A THEORY OF POSITIVE INTEGERS IN FORMAL LOGIC.* PART II.

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15. Formal definition: initial values, induction. If L is an intuitive function which associates well-formed expressions $L(x_1, \cdots, x_n)$ with n-tuples $(x_1, \cdots, x_n)$ of well-formed expressions, then L shall be said to be *defined* (*formally*) by $\boldsymbol{L}$ if $\boldsymbol{L}(x_1, \cdots, x_n)$ conv $L(x_1, \cdots, x_n)$ for each set $(x_1, \cdots, x_n)$ for which L is defined. By the "definition" of a function which correlates intuitive mathematical objects, we shall mean the definition of the function which correlates the corresponding well-formed formulas, in case corresponding formulas have been designated. By the "definition" of a sequence $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \cdots$, we shall mean the definition of a function L whose values for the arguments $1, 2, 3, \cdots$ are $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \cdots$, respectively. That is, $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \cdots$ shall be defined (formally) by $\boldsymbol{L}$, if $\boldsymbol{L}(\boldsymbol{i})$ conv $\boldsymbol{A}_i$ $(i = 1, 2, 3, \cdots)$.

Closely connected with the formal theory of this paper, there is an intuitive theory concerning the formal definition of the functions involved. For the preceding sections, this may be summarized by the following theorem, each part of which can be established, either directly, with the aid of the first, or by means of considerations used above in formal proofs.†

15I. *Suppose that x and y are given positive integers of intuitive logic.* a. $\boldsymbol{x}$ *conv* $\lambda fa \cdot f(\cdots x \text{ times} \cdots f(a) \cdots)$. b. *If* $x + y = z$, $\boldsymbol{x} + \boldsymbol{y}$ *conv* $\boldsymbol{z}$. c. *If* $xy = z$, $\boldsymbol{xy}$ *conv* $\boldsymbol{z}$. d. $\boldsymbol{F}^{\boldsymbol{x}}(\boldsymbol{A})$ *conv* $\boldsymbol{F}(\cdots x \text{ times} \cdots \boldsymbol{F}(\boldsymbol{A}) \cdots)$. e. $I^{\boldsymbol{x}}$ *conv* I; $I(\boldsymbol{A})$ *conv* $\boldsymbol{A}$. f. *If* $x^y = z$, $\boldsymbol{x}^{\boldsymbol{y}}$ *conv* $\boldsymbol{z}$. g. *If* $x > y$ *and* $x - y = z$, $\boldsymbol{x} - \boldsymbol{y}$ *conv* $\boldsymbol{z}$; *if* $x \leq y$, $\boldsymbol{x} - \boldsymbol{y}$ *conv* 1. h. *If* $x \leq y$, $\min(\boldsymbol{x}, \boldsymbol{y})$ *conv* $\min(\boldsymbol{y}, \boldsymbol{x})$ *conv* $\boldsymbol{x}$. i. *If* $x \geq y$, $\max(\boldsymbol{x}, \boldsymbol{y})$ *conv* $\max(\boldsymbol{y}, \boldsymbol{x})$ *conv* $\boldsymbol{x}$. j. $1 \circ 1$ *conv* $1 \circ 2$ *conv* $2 \circ 1$ *conv* 1; $2 \circ 2$ *conv* 2. k. *If* $x \leq y$, $\epsilon^{\boldsymbol{x}}_{\boldsymbol{y}}$ *conv* 1; *if* $x > y$, $\epsilon^{\boldsymbol{x}}_{\boldsymbol{y}}$ *conv* 2. l. *If* $x \neq y$, $\delta^{\boldsymbol{x}}_{\boldsymbol{y}}$ conv 1; *if* $x = y$, $\delta^{\boldsymbol{x}}_{\boldsymbol{y}}$ *conv* 2.‡

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† 15I is stated with the aid of the convention that if n represents a positive integer of intuitive logic, then $\boldsymbol{n}$ shall represent the corresponding positive integer $S(\cdots n - 1 \text{ times} \cdots S(1) \cdots)$ of our formal theory.

