UC DAVIS, APPLIED MATHEMATICS (MAT207C), SPRING 2016

Homework #6

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Problem 1

The dimensionless equation of motion of a frictionless pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sin\theta = 0.$$

In the limit of small amplitude, the period is 2π to leading order. Compute the next term in the expansion of the period for small amplitude.

Proof. Let $\tau = \omega t$ where $\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$. $\omega_0 = 1$ since we are assuming $\lim_{\varepsilon\to 0}\tau = t$. Then set $\theta(t) = v(\tau)$ where $v(\tau) = \varepsilon v_0(\tau) + \varepsilon^2 v_1(\tau) + \dots$. We do not have an $\mathcal{O}(1)$ term in the expansion of v since we are considering the soluton in the limit of small amplitude. Then

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \omega^2\frac{\mathrm{d}^2 v}{\mathrm{d}\tau^2} + \sin v = \omega^2\frac{\mathrm{d}^2 v}{\mathrm{d}\tau^2} + v - \frac{v^3}{3!} + \frac{v^5}{5!} - \dots = 0.$$

We can employ the MacLaurin series of $\sin v$ since $v \approx 0$. Then

$$\left(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2\right)\left(\varepsilon v_0'' + \varepsilon^2 v_1'' + \dots\right) + \left(\varepsilon v_0 + \varepsilon^2 v_1 + \dots\right) - \frac{\left(\varepsilon v_0 + \varepsilon^2 v_1 + \dots\right)}{3!} + \mathcal{O}\left(\varepsilon^5\right) = 0.$$

There is no $\mathcal{O}(1)$ component to this equation since $v \approx 0$. The $\mathcal{O}(\varepsilon)$ equation is

$$v_0'' + v_0 = 0,$$

which has solution $v_0 = A\cos\left(\tau + \phi\right)$. However, we can neglect phase shift since it has no effect on the period of oscillations. Also, we can redefine ε as $\varepsilon = A\varepsilon$ and re-do the entire expansion to get $v_0 = \cos(\tau)$. The $\mathcal{O}\left(\varepsilon^2\right)$ equation is

$$v_1'' + v_1 = -2\omega_1 v_0'' = 2\omega_1\cos(\tau).$$

In order to prevent forcing at resonant frequency, we force $\omega_1 = 0$. The $\mathcal{O}\left(\varepsilon^3\right)$ equation then reduces to

$$v_2'' + v_2 = -2\omega_2 v_0'' - \frac{\cos^3(\tau)}{3!} = 2\omega_2\cos(\tau) - \frac{1}{6}\left(\frac{1}{4}\cos(3\tau) - \frac{3}{4}\cos(\tau)\right) = \left(2\omega_2 + \frac{1}{8}\right)\cos(\tau) - \frac{1}{24}\cos(3\tau)$$

In order to prevent forcing at resonant frequency, we force $2\omega_2 + \frac{1}{8} = 0$, or $\omega_2 = -\frac{1}{16}$. Thus $\tau = 1 - \frac{1}{16}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right)$. Then the period $T = \dfrac{2\pi}{\tau}$, i.e.

$$\boxed{T = \frac{2\pi}{1 - \frac{1}{16}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right)}}$$

$\square$

Problem 2

Find the first term approximation valid for long time to the initial value problem

$$\ddot{u} + \varepsilon\left(u^2 - 1\right)\dot{u} + u = 0$$
$$u(0) = 0, \qquad \dot{u}(0) = 1.$$

Proof. Let $\tau = \varepsilon t$ and $u(t) = v(t,\tau) = v_0(t,\tau) + \varepsilon v_1(t,\tau) + \mathcal{O}\left(\varepsilon^2\right)$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial v}{\partial t} + \varepsilon\frac{\partial v}{\partial \tau} \qquad \text{and} \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \frac{\partial^2 v}{\partial t^2} + 2\varepsilon\frac{\partial^2 v}{\partial t \partial \tau} + \varepsilon^2\frac{\partial^2 v}{\partial \tau^2}.$$

Then

$$\ddot{u} + \varepsilon(u^2 - 1)\dot{u} + u = 0$$
$$\Longrightarrow \frac{\partial^2 v_0}{\partial t^2} + \varepsilon\left(2\frac{\partial^2 v_0}{\partial t\partial\tau} + \frac{\partial^2 v_1}{\partial t^2}\right) + \mathcal{O}\left(\varepsilon^2\right) + \varepsilon\left(v_0^2 - 1\right)\frac{\partial v_0}{\partial t} + \mathcal{O}\left(\varepsilon^2\right) + v_0 + \varepsilon v_1 = 0.$$

