PBG 200A Notes

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1 Review

1.1 Law of Large Numbers (LLN)

If X_n for $n = 1, 2, \ldots$ are IID random variables with finite expectation ($\mathbb{E}[X_1] < \infty$), then

$$\lim_{N\to\infty} \frac{\sum_{i=1}^N X_i}{N} = \mathbb{E}[X_1] \qquad \text{with probability 1.}$$

1.2 From last time

$N_t \coloneqq \prod_{i=1}^t R_i N_0$. Take logs to get

$$\begin{aligned} \log N_t &= \sum_{i=1}^t \log R_i + \log N_0 \\ \implies \frac{1}{t}\log\frac{N_t}{N_0} &= \frac{\sum_{i=1}^t \log R_i}{t} \to \mathbb{E}[\log R_1] =: r \end{aligned}$$

i.e. $\log\frac{N_t}{N_0} \approx rt$. In the example,

$$r = \frac{1}{2}\log 4 + \frac{1}{2}\log\frac{1}{5} = \frac{1}{2}\log\frac{4}{5} < 0$$

This shows that the populations will tend to ∞.

1.3 Geometric Mean

$\mathbb{E}[\log R_1]$ is the log of the geometric mean of R_1, i.e.

$$\text{geometric mean } R_1 = \exp(\mathbb{E}[\log R_1])$$

Basic fact:

$$\mathbb{E}[R_1] \geq \exp(\mathbb{E}[\log R_1])$$

i.e. the arithmentic mean is always at least as large as the geometric mean

1.4 Small Variance Approximation

Let $R_1 = \bar{R} + \sigma X_1$, where X_1 has mean 0 and variance 1. Reminder:

$$\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$\begin{aligned} \mathbb{E}[\log R_1] &= \mathbb{E}[\log\big(\bar{R} + \sigma X_1\big)] \\ &= \log\bar{R} + \mathbb{E}\Big[\log\Big(1 + \frac{\sigma}{\bar{R}}X_1\Big)\Big] \end{aligned}$$

Recall again

$$\log(1 + x) \approx x - \frac{1}{2}x^2$$

So, setting $x = 1 + \frac{\sigma}{\bar{R}} X_1$,

$$\begin{aligned}\mathbb{E}[\log R_1] &\approx \log \bar{R} + \mathbb{E}\left[\frac{\sigma}{\bar{R}} X_1 - \frac{1}{2}\left(\frac{\sigma}{\bar{R}} X_1\right)^2\right] \\ &= \log \bar{R} - \frac{1}{2} \cdot \underbrace{\left(\frac{\sigma}{\bar{R}}\right)^2}_{\text{coefficient of variation for } R_1}\end{aligned}$$

2 How Stochasticity Influences Life-History Evolution (Bet Hedging)

Bet hedging is evolution of reduced variance σ^2 despite a reduction in mean $\bar{R}$.

2.1 Example: Annual Plants with Seed Banks

Simple model: N is the number of seeds underground. g is probability of germination, which produces yield Y_{t+1}. $1 - g$ is the probability the seed does not germinate. Given the seed does not germinate, S is the probability of survival to the next year.

$$N_{t+1} = N_t \underbrace{[gY_{t+1} + (1-g)S]}_{=R_{t+1}}$$

Is bet hedging possible? We look at the mean and variance:

$$\mathbb{E}[R_1] = \mathbb{E}[gY_1 + (1-g)S] = g\mathbb{E}[Y_1] + (1-g)S \tag{1}$$

So mean is increasing with g since $S < 1$ and we assume $\mathbb{E}[Y_1] \geq 1$.

$$\mathrm{var}[R_1] = g^2 \mathrm{var}[Y_1] \tag{2}$$

So variance is also increasing with g.

This shows there is a tradeoff because mean and variance both increase with g.