MAT 228A Notes

Sam Fleischer

September 22, 2016

1 Introduction

In MAT 228 we will study standard model PDEs, namely:

1. The Avection Equation (the typical hyperbolic equation)

$$u_t + cu_x = 0$$

2. The Diffusion (Heat) Equation (the typical parabolic equation)

$$u_t = Du_{xx}$$

3. The Poisson Equation (the typical elliptic equation)

$$u_{xx} = F$$

2 What we will not focus on in 228A

2.1 The Avection Equation

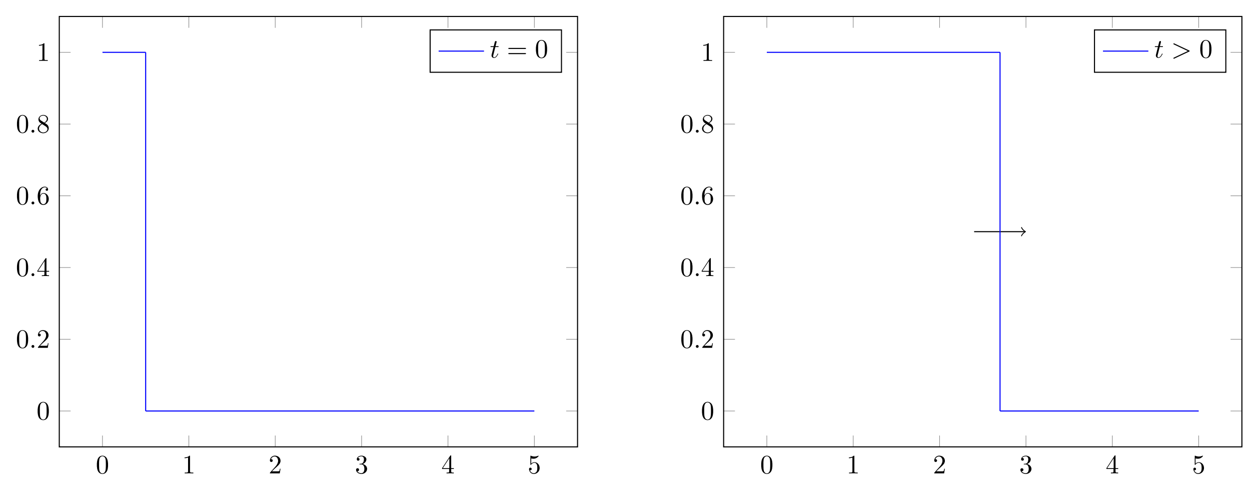
Set $c \equiv 1$, and the domain $x > 0$. Let the boundary condition be $u(0,t) = 1$ and the Initial condition

$$u(x,0) = f(x) \coloneqq \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } x \geq \frac{1}{2} \end{cases}$$

Then for $t \geq 0$, the solution is

$$u(t,x) = \begin{cases} 1 & \text{if } 0 \leq x < t + \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

that is, the wave transports to the right.

