# Homework #2

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## Problem 1

Let $i = \sqrt{-1}$ and set

$$A = \begin{bmatrix} i & 0 & -i \\ 0 & i & -i \end{bmatrix}.$$

Using the null space property, show that $\ell_1$-minimization will recover any 1-sparse vector $x$, given $Ax = y$.

*Proof.* Given $x = [x_1, x_2, x_3]^T \in \mathbb{C}^3$, $Ax = i[x_1 - x_3, x_2 - x_3]^T = 0$ if $x_1 = x_2 = x_3$. Thus null$A = \text{span}\big([1,1,1]^T\big)$. Let $h \in \text{null}A$. Then $h = [a, a, a]^T$ for some $a \in \mathbb{C}$. Then choose $S_i = \{i\}$ for $i = 1,2,3$. Then

$$h_{S_1} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \qquad h_{S_2} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}, \qquad \text{and} \qquad h_{S_3} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$

which gives

$$h_{S_1^C} = \begin{bmatrix} 0 \\ a \\ a \end{bmatrix}, \qquad h_{S_2^C} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix}, \qquad \text{and} \qquad h_{S_3^C} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix}$$

Clearly $\|h_{S_i}\|_1 = |a|$ and $\left\|h_{S_i^C}\right\|_1 = 2|a|$ for $i = 1,2,3$. Thus $\|h_S\|_1 \le \left\|h_{S^C}\right\|_1$ for all $h \in \text{null}A$ and all $S \subset \{1,2,3\}$ with $|S| = 1$. This shows the null space property holds and hence $\ell_1$-minimization will recover any 1-sparse vector $x$. □

## Problem 2

On the connection between (in)coherence parameter $\mu$ and restricted isometry constant $\delta_s$: Show that $\delta_1 = 0$, $\delta_2 = \mu$, and $\delta_s \le (s-1)\mu$.

*Proof.* □

## Problem 3

Let $A = \mathbb{R}^{k\times d}$ be a Gaussian random matrix. Given an estimate for the coherence $\mu$ of $A$.

*Proof.* □

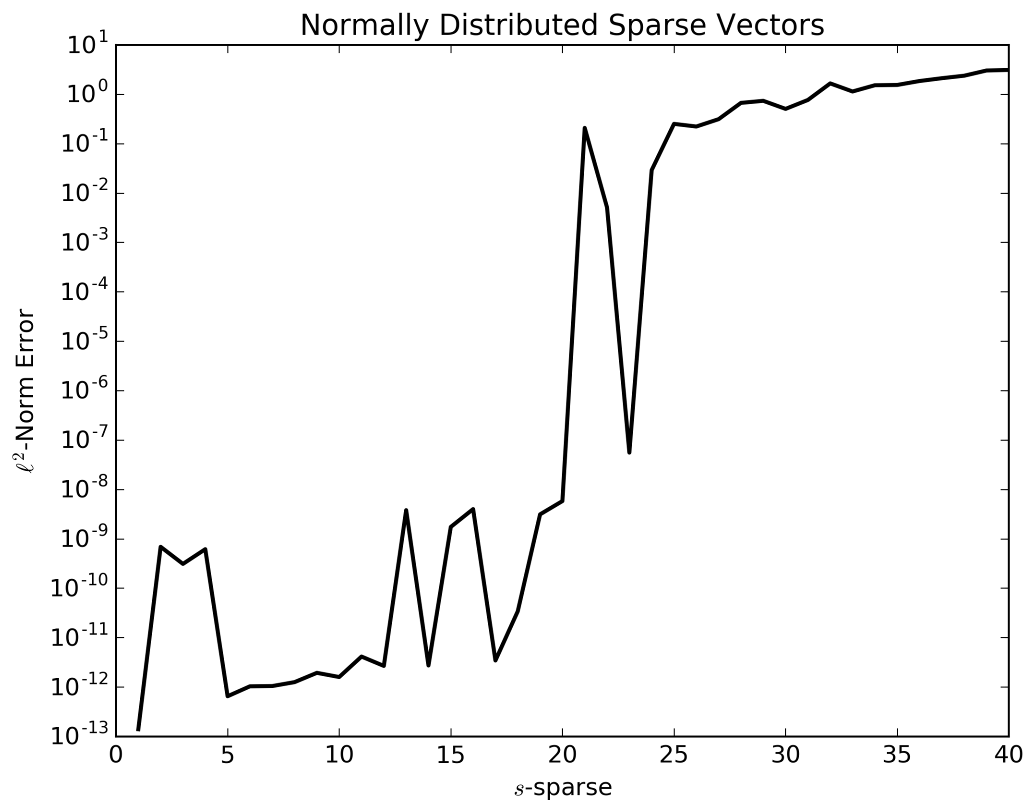
## Problem 4

Consider $y = Ax$, where $A$ is a $100 \times 400$ Gaussian random matrix and $x$ is a $s$-sparse vector of length 400. The locations of the non-zero entries of $x$ are chosen uniformly at random and the non-zero coefficients of $x$ are normal-distributed. For $s = 1, 2, \ldots$, solve

$$\min_z \|z\|_1 \qquad \text{subject to } Az = y,$$

(e.g. using the toolbox CVX). For each fixed $s$ repeat the experiment 10 times. Create a graph plotting $s$ versus the relative reconstruction error (averaged over the ten experiments for each $s$). Starting with which value of $s$ approximately does $\ell_1$-minimization fail to recover $x$?

