MAT 228A Notes

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1 Introduction

In MAT 228 we will study standard model PDEs, namely:

1. The Avection Equation (the typical hyperbolic equation)

$$u_t + cu_x = 0 \tag{1}$$

2. The Diffusion (Heat) Equation (the typical parabolic equation)

$$u_t = Du_{xx} \tag{2}$$

3. The Poisson Equation (the typical elliptic equation)

$$u_{xx} = F \tag{3}$$

2 What we will not focus on in 228A

2.1 The Avection Equation

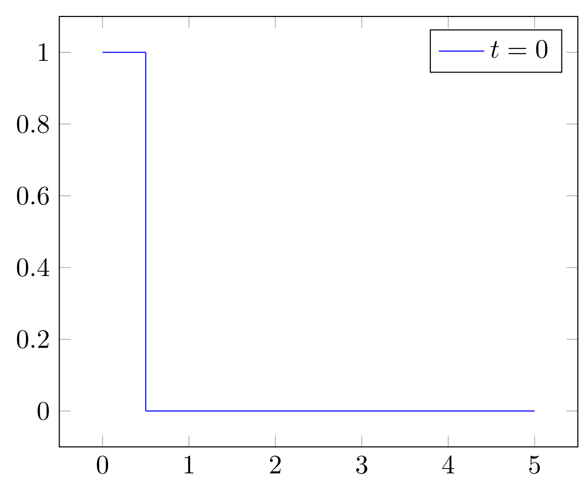
Set $c \equiv 1$, and the domain $x > 0$. Let the boundary condition be $u(0,t) = 1$ and the Initial condition

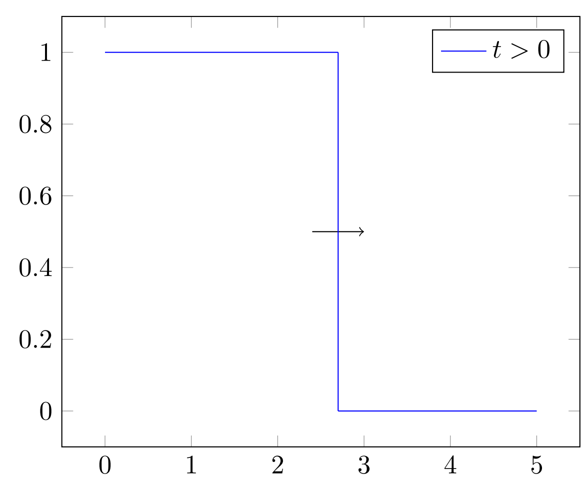
$$u(x,0) = f(x) \coloneqq \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } x \geq \frac{1}{2} \end{cases} \tag{4}$$

Then for $t \geq 0$, the solution is

$$u(t,x) = \begin{cases} 1 & \text{if } 0 \leq x < t + \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

that is, the wave transports to the right.



