

# Games and Boolean models - mid-term exam

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## Instructions:

- Edit your work using the provided tex file. Hand in your work as a LaTeX-generated pdf file attached to an e-mail addressed to roxyjean at gmail.com, by the end (i.e. 24:00) of sunday 12 November 2017. Your name should appear both as the author above, and in the chosen tex/pdf files names.
- All solution methods and corresponding computations have to be carefully commented, either in English or in Italian. Any part of the work consisting of non-commented computations and/or expressions shall be disregarded.

## Notes for the Teacher:

- Answers to exercises are given by proposition environments. When an answer (proposition) has a motivation it is given by a proof environment.

*from Ryser [7]: Our definitions and proofs are concise and they deserve careful scrutiny.*

*But effort and ingenuity lead to mastery, and our subject holds rich for those who learn its secrets.*

## 1 Exercise

For an even integer  $m$ , let  $M = \{1, \dots, m\}$  and define  $f : M \rightarrow M$  by

$$f(k) = \begin{cases} \frac{m}{2} + k & \text{if } 1 \leq k \leq \frac{m}{2}, \\ k - \frac{m}{2} & \text{if } \frac{m}{2} < k \leq m. \end{cases}$$

1. Characterize binary relation  $R^f$  on  $M$  defined by

$$R^f = \{(k, f(k)) : 1 \leq k \leq m\} \subset M \times M$$

in terms of (ir)reflexivity, (a/anti)symmetry, transitivity and completeness. Determine the number of 1s in Boolean matrix  $\mathcal{M}^{R^f} \in \{0, 1\}^{m \times m}$  representing  $R^f$ , i.e.  $\mathcal{M}_{kl}^{R^f} = \begin{cases} 1 & \text{if } (k, l) \in R^f, \\ 0 & \text{if } (k, l) \in M \times M \setminus R^f, \end{cases} \quad 1 \leq k, l \leq m.$

2. Identify a  $(\supseteq)$ -minimal rational preference (binary relation)  $R^{\succsim^*}$  satisfying  $R^{\succsim^*} \supseteq R$ . How many 1s are in Boolean matrix  $\mathcal{M}^{R^{\succsim^*}}$ ? Also determine the corresponding ordered partition  $\mathfrak{P}^{\succsim^*} = (A_1, \dots, A_{|\mathfrak{P}^{\succsim^*}|})$  of  $M$ .

## 1.1 Solution

### 1.1.1 Prerequisites

- Set: an abstract aggregate of elements.
- Mapping: a way to create new sets.
- mapping: a properties holding on element(s) of set.
- Binary relation: see [1], [2] and [5].

### 1.1.2 Notation

- $M$  = a finite set of  $m$  elements.
- $m$  = number of elements of set  $M$ .

### 1.1.3 Analysis of a function $f$

Given the binary relation  $R^f$  as above defined, we'll investigate its properties (symmetry, transitivity, etc.).

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Anzitutto che cos'è la  $f$ ? La  $f$  assegna ad ogni elemento di  $M$  un elemento di stesso, pertanto si potrebbe trattare di una permutazione ovvero

**an elements of the *symmetric group of degree  $n$ , denoted by  $S_n$*  [2].**

Nel nostro caso  $n = |M| = m$ . Quanto detto non è proprio rigoroso in quanto si dovrebbe dimostrare che  $f$  è una permutazione ovvero dovrei fare vedere che la  $f$  è sia iniettiva che suriettiva. Di questo fatto me ne sono accorto svolgendo i calcoli sulla  $f$  ovvero andando a calcolare  $f(0), \dots, f(m)$  per  $|M|$  uguale a 4, 6, 8.

