UC DAVIS, APPLIED MATHEMATICS (MAT207C), SPRING 2016

Homework #1

Sam Fleischer

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Problem 1

Find the two-term asymptotic expansion for small ε for all real roots x of the below equations.

(a) $\varepsilon^2 x^3 - x + \varepsilon = 0$

(b) $\varepsilon \exp(x^2) = 1 + \dfrac{\varepsilon}{1+x^2}$

(a) First note that if $\varepsilon = 0$, then $x = 0$. Let $x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + \ldots$. Then

$$\begin{aligned}
&\varepsilon^2\left(x_0 + x_1\varepsilon + x_2\varepsilon^2 + \ldots\right)^3 - \left(x_0 + x_1\varepsilon + x_2\varepsilon^2 + \ldots\right) + \varepsilon = 0\\
\implies &\varepsilon^2\left(x_0^3 + 3x_0^2x_1\varepsilon + \left(3x_0^2x_2 + 3x_0x_1^2\right)\varepsilon^2 + \ldots\right) - \left(x_0 + x_1\varepsilon + x_2\varepsilon^2 + \ldots\right) + \varepsilon = 0\\
\implies &x_0 = 0 \quad \text{and} \quad x_1 = 1 \quad \text{and} \quad x_2 = 0 \quad \text{and} \quad x_3 = 0 \quad \text{and} \quad x_4 = 0\\
\implies &\varepsilon^2\left(x_1^3\varepsilon^3 + \ldots\right) - \left(x_1\varepsilon + x_5\varepsilon^5 + \ldots\right) + \varepsilon = 0\\
\implies &x_5 = 1\\
\implies &x = \varepsilon + \varepsilon^5 + O\left(\varepsilon^6\right)
\end{aligned}$$

Next we rescale to find the other two roots. Let $x = \varepsilon^{-1}y$. Then

$$\begin{aligned}
&\varepsilon^{-1}y^3 - \varepsilon^{-1}y + \varepsilon = 0\\
\implies &y^3 - y + \varepsilon^2 = 0
\end{aligned}$$

Let $y = y_0 + y_1\varepsilon + y_2\varepsilon^2 + \ldots$. Then

$$\begin{aligned}
&\left(y_0 + y_1\varepsilon + y_2\varepsilon^2 + \ldots\right)^3 - \left(y_0 + y_1\varepsilon + y_2\varepsilon^2 + \ldots\right) + \varepsilon^2 = 0\\
\implies &\left(y_0^3 + 3y_0^2y_1\varepsilon + \left(3y_0^2y_2 + 3y_1^2\right)\varepsilon^2 + \ldots\right) - \left(y_0 + y_1\varepsilon + y_2\varepsilon^2 + \ldots\right) + \varepsilon^2 = 0\\
\implies &y_0^3 - y_0 = 0 \iff y_0 = 0, \pm 1 \quad \text{and} \quad \left(3y_0^2 - 1\right)y_1 = 0 \implies \cancel{y_0^2 = \frac{1}{3}} \text{ or } y_1 = 0\\
\implies &3y_0^2y_2 - y_2 + 1 = 0 \implies y_2 = \frac{1}{1 - 3y_0^2}
\end{aligned}$$

If $y_0 = 0$, then $y_2 = 1$. If $y_0 = \pm 1$, then $y_2 = -\frac{1}{2}$. Thus the asymptotic solutions are

- $y = \varepsilon^2 + O\left(\varepsilon^3\right)$ (this matches with the asymptotic solution reached above).
- $y = \pm 1 - \frac{1}{2}\varepsilon^2 + O\left(\varepsilon^3\right)$. These are impossible to get in the original scaling since as $y \to \pm 1$, $x \to \pm\infty$.