‡ This theorem includes the assertion that the intuitive functions $x + y$, xy, xy, $x - y$, $\min(x, y)$, $\max(x, y)$ are definable (for positive integral arguments and values). Also, constant and identity functions of positive integers are definable: *If* $n, x_1, \cdots, x_n$ *are given positive integers, then* $\mathfrak{C}(\boldsymbol{n}, \boldsymbol{A}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ *conv* $\boldsymbol{A}$ *and* $\mathfrak{I}_{ni}(\mathfrak{I}_n(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n))$ *conv* $\boldsymbol{x}_i$, *where* $\mathfrak{I}_n \rightarrow \lambda \rho_1 \cdots \rho_n f_1 \cdots f_n a \cdot f_1^{\rho_1}(\cdots f_n^{\rho_n}(a))$ *and* $\mathfrak{I}_{ni} \rightarrow \lambda \rho f \cdot \rho(I, \cdots i - 1 \text{ times} \cdots, I, f, I, \cdots n - i \text{ times} \cdots, I)$ (cf. §§ 7, 8).

The remainder of this paper is devoted to further developments of the theories of formal definition and of formal proof in conjunction with each other.

15II. *A necessary condition that a function of positive integers, the values of which are well-formed expressions, be definable is that all the values have the same free symbols.*

This is a consequence of C5VI.

15III(k). *If $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$, $\boldsymbol{F}$ have the same free symbols, then the sequence $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \boldsymbol{F}(1), \boldsymbol{F}(2), \cdots$ is definable by a formula $\mathbf{L}$ such that $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(\boldsymbol{k} + \boldsymbol{X}) =' \boldsymbol{F}(\boldsymbol{X})$.**

Proof. If $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same free symbols, then, by C7I, there exists a formula $\boldsymbol{B}$ such that $\boldsymbol{B}(1)$ conv $\lambda \boldsymbol{n} \cdot I^{\boldsymbol{n}}(\boldsymbol{A})$ and $\boldsymbol{B}(2)$ conv $\boldsymbol{F}$. Let $\boldsymbol{L} \rightarrow \lambda \boldsymbol{n} \cdot \boldsymbol{B}(\min(2, \boldsymbol{n}), \boldsymbol{n} - 1)$.† Then it is clear from 15Ie, g, h that $\boldsymbol{L}$ defines $\boldsymbol{A}$, $\boldsymbol{F}(1)$, $\boldsymbol{F}(2), \cdots$. Also $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(1 + \boldsymbol{X}) =' \boldsymbol{F}(\boldsymbol{X})$, since, assuming $N(\boldsymbol{X})$, we have $\boldsymbol{L}(1 + \boldsymbol{X})$ conv $\boldsymbol{B}(\min(2, S(\boldsymbol{X})), S(\boldsymbol{X}) - 1)$, $= \boldsymbol{B}(2, \boldsymbol{X})$ (13.2, 12.4, 11.2), conv $\boldsymbol{F}(\boldsymbol{X})$. Thus 15III(1) is established. Moreover 15III($k + 1$) is a consequence of 15III(k) and 15III(1).‡ Thus 15III(k) is established by an intuitive induction with respect to k.

COROLLARY. *If $\boldsymbol{A}_{i_1 \ldots i_n}$ $(i_1, \cdots, i_n = 1, \cdots, k)$ have the same free symbols, then a formula $\mathbf{L}$ can be found such that $\mathbf{L}(\boldsymbol{i}_1, \cdots, \boldsymbol{i}_n)$ conv $\boldsymbol{A}_{i_1 \ldots i_n}$.*

This follows from 15III by induction with respect to n, since, given the hypothesis with $n + 1$ replacing n, we can, by using the corollary as stated, find k formulas $\boldsymbol{L}_{i_1}$ such that $\boldsymbol{L}_{i_1}(\boldsymbol{i}_2, \cdots, \boldsymbol{i}_{n+1})$ conv $\boldsymbol{A}_{i_1 \ldots i_{n+1}}$, and then by 15III find an $\boldsymbol{L}$ such that $\boldsymbol{L}(\boldsymbol{i}_1)$ conv $\boldsymbol{L}_{i_1}$.

