13.1: Vector-Valued Functions and Motion in Space, 14.1: Functions of Several Variables, and 14.2: Limits and Continuity in Higher Dimensions

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Section 13.1: Vector-Valued Functions and Motion in Space

Think of a particle's coordinates as a *function of time*.

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$ make up the **curve** in space that we call the particle's **path**. Above is the parametric form. Here is the vector form:

$$\mathbf{r}(t) = \vec{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

from the origin to the particle's position $P(f(t), g(t), h(t))$.

We call $\mathbf{r}$ a **vector-valued function** or **vector function** since its output is a vector.

Example

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

where $f(t) = \cos t$, $g(t) = \sin t$, and $h(t) = t$.

Definition - Limit

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$,

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

Definition - Continuity

A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous over its interval domain.

Definition - Differentiability

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative at** t if f, g, and h have derivatives at t. The derivative is

$$\mathbf{r}'(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Definitions - Velocity and Acceleration Vectors

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = \dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$
4. The unit vector $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is the direction of motion at time t.

Example

The velocity of a particle whose motion in space is governed by the position vector

$$\mathbf{r}(t) = 2\cos t\ \mathbf{i} + 2\sin t\ \mathbf{j} + 5\cos^2 t\ \mathbf{k}$$

is

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= -2\sin t\ \mathbf{i} + 2\cos t\ \mathbf{j} - 10\cos t \sin t\ \mathbf{k} \\ &= -2\sin t\ \mathbf{i} + 2\cos t\ \mathbf{j} - 5\sin 2t\ \mathbf{k}\end{aligned}$$

and the acceleration is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = -2\cos t\ \mathbf{i} - 2\sin t\ \mathbf{j} - 10\cos 2t\ \mathbf{k}$$

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. Constant Function Fule: $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{C} = \mathbf{0}$
2. Scalar Multiple Rules: $\dfrac{\mathrm{d}}{\mathrm{d}t}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{\mathrm{d}}{\mathrm{d}t}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. Sum Rule: $\dfrac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. Difference Rule: $\dfrac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. Dot Product Rule: $\dfrac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. Cross Product Rule: $\dfrac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. Chain Rule $\dfrac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Section 14.1: Functions of Several Variables

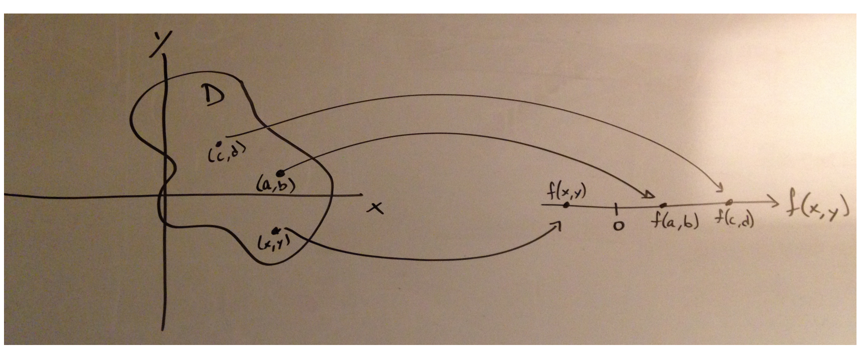
Definition - Real-Valued Function

Suppose D is a set of n-tuples of real numbers $(x_1, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, \ldots, x_n)$$

to each element in D.

Lets start with Two-Dimensional

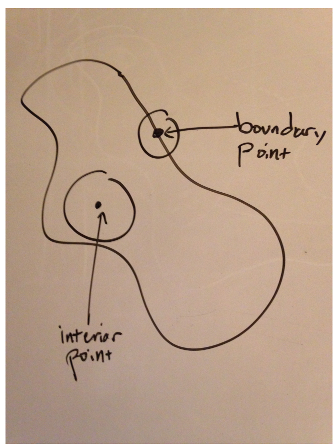


Examples

Function	Domain	Range
$f(x, y) = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty]$
$g(x, y) = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$h(x, y) = \sin(xy)$	Entire Plane	$[-1, 1]$

Definitions - Interior Point, Boundard Point, Open, Closed

- A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R.
- A point (x_0, y_0) is a **boundard point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie inside of R (the boundard point itself need not belong to R.)
- A region is **open** if it consists entirely of interior points.
- A region is **closed** if it contains all its boundary points



Definition - Bounded

A region in the plane is **bounded** if it lies inside a disk of fixed radius. A radius is **unbounded** if it is not bounded.

Example

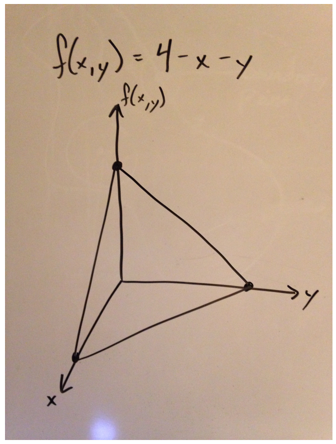
The domain D of the function $f(x, y) = \sqrt{y - x^2}$ is the set of all points in $\mathbb{R}^2$ (2-dimensional Cartesian coordinate plane) such that $y \geq x^2$.

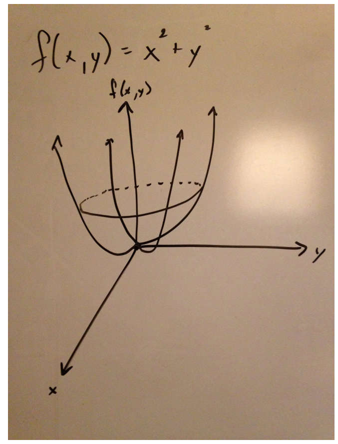
$$D = \{(x, y) \in \mathbb{R}^2 \text{ such that } y \geq x^2\}$$

D is a closed an unbounded set.

Definitions - Level Curve, Surface

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level-curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **surface** of $z = f(x, y)$.





Section 14.2: Limits and Continuity in Higher Dimensions

Definition - Limit

We say that a function $f(x, y)$ approahces the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Theorem 1 - Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = M$$

1. $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) + g(x, y)) = L + M$

2. $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) - g(x, y)) = L - M$

3. $\lim_{(x,y)\to(x_0,y_0)} kf(x, y) = kL$

4. $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) \cdot +g(x, y)) = L \cdot +M$

5. $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \qquad M \neq 0$

6. $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n \qquad n \in \mathbb{Z}_{>0}$

7. $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

Definition - Continuity

A function $f(x,y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Example

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at every point except the origin. This is because **different paths of approach to the origin can lead to different results**. For example,

$$f(x,y)\Big|_{y=mx} = \frac{2xy}{x^2+y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2+(mx)^2} = \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2}$$

So now approach $(0,0)$ along the line $y = mx$...

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} = \lim_{(x,y)\to(0,0)} \left[f(x,y)\Big|_{y=mx} \right] = \frac{2m}{1+m^2}$$

This is a different value for different choices of m, so the limit does not exist, violating criteria 2 and 3 of the definition of continuity. Thus f is not continuous at $(0,0)$.

Two-Path Test for Nonexistence of a Limit

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist. *(These paths do NOT have to be linear!)*

Continuity of Composites

If f is continuous at (x_0,y_0) and g is a single-variable function continuous at $f(x_0,y_0)$, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is continuous at (x_0,y_0).