La  $f$  può essere pensata come suddivisa in due funzioni  $f_{part1}$  e  $f_{part2}$  e pertanto la prima cosa da fare è discernere quale delle due funzioni applicare a  $k$  quando quest'ultimo è passato alla funzione  $f$  in altre parole la scrittura  $f(k)$  si potrebbe leggere come: quale funzione devo applicare a  $k$ ? Ebbene la funzione da applicare dipende da  $k$ , se  $k \leq \frac{m}{2}$  applichiamo la  $f_{part1}$ , altrimenti applichiamo la  $f_{part2}$ . Chiaramente  $f_{part1}$  ed  $f_{part2}$  sono definite come:

$$f(k) = f_{part1}(k) = \frac{m}{2} + k$$

if  $k \leq \frac{m}{2}$  first half elements of  $M$ , and,

$$f(k) = f_{part2}(k) = k - \frac{m}{2}$$

if  $k > \frac{m}{2}$  second half elements of  $M$

Cioè la prima metà di elementi di  $M$  viene calcolata con  $f_{part1}$  mentre la seconda metà di elementi di  $M$  viene calcolata con  $f_{part2}$ .

Proviamo a schematizzare:

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Per  $m = 4$ , ossia  $M = \{1, 2, 3, 4\}$  abbiamo che  $\frac{m}{2} = 2$ ,

$$f(1) = 2 + 1 = 3$$

$$f(2) = 2 + 2 = 4$$

$$f(3) = 3 - 2 = 1$$

$$f(4) = 4 - 2 = 2$$

che posso rappresentare in forma di matrice:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

La matrice precedente la leggiamo in questo modo: nella prima riga ci sono i valori di  $k$ , mentre nella seconda riga sono riportati i valori di  $f(k)$ . Come si può notare da questo primo svolgimento, ma dopo cercheremo di dimostrarlo algebricamente, la relazione  $R^f$  è certamente simmetrica.

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Per  $m = 6$ , (saltiamo da 4 a 6 perchè l'esercizio richiede che  $m$  sia pari), ossia  $M = \{1, 2, 3, 4, 5, 6\}$  abbiamo che  $\frac{m}{2} = 3$ ,

$$f(1) = 3 + 1 = 4$$

$$f(2) = 3 + 2 = 5$$

$$f(3) = 3 + 3 = 6$$

$$f(4) = 4 - 3 = 1$$

$$f(5) = 5 - 3 = 2$$

$$f(6) = 6 - 3 = 3$$

Che possiamo rappresentare sotto forma di matrice come

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

La funzione  $f$  potrebbe essere vista anche come  $k \equiv f(k) \pmod{\frac{m}{2}}$  e con quest'ultima espressione ...

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### 1.1.4 Analysis of $R^f$ properties

**Proposition 1.** *The binary relation  $R^f$  is symmetric, intransitive and incomplete.*

*Proof. Symmetry.* Symmetry seems to be trivial but we need to show that  $(k, f(k)) \in R^f \implies (f(k), k) \in R^f$ .

Thinking  $R^f$  as  $R^f = \{(a, b) \wedge (b, a) : b = a + \frac{m}{2}, \forall a, b \in M, \} \subseteq M \times M$

The reason could be because congruences are symmetric but we need to show to many things in order to prove the proposition.

**Transitivity.** NO, infatti posso trovare due ennuple  $(k, f(k)), (f(k), f(f(k))) \in R^f$

tali che  $(k, f(f(k))) \notin R^f$ . E.g. se prendo  $(1, 3), (3, 1) \in R^{f^4}$ , dove  $f^4$  rappresenta la funzione  $f$  quando  $m = 4$ , la ennupla  $(1, 1) \notin R^{f^4}$ .

