MAT 228B Notes

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1 Amplitude and Phase Errors

- Von Neumann Analysis gives

$$\hat{u}^{n+1} = g(\xi)\hat{u}^n$$

  We look at $|g(\xi)|$ as a function of ξ to quantify amplitude error.

- Set $\theta = \xi\Delta x$. In the limit of small θ, upwinding $|g(\theta)| = 1 - \frac{1}{2}\big(\nu - \nu^2\big)\theta^2 + \mathcal{O}\big(\theta^4\big)$. So this is second-order per step (first-order in amplitude).

- Lax-Windrof $|g(\theta)| = 1 - \frac{1}{8}\big(\nu^2 - \nu^4\big)\theta^4 + \mathcal{O}\big(\theta^6\big)$. So this is fourth-order per step (third-order in amplitude).

- What is the phase error? $u = 1 \cdot e^{-\xi(x-at)}$ (amplitude 1) is a solution to $u_t + au_x = 0$. We have $\mathrm{Re(u)} = A\cos(x - at)$. We have the real phase is translating. In one timestep, how much does the phase change?

- Let $\Delta\phi :=$ phase change per timestep.

$$\Delta\phi = \xi(x - a(t + \Delta t)) - \xi(x - at) = -\xi a\Delta t = -\frac{\theta}{\Delta x}a\Delta t = -\nu\theta$$

- Anyway,

$$g(\theta) = |g|e^{i\phi}$$

  where $\phi = \arg(g)$. So, the numerical scheme changes the phase of a wave with wavenumber θ by some $\phi(\theta)$.

- So, the relative phase of the numerical scheme can be computed by $\frac{\phi(\theta)}{\Delta\phi}$. This is the phase change in the numerical scheme divided by the phase change in the PDE.

- For smooth modes, what does this relative phase look like?
  - For upwinding, relative phase is $1 - \frac{1}{6}(1 - \nu)(1 - 2\nu)\theta^2 + \text{h.o.t.}$.
  - For LW, relative phase is $1 - \frac{1}{6}(1 - \nu^2)\theta^2 + \text{h.o.t.}$.

- So if it's smooth we want to do LW, but if it's not so smooth we should do upwinding. We will get to a method that switches between these two based on the given data.

2 Boundary Conditions

- Look at $u_t + au_x = 0$ on $(0, 1)$ $(a > 0)$ with boundary conditions...

- Regardless of boundary conditions, we should expect things to advect to the right. We need a boundary condition at $x = 0$.

- Set initial condition $u(x, 0) = u_0(x)$ and boundary condition $u(0, t) = g(t)$. We don't need a boundary condition at $x = 1$. No boundary condition at outflow - only at inflow.

- How do we deal with the fact that the solution is given at $x = 0$ and unknown at $x = 1$? Depends on the scheme.

– For upwinding,

$$u_j^{n+1} = u_j^n - \frac{\Delta t a}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

so we don't need to do anything special since u_j does not depend on u_{j+1}.

– Method of Lines upwinding..

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{a\Delta t}{\Delta x}\left(\begin{array}{cccccc} 1 & 0 & & & & \\ -1 & 1 & 0 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 1 & 0 \\ & & & & -1 & 1 \end{array}\right) u + \left(\begin{array}{c} ? \\ ? \\ ? \\ ? \\ ? \end{array}\right)$$

– For Lax-Windroff,

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{a^2\Delta t^2}{2\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

This poses a problem at the outflow boundary since u_j is dependent on u_{j+1}. We need to modify this at the outflow boundary. We could perhaps do upwinding at that point.... or we could extrapolate (linear? constant? something else? upwinding is some kind of extrapolation..) to outside the domain (done most commonly in practice).

3 Systems of Equations

$$\vec{u}_t + A\vec{u}_x = 0$$

e.g. $u_{tt} = c^2 u_{xx}$. We can write that as a system of the above form with the change of variables

$$\vec{u} = \left(\begin{array}{c} u_t \\ u_x \end{array}\right)$$

so

$$\vec{u} + \left(\begin{array}{cc} 0 & c^2 \\ 1 & 0 \end{array}\right)\vec{u} = 0$$

How do we tackle this using the methods we have? Upwinding? Which way is the wind blowing (need eigen-decomposition). Lax Windroff will undergo lots of changes.. next time.