MAT315 - Homework 1 / Homework 3

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3B

7. Pick a basis $\{v_1, v_2, \ldots, v_n\}$ of V and $\{w_1, w_2, \ldots, w_n, \ldots, w_m\}$ of W. ($\dim V \leq \dim W$)

   V has dimension at least two. So define the following two maps on V:

$$f_1\left(\sum_{i=1}^{n} a_i v_i\right) = a_n w_n + \frac{1}{2}\sum_{i=2}^{n-1} a_i w_i$$
$$f_2\left(\sum_{i=1}^{n} a_i v_i\right) = a_1 w_1 + \frac{1}{2}\sum_{i=2}^{n-1} a_i w_i$$

   Again, $n = \dim V \geq 2$ and so these sums on the right are not empty sums. Also, v_1 is distinct from v_n.

   These maps are well-defined because the representation of a vector as a linear combination of basis elements is unique. They are easily checked to be linear; this is done by the proof of Axler 3.4.

   Neither of these maps are injective, because the nonzero vector v_1 gets sent to zero by f_1 and the nonzero vector v_n gets sent to zero by f_2 (basis vectors are nonzero).

   However, their sum is the map:

$$(f_1 + f_2)\left(\sum_{i=1}^{n} a_i v_i\right) = a_1 w_1 + \frac{1}{2}\sum_{i=2}^{n-1} a_i w_i + \frac{1}{2}\sum_{i=2}^{n-1} a_i w_i + a_n w_n = \sum_{i=1}^{n} a_i w_i$$

and this map is injective, because a vector $\sum_{i=1}^{n} a_i v_i$ goes to zero under this map if and only if $\sum_{i=1}^{n} a_i w_i = 0$ (by definition). But since the vectors $\{w_1, \ldots, w_n\}$ are linearly independent (as a subset of the basis vectors), this only happens if $a_i = 0$ for all i, which happens if and only if the input vector $\sum_{i=1}^{n} a_i v_i = 0$. Thus $\ker(f_1 + f_2) = 0$ and so $f_1 + f_2$ is injective by Axler 3.15.

Thus the set of all non-injective linear maps is not closed under addition and so cannot be a subspace of $\mathcal{L}(V, W)$.

12. The null space is the set of all vectors (v_1, v_2, v_3, v_4) with $v_1 = 5v_2$ and $v_3 = 7v_4$.

    I claim this space is spanned by the vectors $5e_1 + e_2$ and $7e_3 + e_4$.

    Suppose $v = (5v_2, v_2, 7v_4, v_4)$ is any vector in the null space; obviously they all can be expressed this way.

    Then $v = v_2(5e_1 + e_2) + v_4(7e_3 + e_4)$, and so v is a linear combination of the two basis vectors. Since v was an arbitrary vector in the null space, the null space is spanned by these two vectors.

    Also, these two vectors are clearly linearly independent. If we have

    $$a(5e_1 + e_2) + b(7e_3 + e_4) = 0$$

    Then the left hand side is the vector $(5a, a, 7b, b)$. Since this vector equals zero, each component must be zero, and so in particular the second coordinate a and the fourth coordinate b must be zero. Thus the only linear combination of these vectors which gives zero is the one with all zero coefficients. Thus these two vectors are linearly independent.

    Thus $\dim \ker T$, the cardinality of a basis for $\ker T$, is 2. By the rank-nullity theorem,

    $$4 = \dim \mathbf{F}^4 = \dim \operatorname{im} T + \dim \ker T = \dim \operatorname{im} T + 2$$

    So $\dim \operatorname{im} T = 2$. But the codomain of T is $\mathbf{F}^2$. Now note $\operatorname{im} T$ is a subspace of $\mathbf{F}^2$ and is spanned by two linearly independent vectors. Since $\mathbf{F}^2$ has dimension 2, any two linearly independent vectors will span $\mathbf{F}^2$ by Axler 2.38. Thus $\operatorname{im} T = \mathbf{F}^2$ and so T is surjective.

