MAT 228A Notes

Sam Fleischer

September 27, 2016

1 Solving the Poisson Equation using Fourier Series

In 1-D, suppose $u_{xx} = f$ on $(0,1)$ with $u(0) = u(1) = 0$. Note that if $f = -(n\pi)^2 \sin(n\pi x)$ then the solution is $u = \sin(n\pi x)$ by observation. Reframe the problem as $Lu = f$ with $L = \dfrac{\partial^2}{\partial x^2}$. Then with $u_n \coloneqq \sin(n\pi x)$, we have

$$Lu_n = -(n\pi)^2 u_n \tag{1}$$

and thus u_n is an eigenfunction of L with eigenvalue $-(n\pi)^2$. We can show they are orthogonal in $L^2(0,1)$ by

$$\langle \sin(n\pi x), \sin(m\pi x)\rangle_{L^2(0,1)} = \int_0^1 \sin(n\pi x)\sin(m\pi x)\mathrm{d}x = \begin{cases} \frac{1}{2} & \text{if } n \neq m \\ 0 & \text{else} \end{cases}. \tag{2}$$

It also turns out these eigenfunctions form a complete set in L^2, that is $\{u_n\}_{n=1}^{\infty}$ is a basis. Thus, for a given $f \in L^2(0,1)$, there are coefficients a_n such that

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) \tag{3}$$

where the convergence is in $\|\cdot\|_{L^2(0,1)}$. The solution u of $Lu = f$ can also be written as a linear combination of u_n,

$$u(x) = \sum_{n=1}^{\infty} \beta_n u_n(x) \tag{4}$$

We can use orthogonality of u_n to explicitly compute a_n, and we obtain

$$a_n = 2\langle f(x), u_n(x)\rangle_{L^2(0,1)} \tag{5}$$

Finally, we have

$$L\Big[\sum \beta_n u_n(x)\Big] = \sum a_n u_n(x), \tag{6}$$

and we exploit orthogonality again, taking an inner product of both sides with u_m, and we obtain

$$\beta_n = -\frac{a_n}{(n\pi)^2} \qquad \text{for } n = 1, 2, \ldots. \tag{7}$$

In 2-D, it is the same basic idea. We have $\nabla^2 u = u_{xx} + u_{yy} = f$ on $(0,1)\times(0,1)$ with $u(x,y) = 0$ on the boundary. The eigenfunctions are given by

$$u_{n,m}(x) = \sin(n\pi x)\sin(m\pi y) \qquad \text{for } n, m = 1, 2, \ldots \tag{8}$$

with eigenvalues $\lambda_{n,m} = -(n^2 + m^2)\pi^2$. The remaining calculations of the general solution to $\nabla^2 u = f$ for any $f \in L^2((0,1)^2)$ are similar to the solution in 1-D.

2 Methods of Solving PDEs Numerically

2.1 Finite Differences

Given a PDE, a domain Ω, and boundary conditions, we take the following steps:

1) Discretize the Domain Ω, that is, represent Ω by a set of points. For example, draw a grid and the points are at the intersections and at the boundary.

2) Represent functions by values at those points.

3) Use discrete values to approximate derivatives using algebraic formulas

The result, assuming the PDE is linear, is an algebraic equation of the form

$$A(\underline{u}) = \underline{b}. \tag{9}$$

2.2 Finite Elements

Reformulate the problem as a variational problem. Rather than solving $\nabla^2 u = f$, define the functional F by

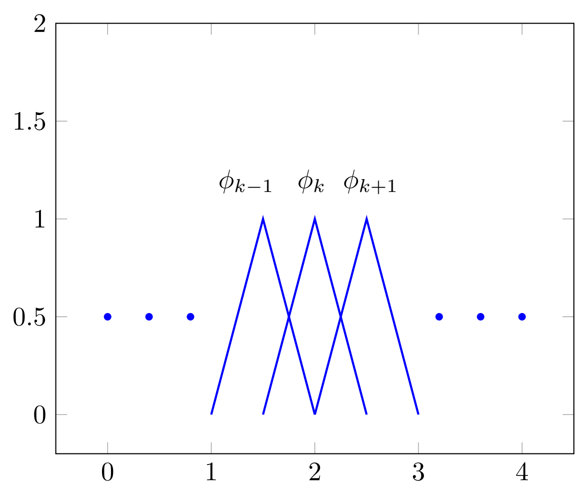
$$F(u) \coloneqq \int_\Omega \frac{1}{2} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u + uf \mathrm{d}x. \tag{10}$$

The minimizer of F also solves $\nabla^2 u = f$ (using the Euler-Lagrange equation). So we want to find $u \in S$ to minimize F where S is the space of "admissible" functions.

Now let's discretize the function space, i.e. choose a subset of the basis elements of S to represent S. Define this subset S_h, where $\dim S_h = N < \infty$. Then for any $u_h \in S_h$,

$$u_h(x) = \sum_{k=1}^{N} a_k \phi_k(x) \tag{11}$$

A good basis $\{\phi_k\}_{k=1}^N$ for S_h are tent functions with small overlap, for example,



It turns out S_h is the space of "connect-the-dot" functions (piecewise linear functions).

Then we can approximate a function $u \in L^2$ by its closest representation using the basis $\{\phi_k\}$.