MAT 228B Notes

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1 Von Neumann Analysis

This is stability analysis of different schemes using Fourier analysis. The diffusion equation,

$$u_t = Du_{xx}$$

Using Fourier analysis in x, we know $\hat{u}_t(\xi, t) = -D\xi^2\hat{u}(\xi, t)$.

Von Neumann Analysis is used to analyze constant-coefficient linear problems on the whole real line or a periodic domain.

1.1 On the Real Line (Infinite 1D Lattice)

Let $x_j = j\Delta x$ where $j \in \mathbb{Z}$. We will use the fact that complex exponentials $v_j = e^{i\xi x_j}$ are eigenfunctions of difference operators. The forward difference operator

$$(D_+v)_j := \frac{v_{j+1} - v_j}{\Delta x}$$

applied to $v_j = e^{i\xi x_j}$ gives

$$(D_+v)_j = \frac{e^{i\xi x_{j+1}} - e^{i\xi x_j}}{\Delta x} = e^{i\xi x_j} \underbrace{\left(\frac{e^{i\xi\Delta x} - 1}{\Delta x}\right)}_{\text{eigenvalue } \lambda_j(\xi) \text{ for } v_j}$$

Note the eigenvalues of v_j are dependent on the wave number ξ. For the second-difference operator,

$$(D^2v)_j = \frac{v_{j-1} + 2v_j + v_{j+1}}{\Delta x^2} = \frac{e^{i\xi x_{j-1}} - 2e^{i\xi x_j} + e^{i\xi x_{j+1}}}{\Delta x^2} = e^{i\xi x_j}\underbrace{\left(\frac{e^{-i\xi\Delta x} - 2 + e^{i\xi\Delta x}}{\Delta x^2}\right)}_{\text{eigenvalue}} = \underbrace{\frac{2}{\Delta x^2}(\cos(\xi\Delta x) - 1)}_{\text{eigenvalue}} v_j$$

Sometimes this is easy to write as $\lambda = -\frac{4}{\Delta x^2}\sin^2(\frac{\xi\Delta x}{2})$.

Let v_j be a discrete function on the discrete lattice $x_j = j\Delta x$. The Fourier transform of the infinite sequence v_j is

$$\hat{v}(\xi) = \frac{\Delta x}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} v_j e^{i\xi x_j}$$

This looks just like a discretized version of the continuous Fourier Transform. This is the projection of the function onto the eigenfunctions. ξ is continuous, but it's bounded. The highest spacial frequency representable on the discrete mesh with Δx is $2\Delta x$, and the wavelength of $e^{i\xi x_j}$ is (Bob talked really fast), anyway, $2\Delta x = \frac{2\pi}{\xi}$, so $|\xi| \leq \frac{\pi}{\Delta x}$.

The inverse Fourier transform is given by

$$v_j = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} \hat{v}(\xi)e^{i\xi x_j}\,\mathrm{d}\xi$$

There is a duality between the spaces $\mathbb{Z}$ and $[a, b]$, and between $\mathbb{R}$ and $\mathbb{R}$. We will use Parseval's relation, which is $\|v(j)\|_2 = \|\hat{v}(\xi)\|_2$. The "two-norm" of the discrete function is

$$\|v(j)\|_2 = \left(\Delta x\sum_{j\in\mathbb{Z}}|v_j|^2\right)^{1/2}$$

We can use this for stability analysis:

$$u^{n+1} = Bu^n$$

Previously, we want to control $\|B\|_2 \leq 1 + \alpha\Delta t$ for stability. Instead, we can show that $\left\|u^{n+1}\right\|_2 \leq (1+\alpha\Delta t)\|u^n\|_2$. Why is this equivalent? Suppose the statement. Then

$$\frac{\left\|u^{n+1}\right\|_2}{\|u^n\|_2} \leq 1 + \alpha\Delta t \implies \|Bu^n\|_2 \|u^n\|_2 \leq 1 + \alpha\Delta t$$

Since this is true for all u, maximize $\|B\|_2 \leq 1 + \alpha\Delta t$.

So we can look at the norms of the discrete functions (and so the norms of their Fourier transforms) in order to show stability of the scheme.

If we can show that

$$\left\|\hat{u}^{n+1}\right\|_2 \leq (1+\alpha\Delta t)\|\hat{u}^n\|_2,$$

then the scheme is stable.

2 Forward Euler for Diffusion

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x^2} D\big(u_{j-1}^n - 2u_j^n + u_{j+1}^n\big).$$

We can represent u_j^n as

$$u_j^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{u}^n(\xi) e^{i\xi x_j} \mathrm{d}\xi$$

But, after some manipulation,

$$u_j^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\Delta x}^{\pi/\Delta x} \left(1 + \frac{\Delta t D}{\Delta x^2}\big(e^{-i\xi\Delta x} - 2 + e^{i\xi\Delta x}\big)\right) \hat{u}^n(\xi) e^{i\xi x_j} \mathrm{d}\xi$$

Now we throw both sides into the Fourier transform:

$$\hat{u}^{n+1}(\xi) = \left(1 + \frac{2\Delta t D}{\Delta x^2}(\cos(\xi\Delta x) - 1)\right)\hat{u}^n(\xi)$$

This is of the form $\hat{u}^{n+1}(\xi) = g(\xi)\hat{u}^n(\xi)$. **Differences have become multiplication**. $g(\xi)$ is called the amplification factor. It is the amount by which the Fourier transform gets multiplied by in each step. In Fourier space,

$$\left\|\hat{u}^{n+1}\right\|_2 = \|g(\xi)\hat{u}^n(\xi)\|_2 \leq \|g(\xi)\|_\infty \|\hat{u}^n(\xi)\|_2 \qquad \text{using Hölder's Inequality}$$

If we can show that $\max_\xi |g(\xi)| \leq 1 + \alpha\Delta t$, then the scheme is stable. So,

$$g(\xi) = 1 - \frac{4D\Delta t}{\Delta x^2}\sin^2\left(\frac{\xi\Delta x}{2}\right)$$

So

$$|g(\xi)| \leq 1 \qquad \text{for all } \xi$$

means

$$-1 \leq 1 - \frac{4D\Delta t}{\Delta x^2}\sin^2\left(\frac{\xi\Delta x}{2}\right) \leq 1.$$

This requires that

$$0 \leq \frac{4D\Delta t}{\Delta x^2}\sin^2\left(\frac{\xi\Delta x}{2}\right) \leq 2$$

This holds for all ξ if

$$\Delta t \leq \frac{\Delta x^2}{2D}$$