Homework #2

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Problem 1

Use the standard 3-point discretization of the Laplacian on a regular mesh to find a numerical solution to the PDEs below. Perform a refinement study using the exact solution to compute the error that shows the rate of convergence for both the 1-norm and the max norm.

(a) $u_{xx} = \exp(x), \qquad u(0) = 0, \qquad u(1) = 1$

(b) $u_{xx} = 2\cos^2(\pi x), \qquad u_x(0) = 0, \qquad u_x(1) = 1$

(a) The discretization for $u_{xx} = f$ with Dirichlet boundary conditions $u(0) = \alpha$ and $u(1) = \beta$ is $A\vec{u} = \vec{b}$ where A is an $N \times N$ matrix of the interior points (this does not include 0 and 1).

$$A = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} f_1 - \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N - \frac{\beta}{h^2} \end{pmatrix}. \tag{0.1}$$

A is invertible, and so we can directly solve for $\vec{u}$ by $\vec{u} = A^{-1}\vec{b}$. In Python I used the fact that A is sparse to speed up the calculations. The exact solution is

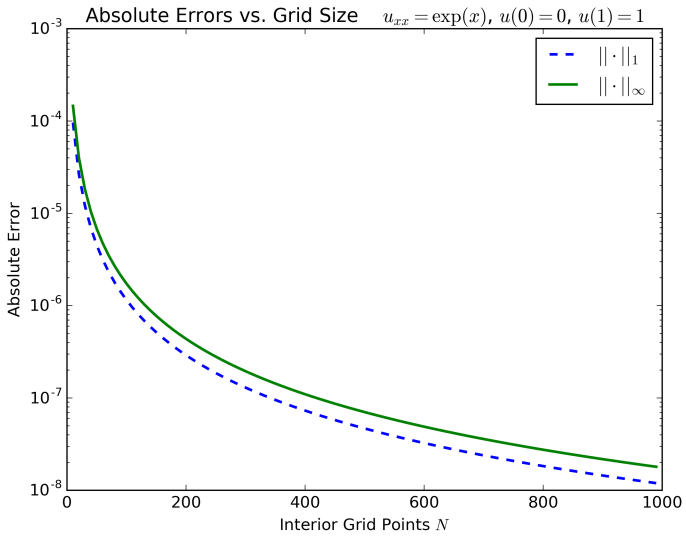
$$u(x) = e^x + (2 - e)x - 1 \tag{0.2}$$

The following is the code I used to generate the graphs.

```
1  def solution(x):
2      return np.exp(x) + (2 - math.exp(1))*x - 1
3
4  def one_norm_error(a,b,h):
5      S = 0
6      for i in range(len(a)):
7          S += abs(a[i]-b[i])
8      return S*h
9
10 def max_norm_error(a,b):
11     S = 0
12     for i in range(len(a)):
13         S = max(S, abs(a[i]-b[i]))
14     return S
15
16 one = []
17 Max = []
18
19 Ns = [10*i for i in range(1,100)]
20
21 for N in Ns:
22     h = 1/(1 + N)
23     sub_diag = 1/(h**2)*np.ones(N)
24     super_diag = 1/(h**2)*np.ones(N)
25     diag = -2/(h**2)*np.ones(N)
26
27     A = np.vstack((sub_diag,diag,super_diag))
28     A = scipy.sparse.dia_matrix((A,[-1,0,1]),shape=(N,N))
29     A = scipy.sparse.csc_matrix(A)
```

```
30
31      grid_pts = np.linspace(h, 1-h, num=N)
32      b = np.exp(grid_pts)
33      b[-1] -= 1/(h**2)
34
35      u = scipy.sparse.linalg.spsolve(A, b)
36
37      tru_sol = solution(grid_pts)
38
39      e_1 = one_norm_error(tru_sol,u,h)
40      e_m = max_norm_error(tru_sol,u)
41
42      one.append(e_1)
43      Max.append(e_m)
```

