Homework #3

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Problem 1

For the age-structured logger head model do the following:

(a) Find the dominant eigenvalue λ, the vector w of reproductive values, an the stable stage distribution v. Plot these and discuss what insights one can gleam from these bar plots that aren't apparent in the 7-stage matrix model used by Crose et al. (1987)

(b) Use w, v, and λ to predict the densities of all ages in 20 years if currently the loggerhead population consists of only 10,000 hatchlings. Simulate the full matrix model, plot the simulation, and compare the predictions at year 20 by plotting them side-by-side using the barplot command. Repeat both computations for 100 years and discuss.

(c) Compute and plot the elasticities of λ to survivorship and fecundity of all age classes. Compare and contrast these elasticities to the Crouse et al. (1987) paper.

(a) The seven stages of this model are

(a) Hatchlings (year 1)
(b) Yearlings (years 2 - 8)
(c) Juveniles (years 9 - 16)
(d) Sub-Adults (years 17 - 22)
(e) First-Time Reproducers (year 23)
(f) Remigrants (year 24)
(g) Mature Adults (years 25 - 55)

If we are just studying the stages, the matrix describing their population shifts after one year are given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 127 & 4 & 80 \\ 0.6747 & 0.737 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0486 & 0.6610 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0147 & 0.6907 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0518 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8091 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8091 & 0.8089 \end{pmatrix}$$

However we are using the 55×55 matrix for the age-structured population model rather than the 7×7 stage-structure population model. The largest eigenvalue of this matrix, $\lambda \approx 0.9644$, corresponds to the eigenvector

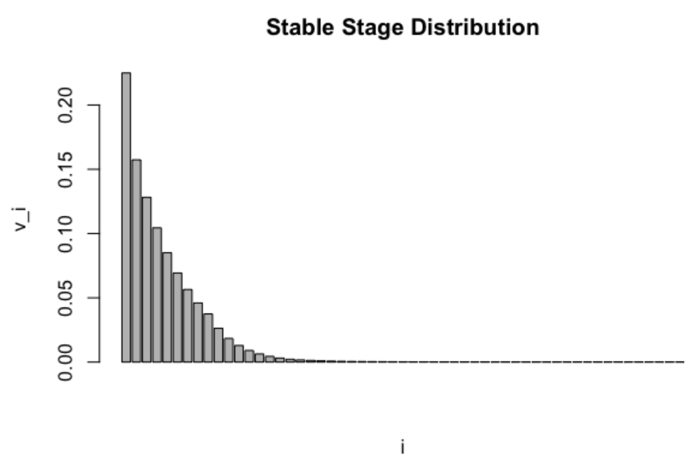
$$v \approx \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

where

$$v_1 = \begin{bmatrix} 2.249330e-01 \\ 1.573566e-01 \\ 1.281925e-01 \\ 1.044336e-01 \\ 8.507811e-02 \\ 6.930994e-02 \\ 5.646420e-02 \\ 4.599926e-02 \\ 3.747387e-02 \\ 2.625837e-02 \\ 1.839953e-02 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1.289276e-02 \\ 9.034103e-03 \\ 6.330298e-03 \\ 4.435711e-03 \\ 3.108153e-03 \\ 2.177918e-03 \\ 1.676713e-03 \\ 1.290851e-03 \\ 9.937877e-04 \\ 7.650873e-04 \\ 5.890177e-04 \end{bmatrix} \quad v_3 = \begin{bmatrix} 4.534671e-04 \\ 3.804249e-04 \\ 3.191479e-04 \\ 2.677412e-04 \\ 2.246147e-04 \\ 1.884349e-04 \\ 1.580828e-04 \\ 1.326196e-04 \\ 1.112579e-04 \\ 9.333701e-05 \\ 7.830275e-05 \end{bmatrix} \tag{0.1}$$

$$v_4 = \begin{bmatrix} 6.569013e-05 \\ 5.510909e-05 \\ 4.623240e-05 \\ 3.878551e-05 \\ 3.253813e-05 \\ 2.729705e-05 \\ 2.290018e-05 \\ 1.921153e-05 \\ 1.611703e-05 \\ 1.352098e-05 \\ 1.134309e-05 \end{bmatrix} \quad v_5 = \begin{bmatrix} 9.516002e-06 \\ 7.983212e-06 \\ 6.697316e-06 \\ 5.618545e-06 \\ 4.713538e-06 \\ 3.954305e-06 \\ 3.317365e-06 \\ 2.783020e-06 \\ 2.334745e-06 \\ 1.958676e-06 \\ 1.643182e-06 \end{bmatrix}. \tag{0.2}$$

