MAT 228B Notes

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1 Flux Limiters

- Set $\tilde{u} = u_j^n + \sigma_j^n(x - x_j)$.
- We have a lot of freedom over how we choose the slope on each cell.
- Consider the advection equation with $a > 0$.
- We compute the flux through the $j - 1/2$ edge.

$$\begin{aligned}
F_{j-1/2}^n &= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(\tilde{u}(x_{j-1/2}, t))\mathrm{d}t \\
&= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} a\tilde{u}(x_{j-1/2}, t)\mathrm{d}t \\
&= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} a\big(u_{j-1}^n + \sigma_{j-1}^n\big(x_{j-1/2} - a(t - t_n) - x_{j-1}\big)\big)\mathrm{d}t \\
&= au_{j-1}^n + a\sigma_{j-1}^n\left(\frac{\Delta x}{2} - a\frac{\Delta t}{2}\right) \\
&= \underbrace{au_{j-1}^n}_{\text{upwind flux}} + \underbrace{\frac{a}{2}(1-\nu)\Delta x\sigma_{j-1}^n}_{\text{second-order correction}}
\end{aligned}$$

  where $\nu = \frac{a\Delta t}{\Delta x}$ is the Courant number.
- For Lax-Wendroff,

$$\Delta x\sigma_{j-1}^n = u_j - u_{j-1} = (\Delta u)_{j-1/2}$$

- For a positive or negative, we can write

$$F_{j-1/2}^n = F_{j-1/2}^{\text{up}} + \frac{|a|}{2}(1-|\nu|)\delta_{j-1/2}^n$$

  where $\delta_{j-1/2}^n$ is a limited difference that depends on the solution.
- We need a way to measure smoothness of the solution in order to decide δ.
- Introduce $\theta_{j-1/2}$ be a ratio of successive differences

$$\theta_{j-1/2} = \frac{\Delta u_{J_{\text{up}}-1/2}}{\Delta u_{j-1/2}}$$

  where

$$J_{\text{up}} = \begin{cases} j-1 & a > 0 \\ j+1 & a < 0 \end{cases}$$

- For smooth functions away from extreme points, we expect $\theta \approx 1$.

- Now we let

$$\delta^n_{j-1/2} = \phi\big(\theta_{j-1/2}\big)(\Delta u)_{j-1/2}$$

and ϕ is called the flux limiter function.

- What are good choices of ϕ?
  - If $\phi = 0$ we get upwinding.
  - If $\phi = 1$ we get Lax-Wendroff.
  - If $\phi(\theta) = \theta$ we get Beam Warming
  - "High resolution schemes" minmod: $\phi(\theta) = \text{minmod}(1, \theta)$. This picks between the above three. The problem is that this is really diffusive.
  - We have the "monotonized-centered" (MC) flux limiter $\phi(\theta) = \max(0, \min(\frac{1+\theta}{2}, 2, 2\theta))$. This is less diffusive.
  - Superbee: $\phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$. This is the most sharpening.
  - Van Leer: $\phi(\theta) = \dfrac{\theta + |\theta|}{1 + |\theta|}$.

2 Simulations

- Upwinding had diffusion but small phase error
- Lax Wendroff has bad (negative) phase error for hogh frequencies with little diffusion.
- Beam Warming has bad (positive) phase error for high frequencies with little diffusion.
- Minmod has much less diffusion than upwinding, very little phase error. The most diffusive of the "high resolution schemes."
- MC limiter has even less diffusion than minmod. Also very little phase error.
- Superbee looks amazing. It preserves sharpness the best. The problem is that superbee sharpens smooth things that shouldn't be sharp. We saw that after testing for 10 periods.
- Van Leer is the most general "middle of the road" limiter function.

All of the high resolution methods are designed to be second-order on smooth data and to avoid introducing unphysical oscillations.

3 Total Variation

The total variation of a grid function

$$\text{TV}(\underline{u}) = \sum_j |u_{j+1} - u_j|$$

The total variation of a differentiable function f is

$$\text{TV}(f) = \int_a^b |f'(x)| \mathrm{d}x$$

So $f(x) = 0$ is no variation. $f(x) = \sin(0.1x)$ is small variation. $f(x) = \sin(10x)$ is high variation.

Let $f_k(x) = e^{ikx}$ on $[0, 2\pi)$. Then

$$\text{TV}(f_k) = \int_0^{2\pi} \left|ike^{ikx}\right| \mathrm{d}x = 2\pi|k|$$

Next class we will derive the hig-res schemes as natural ways to reduce total variation in time.