15IV. *If the free symbols of $\boldsymbol{F}$ are included among those of $\boldsymbol{A}$, then the sequence $\boldsymbol{A}, \boldsymbol{F}(\boldsymbol{A}), \boldsymbol{F}(\boldsymbol{F}(\boldsymbol{A})), \cdots$ is definable by a formula $\mathbf{L}$ such that $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$.*

* For the notation $\vdash' ='$ see the last paragraph of § 2.

† When a heavy-typed letter represents occurrences of a proper symbol in a formula, we shall suppose the symbol to be one whose only occurrences in the formula are those represented by the occurrences of the letter, unless the contrary is implied by the conventions (1) and (2) of § C3. Thus $\boldsymbol{n}$ is here supposed to be distinct from the proper symbols of $\boldsymbol{A}$ and $\boldsymbol{B}$, but in "$\lambda \boldsymbol{n} \cdot \boldsymbol{M}$" $\boldsymbol{n}$ must occur in $\boldsymbol{M}$ as a free symbol in order that $\boldsymbol{M}$ and $\lambda \boldsymbol{n} \cdot \boldsymbol{M}$ be well-formed.

‡ Using the fact that if $\boldsymbol{L}'$ defines $\boldsymbol{A}_2, \cdots, \boldsymbol{A}_{k+1}$, $\boldsymbol{F}(1)$, $\boldsymbol{F}(2), \cdots$, then $\boldsymbol{L}'$ has the same free symbols as $\boldsymbol{A}_2, \cdots, \boldsymbol{A}_{k+1}$ and $\boldsymbol{F}$ (cf. C5VI). Similarly below.

Proof. By 15III, the sequence $\boldsymbol{A}, \boldsymbol{F}^1(\boldsymbol{A}), \boldsymbol{F}^2(\boldsymbol{A}), \cdots$ is definable by an $\boldsymbol{L}$ such that $N(\boldsymbol{Y}) \vdash' \boldsymbol{L}(S(\boldsymbol{Y})) =' \boldsymbol{F}^{\boldsymbol{Y}}(\boldsymbol{A})$. By 15Id, $\boldsymbol{L}$ defines $\boldsymbol{A}$, $\boldsymbol{F}(\boldsymbol{A})$, $\boldsymbol{F}(\boldsymbol{F}(\boldsymbol{A})), \cdots$. Assume $N(\boldsymbol{X})$. Case 1: $\epsilon_1{}^{\boldsymbol{X}} = 1$. Then $\boldsymbol{X} = 1$ (14.8, 12.17, 12.8). Hence $\boldsymbol{L}(S(\boldsymbol{X})) = \boldsymbol{L}(S(1))$, conv $\boldsymbol{F}(\boldsymbol{A})$ (since $\boldsymbol{L}$ defines $\boldsymbol{A}, \boldsymbol{F}(\boldsymbol{A}), \cdots$), conv $\boldsymbol{F}(\boldsymbol{L}(1)), = \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$. Case 2: $\epsilon_1{}^{\boldsymbol{X}} = 2$. Then $\boldsymbol{X} > 1$ (14.9), and $\boldsymbol{X} = S(\boldsymbol{X} - 1)$ (12.5). Hence $\boldsymbol{L}(S(\boldsymbol{X})) = \boldsymbol{L}(S(S(\boldsymbol{X} - 1)))$, $= \boldsymbol{F}^{S(\boldsymbol{X}-1)}(\boldsymbol{A})$ (since $N(\boldsymbol{Y}) \vdash' \boldsymbol{L}(S(\boldsymbol{Y})) =' \boldsymbol{F}^{\boldsymbol{Y}}(\boldsymbol{A})$), conv $\boldsymbol{F}(\boldsymbol{F}^{\boldsymbol{X}-1}(\boldsymbol{A}))$, $= \boldsymbol{F}(\boldsymbol{L}(S(\boldsymbol{X} - 1)))$, $= \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$. Hence, by cases (C9I), $\boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$.*

15V. *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}$, *then the sequence* $\boldsymbol{A}, \boldsymbol{F}(1, \boldsymbol{A}), \boldsymbol{F}(2, \boldsymbol{F}(1, \boldsymbol{A})), \cdots$ *can be defined by a formula* $\boldsymbol{L}$ *such that* $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}(\boldsymbol{X}))$.

