Sept 29 Notes - Section 10.1

Theorems

Theorem 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. Sum Rule: $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. Sum Rule: $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ for any number k
4. Sum Rule: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. Sum Rule: $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{A}{B}$ if $B \neq 0$

Theorem 2 - The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Theorem 3 - The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Theorem 4

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Longrightarrow \qquad \lim_{n\to\infty} a_n = L$$

Theorem 5

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

Theorem 6 - The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

Definitions

Convergence

A sequence $\{a_n\}$ converges to L if for any $\epsilon > 0$ there exists an index N such that if $n \geq N$, then $|a_n - L| < \epsilon$

Limit

The limit of a convergent sequence is the number to which it converges.

Divergence

If a sequence does not converge, it is said to diverge.

Divergence to Infinity/Negative Infinity

A sequence diverges to infinity if for any $M > 0$ there exists an index N such that if $n \geq N$, then $a_n > M$.

Bounded from Above(Below)

A sequence is bounded from above if there exists a number M such that $a_n < M$ for every n.

Bounded/Unbounded

If a sequence is bounded from above and bounded from below, we say the sequence is bounded. If the sequence is not bounded from above or below, the sequence is unbounded.

Upper(Lower) Bound

A number M is an upper bound of a sequence if $M \geq a_n$ for every n.

Least Upper(Greatest Lower) Bound

A number M is the *least* upper bound if

- M is an upper bound
- $M - \epsilon$ is not an upper bound for any $\epsilon > 0$.

Nondecreasing(Nonincreasing)

A sequence is nondecreasing if $a_{n+1} \geq a_n$ for every n.

Monotonic

A sequence is called monotonic if it is either nondreasing or nonincreasing.

Practice Problems

Finding Terms of a Sequence

Find $a_1 \dots a_4$.

5. $a_n = \frac{2^n}{2^{n+1}}$

6. $a_n = \frac{2^n - 1}{2^n}$

10. $a_1 = -2$, $a_{n+1} = \frac{n a_n}{n+1}$

Finding a Sequence's Formula

Find a formula for the nth term of the sequence.

21. $1, 5, 9, 13, 17, \dots$

23. $\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}, \dots$

26. $0, 1, 1, 2, 2, 3, 3, 4, \dots$

Convergence and Divergence

Is the sequence convergent/divergent? Find the limit of convergent sequences.

33. $a_n = \frac{n^2 - 2n + 1}{n - 1}$

43. $a_n = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$

57. $a_n = \left(\frac{3}{n}\right)^{\frac{1}{n}}$

63. $a_n = \frac{n!}{n^n}$ $\quad$ (Compare with $\frac{1}{n}$)

71. $a_n = \left(\frac{x^n}{2n+1}\right)^{\frac{1}{n}}, \ x > 0$

82. $a_n = \frac{1}{\sqrt{n}} \tan^{-1} n$

89. $a_n = \frac{1}{n} \int_1^n \frac{1}{x} dx$

Recursively Defined Sequences

Find the limit of the sequence (trust us, it exists)

94. $a_1 = 0$, $a_{n+1} = \sqrt{8 + 2a_n}$

97. $2, 2 + \frac{1}{2}, 2 + \frac{1}{2 + \frac{1}{2}}, 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \dots$