UC DAVIS, NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS (MAT 228B), WINTER 2017

Homework #3

Sam Fleischer

March 3, 2017

Problem 1

Write programs to solve

$$u_t + au_x = 0,$$

on $[0,1]$ with periodic boundary conditions using upwinding and Lax-Wendroff. For smooth solutions we expect upwinding to be first-order accurate and Lax-Wendroff to be second-order accurate, but it is not clear what accuracy to expect for nonsmooth solutions.

(a) Let $a = 1$ and solve the problem up to time $t = 1$. Perform a refinement study for both upwinding and Lax-Wendroff with $\Delta t = 0.9a\Delta x$ with a smooth initial condition. Compute the rate of convergence in the 1-norm, 2-norm, and max-norm. Note that the exact solution at time $t = 1$ is the initial condition, and so computing the error is easy.

(b) Repeat the previous problem with the discontinuous initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } \left|x - \frac{1}{2}\right| < \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

Here are functions which make a sparse update matrix for upwinding and Law-Wendroff:

```python
def get_upwind_mat(N,nu):
    # Generate the upwinding update matrix of size NxN:
    # 1 - nu on the diagonal, and
    # nu on the sub-diag and upper-right corner (periodicity)
    off_diag = nu*np.ones(N)
    diag = (1-nu)*np.ones(N)
    A = np.vstack((off_diag,diag,off_diag))
    upwind_mat = sp.dia_matrix((A,[N-1,0,-1]),shape=(N,N))
    return upwind_mat

def get_LW_mat(N,nu):
    # Generate the Lax-Wendroff update matrix of size NxN:
    # 1 - nu^2 on the diagonal,
    # (1/2)*(nu + nu^2) on the sub-diag and upper-right corner, and
    # -(1/2)*(nu - nu^2) on the super-diag and bottom-left corner
    diag = (1 - nu**2)*np.ones(N)
    subdiag = (1/2)*(nu + nu**2)*np.ones(N)
    superdiag = -(1/2)*(nu - nu**2)*np.ones(N)
    A = np.vstack((subdiag,superdiag,diag,subdiag,superdiag))
    LW_mat = sp.dia_matrix((A,[N-1,1,0,-1,1-N]),shape=(N,N))
    return LW_mat
```

where numpy is imported as np and `scipy.sparse` is imported as sp. Here are methods which construct functions to apply the upwind or Lax-Wendroff matrices:

```python
def make_upwinding_method(N,dx,dt,transport_coef):
    # Construct a function which applies the upwinding method.

    # Get the upwind update matrix
    nu = transport_coef*dt/dx
    A = get_upwind_mat(N,nu)

    def upwind_step(u):
        # Apply the upwind update matrix to the vector u
        return A.dot(u)

```

```
    return upwind_step

def make_LW_method(N,dx,dt,transport_coef):
    # Construct a function which applies the Lax-Wendroff method.

    # Get the Lax-Wendroff update matrix
    nu = transport_coef*dt/dx
    A = get_LW_mat(N,nu)

    def LW_step(u):
        # Apply the Lax-Wendroff update matrix to the vector u
        return A.dot(u)

    return LW_step
```

Here are functions defining smooth or discontinuous initial conditions given a vector of x values:

```
def smooth_initial_condition(x):
    return (1/2)*np.sin(2*np.pi*x)+(1/2)

def discon_initial_condition(x):
    return np.piecewise(x, [abs(x-0.5)<0.25, abs(x-0.5)>=0.25], [1, 0])
```

Finally, here is sample code to solve a problem:

```
1  # Define Nt, Nx, dt, and dx
2  Nt = 10*(2**5)
3  Nx = int(0.9*Nt)
4  dt = 1/Nt
5  dx = 1/Nx
6  transport_coef = 1
7
8  # Define an update rule.
9  step = make_upwinding_method(Nx,dx,dt,transport_coef)
10
11 # Define initial condition.
12 xaxis = np.linspace(0,1-dx,Nx)
13 u_0 = discon_initial_condition(xaxis)
14
15 # Loop through time and apply the update rule.
16 for t in tqdm(xrange(1,Nt+1)):
17     u_1 = step(u_0)
18     u_0 = u_1 + 0
```

Note that line 9 would change to `step = make_LW_method(Nx,dx,dt,transport_coef)` if we wanted to use Lax-Wendroff instead of upwinding.

