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Introduction to Game Theory

- Game theory "begins" in 1944 with the book [32] Games and Economic Behavior by von-Neumann and Morgestern.
- In 1953 Shapley publishes a fundamental paper [27] defining cooperative games, so that the former ones have been named "non-cooperative" (or strategic) ones thereafter.
- Given a set $N = \{1, ..., n\}$ of n players, a non-cooperative game consists of a product space $S_1 \times ... \times S_n$ of strategies, and n utilities or payoff functions $u_i : S_1 \times ... \times S_n \to R$, $1 \leq i \leq n$ measuring the "goodness" of strategy profiles $s \in S_1 \times ... \times S_n$ to players $i \in N$. This is the branch of game theory where the famous prisoner's dilemma and Nash equilibrium apply. On the other hand, a cooperative (coalitional) game is a set function $v : 2^N \to R_+$ such that $v(0) = 0$, where $2^N = \{A : A \subseteq N\}$ is the 2^n-set of coalitions A or subsets of N. Specifically, $v(A)$ is thought of as the worth of cooperation among all (and only) players $i \in A$ (or coalition members).
    1. Perchè 2?
    2. Ha senso la coalizione in cui sono presenti tutti i players?
- ...

Capitolo 4

Preferences

4.1 Preferences

- In non-cooperative games (see above), the product space $S_1 \times ... \times S_n = X$ of strategies, over which every player $i \in N$ has preferences in the form of a utility function $u_i : S_1 \times ... \times S_n \to R$, is finite (unless otherwise specified).

4.1.1 Rational preferences

- The primitive ingredient of a choice problem is a set X of alternatives, which may be finite or infinite, and in this latter case either countable or uncountable.

  1. L'insieme X di questo punto e the product space $S_1 \times ... \times S_n = X$ sono due cose completamente diverse, giusto?
  2. Qui X è un generico insieme e quindi potrebbe anche essere quello del punto precedente? L'importante, come vedremo, che sia definita una relazione di preferenza?

- When denoting a finite alternative set by $X = X = \{x_1, ..., x_m\}$, the first m natural numbers $1, ..., m \in N$ are used as distinct "names" for the $|X| = m$ distinct alternatives.

4.1.2 Utility representation

Capitolo 5

Randomness

- The next step in the study of non-cooperative games is the understanding of strategies. As we shall see, the famous Nash equilibrium surely exists only when players $i \in N$ may each randomize over their finite strategy sets $S_1 = \{s_i^1, ..., s_i^{|S_i|}\}$.

5.1 Discrete random variables: lotteries

- ...
- By the way, also recall that a probability distribution over X is defined as any set function $p : 2^X \to [0, 1]$ satisfying $p(A) + p(B) = p(A \cap B) + p(A \cup B)$ for any two events or subsets (of elementary mutually exclusive events) $A, B \in 2^X$, as well as $p(X) = 1$, $p(\varnothing) = 0$; then, a main theorem on valuations of distributive lattices (such as Boolean lattice $2^X, \cap, \cup)$) $[1, p.190]$ entails $p(A) = \sum_{i \in A} p(\{i\})$ for all $A \in 2^X$ (this will be detailed when dealing with the *solution* of coalition (cooperative) games $v : 2^N \to R$, see below).

  1. Che significa $p(A) + p(B) = p(A \cap B) + p(A \cup B)$?
  2. $[1, p.190]$ cosa dovrei trovare? non ho capito?

5.2 Probabilities, set functions and voting games

- Recall that the quantitative notion of probability is associated with *events or subsets* $A \in 2^X$ *of elementary, mutually exclusive (atomic) events.* In

Capitolo 6

Strategies v. 10/10/2017

- In simultaneous-move games all player move or take action simultaneously, hence choosing a strategy is the same as choosing an action. This is no longer true in multistage games, where choosing a strategy means choosing a sequence of (conditional) actions. Although in this course non-cooperative games shall be dealt with only in simultaneous-move form, still multistage games are briefly introduced hereafter to formalize both a general definition of strategies and the notion of (in)complete information.
- I - baudo - Multistage games vengono introdotti per generalità e per modellare la nozione di (in)complete information.
- before describing multistage games, the simultaneous-move setting enables to distinguish beetween outcomes of the game and strategy/action profiles. To this end, let each player $i \in N$ choose an action from a finite set $A_i = \{a_i^{1,...,a_i^{|A_i|}}\}$, where $|A_i| \geq 2$ for all $i \in N$. The product space $A = A_1 \times ... \times A_n$ contains all n-tuples or profiles of actions, with generic element $a = (a_1, ..., a_n) \in A$. Preferences

