Games and Boolean models - mid-term exam

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Instructions:

- Edit your work using the provided tex file. Hand in your work as a LaTex-generated pdf file attached to an e-mail addressed to roxyjean at gmail.com, by the end (i.e. 24:00) of sunday 12 November 2017. Your name should appear both as the author above, and in the chosen tex/pdf files names.
- All solution methods and corresponding computations have to be carefully commented, either in English or in Italian. Any part of the work consisting of non-commented computations and/or expressions shall be disregarded.

Notes for the Teacher:

- Answers to execerices are given by proposition environments. When an answer (proposition) has a motivation it is given by a proof environment.

from Ryser [6]: *Our definitions and proofs are concise and they deserve careful scrutiny. But effort and ingenuity lead to mastery, and our subject holds rich for those who learn its secrets.*

1 Exercise

For an even integer m, let $M = \{1, \dots, m\}$ and define $f : M \to M$ by

$$f(k) = \begin{cases} \frac{m}{2} + k \text{ if } 1 \leq k \leq \frac{m}{2}, \\ k - \frac{m}{2} \text{ if } \frac{m}{2} < k \leq m. \end{cases}$$

1. Characterize binary relation R^f on M defined by

   $$R^f = \{(k, f(k)) : 1 \leq k \leq m\} \subset M \times M$$

   in terms of (ir)reflexivity, (a/anti)symmetry, transitivity and completeness. Determine the number of 1s in Boolean matrix $\mathcal{M}^{R^f} \in \{0,1\}^{m \times m}$ representing R^f, i.e. $\mathcal{M}^{R^f}_{kl} = \begin{cases} 1 \text{ if } (k,l) \in R^f, \\ 0 \text{ if } (k,l) \in M \times M \backslash R^f, \end{cases} \quad 1 \leq k,l \leq m.$

2. Identify a ($\supseteq$-)minimal rational preference (binary relation) $R^{\succsim^*}$ satisfying $R^{\succsim^*} \supseteq R$. How many 1s are in Boolean matrix $\mathcal{M}^{R^{\succsim^*}}$? Also determine the corresponding ordered partition $\mathfrak{P}^{\succsim^*} = (A_1, \dots, A_{|\mathfrak{P}^{\succsim^*}|})$ of M.

1.1 Solution

1.1.1 Prerequisites

- Set: an abstract aggregate of elements.
- Mapping: a way to create new sets.
- mapping: a properties holding on element(s) of set.
- Binary relation: see [1], [2] and [5].

1.1.2 Notation

- M = a finite set of m elements.
- m = number of elements of set M.

1.1.3 Analysis of a function f

Given the binary relation R^f as above defined, we'll investigate its properties (symmetry, transitivity, etc.).

Anzitutto che cos'è la f? La f assegna ad ogni elemento di M un elemento di stesso, pertanto si potrebbe trattare di una permutazione ovvero

an elements of the *symmetric group of degree* n*, denoted by* S_n [2].

Nel nostro caso $n = |M| = m$. Quanto detto non è proprio rigoroso in quanto si dovrebbe dimostrare che f è una permutazione ovvero dovrei fare vedere che la f è sia iniettiva che suriettiva. Di questo fatto me ne sono accorto svolgendo i calcoli sulla f ovvero andando a calcolare $f(0), ..., f(m)$ per $|M|$ uguale a $4, 6, 8$.

La f può essere pensata come suddivisa in due funzioni f_{part1} e f_{part2} e pertanto la prima cosa da fare è discernere quale delle due funzioni applicare a k quando quest'ultimo è passato alla funzione f in altre parole la scrittura $f(k)$ si potrebbe leggere come: quale funzione devo applicare a k? Ebbene la funzione da applicare dipende da k, se $k \leq \frac{m}{2}$ applichiamo la f_{part1}, altrimenti applichiamo la f_{part2}. Chiaramente f_{part1} ed f_{part2} sono definite come:

$$f(k) = f_{part1}(k) = \frac{m}{2} + k$$

if $k \leq \frac{m}{2}$ first half elements of M, and,

$$f(k) = f_{part2}(k) = k - \frac{m}{2}$$

if $k > \frac{m}{2}$ second half elements of M

Cioè la prima metà di elementi di M viene calcolata con f_{part1} mentre la seconda metà di elementi di M viene calcolata con f_{part2}.

Proviamo a schematizzare:

Per $m = 4$, ossia $M = \{1, 2, 3, 4\}$ abbiamo che $\frac{m}{2} = 2$,

$$f(1) = 2 + 1 = 3$$
$$f(2) = 2 + 2 = 4$$
$$f(3) = 3 - 2 = 1$$
$$f(4) = 4 - 2 = 2$$

che posso rappresentare in forma di matrice:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

La matrice precedente la leggiamo in questo modo: nella prima riga ci sono i valori di k, mentre nella seconda riga sono riportati i valori di $f(k)$. Come si può notare da questo primo svolgimento, ma dopo cercheremo di dimostrarlo algebricamente, la relazione R^f è certamente simmetrica.