Then the $\mathcal{O}(1)$ equation is a simple harmonic oscillator in t:

$$\frac{\partial^2 v_0}{\partial t^2} + v_0 = 0, \qquad (\mathcal{O}(1)\text{ equation})$$

which implies

$$v_0(t,\tau) = A(\tau)e^{it} + \overline{A}(\tau)e^{-it},$$
$$\frac{\partial v_0}{\partial t}(t,\tau) = i\left[A(\tau)e^{it} - \overline{A}(\tau)e^{-it}\right], \qquad \text{and}$$
$$\frac{\partial^2 v_0}{\partial t\partial\tau}(t,\tau) = i\left[A'(\tau)e^{it} - \overline{A}'(\tau)e^{-it}\right].$$

Then the $\mathcal{O}(\varepsilon)$ equation is a simple harmonic oscillator in t, as well as additional forcing terms determined by v_0:

$$\frac{\partial^2 v_1}{\partial t^2} + v_1 = -2\frac{\partial^2 v_0}{\partial t\partial\tau} - \left(v_0^2 - 1\right)\frac{\partial v_)}{\partial t}, \qquad (\mathcal{O}(\varepsilon)\text{ equation})$$

which implies

$$\frac{\partial^2 v_1}{\partial t^2} + v_1 = -iA^3(\tau)e^{3it} + i\left[2A'(\tau) - A^2(\tau)\overline{A}(\tau) + A(\tau)\right]e^{it} + i\overline{A}^3(\tau)e^{-3it} - i\left[2\overline{A}'(\tau) - \overline{A}^2(\tau)A(\tau) + \overline{A}(\tau)\right]e^{-it}.$$

In order to prevent resonant forcing terms (which are the e^{it} and e^{-it} terms), we require the following:

$$2A'(\tau) - A^2(\tau)\overline{A}(\tau) + A(\tau) = 0. \qquad \text{(dissonance requirement)}$$

To solve this, split $A(\tau)$ into it's magnitutde and argument:

$$A(\tau) = r(\tau)e^{i\theta(\tau)}$$

Then the (dissonance requirement) implies

$$\left(2r' + 2i\theta' r + r^3 - r\right)e^{i\theta} = 0$$
$$\Longrightarrow \begin{cases} 2r' + r^3 - r & = 0 \\ -2\theta' r & = 0 \end{cases}$$

Since $r = 0 \implies A = 0 \implies v_0 = 0 \implies u \approx 0$ is not what we are looking for, this means $\theta' = 0$, i.e. $\theta(\tau) = \theta_0 \in \mathbb{R}$. Also, $2r' + r^3 - r = 0$ implies

$$r(\tau) = \pm\sqrt{\frac{Ke^{\tau}}{1+Ke^{\tau}}}$$

for some $K \in Rl$. Since the magnitude is defined to be positive, we choose the positive branch for r. Then

$$A(\tau) = \sqrt{\frac{Ke^{\tau}}{1+Ke^{\tau}}}e^{i\theta_0},$$

which implies

$$u(t) \approx 2\sqrt{\frac{Ke^{\varepsilon t}}{1+Ke^{\varepsilon t}}}\cos(\theta_0 + t).$$

The initial condition $u(0) = 0$ implies $\theta_0 = \left(\frac{1+2k}{2}\right)\pi$ for any $k \in \mathbb{Z}$. Since $\cos\left(t \pm \frac{\pi}{2}\right) = \mp\sin(t)$, then

$$u(t) \approx \pm 2\sqrt{\frac{Ke^{\varepsilon t}}{1+Ke^{\varepsilon t}}}\sin(t),$$

andthe initial condition $\dot{u}(0) = 1$ subsequentally implies

$$1 = \pm 2\sqrt{\frac{K}{1+K}}, \qquad \text{and thus} \qquad K = \frac{1}{3}.$$

We choose the positive branch to match the initial condition, i.e. restricting the value k to odd integers. Finally, the full first-order solution is

$$\boxed{u(t) \approx 2\sqrt{\frac{e^{\varepsilon t}}{3+e^{\varepsilon t}}}\sin(t)}$$

$\square$