19. Suppose T is injective and W is finite-dimensional. By Axler 3.22, the existence of this requires $\dim V \leq \dim W$.

Pick a basis $\{v_i\}$ of V. Then $\{T(v_i)\}$ is linearly independent in W. To prove this, choose a linear combination $\sum a_i T(v_i)$ summing to zero.

Then

$$\sum a_i T(v_i) = T\left(\sum a_i v_i\right) = 0$$

But because T is injective, the only value mapping to 0 under the map T is zero. Thus $\sum a_i v_i = 0$, and since $\{v_i\}$ is linearly independent, each a_i is zero. Thus we started with a linear combination of the vectors $\{T(v_i)\}$ and showed that if this combination is zero, all the coefficients must be zero.

By Axler 2.32, since W is finite dimensional, the linearly independent set $\{T(v_i)\}$ can be extended to a basis of W. Let this basis of W be composed of the vectors $\{T(v_i)\}$ and some other vectors $\{w_j\}$. Define a map $S : W \to V$ by:

$$S\left(\sum a_i T(v_i) + \sum b_j w_j\right) = \sum a_i v_i$$

This is well-defined because the representation of a vector as a combination of basis vectors is unique, and again the proof of Axler 3.4, with the right substitutions for the variables, shows linearity.

Now suppose a vector v is in V. Write $v = \sum a_i v_i$. Then

$$ST(v) = S\left(T\left(\sum a_i v_i\right)\right) = S\left(\sum a_i T(v_i)\right) = S\left(\sum a_i T(v_i) + \sum 0 v_j\right) = \sum a_i v_i = v$$

and since v was arbitrary, ST is the identity on V. This proves one direction.

Now suppose T has a left inverse S. Then let $u, v \in V$, If $Tu = Tv$ then $STu = STv$ and, since ST is the identity, we conclude $u = v$. Since this holds for any pair of vectors $u, v \in V$, we see T is injective.

27. The rank nullity theorem makes this easy. I'll do it the hard way.

Firstly, we may decompose any vector $v \in V$ as:

$$v = (v - Pv) + Pv$$

The first vector is in $\ker P$ because

$$P(v - Pv) = Pv - P(Pv) = Pv - P^2 v = Pv - Pv = 0$$

The second is clearly in $\operatorname{im} P$.

Thus $V = \ker P + \operatorname{im} P$.

This decomposition is also unique: suppose we have two decompositions of a vector v:

$$v = v_1 + Pv_2 = w_1 + Pw_2$$

where $v_1, w_1 \in \ker P$ (clearly $Pv_2, Pw_2 \in \operatorname{im} P$).

Then

$$Pv = P(v_1 + Pv_2) = Pv_1 + P^2 v_2 = 0 + Pv_2 = Pv_2$$

But also, by the same logic

$$Pv = P(w_1 + Pv_2) = Pw_1 + P^2 w_2 = 0 + Pw_2 = Pw_2$$

and so $Pw_2 = Pv_2$. Thus, by subtracting $Pv_2 = Pw_2$ from both sides, we see $v_1 = w_1$. Thus the two decompositions are the same. Thus every vector $v \in V$ decomposes uniquely into a sum of a vector in $\ker P$ and a vector in $\operatorname{im} P$. Thus $V = \ker P \oplus \operatorname{im} P$.

29. To do this, we will first prove Problem 28: if a linear map $D : P(\mathbf{F}) \to P(\mathbf{F})$ satisfies $\deg Dp = \deg p - 1$ for all non-constant polynomials p, then D is surjective.

Proof: We show by induction that the set $P_n(\mathbf{F})$ of polynomials with degree at most n is in the image of D. Since $P(\mathbf{F}) = \bigcup\limits_{n\in\mathbb{N}} P_n(\mathbf{F})$, this will have shown that the whole $P(\mathbf{F})$ is in $\operatorname{im} D$, and thus D will have been shown to be surjective.