6.0.1 Multistage games

6.0.2 Dominated and dominant strategies

Strategy deletion

Prisoner's dilemma

6.1 Randomization and expected payoffs

6.1.1 Mixed strategies

6.1.2 Domination in mixed strategies

6.1.3 Best responses

6.1.4 Nash equilibrium

6.1.5 Strong equilibrium

Capitolo 7

Strategies v. 26/10/2017

Il concetto di strategia varia a seconda del tipo di gioco. Cosa vuol dire che il concetto varia? Vuol dire che a seconda del gioco è rappresentato da un certo tipo di oggetto matematico. Quasi sempre comunque la strategia è un elemento di un insieme più o meno complesso. Quindi secondo questo ragionamento l'alternative set, strategy set, action profiles, etc. sono tutte strutture matematiche che possono essere annoverate tra quelle che rappresentano il concetto di strategia del mondo reale ed i cui elementi sono, appunto, strategie.

Cominciamo con gli appunti del Prof. Giovanni Rossi.

- In *simultaneous-move games* alla players move only once, simultaneously, hence choosing a strategy is the same as choosing a move. This is no longer true in *multistage games*, where choosing a strategy means choosing a *sequence of (conditional) moves*. Although the non-cooperative games to be dealt with shall be in simultaneous-move form, still multistage games are briefly described below in order to formally define strategies in a most general setting, namely where players have either perfect or else incomplete information, this latter being commonly modeled by means of partitions.

7.1 Information in multistage games

- As the name clearly suggests, multistage games are played in discrete time $t = 0, 1, ..., T$, as $t = 0$ is the starting point or *root of the game tree* (defined hereafter), where some (at last one, and possibly all) players move; next, depending on previous moves, at each $t \geq 1$ a *node* is reached, corresponding either to a moment where at least one player has to move, or else to an end of the game or *leaf*. The concern is only with games where $T < \infty$ (for any leaf).
- Multistage games are thus commonly represented by a *rooted and directed (game) tree* $\mathfrak{T} = (V, E)$

7.2 Dominated and dominant strategies

7.3 Deletion of dominated strategies

7.4 Equilibrium

Capitolo 8

Esercitazione 1

Nell'esercizio vedremo alcuni concetti visti durante la prima parte del corso.

Capitolo 9

Types of cooperative games

- Although in the 70s attention has also been placed on cooperative games with a continuum of players in terms of measure theory (see [5] and related literature), nowadays cooperative games are for the most part dealt with in terms of a finite player set, usually denoted by $N = \{1, ..., n\}$. In particular, these games are approached through discrete mathematics as poset/lattice functions. That is, as real-valued functions defined on finite ordered structures.

  1. Continuum of players vuol dire insieme infinito di giocatori oppure insieme finito/infinito di giocatori che può crescere fino all'infinito?
  2. Nel caso moderno, cioè attraverso l'utilizzo di insiemi finiti e ordinati, nei ragionamenti l'insieme iniziale dei giocatori rimane fisso oppure può crescere?
  3. Measure theory? A measure is a generalization of the concepts of length, area, and volume.
  4. In [5] what's "value concept"?
  5. Edgeworthian?
  6. Per il momento questo mi basta!!! Il punto introduce quello che servirà sapere in seguito: ordered set/lattices e funzioni definite a partire da questi insiemi all'insieme dei numeri reali. Per ulteriori approfondimenti sui giochi with a continuum of players vedi [5] e letteratura affine.
  7. Since about 1960, attention has focused more and more on games with large masses of players, in which no individual player can affect the overall outcome. Such games arise naturally in the social sciences, as models for situations in which there are large numbers of very "small" individuals, like consumers in an economy or voters in an election. Mathematically, it is often convenient to represent these games with the aid of a "continuum" of players - like the continuum of points on a line or the continuum of drops in a liquid. Represented thus, such games are called

     non-atomic.[5]

     Quindi i nostri giochi sono atomici?

