MAT 228A Notes

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1 Last Time - Iterative Solvers

- Jacobi Iterations

$$u_{i,j}^{k+1} = \frac{1}{4}\big(u_{i-1,j}^{k} + u_{i,j-1}^{k} + u_{i+1,j}^{k} + u_{i,j+1}^{k} - h^2 f_{i,j}\big)$$

- GS Lex

$$u_{i,j}^{k+1} = \frac{1}{4}\big(u_{i-1,j}^{k+1} + u_{i,j-1}^{k+1} + u_{i+1,j}^{k} + u_{i,j+1}^{k} - h^2 f_{i,j}\big)$$

2 Analysis of Jacobi

$$u^{k+1} = u^k + Br^k \qquad \text{where} \qquad B \approx A^{-1}$$

Now lets take $B = D^{-1} = -\frac{h^2}{4}I$ where I is the identity.

$$u^{k+1} = u^k - \frac{h^2}{4}\big(f - Au^k\big) = \left(I + \frac{h^2}{4}A\right)u^k - \frac{h^2}{4}$$

Both Jacobi and GS $u^{k+1} = Tu^k + c$ converges iff $\rho(T) < 1$. Set $T_J = I + \frac{h^2}{4}A$. So the spectrum of T_J is the rescaled and shifted spectrum of A. If λ is an eigenvalue of A, then $1 + \frac{h^2}{4}\lambda$ is an eigenvalue of T_J. The eigenfunctions of A are

$$u_{i,j}^{\ell m} = \sin(\ell\pi x_i)\sin(m\pi y_j)$$

with eigenvalues

$$\lambda^{\ell m} = \frac{2}{h^2}(\cos(\ell\pi h) + \cos(m\pi h) - 2)$$

Thus the eigenvalues for T_J, $\mu^{\ell m}$ are

$$\mu^{\ell m} = \frac{1}{2}(\cos(\ell\pi h) + \cos(m\pi h))$$

Since $h = \frac{1}{n+1}$ and $\ell, m = 1, 2, \ldots, n$, we have $\left|\mu^{\ell m}\right| < 1$. So the spectral radius is $\cos(\pi h)$. However,

$$\cos(\pi h) = 1 - \frac{\pi^2 h^2}{2} + \mathcal{O}\big(h^4\big)$$

As $h \to 0$, $\rho(T_J) \to 0$, i.e. convergence slows down.

3 GS Lex

When $f = 0$,

$$u_{i,j}^{k+1} = \frac{1}{4}\big(u_{i-1,j}^{k+1} + u_{i,j-1}^{k+1} + u_{i+1,j}^{k} + u_{i,j+1}^{k}\big)$$

So $T_{GS} = (D-L)^{-1}U$. Let v^k be an eigenvector of T_{GS} with eigenvalue λ. So $v^{k+1} = \lambda v^k$.

$$\lambda v_{i,j} = \frac{1}{4}(\lambda v_{i-1,j} + \lambda v_{i,j-1} + v_{i+1,j} + v_{i,j+1})$$

Now change variables: let $v_{i,j} = \lambda^{\frac{i+j}{2}} u_{i,j}$. Plugging this in above, we get

$$\begin{aligned}
\lambda^{\frac{i+j}{2}+1} u_{i,j} &= \frac{1}{4}\Big(\lambda^{\frac{i+j-1}{2}+1} u_{i-1,j} + \lambda^{\frac{i+j-1}{2}+1} u_{i,j-1} + \lambda^{\frac{i+j+1}{2}} u_{i+1,j} + \lambda^{\frac{i+j+1}{2}} u_{i,j+1}\Big) \\
\implies \lambda^{\frac{1}{2}} u_{i,j} &= \frac{1}{4}(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1})
\end{aligned}$$

Thus $\lambda^{\frac{1}{2}} = \mu$ where μ is an eigenvalue of the Jacobi iteration. So for μ an eigenvalue of Jacobi, we have μ^2 is an eigenvalue of GS Lex. Thus $\rho(T_S) = \cos(\pi h) \implies \rho(T_{GS}) = \cos^2(\pi h) = (1 - \frac{\pi^2 h^2}{2} + \mathcal{O}(h^4))^2 = 1 - \pi^2 h^2 + \mathcal{O}(h^4)$.

4 Error Analysis

Consider an iteration matrix T with a complete set of eigenvectors (diagonalizable) with eigenvalues $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_N|$. Then recall $e^{k+1} = Te^k$ where e^k is alebraic error. Then $e^k = T^k e^0$. We can express e^0 as a linear combination of the eigenvectors,

$$e^0 = \sum_{j=1}^{N} \alpha_j v_j$$

where v_j is the jth eigenvector of T. Thus $e^k = \sum_{j-1}^{N} \lambda_j^k \alpha_j v_j$.

$$e^k = \lambda_1^k \alpha_1 v_1 + \sum_{j=2}^{N} \lambda_j^k \alpha_j v_j = \lambda_1^k \left(\alpha_1 v_1 + \sum_{j=2}^{N} \left(\frac{\lambda_j}{\lambda_1}\right)^k \alpha_j v_j\right)$$

So for k large, we get $e^k \approx \lambda_1^k a_1 v_1$. Lots of iterations? For k large,

$$\frac{\|e^{k+1}\|}{\|e^k\|} \approx \lambda_1 = \rho$$

How many iterations to reduce the error by ε? $\rho^k = \varepsilon$. So $k = \dfrac{\log \varepsilon}{\log \rho}$. How many iterations per digit of accuracy? So for $\varepsilon = 0.1$? $\dfrac{\log(10^{-1})}{\log \rho} = \frac{1}{10 \log \rho}$. So

grid	Jacobi	GS Lex
32×32	507	254
64×64	1971	985
128×128	7764	3882
256×256	30818	15409

Spectral radius of Jacobi

$$\ln \rho_J \approx \ln\left(1 - \frac{\pi^2 h^2}{2}\right) = -\frac{\pi^2 h^2}{2} + \text{higher order terms}$$

Spectral radius of GS Lex

$$\ln \rho_{GS} \approx -\pi^2 h^2 + \text{higher order terms} \tag{1}$$

Work scaling to reduce the error by a factor of Ch^2? So $\rho^k = Ch^2$. Thus $k = \dfrac{\ln(Ch^2)}{\ln \rho} \approx \dfrac{\ln C + 2\ln}{-\pi^2 h^2} \approx B \ln(h) h^{-2}$.

Asymptotically, $h \sim \dfrac{1}{n}$. Thus $k = \mathcal{O}(n^2 \ln(n^2)) = \mathcal{O}(N \ln N)$. This is the work for iteration. The total work to solve is $\mathcal{O}(N^2 \ln N)$. This is like, or a little slower than, a banded factorization. This is slow, but uses very little memory. One does not just use these in research. But they are part of better methods.