

## Math 228B

### Homework 4

Due Wednesday, 3/15/17.

1. Write programs to solve the advection equation

$$u_t + au_x = 0,$$

on  $[0, 1]$  with periodic boundary conditions using upwinding and Lax-Wendroff. For smooth solutions we expect upwinding to be first-order accurate and Lax-Wendroff to be second-order accurate, but it is not clear what accuracy to expect for nonsmooth solutions.

- (a) Let  $a = 1$  and solve the problem up to time  $t = 1$ . Perform a refinement study for both upwinding and Lax-Wendroff with  $\Delta t = 0.9a\Delta x$  with a smooth initial condition. Compute the rate of convergence in the 1-norm, 2-norm, and max-norm. Note that the exact solution at time  $t = 1$  is the initial condition, and so computing the error is easy.
- (b) Repeat the previous problem with the discontinuous initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } |x - 1/2| < 1/4 \\ 0 & \text{otherwise} \end{cases}$$

2. For solving the heat equation we frequently use Crank-Nicolson, which is trapezoidal rule time integration with a second-order space discretization. The analogous scheme for the linear advection equation is

$$u_j^{n+1} - u_j^n + \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = 0,$$

where  $\nu = a\Delta t/\Delta x$ .

- (a) Use von Neumann analysis to show that this scheme is unconditionally stable and that  $\|u^n\|_2 = \|u^0\|_2$ . This scheme is said to be nondissipative – i.e. there is no amplitude error. This seems reasonable because this is a property of the PDE.
- (b) Solve the advection equation on the periodic domain  $[0, 1]$  with the initial condition from problem 1b. Show the solution and comment on your results.
- (c) Compute the relative phase error as  $\arg(g(\theta))/(-\nu\theta)$ , where  $g$  is the amplification factor and  $\theta = \xi\Delta x$ , and plot it for  $\theta \in [0, \pi]$ . How does the relative phase error and lack of amplitude error relate to the numerical solutions you observed in part (b).