Math 207C
Homework 5
Due Monday, May 9th

1. For the van der Pol equation

$$\epsilon \dot{u} = v + u - \frac{u^3}{3}$$

$$\dot{v} = -u,$$

we found equations for the inner and outer solutions in class. On the inner and outer layers, one of the two equations above is in pseudo-steady-state. Identify the values of u and v where there are corner layers. Find the equations for the corner layer and identify the thickness of the layer. Use a balance argument which involves retaining the time derivatives of both state variables.

2. The FitzHugh-Nagumo equations are a model system from electrophysiology to describe the cross membrane electrical potential in neurons:

$$\epsilon \dot{v} = v(a - v)(v - 1) - w$$

$$\dot{w} = v - \gamma w,$$

The variable v is the voltage and w is called a recovery variable. Suppose that $\epsilon > 0$ is small, $0 < a < 1$, and γ is sufficiently large that $v = 0$, $w = 0$ is the only equilibrium.

The single equilibrium is a global attactor: all trajectories approach it. For some initial conditions, the solution quickly approaches the rest state. For others the voltage rapidly rises to a large value for some period of time, then overshoots the equilibrium value, and finally approaches the rest state. This second response is called an action potential. See the below figures for examples, and note the difference in scales of the vertical axes. The MATLAB code to solve the equations and plot the solution is given below.

Analyze the inner and outer layer structures of this system to characterize which initial conditions lead an action potential, and describe the structure of the action potential using a layer analysis.
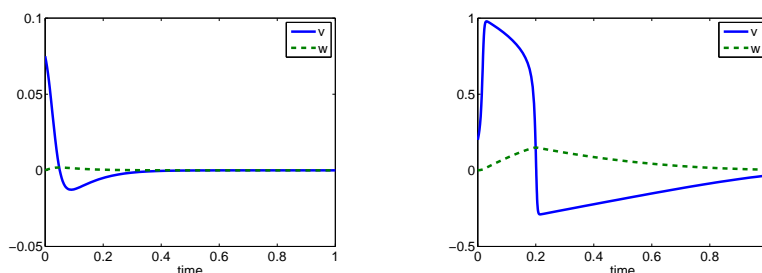

Figure 1: Solution to FitzHugh-Nagumo equations for different initial conditions for $a = 0.1$, $\gamma = 0.5$, and $\epsilon = 0.002$. The left plot shows the response to a subthreshold stimulus, and the right shows that action potential that results from a superthreshold stimulus.

```
% solve the FitzHugh-Nagumo equations
%   ep*v' = v*(a-v)*(v-1) - w
%      w' = v - g*w
% and plot the solution
%


% parameters
%
a  = 0.1;
g  = 0.5;
ep = 2e-3;

tend = 1;
v0   = 0.2;
w0   = 0.0;

% make ode-rhs function and solve
%
dydt= @(t,y)([(y(1).*(a-y(1)).*(y(1)-1)  -y(2) )/ep; y(1)-g*y(2)]);
[t,y]=ode23s(dydt,[0:tend],[v0,w0]);

% make the plot
%
figure;
hp=plot(t,y,'linewidth',3);
axis([ 0 1 -0.5 1]);
set(hp(2),'linestyle','--');
set(gca,'fontsize',16);
xlabel('time');
legend('v','w');
```