Oct 20 Notes - Convergence of Taylor Series and Intro to Vector Calculus

Section 10.9: Convergence of Taylor Series

Theorem 23 - Taylor's Theorem

If f and its first n derivatives, $f', \ldots, f^{(n)}$ are continuous on the closed interval $[a, b]$, and $f^{(n)}$ is differentiable on the open interval (a, b), then $\exists c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}$$

for some $c \in (a, x)$.

Theorem 24 - The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)(t)}| \leq M$ for all $t \in [a, b]$, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies

$$R_n(x) \leq M\frac{|x-a|^{n+1}}{(n+1)!}$$

If this inquality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Section 12.1: Three Dimensional Coordinate Systems

"Cartesian" (rectangular) Coordinates (x, y, z)

Distance between points $P_1(x_1, y_1, z_1)$ **and** $P_2(x_2, y_2, z_2)$

$$d(P_1, P_2) = \sqrt((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)$$

Midpoint between two points $P_1(x_1, y_1, z_1)$ **and** $P_2(x_2, y_2, z_3)$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Standard Equation for the Sphere of Radius a **and Center** (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

Section 12.2: Vectors

Definitions

A **vector** is a directed line segment.

The vector represented by the directed line segment $\vec{AB}$ has **initial point** A and **terminal point** B and its **length** (or **magnitude**) is denoted by $|\vec{AB}|$. Two vectors are **equal** if the have the same length and direction.

A vector whose initial point is the origin $(0, 0, 0)$ is in **standard position**.

A vector $\mathbf{v}$ in standard position and terminal point at (v_1, v_2, v_3) can also be denoted in **component form** as

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

The magnitude or length of a vector is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Vector Algebra Operations

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors and let k be a scalar (k is a number).

$$\begin{aligned} \text{Addition}: &\quad \mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \\ \text{Multiplication}: &\quad k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle \end{aligned}$$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and a and b, scalars (numbers).

Additive Properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + \mathbf{-u} = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$

Scalar Multiplicative Properties:

1. $1\mathbf{u} = \mathbf{u}$
2. $a(b\mathbf{u}) = (ab)\mathbf{u}$
3. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
4. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

Vectors of magnitude 1 are called **unit vectors**.

Three **Standard Unit Vectors**:

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$