MAT 228B Notes

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January 9, 2016

1 Review

- 228A
  - $\nabla^2 u = f$
  - how to discretize (mostly finite difference methods)
  - Then, how to solve the discretized algebraic equations
  - There was some theory about accuracy and convergence
- 228B
  - time-dependent problems
  - $u_t = Du_{xx}$ (diffusion or heat equation)
  - $u_t + au_x = 0$ (advection equation)
  - $u_{tt} = c^2 u_{xx}$ (wave equation, really equivalent to solving two advection equations)
  - Mixed Equations (e.g. advection-diffusion, etc.)
    * $u_t + au_x = Du_{xx} + R(u)$ (advection-diffusion-reaction equations)
    * schemes used to attack problems depend on parameters.. does it look more like one or the other
  - $u_t + uu_x = Du_{xx}$ (Burger's Equation, i.e. nonlinear advection). With $D = 0$ (inviscid), the solution can develop jumps in finite time from smooth initial data.
  - $\rho(u_t + u \cdot \boldsymbol{\nabla} u) = -\boldsymbol{\nabla} p + \mu \nabla^2 u$ with $\boldsymbol{\nabla} \cdot u = 0$ (Incompressible Navier-Stokes)

2 Conservation Laws

- $u_t + (f(u))_x = 0$ (1D conservation law)
- $u_t + \boldsymbol{\nabla} \cdot (F(u)) = 0$ (higher dimension conservation law)

Let $\rho(x,t)$ be a density, e.g. $\frac{\text{mass}}{\text{length}}$ where length is 1D volume. Let $f(\rho)$ be a flux function, i.e. the rate of stuff moving through surface (in 1D, "surface" is point). In 1D, we have $\frac{\text{mass}}{\text{time}}$ moving through a point. In 3D, we have $\frac{\text{mass}}{\text{time}\cdot\text{area}}$.

Consider an interval $[x_1, x_2]$. Let A be the amount of stuff in $[x_1, x_2]$. Then

$$A(t) = \int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x.$$

So we can derive an ODE for A, by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x.$$

By sign convention, positive is to the right, so

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x = -f(\rho(x_2,t)) + f(\rho(x_1,t)).$$

Then the fundamental theorem of calculus gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\int_{x_1}^{x_2} (f(\rho))_x \mathrm{d}x.$$

We are assuming ρ is nice enough to bring $\frac{\mathrm{d}}{\mathrm{d}t}$ through the integral, so

$$\int_{x_1}^{x_2} (\rho_t + (f(\rho))_x)\mathrm{d}x = 0,$$

which is the integral form of a conservation law. Since x_1 and x_2 is arbitrary, we can argue, provided ρ is nice enough (continuously differentiable will do it, for example), then we can drop the integral to get

$$\rho_t + (f(\rho))_x = 0$$

is the differential form of a conservation law.

Let u be a chemical concentration. Suppose u is transported by a velocity a. Then the flux function is, in this case, is $f(u) = au$ where $[a] = \frac{\text{length}}{\text{time}}$ and $[u] = \frac{\text{amount}}{\text{length}}$. Then $u_t + (au)_x = 0$, which is the advection equation (if a is constant, we can get $u_t + au_x = 0$).

If there is no background flow (advection) but there is diffusion, then $f(u) = -Du_x$, which says that things move down the gradient. This is diffusive flux. Putting this in the conservation law gives

$$u_t + (-Du_x)_x = 0$$

So if D is constant, we get $u_t = Du_{xx}$.

2.1 An example of a system of nonlinear conservation laws

Euler equations

- conservation of mass $\rho_t + (v\rho)_x = 0$
- conservation of momentum $(\rho v)_t + \left(\rho v^2 + \rho\right)_x = 0$
- conservation of energy $E_t + (v(E + \rho))_x = 0$