Oct 27 Notes - Dot Product, Cross Product, and Lines and Planes in Space

Section 12.3: The Dot Product

Definition: Dot Product

The **dot product** $\mathbf{u} \cdot \mathbf{v}$ of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the * * * **scalar** * * *

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Theorem 1 - Angle Between Two Vectors

The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}|\,|\mathbf{v}|} \right) = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \right)$$

Definition: Orthogonal

Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$

Vector Projections

The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}$$

The scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$|\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

Definition: Work

The **work** done by a constant force $\mathbf{F}$ acting through a displacement $D = \vec{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D}$$

Section 12.4: The Cross Product

Definition: Cross Product

The **cross product $\mathbf{u} \times \mathbf{v}$** is the * * * **vector** * * *

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\ |\mathbf{v}|\ \sin\theta)\,\mathbf{n}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{n}$ is the **unit normal vector** to the plane generated by $\mathbf{u}$ and $\mathbf{v}$, following the **right hand rule**.

Parallel Vectors

Nonzero vetors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$ 2. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ 4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$

5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$ 6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

Cross Products of the Standard Unit Vectors

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Area of a Parallelogram

The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is

$$A = |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\ |\mathbf{v}| \sin\theta$$

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (in that order).

Volume of a Parallelepiped

The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

$$V = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Section 12.5: Lines and Planes in Space

Vector Equation for a Line

A vector equation for line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

Distance from a Point S to a Line P parallel to $\mathbf{v}$

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector Equation:	$\mathbf{n} \cdot \vec{P_0P} = 0$
Component Equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component Equation Simplified:	$Ax + By + Cz = D$, where
	$D = Ax_0 + By_0 + Cz_0$

Distance from a Point to a Plane

$$d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$

Angles between planes

The angle between planes is defined to be the angle between the normal vectors.