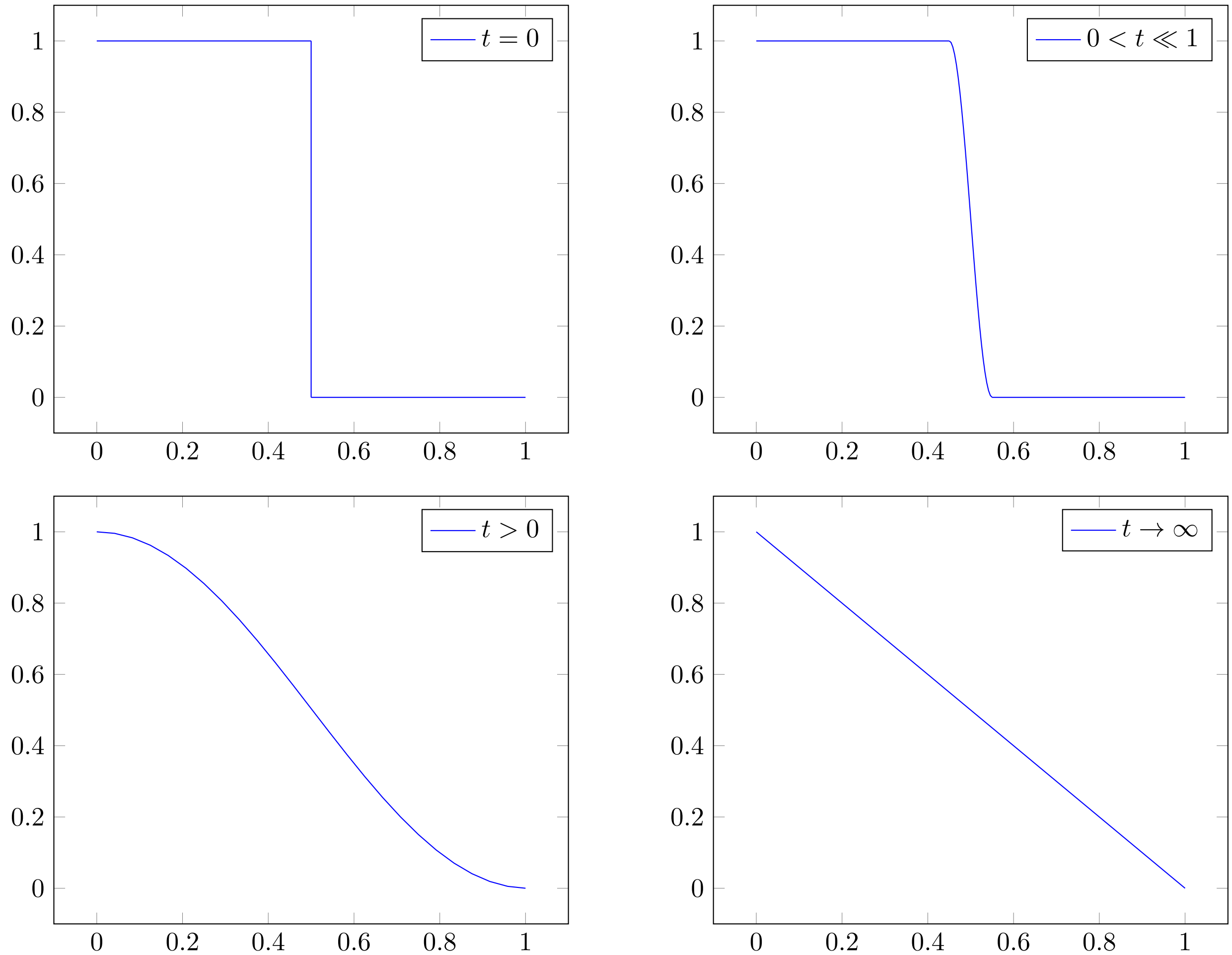


2.2 The Heat Equation

Set $D \equiv 1$, and the domain $0 \leq x \leq 1$. Set the boundary conditions:

$$u(0,t) = 1 \qquad u(1,t) = 0$$

and initial condition $f(x)$ defined above. Then the discontinuity instantaneously smooths out so that $u(t,x) \in C^\infty$ for all $t > 0$, and the equilibrium state is stable and linear.



3 What we will focus on in 228A

We are focusing on the odd one out - it is time-independent. Where do Poisson Equations show up?

