MAT 228A Notes

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1 1-Dimensional Poisson Equation

1.1 Recall

$$u_{xx} = f \qquad u(0) = \alpha \qquad u(1) = \beta \tag{1}$$

Uniform spacing between (and including) 0 and 1 with $N + 2$ points and $h = \frac{1}{N+1}$. This yielded the finite-difference approximation

$$A\underline{u}^h = \underline{b} \tag{2}$$

How big is the error?

$$\underline{e}^h = \underline{u}^h - \underline{u}_{\text{sol}}, \qquad \text{where} \qquad (u_{\text{sol}})_j = u(x_j) \tag{3}$$

We hope that $\left\|\underline{e}^h\right\| = \mathcal{O}\big(h^2\big)$. So the general question is "How do errors in our operators relate to errors in the discrete solution?"

1.2 How is the size of this error related to discretization error?

In general, $\underline{u}_{\text{sol}}$ is not a solution to $A\underline{u}^h = \underline{b}$. We define the local trunctation error $\underline{\tau}^h$

$$\underline{\tau}^h = A\underline{u}_{\text{sol}} - \underline{b} \tag{4}$$

So for $j = 2, \dots, N - 1$,

$$\begin{aligned}
\tau_j^h &= \frac{1}{h^2}[u(x_{j-1}) - 2u(x_j) + u(x_{j+1})] - f(x_j) && (5)\\
&= \frac{1}{h^2}[u(x_j - h) - 2u(x_j) + u(x_j + h)] - f(x_j) && (6)\\
&= \underline{u_x x(x_j)} + \frac{h^2}{12}u^{(4)}(x_j) + \text{higher order terms} - \underline{f(x_j)} \qquad \text{(using Taylor expansions)} && (7)
\end{aligned}$$

The underlined terms balance, so

$$\tau_j^h = \frac{h^2}{12}u^{(4)}(x_j) + \mathcal{O}\big(h^4\big) \tag{8}$$

Now write

$$\begin{aligned}
A\underline{u}_{\text{sol}} &= \underline{b} + \underline{\tau}^h && (9)\\
A\underline{u}_h &= \underline{b} && (10)
\end{aligned}$$

Subtracting these equations, and usin the fact that $\underline{e}^h = \underline{u}^h - \underline{u}_{\text{sol}}$, gives

$$\begin{aligned}
A(\underline{u}^h - \underline{u}_{\text{sol}}) &= -\underline{\tau}^h && (11)\\
A\underline{e}^h &= -\underline{\tau}^h && (12)\\
\underline{e}^h &= -A^{-1}\underline{\tau}^h && (13)
\end{aligned}$$

1.3 Consistency vs. Convergence

A numerical scheme is "consistent" if the trunctation error goes to 0, i.e. if $\underline{\tau}^h \to 0$ as $h \to 0$. A numerical scheme is "convergence" if $\underline{e}^h \to 0$ as $h \to 0$. A scheme can be consistent and not convergent if $\|A^{-1}\| = \infty$.

For linear schemes applied to linear PDEs we have the Lax-Equivalence Theorem:

- If a scheme is consistent and stable, then it is convergent.

1.4 Stability

"Stable" in our case means that

$$\|(A^h)^{-1}\| \leq C \qquad \text{for all } h \leq h_0 \text{ where } C \text{ is independent of } h \tag{14}$$

$$\|\underline{e}^h\| = \|A^{-1}\underline{\tau}^h\| \leq \|A^{-1}\|\|\underline{\tau}^h\| \leq C\|\underline{\tau}^h\|b \tag{15}$$

1.4.1 Stability in the 2-norm

Remember A is symmetric, and that $\|A\|_2 = \rho(A) = \max|\lambda_i|$. $\sqrt{\rho(AA^*)} = \sqrt{\rho(A^2)} = \rho\left(\sqrt{A^2}\right) = \rho(A)$. Then

$$\|A^{-1}\|_2 = \rho(A^{-1}) = \max\left|\frac{1}{\lambda_j}\right| = \frac{1}{\min|\lambda_j|} \tag{16}$$

1.4.2 Explicitly Calculating the Eigenvalues of A

Recall $u^k(x) = \sin(k\pi x)$ is an eigenfunction of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on functions zero at 0 and 1. It turns out the eigenvectors of A are these eigenfunctions evaluated on the grid points. So we claim

$$u_j^k = \sin(k\pi x_i), \qquad 1, \dots, N \tag{17}$$

are eigenvectors of A. Let's explicitly show this.

$$\frac{1}{h^2}[\sin(k\pi x_{j-1}) - 2\sin(k\pi x_j) + \sin(k\pi x_{j+1})] = \frac{1}{h^2}[\sin(k\pi(x_j - h)) - 2\sin(k\pi x_j) + \sin(k\pi x_j + h)] \tag{18}$$

$$= \frac{1}{h^2}[\sin(k\pi x_j)\cos(k\pi h) - \sin(k\pi h)\cos(k\pi x_j)] \tag{19}$$

$$- \frac{1}{h^2}[-2\sin(k\pi x_j)] \tag{20}$$

$$+ \frac{1}{h}^2[\sin(k\pi x_j)\cos(k\pi h) + \sin(k\pi h)\cos(k\pi x_j)] \tag{21}$$

$$= \frac{1}{h^s}(2\cos(k\pi h) - 2)\sin(k\pi x_j) \tag{22}$$

Thus we have eigenvalues

$$\lambda_k = \frac{2}{h^2}(\cos(k\pi h) - 1), \qquad k = 1, \dots, N \tag{23}$$

$$= -\frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right) \tag{24}$$

As k goes from 1 to N, $k\pi h$ goees from πh to $N\pi h = \frac{N\pi}{N+1}$. So, the smallest magnitude eigenvalue is

$$\lambda_1 = \frac{2}{h^2}(\cos(\pi h) - 1) \tag{25}$$

$$= \frac{2}{h^2}\left(1 - \frac{1}{2}\pi^2 h^2 + \mathcal{O}(h^4) - 1\right) \tag{26}$$

$$= -\pi^2 + \mathcal{O}(h^2) \tag{27}$$

The following is in red since I was falling asleep and cannot vouch for accuracy...

So we have control of this inverse since

$$\|A^{-1}\|_2 = \frac{1}{\pi^2} + \mathcal{O}(h^2) \tag{28}$$

and thus

$$
\begin{aligned}
\underline{e}^h &= -\left\|A^{-1}\right\|_2 \left\|\underline{\tau}^h\right\|_2 && (29)\\
&= \left(\frac{1}{\pi^2} + \mathcal{O}\left(h^2\right)\right)(\mathcal{O}(h@2)) && (30)
\end{aligned}
$$

1.5 Eigenvalues of the Continuous space operator are

$$\lambda_k^c = -k^2\pi^2 \tag{31}$$