

Math 228A
Homework 1
Due Friday, 10/14/16

1. Let L be the linear operator $Lu = u_{xx}$, $u_x(0) = u_x(1) = 0$.
 - (a) Find the eigenfunctions and corresponding eigenvalues of L .
 - (b) Show that the eigenfunctions are orthogonal in the $L^2[0, 1]$ inner product:

$$\langle u, v \rangle = \int_0^1 uv \, dx.$$

- (c) It can be shown that the eigenfunctions, $\phi_j(x)$, form a complete set in $L^2[0, 1]$. This means that for any $f \in L^2[0, 1]$, $f(x) = \sum_j \alpha_j \phi_j(x)$. Express the solution to

$$u_{xx} = f, \quad u_x(0) = u_x(1) = 0, \tag{1}$$

as a series solution of the eigenfunctions.

- (d) Note that equation (1) does not have a solution for all f . Express the condition for existence of a solution in terms of the eigenfunctions of L .
2. Define the functional $F : X \rightarrow \mathbb{R}$ by

$$F(u) = \int_0^1 \frac{1}{2} (u_x)^2 + fu \, dx,$$

where X is the space of real valued functions on $[0, 1]$ that have at least one continuous derivative and are zero at $x = 0$ and $x = 1$. The Frechet derivative of F at a point u is defined to be the linear operator $F'(u)$ for which

$$F(u + v) = F(u) + F'(u)v + R(v),$$

where

$$\lim_{\|v\| \rightarrow 0} \frac{\|R(v)\|}{\|v\|} = 0.$$

One way to compute the derivative is

$$F'(u)v = \lim_{\epsilon \rightarrow 0} \frac{F(u + \epsilon v) - F(u)}{\epsilon}.$$

Note that this looks just like a directional derivative.

- (a) Compute the Frechet derivative of F .
 - (b) $u \in X$ is a critical point of F if $F'(u)v = 0$ for all $v \in X$. Show that if u is a solution to the Poisson equation

$$u_{xx} = f, \quad u(0) = u(1) = 0,$$

then it is a critical point of F .

- (c) Let X_h be a finite dimensional subspace of X , and let $\{\varphi_i(x)\}$ be a basis for X_h . This means that all $u_h \in X_h$ can be expressed as $u_h(x) = \sum_i u_i \varphi_i(x)$ for some constants u_i . Thus we can identify the elements of X_h with vectors \vec{u} that have components u_i . Let $G(\vec{u}) = F(u_h)$. Show that the gradient of G (whose components are $(\nabla G)_j = \frac{\partial G}{\partial u_j}$) is of the form $\nabla G(\vec{u}) = A\vec{u} + \vec{b}$, and write expressions for the elements of the matrix A and the vector \vec{b} .
- (d) Divide the unit interval into a set of $N + 1$ equal length intervals $I_i = (x_i, x_{i+1})$ for $i = 0, \dots, N$. The endpoints of the intervals are $x_i = ih$, where $h = 1/(N + 1)$. Let X_h be the subspace of X such that the elements u_h of X_h are linear on each interval, continuous on $[0,1]$, and satisfy $u_h(0) = u_h(1) = 0$. X_h is an N dimensional space with basis elements

$$\varphi_i(x) = \begin{cases} 1 - h^{-1}|x - x_i| & \text{if } |x - x_i| < h \\ 0 & \text{otherwise} \end{cases},$$

for $i = 1, \dots, N$. Compute the matrix A from the previous problem that appears in the gradient.

Finite element methods are based on these “weak formulations” of the problem. The Ritz method is based on minimizing F and the Galerkin method is based on finding the critical points of $F'(u)$.

3. (a) Using a Taylor expansion, derive the finite difference formula to approximate the second derivative at x using function values at $x - h/2$, x , and $x + h$. How accurate is the finite difference approximation?
- (b) Perform a refinement study to verify the accuracy of the difference formula you derived.
- (c) Derive an expression for the quadratic polynomial that interpolates the data $(x - h/2, u(x - h/2))$, $(x, u(x))$, $(x + h, u(x + h))$. How is the finite difference formula you derived in problem 3a related to the interpolating polynomial?