Proof. Let $\boldsymbol{A}' \to \lambda \boldsymbol{x} \cdot I^{\boldsymbol{x}}(\boldsymbol{A})$ and $\boldsymbol{F}' \to \lambda \boldsymbol{p}\boldsymbol{x} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x} - 1))$. By 15IV, $\boldsymbol{A}'$, $\boldsymbol{F}'(\boldsymbol{A}')$, $\boldsymbol{F}'(\boldsymbol{F}'(\boldsymbol{A}')), \cdots$ is definable by a formula $\boldsymbol{L}'$ such that $N(\boldsymbol{X}) \vdash' \boldsymbol{L}'(S(\boldsymbol{X})) =' \boldsymbol{F}'(\boldsymbol{L}'(\boldsymbol{X}))$. Let $\boldsymbol{L} \to \lambda \boldsymbol{x} \cdot \boldsymbol{L}'(\boldsymbol{x}, \boldsymbol{x} - 1)$. Then, assuming $N(\boldsymbol{X})$, $\boldsymbol{L}(S(\boldsymbol{X}))$ conv $\boldsymbol{L}'(S(\boldsymbol{X}), S(\boldsymbol{X}) - 1)$, $= \boldsymbol{L}'(S(\boldsymbol{X}), \boldsymbol{X})$ (11.2), $= \boldsymbol{F}'(\boldsymbol{L}'(\boldsymbol{X}), \boldsymbol{X})$, conv $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}'(\boldsymbol{X}, \boldsymbol{X} - 1))$, conv $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}(\boldsymbol{X}))$. Thus $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}(\boldsymbol{X}))$. Similarly, using 15Ig, $\boldsymbol{L}(S(\boldsymbol{i}))$ conv $\boldsymbol{F}(\boldsymbol{i}, \boldsymbol{L}(\boldsymbol{i}))$ $(i = 1, 2, \cdots)$. Also $\boldsymbol{L}(1)$ conv $\boldsymbol{L}'(1, 1 - 1)$, conv $\boldsymbol{L}'(1, 1)$, conv $\boldsymbol{A}'(1)$, conv $I^1(\boldsymbol{A})$, conv $\boldsymbol{A}$. Hence, by intuitive induction with respect to i, $\boldsymbol{L}$ defines $\boldsymbol{A}, \boldsymbol{F}(1, \boldsymbol{A}), \cdots$.

15VI(k). *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$, *and* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$ *have the same free symbols, then the sequence* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \boldsymbol{F}(1, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \boldsymbol{F}(2, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_{k+1}), \cdots$, *where* $\boldsymbol{A}_i$ *denotes the i-th member of the sequence, is definable.*

Proof. For $k = 1$, this is included in 15V.

Suppose k a given positive integer ≥ 2, and let

$$\begin{aligned}
&\mathfrak{A}_{k1} \to \lambda f a_1 \cdots a_k \cdot f(1, a_1, \cdots, a_k),\\
&\mathfrak{A}_{k2} \to \lambda f a_1 \cdots a_k \cdot f(2, a_2, \cdots, a_k, \mathfrak{A}_{k1}(f, a_1, \cdots, a_k)),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\mathfrak{A}_{kk} \to \lambda f a_1 \cdots a_k \cdot f(\boldsymbol{k}, a_k, \mathfrak{A}_{k1}(f, a_1, \cdots, a_k), \cdots, \mathfrak{A}_{k,k-1}(f, a_1, \cdots, a_k)),\\
&\mathfrak{D}_{k1} \to \lambda \rho \cdot \rho(\lambda t_1 \cdots t_k \cdot t_2(I, \cdots k+1 \text{ times} \cdots, I, t_3(I, \cdots, I, \cdots,\\
&\qquad\qquad t_k(I, \cdots, I, t_1) \cdots))),
\end{aligned}$$

* In C9I, $\vdash \boldsymbol{C}$ may be replaced by $\vdash' \boldsymbol{M} =' \boldsymbol{N}$, since C9I may be applied with $\boldsymbol{T}(\boldsymbol{N})$ taken as $\boldsymbol{C}$ and $\vdash$ replaced by $\boldsymbol{T}(\boldsymbol{M}) \vdash$, and with $\boldsymbol{T}(\boldsymbol{M})$ taken as $\boldsymbol{C}$ and $\vdash$ replaced by $\boldsymbol{T}(\boldsymbol{N}) \vdash$.

