MAT 228B Notes

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1 Fractional Stepping

$u_t = A(u) + B(u)$. We have u^n.

1. Solve $u_t = A(u)$ beginning at u^n for time length Δt to get u^*.
2. Solve $u_t = B(u)$ beginning at u^* for time length Δt to get u^{n+1}.

We can analyze independent of the schemes for steps 1 and 2. Let's first consider the linear problem:

1.1 The Linear Problem

$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + Bu$. Say we have $u(t_n)$. The solution of $u(t_{n+1}) = \exp[(A+B)\Delta t]u(t_n)$ where $\exp[(A+B)\Delta t]$ is a matrix exponential. Now let's apply fractional stepping. Set $u^n = u(t_n)$.

1. Solve $\frac{\mathrm{d}u}{\mathrm{d}t} = Au$. The solution is $u^* = \exp[A\Delta t]u^n$.
2. Solve $\frac{\mathrm{d}u}{\mathrm{d}t} = Bu$. The solution is $u^{n+1} = \exp[B\Delta t]u^* = \exp[B\Delta t]\exp[A\Delta t]u^n$. This is not necessarily the same as the exact solution unless A and B commute.

The single-step error of fractional stepping is

$$u(t_{n+1}) - u^{n+1} = (\exp[(A+B)\Delta t] - \exp[B\Delta t]\exp[A\Delta t])u^n$$

So after Taylor expanding,

$$\begin{aligned}
\exp[(A+B)\Delta t] &= I + \Delta t(A+B) + \frac{\Delta t^2}{2}(A+B)^2 + \mathcal{O}\big(\Delta t^3\big) \\
&= I + \Delta t(B+A) + \frac{\Delta t^2}{2}\big(B^2 + BA + AB + A^2\big) + \mathcal{O}\big(\Delta t^3\big) \\
\exp[B\Delta t]\exp[A\Delta t] &= \left(I + \Delta t B + \frac{\Delta t^2}{2}B^2 + \mathcal{O}\big(\Delta t^2\big)\right)\left(I + \Delta t A + \frac{\Delta t^2}{2}A^2 + \mathcal{O}\big(\Delta t^3\big)\right) \\
&= I + \Delta t(B+A) + \frac{\Delta t^2}{2}\big(B^2 + 2BA + A^2\big) + \mathcal{O}\big(\Delta t^3\big)
\end{aligned}$$

So, the single step error of fractional stepping is

$$(\exp[(A+B)\Delta t] - \exp[B\Delta t]\exp[A\Delta t])u^n = \frac{\Delta t^2}{2}(BA - AB) + \mathcal{O}\big(\Delta t^3\big) = \frac{\Delta t^2}{2}[B,A] + \mathcal{O}\big(\Delta t^3\big)$$

where $[B,A] \coloneqq BA - AB$ is the commutator of B and A. This says that the single-step error is $\mathcal{O}\big(\Delta t^2\big)$. But we take $\mathcal{O}\big(\Delta t^{-1}\big)$ steps, so the method has $\mathcal{O}(\Delta t)$ error. The error of fractional stepping for the ending solution is $\mathcal{O}(\Delta t)$. This shows that refining the time step gives more accurate solutions (which damn well better happen).

1.2 How do we Get 2nd Order in Time?

Let's try taking three fractional steps per timestep. The following method is called the "Strang Splitting."

1. Solve $u_t = A(u)$ beginning with u_n for timelength $\frac{\Delta t}{2}$ to get u^*.
2. Solve $u_t = B(u)$ beginning with u^* for timelength Δt for u^{**}.

3. Solve $u_t = A(u)$ beginning with u^{**} for timelength $\frac{\Delta t}{2}$ to get u^{n+1}.

The product of the exponentials would be $\exp\left[\frac{\Delta t}{2}A\right]\exp[\Delta t B]\exp\left[\frac{\Delta t}{2}A\right]$. This will be equal to $\exp[\Delta t(A+B)] + \mathcal{O}\left(\Delta t^3\right)$. The cost is that there are three steps.

Another way to get second order is, for odd timesteps,

1. $u_t = A(u)$
2. $u_t = B(u)$

and for even timesteps,

1. $u_t = B(u)$
2. $u_t = A(u)$

but this method is just Strang Splitting over $2\Delta t$, so it's second order accurate.

2 IMEX Methods

- Let $u_t = A(u) + B(u)$. Suppose A is stiff and B is not. That is, we would want to use an implicit scheme for A but we are happy to use an explicit scheme for B. An example is Navier Stokes $u_t + \underbrace{u \cdot \boldsymbol{\nabla} u}_{\text{not stiff}} = \underbrace{\nu \nabla^2 u}_{\text{stiff}} - \boldsymbol{\nabla} p + f$.

- So how about CN/AB2?

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\left(A(u^n) + A(u^{n+1})\right) + \underbrace{\frac{3}{2}B(u^n) - \frac{1}{2}B(u^{n-1})}_{\substack{\text{2 past time points and extrapolate to } t_{n+1/2} \\ = B(u^{n+1/2}) + \mathcal{O}\left(\Delta t^2\right)}}$$

  What if

$$\frac{3u^{n+1} - 4u^n + u^{n+1}}{2\Delta t} = A(u^{n+1}) + \underbrace{2B(u^n) - B(u^{n-1})}_{B(u^{n+1}) + \mathcal{O}(\Delta t^2)}$$

  This is better since the stability properties are better.