On Phase Response Function Based Decentralized Phase Desynchronization

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***Abstract*—Desynchronization is a powerful tool to achieve round-robin scheduling, which is crucial in applications as diverse as media access control of communication networks, realization of analog-to-digital converters, and scheduling of traffic flows in intersections. Due to the remarkable scalability and simplicity, pulse-coupled oscillators based desynchronization is receiving increased attention. In this paper, we systematically characterize pulse-coupled oscillators based decentralized phase desynchronization and propose an interaction function that is more general than the existing results. Numerical simulations show that the proposed pulse-based interaction function also has better robustness to pulse losses, time delays, and frequency errors than the existing results.**

***Index Terms*—Pulse-coupled oscillators, phase response function, phase desynchronization.**

I. Introduction

PULSE-COUPLED oscillators (PCOs) were originally proposed to model synchronization in biological systems such as flashing fireflies [1], [2] and firing neurons [3], [4]. In recent years, with proven scalability, simplicity, accuracy, and robustness, the PCO based synchronization strategy has become a powerful clock synchronization primitive for wireless sensor networks [5]–[9].

A less explored property of pulse-coupled oscillators is desynchronization, which spreads the phase variables of all PCOs uniformly apart (with equal difference between neighboring phases). Desynchronization has been found in many biological phenomena, such as neuron spiking [10] and fish signaling [11]. What's more, desynchronization is also very important for Deep Brain Stimulation (DBS) which has been proven an effective treatment for Parkinson's disease [12]. Recently, phase desynchronization has also been employed to perform time-division multiple access (TDMA), a medium access control (MAC) protocol for communications [13]–[16].

In the literature, a number of papers have emerged on PCO based desynchronization. Based on the PCO model in [1], the authors in [17] proposed a desynchronization algorithm (INVERSE-MS) for an all-to-all network. The convergence properties of INVERSE-MS were further explored in [18], [19], and [20], using an algebraic framework and a hybrid systems framework, respectively. However, these results are about the achievement of uniform firing time interval (equal time interval between two consecutive firings), which is referred to as weak desynchronization [17], [18]. Weak desynchronization relies on persistent phase jumps to maintain equal firing intervals, and hence cannot guarantee a uniform spread of phases. Furthermore, it is sensitive to disturbances such as pulse loss and time delay because a lost or delayed pulse will directly lead to errors in the spread of firing time instants.

Recently, algorithms also emerged for phase desynchronization which is also referred to as strict desynchronization. Existing phase desynchronization algorithms can be divided into two categories based on the employed interaction mechanism. In the first mechanism, an oscillator adjusts its phase according to the firing information of its two immediate firing neighbors (the one fires before it and the one after). Typical examples include [17], [21]–[26]. Generally speaking, performance of these desynchronization algorithms are difficult to rigorously analyze since an oscillator can never know the exact current phases of its two immediate neighbors (the one fires before it and the one after). Furthermore, because each oscillator only updates once during its cycle, such desynchronization algorithms tend to have very slow convergence rates, as confirmed by our numerical results in Section IV-B.

The second mechanism is using phase response function (PRF) based interaction. In this mechanism, each oscillator will make phase adjustments every time it receives a pulse, and the adjustment is determined by the phase response function which describes the phase shift induced by a pulse. As in an all-to-all network with N PCOs, every oscillator will receive $N-1$ pulses when its phase evolves one cycle, and will make $N-1$ adjustments during its phase cycle, which significantly improves the convergence speed. Existing results [18], [23], [24], [27], [28] fall within this category.

In this paper, we rigorously analyze the category of phase response function (PRF) based phase desynchronization algorithms. More specifically, we rigorously characterize the decentralized phase desynchronization process and propose a general phase-desynchronizing PRF that includes previous results as special cases. More interestingly, the proposed phase response function provides high robustness to pulse losses, time delays, and frequency errors which will significantly degrade the performance of all existing phase desynchronization approaches,

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as illustrated in the numerical simulations in Section IV. Furthermore, numerical simulations also show that the proposed PRF can significantly improve convergence speed compared with existing results.

The remainder of this paper is organized as follows. In Section II, we present the PCO model and propose a general phase response function for phase desynchronization. Rigorous analysis of the convergence to desynchronization is provided in Section III. In Section IV, the effectiveness and robustness properties of the proposed phase desynchronization algorithm are verified through numerical simulation results. Finally, we conclude in Section V.

II. PCO Based Phase Desynchronization

In this section, we will first introduce the PCO model, and then we will propose a new phase response function for phase desynchronization.

A. PCO Model

We consider a network of N PCOs with an all-to-all communication pattern. Each oscillator has a phase variable $\phi_k \in \mathbb{S}^1$ $(k = 1, 2, \ldots, N)$ where $\mathbb{S}^1$ denotes the one-dimensional torus. Each phase variable ϕ_k evolves continuously from 0 to 2π with a constant speed determined by its natural frequency ω_k. In this paper (except Section IV-E), the natural frequencies are assumed identical, i.e., $\omega_1 = \omega_2 = \ldots = \omega_N = \omega$. When an oscillator's phase reaches 2π, it fires (emits a pulse) and resets its phase to 0, after which the cycle repeats. When an oscillator receives a pulse from a neighboring oscillator, it shifts its phase according to a pulse based interaction function, i.e., the phase response function, which is defined below:

Definition 1: Phase response function (PRF) $F(\phi_k)$ is defined as the phase shift (or jump) induced by a pulse as a function of phase at which the pulse is received [29].

Therefore, the interaction mechanism of PCOs can be described as follows:

1) Each PCO has a phase variable $\phi_k \in \mathbb{S}^1$ with initial value set to $\phi_k(0)$. ϕ_k evolves continuously from 0 to 2π with a constant speed (natural frequency) ω;
2) When the phase variable ϕ_k of PCO k reaches 2π, this PCO fires, i.e., emits a pulse, and simultaneously resets ϕ_k to 0. Then the same process repeats;
3) When a PCO receives a pulse from a neighboring PCO, it updates its phase variable according to the phase response function (PRF) $F(\phi_k)$:

$$\phi_k^+ = \phi_k + F(\phi_k) \tag{1}$$

where ϕ_k^+ and ϕ_k denote the phases of the kth oscillator after and before receiving the pulse, respectively.

B. Phase Response Function

It is already well-known that if the phase response function is chosen appropriately, pulse-coupled oscillators can achieve synchronization. For example, in [9], [30] we showed that using a delay-advance phase response function in which the value of phase shift is negative in the interval $(0, \pi]$, positive in the interval $(\pi, 2\pi)$, and zero at 0 and 2π, oscillator phases can achieve synchronization.

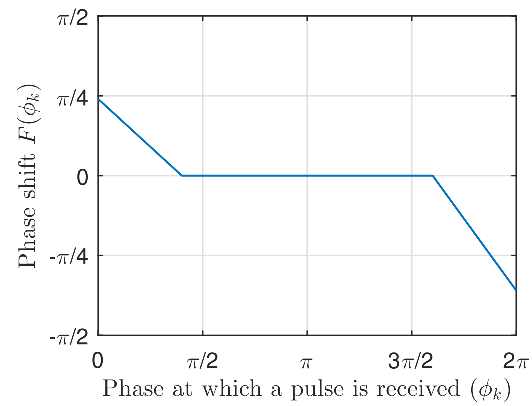

Fig. 1. Proposed phase response function $F(\phi_k)$ in (2) for phase desynchronization ($N = 5, l_1 = 0.6$, and $l_2 = 0.9$).

First, phase desynchronization is defined as follows:

Definition 2: For a network of N oscillators, phase desynchronization denotes the state on which all phases are distributed evenly on the unit circle with identical differences $\frac{2\pi}{N}$ between two neighboring phases.

As discussed earlier, in PCO networks, phase desynchronization is more stringent than weak desynchronization [17], [18] which uniformly spreads firing time instants of constituent oscillators. This is because weak desynchronization can be realized using persistent phase jumps (caused by pulse interactions), which are not permitted by phase desynchronization; whereas weak synchronization follows naturally if phase desynchronization is achieved.

We propose the following phase response function $F(\phi_k)$:

$$F(\phi_k) = \begin{cases} -l_1(\phi_k - \dfrac{2\pi}{N}) & 0 < \phi_k < \dfrac{2\pi}{N} \\ 0 & \dfrac{2\pi}{N} \le \phi_k \le 2\pi - \dfrac{2\pi}{N} \\ -l_2(\phi_k - (2\pi - \dfrac{2\pi}{N})) & 2\pi - \dfrac{2\pi}{N} < \phi_k < 2\pi \end{cases} \tag{2}$$

where $0 \le l_1 < 1$ and $0 \le l_2 < 1$ denote the strengths of coupling (interaction). It is worth noting that l_1 and l_2 can not be zero at the same time. According to this PRF, PCO k updates its phase variable ϕ_k (upon receiving a pulse) only when ϕ_k is within the interval $(0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$ as illustrated in Fig. 1. Therefore, the phase update rule (1) for PCO k can be rewritten as:

$$\phi_k^+ = \begin{cases} (1 - l_1)\phi_k + l_1 \dfrac{2\pi}{N} & 0 < \phi_k < \dfrac{2\pi}{N} \\ \phi_k & \dfrac{2\pi}{N} \le \phi_k \le 2\pi - \dfrac{2\pi}{N} \\ (1 - l_2)\phi_k + l_2(2\pi - \dfrac{2\pi}{N}) & 2\pi - \dfrac{2\pi}{N} < \phi_k < 2\pi \end{cases} \tag{3}$$

According to (3), $\phi_k^+ \in (0,\, 2\pi)$ is a monotonically increasing function of ϕ_k when ϕ_k resides in $(0,\, 2\pi)$, as shown in Fig. 2.