**Completeness.** NO, infatti  $(1, 2) \wedge (2, 1) \notin R^{f^4}$ . □

### 1.1.5 Number of 1s in Boolean matrix representing $R^f$

**Proposition 2.** *There are  $m$  1s in the boolean matrix representing  $R^f$ .*

## 2 Exercise

For player set  $N = \{1, \dots, n\}$  and strategy set  $S_i = \{0, 1\}$  for all  $i \in N$ , let

$$u_i(s) = u_i(s_i, s_{-i}) = \left( s_i - \sum_{j \in N} \frac{s_j}{n} \right)^2 \text{ for all strategy profiles } s \in \{0, 1\}^n.$$

1. Is this a interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles  $s, s' \in \{0, 1\}^n$  provide the same aggregate payoff, that is to say  $\sum_{i \in N} u_i(s) = \sum_{i \in N} u_i(s')$ . If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then determine all pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.
2. Regarding this as a congestion game with a 2-set  $\{0, 1\}$  of facilities, denote by  $u_0(k)$  the utility attained by playing 0 when the number of those playing 0 is  $k$  and by  $u_1(k)$  the utility attained by playing 1 when the number of those playing 1 is  $k$ . Verify whether the game is monotone and, in particular, whether

$$u_0(k) - u_0(k+1) = u_1(k) - u_1(k+1)$$

for all  $1 \leq k < n$ . For  $1 < k < n$ , denote by  $s_0^k \in \{0, 1\}^n$  any of the  $\binom{n}{k}$  strategy profiles where  $k = |\{i : s_i = 0\}|$ , and by  $\mathbf{P} : \{0, 1\}^n \rightarrow \mathbb{R}$  the exact potential function. Determine  $\mathbf{P}(s_0^k)$ . Is there any relation between the set of strong equilibria and the set of equilibria (with non-random strategies)? How many equilibria are there?

3. Verify whether the  $n$ -tuple of random strategies  $\frac{1}{2} \in [0, 1]^n$  where every  $i \in N$  plays both 0 and 1 with equal probability, i.e.  $\frac{1}{2}$ , is an equilibrium.

## 2.1 Solution

### 2.1.1 Prerequisites

- See [3], [4], [5], [8]
- Preference relation. reflexive, transitive and complete.
- Preference aggregation. mainly [5];
- Common interest game. mainly [5];
- Potential game.
- Congestion game.
- Dominance.

### 2.1.2 Notation

- $\Gamma = (\mathbb{N}, \mathbb{S}, u_i)$ .  $\Gamma$  è il gioco definito dall'esercizio.
- $\mathbb{N} = \{1, \dots, n\}$  = A set of  $n$  elements called players.
- $\mathbb{S}_i = \{0, 1\}$  = A set of 2 elements called strategies. A strategy can have many levels, in fact an element of  $\mathbb{S}_i$  can be another set of strategies and so on. For flat strategy set we use the name *alternative*. In the our game there are  $n$  strategy sets. Each element of the strategy set  $\mathbb{S}_i$  has value 0 or 1. Nevertheless, the process of value assignment can continue to infinity if we look at 0 and 1 not as number or as value of real set  $\mathbb{R}$  but as a name indicating a choice.  
In altre parole, assumiamo che gli elementi di  $\mathbb{S}_i$  siano i numeri reali 0, 1  $\in \mathbb{R}$ .
- $\mathbb{S} = \mathbb{S}_1 \times \dots \times \mathbb{S}_n$ . Strategy profiles set. Insieme di ennuple  $(a_1, \dots, a_n)$  con  $a_1, a_n \in \{0, 1\} \subseteq \mathbb{R}$ . Insieme degli outcomes. Insieme dei prospetti. In condizione di completa informazione ogni giocatore conosce tutti i prospetti ed il rispettivo valore dato dalla sua funzione di utilità  $u_i$ .
- $s \in \mathbb{S}$ ,  $s^* \in \mathbb{S}$

- $s_i$ . Sia data  $s \in \mathbb{S} = (s_1, \dots, s_n)$  una tupla, allora  $s_i$  indica l'iesimo elemento all'interno della tupla  $s$ . E.g.  $s = (3, 6, 9, 45)$  allora  $s_2 = 6$ . Per fortuna tutti gli indici iniziano da 1. Sottolineiamo questo fatto perchè molto spesso in computer science and specifically in programming languages indices start from 0.
- $s_{-i}$  = E.g.  $s_{-2} = (3, 9, 45)$ . Questa notazione serve per poter suddividere le componenti o coordinate del generico settore. Una volta distinte da diversi nomi le coordinate possono essere utilizzate nella definizione della funzione stessa.
- $(s_i, s_{-i})$  = E.g.  $(6, (3, 9, 45)) = (3, 6, 9, 45)$ .
- $u_i(s)$  = funzione di utilità dell'iesimo giocatore.