$$\mathfrak{D}_{k2} \to \lambda\rho \cdot \rho(\lambda t_1 \cdots t_k \cdot t_1(I, \cdots, I, t_3(I, \cdots, I, \cdots, t_k(I, \cdots, I, t_2) \cdots))),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathfrak{D}_{kk} \to \lambda\rho \cdot \rho(\lambda t_1 \cdots t_k \cdot t_1(I, \cdots, I, t_2(I, \cdots, I, \cdots, t_{k-1}(I, \cdots, I, t_k) \cdots))),$$

$$\mathfrak{F}_k \to \lambda\sigma\rho\mu \cdot \mu(\mathfrak{D}_{k2}(\rho), \cdots, \mathfrak{D}_{kk}(\rho), \lambda f a_1 \cdots a_k \cdot f(\boldsymbol{k} + \sigma, \mathfrak{D}_{k1}(\rho, f, a_1, \cdots, a_k), \cdots, \mathfrak{D}_{kk}(\rho, f, a_1, \cdots, a_k))),$$

$$\mathfrak{A}_{k,k+i} \to \lambda f a_1 \cdots a_k \cdot f(\boldsymbol{k} + \boldsymbol{i}, \mathfrak{A}_{ki}(f, a_1, \cdots, a_k), \cdots, \mathfrak{A}_{k,i+k-1}(f, a_1, \cdots, a_k)),$$

$$\mathfrak{B}_{ki} \to \lambda\mu \cdot \mu(\mathfrak{A}_{ki}, \cdots, \mathfrak{A}_{k,i+k-1}) \qquad (i = 1, 2, \cdots).$$

By 15V, there exists a $\mathfrak{M}_k$ which defines $\mathfrak{F}_k(1, \mathfrak{B}_{k1}), \mathfrak{F}_k(2, \mathfrak{F}_k(1, \mathfrak{B}_{k1})), \cdots$.* We observe for $i = 1, \cdots, k$ successively that $\mathfrak{A}_{ki}(I, \cdots k+1$ times $\cdots, I)$ conv I. Assuming that $\mathfrak{A}_{ki}(I, \cdots, I)$ conv I for $i = p, \cdots, p + k - 1$, it follows that $\mathfrak{A}_{k,p+k}(I, \cdots, I)$ conv I. Hence, by intuitive induction, $\mathfrak{A}_{ki}(I, \cdots, I)$ conv I $(i = 1, 2, \cdots)$. Consequently $\mathfrak{D}_{kq}(\mathfrak{B}_{ki})$ conv $\mathfrak{A}_{k,i+q-1}$ $(i = 1, 2, \cdots;\ q = 1, \cdots, k)$. It follows that $\mathfrak{F}_k(\boldsymbol{i}, \mathfrak{B}_{ki})$ conv $\mathfrak{B}_{k,i+1}$ $(i = 1, 2, \cdots)$. Hence $\mathfrak{M}_k$ defines $\mathfrak{B}_{k2}, \mathfrak{B}_{k3}, \cdots$. By 15III, there exists an $\mathfrak{N}_k$ which defines $\mathfrak{A}_{k1}, \cdots, \mathfrak{A}_{kk}, \mathfrak{D}_{kk}(\mathfrak{M}_k(1)), \mathfrak{D}_{kk}(\mathfrak{M}_k(2)), \cdots$. Since $\mathfrak{D}_{kk}(\mathfrak{B}_{ki})$ conv $\mathfrak{A}_{k,i+k-1}$, $\mathfrak{N}_k$ defines $\mathfrak{A}_{k1}, \cdots, \mathfrak{A}_{kk}, \mathfrak{A}_{k,k+1}, \cdots$. By 15III, there exists an $\boldsymbol{L}$ which defines the sequence $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \mathfrak{N}_k(1, \boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \mathfrak{N}_k(2, \boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \cdots$, and hence the sequence $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \mathfrak{A}_{k1}(\boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \mathfrak{A}_{k2}(\boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \cdots$. The $\mathfrak{A}_{ki}$ were so chosen that $\mathfrak{A}_{ki}(\boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k)$ conv $\boldsymbol{A}_{k+i}$. Hence $\boldsymbol{L}(\boldsymbol{i})$ conv $\boldsymbol{A}_i$ $(i = 1, 2, \cdots)$.