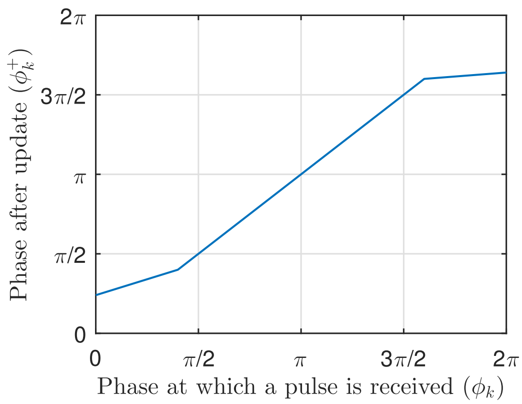

Fig. 2. Proposed phase update rule in (3) for phase desynchronization ($N = 5$, $l_1 = 0.6$, and $l_2 = 0.9$).

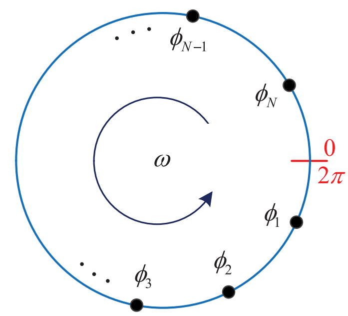

Fig. 3. Initially (at $t = 0$), the phases of PCOs are arranged in a way such that $\phi_1(0) > \phi_2(0) > \ldots > \phi_N(0)$ holds.

Remark 1: Our proposed phase update rule (3) is more general than [18], which in fact is a special case of our phase update rule (3) by setting $l_1 = 0$.

Remark 2: The phase response function in (2) with $l_1 > 0$ allows non-zero interaction when oscillator phases are within the interval $(0, \frac{2\pi}{N})$, which is key to improve the robustness to time delays and contributes to a significant advantage over the results in [17], [18], [27], and [28], as illustrated by numerical simulations in Fig. 14.

III. Convergence Properties of the Proposed Phase Desynchronization Algorithm

In this section, we rigorously prove that the phase update rule (3) can guarantee phase desynchronization. To this end, we will first introduce Lemma 1 on the firing order of PCOs.

Lemma 1: For a network of N PCOs with no two PCOs having equal initial phases, the firing order of PCOs is time-invariant under the phase update rule (3), i.e., if at any time instant t, we have $0 < \phi_{i_1} < \phi_{i_2} < \cdots < \phi_{i_N} \leq 2\pi$ for some sequence of nonrepeated elements $\{i_1, i_2, \ldots, i_N\}$ of $I = \{1, 2, \ldots, N\}$ (i.e., a reordering of the elements of I), then after N pulses, $0 < \phi_{i_1} < \phi_{i_2} < \cdots < \phi_{i_N} \leq 2\pi$ still holds.

Proof: The proof is given in Appendix A. ∎

In order to rigorously analyze the convergence process, we also need a measure to quantify the degree of achievement of desynchronization. Without loss of generality, we denote the initial time instant as $t = 0$ and assume at this time instant the phases of PCOs are arranged in a way such that $\phi_1(0) > \phi_2(0) > \ldots > \phi_N(0)$ holds, as illustrated in Fig. 3. (Note that here we assume that no two PCOs' initial phases are equal.) From Lemma 1, we know that the firing order of PCOs will not be affected by the pulse-induced update. So if ϕ_k is the immediate follower (anti-clockwisely) of ϕ_{k-1} on the unit circle $\mathbb{S}^1$ at $t = 0$, it will always be the immediate follower (anti-clockwisely) of ϕ_{k-1} on $\mathbb{S}^1$. Therefore, the phase differences between neighboring PCOs (in terms of phase) can always be expressed as:

$$\begin{cases} \Delta_k = (\phi_k - \phi_{k+1}) \bmod 2\pi, & k = 1, 2, \ldots, N-1 \\ \Delta_N = (\phi_N - \phi_1) \bmod 2\pi \end{cases} \tag{4}$$

According to Definition 2, phase desynchronization implies that the phase differences between neighboring (in terms of phase) oscillators are equal to $\frac{2\pi}{N}$. Therefore, in order to quantify the degree of achievement of phase desynchronization, we introduce a measure P based on phase differences as follows:

$$P \triangleq \sum_{k=1}^{N} |\Delta_k - \frac{2\pi}{N}| \tag{5}$$

When phase desynchronization is achieved, the phase differences between neighboring PCOs are equal to $\frac{2\pi}{N}$, so P in (5) will reach its minimum 0. It can also be easily verified that P equals 0 only when phase desynchronization is achieved.

Therefore, from the relationship between phase desynchronization and P, to prove the achievement of phase desynchronization, we need to prove that P will converge to 0. Since P will not change between two consecutive pulses, we only need to concentrate on firing events.

To analyze the changes of P caused by firing events (or pulses), we define "active pulse" and "silent pulse" as follows:

Definition 3: A pulse is called an "active pulse" if there exists at least one $k \in \{1, 2, \ldots, N\}$ such that $\phi_k \in (0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$ holds when the pulse is emitted.

Definition 4: A pulse is called a "silent pulse" if there does not exist any $k \in \{1, 2, \ldots, N\}$ such that $\phi_k \in (0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$ holds when the pulse is emitted.

According to Definition 3 and Definition 4, a pulse is either a "silent pulse" or an "active pulse." Since no oscillator phases reside in $(0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$ when a "silent pulse" is emitted, no phase variables are affected according to (2). Therefore, a "silent pulse" will not affect phase differences and the measure P. Similarly, an "active pulse" may change the measure P since the phase variables residing in $(0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$ will be affected by the pulse-induced update.

Next we will introduce Lemma 2 on the lack of existence of N consecutive "silent pulses" before achieving phase desynchronization.

Lemma 2: For a network of N PCOs with no two PCOs having equal initial phases, there cannot be N consecutive "silent pulses" unless phase desynchronization is achieved.

Proof: The proof is given in Appendix B. ∎

Using Lemma 2, the existence of "active pulses" before achieving phase desynchronization can be guaranteed. Further

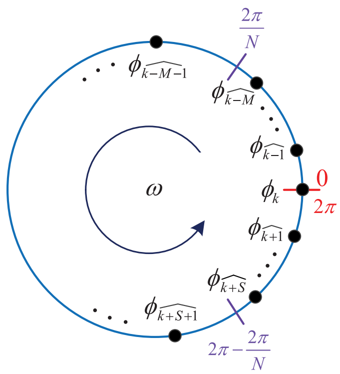

Fig. 4. The phase variables $\phi_{\widehat{k-1}}, \ldots, \phi_{\widehat{k-M}}$ and $\phi_{\widehat{k+1}}, \ldots, \phi_{\widehat{k+S}}$ reside in $(0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$ when oscillator k sends the first "active pulse" at $t = t_k$.

taking into account the fact that only "active pulses" may change P, we can infer that the evolution of P only depends on the changes caused by "active pulses."

Now, we introduce our main result.

Theorem 1: For a network of N PCOs with no two PCOs having equal initial phases, the PCOs will achieve phase desynchronization if the phase response function $F(\phi_k)$ is given by (2) for $0 < l_1 < 1$ and $0 < l_2 < 1$.

Proof: In order to prove the achievement of phase desynchronization, we need to prove that P will converge to 0. Further taking into account the fact that the evolution of P only depends on "active pulses," without loss of generality, we assume that oscillator k emits an "active pulse" at time instant $t = t_k$. According to Definition 3, there is at least one phase variable within $(0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$ when the pulse is sent. Without loss of generality, we assume that there are M phase variables within $(0, \frac{2\pi}{N})$ and S phase variables within $(2\pi - \frac{2\pi}{N}, 2\pi)$, where M and S are positive integers satisfying $2 \leq M + S \leq N - 1$. The M and S phase variables are represented as $\phi_{\widehat{k-1}}, \ldots, \phi_{\widehat{k-M}}$ and $\phi_{\widehat{k+1}}, \ldots, \phi_{\widehat{k+S}}$, respectively, where the superscript "$\widehat{\ }$" represents modulo operation on N, i.e., $\widehat{\bullet} \triangleq (\bullet) \bmod N$, as illustrated in Fig. 4. According to the assumption, we have $\phi_{\widehat{k-M}} < \frac{2\pi}{N} \leq \phi_{\widehat{k-M-1}}$ and $\phi_{\widehat{k+S+1}} \leq 2\pi - \frac{2\pi}{N} < \phi_{\widehat{k+S}}$. Since $\phi_{\widehat{k-1}}, \ldots, \phi_{\widehat{k-M}}$ and $\phi_{\widehat{k+1}}, \ldots, \phi_{\widehat{k+S}}$ reside in $(0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$, they will update their values after receiving the pulse from oscillator k according to the phase update rule in (3) as follows:

$$\begin{cases} \phi^+_{\widehat{k-i}} = (1-l_1)\phi_{\widehat{k-i}} + l_1 \dfrac{2\pi}{N}, & i = 1, \ldots, M \\ \phi^+_{\widehat{k+j}} = (1-l_2)\phi_{\widehat{k+j}} + l_2(2\pi - \dfrac{2\pi}{N}), & j = 1, \ldots, S \end{cases} \tag{6}$$

Note that we also have $\phi^+_k = 0$ and $\phi^+_{\widehat{k+q}} = \phi_{\widehat{k+q}}$ for $q = S+1, \ldots, N-M-1$ (because $\phi_{\widehat{k+q}}$ for $q = S+1, \ldots, N-M-1$ reside in $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$ and thus will not be changed according to the PRF in (2)).

