MAT 228B Notes

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1 Forward-time Centered-Space Discretization

1.1 Diffusion

Solve $u_t = Du_{xx}$ on $(0,1)$ with $u(0) = 0 = u(1)$ and $u(x,0) = g(x)$. We need to discretize space. Take $(0,1)$ and divide it up in to equally-spaced points. $x_j = j\Delta x$ where $j = 0, \dots, N+1$, so $\Delta x = \frac{1}{N+1}$. Then define $u_j(t) \approx u(x_j, t)$. Let

$$u_{xx}(x_j, t) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{(\Delta x)^2}$$

So we've turned a PDE into a set of ODEs. So,

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = D\left(\frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{(\Delta x)^2}\right) \qquad j = 1, \dots, N$$

and $u_j(0) = g(x_j)$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = DLu$$

Then use ODE solvers on this equation. This is called the Method of Lines. We have to be careful about the choice of method. Often, the method designed for the PDE is more efficient.

1.1.1 Simplest ODE method is forward-Euler

Divide time into equally-spaced points Δt apart. Then $t_n = n\Delta t$. Then if $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$, we get $\frac{y^{n+1} - y^n}{\Delta t} = f(y_n)$ where $y^n \approx y(t_n)$ and $y^{n+1} = y^n = \Delta t g(y_n)$.

For the diffusion equation, $u_j^m \approx u(x_j, t_n)$.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) \tag{1}$$

and

$$u_j^{n+1} = u_j^n + \frac{D\Delta t}{(\Delta x)^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) \tag{2}$$

1.2 Advection Equation

$u_t + au_x = 0$ on $(0,10)$ periodic. Let's use a second-order centered difference operator