phi}(t)\Big) - 1 \\ -\frac{1}{2}\sin\Big(2\overline{\phi}(t)\Big) \end{bmatrix}}$$

We can solve this system numerically and plot the solution $u = \overline{a}\cos\Big(t + \overline{\phi}\Big)$, as seen in the figure. For reference, we also solve the system $\ddot{v} + 2\varepsilon\dot{v} + v = 0$ with initial condition $v(0) = 1$ and $v'(0) = 0$:

$$v(t) = \cos\Big(\Big(\varepsilon - \sqrt{1-\varepsilon^2}\Big)t\Big)$$

$\square$