MAT 228B Notes

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1 CFL Condition

This means that the analytic domain of dependence is contained within the numerical Domain of Depdendence. The CFL condition is a necessary condition for convergence.
The numerical Domain of Depdendence for explicit / 3 point centered scheme..

- Consider a space-time lattice.. the point (x_j, t_n) depends on the points $(x_j - \Delta x, t_{n-1}), (x_j, t_{n-1}), (x_j + \Delta x, t_{n-1})$.
- So the Domain of Depdendence is cone-shaped.
- So the points at $t = t_0$ which (x_j, t_n) depend on are all the points between $(x_j - n\Delta x, 0)$ and $(x_j + n\Delta x, 0)$. But remember $n = {}^t/_{\Delta t}$. So it's all the points between $(x_j - \frac{\Delta x}{\Delta t}t, 0)$ and $(x_j + \frac{\Delta x}{\Delta t}t, 0)$.
- When we refine the mesh, we make sure $\Delta t \propto \Delta x$. So the Domain of Depdendence doesn't change shape as Δt and Δx are refined. We also get more points in the Domain of Depdendence.

What do we need for the CFL condition? The CFL for advection eqatuion is

$$\begin{aligned} x - rt \leq x - at &\leq x + rt \\ -1 \leq \frac{a}{r} &\leq 1 \end{aligned}$$

were $-1 = -\frac{a\Delta t}{\Delta x} \leq 1$, that is,

$$\nu = \frac{|a|\Delta t}{\Delta x} \leq 1.$$

For forward-time centered-space, we need this, but it is not sufficient for convergence.

2 Be Smart About Your Scheme!

Suppose $u_t + au_x = 0$ and $a > 0$. We only need information from past times to the left. Why use a centered scheme? Instead we can use the "upwind method."

Consider the point $P = (x, t)$ in space time. We can track back to $Q = (x - a\Delta t, t - \Delta t)$, which is the point in space time at time $t - \Delta t$ along the characteristic curve $x - at = \text{const.}$. Then let A and B be the points to the left and right of point Q, so $A = (x - \Delta x, t - \Delta t)$ and $B = (x + \Delta x, t - \Delta t)$. So,

$$\begin{aligned} |QA| &= x - a\Delta t - (x - \Delta x) = \Delta x - a\Delta t = \Delta x(1 - \nu) \\ |QB| &= x - (x - a\Delta t) = a\Delta t = \Delta x\nu \\ \implies u(Q) &\approx \frac{\Delta x(1 - \nu)u(B) + \Delta x\nu u(A)}{\Delta x(1 - \nu) + \Delta x\nu} = (1 - \nu)u(B) + \nu u(A) \end{aligned}$$

So the upwind method for $a > 0$ is

$$\begin{aligned} u_j^{n+1} &= (1 - \nu)u_j^n + \nu u_{j-1}^n \\ &= u_j^n - \nu\big(u_j^n - u_{j-1}^n\big) \\ &= u_j^n - \frac{a\Delta t}{\Delta x}\big(u_j^n - u_{j-1}^n\big) \\ \implies \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} &= 0 \end{aligned}$$

This is a consistent discretization but we used standard forward-time, backward-space, so this is first order in time and space.

More generally, if $a < 0$, we use forward-space difference since that is the upwind direction. In particular,

$$u_j^{n+1} = \begin{cases} u_j - \dfrac{a\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) & \text{if } a \geq 0 \\[2ex] u_j - \dfrac{a\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right) & \text{if } a < 0 \end{cases}$$

As an aside, these are schemes for $u_t + au_x = 0$. If we had something in conservation form $u_t + \left(a(x)u\right)_x = 0$.

Anyway, we need to check the stability of upwind ($a > 0$)..

$$g(\xi) = 1 - \nu\left(1 - e^{i\xi\Delta x}\right) = (1-\nu) + \nu e^{-i\xi\Delta x}$$

We know $\nu \leq 1$ by CFL (necessary for convergence). This is a circle at center $1-\nu$ with radius ν. We need $\nu \leq 1$ for stability.

Upwinding has a nice maximum principle associated with it..

$$u_j^{n+1} = (1-\nu)u_j^n + \nu u_{j-1}^n \qquad \text{for } a > 0$$

and let $m^n = \max_j |u^n|$. So,

$$m^{n+1} \leq (1-\nu)m^n + \nu m^n = m^n$$