Then I plotted the results and generated a table:



N	$\|\cdot\|_1$ errors	$\|\cdot\|_1$ ratios	$\|\cdot\|_\infty$ errors	$\|\cdot\|_\infty$ ratios
2	0.00115081	-	0.00182123	-
4	0.000449552	0.39064	0.000694778	0.381487
8	0.00014301	0.318117	0.000217656	0.313274
16	4.04669e-05	0.282966	6.10464e-05	0.280472
32	1.07684e-05	0.266103	1.62108e-05	0.265549
64	2.77758e-06	0.257939	4.17864e-06	0.25777
128	7.05337e-07	0.253939	1.06096e-06	0.2539
256	1.77718e-07	0.251961	2.6731e-07	0.251951
512	4.46033e-08	0.250979	6.70883e-08	0.250976
1024	1.11727e-08	0.250491	1.68049e-08	0.25049

Notice I am doubling the grid size N and both $\|\cdot\|_1$ and $\|\cdot\|_\infty$ errors are decreasing by a factor of 4. Thus this method is $\mathcal{O}(h^2)$.

(b) The discretization for $u_{xx} = f$ with Neumann boundary conditions $u_x(0) = \alpha$ and $u_x(1) = \beta$ is different than before. We must first note that this operator is singular and thus $u_{xx} = f$ with these boundary conditios is only solvable if $f \in \text{ran } L$ where $L = \frac{\partial^2}{\partial x^2}$ with these boundary conditions. So first we must ensure that

$2\cos^2(\pi x) \in \text{ran } L$, which is trivially true. However if u is a solution of $u_{xx} = f$, it may not be a solution of a discritization $A\vec{u} = \vec{b}$. Let $\vec{v}$ span $\ker[A]$. Then $A\vec{u} = b - \lambda\vec{v}$ is solvable (where λ is the coefficient of the projection b on to $\ker[A]$). Then we are solving

$$A\vec{u} + \lambda\vec{v} = \vec{b} \tag{0.3}$$

which is an $(N+3)$ dimensional linear system where N is the number of interior points, not including 0 and 1. We must impose a condition on $\vec{u}$ in order to get another equation, for example, $\sum_i u_i = 0$. This gives us the discritization $M\vec{q} = \vec{d}$ where

$$M = \left(\begin{array}{c|c} A & \vec{v} \\ \hline \vec{1}^T & 0 \end{array}\right) \quad \text{and} \quad \vec{q} = \left(\begin{array}{c} \vec{u} \\ \hline \lambda \end{array}\right) \quad \text{and} \quad \vec{d} = \left(\begin{array}{c} \vec{b} \\ \hline 0 \end{array}\right) \tag{0.4}$$

where $\vec{v} = (1 \;\; 2 \;\; 2 \;\; \dots \;\; 2 \;\; 1)^T$ and

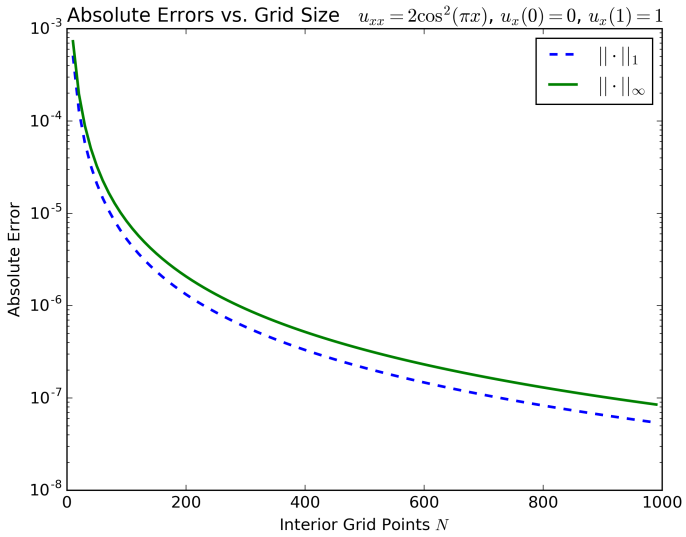
$$A = \begin{pmatrix} -2 & 2 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 2 & -2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} f_0 + \frac{2\alpha}{h} \\ f_2 \\ \vdots \\ f_{N-1} \\ f_{N+1} - \frac{\beta}{2h} \end{pmatrix}. \tag{0.5}$$

