MAT 228B Notes

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1 PDE Theory - Consistency, Convergence, and Stability

1.1 Convergence

- In a nutshell, a convergent scheme gives more accurate answers for finer meshes.
- Definition: A numerical method is "convergent" if $\forall(x^*, t^*)$ in the domain, $\left\|u_j^n - u(x_j, t_n)\right\| \to 0$ as $\Delta x, \Delta t \to 0$ and $x_j \to x^*$ and $t_n \to t^*$.
- Note: sometimes we must constrain Δx and Δt as they go to 0. An example is the Forward Euler for diffusion, where we require $\Delta t \leq \dfrac{(\Delta x)^2}{2D}$ so the solution doesn't blow up.
- The main point is that the timestep and spacestep are related by a constraint that ensures the solution doesn't dlow up.
- The definition requires specifying a norm, like the discrete 1-, 2-, or ∞-norms. We'll see examples which converge in one norm but not others.
- Convergence looks at the error in the solution of a particular problem, while...

1.2 Consistency

- Local Truncation Error (LTE) is the discretization error, that is, the error of the discretization scheme itself. It is how well differences approximate derivatives.
- Consistency looks at the error in the numerical scheme, not an individual solution to a problem.
- Definition, a scheme is "consistent" if the LTE $\to 0$ as $\Delta x, \Delta t \to 0$.
- Example: Forward Euler for Diffusion:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right)$$

Let $u(x,t)$ solve $u_t = Du_{xx}$. Let τ_j^n be the LTE at (x_j, t_n), that is,

$$\tau_j^n = \frac{u(x_j, t_n + \Delta t) - u(x_j, t_n)}{\Delta t} - \frac{D}{(\Delta x)^2}(u(x_j - \Delta x, t_n) - 2u(x_j, t_n) + u(x_j + \Delta x, t_n)). \tag{1}$$

We are interested in how this acts as $\Delta x, \Delta t \to 0$, so we look at the Taylor expansions and see that

$$\tau_j^n = \frac{\Delta t}{2}u_{tt}(x_j, t_n) - \frac{D}{12}(\Delta x)^2 u_{xxxx}(x_j, t_n) + \text{h.o.t.} \tag{2}$$

The take-away:

$$\tau = \mathcal{O}(\Delta t) + \mathcal{O}\left((\Delta x)^2\right). \tag{3}$$

1.3 Stability

- How do we relate LTE (consistency) and convergence? Through "stability."
- The Lax-Equivalence Theorem (a.k.a. the Fundamental Theorem of Finite Differences) says:
  - A linear, consistent difference scheme to a well-posed linear PDE is convergent if and only if it is stable.
  - In practice, we will use

$$\text{stability} + \text{consistency} \implies \text{convergence} \qquad \text{(for linear PDEs and linear schemes)} \tag{4}$$

- Definition: Consider the linear update rule $u^{n+1} = Bu^n + b^n$. Let u^n and v^n be two different solutions to the difference scheme (that is, derived from u^0 not necessarily equal to v^0). The method is "stable" if $\forall T > 0$, $\exists$ a constant K_T such that $\|u^n - v^n\| \leq K_T \|u^0 - v^0\|$, independent of u^0, v^0, for all $n\Delta t \leq T$.
  - Note: This is independent of the number of timesteps needed to reach T, which is an actual time.
  - Another note: The scheme could be more general, i.e. $B_1 u^{n+1} = B_2 u^n + \Delta t f^n$, but if either B_1 or B_2 is invertible, it is equivalent.
- Alternate Definition (Lax-Richtmyer): The scheme is "stable" if $\forall T > 0$, $\exists$ a constant $C_T > 0$ independent of Δt such that $\|B^n\| \leq C_T$ dor all $n\Delta t \leq T$ (remember B depends on Δt and Δx).