MAT 228B Notes

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1 Last Time

For smooth data, Upwinding dampened a lot less than Lax-Friedrichs. Lax-Windroff didn't damp but there was a phase shift. For discontinuous data, Lax-Windroff created wiggles to the left of the discontinuity and those wiggles didn't seem to disappear as the mesh was refined.

2 Modified Equations

We take a PDE and modify it to be a difference equation. We make a scheme to solve them and observed behavior in the difference soltions not present in the solution of the PDE. So what is the PDE which describes the behavior of the difference equations (weird..)? This is the modified equation.

3 Upwinding for $a > 0$

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{\Delta x}\left(u_j^n - u_{j-1}^n\right) = 0$$

Now, let $v(x,t)$ be a smooth function which satisfies the difference equations. This means

$$\frac{v(x,t+\Delta t) - v(x,t)}{\Delta t} + a\left(\frac{v(x,t) - v(x-\Delta x,t)}{\Delta x}\right) = 0$$

Expanding v for small Δx and Δt gives

$$v_t + \frac{1}{2}\Delta t v_{tt} + \frac{1}{6}\Delta t^2 v_{ttt} + \mathcal{O}\left(\Delta t^3\right) + a\left(v_x - \frac{\Delta x}{2}v_{xx} + \frac{\Delta x^2}{6}v_{xxx} + \mathcal{O}\left(\Delta x^3\right)\right) = 0 \tag{1}$$

Assuming $\Delta t = \mathcal{O}(\Delta x)$, we get

$$v_t + av_x + \left(\frac{1}{2}\Delta t v_{tt} - \frac{a\Delta x}{2}v_{xx}\right) + \left(\frac{1}{6}\Delta t^2 v_{ttt} + \frac{a\Delta x^2}{6}v_{xxx}\right) + \left(\mathcal{O}\left(\Delta t^3\right) + \mathcal{O}\left(\Delta x^2\right)\right) = 0 \tag{2}$$

If we truncate to first-order, we get

$$v_t + av_x = -\frac{1}{2}(\Delta t v_{tt} - a\Delta x v_{xx}) \tag{3}$$

Upwinding is a first order approximation to $u_t + au_x = 0$ but it is a second-order approximation. Taking some derivatives give

$$v_{tt} = -av_{tx} + \mathcal{O}(\Delta t) \qquad\qquad v_{tx} = -av_{xx} + \mathcal{O}(\Delta t) \tag{4}$$

So,

$$v_{tt} = a^2 v_{xx} + \mathcal{O}(\Delta t) \tag{5}$$

Finally,

$$v_t + av_x = \frac{1}{2}\big(a\Delta x - \Delta t a^2\big)v_{xx} + \mathcal{O}\big(\Delta t^2\big) \tag{6}$$

$$v_t + av_x = \underbrace{\frac{a\Delta x}{2}(1-\nu)}_{\text{explains the damning and smearing in upwinding}} v_{xx} \tag{7}$$

Assinging $D_{\text{up}} = \frac{a\Delta x}{2}(1-\nu)$, then as $\Delta t, \Delta x \to 0$, ThenFor D_{up}

4 Lax-Freidrichj

We get

$$v_t + av_x + \frac{\Delta x^2}{2\Delta t}\big(1-\nu^2\big)V_{xx} = D_{\text{LF}} v_{xx}$$

Then since this close to 1, we find $D_{up} \approx 2.0$