UC DAVIS, APPLIED MATHEMATICS (MAT207C), SPRING 2016

Homework #8

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Problem 1

Derive the leading order approximation to the general solution of

$$\varepsilon^3 u''' - q(x)u = 0 \qquad q(0) = 0$$

using WKB in the limit of small ε.

Proof. First suppose the solution u is of the form

$$u(x) = \exp\left[\frac{1}{\delta(\varepsilon)} S_0(x) + S_1(x) + \delta(\varepsilon) S_2(x) + \dots\right].$$

Then

$$\varepsilon^3\left[\left(\frac{1}{\delta}\dddot{S}_0 + \dddot{S}_1 + \delta\dddot{S}_2 + \dots\right) + \left(\frac{1}{\delta}\dot{S}_0 + \dot{S}_1 + \delta\dot{S}_2 + \dots\right)^3 + 3\left(\frac{1}{\delta}\ddot{S}_0 + \ddot{S}_1 + \delta\ddot{S}_2 + \dots\right)\left(\frac{1}{\delta}\dot{S}_0 + \dot{S}_1 + \delta\dot{S}_2 + \dots\right)\right] u = qu.$$

We can cancel $u(x)$ on each side since exponentials are nonzero. Also, we force $\delta(\varepsilon) = \varepsilon$ in order to match at leading order (which is $\mathcal{O}(1)$). Then the $\mathcal{O}(1)$ equation is

$$\left(\dot{S}_0\right)^3 = q \qquad \Longleftrightarrow \qquad \dot{S}_0 = \exp[i\theta]\sqrt[3]{q},$$

where $\theta = 0$, $\frac{2\pi}{3}$, or $\frac{-2\pi}{3}$. The $\mathcal{O}(\varepsilon)$ equation is

$$3\ddot{S}_0\dot{S}_0 + 3\left(\dot{S}_0\right)^2\dot{S}_1 = 0 \qquad \Longleftrightarrow \qquad \dot{S}_1 = -\frac{\ddot{S}_0}{\dot{S}_0} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\dot{S}_0\right)$$

which implies

$$S_1 = -\ln\dot{S}_0 + K = -\ln[\exp[i\theta]\sqrt[3]{q}] + K = -\frac{1}{3}\ln q - i\theta + K = -\frac{1}{3}\ln q + \tilde{K}.$$

Finally,

$$\begin{aligned}
u(x) &= \exp\left[\frac{1}{\varepsilon}S_0(x) + S_1(x) + \varepsilon S_2(x) + \dots\right] \\
&= \exp\left[\frac{1}{\varepsilon}\int_{-\infty}^{x}\sqrt[3]{q(s)}\,\mathrm{d}s - \frac{1}{3}\ln q + \tilde{K}\right] \\
&= \exp\left[\frac{1}{\varepsilon}\int_{-\infty}^{x}\sqrt[3]{q(s)}\,\mathrm{d}s\right]\exp\left[\ln\frac{1}{\sqrt[3]{q}}\right]\exp[\tilde{K}] \\
&= \frac{\hat{K}}{\sqrt[3]{q(x)}}\exp\left[\frac{1}{\varepsilon}\int_{-\infty}^{x}\sqrt[3]{q(s)}\,\mathrm{d}s\right]
\end{aligned}$$

□

Problem 2

Derive connection formulas for

$$\varepsilon^2 u'' - q(x)u = 0,$$

where

$$q(x) > 0 \text{ for } x > 0$$
$$q(x) < 0 \text{ for } x < 0$$
$$\lim_{x\to 0^+} = a^2 > 0$$
$$\lim_{x\to 0^-} = -b^2 < 0.$$

and give an expansion for the leading order general solution in the limit of small ε.

