
Homework #4

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Problem 1

Consider the Lagerstrom model for Low Reynolds number flow:

$$\begin{aligned} U'' + \frac{2}{R}U' + \varepsilon UU' &= 0 \\ U(1) &= 0 \\ U(\infty) &= 1. \end{aligned}$$

Compute the leading order expansion of $U'(1)$ in the limit of small ε . This is the analog to Stokes' original calculation for the force on a translating sphere in 3D.

Proof. To compute the leading order expansion, ignore all terms smaller than $\mathcal{O}(1)$. That is, assume $U = U_0 + \mathcal{O}(\varepsilon)$. Then

$$\begin{aligned} U_0'' + \frac{2}{R}U_0' &= 0 \\ U_0(1) &= 0 \\ U_0(\infty) &= 1. \end{aligned}$$

The differential equation has solution $U_0 = \frac{A}{R} + B$. Then $U_0(\infty) = 1$ implies $B = 1$. And thus $U_0(1) = 0$ implies $A = -1$. That is,

$$U_0(R) = 1 - \frac{1}{R}.$$

And so,

$$U(R) = 1 - \frac{1}{R} + \mathcal{O}(\varepsilon) \quad \Rightarrow \quad U'(R) \approx \frac{1}{R^2} \quad \Rightarrow \quad U'(1) \approx 1$$

□

Problem 2

Compute the expansion of $U'(1)$ up to order ε . You will encounter a problem similar to the “Whitehead paradox” which you will resolve using intermediate scale matching.

You may find the following asymptotic expansion useful for small r :

$$\int_r^\infty \frac{e^{-x}}{x^2} dx = \frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r),$$

where γ is Euler's constant.

Proof. Now assume $U = U_0 + \varepsilon U_1 + \mathcal{O}(\varepsilon^2)$. By problem 1, $U_0 = 1 - \frac{1}{R}$. Then

$$\begin{aligned} U_1'' + \frac{2}{R}U_1' &= -U_0U_0' \\ U_1(1) &= 0 \\ U_1(\infty) &= 0. \end{aligned}$$

The differential equation has the solution $U_1 = \frac{C}{R} + D - \frac{\log(R)}{R} - \log(R)$. It is possible for $U_1(1) = 0$, but $U_1(\infty)$ is divergent and hence cannot equal 0. Thus U is only a solution of some inner layer in which R is small. This means we cannot apply the condition at infinity to U . So for U_0 from problem 1, we only apply the condition at 1. That is $U_0 = \frac{A}{R} + B$ and $U_0(1) = 0$. This implies $A = -B$, and thus $U_0 = A\left(1 - \frac{1}{R}\right)$. Thus

$$U_1'' + \frac{2}{R}U_1' = -U_0U_0' = -A^2\left(1 - \frac{1}{R}\right)\left(\frac{1}{R^2}\right)$$

$$U_1(1) = 0.$$

The differential equation has the solution $U_1 = -A^2\left[\frac{\log(R)}{R} + \log(R)\right] + \frac{C}{R} + D$. Then $U_1(1) = 0$ implies $C = -D$, and thus

$$U_1 = -A^2\left[\log(R)\left(\frac{1}{R} + 1\right)\right] + C\left(\frac{1}{R} - 1\right)$$

Thus the two-term expansion of the outer solution is

$$U(R) = A\left(1 - \frac{1}{R}\right) + \varepsilon\left[-A^2\left[\log(R)\left(\frac{1}{R} + 1\right)\right] + C\left(\frac{1}{R} - 1\right)\right]$$

To find the inner solution, define $r = \varepsilon R$ and let $u(r) = U(R)$. Then

$$\frac{dU}{dR} = \varepsilon u' \quad \text{and} \quad \frac{d^2U}{dR^2} = \varepsilon^2 u''.$$

Then

$$u'' + \frac{2}{r}u' + uu' = 0$$

$$u(\infty) = 1$$

Supposing $u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$, then since $u_0(\infty) = 1$, then $u_0 \equiv 1$. Then $u = 1 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$, and so

$$\varepsilon u_1'' + \frac{2}{r}\varepsilon u_1' + (1 + u_1')\varepsilon u_1' + \mathcal{O}(\varepsilon^2) = 0$$

$$\Rightarrow \varepsilon\left[u_1'' + \frac{2}{r}u_1' + u_1'\right] + \mathcal{O}(\varepsilon^2) = 0$$

$$\Rightarrow u_1'' + \left(\frac{2}{r} + 1\right)u_1' = 0.$$

with boundary condition $u_1(\infty) = 0$. Let $v = u_1'$. Then the differential equation has the form

$$v' + \left(\frac{2}{r} + 1\right)v = 0,$$

which has the solution

$$v(r) = E \frac{e^{-r}}{r^2}$$

and thus u_1 has the solution

$$u_1(r) = \int_r^\infty v(t) dt + F$$

The boundary condition $u_1(\infty) = 0$ implies $F = 0$ since the integral must converge to 0 as $r \rightarrow \infty$. Thus

$$u_1(r) = E \int_r^\infty \frac{e^{-t}}{t^2} dt = E \left[\frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r) \right]$$

