

Games and Boolean models - mid-term exam

Giuseppe Baudo

LM Informatics

University of Bologna

November 6, 2017

Instructions:

- Edit your work using the provided tex file. Hand in your work as a LaTeX-generated pdf file attached to an e-mail addressed to roxyjean at gmail.com, by the end (i.e. 24:00) of sunday 12 November 2017. Your name should appear both as the author above, and in the chosen tex/pdf files names.
- All solution methods and corresponding computations have to be carefully commented, either in English or in Italian. Any part of the work consisting of non-commented computations and/or expressions shall be disregarded.

Notes for the Teacher:

- Answers to exercises are given by proposition environments. When an answer (proposition) has a motivation it is given by a proof environment.

1 Exercise

For an even integer m , let $M = \{1, \dots, m\}$ and define $f : M \rightarrow M$ by

$$f(k) = \begin{cases} \frac{m}{2} + k & \text{if } 1 \leq k \leq \frac{m}{2}, \\ k - \frac{m}{2} & \text{if } \frac{m}{2} < k \leq m. \end{cases}$$

1. Characterize binary relation R^f on M defined by

$$R^f = \{(k, f(k)) : 1 \leq k \leq m\} \subset M \times M$$

in terms of (ir)reflexivity, (a/anti)symmetry, transitivity and completeness. Determine the number of 1s in Boolean matrix $\mathcal{M}^{R^f} \in \{0, 1\}^{m \times m}$ representing R^f , i.e. $\mathcal{M}_{kl}^{R^f} = \begin{cases} 1 & \text{if } (k, l) \in R^f, \\ 0 & \text{if } (k, l) \in M \times M \setminus R^f, \end{cases} \quad 1 \leq k, l \leq m.$

2. Identify a (\supseteq)-minimal rational preference (binary relation) R^{\succsim^*} satisfying $R^{\succsim^*} \supseteq R$. How many 1s are in Boolean matrix $\mathcal{M}^{R^{\succsim^*}}$? Also determine the corresponding ordered partition $\mathfrak{P}^{\succsim^*} = (A_1, \dots, A_{|\mathfrak{P}^{\succsim^*}|})$ of M .

1.1 Solution

1.1.1 Prerequisites

- Set: an abstract aggregate of elements.
- Mapping: a way to create new sets.
- mapping: a properties holding on element(s) of set.
- Binary relation: see [1], [2] and [5].

1.1.2 Notation

- M = a finite set of m elements.
- m = number of elements of set M .

1.1.3 Analysis of a function f

Given the binary relation R^f as above defined, we'll investigate its properties (symmetry, transitivity, etc.).

Anzitutto che cos'è la f ? La f assegna ad ogni elemento di M un elemento di stesso, pertanto si potrebbe trattare di una permutazione ovvero

an elements of the *symmetric group of degree n , denoted by S_n* [2].

Nel nostro caso $n = |M| = m$. Quanto detto non è proprio rigoroso in quanto si dovrebbe dimostrare che f è una permutazione ovvero dovrei fare vedere che la f è sia iniettiva che suriettiva. Di questo fatto me ne sono accorto svolgendo i calcoli sulla f ovvero andando a calcolare $f(0), \dots, f(m)$ per $|M|$ uguale a 4, 6, 8.

La f può essere pensata come suddivisa in due funzioni f_{part1} e f_{part2} e pertanto la prima cosa da fare è discernere quale delle due funzioni applicare a k quando quest'ultimo è passato alla funzione f in altre parole la scrittura $f(k)$ si potrebbe leggere come: quale funzione devo applicare a k ? Ebbene la funzione da applicare dipende da k , se $k \leq \frac{m}{2}$ applichiamo la f_{part1} , altrimenti applichiamo la f_{part2} . Chiaramente f_{part1} ed f_{part2} sono definite come:

$$f(k) = f_{part1}(k) = \frac{m}{2} + k$$

if $k \leq \frac{m}{2}$ first half elements of M , and,

$$f(k) = f_{part2}(k) = k - \frac{m}{2}$$

if $k > \frac{m}{2}$ second half elements of M

Cioè la prima metà di elementi di M viene calcolata con f_{part1} mentre la seconda metà di elementi di M viene calcolata con f_{part2} .

