UC DAVIS, APPLIED MATHEMATICS (MAT207C), SPRING 2016

Homework #4

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Problem 1

Consider the Lagerstrom model for Low Reynolds number flow:

$$U'' + \frac{2}{R}U' + \varepsilon UU' = 0$$
$$U(1) = 0$$
$$U(\infty) = 1.$$

Compute the leading order expansion of $U'(1)$ in the limit of small ε. This is the analog to Stokes' original calculation for the force on a translating sphere in 3D.

Proof. To compute the leading order expansion, ignore all terms smaller than $\mathcal{O}(1)$. That is, assume $U = U_0 + \mathcal{O}(\varepsilon)$. Then

$$U_0'' + \frac{2}{R}U_0' = 0$$
$$U_0(1) = 0$$
$$U_0(\infty) = 1.$$

The differential equation has solution $U_0 = \dfrac{A}{R} + B$. Then $U_0(\infty) = 1$ implies $B = 1$. And thus $U_0(1) = 0$ implies $A = -1$. That is,

$$U_0(R) = 1 - \frac{1}{R}.$$

And so,

$$U(R) = 1 - \frac{1}{R} + \mathcal{O}(\varepsilon) \qquad \Longrightarrow \qquad U'(R) \approx \frac{1}{R^2} \qquad \Longrightarrow \qquad U'(1) \approx 1$$

□

Problem 2

Compute the expansion of $U'(1)$ up to order ε. You will encounter a problem similar to the "Whitehead paradox" which you will reoslve using intermediate scale matching.
You may find the following asymptotic expansion useful for small r:

$$\int_r^\infty \frac{e^{-x}}{x^2}\,\mathrm{d}x = \frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r),$$

where γ is Euler's constant.

Proof. Now assume $U = U_0 + \varepsilon U_1 + \mathcal{O}\left(\varepsilon^2\right)$. By problem 1, $U_0 = 1 - \dfrac{1}{R}$. Then

$$U_1'' + \frac{2}{R}U_1' = -U_0U_0'$$
$$U_1(1) = 0$$
$$U_1(\infty) = 0.$$

The differential equation has the solution $U_1 = \dfrac{C}{R} + D - \dfrac{\log(R)}{R} - \log(R)$. It is possible for $U_1(1) = 0$, but $U_1(\infty)$ is divergent and hence cannot equal 0. Thus U is only a solution of some inner layer in which R is small. This means we cannot apply the condition at infinity to U. So for U_0 from problem 1, we only apply the condition at 1. That is $U_0 = \dfrac{A}{R} + B$ and $U_0(1) = 0$. This implies $A = -B$, and thus $U_0 = A\left(1 - \dfrac{1}{R}\right)$. Thus

$$U_1'' + \frac{2}{R}U_1' = -U_0U_0' = -A^2\left(1 - \frac{1}{R}\right)\left(\frac{1}{R^2}\right)$$
$$U_1(1) = 0.$$

The differential equation has the solution $U_1 = -A^2\left[\dfrac{\log(R)}{R} + \log(R)\right] + \dfrac{C}{R} + D$. Then $U_1(1) = 0$ implies $C = -D$, and thus

$$U_1 = -A^2\left[\log(R)\left(\frac{1}{R} + 1\right)\right] + C\left(\frac{1}{R} - 1\right)$$

Thus the two-term expansion of the outer solution is

$$U(R) = A\left(1 - \frac{1}{R}\right) + \varepsilon\left[-A^2\left[\log(R)\left(\frac{1}{R} + 1\right)\right] + C\left(\frac{1}{R} - 1\right)\right]$$

To find the inner solution, define $r = \varepsilon R$ and let $u(r) = U(R)$. Then

$$\frac{\mathrm{d}U}{\mathrm{d}R} = \varepsilon u' \qquad \text{and} \qquad \frac{\mathrm{d}^2U}{\mathrm{d}R} = \varepsilon^2 u''.$$

Then

$$u'' + \frac{2}{r}u' + uu' = 0$$
$$u(\infty) = 1$$

Supposing $u = u_0 + \varepsilon u_1 + \mathscr{O}(\varepsilon^2)$, then since $u_0(\infty) = 1$, then $u_0 \equiv 1$. Then $u = 1 + \varepsilon u_1 + \mathscr{O}(\varepsilon^2)$, and so

$$\varepsilon u_1'' + \frac{2}{r}\varepsilon u_1' + (1 + u_1')\varepsilon u_1' + \mathscr{O}(\varepsilon^2) = 0$$
$$\implies \varepsilon\left[u_1'' + \frac{2}{r}u_1' + u_1'\right] + \mathscr{O}(\varepsilon^2) = 0$$
$$\implies u_1'' + \left(\frac{2}{r} + 1\right)u_1' = 0.$$

with boundary condition $u_1(\infty) = 0$. Let $v = u_1'$. Then the differential equation has the form

$$v' + \left(\frac{2}{r} + 1\right)v = 0,$$

which has the solution

$$v(r) = E\frac{e^{-r}}{r^2}$$

and thus u_1 has the solution

$$u_1(r) = \int_r^\infty v(t)\mathrm{d}t + F$$

The boundary condition $u_1(\infty) = 0$ implies $F = 0$ since the integral must converge to 0 as $r \to \infty$. Thus

$$u_1(r) = E\int_r^\infty \frac{e^{-t}}{t^2}\mathrm{d}t = E\left[\frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r)\right]$$