According to (4) and (6), Δ^+_k for $k = 1, \ldots, N$ can be directly obtained. For a detailed expression of Δ^+_k, please see Appendix C. The new P (denote it as P^+) after the update is given by:

$$P^+ = \sum_{k=1}^{N} |\Delta^+_k - \frac{2\pi}{N}| \tag{7}$$

To show the change of measure P caused by the "active pulse" from oscillator k, we calculate the difference of P before and after the pulse-induced update:

$$P^+ - P = \sum_{k=1}^{N} |\Delta^+_k - \frac{2\pi}{N}| - \sum_{k=1}^{N} |\Delta_k - \frac{2\pi}{N}| \tag{8}$$

$P^+ - P$ can be further simplified as follows, with the detailed procedure given in Appendix C:

$$\begin{aligned} &P^+ - P \\ &= \underbrace{|\Delta^+_{\widehat{k-M-1}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k-M-1}} - \frac{2\pi}{N}| + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N})}_{\textbf{PartA}} \\ &+ \underbrace{|\Delta^+_{\widehat{k+S}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k+S}} - \frac{2\pi}{N}| + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}})}_{\textbf{PartB}} \end{aligned} \tag{9}$$

Next, we discuss the value of **Part A** in (9) under three different cases:

Case 1': If $\Delta_{\widehat{k-M-1}} > \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k-M-1}} \geq \frac{2\pi}{N}$ hold, **Part A** in (9) can be rewritten as:

$$\begin{aligned} \textbf{PartA} &= \Delta^+_{\widehat{k-M-1}} - \Delta_{\widehat{k-M-1}} + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \\ &= \phi_{\widehat{k-M-1}} - (1-l_1)\phi_{\widehat{k-M}} - l_1\frac{2\pi}{N} \\ &\quad - \phi_{\widehat{k-M-1}} + \phi_{\widehat{k-M}} + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \\ &= 2l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \ < 0 \end{aligned} \tag{10}$$

Case 2': If $\Delta_{\widehat{k-M-1}} > \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k-M-1}} < \frac{2\pi}{N}$ hold, we have $\phi_{\widehat{k-M}} - \phi_{\widehat{k-M-1}} + \frac{2\pi}{N} < 0$. Then **Part A** in (9) can be rewritten as:

$$\begin{aligned} \textbf{Part A} &= \frac{2\pi}{N} - \Delta^+_{\widehat{k-M-1}} - \Delta_{\widehat{k-M-1}} + \frac{2\pi}{N} \\ &\quad + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \\ &= \frac{2\pi}{N} - \phi_{\widehat{k-M-1}} + (1-l_1)\phi_{\widehat{k-M}} + l_1\frac{2\pi}{N} \\ &\quad - \phi_{\widehat{k-M-1}} + \phi_{\widehat{k-M}} + \frac{2\pi}{N} + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \\ &= 2(\phi_{\widehat{k-M}} - \phi_{\widehat{k-M-1}} + \frac{2\pi}{N}) \ < 0 \end{aligned} \tag{11}$$

Case 3': If $\Delta_{\widehat{k-M-1}} \leq \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k-M-1}} < \frac{2\pi}{N}$ hold, **Part A** in (9) can be rewritten as:

$$\begin{aligned}\mathbf{PartA} &= -\Delta^+_{\widehat{k-M-1}} + \Delta_{\widehat{k-M-1}} + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \\ &= -\phi_{\widehat{k-M-1}} + (1-l_1)\phi_{\widehat{k-M}} + l_1\frac{2\pi}{N} \\ &\quad + \phi_{\widehat{k-M-1}} - \phi_{\widehat{k-M}} + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \\ &= 0 \end{aligned} \tag{12}$$

According to (2), we cannot have a fourth case where $\Delta_{\widehat{k-M-1}} \leq \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k-M-1}} \geq \frac{2\pi}{N}$ hold because of the following constraint:

$$\begin{aligned}&\Delta^+_{\widehat{k-M-1}} - \Delta_{\widehat{k-M-1}} \\ &= \phi_{\widehat{k-M-1}} - (1-l_1)\phi_{\widehat{k-M}} - l_1\frac{2\pi}{N} - \phi_{\widehat{k-M-1}} + \phi_{\widehat{k-M}} \\ &= l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \ < 0 \end{aligned} \tag{13}$$

It is worth noting that in (13) we used the initial assumption $\phi_{\widehat{k-M}} < \frac{2\pi}{N}$ and the inequality $0 < l_1 < 1$.

Similarly, we also discuss the value of **Part B** in (9) under three different cases:

Case 1": If $\Delta_{\widehat{k+S}} > \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k+S}} \geq \frac{2\pi}{N}$ hold, **Part B** in (9) can be rewritten as:

$$\begin{aligned}\mathbf{PartB} &= \Delta^+_{\widehat{k+S}} - \Delta_{\widehat{k+S}} + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \\ &= (1-l_2)\phi_{\widehat{k+S}} + l_2(2\pi - \frac{2\pi}{N}) - \phi_{\widehat{k+S+1}} \\ &\quad - \phi_{\widehat{k+S}} + \phi_{\widehat{k+S+1}} + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \\ &= 2l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \ < 0 \end{aligned} \tag{14}$$

Case 2": If $\Delta_{\widehat{k+S}} > \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k+S}} < \frac{2\pi}{N}$ hold, we have $\phi_{\widehat{k+S+1}} - \phi_{\widehat{k+S}} + \frac{2\pi}{N} < 0$. Then **Part B** in (9) can be rewritten as:

$$\begin{aligned}\mathbf{PartB} &= \frac{2\pi}{N} - \Delta^+_{\widehat{k+S}} - \Delta_{\widehat{k+S}} + \frac{2\pi}{N} + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \\ &= \frac{2\pi}{N} - (1-l_2)\phi_{\widehat{k+S}} - l_2(2\pi - \frac{2\pi}{N}) + \phi_{\widehat{k+S+1}} \\ &\quad - \phi_{\widehat{k+S}} + \phi_{\widehat{k+S+1}} + \frac{2\pi}{N} + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \\ &= 2(\phi_{\widehat{k+S+1}} - \phi_{\widehat{k+S}} + \frac{2\pi}{N}) \ < 0 \end{aligned} \tag{15}$$

TABLE I
FLOW OF THE PROOF OF THEOREM 1

Part A	**Case 1':** $\Delta_{\widehat{k-M-1}} > \frac{2\pi}{N}$, $\Delta^+_{\widehat{k-M-1}} \geq \frac{2\pi}{N}$	**Part A** < 0
	Case 2': $\Delta_{\widehat{k-M-1}} > \frac{2\pi}{N}$, $\Delta^+_{\widehat{k-M-1}} < \frac{2\pi}{N}$	**Part A** < 0
	Case 3': $\Delta_{\widehat{k-M-1}} \leq \frac{2\pi}{N}$, $\Delta^+_{\widehat{k-M-1}} < \frac{2\pi}{N}$	**Part A** $= 0$
Part B	**Case 1":** $\Delta_{\widehat{k+S}} > \frac{2\pi}{N}$, $\Delta^+_{\widehat{k+S}} \geq \frac{2\pi}{N}$	**Part B** < 0
	Case 2": $\Delta_{\widehat{k+S}} > \frac{2\pi}{N}$, $\Delta^+_{\widehat{k+S}} < \frac{2\pi}{N}$	**Part B** < 0
	Case 3": $\Delta_{\widehat{k+S}} \leq \frac{2\pi}{N}$, $\Delta^+_{\widehat{k+S}} < \frac{2\pi}{N}$	**Part B** $= 0$

Case 3": If $\Delta_{\widehat{k+S}} \leq \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k+S}} < \frac{2\pi}{N}$ hold, **Part B** in (9) can be rewritten as:

$$\begin{aligned}\mathbf{PartB} &= -\Delta^+_{\widehat{k+S}} + \Delta_{\widehat{k+S}} + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \\ &= -(1-l_2)\phi_{\widehat{k+S}} - l_2(2\pi - \frac{2\pi}{N}) + \phi_{\widehat{k+S+1}} \\ &\quad + \phi_{\widehat{k+S}} - \phi_{\widehat{k+S+1}} + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \\ &= 0 \end{aligned} \tag{16}$$

According to (2), we cannot have a fourth case where $\Delta_{\widehat{k+S}} \leq \frac{2\pi}{N}$ and $\Delta^+_{\widehat{k+S}} \geq \frac{2\pi}{N}$ hold due to the following constraint:

$$\begin{aligned}&\Delta^+_{\widehat{k+S}} - \Delta_{\widehat{k+S}} \\ &= (1-l_2)\phi_{\widehat{k+S}} + l_2(2\pi - \frac{2\pi}{N}) - \phi_{\widehat{k+S+1}} - \phi_{\widehat{k+S}} + \phi_{\widehat{k+S+1}} \\ &= l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \ < 0 \end{aligned} \tag{17}$$

where we used the initial assumption $\phi_{\widehat{k+S}} > 2\pi - \frac{2\pi}{N}$ and the inequality $0 < l_2 < 1$.

To make the proof easy to follow, we use Table I to show the flow of the proof.

From the above analysis, we have $P^+ - P \leq 0$, meaning that the value of P will be decreased or unchanged by each "active pulse." According to Lemma 3 in Appendix D, **Case 3'** and **Case 3"** above cannot always exist before phase desynchronization is achieved, i.e., $\Delta_{\widehat{k-M-1}} \leq \frac{2\pi}{N}$ and $\Delta_{\widehat{k+S}} \leq \frac{2\pi}{N}$ cannot always be true before the achievement of phase desynchronization. Consequently, P will not be retained at a non-zero value, and will keep decreasing until it reaches 0, i.e., until phase desynchronization is achieved. Therefore, the PCOs will achieve phase desynchronization under the PRF (2) for $0 < l_1 < 1$ and $0 < l_2 < 1$. ■

Remark 3: In the above proof, in order to obtain the expression of $P^+ - P$ in (9), we only considered the situation where there is at least one phase variable within each of the intervals

$(0, \frac{2\pi}{N})$, $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, and $(2\pi - \frac{2\pi}{N}, 2\pi)$ when an oscillator fires. If one or two of the intervals do not contain any phase variables, the same conclusion can be drawn, as detailed below. Note that when one oscillator fires, if all the other $N-1$ phase variables are within $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, then this pulse is a "silent pulse" which will not cause any change on P, and we have proved that there cannot be N consecutive "silent pulses" unless phase desynchronization is achieved. Therefore, there are five more situations that need to be taken into consideration. Using the same line of reasoning as above and assuming that the update of P is triggered by the pulse of oscillator k, we have the expression of $P^+ - P$ under the five situations as follows:

Situation 1 (there are M, $N-M-1$, and 0 phase variables within $(0, \frac{2\pi}{N})$, $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, and $(2\pi - \frac{2\pi}{N}, 2\pi)$, respectively): In this case, we have

$$\begin{aligned} &P^+ - P \\ &= |\Delta^+_{\widehat{k-M-1}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k-M-1}} - \frac{2\pi}{N}| + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) \end{aligned} \tag{18}$$

which is the same as **Part A** in (9).