Thus, the asymptotic solutions are

$$\boxed{x \approx \varepsilon + \varepsilon^5 \qquad \text{and} \qquad x \approx \frac{\pm 1 - \frac{1}{2}\varepsilon^2}{\varepsilon}}$$

(b) First, make the substitution $y = x^2$:

$$\varepsilon \exp(y) = 1 + \frac{\varepsilon}{1+y}$$

and note that as $\varepsilon \to 0$, we expect $y \to \infty$. Thus $1 + \frac{\varepsilon}{1+y} \approx 1$. Thus,

$$\begin{aligned}
&\varepsilon \exp(y) \approx 1\\
\implies &y \approx \log\left(\varepsilon^{-1}\right)
\end{aligned}$$

Thus $y = \log\left(\varepsilon^{-1}\right) + g(\varepsilon)$ where $g(\varepsilon) = o\left(\log\left(\varepsilon^{-1}\right)\right)$. Then

$$\begin{aligned}
\varepsilon \exp\left(\log\left(\varepsilon^{-1}\right) + g(\varepsilon)\right) &= 1 + \frac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)} \\
\implies \varepsilon \exp\left(\log\left(\varepsilon^{-1}\right)\right) \exp\left(g(\varepsilon)\right) &= 1 + \frac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)} \\
\implies \exp\left(g(\varepsilon)\right) &= 1 + \frac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)} \\
\implies g(\varepsilon) &= \log\left(1 + \frac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)}\right) \\
&= \frac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)} - \frac{1}{2}\left(\frac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)}\right)^2 + \dots
\end{aligned}$$

We can use the Taylor expansion of $\log(1+x)$ since $\dfrac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)} \to 0$ as $\varepsilon \to 0$. Then

$$\begin{aligned}
g(\varepsilon) &\approx \frac{\varepsilon}{1 + \log\left(\varepsilon^{-1}\right) + g(\varepsilon)} \\
&\approx \frac{\varepsilon}{\log\left(\varepsilon^{-1}\right)}
\end{aligned}$$

Note $\dfrac{\varepsilon}{\log\left(\varepsilon^{-1}\right)} = o\left(\log\left(\varepsilon^{-1}\right)\right)$ since

$$\lim_{\varepsilon \to 0} \frac{\frac{\varepsilon}{\log\left(\varepsilon^{-1}\right)}}{\log\left(\varepsilon^{-1}\right)} = \frac{\varepsilon}{\left(\log\left(\varepsilon^{-1}\right)\right)^2} = 0$$

Thus $y \approx \log\left(\varepsilon^{-1}\right) + \dfrac{\varepsilon}{\log\left(\varepsilon^{-1}\right)}$, and hence the asymptotic solutions are

$$\boxed{x \approx \pm\sqrt{\log\left(\varepsilon^{-1}\right) + \frac{\varepsilon}{\log\left(\varepsilon^{-1}\right)}}}$$

Problem 2

The Exponential integral function is defined as

$$\mathrm{Ei}(x) = \int_x^\infty \frac{\exp(-s)}{s}\,\mathrm{d}s.$$

Derive an asymptotic expansion for $\mathrm{Ei}(x)$ for large x. Use a computer (e.g. expint(x) in MATLAB) to check the accuracy of your expansion for different values of x and for different numbers of terms. Discuss your results.

By integration by parts,

$$\begin{aligned}
\int_x^\infty \frac{\exp(-s)}{s}\,\mathrm{d}s &= \left.\frac{-\exp(-s)}{s}\right|_x^\infty - \int_x^\infty \frac{\exp(-s)}{s^2}\,\mathrm{d}s \\
&= \frac{\exp(-x)}{x} - \left[\left.\frac{-\exp(-s)}{s^2}\right|_x^\infty - 2\int_x^\infty \frac{\exp(-s)}{s^3}\,\mathrm{d}s\right]
\end{aligned}$$

$$\vdots$$

$$= \exp(-x)\left[\frac{(-1)^0 0!}{x} + \frac{(-1)^1 1!}{x^2} + \frac{(-1)^2 2!}{x^3} + \cdots + \frac{(-1)^n n!}{x^{n+1}}\right] + R_n(x)$$

where

$$R_n(x) = (-1)^{n+1}(n+1)! \int_x^\infty \frac{\exp(-s)}{s^{n+2}} \mathrm{d}s$$

Thus the asymptotic expansion for $\mathrm{Ei}(x)$ for large x is

$$\mathrm{Ei}(x) \sim \exp(-x) \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}.$$

This is a divergent series in n, but converges to 0 as $x \to \infty$ when n is fixed. The following graph shows the error between the asymptotic expansion and the exponential integral function, as calculated by MATLAB and graphed in Python.