(a)

Upwinding, $a = 1$, $N_x = 0.9aN_t$, smooth initial condition: $u_0(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{2}$

N_x	$\lVert u^{N_t} - u^0 \rVert_1$	$\frac{\lVert u_{\text{fine}}^{N_t} - u_{\text{fine}}^0 \rVert_1}{\lVert u_{\text{coarse}}^{N_t} - u_{\text{coarse}}^0 \rVert_1}$	$\lVert u^{N_t} - u^0 \rVert_2$	$\frac{\lVert u_{\text{fine}}^{N_t} - u_{\text{fine}}^0 \rVert_2}{\lVert u_{\text{coarse}}^{N_t} - u_{\text{coarse}}^0 \rVert_2}$	$\lVert u^{N_t} - u^0 \rVert_\infty$	$\frac{\lVert u_{\text{fine}}^{N_t} - u_{\text{fine}}^0 \rVert_\infty}{\lVert u_{\text{coarse}}^{N_t} - u_{\text{coarse}}^0 \rVert_\infty}$
9	0.105829	-	0.0390366	-	0.165393	-
18	0.0462153	0.436698	0.0120953	0.309844	0.0722268	0.436698
36	0.0205959	0.445651	0.00381168	0.315138	0.0323139	0.447395
72	0.00955095	0.463731	0.0012501	0.327967	0.0149975	0.46412
144	0.00457416	0.478922	0.000423374	0.338671	0.00718442	0.479041
288	0.00223496	0.488606	0.000146277	0.345503	0.00351058	0.488638
576	0.00110423	0.494071	5.11036e-05	0.349363	0.00173451	0.49408
1152	0.000548774	0.496975	1.79586e-05	0.351415	0.000862011	0.496977
2304	0.000273548	0.498472	6.32992e-06	0.352473	0.000429689	0.498473
4608	0.000136564	0.499232	2.23453e-06	0.35301	0.000214515	0.499232
9216	6.82295e-05	0.499615	7.89416e-07	0.353281	0.000107175	0.499615
18432	3.41016e-05	0.499807	2.78993e-07	0.353417	5.35667e-05	0.499807
36864	1.70475e-05	0.499904	9.862e-08	0.353485	2.67782e-05	0.499904

N_x	time (in seconds)	ratio of times
9	0.00081	-
18	0.000635	0.783951
36	0.000844	1.32913
72	0.00121	1.43365
144	0.002144	1.7719
288	0.003905	1.82136
576	0.007908	2.0251
1152	0.016155	2.04287
2304	0.047833	2.96088
4608	0.105026	2.19568
9216	0.320367	3.05036
18432	1.20162	3.75075
36864	4.20294	3.49773

Lax-Wendroff, $a = 1$, $N_x = 0.9aN_t$, smooth initial condition: $u_0(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{2}$

N_x	$\lVert u^{N_t} - u^0 \rVert_1$	$\frac{\lVert u_{\text{fine}}^{N_t} - u_{\text{fine}}^0 \rVert_1}{\lVert u_{\text{coarse}}^{N_t} - u_{\text{coarse}}^0 \rVert_1}$	$\lVert u^{N_t} - u^0 \rVert_2$	$\frac{\lVert u_{\text{fine}}^{N_t} - u_{\text{fine}}^0 \rVert_2}{\lVert u_{\text{coarse}}^{N_t} - u_{\text{coarse}}^0 \rVert_2}$	$\lVert u^{N_t} - u^0 \rVert_\infty$	$\frac{\lVert u_{\text{fine}}^{N_t} - u_{\text{fine}}^0 \rVert_\infty}{\lVert u_{\text{coarse}}^{N_t} - u_{\text{coarse}}^0 \rVert_\infty}$
9	0.0502105	-	0.0186936	-	0.0784707	-
18	0.0106443	0.211994	0.00278917	0.149204	0.0166353	0.211994
36	0.00232186	0.218131	0.000429486	0.153984	0.00363858	0.218727
72	0.000532585	0.229379	6.97013e-05	0.16229	0.000836081	0.229782
144	0.000126933	0.238335	1.17484e-05	0.168553	0.000199357	0.238442
288	3.09433e-05	0.243776	2.02521e-06	0.172382	4.86038e-05	0.243803
576	7.63624e-06	0.246782	3.53404e-07	0.174503	1.19949e-05	0.246789
1152	1.89656e-06	0.248363	6.20647e-08	0.17562	2.9791e-06	0.248365
2304	4.72574e-07	0.249175	1.09354e-08	0.176193	7.42318e-07	0.249175
4608	1.17948e-07	0.249585	1.92991e-09	0.176484	1.85272e-07	0.249586
9216	2.94624e-08	0.249792	3.40881e-10	0.17663	4.62795e-08	0.249792
18432	7.36254e-09	0.249896	6.02347e-11	0.176703	1.15651e-08	0.249896
36864	1.84025e-09	0.249948	1.06459e-11	0.17674	2.89066e-09	0.249948