Situation 2 (there are 0, $N-S-1$, and S phase variables within $(0, \frac{2\pi}{N})$, $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, and $(2\pi - \frac{2\pi}{N}, 2\pi)$, respectively): In this case, we have

$$\begin{aligned} &P^+ - P \\ &= |\Delta^+_{\widehat{k+S}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k+S}} - \frac{2\pi}{N}| + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \end{aligned} \tag{19}$$

which is the same as **Part B** in (9).

Situation 3 (there are M, 0, and $N-M-1$ phase variables within $(0, \frac{2\pi}{N})$, $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, and $(2\pi - \frac{2\pi}{N}, 2\pi)$, respectively): In this case, we have

$$\begin{aligned} P^+ - P &= 2l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N}) + 2l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k-M-1}}) \\ &< 0 \end{aligned} \tag{20}$$

where we used $\phi_{\widehat{k-M}} < \frac{2\pi}{N}$ and $\phi_{\widehat{k-M-1}} > 2\pi - \frac{2\pi}{N}$.

Situation 4 (there are $N-1$, 0, and 0 phase variables within $(0, \frac{2\pi}{N})$, $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, and $(2\pi - \frac{2\pi}{N}, 2\pi)$, respectively): In this case, we have

$$P^+ - P = 2l_1(\phi_{\widehat{k+1}} - \frac{2\pi}{N}) < 0 \tag{21}$$

where we used $\phi_{\widehat{k+1}} < \frac{2\pi}{N}$.

Situation 5 (there are 0, 0, and $N-1$ phase variables within $(0, \frac{2\pi}{N})$, $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, and $(2\pi - \frac{2\pi}{N}, 2\pi)$, respectively): In this case, we have

$$P^+ - P = 2l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k-1}}) < 0 \tag{22}$$

where we used $\phi_{\widehat{k-1}} > 2\pi - \frac{2\pi}{N}$.

In summary, we have $P^+ - P < 0$ under **Situations 3, 4, 5**, meaning that the value of P will decrease under these situations. Note that $P^+ - P$ will become the same as **Part A** and **Part B** in (9) under **Situations 1, 2**, respectively. According to the above proof of Theorem 1, **Part A** and **Part B** will be negative unless **Case 3'** and **Case 3''** hold. From Lemma 3 in Appendix D, **Case 3'** and **Case 3''** cannot always be true, it can be inferred that no matter which situation occurs, the value of P will keep decreasing until phase desynchronization is achieved.

Next, we show that phase desynchronization can also be achieved under the PRF (2) with either l_1 or l_2 being zero. It is worth noting that when l_1 is zero, our PRF reduces to the one in [18].

Corollary 1: For a network of N PCOs with no two PCOs having equal initial phases, the PCOs will achieve phase desynchronization if the phase response function $F(\phi_k)$ is given by (2) for $l_1 = 0$ and $0 < l_2 < 1$.

Proof: The proof is given in Appendix E. ■

Corollary 2: For a network of N PCOs with no two PCOs having equal initial phases, the PCOs will achieve phase desynchronization if the phase response function $F(\phi_k)$ is given by (2) for $0 < l_1 < 1$ and $l_2 = 0$.

Proof: Following the same line of reasoning for Corollary 1 in Appendix E, the proof of Corollary 2 can be easily obtained and hence omitted here. ■

Remark 4: If there are two oscillators having equal initial phases, these two PCOs will always have equal phases. This is because they will always make updates simultaneously with identical phase shifts. Therefore, the existence of oscillators having identical phases makes phase desynchronization impossible. In fact, the situation with equal initial phases fails all existing algorithms on phase desynchronization to the best of our knowledge.

IV. Simulation Results

In this section, we use simulation results to verify that the proposed phase desynchronization algorithm has better robustness than existing results.

We first verified the effectiveness of the proposed phase desynchronization algorithm under an ideal condition where all PCOs have identical nature frequency and there is no pulse loss or time delay in Section IV-A. Then under this ideal condition, we compared our algorithm with existing results in terms of convergence speed in Section IV-B.

Given that pulse loss is prevalent in wireless communications due to interferences, congestions, and intermittent faulty hardware, we also compared our results with existing results in the case where pulses are lost randomly in Section IV-C. Note that in this case, the virtual interaction pattern is not all-to-all any more as the firing of one oscillator will not affect oscillators that fail to receive the pulse corresponding to the firing. In fact, in this case the connection becomes multi-hop and time-varying.

Since time delay is not negligible when the order of processing/transmission delays is comparable to the length of the oscillating period, we also compared our results with existing results in the presence of random communication delays in Section IV-D.

Finally, given that there always exists heterogeneity in the natural frequency ω, we simulated and compared our results

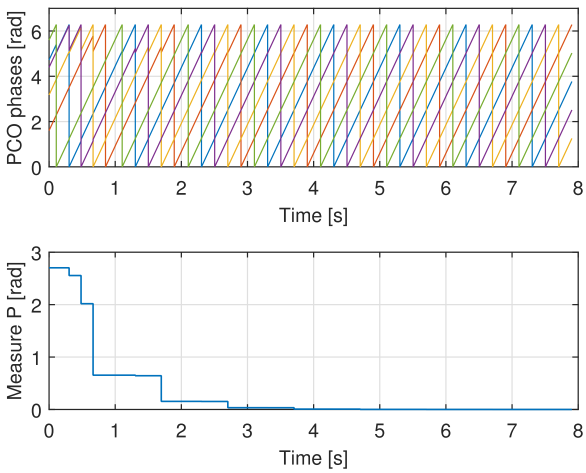

Fig. 5. The evolutions of PCO phases $\phi_k (k = 1, \ldots, N)$ (upper panel) and measure P (lower panel) under the PRF (2) with (l_1, l_2) set to $(0.6, 0.9)$.

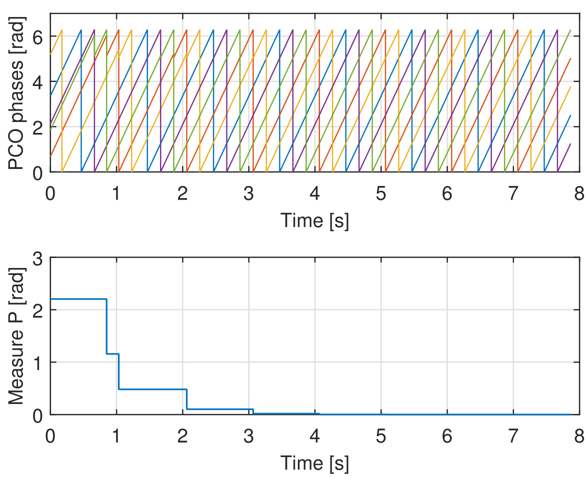

Fig. 6. The evolutions of PCO phases $\phi_k (k = 1, \ldots, N)$ (upper panel) and measure P (lower panel) under the PRF (2) with (l_1, l_2) set to $(0, 0.9)$.

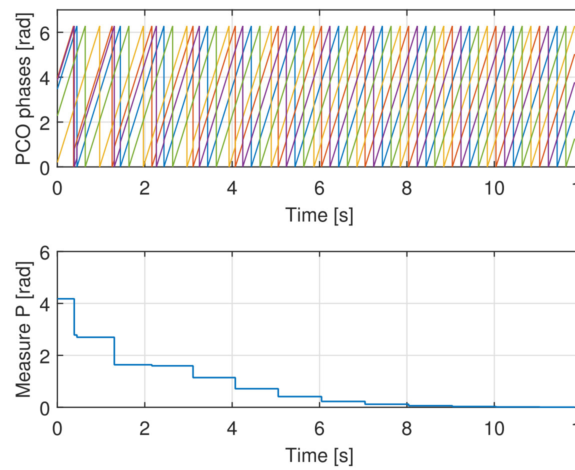

Fig. 7. The evolutions of PCO phases $\phi_k (k = 1, \ldots, N)$ (upper panel) and measure P (lower panel) under the PRF (2) with (l_1, l_2) set to $(0.6, 0)$.

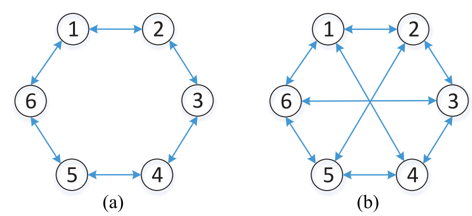

Fig. 8. Ring and circulant symmetric graphs with six oscillators: (a) ring graph; (b) circulant symmetric graph.

with existing results when different oscillators have different frequencies in Section IV-E.

In all simulations, we recorded the convergence time of the achievement of phase desynchronization when $|\Delta_k - \frac{2\pi}{N}| < 10^{-3}$ holds for $k = 1, \ldots, N$.