- Historically, the first cooperative games were defined in 1953 [27] as set functions $v : 2^N \rightarrow R_+, v(0) = 0$, with subsets $A \in 2^N$ referred to as coalitions (of players). These games may thus be called coalition games, although in many articles and books they are simply named cooperative games, as if exhausting the whole class of cooperative games.

  1. Coalition games sono un tipo di cooperative game?
  2. 2^N ? qual è il significato di questa notazione?
  3. Prospetto?

4. Essential games?

- Subsequently, in 1963, a further type of cooperative games entered the picture, involving partitions of players or coalition structures, i.e. partitions $P = \{A_1, ..., A_{|P|}\}$ of N. In particular, these second-generation cooperative games were named games in partition function form, and they are real-valued functions defined on pairs (A, P) such that $A \in 2^N$ and P is a partition of N such that $A \in P$. These pairs (A, P) are now referred to as "embedded coalitions" (or "embedded subsets" [13, 14]). These games pose serious problems in terms of lattice theory, as the corresponding ordered structure (i.e. of embedded subsets) currently needs ad hoc techniques for yielding a lattice (which in any case is not a geometric one, see below).

  1. ...

- Finally, in 1990, a third type of cooperative games was introduced and named "global games" [12]. These are simply real-valued partition functions, but still lead to embarassing results when it comes to define and quantify the so-called "solution". Roughly speaking, a solution of a cooperative games should determine the a priori worth, for each player, of playing the game. Somehow overcoming the mainstream literature, in the sequel we shall interpret solutions of cooperative games (of any kind) in terms Ḿobius inversion and atomic/geometric lattices.

  1. "of any kind" si rifesce ai tre tipi di giochi cooperativi o a tutti i giochi, sia cooperativi che non cooperativi?

Capitolo 10

Order: posets and lattices

10.1 Order: posets and lattices

- ...
- Our concern is only with posets or ordered structures which are:
  1. finite, i.e. $|X| < \infty$,
  2. with a bottom element $x_\perp \in X, i.e. x \geq x_\perp$ for all $x \in X$,
  3. with a top element $x^\top \in X$, i.e. $x^\top \geq x$ for all $x \in X$.

  In particular, in the sequel our main concern shall be with the poset $(X, \geq)$ given by $(2^N, \supseteq)$ for a finite (player set) $N = \{1, ..., n\}$.
  1. Quindi il top del poset $(2^N, \supseteq)$ is equals to 2^N and the bottom of $(2^N, \supseteq)$ is equals to ‰.
- ...
- For all $x, y \in X$, the corresponding ***interval*** (or ***segment*** [25]) is the subset $[x, y] = \{z : x \leq z \leq y\} \subseteq X$; hence $x \leq y \Rightarrow [x, y] \neq$ ‰ while $[y, x] =$ ‰.
- D - (baudo) - A partially ordered set is **locally finite** if each of its intervals has only finitely many elements.
- A chain is a subset $K \subset X$ any two of whose elements are comparable, i.e. for all $x, y \in K$, either $[x, y] \neq$ ‰ or else $[y, x] \neq$ ‰ hold.
- Dually, an antichain is a subset $AK \subset X$ any two of its elements are uncomparable, i.e. for all $x, y \in AK$, both $[x, y] =$ ‰ and $[y, x] =$ ‰ hold.
- The length of a chain $K = \{x_0, ..., x_k\}$ is $|K| - 1 = k$.
- The covering relation, denoted by $> *$, is defined as follows:
  $x > *y \Leftrightarrow [y, x] = \{x, y\}$ (where $\{x, y\} = \{y, x\}$) for all $x, y \in X$.
  1. Vorrei capire la direzione/terminologia. Anche rispetto al libro!?
  2. Todo - copiare appunti dal quaderno
- For $z \geq y$, a $(z - y)$-chain $K_*^{z-y} = \{y = x_0, x_1, ..., x_k = z\}$ is said to be maximal if $x_l > *x_{l-1}$ for all $0 < l \leq k$.
  1. from the free dictionary: A sequence of $n + 1$ subsets of a set of n elements, such that the first member of the sequence is the empty set and each member of the sequence is a proper subset of the next one.
- If for any $y, z \in X$ all maximal $(z - y)$-chains have the same length, then poset $(X, \geq)$ is said to satisfy the Jordan-Dedekind JD condition, in which case for every element $x \in X$ the length of any maximal $(x - x_\perp)$-chain is the *rank* of x. Formally, for any poset $(X, \geq)$ with bottom element $x_\perp$ and satisfying the JD condition, the rank function $r : X \to Z_+$ is defined recursively by

1. $r(x_\perp) = 0$
2. $x > *y \Rightarrow r(x) = r(y) + 1$.