### 2.1.3 Is this a common interest game?

**Definition 1.** *Common interest games* are those where there is a strategy profile  $s^* \in \mathbb{S}$  such that  $s^* \succsim_i s$  for all  $s \in \mathbb{S}$ . [5].

In altre parole, nei common interest games esiste una strategia che è preferita da tutti i giocatori. In altre parole, il best response set di tutti i giocatori è diverso dall'insieme vuoto. E qui sta l'inghippo, l'intersezione non va fatta a gruppi di due giocatori, ma per tutti i giocatori. **Bisogna intersecare il best response set di tutti i giocatori.**

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svolgimento ERRATO

**Definition 2.** A game  $\Gamma$  is a common interest game iif Best response strategy set for player  $i$  intersecting best response strategy set for player  $j$  is not empty.

In altre parole il comune interesse è modellato sull'intersezione di insiemi. Che potremmo assiomatizzare come segue:

Let  $A$  be a set, let  $B$  be a set then if  $A \cap B = \emptyset$  indica che non c'è comune interesse.

Potremmo restringere la definizione sopra e considerare il caso ovvero l'insieme delle strategie in cui la utility function del giocatore  $i$ -esimo restituisce un valore maggiore di zero.

**Proposition 3.**  $\Gamma$  is a common interest game.

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svolgimento CORRETTO

**Proposition 4.**  $\Gamma$  isn't a common interest game.

*Proof.* We'll prove that this proposition is false by constructing a set of strategy profiles that are elements of  $BR_i \cap BR_j$  for all  $i, j \in \mathbb{N}$ . Those construction can be given by strategy profiles in which there are  $k = 2$  quantità di 1s. Either for  $i$  and  $j$  playing 1s is the best responses when  $i$  or  $j$  play 1s and the others play 0. So, for all  $i$  and  $j \in \mathbb{N}$ . So  $\Gamma$  is a common interest game. **Contraddiction.**  $\square$

#### 2.1.4 Is this a constant-sum game?

**Proposition 5.**  $\Gamma$  is a constant-sum game.

*Proof.* For  $n = 3$  players let be  $s, s^*$  two strategy profiles with  $s^* = (0, 0, 0)$  and  $s = (1, 0, 0)$  then the sum of the payoffs of the players is not equals for  $s$  and  $s^*$ .

$$u_1(s^*) + u_2(s^*) + u_3(s^*) \neq u_1(s) + u_2(s) + u_3(s)$$

Contraddiction □

Per i calcoli si rimanda al companion html/javascript file containing some code examples on how to calculate combination on a set of class  $k$ .

#### 2.1.5 Are there Pareto-dominated strategy profiles?

**Proposition 6.**  $\mathbb{S}$  contains pareto-dominated strategy profiles.

*Proof.* For all games strategy profiles in which every player play 0 or every player play 1 are pareto-dominated strategy profiles because if a player deviates from its choice then obtain a plus and no other player can do less than they do. □

