Homework #1

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> # Problem 1
>
> Find the two-term asymptotic expansion for small ε for all real roots x of the below equations.
>
> (a) $\varepsilon^2 x^3 - x + \varepsilon = 0$
>
> (b) $\varepsilon \exp(x^2) = 1 + \dfrac{\varepsilon}{1 + x^2}$

(a) First note that if $\varepsilon = 0$, then $x = 0$. Let $x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + \dots$. Then

$$\varepsilon^2(x_0 + x_1\varepsilon + x_2\varepsilon^2 + \dots)^3 - (x_0 + x_1\varepsilon + x_2\varepsilon^2 + \dots) + \varepsilon = 0$$
$$\implies \varepsilon^2(x_0^3 + 3x_0^2 x_1\varepsilon + (3x_0^2 x_2 + 3x_0 x_1^2)\varepsilon^2 + \dots) - (x_0 + x_1\varepsilon + x_2\varepsilon^2 + \dots) + \varepsilon = 0$$
$$\implies x_0 = 0 \quad \text{and} \quad x_1 = 1 \quad \text{and} \quad x_2 = 0 \quad \text{and} \quad x_3 = 0 \quad \text{and} \quad x_4 = 0$$
$$\implies \varepsilon^2(x_1^3\varepsilon^3 + \dots) - (x_1\varepsilon + x_5\varepsilon^5 + \dots) + \varepsilon = 0$$
$$\implies x_5 = 1$$
$$\implies x = \varepsilon + \varepsilon^5 + O(\varepsilon^6)$$

Next we rescale to find the other two roots. Let $x = \varepsilon^{-1} y$. Then

$$\varepsilon^{-1} y^3 - \varepsilon^{-1} y + \varepsilon = 0$$
$$\implies y^3 - y + \varepsilon^2 = 0$$

Let $y = y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots$. Then

$$(y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots)^3 - (y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots) + \varepsilon^2 = 0$$
$$\implies (y_0^3 + 3y_0^2 y_1\varepsilon + (3y_0^2 y_2 + 3y_1^2)\varepsilon^2 + \dots) - (y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots) + \varepsilon^2 = 0$$
$$\implies y_0^3 - y_0 = 0 \iff y_0 = 0, \pm 1 \quad \text{and} \quad (3y_0^2 - 1)y_1 = 0 \implies y_0^2 \cancel{= \tfrac{1}{3}} \text{ or } y_1 = 0$$
$$\implies 3y_0^2 y_2 - y_2 + 1 = 0 \implies y_2 = \frac{1}{1 - 3y_0^2}$$

If $y_0 = 0$, then $y_2 = 1$. If $y_0 = \pm 1$, then $y_2 = -\frac{1}{2}$. Thus the asymptotic solutions are

- $y = \varepsilon^2 + O(\varepsilon^3)$ (this matches with the asymptotic solution reached above).
- $y = \pm 1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)$. These are impossible to get in the original scaling since as $y \to \pm 1$, $x \to \pm\infty$.

Thus, the asymptotic solutions are

$$\boxed{x \approx \varepsilon + \varepsilon^5 \quad \text{and} \quad x \approx \frac{\pm 1 - \frac{1}{2}\varepsilon^2}{\varepsilon}}$$

(b) First, make the substitution $y = x^2$:

$$\varepsilon \exp(y) = 1 + \frac{\varepsilon}{1 + y}$$

and note that as $\varepsilon \to 0$, we expect $y \to \infty$. Thus $1 + \frac{\varepsilon}{1+y} \approx 1$. Thus,

$$\varepsilon \exp(y) \approx 1$$
$$\implies y \approx \log(\varepsilon^{-1})$$

Thus $y = \log(\varepsilon^{-1}) + g(\varepsilon)$ where $g(\varepsilon) = o(\log(\varepsilon^{-1}))$. Then

$$\varepsilon \exp(\log(\varepsilon^{-1}) + g(\varepsilon)) = 1 + \frac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)}$$

$$\implies \varepsilon \exp(\log(\varepsilon^{-1})) \exp(g(\varepsilon)) = 1 + \frac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)}$$

$$\implies \exp(g(\varepsilon)) = 1 + \frac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)}$$

$$\implies g(\varepsilon) = \log\left(1 + \frac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)}\right)$$

$$= \frac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)} - \frac{1}{2}\left(\frac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)}\right)^2 + \dots$$

We can use the Taylor expansion of $\log(1 + x)$ since $\dfrac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)} \to 0$ as $\varepsilon \to 0$. Then

$$g(\varepsilon) \approx \frac{\varepsilon}{1 + \log(\varepsilon^{-1}) + g(\varepsilon)}$$

$$\approx \frac{\varepsilon}{\log(\varepsilon^{-1})}$$

Note $\dfrac{\varepsilon}{\log(\varepsilon^{-1})} = o(\log(\varepsilon^{-1}))$ since

$$\lim_{\varepsilon \to 0} \frac{\frac{\varepsilon}{\log(\varepsilon^{-1})}}{\log(\varepsilon^{-1})} = \frac{\varepsilon}{(\log(\varepsilon^{-1}))^2} = 0$$

Thus $y \approx \log(\varepsilon^{-1}) + \dfrac{\varepsilon}{\log(\varepsilon^{-1})}$, and hence the asymptotic solutions are

$$\boxed{x \approx \pm\sqrt{\log(\varepsilon^{-1}) + \frac{\varepsilon}{\log(\varepsilon^{-1})}}}$$

Problem 2

The Exponential integral function is defined as

$$\text{Ei}(x) = \int_x^\infty \frac{\exp(-s)}{s}\,ds.$$

Derive an asymptotic expansion for $\text{Ei}(x)$ for large x. Use a computer (e.q. expint(x) in MATLAB) to check the accuracy of your expansion for different values of x and for different numbers of terms. Discuss your results.

By integration by parts,

$$\int_x^\infty \frac{\exp(-s)}{s}\,ds = \left.\frac{-\exp(-s)}{s}\right|_x^\infty - \int_x^\infty \frac{\exp(-s)}{s^2}\,ds$$

$$= \frac{\exp(-x)}{x} - \left[\left.\frac{-\exp(-s)}{s^2}\right|_x^\infty - 2\int_x^\infty \frac{\exp(-s)}{s^3}\,ds\right]$$

$$\vdots$$

$$= \exp(-x)\left[\frac{(-1)^0 0!}{x} + \frac{(-1)^1 1!}{x^2} + \frac{(-1)^2 2!}{x^3} + \cdots + \frac{(-1)^n n!}{x^{n+1}}\right] + R_n(x)$$

where

$$R_n(x) = (-1)^{n+1}(n+1)!\int_x^\infty \frac{\exp(-s)}{s^{n+2}}\,\mathrm{d}s$$

Thus the asymptotic expansion for $\mathrm{Ei}(x)$ for large x is

$$\mathrm{Ei}(x) \sim \exp(-x)\sum_{n=0}^\infty \frac{(-1)^n n!}{x^{n+1}}.$$

This is a divergent series in n, but converges to 0 as $x \to \infty$ when n is fixed. The following graph shows the error between the asymptotic expansion and the exponential integral function, as calculated by MATLAB and graphed in Python.