- Steady State Diffusion Problems ($u_t = 0$)
  - u is a concentration
  - $u_t = \underbrace{Du_{xx}}_{\text{transport by diffusion}} + \underbrace{F}_{\text{input}}$
  - At steady state, $u_t = 0$, and so $-Du_{xx} = F$.
  - $u_t = Du_{xx} \underbrace{-ku}_{\text{loss due to environment... causes exponential decay}} + f$
  - At steady state, $-Du_{xx} + ku = f$, which is a Helmholtz Equation.
- Electrostatics
  - ε is a steady electric field
  - ρ is a charge distribution
  - ε_0 is a parameter

- $\nabla \cdot \varepsilon = \frac{\rho}{\varepsilon_0}$
- Given the charge distribution, we want to find the electric field.
- $\underbrace{\nabla \times \varepsilon}_{\text{curl}} = 0 \implies \exists$ potential function ϕ such that $\varepsilon = -\nabla\phi$.
- So, $\nabla \cdot \varepsilon = \nabla(-\nabla\phi) = \underbrace{-\Delta\phi = \frac{\rho}{\varepsilon_0}}_{\text{Poisson Equation}}$

• Potential Flow

  - $\nabla u = 0$, where u is velocity. $\nabla u = 0$ means it is divergence-free, and thus incompressible.
  - With high Reynolds number it is curl-free, and thus there is a potential function ϕ such that $\Delta\phi = 0$, which is a Poisson equation.
  - With low Reynolds number (usually on very small length scales, i.e. bacterial swimming), it is drag-dominated, effectively no inertia.
  - We have $\mu\Delta u - \nabla p = 0$, so $\nabla \cdot u = 0$, which implies it is imcompressible. These are Stokes Equations.
  - Take Divergence, and assume commutativity with the Laplacian Δ, and thus $-\Delta p = 0$, which is a Poisson Equation.

The Poisson equation is

$$
\begin{aligned}
u_{xx} &= f && \text{in one dimension} \\
u_{xx} + u_{yy} &= f && \text{in two dimensions in the Cartesian coordinates} \\
\Delta u &= f \quad \text{or} \quad (\nabla \cdot \nabla)u = f \quad \text{or} \quad \nabla^2 u = f && \text{in any number of dimensions (finite)}
\end{aligned}
$$

These equations need a defined domain and boundary conditions.

3.1 1-Dimensional Diffusion Equation

Consider a long thin metal tube, which can be represented as a function in one dimension. Let $u(x,t) :=$ the concentration of some chemical (or heat, or whatever) at time t and position on the tube x, where $x \in [a,b]$. Then

$$u_t = Du_{xx} + F$$

What can the boundary conditions be? The easiest condition is to hook one side up to a giant vat with constant concentration so that $u(0,t) = u_0$ is constant, which is called Dirichlet Boundary Condition.

The second type of condition is called a Neumann Boundary Condition, which is a condition on the flux at the boundary. For example, we could cap one end so that no transport can happen through the boundary. Given (a,b), define

$$q(t) := \int_a^b u(x,t)\mathrm{d}x \qquad \text{(this is the total amount of chemical in the interval } (a,b) \text{ at time } t\text{)}$$

So,

$$
\begin{aligned}
u_t &= Du_{xx} \\
\implies \int_a^b u_t \mathrm{d}x &= \int_a^b Du_{xx}\mathrm{d}x \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}q(t) &= \underbrace{D[u_x(b,t) - u_x(a,t)]}_{\text{these terms describe tre transport rate on the boundary}}
\end{aligned}
$$

Define $J := -Du_x$ as the diffusive flux. So if we want no flux (which is the only means of transport), we must have $u_x(0,t) = 0$. Or, given some constant injection on an end point, we could set

$$-Du_x(0,t) = g, \qquad \text{where } g \text{ is constant.}$$

The last type of condition can be likened to a semi-permeable membrane on the boundary. It is called a Robin Boundary Condition, and is simply a linear combination of Dirichlet and Neumann conditions. So,

$$J = \alpha(u_r - u_\ell)$$

The flux balances so that

$$-Du_x(0,t) = \alpha(u(0,t) - u_0)$$

and thus

$$\alpha u(0,t) + Du_x(0,t) = \alpha u_0.$$