

14.7: Extreme Values and Saddle Points, and 14.8: Lagrange Multipliers

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Section 14.7: Extreme Values and Saddle Points

Definitions: Local Maximum/Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b) . Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b) .
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b) .

Theorem 10 - First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Definition: Critical Point

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f .

Note

Not all critical points give rise to local minima/maxima

Definition: Saddle Point

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface.

Theorem 11 - Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Then,

- i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) .
- ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) .
- iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) .
- iv) **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) . In this case, we must find some other way to determine the behavior of f at (a, b) .

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f . It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Example: *Find and classify the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.*

f is a polynomial, and all of its derivatives are polynomials, and there are no domain restrictions on polynomials. So we will only find extreme values f can assume extreme values only when $f_x = 0$ and $f_y = 0$.

$$f_x = 6y - 6x = 0 \quad \text{and} \quad f_y = 6y - 6y^2 + 6x = 0$$

$f_x = 0$ implies $x = y$. Substituting for y in the equation $f_y = 0$ gives us

$$\begin{aligned} 6x - 6x^2 + 6x &= 0 \\ 6x(2 - x) &= 0 \\ x &= 0 \text{ or } 2 \end{aligned}$$

Since $x = y$, the two critical points are $(0, 0)$ and $(2, 2)$.

To classify the critical points, we use Theorem 11. To use Theorem 11, we calculate the second partial derivatives:

$$\begin{aligned} f_{xx} &= -6 \\ f_{xy} &= 6 \\ f_{yy} &= 6 - 12y \end{aligned}$$

So the **Hessian** evaluated at $(0, 0)$ is

$$\begin{aligned} (f_{xx}f_{yy} - f_{xy}^2) \Big|_{(0,0)} &= (-6)(6 - 12(0)) - (6)^2 \\ &= -36 - 36 \\ &= -72 < 0 \end{aligned}$$

thus $(0, 0)$ produces a **Saddle Point** on the surface. The **Hessian** evaluated at $(2, 2)$ is

$$\begin{aligned} (f_{xx}f_{yy} - f_{xy}^2) \Big|_{(2,2)} &= (-6)(6 - 12(2)) - (6)^2 \\ &= 108 - 36 \\ &= 72 > 0 \end{aligned}$$

and $f_{xx} < 0$ thus $(2, 2)$ is a **local maximum** on the surface.

Example: Find the absolute maximum and minimum values of $f(x, y) = 2 + 2x + 4y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Similar to the last example, polynomials do not have domain restrictions, and so (for interior points of the triangular region) we only need to look where $f_x = 0$ and $f_y = 0$.

$$f_x = 2 - 2x = 0 \quad \text{and} \quad f_y = 4 - 2y = 0$$

Thus $x = 1$ and $y = 2$, so $(1, 2)$ is a critical point. Note that

$$f(1, 2) = 7$$

For points on the boundary of the triangle, we consider one side of the triangle at a time. First, we consider points on the line $y = 0$. Then the function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

on the closed interval $0 \leq x \leq 9$. We can find some extreme values on this interval (MAT 021 A) by setting the derivative equal to zero and solving for critical points in the interval. So,

$$f'(x, 0) = 2 - 2x$$

which yields $x = 1$. Since $y = 0$, we test the point $(1, 0)$. We also test the points $(0, 0)$ and $(9, 0)$ because they are endpoints of the region.

$$\begin{aligned} f(0, 0) &= 2 \\ f(1, 0) &= 3 \\ f(9, 0) &= -61 \end{aligned}$$

Next we consider points on the line $x = 0$. Then the function

$$f(x, y) = f(0, y) = 2 + 4y - y^2$$

on the closed interval $0 \leq y \leq 9$. Again finding extreme values for this function on the interval may give us some extreme values

$$f'(0, y) = 4 - 2y$$

which yields $y = 2$. Since $x = 0$, we test the point $(0, 2)$. We also test the points $(0, 0)$ and $(0, 9)$ because they are endpoints of the region.

$$\begin{aligned} f(0, 0) &= 2 \\ f(0, 2) &= 6 \\ f(0, 9) &= -43 \end{aligned}$$

Finally, we consider points on the line $y = 9 - x$. Then the function

$$f(x, y) = f(x, 9 - x) = -43 + 16x - 2x^2$$

on the closed interval $0 \leq x \leq 9$.

$$f'(x, 9 - x) = 16 - 4x$$

which yields $x = 4$. Since $y = 9 - x$, we test the point $(4, 5)$. We also test the points $(0, 9)$ and $(9, 0)$ because they are endpoints of the region.

$$f(0, 9) = -43$$

$$f(4, 5) = -11$$

$$f(9, 0) = -61$$

Whew! In summary, we first checked for extreme values in the interior of R by searching for critical points. Then we looked for local extrema on the boundary of the domain by looking at each line forming the boundary. Here is the summary of the points we checked. This shows that

(a, b)	$f(a, b)$
$(0, 0)$	$f(0, 0) = 2$
$(1, 0)$	$f(1, 0) = 3$
$(9, 0)$	$f(9, 0) = -61$
$(0, 2)$	$f(0, 2) = 6$
$(1, 2)$	$f(1, 2) = 7$
$(4, 5)$	$f(4, 5) = -11$
$(0, 9)$	$f(0, 9) = -43$

the maximum value of f on the triangular region R is 7 and is attained at the point $(1, 2)$ and the minimum value of f on the region R is -61 and is attained at $(9, 0)$.