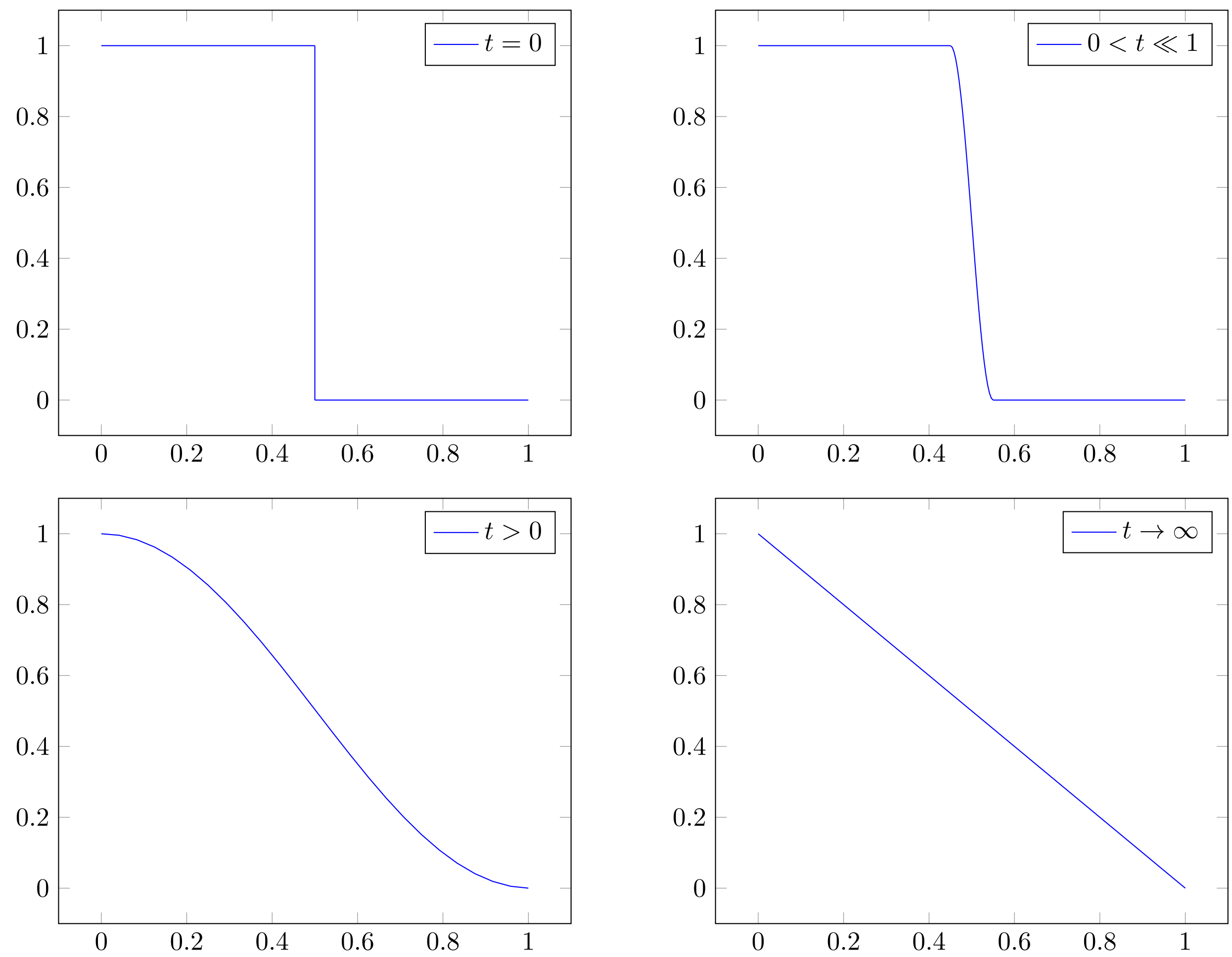


2.2 The Heat Equation

Set $D \equiv 1$, and the domain $0 \le x \le 1$. Set the boundary conditions:

$$u(0,t) = 1 \qquad u(1,t) = 0 \tag{5}$$

and initial condition $f(x)$ defined above. Then the discontinuity instantaneously smooths out so that $u(t,x) \in C^\infty$ for all $t > 0$, and the equilibrium state is stable and linear.



3 What we will focus on in 228A

We are focusing on the odd one out - it is time-independent. Where do Poisson Equations show up?

- Steady State Diffusion Problems ($u_t = 0$)
  - u is a concentration
  - $u_t = \underbrace{Du_{xx}}_{\text{transport by diffusion}} + \underbrace{F}_{\text{input}}$
  - At steady state, $u_t = 0$, and so $-Du_{xx} = F$.
  - $u_t = Du_{xx} \underbrace{-ku}_{\text{loss due to environment... causes exponential decay}} + f$
  - At steady state, $-Du_{xx} + ku = f$, which is a Helmholtz Equation.
- Electrostatics
  - ε is a steady electric field
  - ρ is a charge distribution
  - ε_0 is a parameter

- $\nabla \cdot \varepsilon = \frac{\rho}{\varepsilon_0}$
- Given the charge distribution, we want to find the electric field.
- $\underbrace{\nabla \times \varepsilon}_{\text{curl}} = 0 \implies \exists$ potential function ϕ such that $\varepsilon = -\nabla\phi$.
- So, $\nabla \cdot \varepsilon = \nabla(-\nabla\phi) = \underbrace{-\Delta\phi = \dfrac{\rho}{\varepsilon_0}}_{\text{Poisson Equation}}$

• Potential Flow

- $\nabla u = 0$, where u is velocity. $\nabla u = 0$ means it is divergence-free, and thus incompressible.
- With high Reynolds number it is curl-free, and thus there is a potential function ϕ such that $\Delta\phi = 0$, which is a Poisson equation.
- With low Reynolds number (usually on very small length scales, i.e. bacterial swimming), it is drag-dominated, effectively no inertia.
- We have $\mu\Delta u - \nabla p = 0$, so $\nabla \cdot u = 0$, which implies it is imcompressible. These are Stokes Equations.
- Take Divergence, and assume commutativity with the Laplacian Δ, and thus $-\Delta p = 0$, which is a Poisson Equation.

The Poisson equation is

$$\begin{aligned} u_{xx} &= f && \text{in one dimension} \\ u_{xx} + u_{yy} &= f && \text{in two dimensions in the Cartesian coordinates} \\ \Delta u &= f \quad \text{or} \quad (\nabla \cdot \nabla)u = f \quad \text{or} \quad \nabla^2 u = f && \text{in any number of dimensions (finite)} \end{aligned}$$

These equations need a defined domain and boundary conditions.