M is invertible, and so we can directly solve for $\vec{q}$ by $\vec{q} = M^{-1}\vec{b}$. Note that solving for $\vec{q}$ gives us $\vec{u}$. Since, if u is a solution to $u_{xx} = f$, then $u + C$ is also a solution for any constant C, I subtract the appropriate constant when comparing with any given discrete solution. The following is the code I used to generate the graphs.

```
1  def solution(x,N):
2      mean_zero_sol=0.5*(x**2)-(1/(4*(np.pi**2)))*np.cos(2*np.pi*x)-(1/6)
3      sol = mean_zero_sol - (1/(N+2))*sum(mean_zero_sol)
4      return sol
5
6  def one_norm_error(a,b,h):
7      S = 0
8      for i in range(len(a)):
9          S += abs(a[i]-b[i])
10     return S*h
11
12 def max_norm_error(a,b):
13     S = 0
14     for i in range(len(a)):
15         S = max(S, abs(a[i]-b[i]))
16     return S
17
18 one = []
19 Max = []
20
21 Ns = [10*i for i in range(1,100)]
22
23 for N in Ns:
24     h = 1/(1 + N)
25
26     sub_diag = np.concatenate((1/(h**2)*np.ones(N), [2/(h**2),0]))
27     super_diag = np.concatenate(([0,2/(h**2)], 1/(h**2)*np.ones(N)))
28     diag = -2/(h**2)*np.ones(N+2)
29
```

```
    v = np.append([1], np.append(2*np.ones((N,1)),[1]))
    A = np.vstack((sub_diag,diag,super_diag))
    A = scipy.sparse.dia_matrix((A,[-1,0,1]),shape=(N+2,N+2))
    A = scipy.sparse.hstack([A, v.reshape(N+2,1)])
    A = scipy.sparse.vstack([A, np.concatenate((np.ones(N+2),[0]))])
    A = scipy.sparse.csc_matrix(A)

    grid_pts = np.linspace(0, 1, num=(N+2))
    b = 2*(np.cos(np.pi*grid_pts)**2)
    b[-1] = b[-1] - (2/h)

    u = scipy.sparse.linalg.spsolve(A, np.concatenate((b, [0])))
    u = np.delete(u,N+2)

    tru_sol = solution(grid_pts, N)

    e_1 = one_norm_error(tru_sol,u,h)
    e_m = max_norm_error(tru_sol,u)

    one.append(e_1)
    Max.append(e_m)
```

Then I plotted the results and generated a table:



N	$\|\cdot\|_1$ errors	$\|\cdot\|_1$ ratios	$\|\cdot\|_\infty$ errors	$\|\cdot\|_\infty$ ratios
2	0.0117067	0	0.00878006	0
4	0.00282088	0.240962	0.0035261	0.401603
8	0.000768375	0.272389	0.00109622	0.310888
16	0.000200274	0.260646	0.000301517	0.275051
32	5.10092e-05	0.254697	7.85693e-05	0.26058
64	1.28581e-05	0.252074	2.0009e-05	0.254667
128	3.22669e-06	0.250947	5.04535e-06	0.252154
256	8.08117e-07	0.250447	1.26652e-06	0.251028
512	2.02204e-07	0.250217	3.17266e-07	0.250501
1024	5.05727e-08	0.250107	7.9395e-08	0.250247

Again, notice I am doubling the grid size N and both $\|\cdot\|_1$ and $\|\cdot\|_\infty$ errors are decreasing by a factor of 4. Thus this method is $\mathcal{O}(h^2)$.