A. Effectiveness of the Proposed Phase Desynchronization Algorithm in the Ideal Case

Under ideal condition where the natural frequencies are identical and no pulse loss or time delay exists, we verified that the proposed PRF can indeed achieve phase desynchronization on all-to-all graph. The initial phases of a network of $N = 5$ PCOs were randomly chosen from the interval $[0, 2\pi)$, and the natural frequency ω was set to 2π. The coupling strengths (l_1, l_2) in the PRF (2) were set to $(0.6, 0.9)$, $(0, 0.9)$, and $(0.6, 0)$, respectively. The evolutions of PCO phases and P are given in Fig. 5, Fig. 6, and Fig. 7, respectively. It can be seen that the PCO phases were uniformly spread apart and the measure P converged to 0 in the three cases.

Besides the all-to-all graph, we also considered ring and circulant symmetric graphs (cf. Fig. 8) in the numerical experiments. The initial phases can be randomly chosen but subject to a constraint that the oscillators are indexed in the order of their initial phase magnitude. This constraint is imposed because otherwise two nonadjacent oscillators may converge to the same phase value and become non-separable, making phase desynchronization impossible. In the simulation, the initial phases were set to $\{0.05\pi, 0.26\pi, 0.72\pi, 1.03\pi, 1.24\pi, 1.69\pi\}$, and ω was set to 2π. The coupling strengths (l_1, l_2) in (2) were set to $(0.3, 0.45)$. The evolutions of phases and P are given in Fig. 9 and Fig. 10, respectively, which confirmed the effectiveness of the proposed desynchronization algorithm.

B. Comparison With Existing Results in the Ideal Case

In the ideal case, we compared our algorithm with the DESYNC-STALE algorithm in [17], the desynchronization algorithm in [18], and the FAST-DESYNC algorithm in [26]. For our algorithm, the coupling strengths (l_1, l_2) in (2) were set to $(0.6, 0.9)$. The jump size α in the DESYNC-STALE algorithm in [17], the coupling parameter α in the desynchronization algorithm in [18], and the jump-phase parameter α in the FAST-DESYNC algorithm in [26] were set to 0.95, 0.75, and 0.5, respectively, as used in their own respective papers. The initial phases were randomly chosen from $[0, 2\pi)$, and ω was set to 2π. The results on convergence time under different network sizes are given in Fig. 11 in which the error bars represent the

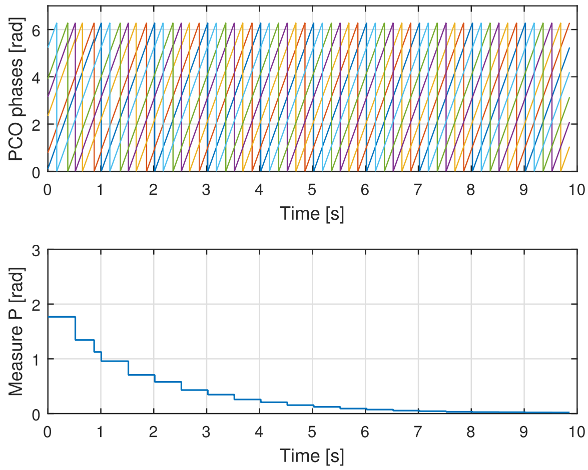

Fig. 9. The evolutions of PCO phases $\phi_k\,(k = 1, \ldots, N)$ (upper panel) and measure P (lower panel) under the PRF (2) with (l_1, l_2) set to $(0.3, 0.45)$ on the ring graph.

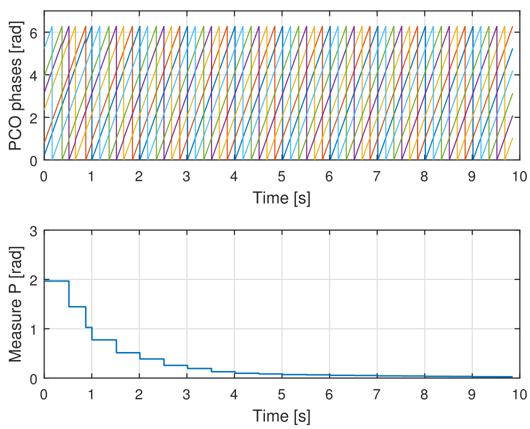

Fig. 10. The evolutions of PCO phases $\phi_k\,(k = 1, \ldots, N)$ (upper panel) and measure P (lower panel) under the PRF (2) with (l_1, l_2) set to $(0.3, 0.45)$ on the circulant symmetric graph.

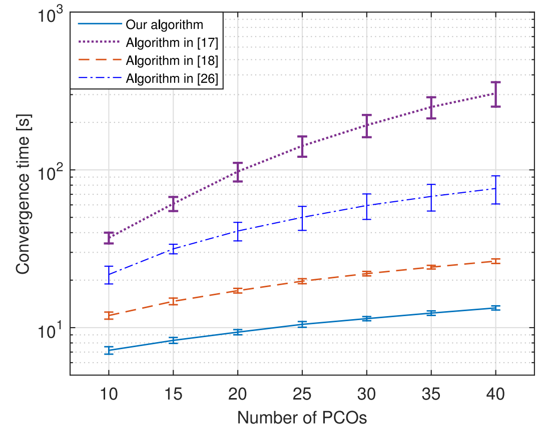

Fig. 11. Comparison of the proposed algorithm with the DESYNC-STALE algorithm [17] and the desynchronization algorithm [18] in the ideal case.

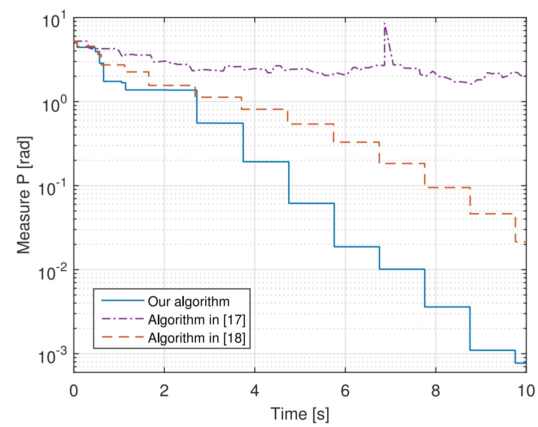

Fig. 12. The evolutions of P for the proposed algorithm, the DESYNC-STALE algorithm in [17], and the desynchronization algorithm in [18] under pulse loss probability $p = 0.05$.

standard variation of 1000 runs. From the simulation results we can see that our algorithm converges faster than the algorithms in [17], [18], and [26].

C. Comparison With Existing Results Under Pulse Losses

In this case, we compared our approach with the DESYNC-STALE algorithm in [17] and the desynchronization algorithm in [18] under pulse losses. The communication links are not reliable and every pulse is transmitted with a failure probability $p\ (0 \leq p < 1)$. For any pulse, it has a probability $1 - p$ to successfully affect an oscillator, and with probability p it will fail to affect the oscillator. Moreover, we assume that the probability for one oscillator to successfully receive a pulse is independent of other oscillators.

For our algorithm, the coupling strengths (l_1, l_2) in (2) were set to $(0.6, 0.9)$. The jump size α in the DESYNC-STALE algorithm in [17] and the coupling parameter α in the desynchronization algorithm in [18] were set to 0.95 and 0.75, respectively, as given in their respective papers. The initial phases of a network of $N = 10$ PCOs were randomly chosen from the interval $[0, 2\pi)$, and ω was set to 2π. The probability p was set to 0.05. The evolutions of P are illustrated in Fig. 12. It can be seen that our algorithm and the one in [18], both of which are PRF based approaches, could still guarantee desynchronization, whereas the DESYNC-STALE algorithm in [17], which relies on the information of two firing neighbors, loses its effectiveness.

We also compared our convergence time with [18] under pulse losses. The probabilities of pulse losses p were set to 0.05 and 0.10, respectively. Other parameters were the same as above. The results on convergence time under different network sizes are given in Fig. 13 where the error bars represent the standard variation of 1000 runs. We can see that the proposed phase desynchronization algorithm converges faster than the algorithm in [18]. However, compared with the ideal communication case (cf. Fig. 11), it is obvious that pulse losses indeed increase the time to convergence for both algorithms.

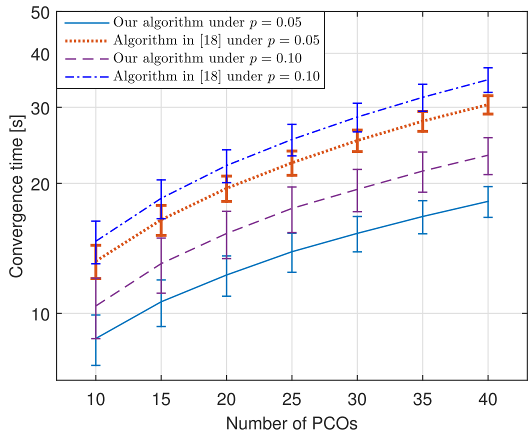

Fig. 13. The convergence time of the proposed algorithm and the desynchronization algorithm in [18] under pulse losses.

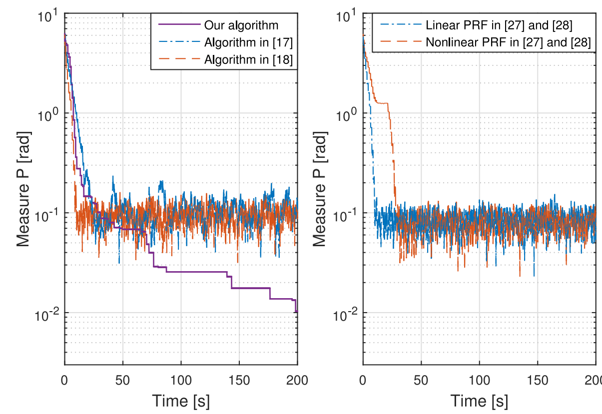

Fig. 14. The evolutions of P for the proposed algorithm, the DESYNC-STALE algorithm in [17], and the desynchronization algorithm in [18] (left panel), and linear and nonlinear PRF based desynchronization algorithm in [27] and [28] (right panel) under time delays uniformly distributed in $[0, 5\,\mathrm{ms}]$.