Thus the rank measures the height of elements (in the Hasse diagram, see above and below).

APPUNTI UTILI PER QUESTA SEZIONE

- Let P be an ordered set. We say P has a **top** element if there exists $\top \in P$ with the property that $x \leq \top$ for all $x \in P$.
- uniquiness of the top (think of duality), antisymmetry etc.? Il top dovrebbe essere unico perchè se supponiamo che esista un altro top $t2$ tale che quindi $x \leq t2$ for all $x \in P$ ma allora si avrebbe $\top \leq t2 \Rightarrow t2! \leq \top$ per la proprietà antisimmetrica pertanto siamo giunti ad una contraddizione perchè avevamo supposto $\top$ essere un top.
- Let P be an ordered set and let $S \subseteq P$. An element $x \in P$ is an **upper bound** of S if $s \leq x$ for all $s \in S$.
- Upper bound is unique when it exists.
- Quindi posso dire che un top è un upper bound che sta dentro l'insieme P? Insomma che differenza cè tra upper bound e top?
- In a partially ordered set, an element p emphcovers an element q when the segment $[q, p]$ contains two elements. $[25, 343]$
- A lattice is a partially ordered set where max and min of two elements (we call them join and meet, as usual, and write $\vee$ and $\wedge$) are defined. $[25, 342]$
- A **segment** $[x, y]$, for x and y in a partially ordered set P, is the set of all elements z between x and y, that is, such that $x \leq z \leq y$. ... . A segment is endowed with the induced order structure; thus, a segment of a lattice is again a lattice. $[25, 342]$
- A partially ordered set is **locally finite** if every segment is finite. $[25, 342]$
- Let P be a non-empty ordered set.
  If $x \vee y$ and $x \wedge y$ exist fora all $x, y \in P$, then P is called a **lattice**.
  If $\vee S$ and $\wedge S$ exist for all $S \subseteq P$, then P is called a **complete lattice**.
- Totally ordered subsets of a poset play an important role in the theory of partial orders.
-

10.1.1 Maximal chains of subsets and permutations

10.1.2 Subset of Boolean lattices

Atomicity

Complementation

Capitolo 11

Möbius inversion

- Möbius inversion applies to any (locally finite) poset, provided a bottom element exists [25]. For the Boolean lattice $(2^N, \cap, \cup)$ of subsets of N ordered by inclusion $\supseteq$ and the geometric lattice $(2^N, \vee, \wedge)$ of partitions of N ordered by coarsening $\geq$ [1, 31], the bottom elements are, respectively, the empty set ‰ and the fines partition $P_\perp = 1, ..., n$.
    1. Locally finite poset - where all intervals are finite
    2. Boolean lattice -
    3. Geometric lattice

Capitolo 12

Incidence algebra

12.0.1 Incidence algebra

12.0.2 Möbius inversion

12.0.3 Vector spaces and bases

12.0.4 Lattice functions

Set functions and Boolean or Pseudo-Boolean functions

Polynomial multilinear extension of set functions

12.1 Formulario di Teoria dei Giochi

A, B, X, R, N, etc. for SETS a, b, x, r, n, etc. for ELEMENTS of above sets i, j, n, x, y, z, etc for INDICES of SETS or ELEMENTS (), for function application and tuples Indices are use to iterate over a set or to name the object to which belongs.

Doesn't exist a way to simulate hash map in mathematics.

Dobbiamo inventare un operatore o una struttura dati in grado di rappresentare in modo funzionale ai calcoli un hash map ovvero un oggetto che possiede, si porta dientro con se a sua volta un insieme.

Vi è quasi una certa ridondanza in questo fatto.