*Proof.* The following graph shows the mean $\ell^2$-norm errors of 10 experiments at each $s$ for $s = 1, 2, \ldots, 40$. The non-zero entries of $x$ are normally distributed around 0 with standard deviation of 1. In this experi-




ment, this method failed for $s \geq 21$. □

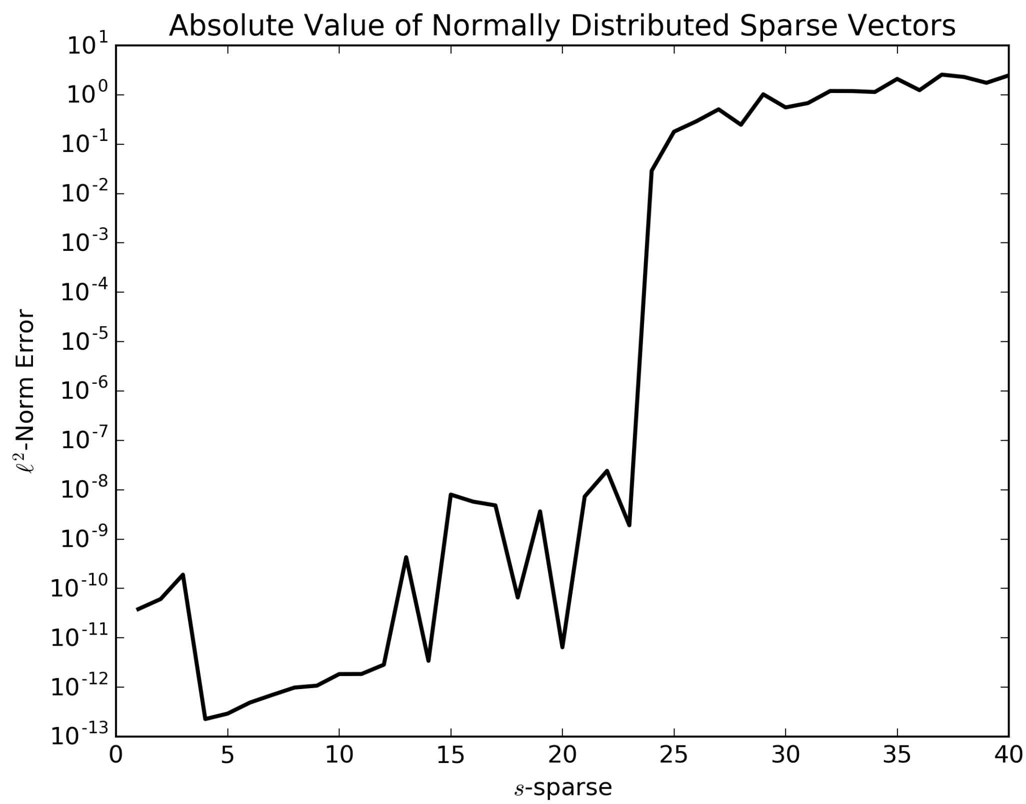
## Problem 5

Same setup as in Problem 4, but now the non-zero entries of $x$ are non-negative. Taking this information into account, we now solve

$$\min_z \|z\|_1 \qquad \text{subject to } Ax = y \text{ and } z \geq 0$$

(here, $z \geq 0$ is meant entrywise, i.e., for each $k$, $z_k \geq 0$). (The positivity constraint is easy to include in CVX). Repeat the simulations as described in Problem 4. Compare your findings to the results from your experiments of Problem 4 and try to quantify the difference regarding the range for $s$ for which recovery is still possible in this case.

*Proof.* The following graph shows the mean $\ell^2$-norm errors of 10 experiments at each $s$ for $s = 1, 2, \ldots, 40$. The non-zero entries of $x$ are the absolute value of a normal distribution around 0 with standard deviation of 1. In this experiment, this method failed for $s \geq 24$. □


Absolute Value of Normally Distributed Sparse Vectors
$\ell^2$-Norm Error
$s$-sparse
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
10^-9
10^-10
10^-11
10^-12
10^-13
0
5
10
15
20
25
30
35
40