#### 2.1.6 Regarding as congestion game

For  $n = 4$

$s$	$(u_1(s), u_2(s), u_3(s), u_4(s))$	$u_1 + u_2 + u_3$	num of 0s	num of 1s
0000	(0, 0, 0, 0, )	0	4	0
0001	(0.0625, 0.0625, 0.0625, 0.5625, )	0.75	3	1
0010	(0.0625, 0.0625, 0.5625, 0.0625, )	0.75	3	1
0011	(0.25, 0.25, 0.25, 0.25, )	1	2	2
0100	(0.0625, 0.5625, 0.0625, 0.0625, )	0.75	3	1
0101	(0.25, 0.25, 0.25, 0.25, )	1	2	2
0110	(0.25, 0.25, 0.25, 0.25, )	1	2	2
0111	(0.5625, 0.0625, 0.0625, 0.0625, )	0.75	1	3
1000	(0.5625, 0.0625, 0.0625, 0.0625, )	0.75	3	1
1001	(0.25, 0.25, 0.25, 0.25, )	1	2	2
1010	(0.25, 0.25, 0.25, 0.25, )	1	2	2
1011	(0.0625, 0.5625, 0.0625, 0.0625, )	0.75	1	3
1100	(0.25, 0.25, 0.25, 0.25, )	1	2	2
1101	(0.0625, 0.0625, 0.5625, 0.0625, )	0.75	1	3
1110	(0.0625, 0.0625, 0.0625, 0.5625, )	0.75	1	3
1111	(0, 0, 0, 0, )	0	0	4

Inoltre,

$$\begin{aligned}
u_0(k) &= u_0(0) = 0 \\
u_0(k) &= u_0(1) = 0.5625 \\
u_0(k) &= u_0(2) = 0.25 \\
u_0(k) &= u_0(3) = 0.0625 \\
u_0(k) &= u_0(4) = 0
\end{aligned}$$

$$\begin{aligned}
&\text{And,} \\
u_1(k) &= u_1(0) = 0 \\
u_1(k) &= u_1(1) = 0.5625 \\
u_1(k) &= u_1(2) = 0.25 \\
u_1(k) &= u_1(3) = 0.0625 \\
u_1(k) &= u_1(4) = 0
\end{aligned}$$

So,

$$\begin{aligned}
u_0(0) - u_0(1) &= 0 - 0.5625 \\
u_1(0) - u_1(1) &= 0 - 0.5625
\end{aligned}$$

Then,  $u_0(0) - u_0(1) = u_1(0) - u_1(1)$

### 3 Exercise

For  $M = \{1, \dots, m\}$ , consider the symmetric congestion game where every player  $i \in N = \{1, \dots, n\}$  has strategy set  $\mathbb{S}_i = \mathcal{K} \subset 2^{2^M}$  consisting of the  $m!$  *maximal chains*  $\{A_0, A_1, \dots, A_{m-1}, A_m\} \in \mathcal{K}$  of subsets of  $M$ . That is,

$$M = A_m \supset^* A_{m-1} \supset^* \dots \supset^* A_1 \supset^* A_0 = \emptyset, \text{ where}$$

$$A_k \supset^* A_{k-1} \Leftrightarrow A_k \supset A_{k-1}, |A_k| = |A_{k-1}| + 1 \quad (1 \leq k \leq m)$$

is the *covering relation*. Hence the set of facilities is  $\{A : \emptyset \subset A \subset M\}$ . For every strategy profile  $s = (s_1, \dots, s_n) \in \mathcal{K}^n$ , denote  $i$ 's strategy ( $i \in N$ ) by

$$s_i = \{A_0, A_1^i, \dots, A_{m-1}^i, A_m\} \in \mathcal{K},$$

and define congestion vector  $\{c_A(s) : \emptyset \subset A \subset M\} \in \mathbb{Z}_+^{2^m-2}$  by

$$c_A(s) = |\{i : A \in s_i\}|.$$

Finally, utilities have form

$$u_i(s) = \sum_{0 \leq k < m} \frac{1}{c_{A_k^i}(s)}.$$

In what follows, distinguish between cases (a)  $n \leq m$  and (b)  $n = m!$ .