This is the stable stage distribution, i.e. as $t \to \infty$, $\dfrac{N_t}{\|N_t\|_1} \to v$ with probability 1.



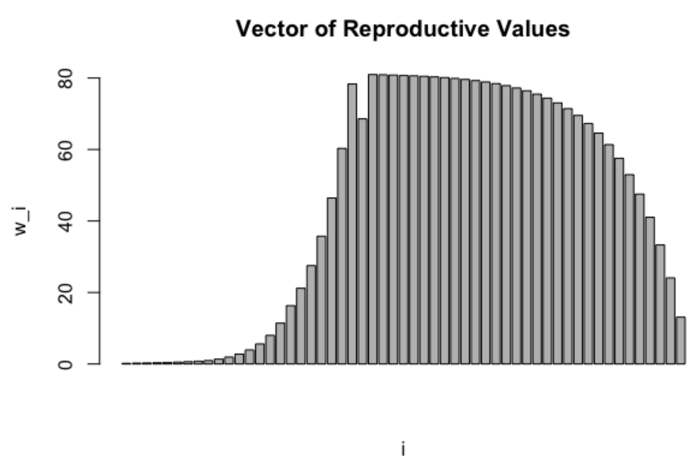
The vector of reproductive values w is the scaled eigenvector of A^T corresponding to the largest eigenvector. The scaling constant is $\dfrac{1}{\|v * w\|_1}$ where $*$ represents component-wise multiplication. We find

$$w \approx \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix}$$

where

$$w_1 = \begin{bmatrix} 0.1579128 \\ 0.2257281 \\ 0.2770818 \\ 0.3401186 \\ 0.4174963 \\ 0.5124777 \\ 0.6290676 \\ 0.7721819 \\ 0.9478551 \\ 1.3527042 \\ 1.9304730 \end{bmatrix} \quad w_2 = \begin{bmatrix} 2.7550191 \\ 3.9317464 \\ 5.6110791 \\ 8.0076905 \\ 11.4279457 \\ 16.3090647 \\ 21.1841807 \\ 27.5165696 \\ 35.7418401 \\ 46.4258138 \\ 60.3034478 \end{bmatrix} \quad w_3 = \begin{bmatrix} 78.3293929 \\ 68.5820615 \\ 80.9692501 \\ 80.9018094 \\ 80.8214200 \\ 80.7255957 \\ 80.6113729 \\ 80.4752192 \\ 80.3129238 \\ 80.1194673 \\ 79.8888669 \end{bmatrix} \tag{0.3}$$

$$w_4 = \begin{bmatrix} 79.6139907 \\ 79.2863379 \\ 78.8957752 \\ 78.4302239 \\ 77.8752858 \\ 77.2137987 \\ 76.4253050 \\ 75.4854191 \\ 74.3650735 \\ 73.0296197 \\ 71.4377565 \end{bmatrix} \quad w_5 = \begin{bmatrix} 69.5402528 \\ 67.2784252 \\ 64.5823229 \\ 61.3685644 \\ 57.5377587 \\ 52.9714318 \\ 47.5283625 \\ 41.0402146 \\ 33.3063312 \\ 24.0875291 \\ 13.0987014 \end{bmatrix}. \tag{0.4}$$

