MAT 228A Notes

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November 17, 2016

1 GS-RB, Full Weighting, Bilinear Interpolation

$\nu =$ total # of smoothing steps, $\nu = \nu_1 + \nu_2$.

ν	ρ
1	0.25
2	0.74
3	0.53
4	0.41

Want to solve to a tolerance, set $\rho^k = \varepsilon$, so $k = {}^{\log\varepsilon}/_{\log\rho}$. That is,

$$k \propto \frac{-1}{\log\rho}$$

1.1 Work

What is the work per iteration? Set WU be the work unit.

$$WU = \underbrace{\nu}_{\mathcal{O}(N),\text{ work from smoothing}} + \underbrace{w}_{\text{work from everything else}}$$

So the total work is proportional to ${}^{-(\nu+w)}/_{\log\rho}$. Assuming w, we get

ν	$WU;\ w=0$	$WU;\ w=1$	$\ldots$	$WU;\ w=5$	$WU;\ w=6$
1	$\boxed{1.66}$	3.32	$\ldots$	9.97	11.63
2	1.77	$\boxed{2.65}$	$\ldots$	$\boxed{6.19}$	7.07
3	2.35	3.14	$\ldots$	6.27	$\boxed{7.05}$
4	2.88	3.60	$\ldots$	6.49	7.21

1.2 How to pick ν_1 and ν_2

Let M be the MG iteration operator. Let $e^1 = Me^0$. So

$$\left\|e^1\right\|_2 \leq \|M\|_2\left\|e^0\right\|_2 \qquad \Longrightarrow \qquad \frac{\left\|e^1\right\|_2}{\left\|e^0\right\|_2} \leq \|M\|_2$$

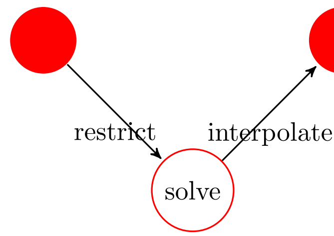
We get

ν_1	ν_2	$\|M\|_2$
1	0	0.559
0	1	1.414
2	0	0.200
1	1	0.141
0	2	1.414
3	0	0.137
2	1	0.081
1	2	0.081
0	3	1.414

A common choice is $(\nu_1, \nu_2) = (1, 1)$ or $(\nu_1, \nu_2) = (2, 1)$.

1.3 Multigrid States

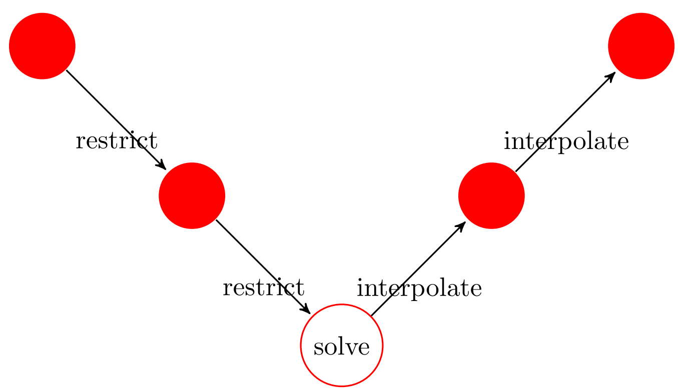
1.3.1 2-Grid



1.3.2 3-Grid

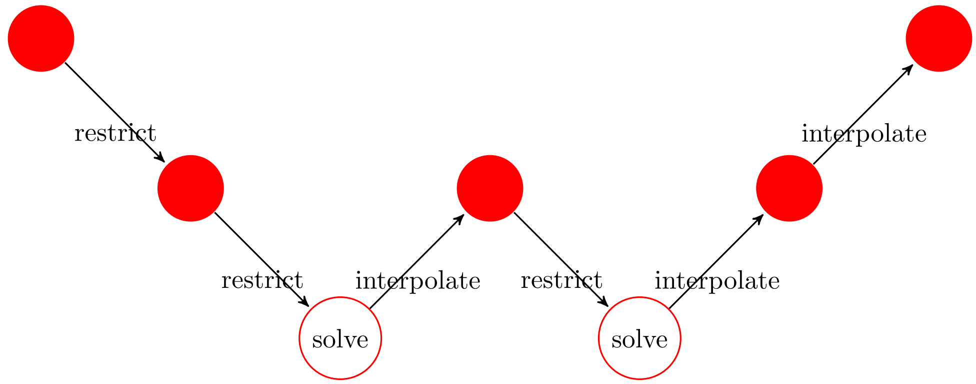
MG Iteration

- Smooth ν_1 times
- Compute r_h
  - restrict to f_{2h}
  - solve $L_{2h}e_{2h} = r_{2h}$
  - smooth $L_{2h}u_{2h} = f_{2h}$, ν_1 times, initial guess $e_{2h} \approx 0$.
  - compute r_{2h}
    - Restrict to f_{4h}
    - SOlve $L_{4h}u_{4h} = f_{4h}$
  - iterpolate and correct $u_{2h} = u_{2h} + I_{4h}^{2h}u_{4h}$
  - smooth ν_2 times
- interpolate and correct $u_h = u_h + I_{2h}^{h}u_{2h}$
- Smooth ν_2 times



1.3.3 Other forms of cycles

Do γ iteration before returning to fine grid. $\gamma = 1$ is called V-cycle, $\gamma = 2$ for three grids is called W-cycle. $\gamma = 2$ for 3 levels looks like



$\gamma = 2$ for 4 levels looks like a fractal.

1.3.4 How much work is there in a V-cycle?

On the fine mesh, smooth, ν times is $\mathcal{O}(N)$. We also have to compute the residual, $r_h = \mathcal{O}(N)$. We also have to restrict, $\mathcal{O}(N)$. We also have to correct $\mathcal{O}(N)$ and interpolate $\mathcal{O}(N)$.

In 2D, the work on a fine mesh is CN. Every time we drop a level is dropping by a factor of 4..

level ℓ	work
$\ell = 1$	CN
$\ell = 2$	$\frac{CN}{4}$
$\ell = 3$	$\frac{CN}{4^2}$
$\vdots$	$\vdots$

The total work in the limit of many levels is

$$\sum_{\ell=1}^{L} CN\left(\frac{1}{4}\right)^{\ell-1} \approx CN\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{4}{3}CN$$

So V-cycle work is not much different than work on the fine level.

1.4 GS-RB, FM, bilinear interpolation, 2D

For $\nu = 2$, 2-grid gives $\rho \approx 0.074$. V-cycle gives $\rho \approx 0.10$.

1.5 Comparisons to SOR

Say we have 64×64 mesh with tolerance of 10^{-6}.

	# iterations	work per iteration	total work
SOR	144	1	144
MG	6	$4/3 \cdot (2+4) = 8$	48

Say we have 265×265 mesh with tolerance of 10^{-7}.

	# iterations	work per iteration	total work
SOR	658	1	658
MG	7	$4/3 \cdot (2+4) = 8$	56