PBG 200A Notes

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1 Reminder of Toy Example from last time

$$N_t = \begin{pmatrix} N_{1,t} \\ N_{2,t} \end{pmatrix} = A \cdot N_t \qquad \text{where} \qquad A = \begin{pmatrix} 0 & 1.1 \\ 0.55 & 0.55 \end{pmatrix} \tag{1}$$

We saw that if $N_0 = \begin{pmatrix} 100 & 100 \end{pmatrix}^T$ then $N_t = 1.1^t N_0$.

Definition: A scalar λ is an eigenvalue for a $k \times k$ matrix A if there is a non-zero vector v such that $Av = \lambda v$. This vector v is called an eigenvector corresponding to the eigenvalue λ.

2 Perron-Frobenius Theorem

If A is a nonnegative, primitive (A^n has all positive entries for some n) matrix, then there exists $\lambda > 0$ and v such that

1. $Av = \lambda v$
2. $\sum_{i=1}^k v_i = 1$ and $v_i > 0$ for all $i = 1, \ldots, k$.
3. Let $n(t) = \sum_{i=1}^k N_i(t)$. Then
$$\frac{1}{t} \log n(t) \to \log \lambda =: r \qquad \text{as} \qquad t \to \infty$$
whenever $N_i(0) \geq 0$ for $i = 1, \ldots, k$ and $n(0) > 0$.
4. $\dfrac{N_i(t)}{n(t)} \to v_i$ as $t \to \infty$. The v_i's are called the "stable stage distribution."

An example of a nonprimitive matrix is

$$A = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \tag{2}$$

which at odd n,

$$A^n = \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix} \tag{3}$$

and at even n,

$$A^n = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \tag{4}$$

The theorem is roughly saying

$$N(t) \approx C\lambda^t v \qquad \text{for large enough } t \tag{5}$$

and C is dependent on $N(0)$.

What is C? First, **Definition:** Let $w = [w_1, \ldots, w_k]$ be such that $wA = \lambda w$ and such that $\sum_{i=1}^k w_i v_i = 1$.

$$
\begin{aligned}
N(t) &\approx C\lambda^t v && (6)\\
wN(t) &\approx wC\lambda^t v && (7)\\
wN(t) &\approx C\lambda^t \cancelto{1}{wv} && (8)\\
wN(t) = wA^t N(0) &= w\lambda^t N(0) && (9)\\
&= \lambda^t wN(0) && (10)\\
\implies C &= wN(0) && (11)
\end{aligned}
$$

which implies w is the vector of reproductive values of each stage. It represents the amount an individual in any stage contributes to the population.

3 Sensitivity and Elasticity Analysis

Sensitivity of λ to the $i-j^{\text{th}}$ entry of A (a_{ij}), denoted S_{ij}, is $\dfrac{\partial\lambda}{\partial a_{ij}}$. It turns out

$$\frac{\partial\lambda}{\partial a_{ij}} = w_i v_j =: S_{ij} \tag{12}$$

In `R`, type `w%o%v` where `%o%` is the outer product.

Elasticity of λ to a_{ij}, denoted E_{ij} is

$$E_{ij} :== \frac{\partial\lambda}{\partial a_{ij}} \cdot \frac{a_{ij}}{\lambda} \left(\approx \frac{\frac{\partial\lambda}{\lambda}}{\frac{\partial a_{ij}}{a_{ij}}} \right) = S_{ij}\frac{a_{ij}}{\lambda} \tag{13}$$

In `R`, type `E = S*A/l` where `l` is λ. Note that `*` in `R` is elementwise multiplication, whereare `%*%` is matrix multiplication.

Sensitivity is about *absolute* (think $+$) changes, whereas elasticity is about *relative* (think $\cdot$) changes.