Proof. In class we showed the WKB approximation is

$$u(x) = \begin{cases} u_L(x) & \text{if } x < 0 \\ u_R(x) & \text{if } x > 0 \end{cases} = \begin{cases} \left|q(x)\right|^{-\frac{1}{4}}\left[A_L \exp\left[-\frac{1}{\varepsilon}\int_x^0 \sqrt{q(s)}\mathrm{d}s\right] + B_L \exp\left[-\frac{1}{\varepsilon}\int_x^0 \sqrt{q(s)}\mathrm{d}s\right]\right] & \text{if } x < 0 \\ q(x)^{-\frac{1}{4}}\left[A_R \exp\left[\frac{1}{\varepsilon}\int_0^x \sqrt{q(s)}\mathrm{d}s\right] + B_R \exp\left[\frac{1}{\varepsilon}\int_0^x \sqrt{q(s)}\mathrm{d}s\right]\right] & \text{if } x > 0 \end{cases}$$

There is an inner layer located at $x = 0$, so we define $X = \varepsilon^{-\alpha}x$ with $U(X) = u(x)$. Additionally, we can Taylor expand q on the left and right, and so, since $\lim_{x\to 0^-} q_L(x) = -b^2$ and $\lim_{x\to 0^+} q_R(x) = a^2$,

$$\begin{aligned} &\varepsilon^{2-2\alpha}\ddot{U}_L - \left(-b^2 + \dot{q}_L(0)\varepsilon^\alpha X + \ldots\right)U_L = 0, \qquad \text{and} \\ &\varepsilon^{2-2\alpha}\ddot{U}_R - \left(a^2 + \dot{q}_R(0)\varepsilon^\alpha X + \ldots\right)U_R = 0. \end{aligned}$$

Similar to Problem 1, matching at leading order (which is $\mathcal{O}(1)$) forces $\alpha = 1$. Thus the $\mathcal{O}(1)$ equations are

$$\ddot{U}_L + b^2 U_L = 0,$$
$$\ddot{U}_R - a^2 U_R = 0,$$

which has solution

$$U(X) \approx \begin{cases} A_1\cos(bX) + B_1\sin(bX) & \text{if } X < 0 \\ A_2\exp(aX) + B_2\exp(-aX) & \text{if } X > 0. \end{cases}$$

To ensure U is continuous and has a continuous first derivative, we must require $A_1 = A_2 + B_2$ and $B_1 = \frac{a}{b}(A_2 - B_2)$. Defining $A := A_2$ and $B := B_2$ gives us

$$U(X) \approx \begin{cases} (A+B)\cos(bX) + \frac{a}{b}(A-B)\sin(bX) & \text{if } X < 0 \\ A\exp(aX) + B\exp(-aX) & \text{if } X > 0. \end{cases}$$

Connecting U_L with u_L and U_R with u_R will ensure a connected u since we have already ensured U is continuous at 0. Thus, we define an intermediate scale $x = \eta x_\eta$ (and thus $X = \frac{\eta x_\eta}{\varepsilon}$).

- Lets first match the left hand side.

$$u_L(x) = u_L(\eta x_\eta) = \left|q(\eta x_\eta)\right|^{-\frac{1}{4}}\left[A_L\exp\left[-\frac{1}{\varepsilon}\int_{\eta x_\eta}^0 \sqrt{q(s)}\mathrm{d}s\right] + B_L\exp\left[\frac{1}{\varepsilon}\int_{\eta x_\eta}^0 \sqrt{q(s)}\mathrm{d}s\right]\right]$$
$$U_L(X) = U_L\left(\frac{\eta x_\eta}{\varepsilon}\right) = (A+B)\cos\left(\frac{b\eta x_\eta}{\varepsilon}\right) + \frac{a}{b}\sin\left(\frac{b\eta x_\eta}{\varepsilon}\right)$$

Next we can Taylor expand each integral in u_L, yielding

$$u_L(x) = \left|q(\eta x_\eta)\right|^{-\frac{1}{4}}\left[A_L \exp\left[\frac{\eta x_\eta}{\varepsilon}\sqrt{q(\eta x_\eta)}\right] + B_L \exp\left[-\frac{\eta x_\eta}{\varepsilon}\sqrt{q(\eta x_\eta)}\right]\right].$$

To perform simple matching, consider $\eta x_\eta \to 0$ (and since q is smooth, $q(\eta x_\eta) \to -b^2$).