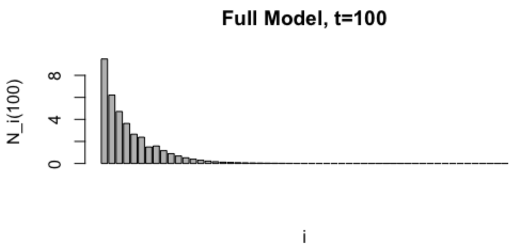


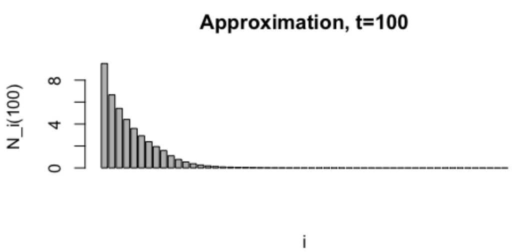
The stable-stage distribution shows us what we should expect in the long term if A stays constant over time. That is, that most of the individuals in the population should be hatchlings and yearlings, and there are less individuals at older ages. Naïvely, this might lead us to focus conservation efforts on the hatchlings and yearlings to preserve the distribution. However, the vector of reproductive values tells us the relative value of each of these stages in terms of what they contribute to future generations. The mature adults, remigrants, and first-time reproducers have the highest reproductive value, and thus are the most important to reproduction. Conservation efforts, then, should be focused on preserving what few adults are in the population. Even though many hatchlings are lost, they have little to no reproductive value anyway. This is intuitive since based on the matrix A, it is highly unlikely that an individual makes it to adulthood. If the individual makes it to adulthood, however, it is extremely valuable in terms of how many new individuals it can produce.

(b) Using $N_0 = \begin{bmatrix} 10,000 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}^T$, and λ, v, w found above, we use the approximation

$$N_t \approx 10^4 w_1 \lambda^t v$$

where $w_1 = 0.2469775$ is the reproductive number of the hatchlings. Here are the plots for $t = 20$ and $t = 100$:





For $t = 20$, there is an error of about 37 in the approximation of the yearlings, about 20 in the approximation of the hatchlings, and about 1 in the approximation of the juveniles. All other errors are $\mathcal{O}(0.1)$. For $t = 100$, all errors are $\mathcal{O}(0.1)$ (the largest error is in the approximation of the hatchlings: about 0.34).

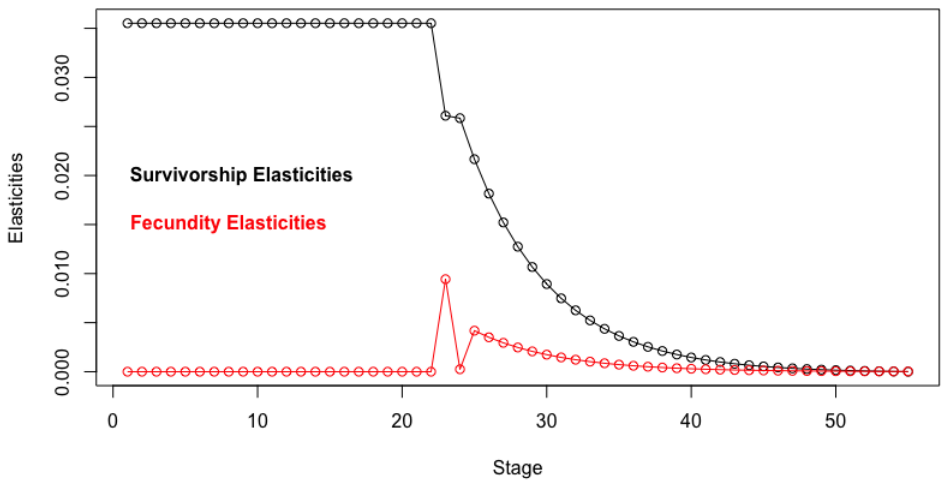
(c) Sensitivities of λ to changes in A (notated S) are given by the outer product of w with v, that is