Firstly, $P_0(\mathbf{F})$, the set of the constants, is in the image. To see this, note that x has degree 1 and is not constant. So Dx has degree zero, thus is a constant (and a nonzero constant at that, by our definition of degree!).

So now pick any constant polynomial a, which is canonically identified with the *number* a in $\mathbf{F}$. Since $\mathbf{F}$ is a field and $Dx \neq 0$, we can consider the multiplicative inverse $(Dx)^{-1}$.

Then $D\Big(a(Dx)^{-1}x\Big) = a(Dx)^{-1} \cdot Dx$ by linearity, and so

$$D(a(Dx)^{-1}x) = a$$

Thus for an arbitrary constant polynomial a, we found a polynomial $a(Dx)^{-1}x$ so that D sends $a(Dx)^{-1}x$ to a.

Now suppose $P_{n-1}(\mathbf{F}) \subset \operatorname{im} D$. Choose any polynomial r of degree $n+1$. Dr has degree n. Write $Dr = bx^n + s(x)$ where s has degree $n-1$. Choose a polynomial $\tilde{s}$ with $D\tilde{s} = s$; we can do this by the induction assumption. Clearly $\deg s = \deg \tilde{s} - 1$ and so $\deg \tilde{s} = n < n+1$. Thus $r - \tilde{s}$ has degree n. Furthermore,

$$D(r - \tilde{s}) = bx^n + s - D\tilde{s} = bx^n + s - s = bx^n$$

We know $b \neq 0$ because otherwise $\deg Dr < n$, and we assumed $\deg r = n+1$ and so $\deg Dr$ must be exactly n. Since $\mathbf{F}$ is a field, we may consider the number b^{-1} and note that:

$$D\Big(b^{-1}(r - \tilde{s})\Big) = b^{-1}(bx^n) = \Big(b^{-1}b\Big)x^n = 1x^n = x^n$$

Let $g = b^{-1}(r - \tilde{s})$. Thus we have $Dg = x^n$.

Now let $f = ax^n + p(x)$ with $\deg p \leq n-1$. Clearly any polynomial of degree n can be written this way. Now since $\deg p \leq n-1$, by the induction assumption $p \in \operatorname{im} D$. Pick $q \in P(\mathbf{F})$ so that $Dq = p$. Then note that:

$$D(ag + q) = aDg + Dq = ax^n + p = f$$

Thus we have shown that any polynomial f of degree n is in $\operatorname{im} D$. Since any polynomial of degree $\leq n-1$ was assumed to be in $\operatorname{im} D$, we conclude that if $\deg f \leq n$ then $f \in \operatorname{im} D$. Thus $P_n(\mathbf{F}) \subset \operatorname{im} D$, as required.

By induction, $P_n(\mathbb{F}) \subset \operatorname{im} D$ for all n, and so $\operatorname{im} D$ contains all of $P(\mathbf{F})$. $\operatorname{im} D \subset P(\mathbf{F})$, and so $\operatorname{im} D = P(\mathbf{F})$. Thus D is surjective.

Now we prove Problem 29.

Firstly, the map $E : q \mapsto 5q'' + 3q'$ is linear.

Proof:

$$\begin{multline*} E(aq + br) = 5(aq + br)'' + 3(aq + br)' = 5(aq' + br')' + 3(aq' + br') \\ = 5(aq'' + br'') + 3(aq' + br') = a(5q'') + b(5r'') + a(3q') + b(3r') \\ = a(5q'' + 3q') + b(5r'' + 3r') \end{multline*}$$

E also lowers the degree of non-constant polynomials by 1. This is because if q is not constant and has degree k, then q' has degree $k-1$. q'' will have degree at most $k-2$—either $k-2$ exactly if q' is not constant, or $-\infty$ (i.e. $q'' = 0$) if q' is constant. In either case, $3q'$ has degree $k-1$, and $5q''$ has degree at most $k-2$, and so their sum will have degree $k-1$.

Thus, E is surjective by Problem 28, and so for any p we can find a polynomial q so that $E(q) = 5q'' + 3q' = p$.