N_x	time (in seconds)	ratio of times
9	0.000962	-
18	0.000733	0.761954
36	0.0012	1.63711
72	0.001797	1.4975
144	0.002363	1.31497
288	0.003883	1.64325
576	0.009921	2.55498
1152	0.020166	2.03266
2304	0.050557	2.50704
4608	0.126219	2.49657
9216	0.401252	3.17901
18432	1.49188	3.71805
36864	5.44029	3.64661

These results show that for upwinding with continuous data, the 1-norm and ∞-norm converge to 0 on the scale of the mesh refinement, that is, the ratios of norms approach $\frac{1}{2}$. The 2-norm ratios converge to $\frac{1}{2\sqrt{2}}$. Using Lax-Wendroff, we double the rate of convergence across the board. The 1-norm and ∞-norm converge with a ratio of $\frac{1}{4}$ and the 2-norm converges with a ratio of $\frac{1}{4\sqrt{2}}$. There is a slight trade-off, however, since the Lax-Wendroff matrix has a non-zero band on the super-diagonal and an element in the bottom left corner, so the computations are slightly more expensive.

(b)

Upwinding, $a = 1$, $N_x = 0.9aN_t$, $u_0(x) = \mathscr{X}_{(\frac{1}{4},\frac{3}{4})}$

N_x	$\left\Vert u^{N_t} - u^0 \right\Vert_1$	$\frac{\left\Vert u^{N_t}_{\text{fine}} - u^0_{\text{fine}} \right\Vert_1}{\left\Vert u^{N_t}_{\text{coarse}} - u^0_{\text{coarse}} \right\Vert_1}$	$\left\Vert u^{N_t} - u^0 \right\Vert_2$	$\frac{\left\Vert u^{N_t}_{\text{fine}} - u^0_{\text{fine}} \right\Vert_2}{\left\Vert u^{N_t}_{\text{coarse}} - u^0_{\text{coarse}} \right\Vert_2}$	$\left\Vert u^{N_t} - u^0 \right\Vert_\infty$	$\frac{\left\Vert u^{N_t}_{\text{fine}} - u^0_{\text{fine}} \right\Vert_\infty}{\left\Vert u^{N_t}_{\text{coarse}} - u^0_{\text{coarse}} \right\Vert_\infty}$
9	0.216964	-	0.0836486	-	0.386057	-
18	0.13819	0.636925	0.046883	0.560475	0.405849	1.05127
36	0.0913189	0.660821	0.0270755	0.577511	0.429004	1.05705
72	0.0621022	0.680059	0.0158358	0.584877	0.447831	1.04388
144	0.0429977	0.69237	0.00933364	0.589401	0.462319	1.03235
288	0.0300724	0.699397	0.00552469	0.591912	0.473056	1.02322
576	0.0211458	0.703161	0.00327744	0.593235	0.480838	1.01645
1152	0.0149101	0.70511	0.00194651	0.593913	0.486411	1.01159
2304	0.0105281	0.706103	0.00115673	0.594257	0.490377	1.00815
4608	0.00743916	0.706603	0.000687595	0.59443	0.493191	1.00574
9216	0.0052584	0.706855	0.000408786	0.594517	0.495183	1.00404
18432	0.00371759	0.706981	0.000243048	0.59456	0.496593	1.00285
36864	0.0026285	0.707044	0.000144512	0.594582	0.497591	1.00201

N_x	time (in seconds)	ratio of times
9	0.000796	-
18	0.00076	0.954774
36	0.000954	1.25526
72	0.001379	1.44549
144	0.002448	1.7752
288	0.00472	1.9281
576	0.008401	1.77987
1152	0.019004	2.26211
2304	0.050015	2.63181
4608	0.121901	2.43729
9216	0.389288	3.19348
18432	1.71821	4.41373
36864	6.93595	4.03672

