
Homework #1

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Problem 1

Let L be the linear operator $Lu = u_{xx}$, $u_x(0) = u_x(1) = 0$.

- (a) Find the eigenfunctions and corresponding eigenvalues of L .
 (b) Show that the eigenfunctions are orthogonal in the $L^2[0, 1]$ inner product

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx.$$

- (c) It can be shown that the eigenfunctions $\phi_j(x)$, form a complete set in $L^2[0, 1]$. This means that for any $f \in L^2[0, 1]$, $f(x) = \sum_j \alpha_j \phi_j(x)$. Express the solution to

$$u_{xx} = f, u_x(0) = u_x(1) = 0,$$

as a series solution of the eigenfunctions.

- (d) Note that this BVP does not have a solution for all f . Express the condition for existence of a solution in terms of the eigenfunctions of L .

- (a) Let $Lu = \lambda u$. Then

$$\begin{aligned} u_{xx} - \lambda u &= 0 \\ \Rightarrow u(x) &= A \exp[\sqrt{\lambda}x] + B \exp[-\sqrt{\lambda}x] \end{aligned}$$

If $\lambda > 0$ then $u_x(x) = \sqrt{\lambda} [A \exp[\sqrt{\lambda}x] - B \exp[-\sqrt{\lambda}x]]$ and the boundary condition $u_x(0) = 0$ implies $A = B$ and so $u_x(x) = A\sqrt{\lambda} [\exp[\sqrt{\lambda}x] - \exp[-\sqrt{\lambda}x]]$. Then the boundary condition $u_x(1) = 0$ implies $A = 0$, thus there are no solutions.

If $\lambda = 0$ then $u(x) = A + Bx$ and the boundary conditions implies $B = 0$. Thus $u(x) = A$ where A is a constant is an eigenfunction.

If $\lambda < 0$ then $u(x) = A \sin(\sqrt{-\lambda}x) + B \cos(\sqrt{-\lambda}x)$ and thus $u_x(x) = \sqrt{-\lambda} [A \cos(\sqrt{-\lambda}x) - B \sin(\sqrt{-\lambda}x)]$. Then the boundary conditions imply $A = 0$ and $\sqrt{-\lambda} = k\pi$ for $k \in \mathbb{N}$. Thus the eigenfunctions are

$$u_k(x) = \cos(k\pi x) \quad \text{for } k = 0, 1, 2, \dots \text{ with corresponding eigenvalues } \lambda_k = -k^2\pi^2$$

- (b) Assume $k \neq j$. Then

$$\begin{aligned} \langle \cos(k\pi x), \cos(j\pi x) \rangle &= \int_0^1 \cos(k\pi x) \cos(j\pi x) dx \\ &= \frac{1}{2} \int_0^1 \cos((k+j)\pi x) + \cos((k-j)\pi x) dx \\ &= \frac{1}{2} \left[\frac{\sin((k+j)\pi x)}{(k+j)\pi} \Big|_0^1 + \frac{\sin((k-j)\pi x)}{(k-j)\pi} \Big|_0^1 \right] \\ &= \frac{1}{2(k+j)\pi} [\sin((k+j)\pi)] + \frac{1}{2(k-j)\pi} [\sin((k-j)\pi)] \end{aligned}$$

Since $k \neq j$ then $k+j$ and $k-j$ are nonzero integers and thus $\langle u_k, u_j \rangle = 0$. If $k = j$, then

$$\langle \cos(k\pi x), \cos(k\pi x) \rangle = \int_0^1 \cos^2(k\pi x) dx = \frac{1}{2} \int_0^1 1 + \cos(2k\pi x) dx = \frac{1}{2}$$

Thus $\sqrt{2}u_k$ are orthonormal.

(c) Let $u = \sum_j \alpha_j \cos(j\pi x)$, and let $f = \sum_j \beta_j \cos(j\pi x)$. Then $u_{xx} = -\sum_j \alpha_j j^2 \pi^2 \cos(j\pi x)$. Then

$$-\sum_j \alpha_j j^2 \pi^2 \cos(j\pi x) = \sum_j \beta_j \cos(j\pi x)$$

Since $\cos(j\pi x)$ are orthonormal, each term in the series must match, and thus

$$\alpha_j = -\frac{\beta_j}{j^2 \pi^2}$$

So, the solution to $Lu = f$ where $f = \sum_j \beta_j \cos(j\pi x)$ is

$$u(x) = -\sum_j \frac{\beta_j}{j^2 \pi^2} \cos(j\pi x)$$

(d) Since L is self-adjoint, then $Lu = f$ is solvable if $f \perp \ker L$ where $\ker L = [1]$, that is, the kernel of L is spanned by the constant function 1. $f \perp \ker L$ if