Problem 2

As a general rule, we usually think that an $\mathcal{O}(h^p)$ local truncation error (LTE) leads to an $\mathcal{O}(h^p)$ error. However, in some cases the LTE can be lower order at some points without lowering the order of the error. Consider the standard second-order discretization of the Poisson equation on $[0,1]$ with homogeneous boundary conditions. The standard discretization of this problem gives an $\mathcal{O}(h^p)$ LTE provided the the solution is at least C^4 . The LTE may be lower order because the solution is not C^4 or because we use a lower order discretization at some points.

(a) Suppose that the LTE is $\mathcal{O}(h^p)$ at the first grid point ($x_1 = h$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error? Hint: Use equation (2.46) From LeVeque to aid in your argument.

(b) Suppose that the LTE is $\mathcal{O}(h^p)$ at an interior point (i.e. a point that does not limit to the boundary as $h \to 0$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error?

(c) Verify the results of your analysis from parts (a) and (b) using numerical tests.

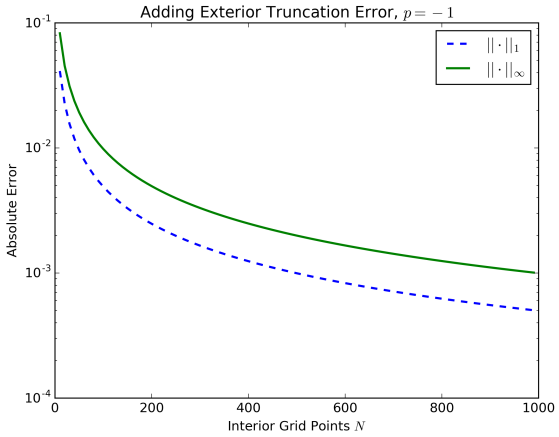
(a) The Green's function is

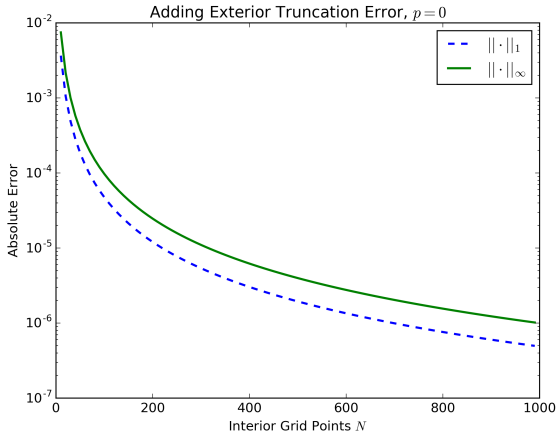
$$B_{ij} = \begin{cases} h(x_j - 1)x_i & \text{if } i = 1,2,\ldots,j \\ h(x_i - 1)x_j & \text{if } i = j+1,\ldots,N \end{cases} = \begin{cases} ijh^3 - ih^2 & \text{if } i = 1,2,\ldots,j \\ ijh^3 - jh^2 & \text{if } i = j+1,\ldots,N \end{cases} \tag{0.6}$$

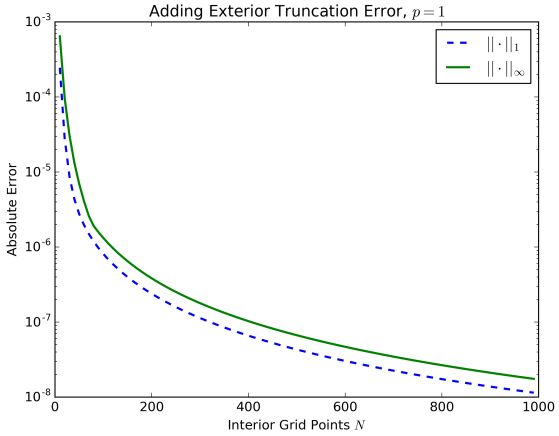
where $x_i = ih$ and $x_j = jh$. Near the exterior, for example at $j = 1$, the maximum of B is a constant multiple of h^2, and thus is $\mathcal{O}(h^2)$. Thus even truncation error of h^p where $p = 0$ doesn't affect the $\mathcal{O}(h^2)$ convergence of the solution.

