MAT 228A Notes

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1 Using Finite Difference Methods to Solve Poisson Equations (1D)

Suppose $u_{xx} = f$, $u(0) = \alpha$, $u(1) = \beta$. Let's discretize the domain ($[0, 1]$) by choosing equally spaced points. Let $x_j = jh$ where h is the spacing between points. So $0 = x_0$ and let $1 = x_{N+1}$ so that the interior points are $x_1, \ldots, x_N$. So $h = \frac{1}{N+1}$.

Notation: use $u_j \approx u(x_j)$.

Next we replace the Laplacian $\frac{\partial^2}{\partial x^2}$ with a difference operator D.

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = f_j \coloneqq f(x_j) \tag{1}$$

This is a linear algebra problem with N unknowns. Collect them into a vector $\underline{u}$:

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \tag{2}$$

The above equations are of the form $A\underline{u} = \underline{b}$. What are A and $\underline{b}$?

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} f_1 - \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N - \frac{\beta}{h^2} \end{pmatrix} \tag{3}$$

So A is the above tri-diagonal matrix and $\underline{b}$ is

$$\underline{b} = \begin{pmatrix} f_1 - \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N - \frac{\beta}{h^2} \end{pmatrix}. \tag{4}$$

What if

$$\underline{u} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \\ u_{N+1} \end{pmatrix}? \tag{5}$$

Then

$$A = \frac{1}{h^2}\begin{pmatrix} h^2 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & h^2 \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} \alpha \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \\ \beta \end{pmatrix} \tag{6}$$

1.1 Errors

Anyway, how close is u_j to $u(x_j)$ (which is the solution to the PDE)? The error at a point is

$$e_j^h = u_j^h - u(x_j) \tag{7}$$

which can be put into a vector

$$\underline{e}^h = \underline{u}^h - \underline{u}_{\text{sol}} \tag{8}$$

where

$$(\underline{u}_{\text{sol}})_j = u(x_j). \tag{9}$$

We would like

$$\left\|\underline{e}^h\right\| \to 0 \qquad \text{as} \qquad h \to 0 \tag{10}$$

where $\|\cdot\|$ is the appropriate norm. If the error goes to 0 as the mesh spacing h goes to zero, the method (numerical scheme) is called a "convergent scheme". What norm should we use to measure the errors?

1.2 Vector, Matrix, and Function norms

Let $\underline{x} \in \mathbb{R}^n$. Then

$$\|\underline{x}\|_2^2 = \sum_{j=1}^{n} x_j^2 \qquad \|\underline{x}\|_1 = \sum_{j=1}^{n} |x_j| \qquad \|\underline{x}\|_\infty = \max_{j=1,\ldots,n} |x_j| \tag{11}$$

Let $u : (a,b) \to \mathbb{R}$. Then

$$\|u\|_2^2 = \int_a^b |u(x)|^2 \mathrm{d}x \qquad \|u\|_1 = \int_a^b |u(x)| \mathrm{d}x \qquad \|u\|_\infty = \operatorname*{ess\,sup}_{x \in (a,b)} |u(x)| \tag{12}$$

Example: $u(x) = 1$ on $(0,1)$

$$\|u\|_2 = \|u\|_1 = \|u\|_\infty = 1 \tag{13}$$

But if we sample on the mesh, where $(\underline{u})_j = u(x_j)$ is a vector of 1's, we have

$$\|\underline{u}\|_2 = \sqrt{N} \qquad \|\underline{u}\|_1 = N \qquad \|\underline{u}\|_\infty = 1 \tag{14}$$

So we should discretize the function norms, rather than using vector norms on finite-dimensional spaces. How?

1.2.1 Discretized Function norms

$\underline{e}_h$ is a grid function. Define

$$\|\underline{e}_h\|_2^2 = h \sum_{j=1}^{N} e_j^2 \tag{15}$$

Similarly,

$$\|\underline{e}_h\|_1 = h \sum_{j=1}^{N} |e_j| \tag{16}$$

The norm is mesh-dependent. The tricky thing here is the notation of the norm. $\|\cdot\|_2$ denotes the specific norm on the specific mesh space created.

1.2.2 Induced Matrix Norms

Let A be some matrix, $A : \mathbb{R}^n \to \mathbb{R}^n$ a linear operator on $\mathbb{R}^n$. Then the definition of the operator norm is

$$\|A\| := \sup_{\|x\|=1} \|Ax\| \tag{17}$$

Supposing $\mathbb{R}^n$ is equipped with $\|\cdot\|_\infty$, then define

$$\|A\|_\infty = \max_i \qquad \underbrace{\sum_{j=1}^{n} |a_{ij}|}_{\text{sum over the } i^{\text{th}} \text{ row}} \qquad \text{(max row sum)} \tag{18}$$

$$\tag{19}$$

If $\mathbb{R}^n$ is equipped with $\|\cdot\|_1$, then

$$\|A\|_1 = \max_j \qquad \underbrace{\sum_{i=1}^{n} |a_{ij}|}_{\text{sum over the } j^{\text{th}} \text{ column}} \qquad \text{(max column sum)} \tag{20}$$

$$\tag{21}$$

If $\mathbb{R}^n$ is equipped with $\|\cdot\|_2$, then

$$\|A\|_2 = \sqrt{\rho(A^*A)} \qquad \text{(largest singular value)} \tag{22}$$

where ρ denotes the spectal radius and A^* is the conjugate transpose of A. The spectral radius is defined as the modulus of the largest eigenvalue.

These are proved in Hunter-Nachtergaele. For example, to prove $\|A\|_\infty$ is the maximum row sum, first we prove boundedness of $\|\cdot\|_\infty$.

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{\|x\|=1} \max_i \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \max_{\|x\|=1} \max_i \sum_{j=1}^{n} |a_{ij}||x_j| \leq \max_i \sum_{j=1}^{n} |a_{ij}| \tag{23}$$

Then we acheive $\|\cdot\|_\infty$. Let I be the index of the row where $\|A\|_\infty$ is maximized. Then

$$\|A\|_\infty = \sum_{j=1}^{n} |a_{Ij}| \tag{24}$$

So picking $(x)_j = \text{sgn}(a_{Ij})$ yields equality.

In general, for any induced matrix norm,

$$\frac{\|Ax\|}{\|x\|} \leq \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} =: \|A\| \tag{25}$$

So by the definition of the matrix norm, we have the inequality

$$\|Ax\| \leq \|A\|\|x\|. \tag{26}$$