MAT 228A Notes

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1 ω-Jacobi in 1D

Over/under relaxation in GS leads to SOR. What happens when we add a parameter to Jacobi method?

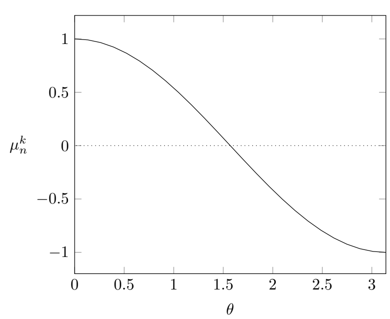
- $u_j^{k+1} = \omega/2\big(u_{j-1}^k + u_{j+1}^k - ?\big) + (1-\omega)u_j^k$
- Update matrix is $\omega T_J + (1-\omega)I$ where T_J is the update matrix of the Jacobi method. The eigenvalues are rescaled and shifted from the eigenvalues of T_J. Eigenvectors are

$$u_{j,\ell} = \sin(\ell\pi x_j)$$

  Eigenvalues are

$$\mu_\ell = \omega\cos(\ell\pi h) + (1-\omega)$$

- For $\omega = 1$, we get



  We see slow damping of the low and high frequencies, but rapid damping of the mid frequencies. For $k = 1$,

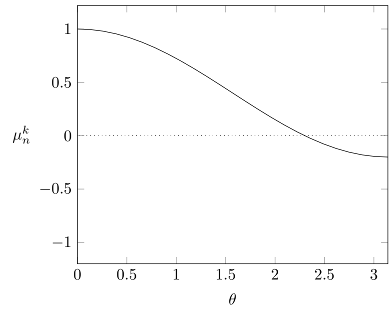
$$\mu_1 = \omega\left(1 - \frac{\pi^2 h^2}{2} + \dots\right) + (1-\omega) \approx 1 - \omega\frac{\pi^2 h^2}{2}$$

  For $k = n$,

$$\mu_n = 1 - 2\omega + \text{(higher order terms)}$$

  This tells us $\omega \leq 1$ for convergence ($\omega = 1$ is fine when you consider the higher order terms). And $\omega \geq 0$ in order for μ_1 to converge.

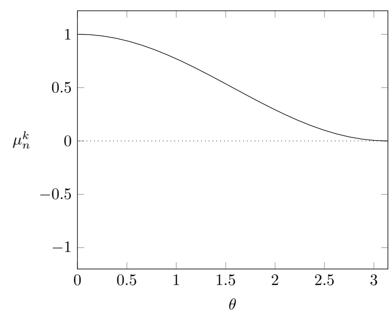
- For $1/2 < \omega < 1$ we get



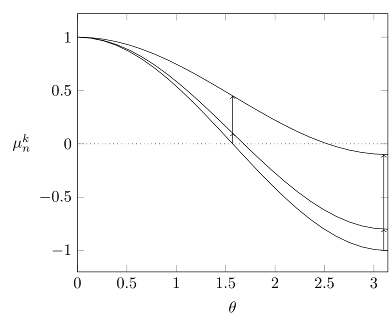
The big idea of multigrid is to combine smoothing and coarse grid.

2 Choosing ω to wipe out the highest frequencies

Choosing $\omega = 1/2$ gives us



$\omega = 1/2$ is most effective on highest frequencies. We want to reduce frequencies with wave greater than or equal to $n/2$. This gets rid of the highest frequencies but made convergence of midrange fequencies much worse. What value of ω is optimal to reduce high-range frequencies?



ω_{opt} is the omega which makes

$$\omega_{\text{opt}} \cos(\ell \pi N) + (1 - \omega_{\text{opt}}) = -\left[\omega_{\text{opt}} \cos\left(\ell \pi \frac{N}{2}\right) + (1 - \omega_{\text{opt}})\right]$$

So we want $1 - \omega^* = 1 - 2\omega^*$, i.e. $\omega^* = 2/3$, so we get $\mu^* = 1 - \omega^* = 1/3$. So all mid-high level frequencies are reduced by a factor of e per iteration. Just 2 iterations reduces error by approximately an order of magnitude. Set μ equal to

$$\mu \coloneqq \max_{n/2 \leq k \leq n} |\mu_k|$$

μ is called the smoothin factor. It is the largest factor by which the high frequency ($k \geq n/2$) modes are reduced with one application of the smoother.

2.1 Smoothing Factors in multiple dimensions..

	ω^*	$\mu_{\omega-\text{Jacobi}}$	$\mu_{\text{GS-lex}}$	$\mu_{\text{GS-RB}}$
1D	$^2/_3$	$^1/_3$	0.45	0.125
2D	$^4/_5$	$^3/_5$	0.5	0.25
3D	$^6/_7$	$^5/_7$	0.567	0.445

Clearly GS-RB is the winner in higher dimensions.

3 GS-Lex analysis of smoothing

Eigenvectors of the iteration matrix are not sin waves.

3.1 Local Fourier Analysis

Lets ignore boundaries and analyze the problem on the infinite domain. We still have a discrete domain but it is the whole real line:

$$x_j = jh \qquad j \in \mathbb{Z} \tag{1}$$

The eigenfunctions are of the form

$$u_\ell^k = \exp(ikx_\ell) \tag{2}$$

What values does k take? Using duality of the Fourier transform, we know that the highest k corresponds to oscillations between 1 and -1 at each grid point. It has period $2h$. So $k_{\max}h = \pi$. $k_{\max} = {}^{\pi}/_{h}$. Thus

$$-\frac{\pi}{h} < k \leq \frac{\pi}{h} \tag{3}$$

are the meaningful values of k, that is

$$-\pi < kh \leq \pi \tag{4}$$

but let $\theta := kh$, so

$$-\pi < \theta \leq \pi \tag{5}$$

That is, θ is the continuous wave number. So,

$$u_\ell(\theta) = \exp[ikx_\ell] = \exp[ikh\ell] = \underbrace{\exp[i\theta\ell]}_{\text{continuum of eigenfunctions}} \tag{6}$$

since $x_\ell = h\ell$ and $kh = \theta$. We have a bounded continuous spectrum.

Let's define high-frequency as ${}^{\pi}/_{2} \leq |\theta| \leq \pi$. What is the worst eigenvalue?

3.2 GS error

$$e^{k+1} = Te^k \tag{7}$$

$$e_\ell^{k+1} = \frac{1}{2}\left(e_{\ell-1}^{k+1} + e_{\ell+1}^k\right) \tag{8}$$

Fix θ..

$$e^k = \exp(i\ell\theta) \tag{9}$$

$$\implies e^{k+1} = Te^k = ae^k \tag{10}$$

where a is called the amplification factor (the eigenvalue). So we get

$$ae_\ell^k = \frac{1}{2}\left(ae_{\ell+1}^k + e_{\ell+1}^k\right) \tag{11}$$

$$e_{\ell-1} = e_\ell \exp(-i\theta) \tag{12}$$

$$e_{\ell+1} = e_\ell \exp(i\theta) \tag{13}$$

$$\implies a = \frac{1}{2}(a \exp(-i\theta) + \exp(i\theta)) \tag{14}$$

$$\implies a = \frac{\exp(i\theta)}{2 - \exp(-i\theta)} \tag{15}$$

The smoothing factor

$$\mu = \max_{|\theta| \geq \pi/2} |a(\theta)| = \frac{1}{\sqrt{5}} \approx 0.45 \tag{16}$$

$|a|$ as a function of θ that looks kind of Gaussian.