Proviamo a schematizzare:

Per $m = 4$, ossia $M = \{1, 2, 3, 4\}$ abbiamo che $\frac{m}{2} = 2$,

$$f(1) = 2 + 1 = 3$$

$$f(2) = 2 + 2 = 4$$

$$f(3) = 3 - 2 = 1$$

$$f(4) = 4 - 2 = 2$$

che posso rappresentare in forma di matrice:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

La matrice precedente la leggiamo in questo modo: nella prima riga ci sono i valori di k , mentre nella seconda riga sono riportati i valori di $f(k)$. Come si può notare da questo primo svolgimento, ma dopo cercheremo di dimostrarlo algebricamente, la relazione R^f è certamente simmetrica.

Per $m = 6$, (saltiamo da 4 a 6 perchè l'esercizio richiede che m sia pari), ossia $M = \{1, 2, 3, 4, 5, 6\}$ abbiamo che $\frac{m}{2} = 3$,

$$f(1) = 3 + 1 = 4$$

$$f(2) = 3 + 2 = 5$$

$$f(3) = 3 + 3 = 6$$

$$f(4) = 4 - 3 = 1$$

$$f(5) = 5 - 3 = 2$$

$$f(6) = 6 - 3 = 3$$

Che possiamo rappresentare sotto forma di matrice come

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

La funzione f potrebbe essere vista anche come $k \equiv f(k) \pmod{\frac{m}{2}}$ e con quest'ultima espressione ...

1.1.4 Analysis of R^f properties

Proposition 1. *The binary relation R^f is symmetric, intransitive and incomplete.*

Proof. Symmetry. Symmetry seems to be trivial but we need to show that $(k, f(k)) \in R^f \implies (f(k), k) \in R^f$.

Thinking R^f as $R^f = \{(a, b) \wedge (b, a) : b = a + \frac{m}{2}, \forall a, b \in M, \} \subseteq M \times M$

The reason could be because congruences are symmetric but we need to show to many things in order to prove the proposition.

Transitivity. NO, infatti posso trovare due ennuple $(k, f(k)), (f(k), f(f(k))) \in R^f$

tali che $(k, f(f(k))) \notin R^f$. E.g. se prendo $(1, 3), (3, 1) \in R^{f^4}$, dove f^4 rappresenta la funzione f quando $m = 4$, la ennupla $(1, 1) \notin R^{f^4}$.

Completeness. NO, infatti $(1, 2) \wedge (2, 1) \notin R^{f^4}$. □

1.1.5 Number of 1s in Boolean matrix representing R^f

Proposition 2. *There are m 1s in the boolean matrix representing R^f .*

2 Exercise

For player set $N = \{1, \dots, n\}$ and strategy set $S_i = \{0, 1\}$ for all $i \in N$, let

$$u_i(s) = u_i(s_i, s_{-i}) = \left(s_i - \sum_{j \in N} \frac{s_j}{n} \right)^2 \text{ for all strategy profiles } s \in \{0, 1\}^n.$$

1. Is this a common interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles $s, s' \in \{0, 1\}^n$ provide the same aggregate payoff, that is to say $\sum_{i \in N} u_i(s) = \sum_{i \in N} u_i(s')$. If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then determine all pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.
2. Regarding this as a congestion game with a 2-set $\{0, 1\}$ of facilities, denote by $u_0(k)$ the utility attained by playing 0 when the number of those playing 0 is k and by $u_1(k)$ the utility attained by playing 1 when the number of those playing 1 is k . Verify whether the game is monotone and, in particular, whether

$$u_0(k) - u_0(k+1) = u_1(k) - u_1(k+1)$$

for all $1 \leq k < n$. For $1 < k < n$, denote by $s_0^k \in \{0, 1\}^n$ any of the $\binom{n}{k}$ strategy profiles where $k = |\{i : s_i = 0\}|$, and by $\mathbf{P} : \{0, 1\}^n \rightarrow \mathbb{R}$ the exact potential function. Determine $\mathbf{P}(s_0^k)$. Is there any relation between the set of strong equilibria and the set of equilibria (with non-random strategies)? How many equilibria are there?