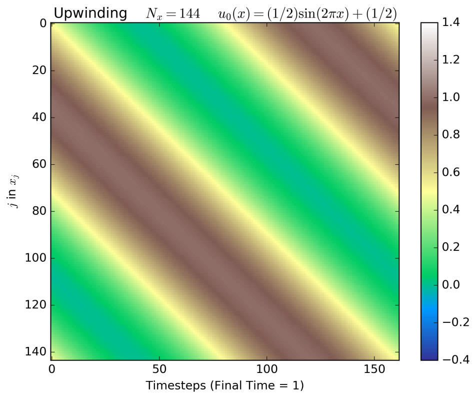
Lax-Wendroff, $a = 1$, $N_x = 0.9aN_t$, $u_0(x) = \mathscr{X}_{(\frac{1}{4},\frac{3}{4})}$

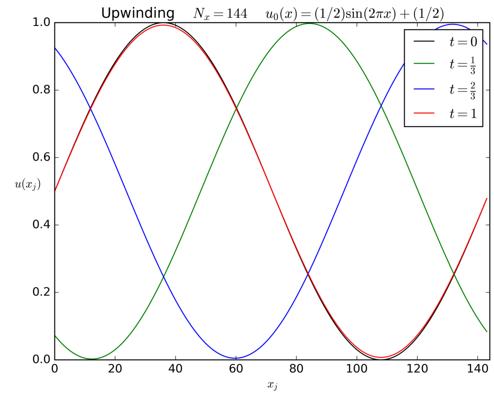
N_x	$\left\Vert u^{N_t} - u^0 \right\Vert_1$	$\frac{\left\Vert u^{N_t}_{\text{fine}} - u^0_{\text{fine}} \right\Vert_1}{\left\Vert u^{N_t}_{\text{coarse}} - u^0_{\text{coarse}} \right\Vert_1}$	$\left\Vert u^{N_t} - u^0 \right\Vert_2$	$\frac{\left\Vert u^{N_t}_{\text{fine}} - u^0_{\text{fine}} \right\Vert_2}{\left\Vert u^{N_t}_{\text{coarse}} - u^0_{\text{coarse}} \right\Vert_2}$	$\left\Vert u^{N_t} - u^0 \right\Vert_\infty$	$\frac{\left\Vert u^{N_t}_{\text{fine}} - u^0_{\text{fine}} \right\Vert_\infty}{\left\Vert u^{N_t}_{\text{coarse}} - u^0_{\text{coarse}} \right\Vert_\infty}$
9	0.158161	-	0.070497	-	0.391898	-
18	0.123856	0.783104	0.041995	0.5957	0.435822	1.11208
36	0.0776195	0.62669	0.0238306	0.567462	0.475573	1.09121
72	0.049597	0.638977	0.0136343	0.572137	0.513524	1.0798
144	0.0321487	0.648197	0.00782582	0.573979	0.545926	1.0631
288	0.0212826	0.662006	0.00449387	0.574236	0.572148	1.04803
576	0.014095	0.662279	0.00257826	0.573728	0.592811	1.03611
1152	0.00929647	0.659557	0.00147713	0.572916	0.608913	1.02716
2304	0.00615957	0.662571	0.000844953	0.572024	0.621425	1.02055
4608	0.00406969	0.660709	0.000482597	0.571152	0.631154	1.01566
9216	0.00269564	0.66237	0.000275243	0.570338	0.638737	1.01201
18432	0.00178031	0.66044	0.000156777	0.569594	0.644664	1.00928
36864	0.00117545	0.660252	8.91934e-05	0.56892	0.649308	1.0072

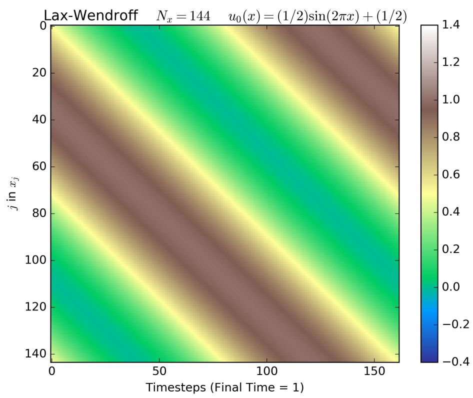
N_x	time (in seconds)	ratio of times
9	0.000949	-
18	0.000746	0.786091
36	0.000904	1.2118
72	0.001327	1.46792
144	0.002165	1.6315
288	0.004459	2.05958
576	0.007954	1.78381
1152	0.017039	2.14219
2304	0.048265	2.83262
4608	0.153822	3.18703
9216	0.664133	4.31754
18432	3.42794	5.16153
36864	15.8522	4.62441

