
Homework #7

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Problem 1

Consider a one-dimensional layered medium with a periodic substructure of alternate layers of material one with thickness $\varepsilon\phi_1$ and diffusion coefficient D_1 and material two with thickness $\varepsilon\phi_2$ and diffusion coefficient D_2 . Without loss of generality take $\phi_2 = 1 - \phi_1$ so that ϕ_i represents the volume fraction of material i and the length of the periodic cell is ε .

The steady-state diffusion equation is

$$\partial_x(D(x)\partial_x u) = f(x),$$

where $D(x) = D_i$ in material i . Let x_* represent a point on the interface between two layers. At such points we require

$$\begin{aligned}\lim_{x \rightarrow x_*^-} u(x) &= \lim_{x \rightarrow x_*^+} u(x) \\ \lim_{x \rightarrow x_*^-} D u_x &= \lim_{x \rightarrow x_*^+} D u_x,\end{aligned}$$

which enforce continuity of the solution and continuity of the flux. Derive a homogenized steady-state diffusion equation.

Proof. First let $y = \frac{x}{\varepsilon}$. Without loss of generality, rescale our equations such that the periodic structure has boundaries at each integer. Then

$$D(y) = \begin{cases} D_1, & \text{if } y \in (0, \phi) \\ D_2, & \text{if } y \in (\phi, 1). \end{cases}$$

Next let $u(x) = v(x, y) = v_0(x, y) + v_1(x, y) + \dots$. Then

$$\partial_x u = \left(\partial_x + \frac{1}{\varepsilon} \partial_y \right) v,$$

which shows

$$\left(\partial_x + \frac{1}{\varepsilon} \partial_y \right) \left(D(y) \left(\partial_x + \frac{1}{\varepsilon} \partial_y \right) (v_0 + \varepsilon v_1 + \dots) \right) = f(x).$$

The highest order is $\frac{1}{\varepsilon^2}$, and the $\mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$ equation is

$$\begin{aligned}\partial_y(D(y)\partial_y v_0) &= 0 \\ \implies \begin{cases} \partial_y(D_1\partial_y v_0) = 0, & \text{if } y \in (0, \phi) \\ \partial_y(D_2\partial_y v_0) = 0, & \text{if } y \in (\phi, 1) \end{cases} \\ \implies v_0(x, y) &= \begin{cases} a_1(x) + b_1(x)y, & \text{if } y \in (0, \phi) \\ a_2(x) + b_2(x)y, & \text{if } y \in (\phi, 1). \end{cases}\end{aligned}$$

Continuity of the solution and of the flux imply

$$\underbrace{\begin{cases} a_1(x) + b_1(x)\phi = a_2(x) + b_2(x)\phi \\ a_1(x) = a_2(x) + b_2(x) \end{cases}}_{\text{continuity of the solution}} \quad \text{and} \quad \underbrace{\{D_1 b_1(x) = D_2 b_2(x)\}}_{\text{continuity of the flux}}.$$

Continuity of the solution shows us

$$b_1(x) = \frac{\phi - 1}{\phi} b_2(x),$$

and thus

$$\left[D_1 \frac{\phi - 1}{\phi} - D_2 \right] b_2(x) = 0 \quad \Rightarrow \quad b_2(x) = 0 \quad \Rightarrow \quad b_1(x) = 0 \quad \Rightarrow \quad a_1(x) = a_2(x)$$

since $D_1, D_2 > 0$ and $\phi < 1$. After defining $a(x) := a_1(x) = a_2(x)$, we get $v_0(x, y) = a(x)$, which shows $\partial_y v_0 = 0$, $\partial_x v_0 = a'(x)$, and $\partial_{xx} v_0 = a''(x)$. The $\mathcal{O}(\frac{1}{\epsilon})$ equation is

$$\begin{aligned} \cancel{\partial_x(D(y)\partial_y v_0)} + \partial_y(D(y)\partial_x v_0) + \partial_y(D(y)\partial_y v_1) &= 0 \\ \begin{cases} D_1 \left(\cancel{\partial_y a'(x)} + \partial_{yy} v_1 \right) = 0, & \text{if } y \in (0, \phi) \\ D_2 \left(\cancel{\partial_y a'(x)} + \partial_{yy} v_1 \right) = 0, & \text{if } y \in (\phi, 1) \end{cases} \\ \Rightarrow v_1(x, y) = \begin{cases} c_1(x) + d_1(x)y, & \text{if } y \in (0, \phi) \\ c_2(x) + d_2(x)y, & \text{if } y \in (\phi, 1). \end{cases} \end{aligned}$$

Continuity of the solution and of the flux imply

$$\underbrace{\begin{cases} c_1(x) + d_1(x)\phi = c_2(x) + d_2(x)\phi \\ c_1(x) = c_2(x) + d_2(x) \end{cases}}_{\text{continuity of the solution}} \quad \text{and} \quad \underbrace{\left\{ D_1[a'(x) + d_1(x)] = D_2[a'(x) + d_2(x)] \right\}}_{\text{continuity of the flux}}$$

