# PBG 200A Notes

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## 1 Positive and Negative Density Dependence

Example $\left(\text{assume } \frac{b}{S} > d_2\right)$:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N\left[\frac{bN}{N+S} - d_1 - d_2N\right] = Nr(N)$$

If $d_1 = 0$, then $r(0) = 0$, and it looks (roughly) like a downward facing parabola (except linear asymptotic on the right). Then for all initial conditions (except 0), solutions approach the positive equilibrium. Increasing $d_1$ moves the parabola down (or moving the $N$ axis up). For small $d_1$, there are three equilibria (two positive (one stable and one unstable), and the no-cats-no-kittens equilibrium, 0, which is now stable). There are two alternative stable states. Increasing $d_1$ to the bifurcation value gives two equilibria - one stable and one semistable. Past this value gives only one equilibrium (0), and it is stable. The Bifurcation plot shows a saddle-node bifurcation. *Note: adding in small amounts of immigration gives the classic hysteresis "S" curve bifurcation plot which can lead to fast-slow cycles.*

## 2 An example - Atlantic Cod

Fisheries caused the collapse of the cod population. One possible explanation is the Trophic Triangle: Larval cod $\rightarrow$ herring $\rightarrow$ Adult cod.

Size-structured populations, the rate at which they grow is dependent on the amount of food they eat. This can lead to feedbacks which generate alternative stable state.

Read Peter Abram's paper of the Hydra effect (madness)!

## 3 Temporal Heterogeneity (part II)

- Well-mixed flask
- Closed flask
- candle altering the environment
- Example:
  - Temperature changes (and abiotic changes in general) can affect the speed at which frogs close their mouths, which affect their successful attack rates.

$$N_{t+1} = R_{t+1}N_t$$

$R$ is subscripted with $R_{t+1}$ because it represents all the variation *after* time $t$. So, given $N_0$,

$$\begin{aligned}
N_1 &= R_1 \cdot N_0 \\
N_2 &= R_2 \cdot N_1 = R_2 \cdot R_1 \cdot N_0 \\
&\vdots \\
N_t &= R_t \cdot N_{t-1} = R_t \cdot R_{t-1} \cdot \cdots \cdot R_1 \cdot N_0 = \prod_{i=1}^{t} R_i \cdot N_0 \\
&\vdots
\end{aligned}$$

We assume (for now) that the $R_i$ are i.i.d. (independent and identically distributed).

## 3.1 Review of probabalistic concepts

A discrete random variable $X$ taking on values $x_1, x_2, \ldots, x_k$ is characterized by its PMF (probability mass function). Namely,

$$\mathbb{P}[X = x_i] = \text{probability that } X \text{ attains } x_i =: p_i$$

### 3.1.1 Example - Dice

Let $X$ be the outcome of rolling a six-sided die. Then define $X_n = n$ for $n = 1, \ldots, 6$. Then assuming a fair die,

$$p_1 = \cdots = p_6 = \frac{1}{6}$$

Let $X$ be the number of heads from flipping a fair coin twice. Then define $X_n = n$ for $n = 0, 1, 2$. Then

$$p_0 = \frac{1}{4}, \qquad p_1 = \frac{1}{2}, \qquad p_2 = \frac{1}{4}$$

### 3.1.2 Continuous Random Variables

A continuous random variable $X$ is characterized by its PMF $p(x)$.

$$\mathbb{P}[a \leq X \leq b] = \int_a^b p(x)\mathrm{d}x$$

### 3.1.3 Example - Uniform Distribution on $[0, 1]$

$p(x) = 1$. So, $\mathbb{P}[a \leq X \leq b] = b - a$ for all $0 \leq a \leq b \leq 1$. The `R` command is `runif()`.

### 3.1.4 Example - Normal Distribution

The mean is $\mu$ with standard deviation $\sigma$. The `R` command is `rnorm()`.

### 3.1.5 Example - Exponential Distribution for $x \geq 0$

$p(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Time to the next earthquake is exponentially distributed. The `R` command is `rexp()`.

### 3.1.6 i.i.d.

A set of random variables $X_1, X_2, \ldots, X_k$ are characterized by the joint probabilities:

$$\mathbb{P}[a_1 \leq X_1 \leq b_1 \ , \ \ldots \ , \ a_k \leq X_k \leq b_k]$$

These random variables are "independent" if

$$\mathbb{P}[a_1 \leq X_1 \leq b_1 \ , \ \ldots \ , \ a_k \leq X_k \leq b_k] = \prod_{i=1}^{k} \mathbb{P}[a_i \leq X_i \leq b_i]$$

These random variables are "identically distributed" if all the $X_i$ have the same PMF (continuous) or PDF (discrete).

### 3.1.7 Expectation

The "expectation" of a discrete random variable $X$ which takes $N$ values is a weighted average over those values by their probabilities.

$$\mathbb{E}[X] = \sum_{i=1}^{N} p_i x_i \tag{1}$$

For a continuous random variable $X$ we have

$$\mathbb{E}[X] = \int_\Omega p(x) x \mathrm{d}x \tag{2}$$

where $\Omega$ is the measure space.

### 3.2 Correlation

If $X$ and $Y$ are uncorrelated,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \tag{3}$$

If $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] > 0$, then $X$ and $Y$ are positively correlated.

### 3.3 Back to the Basic Model

$$N_{t+1} = R_{t+1}N_t \tag{4}$$

$$N_t = R_t \dots R_1 N_0 \tag{5}$$

$$\mathbb{E}[N_t] = \mathbb{E}[R_t \dots R_1 N_0] = N_0 \mathbb{E}[R_t \dots R_1] \tag{6}$$

Supposing $R_t$ are i.i.d., then they are uncorrelated, and thus

$$\mathbb{E}[N_t] = N_0 \mathbb{E}[R_t] \dots \mathbb{E}[R_1] = N_0 \mathbb{E}[R_1]^t \tag{7}$$

because they are identical.

Example:

$$R_1 = \begin{cases} 4 & \text{with probability } \frac{1}{2} \\ \frac{1}{5} & \text{with probability } \frac{1}{2} \end{cases} \tag{8}$$

and thus $\mathbb{E}[R_1] = 2.1$.