3. Verify whether the n -tuple of random strategies $\frac{1}{2} \in [0, 1]^n$ where every $i \in N$ plays both 0 and 1 with equal probability, i.e. $\frac{1}{2}$, is an equilibrium.

2.1 Solution

2.1.1 Prerequisites

- See [3], [4], [5], [6]
- Preference relation. reflexive, transitive and complete.
- Preference aggregation. mainly [5];
- Common interest game. mainly [5];
- Potential game.
- Congestion game.
- Dominance.

2.1.2 Notation

- $\Gamma = (\mathbb{N}, \mathbb{S}, u_i)$. Γ è il gioco definito dall'esercizio.
- $\mathbb{N} = \{1, \dots, n\}$ = A set of n elements called players.
- $\mathbb{S}_i = \{0, 1\}$ = A set of 2 elements called strategies. A strategy can have many levels, in fact an element of \mathbb{S}_i can be another set of strategies and so on. For flat strategy set we use the name *alternative*. In the our game there are n strategy sets. Each element of the strategy set \mathbb{S}_i has value 0 or 1. Nevertheless, the process of value assignment can continue to infinity if we look at 0 and 1 not as number or as value of real set \mathbb{R} but as a name indicating a choice.
In altre parole, assumiamo che gli elementi di \mathbb{S}_i siano i numeri reali 0, 1 $\in \mathbb{R}$.
- $\mathbb{S} = \mathbb{S}_1 \times \dots \times \mathbb{S}_n$. Strategy profiles set. Insieme di ennuple (a_1, \dots, a_n) con $a_1, a_n \in \{0, 1\} \subseteq \mathbb{R}$. Insieme degli outcomes. Insieme dei prospetti. In condizione di completa informazione ogni giocatore conosce tutti i prospetti ed il rispettivo valore dato dalla sua funzione di utilità u_i .
- $s \in \mathbb{S}$, $s^* \in \mathbb{S}$

- s_i . Sia data $s \in \mathbb{S} = (s_1, \dots, s_n)$ una tupla, allora s_i indica l'iesimo elemento all'interno della tupla s . E.g. $s = (3, 6, 9, 45)$ allora $s_2 = 6$. Per fortuna tutti gli indici iniziano da 1. Sottolineiamo questo fatto perchè molto spesso in computer science and specifically in programming languages indices start from 0.
- s_{-i} = E.g. $s_{-2} = (3, 9, 45)$. Questa notazione serve per poter suddividere le componenti o coordinate del generico settore. Una volta distinte da diversi nomi le coordinate possono essere utilizzate nella definizione della funzione stessa.
- (s_i, s_{-i}) = E.g. $(6, (3, 9, 45)) = (3, 6, 9, 45)$.
- $u_i(s)$ = funzione di utilità dell'iesimo giocatore.

2.1.3 Is this a common interest game?

Definition 1. *Common interest games* are those where there is a strategy profile $s^* \in \mathbb{S}$ such that $s^* \succsim_i s$ for all $s \in \mathbb{S}$. [5].

In altre parole, nei common interest games esiste una strategia che è preferita da tutti i giocatori. In altre parole, il best response set di tutti i giocatori è diverso dall'insieme vuoto. E qui sta l'inghippo, l'intersezione non va fatta a gruppi di due giocatori, ma per tutti i giocatori. **Bisogna intersecare il best response set di tutti i giocatori.**

svolgimento ERRATO

Definition 2. A game Γ is a common interest game iif Best response strategy set for player i intersecting best response strategy set for player j is not empty.

In altre parole il comune interesse è modellato sull'intersezione di insiemi. Che potremmo assiomatizzare come segue:

Let A be a set, let B be a set then if $A \cap B = \emptyset$ indica che non c'è comune interesse.

