MAT 228B Notes

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1 Review

- Trapezoidal Rule

$$\frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2}\big(f(y^n) + f(y^{n+1})\big)$$

This is actually a centered difference around the half-time level.

$$\frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2}\big(f(y^n) + f(y^{n+1})\big) = f(y^{n+1/2}) + \mathcal{O}\big((\Delta t)^2\big)$$

using Taylor Series.

- BDF-2

$$\frac{3y^{n+1} - 4y^n + y^{n-1}}{2\Delta t} = f(y^{n+1})$$

- Both of these are A-Stable (both of their regions of absolute stability contain the left half of the complex plane, i.e. if the solution *should* decay, it does, independent of timestep).

2 Region of Absolute Stability for the Trapezoidal Rule

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$$\begin{aligned}
\frac{y^{n+1} - y^n}{\Delta t} &= \frac{\lambda}{2}\big(y^n + y^{n+1}\big) \\
\Big(1 - \frac{z}{2}\Big)y^{n+1} &= \Big(1 + \frac{z}{2}\Big)y^n \\
y^{n+1} &= \left(\frac{2+z}{2-z}\right)y^n
\end{aligned}$$

So the region of absolute stability is defined by

$$\begin{gathered}
\left|\frac{2+z}{2-z}\right| < 1 \\
|2+z| < |2-z|
\end{gathered}$$

This is the left half of the complex plane because the above says the distance to -2 must be less than the distance to 2. This is the minimal amount you need to be A-Stable.

- Trapezoidal Rule for Diffusion with standard 3-point second order (1D) spacial discretization is called "Crank-Nicolson." The scheme is 2nd order in space and time and is unconditionally stable (pick any timestep you want).

3 Runga-Kutta Methods vs. Linear Multistep Methods

Runga-Kutta are single-step multi-stage methods, whereas Linear Multistep Methods use information from past time.

3.1 Runga-Kutta Methods

- 2nd Order "Improved Euler" Runga-Kutta Method

$$y^* = y^n + \Delta t f(y^n)$$
$$y^{n+1} = y^n + \frac{Dt}{2}(f(y^n) + y(y^*))$$

Note we are only using y^n, not any other past timesteps. We are using stages to get y^{n+1}.

- General form of an r-stage RK Method:

$$y' = f(t, y)$$
$$y_i^* = y^n + \Delta t \sum_{j=1}^{r} A_{ij} f(t_n + c_j \Delta t, y_j^*) \qquad i = 1, \dots, r$$
$$y^{n+1} = y^n + \Delta t \sum_{j=1}^{r} b_j f(t_n + c_j \Delta t, y_j^*)$$

This is an implicit Runga-Kutta scheme.. Explicit RK schemes sum from $j = 1$ to $i - 1$ rather than r.

- A is called the Runga-Kutta Matrix, b are called the RK weights, and c are called the RK nodes. These are expressed it Butcher Tables in the form

$$\begin{array}{c|c} \underline{c} & A \\ \hline & \underline{b}^T \end{array}$$

$$y_1^* = y^n$$
$$y_2^* = y^n + \Delta t f(t_n, y_1^*)$$
$$y^{n+1} = y^n + \frac{\Delta t}{2}(f(t_n, y^*) + f(t_n + \Delta t, y_2^*))$$

gives the Butcher Table

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

- Classical RK4 Scheme

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

If A is strictly lower-triangular, it is an explicit time method. It doesn't have to be, though, and is very high-order accurate in time, but it is very expensive.

- DIRK are "diagonally implicit" RK methods, which include nonzero elements on the diagonals:
  - TR-DBF2 Method

$$y^* = y^n + \frac{\Delta t}{4}(f(y^n) + f(y^*))$$
$$y^{n+1} = \frac{1}{3}\big(4y^* - y^n + \Delta t f(y^{n+1})\big)$$

This is a combination of the Trapezoidal Rule and BDF2. This tries to blend the best of both worlds. The Butcher Table is

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1 & 1/3 & 1/3 & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 \end{array}$$

3.2 Linear Multistep Methods

- "Adams Bashforth 2" or "AB2"

$$\frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2}\big(3f(y^n) - f(y^{n-1})\big)$$

  This uses data at y^n and y^{n-1} to get y^{n+1}. This is an extrapolation in time.

- BDF-2 is an implicit time method.

- General form of an r-step Method

$$\sum_{j=0}^{r} \alpha_j y^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j f(y^{n+j})$$

- The Adams Methods

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$$y^{n+r} - y^{n+r-1} = \Delta t \sum_{j=0}^{r} \beta_j f(y^{n+j})$$

  – If $\beta_r = 0$, this is called Adams-Bashforth (eg. Forward Euler is an AB method)

$$y^{n+1} - y^n = \Delta t f(y^n)$$

  and AB2 is

$$y^{n+1} - y^n = \Delta t \frac{1}{2}\big(3f(y^n) - f(y^{n-1})\big)$$

  – If $\beta_r \neq 0$, Adams Moulton Method implicit time (e.g. trapezoidal rule)

$$y^{n+1} - y^n = \Delta t \frac{1}{2}\big(f(y^n) + f(y^{n+1})\big)$$

- BDF-Backward (difference formulas)

$$\sum_{j=0}^{r} \alpha_j y^{n+j} = \Delta t \beta_r f(y^{n+r})$$

  These are good for stiff equations. Backward Euler is BDF1. High order BDFs are great if you have lots of eigenvalues clustered along the negative real axis.