# Games and Boolean models - mid-term exam

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LM Informatics or Mathematics or other (specify)

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*Instructions:*

- Edit your work using the provided tex file. Hand in your work as a LaTex-generated pdf file attached to an e-mail addressed to roxyjean at gmail.com, by the end (i.e. 24:00) of sunday 12 November 2017. Your name should appear both as the author above, and in the chosen tex/pdf files names.
- All solution methods and corresponding computations have to be carefully commented, either in English or in Italian. Any part of the work consisting of non-commented computations and/or expressions shall be disregarded.

## 1 Exercise

For an even integer $m$, let $M = \{1, \ldots, m\}$ and define $f : M \to M$ by

$$f(k) = \begin{cases} \frac{m}{2} + k \text{ if } 1 \leq k \leq \frac{m}{2}, \\ k - \frac{m}{2} \text{ if } \frac{m}{2} < k \leq m. \end{cases}$$

1. Characterize binary relation $R^f$ on $M$ defined by

$$R^f = \{(k, f(k)) : 1 \leq k \leq m\} \subset M \times M$$

in terms of (ir)reflexivity, (a/anti)symmetry, transitivity and completeness. Determine the number of 1s in Boolean matrix $\mathcal{M}^{R^f} \in \{0,1\}^{m \times m}$ representing $R^f$, i.e. $\mathcal{M}^{R^f}_{kl} = \begin{cases} 1 \text{ if } (k,l) \in R^f, \\ 0 \text{ if } (k,l) \in M \times M \backslash R^f, \end{cases} \quad 1 \leq k, l \leq m.$

2. Identify a ($\supseteq$-)minimal rational preference (binary relation) $R^{\succsim^*}$ satisfying $R^{\succsim^*} \supseteq R$. How many 1s are in Boolean matrix $\mathcal{M}^{R^{\succsim^*}}$? Also determine the corresponding ordered partition $\mathfrak{P}^{\succsim^*} = (A_1, \ldots, A_{|\mathfrak{P}^{\succsim^*}|})$ of $M$.

## 2 Exercise

For player set $N = \{1, \ldots, n\}$ and strategy set $\mathbb{S}_i = \{0, 1\}$ for all $i \in N$, let

$$u_i(s) = u_i(s_i, s_{-i}) = \left( s_i - \sum_{j \in N} \frac{s_j}{n} \right)^2 \quad \text{for all strategy profiles } s \in \{0,1\}^n.$$

1. Is this a common interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles $s, s' \in \{0,1\}^n$ provide the same aggregate payoff, that is to say $\sum_{i\in N} u_i(s) = \sum_{i\in N} u_i(s)$. If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then determine all pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.

2. Regarding this as a congestion game with a 2-set $\{0,1\}$ of facilities, denote by $u_0(k)$ the utility attained by playing 0 when the number of those playing 0 is $k$ and by $u_1(k)$ the utility attained by playing 1 when the number of those playing 1 is $k$. Verify whether the game is monotone and, in particular, whether
$$u_0(k) - u_0(k+1) = u_1(k) - u_1(k+1)$$
for all $1 \leq k < n$. For $1 < k < n$, denote by $s_0^k \in \{0,1\}^n$ any of the $\binom{n}{k}$ strategy profiles where $k = |\{i : s_i = 0\}|$, and by $\mathbf{P} : \{0,1\}^n \to \mathbb{R}$ the exact potential function. Determine $\mathbf{P}(s_0^k)$. Is there any relation between the set of strong equilibria and the set of equilibria (with non-random strategies)? How many equilibria are there?

3. Verify whether the $n$-tuple of random strategies $\mathbf{\frac{1}{2}} \in [0,1]^n$ where every $i \in N$ plays both 0 and 1 with equal probability, i.e. $\frac{1}{2}$, is an equilibrium.

# 3 Exercise

For $M = \{1,\ldots,m\}$, consider the symmetric congestion game where every player $i \in N = \{1,\ldots,n\}$ has strategy set $\mathbb{S}_i = \mathcal{K} \subset 2^{2^M}$ consisting of the $m!$ *maximal chains* $\{A_0, A_1, \ldots, A_{m-1}, A_m\} \in \mathcal{K}$ of subsets of $M$. That is,
$$M = A_m \supset^* A_{m-1} \supset^* \cdots \supset^* A_1 \supset^* A_0 = \emptyset, \text{ where}$$
$$A_k \supset^* A_{k-1} \Leftrightarrow A_k \supset A_{k-1}, |A_k| = |A_{k-1}| + 1 \ (1 \leq k \leq m)$$
is the *covering relation*. Hence the set of facilities is $\{A : \emptyset \subset A \subset M\}$. For every strategy profile $s = (s_1, \ldots, s_n) \in \mathcal{K}^n$, denote $i$'s strategy $(i \in N)$ by
$$s_i = \{A_0, A_1^i, \ldots, A_{m-1}^i, A_m\} \in \mathcal{K},$$
and define congestion vector $\{c_A(s) : \emptyset \subset A \subset M\} \in \mathbb{Z}_+^{2^m-2}$ by
$$c_A(s) = |\{i : A \in s_i\}|.$$

Finally, utilities have form

$$u_i(s) = \sum_{0<k<m} \frac{1}{c_{A_k^i}(s)}.$$

In what follows, distinguish between cases (a) $n \leq m$ and (b) $n = m!$.

1. Is this a common interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles $s, s' \in \{0,1\}^n$ provide the same aggregate payoff, that is to say $\sum_{i \in N} u_i(s) = \sum_{i \in N} u_i(s)$. If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then provide examples of pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.

2. Characterize the set of equilibria and the set of strong equilibria (with non-random strategies). Compute the value $\mathbf{P}(s)$ taken by the exact potential $\mathbf{P}$ at any equilibrium $s$.

3. Verify whether the random strategy profile consisting of $n$ uniform distributions over the $m!$-set $\mathcal{K}$ of maximal chains is an equilibrium or not.

# 4 Exercise

Let $M = \{1, \ldots, 10\}$ and define $f : M \to \{0,1\}$ by $f(i) = \left\{ \begin{array}{c} 1 \text{ if } i \text{ is a prime,} \\ 0 \text{ otherwise.} \end{array} \right.$
Compute the discrete Choquet integral $E_\eta^C(f)$ of $f$ with respect to fuzzy probability $\eta : 2^M \to [0,1]$ defined by

$$\eta(A) = \binom{11}{2}^{-2} \left( \sum_{i \in A} i \right)^2 \quad \text{for all } A \in 2^M.$$