15VII. *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}_1$, *then the sequence* $\boldsymbol{A}_1, \boldsymbol{F}(1, \boldsymbol{A}_1), \boldsymbol{F}(2, \boldsymbol{A}_2, \boldsymbol{A}_1), \boldsymbol{F}(3, \boldsymbol{A}_3, \boldsymbol{A}_2, \boldsymbol{A}_1), \cdots$, *where* $\boldsymbol{A}_i$ *denotes the i-th member of the sequence, is definable.*

Proof. Let $\mathfrak{V} \to \lambda\mu f a \cdot \mu(2, f(1, a), a)$ and $\mathfrak{W} \to \lambda\sigma\rho\mu f a \cdot \rho(\lambda\tau \cdot I^\tau(\mu(S^2(\sigma), \rho(f, f, a))), f, a)$. By 15V, there exists a formula $\mathfrak{K}$ which defines $\mathfrak{V}, \mathfrak{W}(1, \mathfrak{V}), \mathfrak{W}(2, \mathfrak{W}(1, \mathfrak{V})), \cdots$. $\mathfrak{K}(2)$ conv $\mathfrak{W}(1, \mathfrak{V})$ (by the def. of $\mathfrak{K}$), conv $\lambda\mu f a \cdot \mathfrak{V}(\lambda\tau \cdot I^\tau(\mu(S^2(1), \mathfrak{V}(f, f, a))), f, a)$ (using the def. of $\mathfrak{W}$), conv $\lambda\mu f a \cdot \{\lambda\tau \cdot I^\tau(\mu(S^2(1), \mathfrak{V}(f, f, a)))\}(2, f(1, a), a)$ (using the def. of $\mathfrak{V}$), conv $\lambda\mu f a \cdot \mu(3, \mathfrak{V}(f, f, a), f(1, a), a)$, conv $\lambda\mu f a \cdot \mu(3, \mathfrak{K}(1, f, f, a), f(1, a), a)$ (by the def. of $\mathfrak{K}$). Assume $\mathfrak{K}(\boldsymbol{i} + 1)$ conv $\lambda\mu f a \cdot \mu(\boldsymbol{i} + 2, \mathfrak{K}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{K}(1, f, f, a), f(1, a), a$. Then $\mathfrak{K}(\boldsymbol{i} + 2)$ conv $\mathfrak{W}(\boldsymbol{i} + 1, \mathfrak{K}(\boldsymbol{i} + 1))$, conv $\lambda\mu f a \cdot \mathfrak{K}(\boldsymbol{i} + 1, \lambda\tau \cdot I^\tau(\mu(S^2(\boldsymbol{i} + 1), \mathfrak{K}(\boldsymbol{i} + 1, f, f, a))), f, a)$, conv $\lambda\mu f a \cdot \{\lambda\mu f a \cdot \mu(\boldsymbol{i} + 2, \mathfrak{K}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{K}(1, f, f, a), f(1, a), a)\}(\lambda\tau \cdot I^\tau(\mu(S^2(\boldsymbol{i} + 1), \mathfrak{K}(\boldsymbol{i} + 1, f, f, a))), f, a)$ (by

* For $\mathfrak{F}_k(1, \mathfrak{B}_{k1}), \mathfrak{F}_k(2, \mathfrak{F}_k(1, \mathfrak{B}_{k1})), \cdots$ conv $\boldsymbol{A}', \boldsymbol{F}'(1, \boldsymbol{A}'), \cdots$, respectively, where $\boldsymbol{A}' \to \mathfrak{F}_k(1, \mathfrak{B}_{k1})$ and $\boldsymbol{F}' \to \lambda x p \cdot \mathfrak{F}_k(x + 1, p)$.

