MAT 228B Notes

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1 ADI Scheme

$$\left(I - \frac{\Delta tb}{2}L_x\right)u^* = \left(I + \frac{\Delta tb}{2}L_y\right)u^n$$
$$\left(I - \frac{\Delta tb}{2}L_y\right)u^{n+1} = \left(I + \frac{\Delta tb}{2}L_x\right)u^*$$

which is

$$\left(I - \frac{\Delta tb}{2}L_x\right)\left(I - \frac{\Delta tb}{2}L_y\right)u^{n+1} = \left(I + \frac{\Delta tb}{2}L_x\right)\left(I + \frac{\Delta tb}{2}L_y\right)u^n$$

vs. Crank Nicolson

$$\left(I - \frac{\Delta tb}{2}L_x - \frac{\Delta tb}{2}L_y\right)u^{n+1} = \left(I + \frac{\Delta tb}{2}L_x + \frac{\Delta tb}{2}L_y\right)u^n$$

so ADI is a small perturbation of Crank Nicolson, but a lot faster. What about in 3D? Start with a scheme like

$$\left(I - \frac{\Delta tb}{2}L_x\right)\left(I - \frac{\Delta tb}{2}L_y\right)\left(I - \frac{\Delta tb}{2}L_z\right)u^{n+1} = \left(I + \frac{\Delta tb}{2}L_x\right)\left(I + \frac{\Delta tb}{2}L_y\right)\left(I + \frac{\Delta tb}{2}L_z\right)u^n$$

just by analogy with the 2D scheme.

2 Backward-Euler-like Schemes

In 2D, might try

$$(I - \Delta tbL_x - \Delta tbL_y)u^{n+1} = u^n$$

so,

$$(I - \Delta tbL_x)(I - \Delta tbL_y)u^{n+1} = u^n + \Delta t^2b^2L_xL_yu^n$$

3 Fractional-Step Methods

Consider the PDE $u_t = b\nabla^2 u + f(u)$.

- For example, $u_t = b\nabla^2 u + Ku(1-u)$

How would we tackle this?

- method of lines?
- IM-EX (Implicit-Explicit) Scheme, i.e. some terms are implicit and some terms are explicit.
- Fractional Step (different solvers for different terms)

3.1 Method of Lines

Let's use the trapezoidal rule:

$$\frac{u^{n+1} - u_n}{\Delta t} = \frac{b}{2}\left(Lu^n + Lu^{n+1}\right) + \frac{1}{2}\left(f(u^n) + f(u^{n+1})\right)$$
$$u^{n+1} - \frac{b\Delta t}{2} - Lu^{n+1} - \frac{\Delta t}{2}f(u^{n+1}) = u^n + \frac{b\Delta t}{2}Lu^n + \frac{\Delta t}{2}f(u^n)$$

If f is nonlinear, we have to do a nonlinear solve every step (expensive). Perhaps we use Newton's method?

Newton's method for a scalar equation.. trying to solve $F(x) = 0$ for x.

$$f'(x^k)\left(x^{k+1} - x_k\right) = 0 - f(x^k)$$
$$x^{k+1} = x^k - \frac{f(x^k)}{f(x^{k+1})}$$

What changes with a vector instead of a scalar? Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$, and we want to solve $f(u) = 0$. The "derivative" of a vector-to-vector function is the Jacobian. It's the first-order approximation of a nonlinear vector function.

$$L(u^k)\left(u^{k+1} - u^k\right) = -f(u^k)$$

- Solve $L\delta = -f(u^k)$ for δ.
- Set $u^{k+1} = u^k + \delta$.

So we would have to run Newton's method each iteration.

3.1.1 Fractional Stepping

Discretize Space:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + f(u)$$

How to get u^{n+1} from u^n? Solve $\frac{\mathrm{d}u}{\mathrm{d}t} = Lu$ for time length Δt to get u^*, where $u(t_n) = u^n$. Then solve $\frac{\mathrm{d}u}{\mathrm{d}t} = f(u)$ for time length Δt to get u^{n+1}, where $u(t_n) = u^*$.