MAT 228B Notes

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$$\begin{aligned} F_{j-1/2} &= F^{\text{up}}_{j-1/2} + \frac{|a|}{2}(1-|\nu|)\delta_{j-1/2} \\ \delta_{j-1/2} &= \phi(\theta_{j-1/2})(u_j - u_{j-1}) \\ \theta_{j-1/2} &= \frac{\Delta u_{J_{\text{up}}-1/2}}{\Delta u_{j-1/2}} \end{aligned}$$

Define total variation TV as

$$\text{TV}(\vec{u}) = \sum_j |u_{j+1} - u_j|$$

A two level in time scheme is total variation diminishing (TVD) if $\text{TV}(u^{n+1}) \leq \text{TV}(u^n)$ for all n. One can show that TVD $\implies$ monotonicity-preserving. One can also show that upwinding is TVD but Lax-Wendroff and Beam-Warming are not TVD.

We want to design ϕ to give a TVD scheme, but we also want 2nd order for smooth data.

For second order, we require that $\phi(1) = 1$ and ϕ be Lipschitz-continuous at $\theta = 1$ (Lipschitz means bounded derivatives).

For $a > 0$,

$$u_j^{n+1} = \underbrace{u_j^n - \nu\big(u_j^n - u_{j-1}^n\big)}_{\text{upwinding}} - \frac{\nu(1-\nu)}{2}\Big(\phi\big(\theta_{j+1/2}\big)\big(u_{j+1}^n - u_j^n\big) - \phi\big(\theta_{j-1/2}\big)\big(u_j^n - u_{j-1}^n\big)\Big)$$

This is a three-point scheme.

$$u_j^{n+1} = u_j^n - C_{j-1}^n\big(u_j^n - u_{j-1}^n\big) + D_j^n\big(u_{j+1}^n - u_j^n\big)$$

A scheme of the above form is TVD if

- $C_{j-1} \geq 0$
- $D_j \geq 0$
- $C_j^n + D_j^n \leq 1$

We are tempted to write

$$C_{j-1} = \nu - \frac{\nu(1-\nu)}{2}\phi(\theta_{j-1/2})$$

and

$$D_j = -\frac{\nu(1-\nu)}{2}\phi(\theta_{j+1/2})$$

But D_j is generally negative, which doesn't ensure TVD. The trick is to think about the nonlinear scheme by writing

$$u_{j+1} - u_j = \frac{u_j - u_{j-1}}{\theta_{j+1/2}}$$

This allows us to write

$$C_{j-1} = \nu + \frac{\nu(1-\nu)}{2}\left(\frac{\phi(\theta_{j+1/2})}{\theta_{j+1/2}} - \phi(\theta_{j-1/2})\right)$$
$$D_j = 0$$

Now for TVD we require $C_{j-1} \in [0,1]$. Forcing these inequalities give

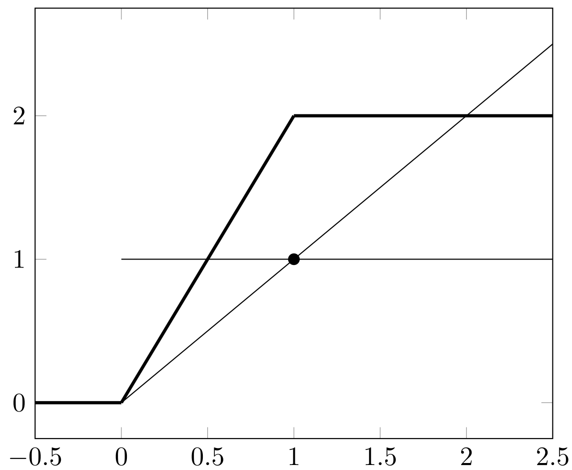
$$\nu \leq 1 \qquad \text{for CFL condition}$$
$$\left|\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right| \leq 2 \qquad \text{for all } \theta_1, \theta_2 \geq 0$$

We also require $\phi = 0$ for $\theta \leq 0$. $\theta \leq 0$ means we have a max or a min and we don't know whether this is smooth or not at those points.

We also require

$$0 \leq \frac{\phi(\theta)}{\theta} \leq 2 \qquad \text{and} \qquad 0 \leq \phi(\theta) \leq 2$$

for all $\theta > 0$. These two inequalities give us TVD for $\nu \leq 1$.



Minmod follows the bottom of the Sweby Region, Superbee follows the top of the Sweby region, and MC and Van Leer go in between.