Per $m = 6$, (saltiamo da 4 a 6 perchè l'esercizio richiede che m sia pari), ossia $M = \{1, 2, 3, 4, 5, 6\}$ abbiamo che $\frac{m}{2} = 3$,

$$f(1) = 3 + 1 = 4$$
$$f(2) = 3 + 2 = 5$$
$$f(3) = 3 + 3 = 6$$
$$f(4) = 4 - 3 = 1$$
$$f(5) = 5 - 3 = 2$$
$$f(6) = 6 - 3 = 3$$

Che possiamo rappresentare sotto forma di matrice come

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

La funzione f potrebbe essere vista anche come $k \equiv f(k) \pmod{\frac{m}{2}}$ e con quest'ultima espressione ...

1.1.4 Analysis of R^f properties

Proposition 1. *The binary relation R^f is symmetric, intransitive and incomplete.*

Proof. **Symmetry**. Symmetry seems to be trivial but we need to show that $(k, f(k)) \in R^f \implies (f(k), k) \in R^f$.
Thinking R^f as $R^f = \{(a,b) \wedge (b,a) : b = a + \frac{m}{2}, \forall a, b \in M, \} \subseteq M \times M$
The reason could be because congruences are symmetric but we need to show to many things in order to prove the proposition.

Transitivity. NO, infatti posso trovare due ennuple $(k, f(k)), (f(k), f(f(k))) \in R^f$
tali che $(k, f(f(k))) \notin R^f$. E.g. se prendo $(1,3),(3,1) \in R^{f^4}$, dove f^4 rappresenta la funzione f quando $m = 4$, la ennupla $(1,1) \notin R^{f^4}$.

Completeness. NO, infatti $(1,2) \wedge (2,1) \notin R^{f^4}$. □

1.1.5 Number of 1s in Boolean matrix representing R^f

Proposition 2. *There are m 1s in the boolean matrix representing R^f.*

2 Exercise

For player set $N = \{1, \dots, n\}$ and strategy set $\mathbb{S}_i = \{0,1\}$ for all $i \in N$, let

$$u_i(s) = u_i(s_i, s_{-i}) = \left(s_i - \sum_{j \in N} \frac{s_j}{n} \right)^2 \text{ for all strategy profiles } s \in \{0,1\}^n.$$

1. Is this a common interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles $s, s' \in \{0,1\}^n$ provide the same aggregate payoff, that is to say $\sum_{i \in N} u_i(s) = \sum_{i \in N} u_i(s)$. If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then determine all pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.

2. Regarding this as a congestion game with a 2-set $\{0,1\}$ of facilities, denote by $u_0(k)$ the utility attained by playing 0 when the number of those playing 0 is k and by $u_1(k)$ the utility attained by playing 1 when the number of those playing 1 is k. Verify whether the game is monotone and, in particular, whether

$$u_0(k) - u_0(k+1) = u_1(k) - u_1(k+1)$$

for all $1 \leq k < n$. For $1 < k < n$, denote by $s_0^k \in \{0,1\}^n$ any of the $\binom{n}{k}$ strategy profiles where $k = |\{i : s_i = 0\}|$, and by $\mathbf{P} : \{0,1\}^n \to \mathbb{R}$ the exact potential function. Determine $\mathbf{P}(s_0^k)$. Is there any relation between the set of strong equilibria and the set of equilibria (with non-random strategies)? How many equilibria are there?

3. Verify whether the n-tuple of random strategies $\mathbf{\frac{1}{2}} \in [0,1]^n$ where every $i \in N$ plays both 0 and 1 with equal probability, i.e. $\frac{1}{2}$, is an equilibrium.

2.1 Solution

2.1.1 Prerequisites

- See [3], [4], [5], [7]
- Preference relation. reflexive, transitive and complete.
- Preference aggregation. mainly [5];
- Common interest game. mainly [5];
- Potential game.
- Congestion game.
- Dominance.