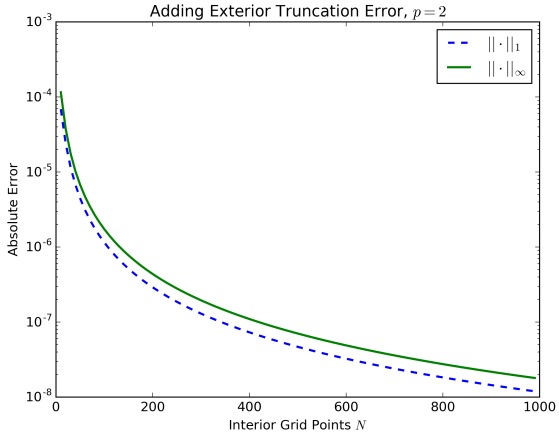
(b) Near the interior, on the other hand, for example at $j = \frac{N}{2}$, the maximum of B is the $\frac{N}{2}$nd term in the column, which is $N^2h^3 - Nh = \mathcal{O}(h)$. Thus, truncation error of h^p where $p = 0$ affects the $\mathcal{O}(h^2)$ convergence of the solution by a factor of h. For all $p \geq 1$, however, the rate of convergence is unaffected.

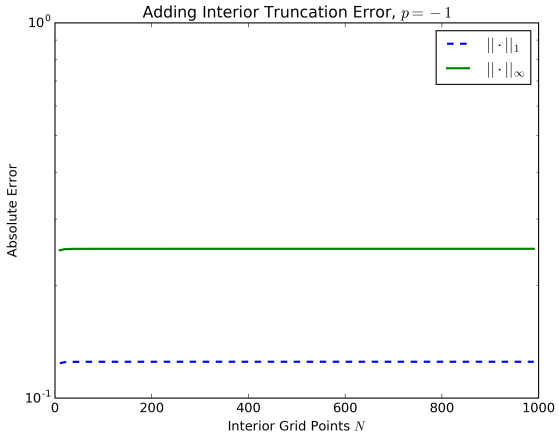
(c) The following are plots and table generated by similar Python code to problem 1. Notice for an exterior point, $p \geq 0$ produces $\mathcal{O}(h^2)$ error, but $p = -1$ produces $\mathcal{O}(h)$ error. For an interior point, $p \geq 1$ produces $\mathcal{O}(h^2)$ error, $p = 0$ produces $\mathcal{O}(h)$ error, and $p = -1$ doesn't allow for convergence at all.

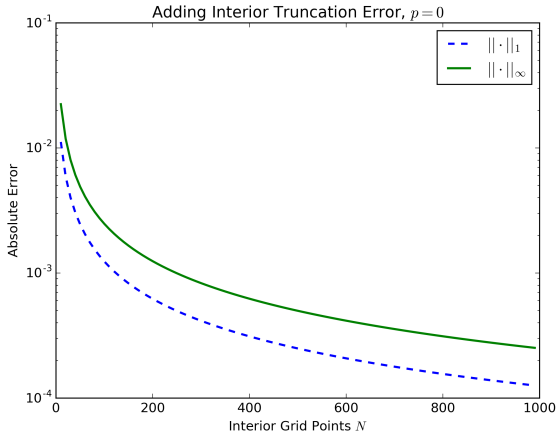

Adding Exterior Truncation Error, $p=-1$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$

