MAT 228B Notes

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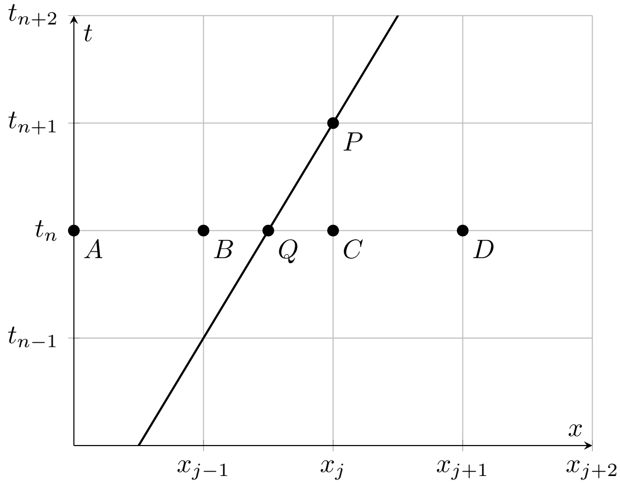
1 Second Order Methods

Consider a space-time grid and the point (x_j, t_{n+1}). Assume $a > 0$ and $\nu \leq 1$.



Use 3 points and quadratic interpolation. Use points B, C, and D and get Lax-Wendroff. or use A, B, and C to get Mean-Warming which gives 2nd order upwinding scheme and/or 1-sided LW.

Derive LW/BW from Taylor expansion

$$u(x, t + \Delta t) = u(x + t) + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \mathcal{O}\big(\Delta t^3\big)$$

Then use the PDE to express time derivatives as space derivatives, so

$$u_t + au_x = 0 \qquad \text{means} \qquad u_t = -au_x$$

and so $u_{tt} = -au_{tx} = a^2 u_{xx}$, so

$$u(x, t + \Delta t) = u(x, t) - a\Delta t u_x + \frac{a^2 \Delta t^2}{2} u_{xx} + \mathcal{O}\big(\Delta t^3\big)$$

Now use finite differences to approximate the spatial derivatives. If we use centered second-order differences, we get the Lax-Wendroff method:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\big(u_{j+1}^n - u_{j-1}^n\big) + \frac{a^2 \Delta t^2}{2\Delta x^2}\big(u_{j+1}^n - 2u_j^n + u_{j-1}^n\big)$$

We can use one-sided differences (upwind) for $a > 0$. So we get the Beam Warming scheme.

LW is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2\Delta x}\big(u_{j+1}^n - u_{j-1}^n\big) = \frac{a^2 \Delta t}{2\Delta x^2} = \frac{a^2 \Delta t}{2\Delta x^2}\big(u_{j-1}^n - 2u_j^n + u_{j+1}^n\big)$$

Truncation error analysis:

$$u_t + \frac{\Delta t}{2} u_{tt} + \mathcal{O}\big(\Delta t^2\big) + au_x + \mathcal{O}\big(\Delta x^2\big) = \frac{a^2}{\Delta t} 2u_{xx} + \mathcal{O}\big(\Delta t \Delta x^2\big)$$

$$u_{tt} = a^2 u_{xx}$$

and the LTE $= \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$.

For Beam Warming,

$$u_t = \frac{\Delta t}{2} u_{tt} + \mathcal{O}(\Delta t^2) + a u_x + \mathcal{O}(\Delta x^2) = \frac{a^2 \Delta t}{2} u_{xx} + \mathcal{O}(\Delta t \Delta x)$$

and so

$$\text{LTE} = \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t \Delta x)$$

Stability of Lax-Windroff:

$$|g(\xi)|^2 = \left| -4\nu^2 (1 - \nu^2) \sin^4 (\frac{\xi \Delta x}{2}) \right|$$

So we require $4\nu^2(1 - \nu^2) \geq 0$ for stability. So we need $\nu^2 \leq 1$. And again, we get $\frac{\Delta t |a|}{\Delta x} \leq 1$.