Potremmo restringere la definizione sopra e considerare il caso ovvero l'insieme delle strategie in cui la utility function del giocatore i -esimo restituisce un valore maggiore di zero.

Proposition 3. Γ is a common interest game.

svolgimento CORRETTO

Proposition 4. Γ isn't a common interest game.

Proof. We'll prove that this proposition is false by constructing a set of strategy profiles that are elements of $BR_i \cap BR_j$ for all $i, j \in \mathbb{N}$. Those construction can be given by strategy profiles in which there are $k = 2$ quantità di 1s. Either for i and j playing 1s is the best responses when i or j play 1s and the others play 0. So, for all i and $j \in \mathbb{N}$. So Γ is a common interest game. **Contraddiction.** \square

2.1.4 Is this a constant-sum game?

Proposition 5. Γ is a constant-sum game.

Proof. For $n = 3$ players let be s, s^* two strategy profiles with $s^* = (0, 0, 0)$ and $s = (1, 0, 0)$ then the sum of the payoffs of the players is not equals for s and s^* .

$$u_1(s^*) + u_2(s^*) + u_3(s^*) \neq u_1(s) + u_2(s) + u_3(s)$$

Contraddiction □

Per i calcoli si rimanda al companion html/javascript file containing some code examples on how to calculate combination on a set of class k .

2.1.5 Are there Pareto-dominated strategy profiles?

Proposition 6. \mathbb{S} contains pareto-dominated strategy profiles.

Proof. For all games strategy profiles in which every player play 0 or every player play 1 are pareto-dominated strategy profiles because if a player deviates from its choice then obtain a plus and no other player can do less than they do. □

2.1.6 Regarding as congestion game

For $n = 4$

| s | $(u_1(s), u_2(s), u_3(s), u_4(s))$ | $u_1 + u_2 + u_3$ | num of 0s | num of 1s |
|------|------------------------------------|-------------------|-----------|-----------|
| 0000 | (0, 0, 0, 0,) | 0 | 4 | 0 |
| 0001 | (0.0625, 0.0625, 0.0625, 0.5625,) | 0.75 | 3 | 1 |
| 0010 | (0.0625, 0.0625, 0.5625, 0.0625,) | 0.75 | 3 | 1 |
| 0011 | (0.25, 0.25, 0.25, 0.25,) | 1 | 2 | 2 |
| 0100 | (0.0625, 0.5625, 0.0625, 0.0625,) | 0.75 | 3 | 1 |
| 0101 | (0.25, 0.25, 0.25, 0.25,) | 1 | 2 | 2 |
| 0110 | (0.25, 0.25, 0.25, 0.25,) | 1 | 2 | 2 |
| 0111 | (0.5625, 0.0625, 0.0625, 0.0625,) | 0.75 | 1 | 3 |
| 1000 | (0.5625, 0.0625, 0.0625, 0.0625,) | 0.75 | 3 | 1 |
| 1001 | (0.25, 0.25, 0.25, 0.25,) | 1 | 2 | 2 |
| 1010 | (0.25, 0.25, 0.25, 0.25,) | 1 | 2 | 2 |
| 1011 | (0.0625, 0.5625, 0.0625, 0.0625,) | 0.75 | 1 | 3 |
| 1100 | (0.25, 0.25, 0.25, 0.25,) | 1 | 2 | 2 |
| 1101 | (0.0625, 0.0625, 0.5625, 0.0625,) | 0.75 | 1 | 3 |
| 1110 | (0.0625, 0.0625, 0.0625, 0.5625,) | 0.75 | 1 | 3 |
| 1111 | (0, 0, 0, 0,) | 0 | 0 | 4 |

Inoltre,

$$\begin{aligned}
u_0(k) &= u_0(0) = 0 \\
u_0(k) &= u_0(1) = 0.5625 \\
u_0(k) &= u_0(2) = 0.25 \\
u_0(k) &= u_0(3) = 0.0625 \\
u_0(k) &= u_0(4) = 0
\end{aligned}$$

$$\begin{aligned}
&\text{And,} \\
u_1(k) &= u_1(0) = 0 \\
u_1(k) &= u_1(1) = 0.5625 \\
u_1(k) &= u_1(2) = 0.25 \\
u_1(k) &= u_1(3) = 0.0625 \\
u_1(k) &= u_1(4) = 0
\end{aligned}$$