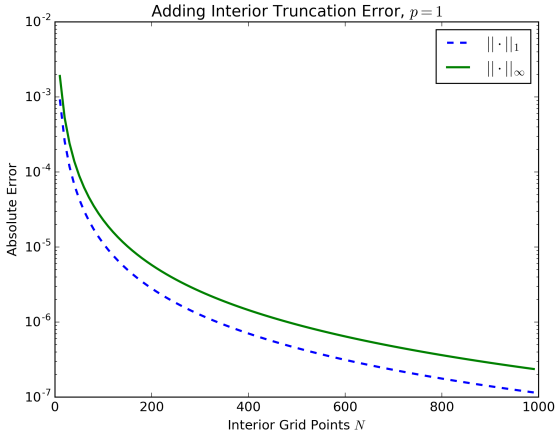

Adding Exterior Truncation Error, $p=0$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$


Adding Exterior Truncation Error, $p=1$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$


Adding Exterior Truncation Error, $p=2$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$


Adding Interior Truncation Error, $p=-1$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$


Adding Interior Truncation Error, $p=0$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$


Adding Interior Truncation Error, $p=1$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$

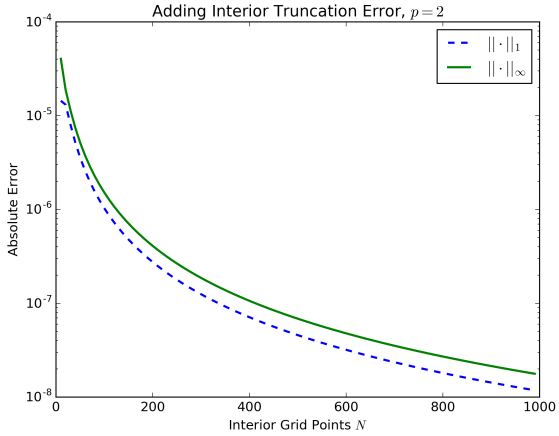

Adding Interior Truncation Error, $p=2$
Absolute Error
Interior Grid Points N
$||\cdot||_1$
$||\cdot||_\infty$

Table 0.1: Exterior Point, $p = -1$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.10996	0	0.220591	0
4	0.0795504	0.723447	0.159593	0.72348
8	0.0492397	0.618975	0.09869	0.618384
16	0.0276412	0.56136	0.0553516	0.560864
32	0.0146816	0.53115	0.0293831	0.530845
64	0.00757119	0.515692	0.0151477	0.515524
128	0.00384522	0.507875	0.00769182	0.507787
256	0.00193778	0.503945	0.00387591	0.5039
512	0.000972714	0.501974	0.00194552	0.501952
1024	0.000487318	0.500988	0.000974658	0.500976

Table 0.2: Exterior Point, $p = 0$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.0358862	0	0.0724429	0
4	0.0155504	0.433326	0.0315933	0.436113
8	0.00534396	0.343653	0.0108985	0.344961
16	0.00158787	0.297133	0.00324499	0.297748
32	0.000434455	0.273609	0.000888817	0.273904
64	0.000113745	0.261811	0.000232829	0.261954
128	2.9108e-05	0.255906	5.9599e-05	0.255977
256	7.36296e-06	0.252953	1.50778e-05	0.252988
512	1.85161e-06	0.251477	3.79199e-06	0.251494
1024	4.6427e-07	0.250738	9.5083e-07	0.250747

Table 0.3: Exterior Point, $p = 1$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.0111949	0	0.0230602	0
4	0.00275045	0.245688	0.0059933	0.259898
8	0.000466653	0.169664	0.00114386	0.190856
16	5.67118e-05	0.121529	0.000179896	0.157271
32	7.34304e-06	0.12948	2.53532e-05	0.140933
64	1.74856e-06	0.238125	3.3697e-06	0.13291
128	5.26978e-07	0.301378	8.5965e-07	0.255112
256	1.5185e-07	0.288152	2.41082e-07	0.280442
512	4.11298e-08	0.270858	6.37409e-08	0.264395
1024	1.07229e-08	0.260708	1.6382e-08	0.257009

