Homework #5

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Problem 1

For the Van del Pol equation

$$\varepsilon \dot{u} = v + u - \frac{u^3}{3}$$

$$\dot{v} = -u$$

we found equations for the inner an outer solutions in class. On the inner and outer layers, one of the two equations above is in pseudo-steady-state. Identify the values of u and v where there are corner layers. Find the equations for the corner layer and identify the thickness of the layer. Use a balance argument which involves retaining the time derivatives of both state variables.

Proof. The following system is considered to be the "outer" solution since $\dot{u} = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ and $\dot{v} = \mathcal{O}(1)$, that is, u changes very rapidly and v changes at a "normal" rate.

$$\varepsilon \dot{u} = v + u - \frac{u^3}{3}$$

$$\dot{v} = -u. \tag{outer}$$

We can rescale the system by setting $T = \varepsilon t$, $U(T) = u(t)$, and $V(T) = v(t)$. This is considered to be the "inner" solution since $\dot{U} = \mathcal{O}(1)$ and $\dot{V} = \mathcal{O}(\varepsilon)$.

$$\dot{U} = V + U - \frac{U^3}{3}$$

$$\dot{V} = -\varepsilon U. \tag{inner}$$

The $\mathcal{O}(1)$ equations of the outer problem are

$$0 = v + u - \frac{u^3}{3}$$

$$\dot{v} = -u \tag{outer $\mathcal{O}(1)$}$$

and the $\mathcal{O}(1)$ equations of the inner problem are

$$\dot{U} = V + U - \frac{U^3}{3}$$

$$\dot{V} = 0. \tag{inner $\mathcal{O}(1)$}$$

The slow manifold is $v = \frac{u^3}{3} - u$. This is a positive cubic with zeros $u = (0,0), (\pm\sqrt{3}, 0)$. There is no $\dot{u}$ term (and hence no explicit horizontal flow) in (outer $\mathcal{O}(1)$), but $\dot{v} = -u$ shows us flow is in the negative v direction on the positive u plane and in the positive v direction on the negative u plane. The flow is toward the maximum, $\left(-1, \frac{2}{3}\right)$, and the mimimum $\left(1, -\frac{2}{3}\right)$, and away from the origin. However the only equilibrium point is $(0,0)$, so we must look at the (inner $\mathcal{O}(1)$) system.

The fast flow is purely in the U directior since $\dot{V} = 0$. Above the slow manifold there is leftward fast flow, and below the manifold is rightward fast flow. Since no flow approaches the equilibrium, we are expecting a limit cycle. There is an inner layer whenever $(u, v) = \left(\pm 1, \mp \frac{2}{3}\right)$. In u this is a large sudden shift is value and in derivative. In v there is little shift in value but large shift in derivative. This is the corner.

Next make the following variable shifts:

$$u = 1 + \varepsilon^\gamma \bar{u} \qquad v = -\frac{2}{3} + \varepsilon^\beta \bar{v} \qquad t = t_0 + \varepsilon^\alpha \bar{t}.$$

Since $\bar{t}$ is arbitrary, set $t_0 = 0$. Then

$$\varepsilon^{1+\gamma-\alpha}\,\dot{\bar{u}} = \left(\varepsilon^\beta\,\bar{v} - \frac{2}{3}\right) + \left(\varepsilon^\gamma\,\bar{u} + 1\right) - \frac{\left(\varepsilon^\gamma\,\bar{u}+1\right)^3}{3} = \varepsilon^\beta\,\bar{v} - \frac{\left(\varepsilon^{3\gamma}\,\bar{u}^3 + 3\varepsilon^{2\gamma}\,\bar{u}^2\right)}{3}$$
$$\varepsilon^{\beta-\alpha}\,\dot{\bar{v}} = -(\varepsilon^\gamma\,\bar{u} + 1).$$