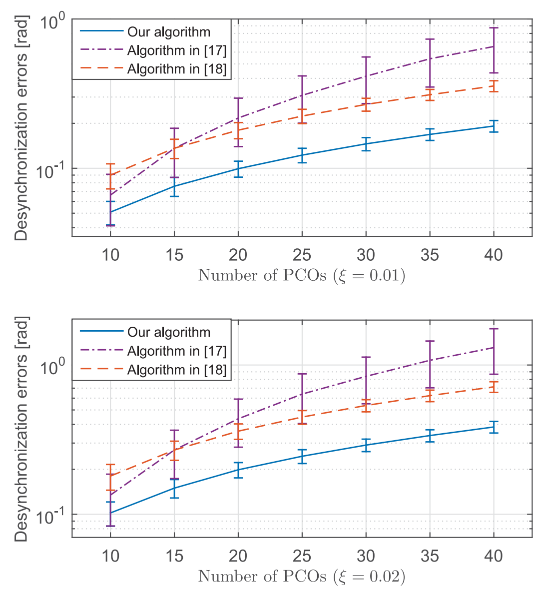

Fig. 15. The desynchronization errors of the proposed algorithm, the DESYNC-STALE algorithm in [17], and the desynchronization algorithm in [18] in the constant frequency error case.

D. Comparison With Existing Results Under Random Time Delays

In this case, we assume that there is a random delay associated with each communication link, which is uniformly distributed in $[0, \tau]$ with τ denoting the maximal delay. Moreover, we assume that delays on different links are independent of each other.

We compared the proposed phase desynchronization algorithm with the DESYNC-STALE algorithm in [17] and the desynchronization algorithms in [18], [27], and [28] on a network of $N = 10$ PCOs. The initial phases were randomly chosen from $[0, 2\pi)$ and ω was set to 2π. The maximal delay τ was set to 0.5% of the free-running firing period. The coupling strengths (l_1, l_2) in (2) were set to $(0.9, 0)$. The jump size α in the DESYNC-STALE algorithm in [17] and the coupling parameter α in the desynchronization algorithm in [18] were set to 0.95 and 0.75, respectively, as used in their own respective papers. Both linear and nonlinear realizations of phase response functions in F. Ferrante's work [27] and [28] were considered, which are given by

$$F(\phi_k) = \begin{cases} 0 & \phi_k \le 2\pi - \dfrac{2\pi}{N} \\ -l(\phi_k - 2\pi + \frac{2\pi}{N}) & \phi_k > 2\pi - \dfrac{2\pi}{N} \end{cases} \tag{23}$$

and

$$F(\phi_k) = \begin{cases} 0 & \phi_k \le 2\pi - \dfrac{2\pi}{N} \\ l\dfrac{N}{4\pi}\left[(\phi_k - 2\pi)^2 - (\dfrac{2\pi}{N})^2\right] & \phi_k > 2\pi - \dfrac{2\pi}{N} \end{cases} \tag{24}$$

The coupling strength l was set to 0.7, as used in [27] and [28]. The evolutions of P are given in Fig. 14. It can be seen that our phase desynchronization algorithm can still achieve desynchronization under time delays, whereas none of the algorithms in [17], [18], [27], or [28] works anymore.

E. Comparison With Existing Results Under Random Frequency Errors

Given that there always exist errors on the natural frequency ω, we compared the proposed phase desynchronization algorithm with the DESYNC-STALE algorithm in [17] and the desynchronization algorithm in [18] under constant and time-varying frequency errors.

In the constant frequency error case, the natural frequencies of oscillators were assumed to be independently and uniformly distributed in $[2\pi - \xi, 2\pi + \xi]$ where ξ denotes the maximal error. The initial phases were randomly chosen from $[0, 2\pi)$. The coupling strengths (l_1, l_2) in (2) were set to $(0.6, 0.9)$. The jump size α in the DESYNC-STALE algorithm in [17] and the coupling parameter α in the desynchronization algorithm in [18] were set to 0.95 and 0.75, respectively, as given in their respective papers. It is worth noting that frequency errors lead

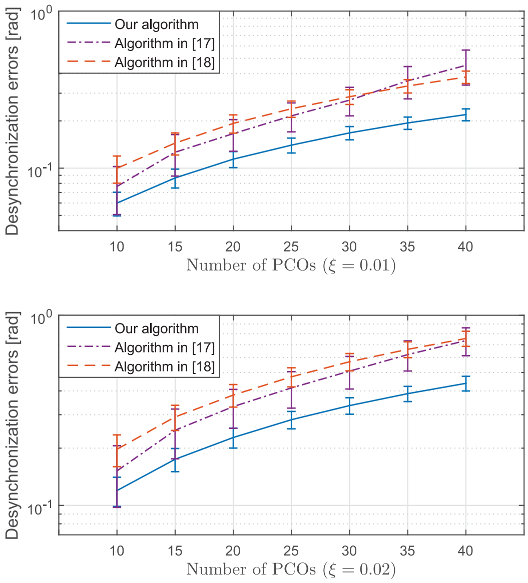

Fig. 16. The desynchronization errors of the proposed algorithm, the DESYNC-STALE algorithm in [17], and the desynchronization algorithm in [18] in the time-varying frequency error case.

to desynchronization errors in all three algorithms. We recorded the mean values of P over one round of firings after the transient period and plotted the results in Fig. 15 where ξ was set to 0.01 and 0.02, respectively. The error bars represent the standard variation of 1000 runs. The results show that our approach has less desynchronization error and thus better robustness than the results in [17] and [18].

In the time-varying frequency error case, the natural frequencies were assumed to be of the form $\omega_k = 2\pi + \xi \sin(0.1t + \vartheta_k)$ for $k = 1, \ldots, N$, where ξ and ϑ_k denote the maximal error and initial frequency offset, respectively. ξ was set to 0.01 and 0.02, respectively, and ϑ_k was randomly chosen from $[0, 2\pi)$. Other parameters were the same as the constant frequency error case. The desynchronization errors of all three algorithms were illustrated in Fig. 16. It can be seen that our proposed algorithm has better robustness than the results in [17] and [18].

V. Conclusions and Discussions

Phase response function based decentralized desynchronization is rigorously analyzed and a phase-desynchronizing phase response function is given, which includes existing results as special cases. Simulation results show that the proposed phase response function can achieve better convergence speed and robustness to pulse losses, time delays, and frequency errors than existing results. In future work, we plan to analytically explore the robustness of the proposed algorithm using the hybrid systems framework.

Appendix A
Proof of Lemma 1

Proof: Assume that at any time instant t, the phases satisfy the following relationship $0 < \phi_{i_1} < \phi_{i_2} < \cdots < \phi_{i_N} \leq 2\pi$. Since ϕ_{i_N} is the largest, it will reach 2π first and send a pulse that will be received by all the other PCOs. After receiving this pulse, all the other PCOs update their phases according to the phase update rule (3). Since $\phi_k^+ \in (0, 2\pi)$ is a monotonically increasing function of ϕ_k when ϕ_k resides in $(0, 2\pi)$, we have $0 = \phi_{i_N} < \phi_{i_1} < \cdots < \phi_{i_{N-1}} < 2\pi$ after this update. Following the same line of reasoning, it follows that after N pulses, $0 < \phi_{i_1} < \phi_{i_2} < \cdots < \phi_{i_N} \leq 2\pi$ holds, which means that the firing order of PCOs is time-invariant. ■

Appendix B
Proof of Lemma 2

Proof: We use proof of contradiction. Assume that N consecutive pulses are all "silent pulses" but phase desynchronization has not been achieved. From Lemma 1, the firing order of oscillators is time-invariant, so the N consecutive pulses must be from N different oscillators. For a pulse from oscillator i to be a "silent pulse," the phase variable of the oscillator who sends a pulse immediately before oscillator i must be no less than $\frac{2\pi}{N}$, and the phase variable of the oscillator who sends a pulse immediately after oscillator i must be no greater than $2\pi - \frac{2\pi}{N}$, which means that the phases of all the other oscillators are outside $(0, \frac{2\pi}{N}) \cup (2\pi - \frac{2\pi}{N}, 2\pi)$. Therefore, for the N consecutive "silent pulses," the phase differences between any two neighboring phases are no less than $\frac{2\pi}{N}$. Given that the sum of all phase differences has to be 2π, we have all phase differences being equal to $\frac{2\pi}{N}$, meaning that phase desynchronization is achieved, which contradicts the initial assumption. Therefore, there cannot be N consecutive "silent pulses" unless phase desynchronization is achieved. ■

Appendix C
Simplification Process of $P^+ - P$

According to (4) and (6), phase differences after the update caused by the "active pulse" from oscillator k are given by:

$$\begin{cases} \Delta^+_{\widehat{k-M-1}} &= \phi^+_{\widehat{k-M-1}} - \phi^+_{\widehat{k-M}} \\ &= \phi_{\widehat{k-M-1}} - (1-l_1)\phi_{\widehat{k-M}} - l_1 \dfrac{2\pi}{N} \\ \Delta^+_{\widehat{k-i}} &= \phi^+_{\widehat{k-i}} - \phi^+_{\widehat{k-i+1}} \\ &= (1-l_1)(\phi_{\widehat{k-i}} - \phi_{\widehat{k-i+1}}),\ i = 2, \ldots, M \\ \Delta^+_{\widehat{k-1}} &= \phi^+_{\widehat{k-1}} - \phi^+_{k} \\ &= (1-l_1)\phi_{\widehat{k-1}} + l_1 \dfrac{2\pi}{N} \\ \Delta^+_{k} &= \phi^+_{k} - \phi^+_{\widehat{k+1}} + 2\pi \\ &= 2\pi - (1-l_2)\phi_{\widehat{k+1}} - l_2(2\pi - \dfrac{2\pi}{N}) \\ \Delta^+_{\widehat{k+j}} &= \phi^+_{\widehat{k+j}} - \phi^+_{\widehat{k+j+1}} \\ &= (1-l_2)(\phi_{\widehat{k+j}} - \phi_{\widehat{k+j+1}}),\ j = 1, \ldots, S-1 \\ \Delta^+_{\widehat{k+S}} &= \phi^+_{\widehat{k+S}} - \phi^+_{\widehat{k+S+1}} \\ &= (1-l_2)\phi_{\widehat{k+S}} + l_2(2\pi - \dfrac{2\pi}{N}) - \phi_{\widehat{k+S+1}} \\ \Delta^+_{\widehat{k+q}} &= \phi^+_{\widehat{k+q}} - \phi^+_{\widehat{k+q+1}} \\ &= \phi_{\widehat{k+q}} - \phi_{\widehat{k+q+1}},\ q = S+1, \ldots, N-M-2 \end{cases} \tag{25}$$

