MAT 228B Notes

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1 Why was Crank Nicolson Bad for $u_t = u_{xx}$ with $u(0,x) = \mathcal{X}_{[0,0.5]}(x)$?

For $y' = \lambda y$, you get $y^{n+1} = R(z)y^n$. With trapezoidal rule, you get $R(z) = \dfrac{1+\frac{z}{2}}{1-\frac{z}{2}}$. With backward Euler, you get $R(z) = \dfrac{1}{1-z}$. Both of these methods are A-Stable, which means $|R(z)| < 1$ for all $z = a + bi$ with $a < 0$ (left half-plane).

1.1 What happens as $z \to -\infty$ along the real axis?

The Trapezoidal rule: $R(z) \to -1$. Backward Euler $R(z) \to 0$. This is saying, for large $-\lambda$, the trapezoidal rule gives slowly damped oscillations. Backward Euler gives quickly damped solutions without oscillations (rapid monotonic decay to 0). The behavior of the Backward Euler method matches the behavior of the ODE.

Both BE and TR are A-Stable, i.e. $R(z) < 1$ for all z in the left half plane. BE is also L-stable (if A-stable and $|R(z)| \to 0$ as $z \to \infty$).

1.2 So,

Using CN on $u_t = u_{xx}$ with step function initial condition.. use

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu$$

The eigenvalues of L are $\lambda_k = \frac{2D}{\Delta x^2}(\cos(k\pi\Delta x) - 1)$. How big are these eigenvalues? For k small,

$$\lambda_k = -k^2\pi^2 D + \mathcal{O}\big(\Delta x^2\big)$$

These agree with the continuous operator. For k large,

$$\lambda_k \approx -\frac{4D}{\Delta x^2} \to \infty \text{ as } \Delta x \to 0.$$

Large k give $z_k = \Delta t \lambda_k \approx -\frac{4D\Delta t}{\Delta x^2}$. We could refine the time scale, but then we might as well use an explicit method.

The eigenvectors are ${}^k v_j = \sin(k\pi x_j)$.

For discontinuous initial data, the Fourier coefficients decay like $\frac{1}{k}$. For Crank-Nicolson, $u^{n+1} = Bu^n + b$, and $\mu_k = \dfrac{1 + \frac{\lambda_k \Delta t}{2}}{1 - \frac{\lambda_k \Delta t}{2}}$ where λ_k are the eigenvalues of the Laplacian.

$\boxed{\text{If you are expecting sharp frons, don't use CN.}}$ Backward Euler is OK, but it is only 1st order accurate. You could use BDF2 - it is second order and L-stable. It is the same work, but more storage, but storage is not usually a big deal.

2 TR-BDF2

This is diagonally implicit RK, 2-stage, and L-stable. The first step is half-step trapezoidal rule. The second step is BDF2 on the initial and half-step.

$$u^* = u^n + \frac{\Delta t}{4}(f(u^n) + f(u^*))$$
$$u^{n+1} = \frac{1}{3}\big(4u^* - u^n + \Delta t g(u^{n+1})\big)$$