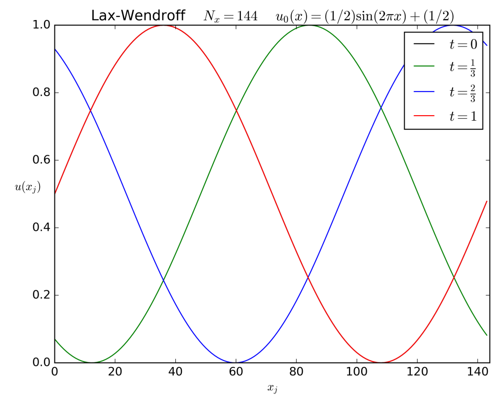
These results show that for both upwinding and Lax-Wendroff, while the 1-norm and 2-norm ratios approach a number less than 1 (suggesting convergence), the ∞-norm does not converge. This is because there is a phase-shift for the high frequencies captured at the discontinuities.

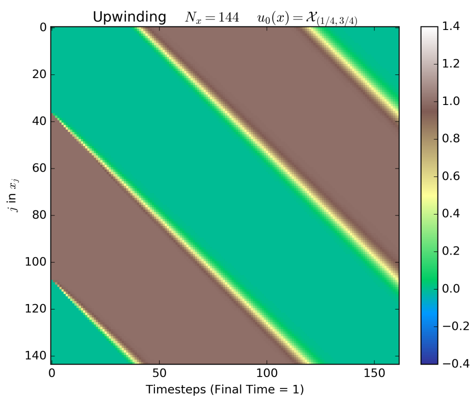
Below are images of the solutions. The horizontal axis is time, with $t = 0$ on the left and $t = 1$ on the right. The vertical axis is space, with $x_j = x_0$ on top and $x_j = x_{N_x}$ on bottom. The first four images depict smooth initial data and the next four depict discontinuous initial data. Notice the diffusion in the upwinding technique. Even for smooth data, we see a loss of amplitude at $t = 1$. Lax-Wendroff is very suited for continuous data - the red curve completely covers up the black curve. When we use Lax-Wendroff for discontinuous data, we see a dark blue band and a white band moving slower (initially) than the rest of the data. This is because high frequencies have a significant lag in the Lax-Wendroff method. We don't see as much lag in the upwinding scheme for discontinuous data, but we do see significant diffusion because $g_{\text{up-wind}}(\xi)$ is relatively small.

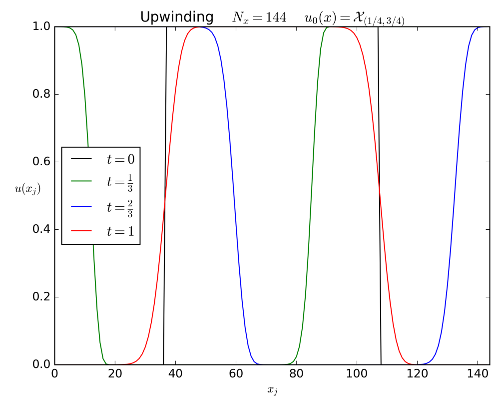

Upwinding $N_x = 144$ $u_0(x) = (1/2)\sin(2\pi x) + (1/2)$
j in x_j
Timesteps (Final Time = 1)

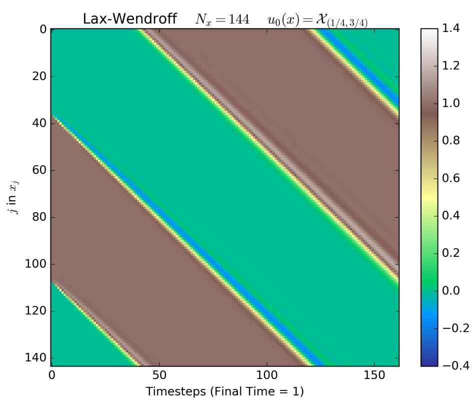

Upwinding $N_x = 144$ $u_0(x) = (1/2)\sin(2\pi x) + (1/2)$
$t = 0$
$t = \frac{1}{3}$
$t = \frac{2}{3}$
$t = 1$
$u(x_j)$
x_j

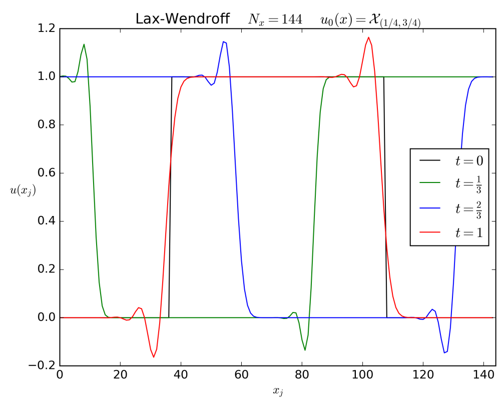

Lax-Wendroff $N_x = 144$ $u_0(x) = (1/2)\sin(2\pi x) + (1/2)$
j in x_j
Timesteps (Final Time = 1)


