MAT 228B Notes

Sam Fleischer

February 10, 2017

1 Last Time

- Laplacian in 2D: $\nabla^2 u = u_{xx} + u_{yy} \qquad L = L_x + L_y$.
- LOD method.. think about as a fractional stepping approach
- Method of lines: $\frac{\mathrm{d}u}{\mathrm{d}t} = L_x u + L_y u$. Solve $\frac{\mathrm{d}u}{\mathrm{d}t} = L_x u$ and then $\frac{\mathrm{d}u}{\mathrm{d}t} = L_y u$.

2 ADI (class of) Schemes

- This is another approach "Alternating Direction Implicit"
- For $2D$ diffusion, a common (Peaceman-Rachford) ADI scheme is

$$\left(I - \frac{d\Delta t}{2}L_x\right)u^* = \left(I + \frac{d\Delta t}{2}L_y\right)u^n$$

- Kind of looks like Crank Nicolson.. but y explicit and x implicit at the same time (for the intermediate step)
- The follow this up with the secondary step:

$$\left(I - \frac{d\Delta t}{2}L_y\right)u^{n+1} = \left(I + \frac{d\Delta t}{2}L_x\right)u^*.$$

- What is the work?
  - Step 1: solve N_y tridiagonal systems of size N_x (work is $\mathcal{O}(N_xN_y) = \mathcal{O}(N)$, which is optimal, ($N$ is the number of gridpoints in $2D$))
  - Step 2:solve N_x tridiagonal systems of size N_y (work is $\mathcal{O}(N_xN_y) = \mathcal{O}(N)$)
  - So the total work is $\mathcal{O}(N)$.
- Is it stable?
  - Doing the discrete Fourier transform gives

$$\left(1 + 2b\Delta t\sin^2\left(\frac{\xi_1\Delta x}{2}\right)\right)\hat{u^*} = \left(1 - 2b\Delta t\sin^2\left(\frac{\xi_2\Delta y}{2}\right)\right)\hat{u^n}$$
$$\left(1 + 2b\Delta t\sin^2\left(\frac{\xi_1\Delta y}{2}\right)\right)\hat{u^{n+1}} = \left(1 - 2b\Delta t\sin^2\left(\frac{\xi_2\Delta x}{2}\right)\right)\hat{u^*}$$

  - Solving this gives

$$\hat{u^{n+1}} = \underbrace{\frac{\left(1 - 2b\Delta t\sin^2(\frac{\xi_1\Delta x}{2})\right)\left(1 - 2b\Delta t\sin^2(\frac{\xi_2\Delta y}{2})\right)}{\left(1 + 2b\Delta t\sin^2(\frac{\xi_2\Delta y}{2})\right)\left(1 + 2b\Delta t\sin^2(\frac{\xi_2\Delta x}{2})\right)}}_{g(\xi_1,\xi_2)}\hat{u^n} \tag{1}$$

  - Note that $g(\xi_1, \xi_2) = g_1(\xi_1)g_2(\xi_2)$ where g_1 and g_2 are the ampification factors of Crank-Nicolson.
  - So, $|g(\xi_1, \xi_2)| = |g_1(\xi_1)| \cdot |g_2(\xi_2)| \leq 1$, so it is unconditionally stable.

- Is it consistent?
  - Multiply first equation by $\left(I + \frac{b\Delta t}{2} L_x\right)$, so.

$$\underbrace{\left(I + \frac{b\Delta t}{2} L_x\right)\left(I - \frac{b\Delta t}{2} L_x\right)}_{\text{these commute}} u^* = \left(I + \frac{b\Delta t}{2} L_x\right)\left(I + \frac{b\Delta t}{2} L_y\right) u^n \tag{2}$$

  - So,

$$\left(I - \frac{b\Delta t}{2} L_x\right)\left(I + \frac{b\Delta t}{2} L_x\right) u^* = \left(I + \frac{b\Delta t}{2} L_x\right)\left(I + \frac{b\Delta t}{2} L_y\right) u^n \tag{3}$$

  - Then we eliminate u^* by using the second step of the scheme.

$$\left(I - \frac{b\Delta t}{2} L_x\right)\left(I - \frac{b\Delta t}{2} L_y\right) u^{n+1} = \left(I + \frac{b\Delta t}{2} L_x\right)\left(I + \frac{b\Delta t}{2} L_y\right) u^n \tag{4}$$

  - Expanding this gives

$$\left(I - \frac{b\Delta t}{2} L_x - \frac{b\Delta t}{2} L_y + \frac{b^2\Delta t^2}{4} L_x L_y\right) u^{n+1} = \left(I + \frac{b\Delta t}{2} L_x + \frac{b\Delta t}{2} L_y + \frac{b^2\Delta t^2}{4} L_x L_y\right) u^n \tag{5}$$

  - So this is a small perturbation of Crank-Nicolson.

$$\underbrace{\left(I - \frac{b\Delta t}{2} L\right) u^{n+1} = \left(I + \frac{b\Delta t}{2} L\right) u^n}_{\text{Crank-Nicolson, where } L \text{ is the discrete 2D Laplacian}} \quad - \frac{b^2\Delta t^2}{4} L_x L_y \left(u^{n+1} - u^n\right) \tag{6}$$

  - So,

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{b}{2} L u^n + \frac{b}{2} L u^{n+1} - \frac{b^2 \Delta t^2}{4} L_x L_y \left(\frac{u^{n+1} - u^n}{\Delta t}\right) \tag{7}$$

  - How big is this (do a Taylor Series)?
  - We know plugging in the solution to the PDE into

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{b}{2} L u^n + \frac{b}{2} L u^{n+1} \tag{8}$$

    gives us LTE for CN (which is second-order).
  - In the limit of $\Delta t, \Delta x \to 0$,

$$\Delta t^2 L_x L_y \left(\frac{u^{n+1} - u^n}{\Delta t}\right) \to \Delta t^2 \frac{\partial^5}{\partial y^2 \partial x^2 \partial t} u \tag{9}$$

  - These extra terms contribute $\mathcal{O}\left(\Delta t^2\right)$ to the truntation error in addition to the LTE of CN.
- Thus, the Lax Equivalence Theorem says this is convergent. It is second-order in space and time and unonditionally stabl $\mathcal{O}(N)$ work per timestep.
- What are the boundary conditions shoud hold for u^*?
  - The first step is Forward Euler in y direction for half a timestep, and
  - Backward Euler in the x direction for half a timestep.
  - It's a consistent discretization of the PDE to take step length $\frac{\Delta t}{2}$.
  - So, $u^* = u^{n+1/2} + \mathcal{O}\left(\Delta t^2\right)$.
  - This means the single-step error in Forward or Backward Euler is $\mathcal{O}\left(\Delta t^2\right)$.