Continuity of the solution shows us

$$d_1(x) = \frac{\phi - 1}{\phi} d_2(x),$$

and thus

$$\begin{aligned} D_1 \left[a'(x) + \frac{\phi - 1}{\phi} d_2(x) \right] &= D_2[a'(x) + d_2(x)] \\ \Rightarrow d_2(x) &= \frac{\phi(D_2 - D_1)a'(x)}{(D_1 - D_2)\phi - D_1} \quad \Rightarrow \quad d_1(x) = \frac{(\phi - 1)(D_2 - D_1)a'(x)}{(D_1 - D_2)\phi - D_1}. \end{aligned}$$

Finally, the $\mathcal{O}(1)$ equation is

$$\begin{aligned} \partial_x(D(y)\partial_x v_0) + \partial_x(D(y)\partial_y v_1) + \partial_y(D(y)\partial_x v_1) + \partial_y(D(y)\partial_y v_2) &= f(x) \\ \Rightarrow \int_0^1 D(y)(a''(x) + \partial_{xy} v_1(x, y)) dy + \underbrace{D(y)[\partial_x v_1(x, y) + \partial_y v_2(x, y)] \Big|_{y=0}^{y=1}}_{\text{cancels due to periodicity in } y} &= f(x) \\ \Rightarrow D_1 a''(x) \int_0^\phi dy + D_2 a''(x) \int_\phi^1 dy + D_1 d_1'(x) \int_0^\phi dy + D_2 d_2'(x) \int_\phi^1 dy &= f(x) \\ \Rightarrow D_1(a''(x) + d_1'(x))\phi + D_2(a''(x) + d_2'(x))(1 - \phi) &= f(x). \end{aligned}$$

Since

$$d_1''(x) = \frac{(\phi - 1)(D_2 - D_1)}{(D_1 - D_2)\phi - D_1} a''(x) \quad \text{and} \quad d_2''(x) = \frac{\phi(D_2 - D_1)}{(D_1 - D_2)\phi - D_1} a''(x),$$

then

$$D^*(x) := \frac{a''(x)D_1D_2}{D_1(1 - \phi) + D_2\phi} = f(x).$$

Finally, the homogenized equation is

$$\boxed{\partial_x(D^*(x)\partial_x u) = f(x)}$$

□

Problem 2

Express the below equation in the standard form to apply the averaging theorem ($\dot{\mathbf{x}} = \varepsilon \mathbf{f}(\mathbf{x}, t)$, \mathbf{f} periodic in time), and give the averaged equations.

$$\ddot{u} + 4\varepsilon(\cos^2 t)\dot{u} + u = 0.$$

Generate a numerical solution to the time-averaged equations and compare it with the solution of

$$\ddot{v} + 2\varepsilon\dot{v} + v = 0,$$

in which the coefficient has been replaced with the time-averaged value without invoking the averaging theorem.

Proof. Let $\varepsilon \rightarrow 0$. Thus $\ddot{u} + u = 0$. This has the solution $u(t) = a(t) \cos(t + \phi(t))$, where a and ϕ are parameters based on the small timescale. Define $u_1 = u$ and $u_2 = \dot{u}$ (this is the derivative with respect to the long timescale. Then

$$\begin{aligned} u_1 &= a(t) \cos(t + \phi(t)), \\ u_2 &= -a(t) \sin(t + \phi(t)), \end{aligned}$$

and thus

$$\begin{aligned} a'(t) \cos(t + \phi(t)) - a(t) \sin(t + \phi(t))(\phi'(t) + 1) &= \dot{u}_1 = u_2 = -a(t) \sin(t + \phi(t)), \\ -a'(t) \sin(t + \phi(t)) - a(t) \cos(t + \phi(t))(\phi'(t) + 1) &= \dot{u}_2 = -4\varepsilon(\cos^2(t))u_1 - u_2 = -4\varepsilon \cos^2(t)[-a(t) \sin(t + \phi(t))] - a(t) \cos(t + \phi(t)) \end{aligned}$$

We can solve for $a'(t)$ and $\phi'(t)$:

$$\begin{aligned} a'(t) &= -4\varepsilon \cos^2(t) \sin^2(t + \phi(t)), \\ \phi'(t) &= -4\varepsilon \cos^2(t) \sin(t + \phi(t)) \cos(t + \phi(t)). \end{aligned}$$

This is of the form $\dot{\mathbf{x}} = \varepsilon \mathbf{f}(\mathbf{x}, t)$ with \mathbf{f} periodic in t , and we can thus employ the averaging theorem:

$$\begin{aligned} \bar{a}'(t) &= \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t a'(t) dt \\ \bar{\phi}'(t) &= \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t \phi'(t) dt. \end{aligned}$$