Lax-Wendroff $N_x = 144$ $u_0(x) = (1/2)\sin(2\pi x) + (1/2)$
$t = 0$
$t = \frac{1}{3}$
$t = \frac{2}{3}$
$t = 1$
$u(x_j)$
x_j


Upwinding $N_x = 144$ $u_0(x) = \mathcal{X}_{(1/4,3/4)}$
j in x_j
Timesteps (Final Time = 1)


Upwinding $N_x = 144$ $u_0(x) = \mathcal{X}_{(1/4,3/4)}$
$t = 0$
$t = \frac{1}{3}$
$t = \frac{2}{3}$
$t = 1$
$u(x_j)$
x_j


Lax-Wendroff $N_x = 144$ $u_0(x) = \mathcal{X}_{(1/4,3/4)}$
j in x_j
Timesteps (Final Time = 1)


Lax-Wendroff $N_x = 144$ $u_0(x) = \mathcal{X}_{(1/4,3/4)}$
$t = 0$
$t = \frac{1}{3}$
$t = \frac{2}{3}$
$t = 1$
$u(x_j)$
x_j

Problem 2

For solving the heat equation we frequently use Crank-Nicolson, which is trapezoidal rule time integration with a second-order space discretization. The analogous scheme for the linear advection equation is

$$u_j^{n+1} - u_j^n + \frac{\nu}{4}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\nu}{4}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = 0,$$

where $\nu = \frac{a\Delta t}{\Delta x}$.

(a) Use von Neumann analysis to show that this scheme is unconditionally stable and that $\|u^n\|_2 = \left\|u^0\right\|_2$. This scheme is said to be nondissipative - i.e. there is no amplitude error. This seems reasonable because this is a property of the PDE.

(b) Solve the advection equation on the periodic domain $[0,1]$ with the initial condition from problem 1b. Show the solution and comment on your results.

(c) Compute the relative phase as $\frac{\arg(g(\theta))}{-\nu\theta}$, where g is the amplification factor and $\theta = \xi\Delta x$, and plot it for $\theta \in [0,\pi]$. How does the relative phase and lack of amplitude error relative to the numerical solutions you observed in part (b)?

First note the code used to compute the matrices needed for this scheme:

```
def get_CN_mats(N,nu):
    # Generate the "Crank-Nicolson for Advection" LHS and RHS
    #   matrices of size NxN:
    # The LHS matrix has 1 on the diagonal,
    # -nu/4 on the sub-diag and upper-right corner, and
    # nu/4 on the super-diag and bottom-left corner.
    # The RHS matrix is the transpose of the LHS matrix
    diag = np.ones(N)
    pos = (nu/4)*np.ones(N)
    neg = -pos
    A_LHS = np.vstack((neg,pos,diag,neg,pos))
    LHS = sp.dia_matrix((A_LHS,[N-1,1,0,-1,1-N]),shape=(N,N))
    RHS = LHS.transpose()
    return (LHS,RHS)
```

And here is the method which constructs functions to apply the scheme:

```
def make_CN_method(N,dx,dt,transport_coef):
    # Construct a function which applies the "Crank-Nicolson
    #   for Advection" method.

    # Get the "Crank Nicolson for Advection" LHS and RHS matrices
    nu = transport_coef*dt/dx
    (LHS, RHS) = get_CN_mats(N,nu)

    def CN_step(u):
        # Apply the RHS CN matrix to the vector u
        right_side = RHS.dot(u)
        # Solve LHS u^(n+1) = RHS u^n for u^(n+1)
        return sp.linalg.spsolve(LHS,right_side)

    return CN_step
```

The code for actually solving a problem using this scheme looks similar to the code given in Problem 1, but `step = make_CN_method(Nx,dx,dt,transport_coef)` would replace the line defining `step` through a different scheme.