The only reasonable option for α, β, and γ is setting $\alpha = \beta = \frac{2}{3}$ and $\gamma = \frac{1}{3}$, which yields

$$\dot{\bar{u}} = \bar{v} - \bar{u}^2 - \frac{1}{3}\varepsilon\bar{u}^3$$
$$\dot{\bar{v}} = -1 - \varepsilon^{\frac{1}{3}}\,\bar{u},$$

and thus the thickness of the layer is $\varepsilon^{\frac{2}{3}}$. $\square$

Problem 2

The FitzHigh-Nagumo equations are a model system from electrophysiology to describe the cross membrane electrical potential in neurons:

$$\varepsilon \dot{v} = v(a - v)(v - 1) - w$$
$$\dot{w} = v - \gamma w,$$

The variable v is the voltage and w is called a recovery variable. Suppose that $\varepsilon > 0$ is small, $0 < a < 1$, and γ is sufficiently large that $v = 0$, $w = 0$ is the only equilibrium.

The single equilibrium is a global attractor: all trajectories approach it. For some initial conditions, the solution quickly approaches the rest state. For others the voltage rapidly rises to a large value for some period of time, then overshoots the equilibrium value, and finally approaches the rest state. This second response is called an action potential.

Analyze the inner and outer later structures of this system to characterize which initial conditions lead to an action potential, and describe the structure of the action potential using a layer analysis.

Proof. Set $g(v) = v(a - v)(v - 1)$. Then the outer $\mathcal{O}(1)$ problem is

$$w = g(v)$$
$$\dot{w} = v - \gamma w$$

and the inner $\mathcal{O}(1)$ problem is

$$\dot{V} = g(V) - W$$
$$\dot{W} = 0$$

The slow manifold is a negative cubic with zeros $(0, 0)$, $(a, 0)$, and $(1, 0)$. $g'(v) = 0$ gives $v_{1,2} = \frac{(a+1)\pm\sqrt{(a+1)^3-3a}}{3}$ where $v_1 < v_2$. Define $w_i = g(v_i)$ for $i = 1, 2$. Thus the minimum is (v_1, w_1) and the maximum is (v_2, w_2). The line $w = \frac{1}{\gamma} v$ separates the slow flow along this manifold. Since the line is steep enough, all points on the manifold with $v > 0$ are beneath the line $x = \frac{1}{\gamma} v$. Flow underneath this line is upward, and flow above this line is downward, so all slow flow approaches the equilibrium $(0, 0)$ and the maximum (v_2, w_2). The slow manifold can be broken into three parts: g_1: the decreasing part from $-\infty$ to v_1, g_2: the increasing part from v_1 to v_2, and g_3: the decreasing part from v_2 to ∞. Trajectories starting above the curve g_2 or above the horizontal lines protruding leftward from the minimum and rightward from the maximum never reach g_3. They quickly approach g_1 and increase or decrease toward the original. Trajectories starting below g_2 or below the horizontal lines protruding leftward from the minimum and rightward from the maximum always reach the g_3 manifold before quickly returning to the g_1 manifold. We expect a corner layer in v and w at (v_2, w_2).

Next make the following variable shifts:

$$v = v_2 + \varepsilon^\phi \bar{v} \qquad w = w_2 + \varepsilon^\beta \bar{w} \qquad t = t_0 + \varepsilon^\alpha \bar{t}.$$

Since $\bar{t}$ is arbitrary, set $t_0 = 0$. Using Maple, with the above substitutions gives

$$\varepsilon^{1+\phi-\alpha} \dot{\bar{v}} = \varepsilon^{3\phi} \bar{v}^3 - \varepsilon^{2\phi} \bar{v}^2 \sqrt{a^2 - a + a} - \varepsilon^\beta \bar{w}$$
$$\varepsilon^{\beta-\alpha} \dot{\bar{w}} = v_2 - \gamma w_2 + \varepsilon^\phi \bar{v} - \gamma \varepsilon^\beta \bar{w}.$$

The only reasonable option for α, β, and ϕ is setting $\alpha = \beta = \phi = 1$, which yields

$$\varepsilon \dot{\bar{v}} = \varepsilon^3 \bar{v}^3 - \varepsilon^2 \bar{v}^2 \sqrt{a^2 - a + a} - \varepsilon \bar{w}$$

$$\dot{\bar{w}} = v_2 - \gamma w_2 + \bar{v} - \gamma \bar{w}.$$

and thus the thickness of the layer is ε^1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$