2.1.2 Notation

- $\Gamma = (\mathbb{N}, \mathbb{S}, u_i)$. Γ è il gioco definito dall'esercizio.
- $\mathbb{N} = \{1, ..., n\}$ = A set of n elements called players.
- $\mathbb{S}_i = \{0, 1\}$ = A set of 2 elements called strategies. A strategy can have many levels, in fact an element of $\mathbb{S}_i$ can be another set of strategies and so on. For flat strategy set we use the name *alternative*. I the our game there are n strategy sets. Each element of the strategy set $\mathbb{S}_i$ has value 0 or 1. Nevertheless, the process of value assignment can continue to infinity if we look at 0 and 1 not as number or as value of real set $\mathbb{R}$ but as a name indicating a choice.
  In altre parole, assumiamo che gli elementi di $\mathbb{S}_i$ siano i numeri reali $0, 1 \in \mathbb{R}$.
- $\mathbb{S} = \mathbb{S}_1 \times, ..., \times \mathbb{S}_n$. Strategy profiles set. Insieme di ennuple $(a_1, ..., a_n)$ con $a_1, a_n \in \{0, 1\} \subseteq \mathbb{R}$. Insieme degli outcomes. Insieme dei prospetti. In condizione di completa informazione ogni giocatore conosce tutti i prospetti ed il rispettivo valore dato dalla sua funzione di utilità u_i.
- $s \in \mathbb{S}$, $s^* \in \mathbb{S}$

- s_i. Sia data $s \in \mathbb{S} = (s_1, ..., s_n)$ una tupla, allora s_i indica l'iesimo elemento all'interno della tupla s. E.g. $s = (3, 6, 9, 45)$ allora $s_2 = 6$. Per fortuna tutti gli indici iniziano da 1. Sottolineamo questo fatto perchè molto spesso in computer science and specifically in programming languages indices start from 0.
- s_{-i} = E.g. $s_{-2} = (3, 9, 45)$. Questa notazione serve per poter suddividere le componenti o coordinate del generico settore. Una volta distinte da diversi nomi le coordinate possono essere utilizzate nella definizione della funzione stessa.
- (s_i, s_{-i}) = E.g. $(6, (3, 9, 45)) = (3, 6, 9, 45)$.
- $u_i(s)$ = funzione di utilità dell'iesimo giocatore.

2.1.3 Is this a common interest game?

Definition 1. *Common interest games are those where there is a strategy profile $s^* \in \mathbb{S}$ such that $s^* \gtrsim_i s$ for all $s \in \mathbb{S}$. [5].*

In altre parole, nei common interest games esiste una strategia che è preferita da tutti i giocatori. In altre parole, il best response set di tutti i giocatori è diverso dall'insieme vuoto. E qui sta l'inghippo, l'intersezione non va fatta a gruppi di due giocatori, ma per tutti i giocatori. Bisogna intersecare il best response set di tutti i giocatori.

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svolgimento ERRATO

Definition 2. *A game Γ is a common interest game iif Best response strategy set for player i intersecanting best response strategy set for player j is not empty.*

In altre parole il comune interesse è modellato sull'intersezione di insiemi. Che potremmo assiomatizzare come segue:

Let A be a set, let B be a set then if $A \cap B = \emptyset$ indica che non c'è comune interesse.

Potremmo restringere la definizione sopra e considerare il caso ovvero l'insieme delle strategie in cui la utility function del giocatore i-esimo restituisce un valore maggiore di zero.

Proposition 3. *Γ is a common interest game.*

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svolgimento CORRETTO

Proposition 4. *Γ isn't a common interest game.*

Proof. We'll prove that this proposition is false by constructing a set of strategy profiles that are elements of $BR_i \cap BR_j$ for all $i, j \in \mathbb{N}$. Those construction can be given by strategy profiles in which there are $k = 2$ quantitá di 1s. Either for i and j playing 1s is the best responses when i or j play 1s and the others play 0. So, for all i and $j \in \mathbb{N}$. So Γ is a common interest game. Contraddiction. $\square$

2.1.4 Is this a constant-sum game?

Proposition 5. Γ *is a constant-sum game.*

Proof. For $n = 3$ players let be s, s^* two strategy profiles with $s^* = (0, 0, 0)$ and $s = (1, 0, 0)$ then the sum of the payoffs of the players is not equals for s and s^*.

$$u_1(s^*) + u_2(s^*) + u_3(s^*) \neq u_1(s) + u_2(s) + u_3(s)$$

Contraddiction □

Per i calcoli si rimanda al companion html/javascript file containing some code examples on how to calculate combination on a set of class k.

2.1.5 Are there Pareto-dominated strategy profiles?

Proposition 6. $\mathbb{S}$ *contains pareto-dominated strategy profiles.*

Proof. For all games strategy profiles in which every player play 0 or every player play 1 are pareto-dominated strategy profiles because if a player deviates from its choice then obtain a plus and no other player can do less than they do. □