We calculate these integrals:

$$\begin{aligned} \bar{a}'(t) &= \varepsilon \lim_{t \rightarrow \infty} \frac{1}{2t} \left[\left(t + \sin(2t) + \frac{1}{4} \sin(4t) \right) \cos(2\bar{\phi}(t)) - 2t - \sin(2t) \right] \\ &= \varepsilon \lim_{t \rightarrow \infty} \left[\left(\frac{1}{2} + \frac{\sin(2t) + \frac{1}{4} \sin(4t)}{2t} \right) \cos(2\bar{\phi}(t)) - 1 - \frac{\sin(2t)}{2t} \right] \\ &= \varepsilon \left[\frac{1}{2} \cos(2\bar{\phi}(t)) - 1 \right] \\ \bar{\phi}'(t) &= \varepsilon \lim_{t \rightarrow \infty} \frac{1}{2t} \left[-\frac{1}{4} (4t + 4\sin(2t) + \sin(4t)) \sin(2\bar{\phi}(t)) \right] \\ &= \varepsilon \lim_{t \rightarrow \infty} \left[\left(-\frac{1}{2} - \frac{\sin(2t) + \frac{1}{4} \sin(4t)}{2t} \right) \sin(2\bar{\phi}(t)) \right] \end{aligned}$$

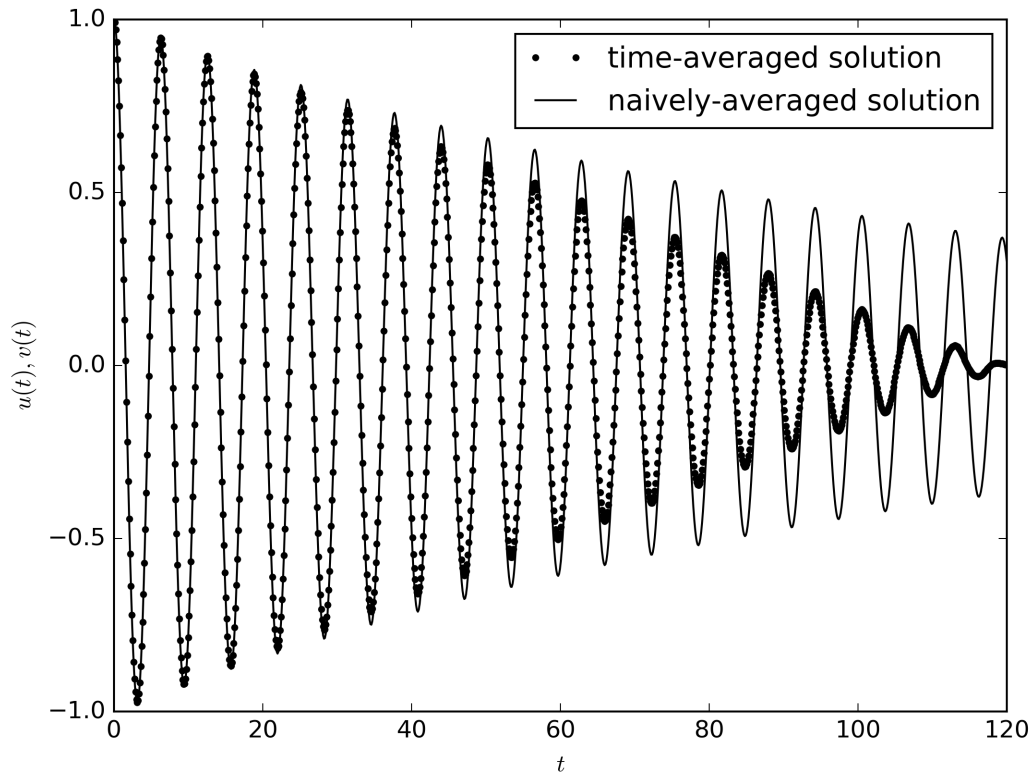
$$= -\varepsilon \frac{1}{2} \sin(2\bar{\phi}(t))$$

and thus the averaged system is

$$\begin{bmatrix} \bar{a} \\ \bar{\phi} \end{bmatrix}' = \varepsilon \begin{bmatrix} \frac{1}{2} \cos(2\bar{\phi}(t)) - 1 \\ -\frac{1}{2} \sin(2\bar{\phi}(t)) \end{bmatrix}$$

We can solve this system numerically and plot the solution $u = \bar{a} \cos(t + \bar{\phi})$, as seen in the figure. For reference, we also solve the system $\ddot{v} + 2\varepsilon \dot{v} + v = 0$ with initial condition $v(0) = 1$ and $v'(0) = 0$:

$$v(t) = e^{-\varepsilon t} \cos\left(\left(\sqrt{1 - \varepsilon^2}\right)t\right)$$



□