MAT 228A Notes

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1 Last Time

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \tag{1}$$

Compute the inverse of A, $A^{-1} = B$:

$$B_{ij} = \begin{cases} h(x_j - 1)x_i & \text{for } i \leq j \\ h(x_i - 1)x_j & \text{for } i > j \end{cases} \tag{2}$$

2 Error analysis

$$A\vec{e} = -\vec{\tau} \tag{3}$$

$$\vec{e} = -A^{-1}\vec{\tau} = -B\vec{\tau} \tag{4}$$

How does error at a point influence overall error?

$$B\tau = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} = \sum_{j=1}^{n} \vec{b}_j \tau_j \tag{5}$$

How much does truncation at a point affect the error?

2.1 Interior point

Consider an interior point. Graphing B_{ij} for fixed j gives a downward facing triangle (discrete Green's function). The tip, b_{jj} is the largest element. We know $b_{jj} = \mathcal{O}(h)$. We also know $\tau_j = \mathcal{O}\big(h^2\big)$ and so each term in $\sum_{j=1}^{n} \vec{b}_j \tau_j$ is $\mathcal{O}\big(h^3\big)$. But there are $\mathcal{O}\big(\frac{1}{h}\big)$ terms in the sum.

2.2 Points near a boundary

What about near a boundary? Let $\vec{b}_1$ be the biggest element.

$$B_1 = h(x_1 - 1)x_1 = h(h-1)h = h^2(h-1) = \mathcal{O}\big(h^2\big) \tag{6}$$

So,

$$\vec{b}_1\tau_1 = \mathcal{O}\big(h^4\big) \tag{7}$$

which is intuitive since there is no error on the boundary, so there is smaller error near the boundary.

3 Neumann Boundary Conditions

Two questions:

1. How to discretize
2. How to solve the linear system

3.1 Left Boundary (Right boundary is analagous)

$x_0 = 0$, $x_1 = h$, $x_2 = 2h$, and so on. Also pin down

$$u_x(0) = g \tag{8}$$

We have

$$\frac{1}{h^2}(u_0 - 2u_1 + u_2) = f \tag{9}$$

We should discretize the boundary condition:

$$\frac{1}{h}(u_1 - u_0) = g \tag{10}$$

Load them up in a matrix:

$$\begin{pmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{11}$$

and

$$\vec{u} = \begin{pmatrix} g \\ f_1 \\ \vdots \end{pmatrix} \tag{12}$$

We can solve the first equation $u_0 = u_1 - hg$ and then plug it in to the equation at the first ineriour park. So,

$$\frac{1}{h^2}(u_1 - hg - u_1 = u_2) = f_1 \tag{13}$$

$$\implies \frac{1}{h^2}(u_2 - u_1) = f_1 + \frac{g}{h} \tag{14}$$

So,

$$A = \frac{1}{h^2}\begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix} \tag{15}$$

$$\vec{u} = \begin{pmatrix} f_1 + \frac{g}{h} \\ f_2 \\ \vdots \end{pmatrix} \tag{16}$$

But this is only first order accurate. Another discretization:

$$\frac{u_1 - u_0}{h} = u_x(0) + \mathcal{O}(h^2) \tag{17}$$

and

$$\frac{u_1 - u_0}{h} = u_x\left(\frac{h}{2}\right) + \mathcal{O}(h^2) \tag{18}$$

Imagine we are extending the domain: ghost point $x_{-1} = -h$. Our equation at $x = 0$ is

$$u_{-1} - 2u_0 + u_1 h^2 = f_0 \tag{19}$$

Discretize about x_0.

$$\frac{u_1 - u_{-1}}{2h} = g \tag{20}$$

which implies

$$u_{-1} = u_1 - 2hg \tag{21}$$

this is a way to extrapolate from the interior. So we get

$$\frac{u_1 - 2hg - 2u_0 + u_1}{h^2} = f_0 \tag{22}$$

$$\frac{-2u_0 + 2u_1}{h_2} = f_0 + \frac{2g}{h} \tag{23}$$

So the second order method is

$$\frac{1}{h^2}\begin{pmatrix} -2 & 2 & & \\ 1 & -2 & 1 & \\ & & \ddots & \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 + \frac{2g}{h} \\ f_1 \\ \vdots \end{pmatrix} \tag{24}$$

But this is not symmetric, so

$$\frac{1}{h^2}\begin{pmatrix} -1 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{2}f_0 + \frac{g}{h} \\ f_1 \\ \vdots \end{pmatrix} \tag{25}$$

In the homework we will get at a finite volume discretization. Very natural with Neumann boundary problems.

3.2 Solvability

$$u_{xx} = f \qquad x \in (0,1) \qquad u_x(0) = \alpha \qquad u_x(1) = \beta \tag{26}$$

Physically, there must be a constraint on f, α, and β. They have to have some steady balance of some sort. We integrate this equation:

$$\int_0^1 u_{xx} \mathrm{d}x = \int_0^1 f(x)\mathrm{d} \implies \boxed{\beta - \alpha = \int_0^1 f(x)\mathrm{d}x} \qquad \text{by the Fundamental Theorem of Calculus} \tag{27}$$

This is a necessary condition for a solution to exist for the problem.

Supposing u is a solution to the problem. Then $u + C$ for any constant is also a solution. This is because u_x and u_{xx} are the same as $(u+C)_x$ and $(u+C)_{xx}$.

Let's discretize this:

$$\frac{1}{h^2}\begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} f_0 + \frac{2\alpha}{n} \\ f_1 \\ \vdots \\ f_n \\ f_{n+1} - \frac{2\beta}{n} \end{pmatrix} \tag{28}$$

It turns out this matrix is singular. Notice if $\vec{u}$ is a solution. Then $\vec{u} + \vec{c} = \vec{u} + c\vec{1}$ is also a solution. It turns out $\vec{1}$ spans the null space, i.e. $\vec{1}$ is the eigenvector corresponding to the eigenvalue 0. $A\vec{1} = \vec{0}$.

$$A\vec{u} = \vec{b} \tag{29}$$

has a solution if $\vec{b} \in \mathrm{ran}(A)$, i.e. $\vec{b} \perp \ker(A^*)$. For matrices, if A is $n \times n$, adn $\vec{b} \in \mathbb{R}^n$. Then

$$\vec{b} = \vec{b}_r + \vec{b}_0 \tag{30}$$

where $\vec{b}_r \in \mathrm{ran}(A)$ and $\vec{b}_0 \in \ker(A^*)$. Furthermore, $\vec{b}_r \cdot \vec{b}_0 = 0$. To guarantee $\vec{b} \in \mathrm{ran}(A)$, just show $\vec{b} \perp \ker(A^*)$.