Therefore, $P^+ - P$ can be divided into 7 parts as follows:

$$
\begin{aligned}
P^+ - P &= \sum_{k=1}^{N} |\Delta_k^+ - \frac{2\pi}{N}| - \sum_{k=1}^{N} |\Delta_k - \frac{2\pi}{N}| \\
&= \underbrace{|\Delta^+_{\widehat{k-M-1}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k-M-1}} - \frac{2\pi}{N}|}_{\textbf{Part 1}} \\
&+ \underbrace{\sum_{i=2}^{M} |\Delta^+_{\widehat{k-i}} - \frac{2\pi}{N}| - \sum_{i=2}^{M} |\Delta_{\widehat{k-i}} - \frac{2\pi}{N}|}_{\textbf{Part 2}} \\
&+ \underbrace{|\Delta^+_{\widehat{k-1}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k-1}} - \frac{2\pi}{N}|}_{\textbf{Part 3}} + \underbrace{|\Delta^+_{k} - \frac{2\pi}{N}| - |\Delta_{k} - \frac{2\pi}{N}|}_{\textbf{Part 4}} \\
&+ \underbrace{\sum_{j=1}^{S-1} |\Delta^+_{\widehat{k+j}} - \frac{2\pi}{N}| - \sum_{j=1}^{S-1} |\Delta_{\widehat{k+j}} - \frac{2\pi}{N}|}_{\textbf{Part 5}} \\
&+ \underbrace{|\Delta^+_{\widehat{k+S}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k+S}} - \frac{2\pi}{N}|}_{\textbf{Part 6}} \\
&+ \underbrace{\sum_{q=S+1}^{N-M-2} |\Delta^+_{\widehat{k+q}} - \frac{2\pi}{N}| - \sum_{q=S+1}^{N-M-2} |\Delta_{\widehat{k+q}} - \frac{2\pi}{N}|}_{\textbf{Part 7}} \qquad (26)
\end{aligned}
$$

Part 2, **Part 3**, **Part 4**, **Part 5**, and **Part 7** in (26) can be simplified as follows:

$$
\begin{aligned}
\textbf{Part 2} &= \sum_{i=2}^{M} \Big\{ |(1-l_1)(\phi_{\widehat{k-i}} - \phi_{\widehat{k-i+1}}) - \frac{2\pi}{N}| \\
&\quad - |(\phi_{\widehat{k-i}} - \phi_{\widehat{k-i+1}}) - \frac{2\pi}{N}| \Big\} \\
&= l_1(\phi_{\widehat{k-M}} - \phi_{\widehat{k-1}}) \qquad (27)
\end{aligned}
$$

where we used the relationships $\phi_{\widehat{k-i}} - \phi_{\widehat{k-i+1}} < \frac{2\pi}{N}$ and $(1 - l_1)(\phi_{\widehat{k-i}} - \phi_{\widehat{k-i+1}}) < \frac{2\pi}{N}$ for $i = 2, \ldots, M$ as $0 < l_1 < 1$.

$$
\begin{aligned}
\textbf{Part 3} &= |(1-l_1)\phi_{\widehat{k-1}} + l_1\frac{2\pi}{N} - \frac{2\pi}{N}| - |\phi_{\widehat{k-1}} - \frac{2\pi}{N}| \\
&= -l_1(\frac{2\pi}{N} - \phi_{\widehat{k-1}}) \qquad (28)
\end{aligned}
$$

In the above derivation we used $\phi_{\widehat{k-1}} < \frac{2\pi}{N}$.

$$
\begin{aligned}
\textbf{Part 4} &= |2\pi - (1-l_2)\phi_{\widehat{k+1}} - l_2(2\pi - \frac{2\pi}{N}) - \frac{2\pi}{N}| \\
&\quad - |2\pi - \phi_{\widehat{k+1}} - \frac{2\pi}{N}| \\
&= l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+1}}) \qquad (29)
\end{aligned}
$$

where we used the inequalities $2\pi - \phi_{\widehat{k+1}} < \frac{2\pi}{N}$ and $2\pi - (1 - l_2)\phi_{\widehat{k+1}} - l_2(2\pi - \frac{2\pi}{N}) < \frac{2\pi}{N}$ due to $0 < l_2 < 1$.

$$
\begin{aligned}
\textbf{Part 5} &= \sum_{j=1}^{S-1} \Big\{ |(1-l_2)(\phi_{\widehat{k+j}} - \phi_{\widehat{k+j+1}}) - \frac{2\pi}{N}| \\
&\quad - |(\phi_{\widehat{k+j}} - \phi_{\widehat{k+j+1}}) - \frac{2\pi}{N}| \Big\} \\
&= l_2(\phi_{\widehat{k+1}} - \phi_{\widehat{k+S}}) \qquad (30)
\end{aligned}
$$

where we used the relationships $\phi_{\widehat{k+j}} - \phi_{\widehat{k+j+1}} < \frac{2\pi}{N}$ and $(1 - l_2)(\phi_{\widehat{k+j}} - \phi_{\widehat{k+j+1}}) < \frac{2\pi}{N}$ for $j = 1, \ldots, S-1$.

$$
\begin{aligned}
\textbf{Part 7} &= \sum_{q=S+1}^{N-M-2} \Big\{ |\phi_{\widehat{k+q}} - \phi_{\widehat{k+q+1}} - \frac{2\pi}{N}| \\
&\quad - |\phi_{\widehat{k+q}} - \phi_{\widehat{k+q+1}} - \frac{2\pi}{N}| \Big\} \\
&= 0 \qquad (31)
\end{aligned}
$$

Combining (26)–(31) leads to:

$$
\begin{aligned}
&P^+ - P \\
&= \underbrace{|\Delta^+_{\widehat{k-M-1}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k-M-1}} - \frac{2\pi}{N}| + l_1(\phi_{\widehat{k-M}} - \frac{2\pi}{N})}_{\textbf{Part A}} \\
&+ \underbrace{|\Delta^+_{\widehat{k+S}} - \frac{2\pi}{N}| - |\Delta_{\widehat{k+S}} - \frac{2\pi}{N}| + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}})}_{\textbf{Part B}} \\
&\qquad (32)
\end{aligned}
$$

Appendix D
Lemma 3

Lemma 3: **Case 3'** and **Case 3"** cannot always exist before phase desynchronization is achieved.

Proof: It can be easily inferred that before achieving phase desynchronization there always exists one phase difference smaller than $\frac{2\pi}{N}$ and one phase difference larger than $\frac{2\pi}{N}$, and in between the two phase differences there may be some phase differences (represent the number as Q, $0 \le Q \le N-2$) that are equal to $\frac{2\pi}{N}$, which is defined as **State A**. Denote the phase difference smaller than $\frac{2\pi}{N}$ and the phase difference larger than $\frac{2\pi}{N}$ as Δ_j and $\Delta_{\widehat{j+Q+1}}$, respectively. There are Q phase differences between Δ_j and $\Delta_{\widehat{j+Q+1}}$ which are equal to $\frac{2\pi}{N}$ and denoted as $\Delta_{\widehat{j+1}}, \ldots, \Delta_{\widehat{j+Q}}$ (cf. Fig. 17).

As illustrated in Fig. 17, if $\phi_{\widehat{j+Q+2}} < \frac{2\pi}{N}$, we have $\Delta_{\widehat{j+Q+2}} < \frac{2\pi}{N}$. Then we can infer that **Case 3'** does not exist when oscillator j fires because $\phi_{\widehat{j+Q+2}}$ resides in $(0, \frac{2\pi}{N})$ and $\Delta_{\widehat{j+Q+1}} > \frac{2\pi}{N}$ is true.

Thus we only need to consider the situation when $\phi_{\widehat{j+Q+2}} \ge \frac{2\pi}{N}$ holds. It can be proven that under this situation **State A** must evolve to **State B** (cf. Fig. 18) after Q pulses. In **State B**, the Q phase differences which were equal to $\frac{2\pi}{N}$ in **State**

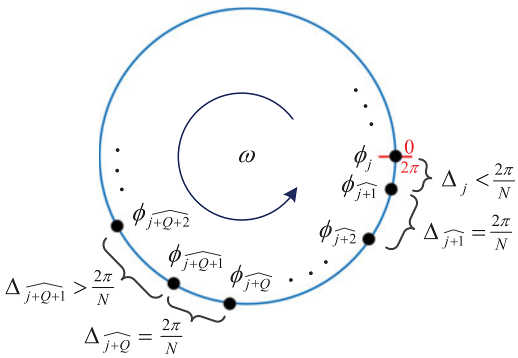

Fig. 17. **State A** on which Q ($0 \leq Q \leq N-2$) phase differences between the smaller phase difference and the larger phase difference are equal to $\frac{2\pi}{N}$.