(a) Let $u_j^n = e^{i\xi x_j}$ and $u_j^{n+1} = g(\xi)u_j^n$. Then

$$g(\xi)e^{i\xi x_j} - e^{i\xi x_j} + \frac{\nu}{4}\left(e^{i\xi x_j}e^{i\xi\Delta x} - e^{i\xi x_j}e^{-i\xi\Delta x}\right) + \frac{\nu}{4}\left(g(\xi)e^{i\xi x_j}e^{i\xi\Delta x} - g(\xi)e^{i\xi x_j}e^{-i\xi\Delta x}\right) = 0$$

$$g(\xi) - 1 + \frac{\nu}{4}\left(e^{i\xi\Delta x} - e^{-i\xi\Delta x}\right) + g(\xi)\frac{\nu}{4}\left(e^{i\xi\Delta x} - e^{-i\xi\Delta x}\right) = 0$$

$$g(\xi) = \frac{1 - \frac{\nu}{2}\cdot\frac{e^{i\xi\Delta x} - e^{-i\xi\Delta x}}{2}}{1 + \frac{\nu}{2}\cdot\frac{e^{i\xi\Delta x} - e^{-i\xi\Delta x}}{2}}$$

Since $i\sin(\theta) = \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right)$, then, with $\theta := \xi\Delta x$,

$$g(\theta) = \frac{1 - i\frac{\nu}{2}\sin(\theta)}{1 + i\frac{\nu}{2}\sin(\theta)} = \frac{\left(1 - i\frac{\nu}{2}\sin(\theta)\right)^2}{1 + \frac{\nu^2}{4}\sin^2(\theta)} = \frac{1 - \frac{\nu^2}{4}\sin^2(\theta)}{1 + \frac{\nu^2}{4}\sin^2(\theta)} - i\frac{\nu\sin(\theta)}{1 + \frac{\nu^2}{4}\sin^2(\theta)}$$

For ease, let $A = 1 - \frac{\nu^2}{4}\sin^2(\theta)$, $B = \nu\sin(\theta)$, and $C = 1 + \frac{\nu^2}{4}\sin^2(\theta)$. Then

$$g(\theta) = \frac{A}{C} - i\frac{B}{C}$$

So,

$$\left|g(\theta)\right|^2 = \frac{A^2 + B^2}{C^2}$$

But since

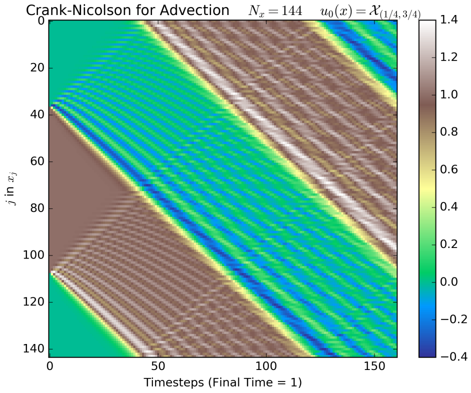
$$A^2 + B^2 = 1 - \frac{\nu^2}{2}\sin^2(\theta) + \frac{\nu^4}{16}\sin^4(\theta) + \nu^2\sin^2(\theta) = 1 + \frac{\nu^2}{2}\sin^2(\theta) + \frac{\nu^4}{16}\sin^4(\theta) = C^2$$

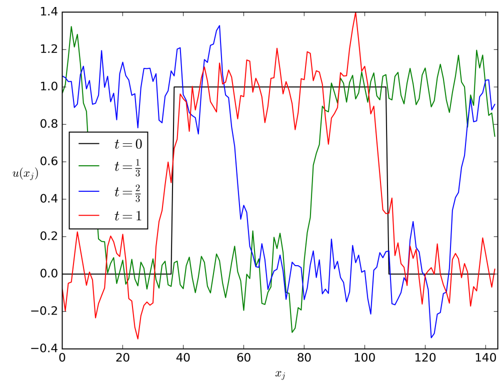
Which proves $\left|g\right|^2 = 1$, showing the scheme is unconditionally stable. This means we can show

$$\left\|u^n\right\|_2 = \left\|g(\xi)^n u^0\right\|_2 = \left|g(\xi)\right|^n\left\|u^0\right\|_2 = 1\cdot\left\|u^0\right\|_2 = \left\|u^0\right\|_2,$$

which implies the scheme is non-dissipative.

(b) Below is an image of the solution of the advection equation ont he periodic domain sing the "Crank Nicolson for Advection" scheme. We see backward propagation of high-frequency modes and forward propagation of the low-frequency modes. This suggests $\frac{\arg(g(\theta))}{-\nu\theta}$ generally decreases with θ, and even becomes negative after certain values of θ.





