MAT 228B Notes

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1 Last Time

We were analyzing stability, and wrote $u^{n+1} = Bu^n + b^n$. We know this is stable if $\|B^n\| \leq C_T$ where C_T is independent of Δt. Last time we showed (for certain examples) $\|B\| \leq 1$. Then $\|B^n\| \leq 1$.

2 What if the Solution is Supposed to Grow in Time?

It would be silly to get a numerical scheme which doesn't allow growth.. we need to cautiously allow some growth. If there is a constant $\alpha \geq 0$ independent of Δt (for Δt small enough) such that $\|B\| \leq 1 + \alpha\Delta t$, then the scheme is Lax-Richtmyer Stable.

> Show this is true? Suppose $\|B\| \leq 1 + \alpha\Delta t$. We want to show $\|B\|^n \leq C_T$. $\|B^n\| \leq \|B\|^n \leq (1+\alpha\Delta t)^n \leq e^{\alpha\Delta tn} = e^{\alpha T}$. α is like the exponential growth rate we are going to allow for the problem.

Consider $u_t = u_{xx} + Ku$. Spatial diffusion and exponential growth (if $K > 0$). Consider Forward Euler Stability in $\|\cdot\|_\infty$. Forward Euler is

$$u^{n+1} = \underbrace{(I + \Delta tL + K\Delta tI)}_{B} u^n$$

We get $\|B\|_\infty = \left|\frac{\Delta t}{\Delta x^2}\right| + \left\|1 - \frac{2\Delta t}{\Delta x^2} + k\Delta x\right\| + \left|\frac{\Delta t}{\Delta x^2}\right| \leq 2\frac{\Delta t}{\Delta x^2}\left|1 - \frac{2\Delta t}{\Delta x^@}\right| + |K|\Delta t$. So, we require $1 - \frac{2\Delta t}{\Delta x^2} \geq 0$, whege $\frac{\Delta t\Delta x^2}{2}$. So, $\|B\|_\infty \leq 1 + |k|\Delta t$. With the restriction, this method is stable.

2.1 If $K < 0$, We Expect the Physical Solution to Decay

We want $\|B\|_\infty \leq 1$. Try Δt such that $1 - \frac{2\Delta t}{\Delta x^2} + k\Delta t \geq 0$, i.e.

$$\Delta t \leq \frac{\Delta x^2}{2 - K\Delta x^2}$$

So,

$$\|B\|_\infty = 1 + K\Delta t$$

Want $0 \leq 1 + K\Delta t \leq 1$, i.e. $-1 \leq K\Delta t \leq 0$, i.e. $\Delta t \leq -\frac{1}{K}$.

3 Variable Coefficient Diffusion

In the conservative form, $u_t = (a(x)u_x)_x$. In multi-D,

$$u_t = -\nabla \cdot J \qquad \text{where} \qquad J = -a(x)\nabla u$$

We'll generally want to discretize the conservative form.

3.1 Discretize the Conservative Form

We discretize space (1D, equally spaced). We have x_j (points) and $x_{j-1/2}$ edges. To approximate the flux J,

$$J = -a(x)u_x \qquad \text{where} \qquad J_{j-1/2} = -\left(a(x_{j-1/2})\left(\frac{u_j - u_{j-1}}{\Delta x}\right)\right)$$

So, for $u_t = -J_x$, (in semi-discrete form) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x_j) = -(\frac{J_{j+1/2} - J_{j-1/2}}{\Delta x})$$

So,

$$\begin{aligned}
[(a(x)u_x)_x]_j = -\left(\frac{J_{j+1/2} - J_{j-1/2}}{\Delta x}\right) &= \frac{a_{j+1/2}\left(\frac{u_{j+1} - u_j}{\Delta x}\right) - a_{j-1/2}\left(\frac{u_j - u_{j-1}}{\Delta x}\right)}{\Delta x} \\
&= \frac{a_{j-1/2}u_{j-1} - \big(a_{j-1/2} + a_{j+1/2}\big)u_j + a_{j+1/2}u_{j+1}}{\Delta x^2}
\end{aligned}$$

and note that if a is constant it reduces to what we had before.

3.2 Stability of Forward Euler in $\|\cdot\|_\infty$

Using a "constant coefficient way of thinking," Bob would guess $\Delta t \leq \dfrac{\Delta x^2}{2\max(a(x))}$..

Forward Euler for this problem is

$$u^{n+1} = \underbrace{(I + \Delta t A)}_{B} u^n$$

$$\|B\|_\infty = \max_j \left(\left|\frac{a_{j-1/2}\Delta t}{\Delta x^2}\right| + \left|1 - \frac{\Delta t\big(a_{j-1/2} + a_{j+1/2}\big)}{\Delta x^2}\right| + \left|\frac{a_{j+1/2}\Delta t}{\Delta x^2}\right| \right)$$

So we pick Δt so that $1 - \frac{\Delta t\left(a_{j-1/2} + a_{j+1/2}\right)}{\Delta x^2} \geq 0$ for all j, which is equivalent to

$$\Delta t \leq \frac{\Delta x^2}{a_{j-1/2} + a_{j+1/2}} \qquad \text{for all } j$$

and stuff cancels and we get $\|B\|_\infty \leq 1$.