MAT 228B Notes

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1 Non-Rectangular Domains

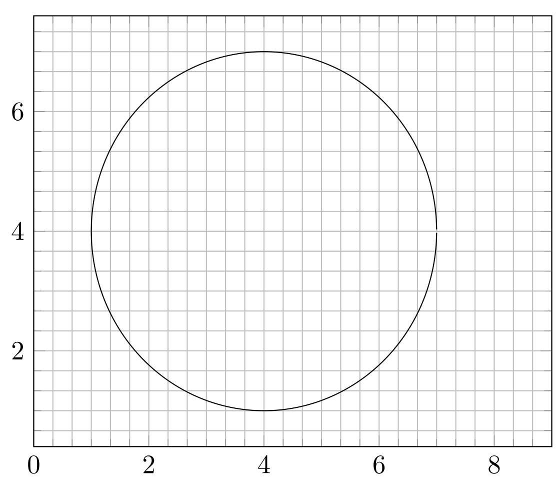
How do we solve $u_t = \nabla^2 u$ on a non-rectangular domain? There are three ways:

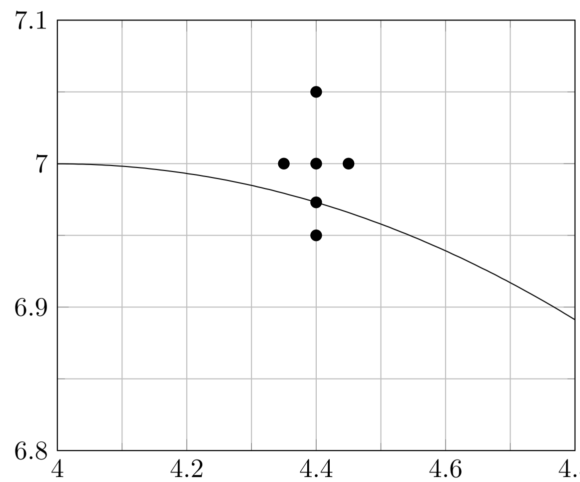
(a) Cartesian mesh (embed the shape in a rectangle, mesh the rectangle). The question is what to do at the points near the boundary.

(b) Body-fitted mesh (warped rectangle.. conformal mapping). The question is how to transform to another coordinate system. And, what happens to the PDE after the coordinate transform?

(c) Unstructured mesh (throw down points on the interior and triangulate somehow with points on the boundary). The question is which points are "next to" which points. This method is used with finite element methods (not finite difference (volume) or spectral)

1.1 Cartesian Grid





For a point near the grid. Suppose P is a point where the points to the east E, west W, and north N are still in the domain, but the point to the south S is outside the domain. Call B the boundary point between P and S. So $\nabla^2 = \partial_{xx} + \partial_{yy}$ where ∂_{xx} is unchanged, but we need to modify the discretization of ∂_{yy}. There are two ways to do this:

(a) We can add an equation at point S to extrapolate a value from inside.

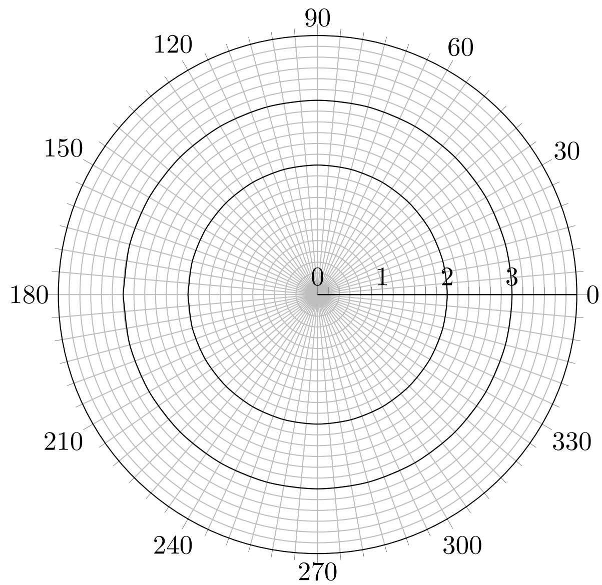
(b) Or, we could exclude S by modifying the stencil of P. Rather than N, E, S, and W, use N, E, B, and W. So, $y_P - y_B = \alpha\Delta x$ for $\alpha \in (0,1)$, so

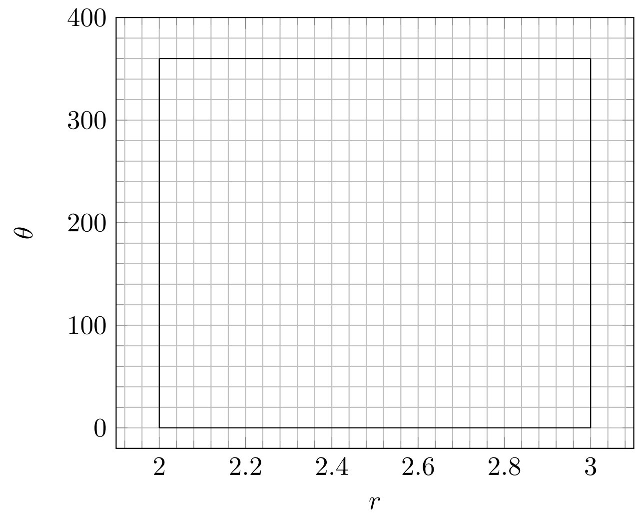
$$u_{yy}(P) \approx \frac{2\alpha u_N - 2(\alpha+1)u_P + 2u_B}{\alpha(\alpha+1)\Delta x^2}.$$

This is first-order accurate.

The heat equation (or related equations) we have $\mathcal{O}(\Delta x)$ truncation errors, but the solution will be $\mathcal{O}(\Delta x^2)$. In 1D, for the Poisson equation, an interior point $\mathcal{O}(\Delta x) = \tau$, but $\mathcal{O}(\Delta x^2)$ error. The boundary point $\mathcal{O}(1)$ but still get $\mathcal{O}(\Delta x^2)$ error.

You could also put a finer mesh inside parts of a coarser mesh, but there are problems at the "coarse-fine interface."





1.2 Boundary-Fitted Meshes

We need a transformation from a rectangle to the physical domain. So we'll call the original domain $\vec{x} = (x, y)$ and the rectangle is $\vec{\xi} = (\xi, \eta)$. So we want a mapping $\vec{x} = \vec{F}(\vec{\xi})$. Just mesh on $\vec{\xi}$.

Example: Let's solve on an annulus in 2D.

- Use polar coordinates. So $r \in (a, b)$ and $\theta \in (0, 2\pi)$. So $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- We get $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ in polar coordinates.

For non-polar coordinates, we have $\vec{x} = \vec{F}(\vec{\xi})$. We want a mesh in $\vec{\xi}$ space. So, define

$$J_{ij} = \frac{\partial \vec{F}_i}{\partial \vec{\xi}_j} \tag{1}$$

and $g = J^T J$, with $|g| = \det(g)$. Then we get

$$\nabla^2 u = \sum_{i,j} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \vec{\xi}_j} \left(\sqrt{|g|} g_{ij}^{-1} \frac{\partial}{\partial \vec{\xi}_j} u \right) \tag{2}$$