$$\langle f, 1 \rangle = \int_0^1 f(x) dx = 0$$

that is, the mean of $f(x)$ is zero. In terms of the eigenfunctions,

$$\langle f, \cos(0\pi x) \rangle = 0.$$

Problem 2

Define the functional $F : X \rightarrow \mathbb{R}$ by

$$F(u) = \int_0^1 \frac{1}{2} (u_x)^2 + f u dx,$$

where X is the space of real-valued functions on $[0, 1]$ that have at least one continuous derivative and are zero at $x = 0$ and $x = 1$. The Frechet derivative of F at a point u is defined to be the linear operator $F'(u)$ for which

$$F(u + v) = F(u) + F'(u)v + R(v),$$

where

$$\lim_{\|v\| \rightarrow 0} \frac{\|R(v)\|}{\|v\|} = 0.$$

One way to compute the derivative is

$$F'(u)v = \lim_{\varepsilon \rightarrow 0} \frac{F(u + \varepsilon v) - F(u)}{\varepsilon}.$$

Note that this looks just like a directional derivative.

- (a) Compute the Frechet derivative of F .
- (b) $u \in X$ is a critical point of F if $F'(u)v = 0$ for all $v \in X$. Show that if u is a solution to the Poisson equation $u_{xx} = f$, $u(0) = u(1) = 0$, then it is a critical point of F .
- (c) Let X_h be a finite dimensional subspace of X , and let $\{\phi_i(x)\}$ be a basis for X_h . This means that all $u_h \in X_h$ can be expressed as $u_h(x) = \sum_i u_i \phi_i(x)$ for some constants u_i . Thus we can identify the elements of X_h with vectors \vec{u} that have components u_i . Let $G(\vec{u}) = F(u_h)$. Show that the gradient of G (whos components are $(\nabla G)_j = \frac{\partial G}{\partial u_j}$) is of the form $\nabla G(\vec{u}) = A\vec{u} + \vec{b}$, and write expressions for the elements of the matrix A and the vector \vec{b} .
- (d) Divide the unit interval into a set of $N + 1$ equal length intervals $I_i = (x_i, x_{i+1})$ for $i = 0, \dots, N$. The endpoints of the intervals are $x_i = ih$, where $h = \frac{1}{N+1}$. Let X_h be the subspace of X such that the elements u_h of X_h are linear on each interval, continuous on $[0, 1]$, and satisfy $u_h(0) = u_h(1) = 0$. X_h is an N dimensional space with basis elements

$$\phi_i(x) = \begin{cases} 1 - h^{-1}|x - x_i| & \text{if } |x - x_i| < h, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \dots, N$. Compute the matrix A from the previous problem that appears in the gradient.

Finite element methods are based on these “weak formulations” of the problem. The Ritz method is based on minimizing F and the Galerkin method is based on finding the critical points of $F'(u)$.

(a)

$$\begin{aligned} F'(u)v &= \lim_{\varepsilon \rightarrow 0} \frac{F(u + \varepsilon v) - F(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\int_0^1 \frac{1}{2} (u_x + \varepsilon v_x)^2 + f(u + \varepsilon v) - \frac{1}{2} u_x^2 - f u dx}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^1 u_x v_x + \frac{1}{2} \varepsilon v_x^2 + f v dx \\ &= \int_0^1 u_x v_x + f v dx \\ &= [v u_x]_0^1 + \int_0^1 v(f - u_{xx}) dx \end{aligned}$$

But $v \in X$, and thus $v(0) = v(1) = 0$. So, for all v ,

$$F'(u)v = \int_0^1 v(f - u_{xx})dx$$

(b) Let u be a solution to the Poisson equation. Then $u_{xx} = f$. Then $u_{xx} - f = 0$. Thus,

$$F'(u)v = \int_0^1 v(f - u_{xx})dx = \int_0^1 v \cdot 0 dx = 0$$

(c)

$$\begin{aligned} G(\vec{u}) &= F(u_h) = \int_0^1 \frac{1}{2} \left(\sum_{i=1}^n u_i \phi'_i(x) \right)^2 + f(x) \sum_{i=1}^n u_i \phi_i(x) dx \\ \Rightarrow (\nabla G)_j &= \frac{\partial}{\partial u_j} G(\vec{u}) = \int_0^1 \left(\sum_{i=1}^n u_i \phi_i(x) \right) \phi'_j(x) + f(x) \phi_j(x) dx \\ &= \left[\phi_j \sum_{i=1}^n u_i \phi'_i \right]_0^1 + \int_0^1 \phi_j(x) \left(f(x) - \sum_{i=1}^n u_i \phi''_i(x) \right) dx \end{aligned}$$