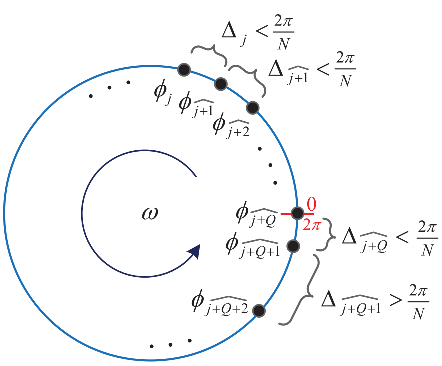

Fig. 18. **State B** on which Q ($0 \leq Q \leq N-2$) phase differences between the smaller phase difference and the larger phase difference become smaller than $\frac{2\pi}{N}$.

A become smaller than $\frac{2\pi}{N}$, meaning that the condition in **Case 3''** is not satisfied when oscillator $\widehat{j+Q}$ fires because phase $\phi_{\widehat{j+Q+1}}$ resides in $(2\pi - \frac{2\pi}{N}, 2\pi)$ and $\Delta_{\widehat{j+Q+1}} > \frac{2\pi}{N}$ is true.

Now we illustrate how those phase differences equal to $\frac{2\pi}{N}$ become smaller than $\frac{2\pi}{N}$ after Q pulses. Suppose that at $t = t_j$, an "active pulse" is emitted by oscillator j. This pulse only affects $\phi_{\widehat{j+1}}$ since only $\phi_{\widehat{j+1}}$ resides in $(2\pi - \frac{2\pi}{N}, 2\pi)$ and it decreases the value of $\phi_{\widehat{j+1}}$ since $\phi^+_{\widehat{j+1}} - \phi_{\widehat{j+1}} = l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{j+1}}) < 0$ holds, which in turn makes $\Delta_{\widehat{j+1}}$ smaller than $\frac{2\pi}{N}$ after this pulse. Because $\phi_{\widehat{j+2}}, \ldots, \phi_{\widehat{j+Q+2}}$ reside in $[\frac{2\pi}{N}, 2\pi - \frac{2\pi}{N}]$, they will keep unchanged, meaning that $\Delta_{\widehat{j+Q+1}}$ is still larger than $\frac{2\pi}{N}$, and $\Delta_{\widehat{j+2}}, \ldots, \Delta_{\widehat{j+Q}}$ are equal to $\frac{2\pi}{N}$. Therefore, after one firing, the number of phase differences equal to $\frac{2\pi}{N}$ is reduced by one to $Q-1$, as illustrated in Fig. 19. Following the same line of reasoning, we can obtain that after Q firings, the phase differences equal to $\frac{2\pi}{N}$ in **State A** become smaller than $\frac{2\pi}{N}$, which means that **State B** is achieved.

So **State A** must evolve to **State B**, and thus the condition in **Case 3''** cannot always exist before achieving phase desynchronization because under **State B** $\Delta_{\widehat{j+Q+1}} > \frac{2\pi}{N}$ will be true when PCO $\widehat{j+Q}$ fires.

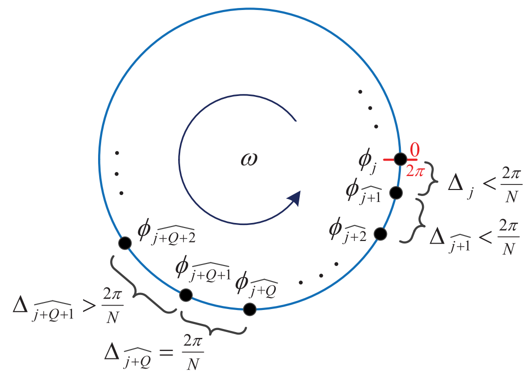

Fig. 19. $Q-1$ ($0 \leq Q \leq N-2$) phase differences between the smaller phase difference and the larger phase difference are equal to $\frac{2\pi}{N}$ after one firing.

Therefore, **Case 3'** and **Case 3''** cannot always exist before phase desynchronization is achieved. ■

Appendix E
Proof of Corollary 1

Proof: Since $l_1 = 0$ and $0 < l_2 < 1$ hold, the PRF $F(\phi_k)$ in (2) and the phase update rule in (3) can be rewritten as:

$$F(\phi_k) = \begin{cases} 0 & 0 < \phi_k \leq 2\pi - \dfrac{2\pi}{N} \\ -l_2(\phi_k - (2\pi - \dfrac{2\pi}{N})) & 2\pi - \dfrac{2\pi}{N} < \phi_k < 2\pi \end{cases} \tag{33}$$

and

$$\phi_k^+ = \begin{cases} \phi_k & 0 < \phi_k \leq 2\pi - \dfrac{2\pi}{N} \\ (1-l_2)\phi_k + l_2(2\pi - \dfrac{2\pi}{N}) & 2\pi - \dfrac{2\pi}{N} < \phi_k < 2\pi \end{cases} \tag{34}$$

It is worth noting that the phase update rule in (34) is the same as the phase update rule in [18]. So the rule in [18] is a special case of our phase update rule in (3) for $l_1 = 0$. As pointed out in [22], there is still a lack of rigorous mathematical proof for the convergence of the desynchronization algorithm in [18] since the proof of Corollary 1 in [18] did not provide clear condition where the cardinality must decrease. Now we give a rigorous proof for the convergence of phase desynchronization under this situation.

Following the same line of reasoning as the proof of Theorem 1, P will keep unchanged between two consecutive pulses, so we only need to concentrate on how P evolves at discrete-time instants when pulses are emitted. If there are no phase variables within $(2\pi - \frac{2\pi}{N}, 2\pi)$, then all the phase differences will not change, neither will P. Thus we only need to consider the situation in which there are some phase variables within $(2\pi - \frac{2\pi}{N}, 2\pi)$. If $N-1$ phase variables are within $(2\pi - \frac{2\pi}{N}, 2\pi)$, then this situation becomes the same as **Situation 5** in Remark 3. According to (22), we have $P^+ - P < 0$, which means P will decrease under this situation. If there are S phase variables within $(2\pi - \frac{2\pi}{N}, 2\pi)$ (cf. Fig. 20), where S satisfies $1 \leq S \leq N-2$, then according to (34), the phase

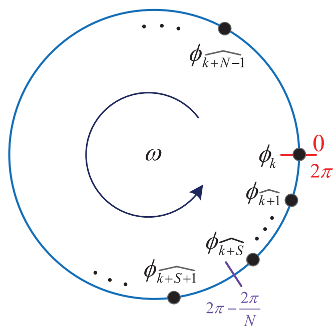

Fig. 20. The phase variables $\phi_{\widehat{k+1}}, \ldots, \phi_{\widehat{k+S}}$ reside in $(2\pi - \frac{2\pi}{N}, 2\pi)$ when oscillator k sends the first "active pulse" at $t = t_k$.

differences after the update can be rewritten as:

$$\begin{cases} \Delta_k^+ & = 2\pi - (1-l_2)\phi_{\widehat{k+1}} - l_2(2\pi - \dfrac{2\pi}{N}) \\ \Delta_{\widehat{k+j}}^+ & = (1-l_2)(\phi_{\widehat{k+j}} - \phi_{\widehat{k+j+1}}),\ j = 1, \ldots, S-1 \\ \Delta_{\widehat{k+S}}^+ & = (1-l_2)\phi_{\widehat{k+S}} + l_2(2\pi - \dfrac{2\pi}{N}) - \phi_{\widehat{k+S+1}} \\ \Delta_{\widehat{k+q}}^+ & = \phi_{\widehat{k+q}} - \phi_{\widehat{k+q+1}},\ q = S+1, \ldots, N-2 \\ \Delta_{\widehat{k+N-1}}^+ & = \phi_{\widehat{k+N-1}} \end{cases} \tag{35}$$

Combining (4), (5), (7), and (35) leads to:

$$P^+ - P$$
$$= |\Delta_{\widehat{k+S}}^+ - \frac{2\pi}{N}| - |\Delta_{\widehat{k+S}} - \frac{2\pi}{N}| + l_2(2\pi - \frac{2\pi}{N} - \phi_{\widehat{k+S}}) \tag{36}$$

which is the same as **Part B** in (9). According to the proof of Theorem 1, P will decrease under **Case 1''** and **Case 2''** and keep unchanged under **Case 3''**.

Now we show that **Case 3''** cannot always exist before the achievement of phase desynchronization. As illustrated in Fig. 17, **State A** always exists unless phase desynchronization is achieved. Note that different from the proof of Lemma 3, we do not need to consider the relationship between $\phi_{\widehat{j+Q+2}}$ and $\frac{2\pi}{N}$ since $l_1 = 0$ holds. When oscillator j fires, it emits an "active pulse." This pulse only affects $\phi_{\widehat{j+1}}$ since only $\phi_{\widehat{j+1}}$ resides in $(2\pi - \frac{2\pi}{N}, 2\pi)$, and it decreases the value of $\phi_{\widehat{j+1}}$ according to (34). Thus $\Delta_{\widehat{j+1}}$ becomes smaller than $\frac{2\pi}{N}$, $\Delta_{\widehat{j+2}}, \ldots, \Delta_{\widehat{j+Q}}$ are still equal to $\frac{2\pi}{N}$, and $\Delta_{\widehat{j+Q+1}}$ is still larger than $\frac{2\pi}{N}$. Therefore, after one firing, the number of phase differences equal to $\frac{2\pi}{N}$ is reduced by one to $Q-1$, as illustrated in Fig. 19. Following the same line of reasoning, we can obtain that after Q firings, the phase differences equal to $\frac{2\pi}{N}$ in the **State A** become smaller than $\frac{2\pi}{N}$, which means that **State B** in Fig. 18 is achieved. Thus the condition in **Case 3''** cannot always exist before achieving phase desynchronization because under **State B** $\Delta_{\widehat{j+Q+1}} > \frac{2\pi}{N}$ will be true when PCO $\widehat{j+Q}$ fires.

Consequently, P will keep decreasing until it reaches 0, i.e., until phase desynchronization is achieved. Therefore, the PCOs will achieve phase desynchronization under the PRF (2) for $l_1 = 0$ and $0 < l_2 < 1$. ∎

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