Table 0.4: Exterior Point, $p = 2$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.00296442	0	0.00659926	0
4	0.000282742	0.0953785	0.000873302	0.132333
8	8.86057e-05	0.31338	0.000152554	0.174686
16	3.48326e-05	0.393119	5.55976e-05	0.364446
32	1.03595e-05	0.29741	1.58275e-05	0.284679
64	2.75001e-06	0.265456	4.15279e-06	0.262378
128	7.03545e-07	0.255834	1.05931e-06	0.255083
256	1.77603e-07	0.252441	2.67204e-07	0.252245
512	4.45961e-08	0.251099	6.70816e-08	0.25105
1024	1.11721e-08	0.250518	1.68043e-08	0.250506

Table 0.5: Interior Point, $p = -1$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.10996	0	0.220401	0
4	0.11955	1.08721	0.239305	1.08577
8	0.123314	1.03148	0.246696	1.03088
16	0.124527	1.00984	0.249074	1.00964
32	0.124874	1.00279	0.249754	1.00273
64	0.124968	1.00075	0.249937	1.00073
128	0.124992	1.00019	0.249984	1.00019
256	0.124998	1.00005	0.249996	1.00005
512	0.124999	1.00001	0.249999	1.00001
1024	0.125	1	0.25	1

Table 0.6: Interior Point, $p = 0$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.0358862	0	0.0722528	0
4	0.0235504	0.656253	0.0473052	0.654718
8	0.0135744	0.576397	0.0272172	0.575353
16	0.00728703	0.536821	0.014594	0.536203
32	0.00377363	0.517856	0.00755263	0.517518
64	0.00191984	0.508752	0.00384108	0.508575
128	0.000968229	0.504327	0.00193681	0.504236
256	0.000486196	0.50215	0.000972482	0.502104
512	0.000243619	0.501072	0.000487261	0.501049
1024	0.00012194	0.500535	0.000243886	0.500523

Table 0.7: Interior Point, $p = 1$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.0111949	0	0.0228701	0
4	0.00435045	0.388611	0.00890522	0.389382
8	0.00138115	0.317473	0.00283066	0.317865
16	0.000390562	0.282781	0.000801012	0.282977
32	0.00010391	0.266053	0.00021319	0.266151
64	2.68012e-05	0.257926	5.49978e-05	0.257975
128	6.80578e-06	0.253936	1.39672e-05	0.25396
256	1.71479e-06	0.251961	3.51936e-06	0.251973
512	4.30375e-07	0.250978	8.83305e-07	0.250985
1024	1.07804e-07	0.250489	2.21261e-07	0.250492

Table 0.8: Interior Point, $p = 2$

N	one-norm	one-norm ratios	max norm	max norm ratios
2	0.00296442	0	0.00640922	0
4	0.000510448	0.172192	0.00122522	0.191166
8	3.12121e-05	0.0611465	0.000121046	0.0987951
16	1.51123e-05	0.484179	2.23931e-05	0.184996
32	7.29326e-06	0.482605	1.03862e-05	0.463811
64	2.32253e-06	0.318448	3.37982e-06	0.325415
128	6.47111e-07	0.278624	9.56452e-07	0.282989
256	1.70354e-07	0.263253	2.53945e-07	0.265507
512	4.36774e-08	0.256393	6.53989e-08	0.257532
1024	1.10565e-08	0.25314	1.65924e-08	0.253711

Problem 3

We typically discretized the interval $[0,1]$ into equally spaced points $x_j = jh$ for $j = 0 \dots N+1$ with $h = \frac{1}{N+1}$. Another common discretization is the *cell centered mesh*, in which $[0,1]$ is discretized into N cells. This approach is commonly used with finite-volume methods. The grid points are placed at centers of the cells: $x_j = (j - \frac{1}{2})h$ for $j = 1 \dots N$ where $h = \frac{1}{N}$. This type of discretization is more natural for some problems, particularly those with Neumann boundary conditions.

