MAT 228B Notes

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1 Systems of Equations

Consider $u_t + Au_x = 0$ where A is a diagonalizable matrix with real eigenvalues. Then $A = V\Lambda V^{-1}$, so

$$\begin{aligned} u_t + V\Lambda V^{-1}u_x &= 0 \\ V^{-1}u_t + \Lambda V^{-1}u_x &= 0 \end{aligned}$$

Defining $w = V^{-1}u$ gives

$$w_t + \Lambda w_x = 0$$

which is a set of de-coupled advection equations (one for each eigenspace). So upwind, use the sign of the elements of λ. Now define Λ^+ and Λ^- as

$$\begin{aligned} \Lambda^+ &= \frac{\Lambda + |\Lambda|}{2} \\ \Lambda^- &= \frac{\Lambda - |\Lambda|}{2} \end{aligned}$$

This helps with notation:

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\Lambda^+\left(w_j^n - w_{j-1}^n\right) - \frac{\Delta t}{\Delta x}\Lambda^-\left(w_{j+1}^n - w_j^n\right)$$

Back to original variables:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}A^+\left(u_j^n - u_{j-1}^n\right) - \frac{\Delta t}{\Delta x}A^-\left(u_{j+1}^n - u_j^n\right)$$

where $A^+ = V\Lambda^+V^{-1}$ and $A^- = V\Lambda^-V^{-1}$. *Note: for nonlinear or variable coefficient problems, we would decompose these differences locally "into left-moving stuff and right-moving stuff".*

2 Conservation Laws and Finite Volume Methods

This is not in the book. The general form of a conservation law is

$$u_t + (f(u))_x = 0$$

where f is a flux function. If $f(u) = au$, we recover the advection equation. If $f(u) = \frac{1}{2}u^2$, then $f'(u) = u$, so we recover $u_t + uu_x = 0$ (inviscid Burgers equation (model problem for nonlinear conservation laws - develops shocks etc. in finite time)).

Remember the integral form of the conservation law is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u\,\mathrm{d}x = f(u(x_1,t)) - f(u(x_2,t))$$

If u is a density, the the integral is the "total amount of stuff" in the interval $[x_1, x_2]$. "Technically, stuff changes only from movement across the boundary." How do we deal with an integral conservation law numerically?

2.1 Finite Volume Methods

In a finite-difference method, when we write u_j, we mean this is an approximation of $u(x_j)$. We are using approximate values of the function in our scheme. In a finite-volume method, we don't divide the space into a set of points. Rather, we divide the domain into a set of volumes c_j. In 1D, for finite-difference methods, we consider points x_j, but for finite-volume methods, we consider volumes $c_j = \left[x_{j-1/2}, x_{j+1/2}\right]$. We will let

$$u_j \approx \frac{1}{|c_j|}\int_{c_j} u(x)\mathrm{d}x,$$

i.e. the average value of the function over the volume c_j. In 1D this looks like

$$u_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x)\mathrm{d}x$$

Note that

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x)\mathrm{d}x = u(x_j) + \mathcal{O}(\Delta x)^2$$

using midpoint rule and numerical quadrature. u at the center is a 2nd-order accurate approximation to the average (and vice-versa).

What is the conservation Law on c_j?

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x)\mathrm{d}x = f(u(x_{j-1/2}), t) + f(u(x_{j+1/2}), t)$$

Integrating in time from t_n to t_{n+1} gives

$$\Delta x\left(u_j^{n+1} - u_j^n\right) = \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) - f(u(x_{j+1/2}, t))\mathrm{d}t$$

Let $F_{j+1/2} := \frac{1}{\Delta t^2}\int_{t_n}^{t_{n+1}} F(u(x_{j+1/2}, t))\mathrm{d}t$. Also, note

$$\begin{aligned}
\Delta x\left(u_j^{n+1} - u_j^n\right) &= \Delta t\left(F_{j-1/2} - F_{j+1/2}\right)\\
u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j-1/2}F_{j+1/2}\right)\\
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1/2} - F_{j-1/2}}{\Delta x} &= 0
\end{aligned}$$

We also have

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\mathrm{d}x = u_j^n.$$

With these averages, this equation is exact - no approximations.

Next, the approximation enters in the numerical flux function. Suppose $f(u) = au$.

2.1.1 Upwinding

$$F_{j+1/2}^n = \begin{cases} au_j & \text{if } a \geq 0 \\ au_{j+1} & \text{if } a \leq 0 \end{cases}$$

Supposing $a > 0$, then

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(au_j - au_{j-1}^n\right)$$

2.1.2 Two-Step Lax-Wendroff

We predict

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n) - f^n(u_j)\right).$$

Anyway,

$$F_{j+1/2}^{n+1/2} = f(u_{j+1/2}^{n+1/2})$$