MAT 228A Notes

Sam Fleischer

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1 Recall from last time

$$u_{xx} = f \qquad u_x(0) = \alpha \qquad u_x(1) = \beta \tag{1}$$

The boundary values are unknown, and we do a second order discretization:

$$\frac{1}{h^2}\begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}\vec{u} = \begin{pmatrix} f_0 + \frac{2\alpha}{h} \\ f_1 \\ \vdots \\ f_N \\ f_{N+1} - \frac{2\beta}{h} \end{pmatrix} \tag{2}$$

The solution is not unique. Condition for a solution:

$$\int_0^1 f(x)\mathrm{d}x = \beta - \alpha \tag{3}$$

(this is derived by requiring f to be orthogonal to the kernel of the adjoint of $\frac{\partial^2}{\partial x^2}$). Let

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix} \tag{4}$$

and note that $A\vec{1} = \vec{0}$ (i.e. the sum of each row is 0) and thus $\vec{1} \in \ker(A)$. In fact, $\ker(A) = \left[\vec{1}\right]$. This means A is singular (which makes sense since the PDE has non-unique solutions).

2 Discretization

Suppose we have $A\vec{u} = \vec{b}$ but A is singular. For this to have a solution, we require $b \in \operatorname{ran}(A)$, i.e. $b \perp \ker(A^*)$. If $\dim\ker(A^*) = n$, this translates to n requirement.

For A given above,

$$A^* = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & & \\ -2 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 2 \\ & & & & & 1 & -2 \end{pmatrix} \tag{5}$$

and note

$$\vec{v} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \frac{1}{2} \end{pmatrix} \tag{6}$$

spans $\ker(A^*)$. So $b \in \operatorname{ran}(A)$ if $\vec{v}^T \cdot \vec{b} = 0$ (i.e. $b \perp \ker(A^*)$). This translates to

$$\frac{1}{2} f_0 + \frac{\alpha}{h} + f_1 + \cdots + f_N + \frac{1}{2} f_{N+1} - \frac{\beta}{h} = 0 \tag{7}$$

$$\frac{h}{2} f_0 + h \sum_{j=1}^{N} f_j + \frac{h}{2} f_{N+1} = \beta - \alpha \tag{8}$$

This precisely matches the continuous condition (it is a trapezoidal approximation to the integral):

$$\frac{h}{2} f_0 + h \sum_{j=1}^{N} f_j + \frac{h}{2} f_{N+1} = \int_0^1 f(x) \mathrm{d}x + \mathcal{O}(h^2) \tag{9}$$

One caveat is that even if $\int_0^1 f(x)\mathrm{d}x = \beta - \alpha$ holds, the discrete condition $\vec{v}^T \cdot \vec{b} = 0$ may not hold.

One way to solve $A\vec{u} = \vec{b}$ is to use an iterative scheme.

$$\vec{u}^{(k+1)} = T\vec{u}^{(k)} + \vec{c} \tag{10}$$

To get a solution, we need $\vec{b} \in \operatorname{ran}(A)$. So we project onto the range: Let P be the orthogonal projection onto the range.

$$P\vec{b} = \vec{b} - \underbrace{\frac{\vec{v}^T \cdot \vec{b}}{\vec{v}^T \cdot \vec{v}}}_{=\mathcal{O}(h^2)} \vec{v} \in \operatorname{ran}(A) \tag{11}$$

3 Direct Solve

Perturbed system looks like

$$A\vec{u} = \vec{b} - \lambda\vec{v} \tag{12}$$

$$A\vec{u} + \lambda\vec{v} = \vec{b} \tag{13}$$

We've added an unknown: λ, so we need a new equation. Let's force the solution with mean 0, i.e.

$$\vec{1}^T \cdot \vec{u} = 0 \tag{14}$$

So $\vec{u}$ and λ are the unknowns:

$$\left(\begin{array}{c} \vec{u} \\ \hdashline \lambda \end{array}\right) = \left(\begin{array}{c:c} A & \vec{v} \\ \hdashline \vec{1}^T & 0 \end{array}\right) \left(\begin{array}{c} \vec{b} \\ \hdashline 0 \end{array}\right) \tag{15}$$

4 2D Equation

$$\nabla^2 u = f \qquad \text{on } (0,1)^2 \tag{16}$$

and Dirichet boundaries. Supposing equal spacing in x and y directions (regular grid),

$$u_{xx} + u_{yy} = f \tag{17}$$

$x_i = ih$, $y_j = jh$, for $i, j = 0, 1, \ldots, n+1$ with $h = \frac{1}{n+1}$. So $u_{ij} = u(x_i, y_j)$.

$$(u_{xx})_{ij} \approx \frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{h^2} \tag{18}$$

$$(u_{yy})_{ij} \approx \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h^2} \tag{19}$$

So the discrete approximate equation is

$$\frac{u_{i-1,j} + u_{i,j-1} - 4u_{ij} + u_{i+1,j} + u_{i,j+1}}{h^2} = f_{ij} \tag{20}$$

Stensil of the difference operator:

$$\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \tag{21}$$

We get n^2 linear equations: $A\vec{u} = \vec{b}$. To use this form we need to arrange u_{ij} into a vector. The standard way is row-wise ordering. For example:

$$\begin{bmatrix} \text{seventh} & \text{eigth} & \text{ninth} \\ \text{fourth} & \text{fifth} & \text{sixth} \\ \text{first} & \text{second} & \text{third} \end{bmatrix} \tag{22}$$

So

$$\vec{u} = \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} \tag{23}$$

What does the matrix look like (in general, not 9×9)? We get a penta-diagonal structure.

$$A = \frac{1}{h^2}\begin{pmatrix} \cdots & 1 & \cdots & 1 & -4 & 1 & \cdots & 1 & \cdots \end{pmatrix} \tag{24}$$

with 1's on the 1^{th} and n^{th} sub- and super-diagonals.

5 3D Equation

The 3D equation is similar, but the diagonal has a septa-diagonal structure:

$$A = \frac{1}{h^2}\begin{pmatrix} \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & -6 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots \end{pmatrix} \tag{25}$$

with 1's on the 1^{th}, n^{th}, and $(n^2)^{\text{th}}$ sub- and super-diagonals. Conceptually there is not much difference between 2D and 3D, but computationally, there is a lot less sparsity in 3D, and so it is much, much more expensive.