MAT 228B Notes

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1 Conservation Laws

1.1 Integral Form

Using the integral form of $u_t + (f(u))_x = 0$, we find

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x} = 0$$

where we are letting

$$u_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t)\mathrm{d}x,$$

where F is time-averagd flux through the cell edge. Schemes of the above form are discretely conservative.

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\Big(F_{j+1/2}^n - F_{j-1/2}^n\Big)$$

Let the cells be indexed by $j = j_1$ to $j = j_2$. The total amount of stuff in these cells is $\Delta x \sum_{j=j_1}^{j_2} u_j^n$ at time t_n.

$$\begin{aligned}
\Delta x \sum_{j=j_1}^{j_2} u_j^{n+1} &= \Delta x \sum_{j=j_1}^{j_2} u_j^n - \Delta t \sum_{j=j_1}^{j_2}\Big(F_{j+1/2}^n - F_{j-1/2}^n\Big) \\
&= \Delta x \sum_{j=j_1}^{j_2} u_j^n - \Delta t\Big(F_{j_2+1/2}^n - F_{j_1-1/2}^n\Big)
\end{aligned}$$

If F's are zero on the ends, $\Delta x \sum_{j=j_1}^{j_2} u_j^{n+1} = \Delta x \sum_{j=j_1}^{j_2} u_j^n$

1.2 Differential Form

Using the differential form of $u_t + (f(u))_x = 0$ i.e. $u_t + f'(u)u_x = 0$. Suppose $f'(u) > 0$.

1.2.1 Upwind Discretization

Because $f'(u) > 0$, an appropriate discretization is

$$\begin{aligned}
\frac{u_j^{n+1} - u_j^n}{\Delta t} + f'(u_j^n)\frac{u_j^n - u_{j-1}^n}{\Delta x} &= 0 \\
u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x} f'(u_j^n)\big(u_j^n - u_{j-1}^n\big)
\end{aligned}$$

$$\begin{aligned}
\Delta x \sum_{j=j_1}^{j_2} u_j^{n+1} &= \Delta x \sum_{j=j_1}^{j_2} u_j^n - \Delta t \sum_{j=j_1}^{j_2} f'(u_j^n)\big(u_j^n - u_{j-1}^n\big) \\
&= \Delta x \sum_{j=j_1}^{j_2} u_j^n - \Delta t\big(f'(u_{j_1}^n)\big(u_{j_1}^n - u_{j_1-1}^n\big) + f'(u_{j_1+1}^n)\big(u_{j_1+1}^n - u_{j_1}^n\big) + \dots\big)
\end{aligned}$$

The series is not telescoping, so this is not discreteley conservative.

2 Using Numerical Flux Functions

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x} = 0$$

with

$$u_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\mathrm{d}x$$

If

$$F_{j+1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\mathrm{d}t,$$

then we cannot compute this exactly. We need to construct "numerical flux functions" for $F_{j+1/2}^n$ which approximate the time-averaged flux.

Suppose $f(u) = au$ and $a > 0$. We can approximate $f(u(x_{j+1/2}, t)) = f(u(x_j, t_n))$ over the timestep. But $f(u(x_j, t_n)) = au_j^n$, so

$$F_{j+1/2}^{\mathrm{up}} = \begin{cases} au_j^n & a > 0 \\ au_{j+1}^n & a < 0 \end{cases}$$

This leads us to...

3 2-Step Lax-Wendroff

We are going to approximate $f(u(x_{j+1/2}, t)) = f(u(x_{j+1/2}, t_{n+1/2}))$ over the time interval. This gives 2nd order time approximation to the average flux. But we need an approximation for $x_{j+1/2}$ and $t_{n+1/2}$. That is,

$$u_{j+1/2}^{n+1/2} = u_{j+1/2}^n - \frac{\Delta t}{2}\left(\frac{f(u_{j+1}^n) - f(u_j^n)}{\Delta x}\right)$$
$$u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^n) - f(u_j^n)\right)$$

Letting $f(u) = au$ gives

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_j^n\right)$$

Then,

$$F_{j+1/2}^{\mathrm{LW}} = f(u_{j+1/2}^{n+1/2}) = au_{j+1/2}^{n+1/2} = \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{a^2\Delta t}{2\Delta x}\left(u_{j+1}^n - u_j^n\right)$$

So,

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^{\mathrm{LW}} - F_{j-1/2}^{\mathrm{LW}}\right) \\ &= u_j^n - \frac{\Delta t}{\Delta x}\left(\frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{a^2\Delta t}{2\Delta x}\left(u_j^n - u_{j-1}^n\right) - \frac{a}{2}\left(u_{j-1}^n + u_j^n\right) + \frac{a^2\Delta t}{2\Delta x}\left(u_j^n - u_{j-1}^n\right)\right) \\ &= u_j^n - \frac{\Delta t}{\Delta x}\left(\frac{a}{2}\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{a^2\Delta t}{2\Delta x}\left(u_{j+1}^n - 2u_j^n + u_{j+1}^n\right)\right) \end{aligned}$$

4 Preview

Suppose $a > 0$.

$$\begin{aligned} F_{j+1/2}^{\mathrm{LW}} &= au_j^n - au_j^n + \frac{a}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{a^2\Delta t}{2\Delta x}\left(u_{j+1}^n - u_j^n\right) \\ &= au_j^n + \frac{a}{2}\left(u_{j+1}^n - u_j^n\right) - \frac{a^2\Delta t}{2\Delta x}\left(u_{j+1}^n - u_j^n\right) \\ &= \underbrace{au_j^n}_{\text{upwinding}} + \underbrace{\frac{a}{2}(1-\nu)\left(u_{j+1} - u_j^n\right)}_{\text{2nd order correction}} \end{aligned}$$

So the general form of a high-resolution scheme is

$$F = F^{\mathrm{up}} + \left(F^{\mathrm{LW}} - F^{\mathrm{UP}}\right)\phi$$

where ϕ is flux limiter that depends on the solution.