1. Is this a common interest game? If yes, then determine the (non-empty) set of strategy profiles where each player attains the maximum payoff. If no, then show that different players have different optimal strategy profiles. Is this a constant-sum game? If yes, then show that any two strategy profiles  $s, s' \in \{0, 1\}^n$  provide the same aggregate payoff, that is to say  $\sum_{i \in N} u_i(s) = \sum_{i \in N} u_i(s')$ . If no, then show that there are different strategy profiles providing different aggregate payoffs. Are there Pareto-dominated strategy profiles? If yes, then provide examples of pairs of strategy profiles one of which Pareto-dominates the other. If no, then show that for any pair of strategy profiles neither one Pareto-dominates the other.
2. Characterize the set of equilibria and the set of strong equilibria (with non-random strategies). Compute the value  $\mathbf{P}(s)$  taken by the exact potential  $\mathbf{P}$  at any equilibrium  $s$ .
3. Verify whether the random strategy profile consisting of  $n$  uniform distributions over the  $m!$ -set  $\mathcal{K}$  of maximal chains is an equilibrium or not.

### 3.1 Solution

#### 3.1.1 Prerequisites

**Definition 3.** For any symmetric congestion game form  $F = (N, M, \times_{j \in N} \mathbb{S}_j)$ , if union  $\cup_{j \in N} \mathbb{S}_j$  displays no bad configuration, then  $SE(\Gamma) = NE(\Gamma)$  for all monotone congestion games  $\Gamma$  derived from  $F$ .

- bad configuration ???
- monotone congestion game ???
- non-singleton strategies

**Definition 4.** Union  $\cup_{j \in N} \mathbb{S}_j$  of all  $n$  strategy sets displays a bad configuration if there are two facilities  $a, a' \in M$  and three strategies  $s, s', s'' \in \cup_{j \in N} \mathbb{S}_j$  such that  $a \in sa'$  and  $a \notin s' \ni a'$  while  $a, a' \in s''$  (this indeed may be regarded as an acyclicity condition for deviations of non-singleton coalitions, to be compared with the finite improvement property of all potential games). [5].

- non-singleton strategies

#### 3.1.2 Notation

- $\mathcal{K}$  = a set.
- $\mathbf{P}(s)$  = funzione potenziale.
- $\mathbf{P}$  = potenziale.

### 3.1.3 Analysis of the game definition

LET  $G = (N, M, u)$  be SYMMETRIC CONGESTION GAME  
 LET  $M$  be SET  
 LET  $m$  be  $|M|$ ,  $m$  also represents the (variable)name of the  $m$ -nth element..  
 LET  $k$  be INDICES of  $M$   
 LET  $A_k$  be SET ??? kappesimo sottoinsieme di di elementi di  $M$   
 LET  $A_k$  IN  $2^M$   
 LET  $|2^M| = 2^m$   
 LET  $A$  be SET  
 LET  $XXX$  be a CHAIN

LET  $K^n$  be SET - of all strategy profiles  
 LET  $s$  be ELEMENTS of  $K^n$

### 3.1.4 Is this a common interest game?

**Definition 5.** *Common interest games* are those where there is a strategy profile  $s^* \in \mathbb{S}$  such that  $s^* \succsim_i s$  for all  $s \in \mathbb{S}$ . [5].

Who are the strategy profiles in this game?

### 3.1.5 Is this a constant-sum game?

### 3.1.6 Are there Pareto-dominated strategy profiles.

### 3.1.7 Characterize the set of equilibria.

### 3.1.8 Characterize the set of strong equilibria

### 3.1.9 Compute the value $P(s)$ taken by the exact potential $P$ at any equilibrium $s$ .

### 3.1.10 Verify whether random strategy profile consisting of $n$ uniform distributions over the $m!$ set $\mathcal{K}$ of maximal chains is an equilibrium or not.

## 4 Exercise

Let  $M = \{1, \dots, 10\}$  and define

Let  $f : M \rightarrow \{0, 1\}$  by

Let  $f(i) = \begin{cases} 1 & \text{if } i \text{ is a prime,} \\ 0 & \text{otherwise.} \end{cases}$  Compute the discrete Choquet integral  $E_\eta^C(f)$

of  $f$  with respect to fuzzy probability  $\eta : 2^M \rightarrow [0, 1]$  defined by

$$\eta(A) = \binom{11}{2}^{-2} \left( \sum_{i \in A} i \right)^2 \text{ for all } A \in 2^M.$$

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