14.6: Tangent Planes and Differentials

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Section 14.6: Partial Derivatives

Definition: Tangent Planes and Normal Lines

The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f\big|_{P_0}$. The **normal line** of the surface at P_0 is the line though P_0 parallel to $\nabla f\big|_{P_0}$.

Using concepts from Section 12.5, we can write the following:

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \tag{1}$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \tag{2}$$

To find equations for tangent planes and normal lines for surfaces of the form $z = f(x, y)$, consider the function $F(x, y, z) = f(x, y) - z$, and so

$$F_x = f_x, \qquad F_y = f_y, \qquad \text{and } F_z = -1$$

Thus,

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0 \tag{3}$$

Note (3) is equivalent to (1) when $z = f(x, y)$.

Example

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

Since $z = f(x, y) = x\cos y - ye^x$, we use (3). So,

$$\begin{aligned} f_x &= \cos y - ye^x \\ \implies f_x(0, 0) &= 1 - 0 = 1 \end{aligned}$$

$$
\begin{aligned}
f_y &= -x\sin y - e^x \\
\implies f_y(0,0) &= 0 - 1 = -1
\end{aligned}
$$

The tangent plane is therefore

$$
\begin{aligned}
1\cdot(x-0) + (-1)\cdot(y-0) - (z-0) &= 0 \\
x - y - z &= 0
\end{aligned}
$$

Example

Find the normal line at the point $P_0(1,-1,3)$ on the surface $x^2+2xy-y^2+z^2=7$.

This is of the form $f(x,y,z)=c$, where $f(x,y,z)=x^2+2xy-y^2+z^2$ and $c=7$, so we use (2).

$$
\begin{aligned}
f_x &= 2x - 2y \\
\implies f_x(1,-1,3) &= 2(1) - 2(-1) = 4 \\
f_y &= 2x - 2y \\
\implies f_y(1,-1,3) &= 2(1) - 2(-1) = 4 \\
f_z &= 2z \\
\implies f_z(1,-1,3) &= 2(3) = 6
\end{aligned}
$$

The normal line is therefore

$$
x = 1+4t, \qquad y = -1+4t, \qquad z = 3+6t
$$

Estimating the Change in f in a direction $\mathbf{u}$

To estimate the change in the value of a differentiable function f when we move a small distance $\mathrm{d}s$ from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$
\mathrm{d}f = \left(\boldsymbol{\nabla} f\big|_{P_0}\cdot\mathbf{u}\right)\,\mathrm{d}s
$$

Example

Estimate how much the value of $f(x,y,z)=y\sin x+2yz$ will change if the point $P(x,y,z)$ moves 0.1 *unit from $P_0(0,1,0)$ straight toward $P_1(2,2,-2)$.*

In order to find the direction from P_0 to P_1, we first must find the vector $\vec{P_0P_1}$.

$$
\vec{P_0P_1} = (2-0)\mathbf{i} + (2-1)\mathbf{j} + (-2-0)\mathbf{k} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}
$$

Now we find the *unit vector* in the direction of $\vec{P_0P_1}$ by dividing each component of $\vec{P_0P_1}$ by its magnitude.

$$
\mathbf{u} = \frac{\vec{P_0P_1}}{\left\|\vec{P_0P_1}\right\|} = \frac{2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+1^2+(-2)^2}} = \frac{2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{3} = \frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}
$$

So, we have our unit direction vector, and now we need our gradient vector $\nabla f\big|_{P_0}$. Recall $f(x, y, z) = y \sin x + 2yz$.

$$
\begin{aligned}
f_x &= y \cos x \\
\implies f_x(0, 1, 0) &= 1 \\
f_y &= \sin x + 2z \\
\implies f_y(0, 1, 0) &= 0 \\
f_z &= 2y \\
\implies f_z(0, 1, 0) &= 2 \\
\implies \nabla f\big|_{P_0} &= \mathbf{i} + 2\mathbf{k}
\end{aligned}
$$

Now we have our gradient, so we can find the dot product $\nabla f\big|_{P_0} \cdot \mathbf{u}$

$$\nabla f\big|_{P_0} \cdot \mathbf{u} = \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) \cdot (\mathbf{i} + 2\mathbf{k}) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}$$

Then since we move 0.1 unit, $\mathrm{d}s = 0.1$, so

$$\mathrm{d}f = \left(\nabla f\big|_{P_0}\right) \mathrm{d}s = -\frac{2}{3} \cdot 0.1 = -\frac{2}{30} \approx -0.067 \text{ unit}$$

Defintion: Linearization

The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

This concept can be thought of as the two-dimensional expansion of Taylor Polynomials of functions of one variable. The standard linear approximation given above is the first order Taylor expansion of a function of two variables. This concept can be expanded to functions of n variables by making use of the binomial theorem.

The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values $|f_{xx}|$, $|f_{yy}|$, and $f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2$$

Again, this concept is an expansion of the error bound in Taylor Polynomials (Theorem 24 from chapter 10.9). Clearly, to make the error small, we must make $|x - x_0|$ and $|y - y_0|$ small. In other words, the approximation is only valid close to the expansion point, just like in Taylor Polynomials from chapter 10.

Definition: Total Differential

If we move from (x_0, y_0) to a point $(x_0 + \mathrm{d}x, y_0 + \mathrm{d}y)$ nearby, the resulting change

$$\mathrm{d}f = f_x(x_0, y_0)\mathrm{d}x + f_y(x_0, y_0)\mathrm{d}y$$

in the linearization of f is called the **total differential** of f.

Exmaple

Suppose that a cylindrical can is designed to have a radius of 1 *inch and a height of* 5 *inches, but that the radius and height are off by the amounts* $\mathrm{d}r = +0.03$ *and* $\mathrm{d}h = -0.1$. *Esimate the resulting absolute change in the volume of the can.*

Recall the formula for volume of a cylinder:

$$V(r, h) = \pi r^2 h$$

So, we find the partial derivatives of V with respect to r and h:

$$\begin{aligned} V_r &= 2\pi r h \\ \implies V_r(1, 5) &= 2\pi(1)(5) = 10\pi \\ V_h &= \pi r^2 \\ \implies V_h(1, 5) &= \pi(1^2) = \pi \end{aligned}$$

Thus, the total differential of V is

$$\begin{aligned} \mathrm{d}V &= V_r(1, 5)\mathrm{d}r + V_h(1, 5)\mathrm{d}h \\ &= 10\pi(0.03) + \pi(-0.1) \\ &= 0.2\pi \\ &\approx 0.63 \text{ inches}^3 \end{aligned}$$