2.1.6 Regarding as congestion game

For $n = 4$

s	$(u_1(s), u_2(s), u_3(s), u_4(s))$	$u_1 + u_2 + u_3$	num of 0s	num of 1s
0000	(0, 0, 0, 0,)	0	4	0
0001	(0.0625, 0.0625, 0.0625, 0.5625,)	0.75	3	1
0010	(0.0625, 0.0625, 0.5625, 0.0625,)	0.75	3	1
0011	(0.25, 0.25, 0.25, 0.25,)	1	2	2
0100	(0.0625, 0.5625, 0.0625, 0.0625,)	0.75	3	1
0101	(0.25, 0.25, 0.25, 0.25,)	1	2	2
0110	(0.25, 0.25, 0.25, 0.25,)	1	2	2
0111	(0.5625, 0.0625, 0.0625, 0.0625,)	0.75	1	3
1000	(0.5625, 0.0625, 0.0625, 0.0625,)	0.75	3	1
1001	(0.25, 0.25, 0.25, 0.25,)	1	2	2
1010	(0.25, 0.25, 0.25, 0.25,)	1	2	2
1011	(0.0625, 0.5625, 0.0625, 0.0625,)	0.75	1	3
1100	(0.25, 0.25, 0.25, 0.25,)	1	2	2
1101	(0.0625, 0.0625, 0.5625, 0.0625,)	0.75	1	3
1110	(0.0625, 0.0625, 0.0625, 0.5625,)	0.75	1	3
1111	(0, 0, 0, 0,)	0	0	4

Inoltre,

$u_0(k) = u_0(0) = 0$
$u_0(k) = u_0(1) = 0.5625$
$u_0(k) = u_0(2) = 0.25$
$u_0(k) = u_0(3) = 0,0625$
$u_0(k) = u_0(4) = 0$

And,
$u_1(k) = u_1(0) = 0$
$u_1(k) = u_1(1) = 0.5625$
$u_1(k) = u_1(2) = 0.25$
$u_1(k) = u_1(3) = 0,0625$
$u_1(k) = u_1(4) = 0$

So,

$u_0(0) - u_0(1) = 0 - 0.5625$
$u_1(0) - u_1(1) = 0 - 0.5625$

Then, $u_0(0) - u_0(1) = u_1(0) - u_1(1)$

3 Exercise

For $M = \{1, \ldots, m\}$, consider the symmetric congestion game where every player $i \in N = \{1, \ldots, n\}$ has strategy set $\mathbb{S}_i = \mathcal{K} \subset 2^{2^M}$ consisting of the $m!$ *maximal chains* $\{A_0, A_1, \ldots, A_{m-1}, A_m\} \in \mathcal{K}$ of subsets of M. That is,

$$M = A_m \supset^* A_{m-1} \supset^* \cdots \supset^* A_1 \supset^* A_0 = \emptyset, \text{ where}$$

$$A_k \supset^* A_{k-1} \Leftrightarrow A_k \supset A_{k-1}, |A_k| = |A_{k-1}| + 1 \ (1 \leq k \leq m)$$

is the *covering relation*. Hence the set of facilities is $\{A : \emptyset \subset A \subset M\}$. For every strategy profile $s = (s_1, \ldots, s_n) \in \mathcal{K}^n$, denote i's strategy ($i \in N$) by

$$s_i = \{A_0, A_1^i, \ldots, A_{m-1}^i, A_m\} \in \mathcal{K},$$

and define congestion vector $\{c_A(s) : \emptyset \subset A \subset M\} \in \mathbb{Z}_+^{2^m-2}$ by

$$c_A(s) = |\{i : A \in s_i\}|.$$

Finally, utilities have form

$$u_i(s) = \sum_{0<k<m} \frac{1}{c_{A_k^i}(s)}.$$

In what follows, distinguish between cases (a) $n \leq m$ and (b) $n = m!$.

1. Is this a common interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles $s, s' \in \{0,1\}^n$ provide the same aggregate payoff, that is to say $\sum_{i\in N} u_i(s) = \sum_{i\in N} u_i(s)$. If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then provide examples of pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.

2. Characterize the set of equilibria and the set of strong equilibria (with non-random strategies). Compute the value $\mathbf{P}(s)$ taken by the exact potential $\mathbf{P}$ at any equilibrium s.

3. Verify whether the random strategy profile consisting of n uniform distributions over the $m!$-set $\mathcal{K}$ of maximal chains is an equilibrium or not.

4 Exercise

Let $M = \{1, \ldots, 10\}$ and define
Let $f : M \to \{0,1\}$ by
Let $f(i) = \begin{cases} 1 \text{ if } i \text{ is a prime,} \\ 0 \text{ otherwise.} \end{cases}$ Compute the discrete Choquet integral $E^C_\eta(f)$ of f with respect to fuzzy probability $\eta : 2^M \to [0,1]$ defined by

$$\eta(A) = \binom{11}{2}^{-2} \left(\sum_{i\in A} i\right)^2 \quad \text{for all } A \in 2^M.$$

References

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[2] Herstein I.N. *Algebra.* Editori riuniti, Roma, 1999.

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