MAT 228A Notes

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1 Iterative Methods for the Poisson Equation

Linear fixed-point iteration problem: $u_{k+1} = Tu_k + c$. When should we stop the iteration? There are two standard ways to figure this out:

1. Stop based on the size of the residual, $r_k = f - Au_k$
2. Stop based on the size of $u_{k+1} - u_k$

Use the residual

- absolute tolerance $\|r_k\| < \text{tol} =: \varepsilon$
- relative tolerance $\|r_k\| \leq \text{tol}\|f\|$

Relative tolerance is better.. if $u_0 = 0$, $r_0 = f$.

Reminder: residual equation: $Ae_k = r_k$ where e_k is the algebraic error on the kth iterate. We want to control e_k,

$$\|e_k\| = \left\|A^{-1}r_k\right\| \leq \left\|A^{-1}\right\|\|r_k\| \leq \left\|A^{-1}\right\|\varepsilon$$

$\varepsilon = Ch^2$... use $u_{k+1} - u_k$

- Abslute: $\|u_{k+1} - u_k\| < \text{tol}$
- Relative: $\|u_{k+1} - u_k\| < \text{tol}\|u_k\|$

2 Jac, GS, SOR, Multigrid

$$\begin{aligned} u_{k+1} &= u_k + Br_k, \qquad B \approx A^{-1} \\ u_{k+1} - u_k &= Br_k \end{aligned}$$

Want to control the error - $\|e_k\| = \left\|A^{-1}r_k\right\| = \left\|A^{-1}B^{-1}(u_{k+1} - u_k\right\| \leq \left\|A^{-1}B^{-1}\right\|\|u_{k+1} - u_k\|$.

2.1 Jacobi

$B = -\frac{h^2}{4}I$, which implies $\left\|B^{-1}\right\| = \frac{4}{h^2}$.

2.2 Ordering of Unknowns

Jacobi does depend on the ordering of unknowns. GS does. We did GS-Lex, which is a sweep through of the unknowns - gives some sort of diagonal structure (with streaks at nth, n^2th, etc. super- and sub-diagonals). Another way to order them is label them red or blue (even/odd) so $i + j$ is either even or odd. Then lexicographic ordering for red points and THEN for blue points. This is called GS-RB. Update red points first, and then blue points.

2.2.1 GS-Lex Pseudocode

- loop k
  - loops i,j
    - $u_{ij} = \frac{1}{4}\left(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - h^2F_{ij}\right)$

2.2.2 GS-RB Pseudocode

- loop k
  - loop red
    - $u_{ij} = \frac{1}{4}\big(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - h^2 F_{ij}\big)$
  - loop blue
    - $u_{ij} = \frac{1}{4}\big(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - h^2 F_{ij}\big)$

2.2.3 Other variations

Block or line relaxation methods - update groups of points at once. For example, in 2D, each row is a group - do a 1D solve on each row of points. This is useful for problems such as

$$u_{xx} + \varepsilon u_{yy} = f$$

3 Successive Over Relaxation (SOR) Method

This is a generalization of GS by including a relaxation parameter.

3.1 GS

- $u_{ij} = \dfrac{1}{4}\big(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - h^2 F_{ij}\big)$

3.2 SOR

- $u_{ij} = \dfrac{\omega}{4}\big(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - h^2 F_{ij}\big) + (1-\omega)u_{ij}$

Choosing $\omega < 1$ is “under-relaxation” and $\omega > 1$ is “over relaxation”. SOR requires $\omega \in (0, 2)$ for convergence.

$$A = M - N$$

where $Mu_{k+1} = Nu_k + F$ and so $u_{k+1}M^{-1}Nu_k + M^{-1}F$. Also,

$$M = \frac{1}{\omega}D - L \qquad \text{and} \qquad N = \frac{1-\omega}{\omega}D + U$$

$$\begin{aligned}
T_{\text{SOR}} = M^{-1}N = \left(\frac{1}{\omega}D - L\right)^{-1}\left(\left(\frac{1-\omega}{\omega}\right)D + U\right) &= (D - \omega L)^{-1}((1-\omega)D + \omega U) \\
\implies \det(T_{\text{SOR}}) = \det\big((D-\omega L)^{-1}\big)\det((1-\omega)D + \omega U) = \frac{\det(1-\omega)D}{\det(D)} &= (1-\omega)^N
\end{aligned}$$

where N is the number of grid points. Therefore we require $|1-\omega| < 1$ (i.e. $\omega \in (0, 2)$).

3.2.1 Convergence analysis

Use the same trick to compute the eigenvalues of the update matrix in terms of the eigenvalues of the Jacobi update. So let μ be an eigenvalue of the Jacobi update. Then

$$\mu = \frac{\lambda + \omega - 1}{\omega \lambda^{\frac{1}{2}}}$$

Rearrange this equation to get

$$\begin{aligned}
\lambda - \omega\mu\lambda^{\frac{1}{2}} + (\omega - 1) &= 0 \\
\implies 2\lambda^{\frac{1}{2}} &= \omega\mu \pm \big(\omega^2\mu^2 - 4(\omega-1)\big)^{\frac{1}{2}}
\end{aligned}$$

As $\omega \to 0$, we see $\lambda \to 1$. As $\omega \to 2$, we have $(4\mu^2 - 4)^{\frac{1}{2}}$ where $\mu < 1$ and thus we have complex eigenvalues. But if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then we have an explicit formula for the modulus of the eigenvalues:

$$\left|\lambda^{\frac{1}{2}}\right| = |\omega - 1|$$

and so we decrease ω for better convergence. If $\lambda \in \mathbb{R}$, then

$$\frac{\partial \lambda^{\frac{1}{2}}}{\partial \omega} < 0$$

which takes some work, but increasing ω gives better convergence. So the optimal ω satisfies

$$(\omega^*)^2 \mu^2 - 4(\omega^* - 1) = 0 \tag{1}$$

$$\implies \omega^* = \frac{2}{1 + (1 - \rho_J^2)^{\frac{1}{2}}} \tag{2}$$

where ρ_J is the spectral radius of the Jacobi update matrix, so $\rho_{\text{SOR}} = 1 - \omega^*$.

$$\rho_J = \cos(\pi h) \tag{3}$$

$$w^* = \frac{2}{1 + (1 - \cos^2(\pi h))^{\displaystyle\frac{1}{2}}} = \frac{2}{1 + \sin(\pi h)} = 2(1 - \pi h) + \mathcal{O}\big(h^2\big) \tag{4}$$

so

$$\rho_{\text{SOR}} = 1 - 2\pi h \tag{5}$$

3.3 Number of iterations to reduce error by 10^{-1}

$N \times N$	GS	SOR
32×32	254	12
64×64	985	24
128×128	3882	47
256×256	15404	94

(6)