MAT 228B Notes

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1 Avoiding Wiggles

- Suppose $u_j^0 \geq u_{j+1}^0$ for all j. If $u_j^n \geq u_{j+1}^n$ for all j and n, then the scheme is monotone-preserving.
- Godunov's Theorem - A linear monotonicity preserving scheme is at most first-order accurate.
- Idea: To get higher-order accuracy and not get wiggles, we need to use a non-linear scheme.
- For example:

$$F_{j+1/2} = F_{j+1/2}^{\text{up}} + \left(F_{j+1/2}^{\text{LW}} - F_{j+1/2}^{\text{up}}\right)\phi(u^n)$$

  where ϕ is a "flux limiter." We have a nonlinear scheme, even for a linear PDE.
- Let $u_t + (f(u))_x = 0$. Godunov's Method (an REA (reconstruct, evolve, average) method) is
  1. Reconstruct a function from cell averages.
     - Given u_j^n, get $\tilde{u}(x, t_n)$.
     - For example: piecewise-constant reconstruction.
  2. Evolve.
     - Solve the PDE exactly using the reconstruction as initial data.
  3. Average on each cell to update the averages.
     - Set $u_J^{n+1} = \dfrac{1}{\Delta x}\displaystyle\int_{x-1/2}^{x+1/2} \tilde{u}(x, t_{n+1})\mathrm{d}x$
- Example: Advection with $a > 0$, $\nu \leq 1$.
  - Remember

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right).$$

  - We only need $u(x_{j+1/2}, t)$ to compute the u_J^{n+1} using the above equation, and $F_{j+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(\tilde{u}(x_{j+1/2}, t))\mathrm{d}t$.
  - So $\tilde{u}(x_{j+1/2}, t) = u_j^n$.
  - $F_{j+1/2}^n = au_j^n$ (this is upwinding)
- We can generalize to a nonlinear problem

$$u_t + (f(u))_x = 0$$

  and

$$u(x, 0) = \begin{cases} u_\ell & x < 0 \\ u_r & x > 0 \end{cases}$$

  This is a Riemann problem.

2 Higher-Order Accuracy

- We can do a more accuracte reconstruction than piecewise constant. How about piecewise linear? But remember we need to preserve the averages.

$$\tilde{u}(x,t_n) = u_j^n + \sigma_j^n \underbrace{(x - x_j)}_{0 \text{ average}} \qquad x_{j-1/2} \leq x \leq x_{j+1/2}$$

- So the average on the cell is independent from σ (independent of slope).
- What slope do we pick?
  - $\sigma_j^n = 0$ this is upwinding
  - ? force continuity?
  - $\sigma_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$ (centered, Fromm's method)
  - $\sigma_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x}$, downward slope (Lax-Wendroff)
  - $\sigma_j^n = \frac{u_j - u_{j-1}}{\Delta x}$ (upwind difference, beam-warming).
- Considering $a > 0$, using LW slope for

$$u_j^n = \begin{cases} 1 & j \leq J \\ 0 & j > J \end{cases}$$

  This immediately introduces an overshoot behijnd the jump, killing monotonicity. So Lax-Wendroff gives wiggles behind, Fromm's method gives wiggles on both siedes (but they're half as off), and Beam-Warming gives wiggles in front of the discontinuity.

- BUT! You can pick different slopes at each interval!
- No wiggles if we pick Lax-Wendroff to the right of the jump and beam-warming to the left. To avoid oscillations, use so-called "limited slope"
- "Minmod" slope

$$u_j^n = \text{minmod}\left(\frac{u_j - u_{j-1}}{\Delta x}, \frac{u_{j-1} - u_j}{\Delta x}\right)$$

  where

$$\text{minmod}(a,b) = \min(|a|,|b|)$$

  A different choice monotonized centered slope

$$\sigma_j^n = \text{minmod}\left(\frac{u_{j+1} - u_{j-1}}{2\Delta x}, 2\left(\frac{u_{j+1} - u_j}{\Delta x}\right), 2\left(\frac{u_j - u_{j-1}}{\Delta x}\right)\right)$$