Absolute Maxima and Minima on Closed Bounded Regions

In general, if you are constraining your search of extrema of $f(x, y)$ to a closed, bounded region R , the use the following procedure to organize your search:

1. *List the interior points of R* where f may have local maxima and minima and evaluate f at these points. These are the critical points of f .
2. *List the boundary points of R* where f has local maxima and minima and evaluate f at these points.
3. *Look through the lists* for the maximum and minimum values of f . These will be the absolute maximum and minimum values of f on R . Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in steps 1 and 2.

Summary of Max-Min Test

The extreme values of $f(x, y)$ can occur only at

- i) **boundary points** of the domain of f

ii) **critical points** (interior points where $f_x = f_y = 0$ or where points f_x or f_y fails to exist.)

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

- i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \implies$ **local maximum**
- ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \implies$ **local minimum**
- iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \implies$ **saddle point**
- iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \implies$ **test is inconclusive**

Section 14.8: Lagrange Multipliers

Imagine a small sphere $f(x, y, z)$ centered at the origin expanding like a soap bubble until it just touches a surface $g(x, y, z)$ in three dimensions. At each point of contact, the surface and the sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 \quad \text{and} \quad g(x, y, z)$$

equal to 0, then the gradients ∇f and ∇g will be parallel, and thus, there must be some scalar λ ("lambda") such that

$$\nabla f = \lambda \nabla g$$

This scalar λ is an example of a **Lagrange Multiplier**.

Theorem 12 - The Orthogonal Gradient Theorem

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

If P_0 is a point on C where f has local maximum or minimum relative to its values on C , then ∇f is orthogonal to C at P_0 .

Corollary

At the points on a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{r}' = 0$.

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x , y , z , and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0$$

For functions of two independent variables, the condition is similar, but without the variable z .

Example: Find the minimum distance from the point $(1, 2, 15)$ to the paraboloid given by the equation $z = x^2 + y^2$.

The distance from a point (x, y, z) to the point $(1, 2, 15)$ is given by the equation

$$\hat{f}(x, y, z) = \sqrt{(x-1)^2 + (y-2)^2 + (z-15)^2}$$

However, since the square root function is an increasing function, we can minimize \hat{f} by minimizing $f = \hat{f}^2$, so

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-15)^2$$

and set the function g as

$$g(x, y, z) = x^2 + y^2 - z$$

which is simply the constraint $(z = x^2 + y^2)$ solved for 0. We can use the method of Lagrange Multipliers by setting $\nabla f = \lambda \nabla g$ and considering $g(x, y, z) = 0$, or more explicitly,

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x, y, z) &= 0 \end{aligned}$$

or

$$\begin{aligned} 2(x-1) &= \lambda(2x) \\ 2(y-2) &= \lambda(2y) \\ 2(z-15) &= \lambda(-1) \\ x^2 + y^2 - z &= 0 \end{aligned}$$

The first two equations give

$$\begin{aligned} (\lambda - 1)x &= 1 \\ (\lambda - 1)y &= 2 \end{aligned}$$

Now we use some of our intuition... If $\lambda = 1$, $x = 0$, or $y = 0$, then these give us $0 = 1$, a contradiction, so $\lambda \neq 1$, $x \neq 0$, and $y \neq 0$. Also, if $\lambda = 0$, then $(x, y, z) = (1, 2, 15)$, which does not satisfy the fourth equation $x^2 + y^2 - z = 0$. So $\lambda \neq 0$. So, we can divide the two above equations, to give us

$$\begin{aligned} \frac{x}{y} &= \frac{1}{2}, \quad \text{or} \\ 2x &= y \end{aligned}$$

Dividing the first and third equation gives us

$$\frac{x-1}{2(z-15)} = -x, \quad \text{or}$$

$$z = 15 - \frac{1}{2} \left(1 - \frac{1}{x} \right)$$

So we have y and z in terms of x , so we can substitute in the constraint equation:

$$x^2 + (2x)^2 - \left[15 - \frac{1}{2} \left(1 - \frac{1}{x} \right) \right] = 0, \quad \text{or} \\ 10x^3 - 29x - 1 = 0$$

This does not factor nicely, so we must numerically solve for x , to get

$$\begin{aligned} x_1 &\approx -1.685, \\ x_2 &\approx -0.034, \quad \text{and} \\ x_3 &\approx 1.720 \end{aligned}$$

Plug these values in to the above equations to get y and z for each x .

$$\begin{aligned} P_1 &\approx (-1.685, -3.371, 14.203), \\ P_2 &\approx (-0.034, -0.069, 0.006), \quad \text{and} \\ P_3 &\approx (1.720, 3.440, 14.791) \end{aligned}$$

Finally, we can find the distance from the point $(1, 2, 15)$ by plugging these points in to \hat{f} .

$$\begin{aligned} d_1 &= \hat{f}(P_1) \approx 6.057, \\ d_2 &= \hat{f}(P_2) \approx 15.171, \quad \text{and} \\ d_3 &= \hat{f}(P_3) \approx 1.623 \end{aligned}$$

Clearly, the minimum distance is approximately 1.62335.

Lagrange Multipliers with Two Constraints

Maybe you are dealing with two constraints, $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$ (where $\nabla g_1 \neq c \nabla g_2$, i.e. their gradients are not parallel). Then we can locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding x, y, z, λ , and μ (“mew”) that simulatenously satisfy

$$\begin{aligned} \nabla f &= \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1(x, y, z) &= 0 \\ g_2(x, y, z) &= 0 \end{aligned}$$