$$S = wv^T = \left(w_i v_j\right)_{i,j=1}^{55}$$

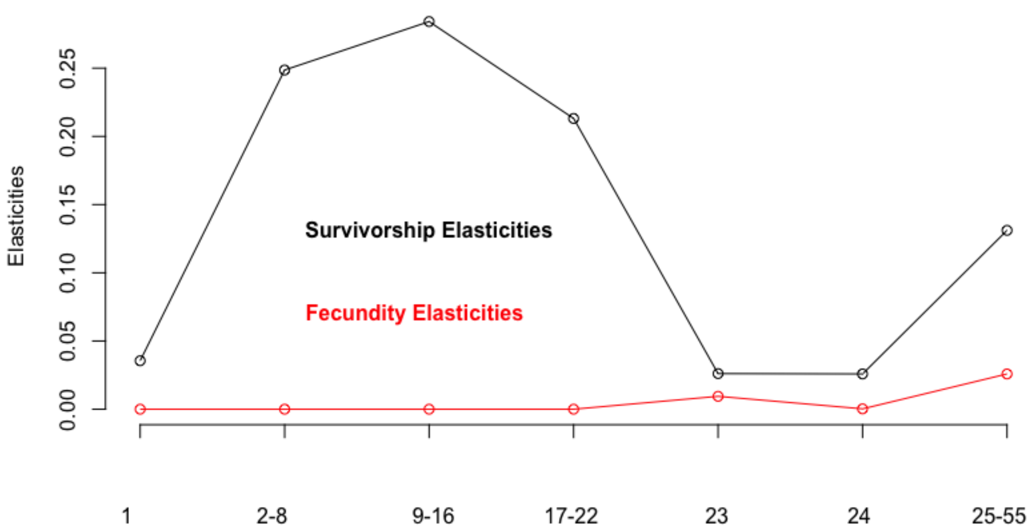
The elasticities of λ to changes in A (notated E) are given by $S * A * \frac{1}{\lambda}$, that is

$$E = \frac{1}{\lambda}\left(a_{ij} w_i v_j\right)_{i,j=1}^{55}$$

The elasticities associated with fecundity are given by the elements in the first row, which correspond to how many births do individuals in any given stage give. The elasticities associated with survivorship are given by the entries on the sub-diagonal since surviving individuals move on to the next age class. Here is the result:



This shows that λ is most sensitive to the survivorship of the hatchlings, yearlings, juveniles, and sub-adults, and is most sensitive to changes in fecundity in first-time reproducers and mature adults. As the individuals get older, they have already given most of what they can to the population, and thus increasing their survivorship or fecundity doesn't make a huge difference. I think it is strange that the survivorship for the first 22 stages are exactly the same. This means $v_{j+1}w_j$ are the same for $j = 1, \ldots, 22$. This seems like a huge coincidence, especially given the nature of w_i and v_i for $i = 1, \ldots, 22$. They must be decaying/growing at exactly the same rate! Although it doesn't look like it, this result matches Fig. 3 in Crouse et al. 1987. By adding the elasticities for a given stage into a single number, we get the following result.



The survivorship elasticities shown above are similar to the component-wise sums of Crouse's P_i and G_i, as shown in her figure below. The age-structured model shows much more overall importance in survivability of yearlings, juveniles, and subadults.

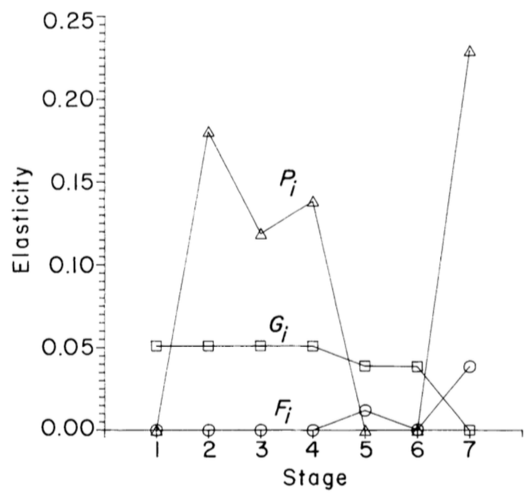


FIG. 3. The elasticity, or proportional sensitivity, of λ_m to changes in fecundity F_i (○), survival while remaining in the same stage P_i (△), and survival with growth G_i (□). Because the elasticities of these matrix elements sum to 1, they can be compared directly in terms of their contribution to the population growth rate r.