Oct 06 Notes - First Half of Chapter 10 Toolkit

Section 10.1: Sequences

Theorem 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. Sum Rule: $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. Sum Rule: $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ for any number k
4. Sum Rule: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. Sum Rule: $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{A}{B}$ if $B \neq 0$

Theorem 2 - The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Theorem 3 - The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Theorem 4

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Longrightarrow \qquad \lim_{n\to\infty} a_n = L$$

Theorem 5

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

Theorem 6 - The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

Section 10.2: Infinite Series

Geometric Series

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \ldots$ converges to $\frac{1}{1-r}$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{1}{1-r}, \quad |r| < 1$$

If $|r| \geq 1$, the series diverges.

Theorem 7 - The nth-Term Test for Divergence

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Theorem 8

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule*: $\sum(a_n + b_n) = A + B$
2. *Difference Rule*: $\sum(a_n - b_n) = A - B$
3. *Constant Multiple Rule*: $\sum(ka_n) = kA$

Section 10.3 The Integral Test

Theorem 9 - The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N} \infty a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge$.

p-Series

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \ldots$, ($p$ a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Section 10.4: Comparison Tests

Theorem 10 - The Comparison Test

Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \quad \text{for all} \quad n > N$$

1. If $\sum c_n$ converges, then $\sum a_n$ also converges.
2. If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Theorem 11 - The Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Section 10.5: Absolute Convergence; The Ratio and Root Tests

Theorem 12 - The Absolute Convergence Test

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Theorem 13 - The Ratio Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

1. If $\rho < 1$, then the $\sum a_n$ converges absolutely.
2. If $\rho > 1$ or ρ is infinite, then the $\sum a_n$ diverges.
3. If $\rho = 1$, then the test is inconclusive.

Theorem 14 - The Root Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

1. If $\rho < 1$, then the $\sum a_n$ converges absolutely.
2. If $\rho > 1$ or ρ is infinite, then the $\sum a_n$ diverges.
3. If $\rho = 1$, then the test is inconclusive.

Section 10.6: Alternating Series and Conditional Convergence

Theorem 15 - The Alternating Series Test

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$$

converges if all three of the following conditions are satisfied:

1. Each u_n is positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Theorem 16 - The Alternating Series Estimation Theorem

If the alternating series $\sum(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error of whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

Theorem 17 - The Rearrangement Theorem for Absolutely Convergent Series

If $\sum a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ sonverges absolutely and $\sum b_n = \sum a_n$.

Summary of Tests

1. **The nth-Term test:** If it is not true that $a_n \to 0$, then the series diverges.
2. **Geometric Series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. p**-Series:** $\sum \dfrac{1}{n^p}$ converges if $p > 1$; otherwise if diverges.
4. **Series with nonnegative terms:** Try the integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating Series:** $\sum a_n$ converges if the series satisfies the three conditions of the Alternating Series Test.