MAT 228B Notes

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1 1D Diffusion Equation

$$\frac{u^{n+1} - u^n}{\Delta t} = Lu^n \tag{1}$$

The eigenvalues of L are $\lambda_k = \dfrac{2D}{(\Delta x)^2}(\cos(k\pi\Delta x) - 1)$, $k = 1, \dots, N$ with eigenfunctions $(v_k)_j = \sin(k\pi x_j)$. Small k gives $\lambda_k = \mathcal{O}(1)$ (low spacial frequencies) and large timescale $\mathcal{O}(1)$. Large k gives $\lambda_k = \mathcal{O}\Big(\frac{1}{(\Delta x)^2}\Big)$ (high scpacial frequencies) and small timescale $\mathcal{O}\big((\Delta x)^2\big)$.

We don't want (generically) the time step on the fastest time scales because they (generically) go away so quickly in time.

For stiff equations, there are methods which perform well.. (jump over points with high frequencies without harm).

1.1 Backward Euler

For $y' = f(t, y)$. Backward Euler is $\frac{y^{n+1} - y^n}{\Delta t} = f(t_{n+1}, y^{n+1})$.

The Region of Absolute Stability for backward Euler satisfies

$$\frac{y^{n+1} - y^n}{\Delta t} = \lambda y^{n+1} \tag{2}$$

$$y^{n+1} - y^n = \Delta t \lambda y^{n+1} \tag{3}$$

$$(1 - z)y^{n+1} = y^n \tag{4}$$

$$y^{n+1} = \frac{1}{1 - z} y^n \tag{5}$$

which means we need $\frac{1}{|1-z|} < 1$ or $|1 - z| > 1$. This is outside the open unit ball around $1 + 0i$. That is, $\mathbb{C} \setminus B((1, 0))$. Backward Euler for diffusion "unconditionally stable". This is an "absolutely stable" or an "A-Stable" method (the entire left plane). A method is "A-Stable" if the entire left half plane of $\mathbb{C}$ is contained in the region of absolute stability, which means that a solution that *should* decay actually *does* in discrete time. The cost to Backward Euler is that the equation is implicit.

1.2 Accuracy

Both Forward Euler and Backward Euler are first-order accurate in time. That means is that the truncation errors or the discretization errors are $\mathcal{O}(\Delta t)$.

Let $u(x, t)$ be the solution to $u_t = Du_{xx}$ with appropriate boundary and initial conditions. The Local Truncation Error τ for Forward Euler for Diffusion is (the amount by which the discrete solution fails to satisfy the actual solution). Out difference equation is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\big(u_{j-1}^n - 2u_j^n + u_{j+1}^n\big) \tag{6}$$

So,

$$\tau_j^n = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} - \frac{D}{(\Delta x)^2}(u(x_{j-1}, t_n) - 2u(x_j, t_n) + u(x_{j+1}, t_n)) \tag{7}$$

Expanding around $x = x_j$, $t = t_n$ as $\Delta x, \Delta t \to 0$. The above equation requires three Taylor expansions, and the result is

$$\tau_j^n = u_t + \frac{\Delta t}{2}u_{tt} + \mathcal{O}\big((\Delta t)^2\big) - D\bigg(u_{xx} + \frac{(\Delta x)^2}{12}u_{xxxx} + \mathcal{O}\Big((\Delta x)^2\Big)\bigg) \tag{8}$$

So as $\Delta x, \Delta t \to 0$,

$$\tau_j^n = \frac{\Delta t}{2}u_{tt} - \frac{D}{12}(\Delta x)^2 u_{xxxx} + \text{h.o.t.} \tag{9}$$

We have

$$u_t = Du_{xx} \tag{10}$$

$$u_{tt} = (u_t)_t = D(u_{xx})_t = D^2 u_{xxxx} \tag{11}$$

So,

$$\tau_j^n = \left(\frac{\Delta t}{2}D^2 - \frac{D}{12}(\Delta x)^2\right)u_{xxxx} + \text{h.o.t.} \tag{12}$$

It is the same analysis for Backward Euler, but

$$\tau_j^n = -\left(\frac{D^2}{2}\Delta t + \frac{D}{12}(\Delta x)^2\right)u_{xxxx} + \text{h.o.t.} \tag{13}$$

How do we get higher-order accuracy? Specifically, how do we get 2nd order in time? If we average the Forward and Backward Euler, the result is a useful scheme. Remember,

$$\text{F.E. is } \frac{y^{n+1} - y^n}{\Delta t} = f(y^n) \qquad \text{B.E. is } \frac{y^{n+1} - y^n}{\Delta t} = f(y^{n+1}) \tag{14}$$

So (trapzoidal rule),

$$\frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2}\big(f(y^n) + f(y^{n+1})\big) \tag{15}$$

or (a midpoint rule, not really used because it have a very tiny stability region),

$$\frac{y^{n+1} - y^{n-1}}{2\Delta t} = f(y^n) \tag{16}$$

or (BDF2, i.e. one-sided backward-difference in time),

$$\frac{3y^{n+1} - 4y^n + y^{n-1}}{2\Delta t} = f(y^{n+1}) \tag{17}$$

These are all relatively second methods to get second-order accuracy in time. It turns both the Trapezoidal Rule and BDF2 are second-order accurate in time A-stable.