And thus the two-term expansion of the inner solution is

$$u(r) = 1 + \varepsilon\left[E\left[\frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r)\right]\right] + \mathcal{O}\left(\varepsilon^2\right)$$

To match the inner and outer solutions, we employ an intermediate timescale $r = \eta r_\eta$ (and hence $R = \frac{\eta r_\eta}{\varepsilon}$). By "intermediate", we mean $\varepsilon \prec \eta \prec 1$. Then

$$\begin{aligned}
u(r) - U(R) &= \underbrace{1 + \varepsilon\left[E\left[\frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r)\right]\right]}_{\text{two-term expansion of } u(r)} - \underbrace{\left(A\left(1 - \frac{1}{R}\right) + \varepsilon\left[-A^2\left[\log(R)\left(\frac{1}{R} + 1\right)\right] + C\left(\frac{1}{R} - 1\right)\right]\right)}_{\text{two-term expansion of } U(R)} + \mathcal{O}\left(\varepsilon^2\right) \\
&= \underbrace{1 + \varepsilon\left[E\left[\frac{1}{\eta r_\eta} + \log\left(\eta r_\eta\right) + \gamma - 1 + \mathcal{O}\left(\eta r_\eta\right)\right]\right]}_{\text{two-term expansion of } u(\eta r_\eta)} \\
&\qquad - \underbrace{\left(A\left(1 - \frac{1}{\frac{\eta r_\eta}{\varepsilon}}\right) + \varepsilon\left[-A^2\left[\log\left(\frac{\eta r_\eta}{\varepsilon}\right)\left(\frac{1}{\frac{\eta r_\eta}{\varepsilon}} + 1\right)\right] + C\left(\frac{1}{\frac{\eta r_\eta}{\varepsilon}} - 1\right)\right]\right)}_{\text{two-term expansion of } U\left(\frac{\eta r_\eta}{\varepsilon}\right)} + \mathcal{O}\left(\varepsilon^2\right) \\
&= \underbrace{1 + \frac{E}{r_\eta}\frac{\varepsilon}{\eta} + E\varepsilon\log\left(\eta r_\eta\right) + E\varepsilon\left(\gamma - 1\right) + \mathcal{O}\left(\varepsilon\eta r_\eta\right)}_{\text{two-term expansion of} u(\eta r_\eta)} \\
&\quad \underbrace{- A + \frac{A}{r_\eta}\frac{\varepsilon}{\eta} + A^2\frac{\varepsilon^2}{\eta r_\eta}\log\left(\eta r_\eta\right) - A^2\frac{\varepsilon^2}{\eta r_\eta}\log(\varepsilon) + A^2\varepsilon\log\left(\eta r_\eta\right) - A^2\varepsilon\log(\varepsilon) - C\frac{\varepsilon^2}{\eta r_\eta} + C\varepsilon}_{\text{two-term expansion of } U\left(\frac{\eta r_\eta}{\varepsilon}\right)} + \mathcal{O}\left(\varepsilon^2\right) \\
&= \Big[1 - A\Big] + \Big[E + A\Big]\frac{\varepsilon}{\eta r_\eta} + \Big[E\left(\gamma - 1\right) + C - A^2\log(\varepsilon)\Big]\varepsilon + o(\varepsilon),
\end{aligned}$$

assuming the order relations $\dfrac{\varepsilon^2}{\eta}\log\varepsilon \prec \varepsilon$ and $\dfrac{\varepsilon^2}{\eta}\log\eta \prec \varepsilon$. Then in order to match u with U, the constants A, C, and E are

$$A = 1 \qquad E = -1 \qquad C = \log(\varepsilon) + \gamma - 1.$$

Thus, define u_{match} as

$$u_{\text{match}} = 1 - \frac{1}{R} + \left(1 - \gamma - \log(\varepsilon R)\right)\varepsilon.$$

Then the complete two-term expansion of the solution of the differential equation is

$$\begin{aligned}
U(R) = &\underbrace{1 - \frac{1}{R} - \varepsilon\left(\log(\varepsilon R) + \gamma - 1\right) + \mathcal{O}(\varepsilon R)]}_{\text{two-term expansion of } u(\varepsilon R)} + \underbrace{\left(\left(1 - \frac{1}{R}\right) + \varepsilon\left[-\log(R)\left(\frac{1}{R} + 1\right) + \left(\gamma - 1\right)\left(\frac{1}{R} - 1\right)\right]\right)}_{\text{two-term expansion of } U(R)} \\
&- \underbrace{\left[1 - \frac{1}{R} + \left(1 - \gamma - \log(\varepsilon R)\right)\varepsilon\right]}_{\text{matched terms}}
\end{aligned}$$

After collecting terms of similar order, the complete two-term expansion of the solution is

$$U(R) = \left[1 - \frac{1}{R}\right] + \varepsilon\left[\frac{\log\left(\frac{\varepsilon}{R}\right)}{R} - \log(\varepsilon R) + \left(\gamma - 1\right)\left(\frac{1}{R} - 1\right)\right] + \mathcal{O}\left(\varepsilon^2\right)$$

Thus the first derivative is approximately

$$U'(R) \approx \frac{1}{R^2} - \varepsilon\left[\frac{-1 - \log\left(\frac{\varepsilon}{R}\right) + 1 - \gamma - R}{R^2}\right] \implies \boxed{U'(1) \approx 1 - \varepsilon\log(\varepsilon) + \varepsilon\left(-1 - \gamma\right)}$$

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