Math 207C
Homework 6
Due Friday, May 27th

1. Consider a one-dimensional layered medium with a periodic substructure of alternate layers of material one with thickness $\epsilon\phi_1$ and diffusion coefficient D_1 and material 2 with thickness $\epsilon\phi_2$ and diffusion coefficient D_2. Without loss of generality take $\phi_2 = 1 - \phi_1$ so that ϕ_i represents the volume fraction of material i and the length of the periodic cell is ϵ.

   The steady-state diffusion equation is

   $$\partial_x \left(D(x)\partial_x u\right) = f(x),$$

   where $D(x) = D_i$ in material i. Let x_* represent a point on the interface between two layers. At such points we require

   $$\begin{aligned} \lim_{x\to x_*^-} u(x) &= \lim_{x\to x_*^+} u(x) \\ \lim_{x\to x_*^-} Du_x &= \lim_{x\to x_*^+} Du_x, \end{aligned}$$

   which enforce continuity of the solution and continuity of the flux. Derive a homogenized steady-state diffusion equation.

2. Express the below equation in the standard form to apply the averaging theorem ($\dot{\mathbf{x}} = \epsilon\mathbf{f}(\mathbf{x}, t)$, $\mathbf{f}$ periodic in time), and give the averaged equations.

   $$\ddot{u} + 4\epsilon\left(\cos^2 t\right)\dot{u} + u = 0$$

   Generate a numerical solution to the time-averaged equations and compare it with the solution of

   $$\ddot{v} + 2\epsilon\dot{v} + v = 0,$$

   in which the coefficient has been replaced with the time-averaged value without invoking the averaging theorem.