But the boundary conditions are 0 since $\phi_j \in X_h \subset X$ and $u(0) = u(1) = 0$ for all $u \in X$. Thus,

$$\begin{aligned} (\nabla G)_j &= - \int_0^1 \left(\sum_{i=1}^n u_i \phi''_i(x) \right) \phi_j(x) dx + \int_0^1 f(x) \phi_j(x) dx \\ &= A\vec{u} + \vec{b} \end{aligned}$$

where $\vec{b} = \begin{pmatrix} \int_0^1 f(x) \phi_1(x) dx \\ \int_0^1 f(x) \phi_2(x) dx \\ \vdots \\ \int_0^1 f(x) \phi_n(x) dx \end{pmatrix}$ and $A = (a_{ij})$ where $a_{ij} = - \int_0^1 \phi_i(x) \phi''_j(x) dx$.

(d) Note that

$$\phi_i(x) = \begin{cases} 1 - h^{-1}|x - x_i| & \text{if } |x - x_i| < h, \\ 0 & \text{otherwise} \end{cases}$$

and thus

$$\phi'_i(x) = \begin{cases} \frac{1}{h} & \text{if } x_{i-1} < x < x_i \\ -\frac{1}{h} & \text{if } x_i < x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \phi''_i(x) = \delta\left(x - \frac{x_i + x_{i+1}}{2}\right)$$

This shows

$$a_{ij} = - \int_0^1 \phi_i(x) \delta\left(x - \frac{x_j + x_{j+1}}{2}\right) dx = -\phi_i\left(\frac{x_j + x_{j+1}}{2}\right) = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and so $A = -I$ where I is the $n \times n$ identity matrix.

Problem 3

- (a) Using a Taylor expansion, derive the finite difference formula to approximate the second derivative at x using function values at $x - \frac{h}{2}$, x , and $x + h$. How accurate is the finite difference approximation?
- (b) Perform a refinement study to verify the accuracy of the difference formula you derived.
- (c) Derive an expression for the quadratic polynomial that interpolates the data $\left(x - \frac{h}{2}, u\left(x - \frac{h}{2}\right)\right)$, $(x, u(x))$, and $(x + h, u(x + h))$. How is the finite difference formula you derived in problem 3a related to the interpolating polynomial?

- (a) Let D^2 be a finite difference formula for the second derivative. Then

$$(D^2 u)_j = au\left(x - \frac{h}{2}\right) + bu(x) + cu(x+h) \quad (0.1)$$

$$= (a+b+c)u(x) + \left(c - \frac{a}{2}\right)hu'(x) + \left(\frac{a}{8} + \frac{c}{2}\right)h^2u''(x) + \left(\frac{c}{6} - \frac{a}{48}\right)h^3u'''(x) + \dots \quad (0.2)$$

The letting $a+b+c=0$, $c=\frac{a}{2}$ and $a=\frac{8}{3h^2}$, we get

$$(D^2 u)_j = u''(x) + \frac{h}{6}u'''(x) + \dots \quad (0.3)$$

where $a=\frac{8}{3h^2}$, $b=-\frac{4}{h^2}$, and $c=\frac{4}{3h^2}$. Thus

$$(D^2 u)_j = \frac{8}{3h^2}u\left(x - \frac{h}{2}\right) - \frac{4}{h^2}u(x) + \frac{4}{3h^2}u(x+h) \quad (0.4)$$

not done...

- (b)

- (c) The polynomial through the points

$$(x, u(x)) \quad (x+h, u(x+h)) \quad \left(x - \frac{h}{2}, u\left(x - \frac{h}{2}\right)\right) \quad (0.5)$$

satisfies

$$u(x) = A + Bx + Cx^2 \quad (0.6)$$

$$u(x+h) = A + B(x+h) + C(x+h)^2 \quad (0.7)$$

$$u\left(x - \frac{h}{2}\right) = A + B\left(x - \frac{h}{2}\right) + C\left(x - \frac{h}{2}\right)^2 \quad (0.8)$$

This is a simple linear algebra problem that can be solved symbolically using a symbolic solver like Maple or Python. We get:

$$A = \frac{1}{3h^2} \left(3h^2 u(x) - 3hu(x)x - hu(x+h)x + 4hu\left(x - \frac{h}{2}\right)x - 6u(x)x^2 + 2u(x+h)x^2 + 4u\left(x - \frac{h}{2}\right)x^2 \right) \quad (0.9)$$

$$B = \frac{1}{3h^2} \left(3hu(x) + hu(x+h) - 4hu\left(x - \frac{h}{2}\right) + 12u(x)x - 4u(x+h)x - 8u\left(x - \frac{h}{2}\right)x \right) \quad (0.10)$$

$$C = \frac{1}{3h^2} \left(-6u(x) + 2u(x+h) + 4u\left(x - \frac{h}{2}\right) \right) \quad (0.11)$$