(c) First note $\arg(g(\theta)) = \arg(\frac{A}{C} - i\frac{B}{C}) = \tan^{-1}\left(-\frac{B}{A}\right) = \tan^{-1}\left(\frac{-\nu\sin(\theta)}{1-\frac{\nu^2}{4}\sin^2(\theta)}\right)$. Using the Maclaurin Series expansions

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots,$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \qquad \text{and}$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots,$$

we get

$$\arg(g(\theta)) = \left(\frac{-\nu\sin(\theta)}{1-\frac{\nu^2}{4}\sin^2(\theta)}\right) - \frac{1}{3}\left(\frac{-\nu\sin(\theta)}{1-\frac{\nu^2}{4}\sin^2(\theta)}\right)^3 + \ldots$$

But since

$$\begin{aligned}
\frac{-\nu\sin(\theta)}{1-\frac{\nu^2}{4}(\sin^2(\theta))} &= (-\nu\sin(\theta))\left(\frac{1}{1-\frac{\nu^2}{4}\sin^2(\theta)}\right) \\
&= -\nu\left(\theta - \frac{\theta^3}{3!} + \ldots\right)\left(1 + \frac{\nu^2}{4}\sin^2(\theta) + \ldots\right) \\
&\approx -\nu\left(\theta - \frac{\theta^3}{3!}\right)\left(1 + \frac{\nu^2}{4}\left(\theta - \frac{\theta^3}{3!}\right)^2\right) \\
&\approx -\nu\theta\left(1 - \frac{\theta^2}{3!}\right)\left(1 + \frac{\nu^2}{4}\theta^2\right) \\
&= -\nu\theta\left(1 + \frac{3\nu^2-2}{12}\theta^2 - \frac{\nu^2}{24}\theta^4\right) \\
&\approx -\nu\theta\left(1 + \frac{3\nu^2-2}{12}\theta^2\right)
\end{aligned}$$

then

$$\begin{aligned}
\arg(g(\theta)) &\approx -\nu\theta\left(1 + \frac{3\nu^2-2}{12}\theta^2\right) - \frac{1}{3}\left(-\nu\theta\left(1 + \frac{3\nu^2-2}{12}\theta^2\right)\right)^3 \\
&= -\nu\theta\left[1 + \frac{3\nu^2-2}{12}\theta^2 - \frac{1}{3}\nu^2\theta^2\left(1 + \frac{3\nu^2-2}{4}\theta^2 + \mathcal{O}(\theta^4) + \mathcal{O}(\theta^6)\right)\right] \\
&\approx -\nu\theta\left[1 - \frac{\nu^2+2}{12}\theta^2\right].
\end{aligned}$$

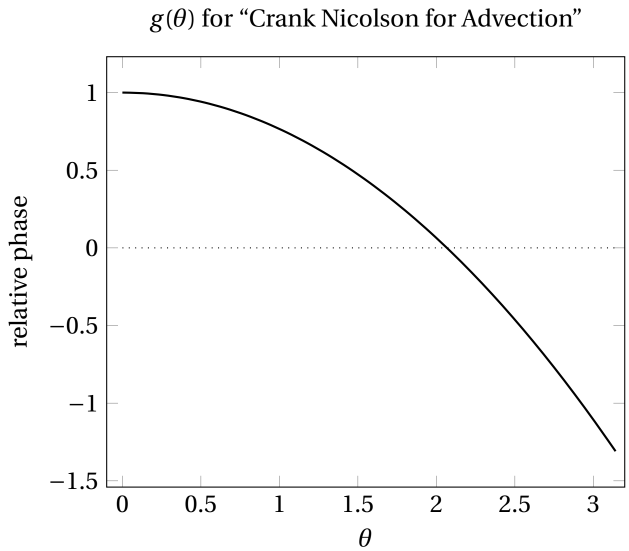
Thus the relative phase is

$$\frac{\arg(g(\theta))}{-\nu\theta} = 1 - \frac{\nu^2+2}{12}\theta^2$$

which for $\nu = 0.9$, is

$$\frac{\arg(g(\theta))}{-\nu\theta} = 1 - \frac{2.81}{12}\theta^2$$

which looks like



and we can see that high frequencies have relative phases near −1. This means high frequencies are propagated backward, which is a big problem since the scheme is non-dissipative. This is what we see in part (b): The high frequencies at the discontinuity are sent in the opposite direction as the general behavior of the low-frequency wave. This method is clearly not suited for discontinuous data because of the high phase error of high frequency modes combined with the fact that it is nondissipative.