So,

$$\begin{aligned}
u_0(0) - u_0(1) &= 0 - 0.5625 \\
u_1(0) - u_1(1) &= 0 - 0.5625
\end{aligned}$$

Then, $u_0(0) - u_0(1) = u_1(0) - u_1(1)$

3 Exercise

For $M = \{1, \dots, m\}$, consider the symmetric congestion game where every player $i \in N = \{1, \dots, n\}$ has strategy set $\mathbb{S}_i = \mathcal{K} \subset 2^{2^M}$ consisting of the $m!$ *maximal chains* $\{A_0, A_1, \dots, A_{m-1}, A_m\} \in \mathcal{K}$ of subsets of M . That is,

$$M = A_m \supset^* A_{m-1} \supset^* \dots \supset^* A_1 \supset^* A_0 = \emptyset, \text{ where}$$

$$A_k \supset^* A_{k-1} \Leftrightarrow A_k \supset A_{k-1}, |A_k| = |A_{k-1}| + 1 \quad (1 \leq k \leq m)$$

is the *covering relation*. Hence the set of facilities is $\{A : \emptyset \subset A \subset M\}$. For every strategy profile $s = (s_1, \dots, s_n) \in \mathcal{K}^n$, denote i 's strategy ($i \in N$) by

$$s_i = \{A_0, A_1^i, \dots, A_{m-1}^i, A_m\} \in \mathcal{K},$$

and define congestion vector $\{c_A(s) : \emptyset \subset A \subset M\} \in \mathbb{Z}_+^{2^m-2}$ by

$$c_A(s) = |\{i : A \in s_i\}|.$$

Finally, utilities have form

$$u_i(s) = \sum_{0 \leq k < m} \frac{1}{c_{A_k^i}(s)}.$$

In what follows, distinguish between cases (a) $n \leq m$ and (b) $n = m!$.

1. Is this a common interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles $s, s' \in \{0, 1\}^n$ provide the same aggregate payoff, that is to say $\sum_{i \in N} u_i(s) = \sum_{i \in N} u_i(s')$. If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then provide examples of pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.
2. Characterize the set of equilibria and the set of strong equilibria (with non-random strategies). Compute the value $\mathbf{P}(s)$ taken by the exact potential \mathbf{P} at any equilibrium s .
3. Verify whether the random strategy profile consisting of n uniform distributions over the $m!$ -set \mathcal{K} of maximal chains is an equilibrium or not.

4 Exercise

Let $M = \{1, \dots, 10\}$ and define $f : M \rightarrow \{0, 1\}$ by $f(i) = \begin{cases} 1 & \text{if } i \text{ is a prime,} \\ 0 & \text{otherwise.} \end{cases}$

Compute the discrete Choquet integral $E_\eta^C(f)$ of f with respect to fuzzy probability $\eta : 2^M \rightarrow [0, 1]$ defined by

$$\eta(A) = \binom{11}{2}^{-2} \left(\sum_{i \in A} i \right)^2 \text{ for all } A \in 2^M.$$

References

- [1] Herstein I.N. *Abstract Algebra*. Prentice-Hall Inc., Upper Saddle River, New Jersey, 1996.
- [2] Herstein I.N. *Algebra*. Editori riuniti, Roma, 1999.
- [3] Holzman R., Law-Yone N. *Strong equilibrium in congestion games*. Games and Economic Behavior, (21):85-101, 1997.
- [4] Rosenthal R.W. *A class of games possessing pure-strategy Nash equilibria*. International Journal of Game Theory, 2: 6567, 1973, MR 0319584, doi:10.1007/BF01737559.
- [5] Rossi G. *Games and Boolean models. Lecture notes*. University of Bologna, Bologna, 2017.

- [6] Voorneveld, M. *Potential games and interactive decisions with multiple criteria*. Tilburg University: CentER, Center for Economic Research, 1999.