Ma in realtà si tratta solo di comprendere il significato dell'applicazione/funzione/mappa che assegna ad ogni elemento di un insieme, un altro insieme.

```
%% ATTENZIONE
l'indice n a volte seve per dare un nome e un numero, altre volte solo per dare un numero,
per esempio nella formula d_1, ..., d_n l'indice n significa prendo enne indici
senza che faccia specificatamente ad n che rappresenta il numero dei giocatori.
n indica due cose differenti!!! non sempre quando trovate n vuol dire che tale n si
intenda il numero di giocatori

%% VALUE
Si definisce value un qualunque numero reale.

    %% VOGLIAMO QUANTIFICARE (VALORIZZARE) - valori di R
    LET R be REAL SET
    LET v in R

    %% VOGLIAMO CONTARE (COUNT)            - valori di N
    LET n in N
    LET A be SET
    LET |A| := n
```

```
    %% VOGLIAMO RANDOMIZZARE. In questo caso occorre il value di un intero insieme.
    LET D be SET
    LET n INDICE of N
    LET D = {d_1, ..., d_n}
    LET [0,1] \in R
    LET d_1, ..., d_n \in [0,1]
    LET d_1 + ... + d_n = 1
    Ora che abbiamo costruito D (o DELTA) ovvero l'insieme randomizzatore
    lo possiamo utilizzare nei nostri calcoli.


LET A, B be SETS
LET B = {b}
LET b in R
LET {B} be an anonymous SET. In this case, {B} is a set with two elements:
B (which is a set) and 0 (the empty set).
LET a elements of A
LET value be a FUNCTION
LET value A \to {B} -- ABSTRACTION
LET value defined as a --> B in words: each elements of A is mapped to the same set B.
    This fact is useful for successive steps, counting how many of...
LET value(a) --> SUM of elements of B
```

Now we are able to define randomness

```
LET A, B_1, ... B_n , {B_1, ..., B_n} be SETS   %% Here n is just an INDICES
LET a elements of A
LET value be a FUNCTION
LET value A \to {B_1, ..., B_n} -- ABSTRACTION
LET value defined as
```

So we are conducted to the definition of value of an element of a set. Whenever I get an element of a set I can ask for its value.

In questo modo non basta più dire che A e B sono insiemi ma occorre anche specificare la funzione value che restituisce il valore di un suo elemento, cioè una funzione che dato in input un elemento dell'insieme restituisce un valore.

Per convenzione e per semplificare i calcoli assumiamo che le funzioni value restituiscano sempre un insieme e che il value è dato dalla somma dei valori presenti nell'insieme restituito.

Let's define *Pareto*

Prima però proviamo a fare il passaggio successivo, prima abbiamo imparata a calcolare il valore del payoff di un giocatore ma questa cosa può anche essere sbagliata dal punto di vista della teoria dei giochi cooperativi. Adesso vogliamo introdurre la nozione di payoff per coalizione che come si può immaginare è la somma dei payoff dei singoli giocatori.

Altra cosa importante sarà quello di attivare il meccanismo della partizione (partizionamento) e della conseguente enumerazione di sotto-insiemi che godono di determinate proprietà.

Ed infine occorrerà passare alle boolean function.

Capitolo 13

Potential games

Un gioco a potenziale, o gioco con potenziale, è un gioco in cui l'incentivo per i giocatori per passare da una strategia ad un'altra puèssere espresso con una singola funzione globale, detta funzione potenziale, richiamando l'omonimo concetto fisico.

Il concetto fu introdotto da Dov Monderer e Lloyd Shapley nel 1996. Vedi [**2**].

La funzione potenziale si rivela uno strumento utile per analizzare gli equilibri di Nash in certi giochi, dato che gli incentivi di tutti i giocatori sono mappati in una singola funzione, e l'insieme degli equilibri di Nash si trova fra gli ottimi locali della funzione potenziale.

I massimi della funzione potenziale sono equilibri di Nash, mentre l'inverso non è sempre vero. L'uso dei massimi della funzione potenziale permette di raffinare l'insieme degli equilibri di Nash.

il [**2**] è un tantino complesso da leggere, pertanto inizierei da qualcosa di more simple.

...e congestion games...

13.1 Congestion games