PBG 200A Notes

Sam Fleischer

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1 From Last Time

$$\log \frac{N_t}{N_0} \approx \text{ normal with mean } rt \text{ and variance } \sigma^2 t$$

and note that $\text{var}[\log R_1] = \sigma^2$.
$N_0 = 100$, $t = 100$, r, σ^2...

$$\mathbb{P}[N_{100} \leq 1] = \mathbb{P}\left[\log \frac{N_{100}}{100} \leq \log \frac{1}{100}\right] = \mathbb{P}\left[\frac{\log \frac{N_{100}}{100} - 100r}{10\sigma} \leq \underbrace{\frac{\log \frac{1}{100} - 100r}{10\sigma}}_{z \text{ score}}\right]$$

2 Correlated Fluctuations

2.1 Recall

X and Y are random variables with means μ_X and μ_Y, respectively, then the "covariance" between X and Y is

$$\text{cov}[X, Y] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

Notice positive covariance means that when X is generally above the mean, so is Y. The correlation is the normalized covariance.

$$\text{cor}[X, Y] = \frac{\text{cov}[X, Y]}{\sqrt{\text{var}[X]\text{var}[Y]}} \in [-1, 1]$$

2.2 Stationary and Ergodic Random Variables

$$N_{t+1} = R_{t+1} N_t$$

Stationarity is when statistical properties like mean, variance, autocorrelation, etc. are constant over time. Ergodicity is
LLN$^+$ X_1, X_2, ... are stationary and ergodic, then

$$\frac{\sum_{i=1}^t}{t} \to \mathbb{E}[X_1] \qquad \text{as } t \to \infty$$

CLT$^+$ $X_1, X_2, \ldots$ as before with mean μ and variance σ^2, then

$$\frac{X_1 + \cdots + X_t - t\mu}{\sqrt{t}\sigma^+} \to \text{standard normal} \qquad \text{as } t \to \infty$$

where

$$\left(\sigma^+\right)^2 = \sigma^2 (1 + 2 \sum_{\tau=1}^{\infty} \text{cor}[X_1, X_{1+\tau}])$$

This means that positive correlations give rise to greater variations in population densities.

3 Density Dependence

$$N_{t+1} = R(N_t, E_{t+1})N_t$$

where E_{t+1} is a sequence of random variables. Define

$$r_0 = \mathbb{E}[logR(0, E_1)] \qquad \text{and} \qquad r_\infty = \lim_{N\to\infty} \mathbb{E}[\log R(N, E_1)]$$

So they are the realized per-capita growth rates at low and high densities.

If there is no density dependence then $r_0 = r_\infty$. Either

- $r \coloneqq r_0 = r_\infty > 0$, then $N_t \to \infty$ (persistence)
- $r \coloneqq r_0 = r_\infty < 0$, then $N_t \to 0$ (boundedness)

If there is negative density dependence, then $r_0 > r_\infty$. Either

- $r_0 < 0$, then $N_t \to 0$.
- $r_\infty > 0$, then $N_t \to \infty$.
- $r_0 > 0$ and $r_\infty < 0$, then persistence and boundedness (regulation)

If there is positive density dependence, then $r_0 < r_\infty$. Either

- $r_\infty < 0$, then $N_t \to 0$.
- $r_0 > 0$, then $N_t \to \infty$.
- $r_0 < 0$ and $r_\infty > 0$, then $N_t \to 0$ or $N_t \to \infty$ with positive probability for any initial condition.

4 Part III - Individual Heterogeneity

4.1 Example - Juveniles and Adults

$N_1(t)$ is the density of juveniles and $N_2(t)$ is the density of adults.

$$N_1(t+1) = 1.1N_2(t) \tag{1}$$
$$N_2(t+1) = 0.55N_1(t) + 0.55N_2(t) \tag{2}$$

So let $N = \begin{pmatrix} N_1 & N_2 \end{pmatrix}^T$ and $A = \begin{pmatrix} 0 & 1.1 \\ 0.55 & 0.55 \end{pmatrix}$. Then $N_{t+1} = AN_t$, i.e.

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{t+1} = \begin{pmatrix} 0 & 1.1 \\ 0.55 & 0.55 \end{pmatrix} \cdot \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_t \tag{3}$$

Then we have $N(1) = AN(0)$, $N(2) = AN_1 = A^2N(0)$, and so on, and

$$N(t) = A^tN(0). \tag{4}$$

Suppose $N(0) = \begin{pmatrix} 100 & 100 \end{pmatrix}^T$. Then $N(1) = \begin{pmatrix} 110 & 110 \end{pmatrix}^T = 1.1N(0)$. Then $N(2) = 1.21N(0)$. “Hmmmmmm!!”