MAT 228B Notes

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1 Von Neumann Analysis

- This is relatively easy for one-step methods (includes only u^n and u^{n+1})
- Assume $u_j^n = e^{i\xi x_j}$ (assume you have an eigenvector).
- Then $u_j^{n+1} = g(\xi)e^{i\xi x_j}$.
- And we want $|g(\xi)| \leq 1 + \alpha\Delta t$ for all ξ.
- $$\begin{aligned} u_j^{n+1} &= u_j^n + \frac{\Delta t D}{\Delta x^2}\left(u_{j-1}^n - 2u_j^n +{}_{j+1}^n\right) \\ u_j^{n+1} &= e^{i\xi x_j} + \frac{\Delta t D}{\Delta x^2}\left(e^{-i\xi(x_j - \Delta x)} - 2e^{i\xi x_j} + e^{i\xi(x_j + \Delta x)}\right) \\ &= \left(1 + \frac{\Delta t D}{\Delta x^2}\left(e^{-i\xi\Delta x} - 2 + e^{i\xi\Delta x}\right)\right)e^{i\xi x_j} \end{aligned}$$

2 Show the Leapfrog Scheme is Unstable for Diffusion

- $$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{D}{\Delta x^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right)$$
- Assume $u_j^n = g^n e^{i\xi x_j}$.
- We get
  $$\frac{g^{n+1} - g^{n-1}}{2\Delta t} = \frac{D}{\Delta x^2}g^n\left(-4\sin^2\left(\frac{\xi\Delta x}{2}\right)\right)$$
- This is a recursion relation for the amplification factor. Dividing by g^{n-1} gives
  $$g^2 - 1 = -\underbrace{\frac{8D\Delta t}{\Delta x^2}\sin^2\left(\frac{\xi\Delta x}{2}\right)}_{\text{define this to be } \beta(\xi)}g$$
  and we can do this because $g = 0$ automatically gives stability ($|g(\xi)| \leq 1 + \alpha\Delta t$).
- We know $\beta \geq 0$ for all ξ. And the product of the roots g_1 and g_2, $g_1 g_2 = -1$. So if one of the roots is close to 0, then one is far from 0.
- Calculating the roots of $g^2 + \beta(\xi)g - 1 = 0$ gives
  $$g_{+/-} = \frac{1}{2}\left(-\beta \pm \left(\beta^2 + 4\right)^{\frac{1}{2}}\right)$$
- In particular, we see
  $$g_- = \frac{1}{2}\left(-\beta - \left(\beta^2 + 4\right)^{\frac{1}{2}}\right)$$
  In particular, $|g_-| > 1$ for $\beta \neq 0$.

3 Forced Diffusion

- $u_t = bu_{xx} + f$.
- Forward Euler presents a timestep restriction.. in 1D, $\Delta t \leq \frac{\Delta x^2}{2b}$.
- We can write

$$u_j^{n+1} = \frac{b\Delta t}{\Delta x^2} u_{j-1}^n + \left(1 - \frac{2b\Delta t}{\Delta x^2}\right) u_j^n + \frac{b\Delta t}{\Delta x^2} u_{j+1}^n$$

- In the ∞-norm analyis, we wanted the diagonal terms to be ≥ 0....
- In 2D, $\Delta t \leq \frac{\Delta x^2}{4b}$.
- In 3D, $\Delta t \leq \frac{\Delta x^2}{6b}$.
- This is because 4 and 6 nearest neighbors in 2 and 3D, respectively.
- In practice, we generally use an implicit time method to avoid these restrictions.
- Let L be the discrete Laplacian. Then

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} &= \frac{b}{2}\left(Lu^n + Lu^{n+1}\right) + f^{n+\frac{1}{2}} \\ \implies \left(I - \frac{d\Delta t}{2} L\right) u^{n+1} &= \left(I + \frac{d\Delta t}{2} L\right) u^n + \Delta t f^{n+\frac{1}{2}} \end{aligned}$$

- We need to solve a linear system each time step.. accounting (how much work)?
  - Grid spacing $\Delta x \propto \frac{1}{N}$.
  - Assume $\Delta t \propto \Delta x$ in the implicit method.
  - We have to take $\mathcal{O}\left(\frac{1}{\Delta x}\right) = \mathcal{O}(N)$ timesteps.
  - Cost per timestep (1D)? Since the matrix is tri-diagonal and diagonally dominant, it's just $\mathcal{O}(N)$ per step.
- So the cost of the implicit method is $\mathcal{O}\left(N^2\right)$.
- What about Forward Euler?
  - The number of timesteps $\propto \frac{T}{\Delta t} = \mathcal{O}\left(\frac{1}{\Delta x^2}\right) = \mathcal{O}\left(N^2\right)$.
  - Work per timestep? It's $\mathcal{O}(N)$ (since it's multiplication by a known sparse, tridiagonal matrix)
- So the total work is $\mathcal{O}\left(N^3\right)$. So asymptotically, Forward Euler is more expensive.
- Using sparse solvers and sparse matrix arithmetic makes all the difference!

4 Next Time?

- $u_t = -ku_{xxxx}$
- Came up in Bob's research about thin filaments (bending structures) in viscous fluids.
- Through the Von Neumann Analysis, you get

$$\Delta t \leq C\Delta x^4 \qquad \text{for stability}$$

- The work for Crank Nicolson is the same.. requires $\mathcal{O}\left(N^2\right)$ steps.
- The explicit time, though, requires $\mathcal{O}\left(N^5\right)$ steps.