$$u_L(\eta x_\eta) \to b^{-\frac{1}{2}}\left[A_L \exp\left[\frac{\eta x_\eta}{\varepsilon} bi\right] + B_L \exp\left[-\frac{\eta x_\eta}{\varepsilon} bi\right]\right]$$

In order to compare terms, we write each sin and cos in U_L in terms of complex exponentials:

$$U_L\left(\frac{\eta x_\eta}{\varepsilon}\right) = \frac{1}{2}\left(A+B+\frac{a}{b}(A-B)\right)\exp\left[\frac{\eta x_\eta}{\varepsilon} bi\right] + \frac{1}{2}\left(A+B-\frac{a}{b}(A-B)\right)\exp\left[-\frac{\eta x_\eta}{\varepsilon} bi\right].$$

Matching requires

$$\boxed{b^{-\frac{1}{2}} A_L = \frac{1}{2}\left(A+B+\frac{a}{b}(A-B)\right) \qquad \text{and} \qquad b^{-\frac{1}{2}} B_L = \frac{1}{2}\left(A+B-\frac{a}{b}(A-B)\right)}$$

- Similarly, on the right side,

$$u_R(x) = u_R(\eta x_\eta) = q(\eta x_\eta)^{-\frac{1}{4}}\left[A_R \exp\left[-\frac{1}{\varepsilon}\int_0^{\eta x_\eta}\sqrt{q(s)}\,\mathrm{d}s\right] + B_R \exp\left[\frac{1}{\varepsilon}\int_0^{\eta x_\eta}\sqrt{q(s)}\,\mathrm{d}s\right]\right]$$
$$U_R(X) = U_R\left(\frac{\eta x_\eta}{\varepsilon}\right) = A\exp\left[\frac{a\eta x_\eta}{\varepsilon}\right] + B\exp\left[-\frac{a\eta x_\eta}{\varepsilon}\right]$$

Next we can Taylor expand each integral in u_R, yielding

$$u_L(x) = q(\eta x_\eta)^{-\frac{1}{4}}\left[A_R \exp\left[-\frac{\eta x_\eta}{\varepsilon}\sqrt{q(\eta x_\eta)}\right] + B_R \exp\left[\frac{\eta x_\eta}{\varepsilon}\sqrt{q(\eta x_\eta)}\right]\right].$$

To perform simple matching, consider $\eta x_\eta \to 0$ (and since q is smooth, $q(\eta x_\eta) \to a^2$).

$$u_R(\eta x_\eta) \to a^{-\frac{1}{2}}\left[A_R \exp\left[-\frac{a\eta x_\eta}{\varepsilon}\right] + B_R \exp\left[\frac{a\eta x_\eta}{\varepsilon}\right]\right]$$

Thus, in order to match u_R with U_R, we require

$$\boxed{A = a^{-\frac{1}{2}} B_R \qquad \text{and} \qquad B = a^{-\frac{1}{2}} A_R.}$$

Finally, the leading order solutioin is

$$\boxed{u(x) = \begin{cases} \frac{\sqrt{b}}{2}\left|q(x)\right|^{-\frac{1}{4}}\Bigg[\left(A+B+\frac{a}{b}(A-B)\right)\exp\left[-\frac{1}{\varepsilon}\int_x^0\sqrt{q(s)}\,\mathrm{d}s\right] & \\ \qquad\qquad + \left(A+B-\frac{a}{b}(A-B)\right)\exp\left[\frac{1}{\varepsilon}\int_x^0\sqrt{q(s)}\,\mathrm{d}s\right]\Bigg] & \text{if } x<0 \\ \sqrt{a}\,q(x)^{-\frac{1}{4}}\left[B\exp\left[-\frac{1}{\varepsilon}\int_0^x\sqrt{q(s)}\,\mathrm{d}s\right] + A\exp\left[\frac{1}{\varepsilon}\int_0^x\sqrt{q(s)}\,\mathrm{d}s\right]\right] & \text{if } x>0 \end{cases}}$$

$\square$