MAT 228A Notes

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0.0.1 Announcement

Moving to Briggs book

1 Last Time - SOR method

$$\omega^* = \frac{2}{1+(1-\rho_J^2)^{\frac{1}{2}}}, \qquad \rho_{\text{SOR}} = \omega^* - 1 \tag{1}$$

$$\omega^* = \frac{2}{1+\sin(\pi h)} = 2(1-\pi h) + \mathcal{O}\big(h^2\big) \tag{2}$$

using Taylor expansions. So,

$$\rho_{\text{SOR}} = 1 - 2\pi h \tag{3}$$

Iteration count for GS: on 256^2 mesh, the number of iterations per digit of accuracy is 15409. SOR requires 94 iterations.

2 Scaling on the work of SOR to converge to a tolerance equal to Ch^2

Assume the spectral radius has a power relationship with Ch^2, i.e.

$$\rho^k = Ch^2 \qquad \Longrightarrow \qquad k = \frac{\ln\big(Ch^2\big)}{\ln(\rho)} \tag{4}$$

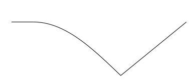
but we have $\rho \approx 1 - 2\pi h \approx -2\pi h$, so

$$k \approx \frac{\ln(C) + \ln\big(h^2\big)}{-2\pi h} \qquad \Longrightarrow \qquad k = \mathcal{O}\big(h^{-1}\ln(h)\big) \tag{5}$$

In 2D, $h \sim \frac{1}{n}$ where n is the number of points in each direction. $N = n^2$ is the total number of points. So

$$k = \mathcal{O}\Big(N^{\frac{1}{2}}\ln(N)\Big) \tag{6}$$

The work per iteration is $\mathcal{O}(N)$, and thus work to solve is $\mathcal{O}\Big(N^{\frac{3}{2}}\ln(N)\Big)$. One drawback is that we need to know ω^*. We expect $\omega^* = \dfrac{2}{1+Ch}$. Graphing spectral radius as a function of ω gives us a corner..



All methods so far

$$\frac{\|e_{k+1}\|}{\|e_k\|} \approx \rho \to 1 \qquad \Longrightarrow \qquad h \to 0 \tag{7}$$

Can we find a method such that

$$\rho < C < 1 \qquad \Longrightarrow \qquad h \to 0 \tag{8}$$

.. an iteration to fixed tolerance independent of the mesh.

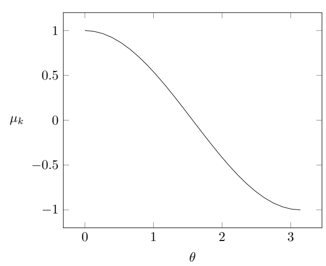
$$\|e_{k+1}\| \|e_k\| \approx \rho \leftarrow \text{applies for large } k \tag{9}$$

What happens for small k?

3 Analyze Jacobi in 1D

$$I_J = I + \frac{h^2}{2} A \tag{10}$$

where $A = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}$. Eigenvalues of A are $\lambda_k = \frac{2}{h^2}(\cos(k\pi h) - 1)$. Eigenvalues of I_J are $\mu_k = 1 + \frac{h^2}{2}\lambda_k = \cos(k\pi h)$. If $k = 1, \dots, n$, with $h = \frac{1}{n+1}$, then $k\pi h = \pi h, \dots, \frac{n\pi}{n+1}$. Set $\theta := k\pi h$. Plotting the specrum as a function of θ gives



This shows low and high spacial frequencies are damped the least. Try to improve convergence/smooting with a parameter..

4 ω-Jacobi

$$u_j^{k+1} = \frac{\omega}{2}\left(u_{j-1}^k + u_{j+1}^k - h^2 f_j\right) + (1-\omega)u_j^k \tag{11}$$

What is the update matrix?

$$T = \omega\left(I + \frac{h^2}{2}A\right) + (1-\omega)I \tag{12}$$

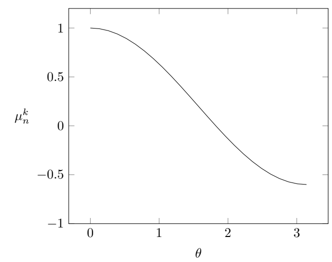
with eigenvalues

$$\mu_k^\omega = \omega\cos(k\pi h) + (1-\omega) \tag{13}$$

What happens to high and low frequencies?

4.1 k small?

For small k, for example $k = 1$, $\mu_1^\omega = \omega\left(1 - \frac{\pi h^2}{2} + \dots\right) + (1-\omega) = 1 - \frac{\omega\pi h^2}{2}$. For large k, for example $k = n$, we have $\mu_n^\omega = 1 - 2\omega + \dots$.



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