And thus the two-term expansion of the inner solution is

$$u(r) = 1 + \varepsilon \left[E \left[\frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r) \right] \right] + \mathcal{O}(\varepsilon^2)$$

To match the inner and outer solutions, we employ an intermediate timescale $r = \eta r_\eta$ (and hence $R = \frac{\eta r_\eta}{\varepsilon}$). By “intermediate”, we mean $\varepsilon < \eta < 1$. Then

$$\begin{aligned} u(r) - U(R) &= 1 + \varepsilon \underbrace{\left[E \left[\frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r) \right] \right]}_{\text{two-term expansion of } u(r)} - \underbrace{\left(A \left(1 - \frac{1}{R} \right) + \varepsilon \left[-A^2 \left[\log(R) \left(\frac{1}{R} + 1 \right) \right] + C \left(\frac{1}{R} - 1 \right) \right] \right)}_{\text{two-term expansion of } U(R)} + \mathcal{O}(\varepsilon^2) \\ &= 1 + \varepsilon \underbrace{\left[E \left[\frac{1}{\eta r_\eta} + \log(\eta r_\eta) + \gamma - 1 + \mathcal{O}(\eta r_\eta) \right] \right]}_{\text{two-term expansion of } u(\eta r_\eta)} \\ &\quad - \underbrace{\left(A \left(1 - \frac{1}{\frac{\eta r_\eta}{\varepsilon}} \right) + \varepsilon \left[-A^2 \left[\log\left(\frac{\eta r_\eta}{\varepsilon}\right) \left(\frac{1}{\frac{\eta r_\eta}{\varepsilon}} + 1 \right) \right] + C \left(\frac{1}{\frac{\eta r_\eta}{\varepsilon}} - 1 \right) \right] \right)}_{\text{two-term expansion of } U\left(\frac{\eta r_\eta}{\varepsilon}\right)} + \mathcal{O}(\varepsilon^2) \\ &= 1 + \underbrace{\frac{E}{r_\eta} \frac{\varepsilon}{\eta} + E \varepsilon \log(\eta r_\eta) + E \varepsilon (\gamma - 1) + \mathcal{O}(\varepsilon \eta r_\eta)}_{\text{two-term expansion of } u(\eta r_\eta)} \\ &\quad - \underbrace{A + \frac{A}{r_\eta} \frac{\varepsilon}{\eta} + A^2 \frac{\varepsilon^2}{\eta r_\eta} \log(\eta r_\eta) - A^2 \frac{\varepsilon^2}{\eta r_\eta} \log(\varepsilon) + A^2 \varepsilon \log(\eta r_\eta) - A^2 \varepsilon \log(\varepsilon) - C \frac{\varepsilon^2}{\eta r_\eta} + C \varepsilon + \mathcal{O}(\varepsilon^2)}_{\text{two-term expansion of } U\left(\frac{\eta r_\eta}{\varepsilon}\right)} \\ &= \left[1 - A \right] + \left[E + A \right] \frac{\varepsilon}{\eta r_\eta} + \left[E(\gamma - 1) + C \right] \varepsilon + \mathcal{O}(\varepsilon), \end{aligned}$$

assuming the order relation $\eta \log \eta < \varepsilon$. Then in order to match u with U , the constants A , C , and E are

$$A = 1 \quad E = -1 \quad C = \gamma - 1.$$

Thus, define u_{match} as

$$u_{\text{match}} = 1 - \frac{1}{R} + (1 - \gamma)\varepsilon.$$

Noting that $\log(\varepsilon R) = \log(\varepsilon) + \log(R)$, then the two-term expansion of $u(\varepsilon R)$ is

$$u(\varepsilon R) = 1 - \frac{1}{R} - \varepsilon \log(\varepsilon) - \varepsilon \log R - \varepsilon(\gamma - 1) + \mathcal{O}(\varepsilon R),$$

then the complete two-term expansion of the solution of the differential equation is

$$U(R) = \underbrace{1 - \frac{1}{R} - \varepsilon \log(\varepsilon) - \varepsilon \log R - \varepsilon(\gamma - 1) + \mathcal{O}(\varepsilon R)}_{\text{two-term expansion of } u(\varepsilon R)} + \underbrace{\left(\left(1 - \frac{1}{R} \right) + \varepsilon \left[-\log(R) \left(\frac{1}{R} + 1 \right) + (\gamma - 1) \left(\frac{1}{R} - 1 \right) \right] \right)}_{\text{two-term expansion of } U(R)}$$

$$-\underbrace{\left[1 - \frac{1}{R} + (1-\gamma)\varepsilon\right]}_{\text{matched terms}}$$

After collecting terms of similar order, the complete two-term expansion of the solution is

$$U(R) = \left[1 - \frac{1}{R}\right] + \left[-2\log R - (\gamma - 1) + \frac{\gamma - 1 - \log R}{R}\right]\varepsilon + \mathcal{O}(\varepsilon^2)$$

Thus

$$U'(R) \approx \frac{1}{R^2} - \frac{2}{R} - \frac{-1 - (\gamma - 1 - \log R)}{R^2} \implies \boxed{U'(1) \approx \gamma - 1}$$

□