MAT 228B Notes

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1 Review

Stability means the amount of growth you get doesn't increase as you refine the mesh.

2 Proof of (one direction of) the Lax Equivalence Theorem

We will show

$$\text{consistent} + \text{stable} = \text{convergent}$$

Set $u^{n+1} = Bu^n + b^n$ (linear scheme for a linear PDE). Let u^n_{sol} be the solution of the PDE sampled on the mesh at time t_n.

We want the error $e^n := u^n - u^n_{\text{sol}} \to 0$ as $\Delta x, \Delta t \to 0$.

Forward Euler is

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} &= Lu^n + f^n \\ \frac{u_{\text{sol}}^{n+1} - u_{\text{sol}}^n}{\Delta t} &= Lu^n_{\text{sol}} + f^n + \underbrace{\tau^n}_{\text{LTE}} \\ u^{n+1}_{\text{sol}} &= (I + \Delta t L)u^n_{\text{sol}} + \Delta t f^n + \Delta t \tau^n \end{aligned}$$

So,

$$u^{n+1}_{\text{sol}} = Bu^n_{\text{sol}} + b^n + \Delta t \tau^n$$

Subtracting $u^{n+1}_{\text{sol}} = Bu^n_{\text{sol}} + b^n + \Delta t \tau^n$ from $u^{n+1} = Bu^n + b^n$ gives

$$e^{n+1} = Be^n - \Delta t \tau^n$$

Assuming $e^0 = 0$ (starting at the correct initial conditions), then $e^1 = Be^0 - \Delta t \tau^0 = -\Delta t \tau^0$ and

$$\begin{aligned} e^2 &= -\Delta t B \tau^0 - \Delta t \tau^1 \\ e^3 &= -\Delta t \big(B^2 \tau^0 + B\tau^1 + \tau^2\big) \\ &\vdots \\ e^n &= -\Delta t \sum_{k=0}^{n-1} B^{n-k-1} \tau^k \end{aligned}$$

So,

$$\begin{aligned} \|e^n\| &= \Delta t \left\| \sum_{k=0}^{n-1} B^{n-k-1} \tau^k \right\| \\ &\leq \Delta t \sum_{k=0}^{n-1} \left\| B^{n-k-1} \tau^k \right\| \\ &\leq \Delta t \sum_{k=0}^{n-1} \left\| B^{n-k-1} \right\| \left\| \tau^k \right\| \end{aligned}$$

We haven't used consistency or stability yet.

From stability, $n-k-1<n$ implies $\left\|B^{n-k-1}\right\| \leq C_T$.

So,

$$\|e^n\| \leq \Delta t C_T \sum_{k=0}^{n-1} \left\|\tau^k\right\|$$

We can bound $\tau \leq \max_k \left\|\tau^k\right\|$. So,

$$\begin{aligned}\|e^n\| &\leq n\Delta t C_T \max_k \left\|\tau^k\right\| \\ &= TC_T \max_k \left\|\tau^k\right\| \to 0 \text{ as } \Delta x, \Delta t \to 0\end{aligned}$$

because it is consistent. $\square$.

3 Stability of Crank-Nicolson for Diffusion in 2-norm

$$\begin{aligned}\frac{u_j^{n+1}-u_j^n}{\Delta t} &= \frac{1}{2}\left(Lu^n + L^{n+1}\right) + \underbrace{f^{n+1/2}}_{\text{or } 1/2(f^n+f^{n+1})} \\ \implies \left(I - \frac{\Delta t}{2}L\right)u^{n+1} &= \left(I + \frac{\Delta t}{2}L\right)u^n + \Delta t f^{n+1/2} \\ u^{n+1} = \left(I + \frac{\Delta t}{2}L\right)\left(I - \frac{\Delta t}{2}L\right)^{-1} u^n &+ \Delta t\left(I - \frac{\Delta t}{2}L\right)^{-1} f^{n+1/2}\end{aligned}$$

Note.. for analysis, the bottom line is fine, but for implementation, solving the second line is better. Note: since the eigenvalue of I is 1, and the eigenvalues of L are nonpositive. So the eigenvalues of $I - \frac{\Delta t}{2}L$ are greater than 0 (greater than or equal to 1).

Set $B \coloneqq \left(I - \frac{\Delta t}{2}L\right)^{-1}\left(I + \frac{\Delta t}{2}L\right)$. Generally, products of symmetric matrices are not symmetric, unless they commute. But they do, so it is. L is symmetric and $\left(I - \frac{\Delta t}{2}L\right)^{-1}$ and $I + \frac{\Delta t}{2}L$ commute, thus B is symmetric.

We know

$$\|B^n\|_2 \leq \|B\|_2^n$$

Let λ_k be an eigenvalue of L. Then the eigenvalues of B are

$$\mu_k = \frac{1 + \frac{\Delta t}{2}\lambda_k}{1 - \frac{\Delta t}{2}\lambda_k}$$

We know $\|B\|_2 = \max_k \left|\frac{1 + \frac{\Delta t}{2}\lambda_k}{1 - \frac{\Delta t}{2}\lambda_k}\right|$. Because $\lambda_k \leq 0$, we know $\|B\|_2 \leq 1$. So, $\|B^n\|_2 \leq 1$.

4 Stability for F.E. for Diffusion in ∞-norm

$$\begin{aligned}\frac{u_j^{n+1}-u_j^n}{\Delta t} &= Lu^n + f^n \\ u^{n+1} &= \underbrace{(I + \Delta t L)}_{B} u^n + \Delta t f^n\end{aligned}$$

We know $\|B^n\|_\infty \leq \|B\|_\infty^n$.

$$\|B\|_\infty = (\text{max row sum})(B)$$

One row of the matrix..

$$u_j^{n+1} = \frac{D\Delta t}{(\Delta x)^2}u_{j-1}^n + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)u_j^n + \frac{D\Delta t}{(\Delta x)^2}u_{j+1}^n + \Delta t f_j^n$$

Now,

$$\|B\|_\infty = \left|\frac{D\Delta t}{(\Delta x)^2}\right| + \left|1 - \frac{2D\Delta t}{(\Delta x)^2}\right| + \left|\frac{D\Delta t}{(\Delta x)^2}\right|$$

If the middle term is positive, i.e. $1 - \frac{2D\Delta t}{(\Delta x)^2} \geq 0$, then

$$\|B\|_\infty = 1, \qquad \text{if } \Delta t \leq \frac{(\Delta x)^2}{2D}$$

This is the same restriction we saw from the absolute stability analysis. So again we get $\|B^n\|_\infty \leq \|B\|_\infty^n \leq 1$.