(a) We may write $u_{xx} = (-J)_x$, where $J = -u_x$ is the diffusive flux. Suppose we discretize this problem by using a centered difference to compute the flux at the cell edges, $J_{j-\frac{1}{2}}$, followed by another centered difference of the flux. Show that at interior points this gives the standard second-order discretization of u_{xx}.

(b) Again using the idea of flux differencing, derive the discrete approximation to u_{xx} at the first interior grid point adjacent to a boundary with Neumann boundary condition $u_x = g$.

(c) The cell-centered discretization seems awkward for Dirichlet boundary conditions because the spacing from the boundary to the first interior grid point is not equal to the spacing between interior grid points. One option for discretizing the second derivative at the first interior point is to use the finite difference stencil you derived in the first homework assignment. Alternately, a common discretization of u_{xx} at x_1 with given boundary value $u(0)$ is

$$\frac{2u(0) - 3u_1 + u_2}{h^2} = f_1.$$

What is the local trucation error of this discretization? What order accuracy do you expect for the numerical solution?

(a) Flux at the cell edges $J_{j-\frac{1}{2}}$ is equal to the negative flux of u evaluated at the $(j-1)$st grid point. That is,

$$J_{j-\frac{1}{2}} = -u_x((j-1)h) \qquad \text{and} \qquad J_{j+\frac{1}{2}} = -u_x(jh) \tag{0.7}$$

and we have finite difference formulas:

$$u_x((j-1)h) \approx \frac{u(x_j) - u(x_{j-1})}{h} \qquad \text{and} \qquad u_x(jh) \approx \frac{u(x_{j+1}) - u(x_j)}{h} \tag{0.8}$$

Thus,

$$u_{xx}(x_j) = (-(J_x))_j \approx \frac{J_{j-\frac{1}{2}} - J_{j+\frac{1}{2}}}{h} \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} \tag{0.9}$$

which is equal to the standard second-order discretization of u_{xx}.

(b) Let the boundary be 0 and so x_1 is the first grid point, located at $\left(1 - \frac{1}{2}\right)h$. If the flux at the boundary is g, i.e. if $u_x(0) = g$, then

$$u_{xx}(x_1) \approx \frac{J_{1-\frac{1}{2}} - J_{1+\frac{1}{2}}}{h} = \approx \frac{-u_x(0) - \dfrac{u(x_1) - u(x_2)}{h}}{h} = \frac{-hg - u(x_1) + u(x_2)}{h^2} \tag{0.10}$$

(c) Note we can write $x_0 = 0 = x_1 - \frac{h}{2}$ and $x_2 = \frac{3h}{2} = x_1 + h$. Thus we can utilize Taylor expansions

$$u(x_0) = u\left(x_1 - \frac{h}{2}\right) = u(x_1) - \frac{h}{2}u'(x_1) + \frac{h^2}{4 \cdot 2!}u''(x_1) - \frac{h^3}{8 \cdot 3!}u'''(x_1) + \frac{h^4}{16 \cdot 4!}u''''(x_1), \qquad \text{and} \tag{0.11}$$

$$u(x_2) = u(x_1 + h) = u(x_1) + hu'(x_1) + \frac{h^2}{2!}u''(x_1) + \frac{h^3}{3!}u'''(x_1) + \frac{h^4}{4!}u''''(x_1) \tag{0.12}$$

Thus,

$$\frac{2u(0) - 3u_1 + u_2}{h^2} = \frac{\frac{3h^2}{4}u''(x_1) + \mathcal{O}(h^3)}{h^2} = \frac{3}{4}u''(x_1) + \mathcal{O}(h) \tag{0.13}$$

So we expect $\mathcal{O}(h)$ for the accuracy of the numerical solution.