the assumption), conv $\lambda\mu fa \cdot \mu(\boldsymbol{i}+3, \mathfrak{R}(\boldsymbol{i}+1, f, f, a), \mathfrak{R}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{R}(1, f, f, a), f(1, a), a)$. Hence, by induction with respect to i, $\mathfrak{R}(\boldsymbol{i}+1)$ conv $\lambda\mu fa \cdot \mu(\boldsymbol{i}+2, \mathfrak{R}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{R}(1, f, f, a), f(1, a), a)$. By 15III, there can be found an expression $\boldsymbol{L}$ which defines $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\mathfrak{R}(1, \boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$, $\mathfrak{R}(2, \boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1), \cdots$. Then $\boldsymbol{L}(3)$ conv $\mathfrak{B}(\boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$, conv $\boldsymbol{A}_3$. Assuming that $\boldsymbol{L}(\boldsymbol{j})$ conv $\boldsymbol{A}_j$ $(j = 1, \cdots, i+2)$, then $\boldsymbol{L}(\boldsymbol{i}+3)$ conv $\mathfrak{R}(\boldsymbol{i}+1, \boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$, conv $\{\lambda\mu fa \cdot \mu(\boldsymbol{i}+2, \mathfrak{R}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{R}(1, f, f, a), f(1, a), a)\}(\boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$ (as shown above), conv $\boldsymbol{F}(\boldsymbol{i}+2, \boldsymbol{L}(\boldsymbol{i}+2), \cdots, \boldsymbol{L}(3), \boldsymbol{L}(2), \boldsymbol{L}(1))$, conv $\boldsymbol{F}(\boldsymbol{i}+2, \boldsymbol{A}_{i+2}, \cdots, \boldsymbol{A}_1)$ (by hyp.), which is $\boldsymbol{A}_{i+3}$. Hence, by induction, $\boldsymbol{L}(\boldsymbol{i})$ conv $\boldsymbol{A}_i$ $(i = 1, 2, \cdots)$.

In 15III-15VII, the expressions $\boldsymbol{F}$, $\boldsymbol{A}$, $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$ of the hypotheses may be replaced by any definable functions of given numbers of positive integers. For example, 15V can be generalized thus: *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}$, *then a formula* $\boldsymbol{L}$ *can be found such that* $\boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, 1)$ *conv* $\boldsymbol{A}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ *and* $\boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{i}+1)$ *conv* $\boldsymbol{F}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{i}, \boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{i}))$ $(x_1, \cdots, y_m = 1, 2, \cdots)$. For if $\boldsymbol{A}' \rightarrow I^{\boldsymbol{b}_1}(\cdots I^{\boldsymbol{b}_m}(\boldsymbol{A}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n))\cdots)$ and $\boldsymbol{F}' \rightarrow I^{\boldsymbol{a}_1}(\cdots I^{\boldsymbol{a}_n}(\boldsymbol{F}(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m))\cdots)$, where $\boldsymbol{a}_1, \cdots, \boldsymbol{b}_m$ are distinct proper symbols, then, by 15V, there exists an expression $\boldsymbol{L}'$ which defines $\boldsymbol{A}'$, $\boldsymbol{F}'(1, \boldsymbol{A}')$, $\cdots$, and we may take for $\boldsymbol{L}$ the function $\lambda \boldsymbol{r}_1 \cdots \boldsymbol{s}_m \boldsymbol{p} \cdot \{\lambda \boldsymbol{a}_1 \cdots \boldsymbol{b}_m \cdot \boldsymbol{L}'(\boldsymbol{p})\}(\boldsymbol{r}_1, \cdots, \boldsymbol{s}_m)$. Any of the parameters $\boldsymbol{x}_1, \cdots, \boldsymbol{y}_m$ of the sequence defined by $\boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{y}_m)$ may be equated, since *a function obtained from a definable function by equating (or interchanging) a pair of variables is definable (provided the domains of the two variables are the same).* For if $\boldsymbol{L}$ defines $L(x, y)$, then $\lambda \boldsymbol{x} \cdot \boldsymbol{L}(\boldsymbol{x}, \boldsymbol{x})$ defines $L'(x)$ where $L'(x) = L(x, x)$ and $\lambda \boldsymbol{xy} \cdot \boldsymbol{L}(\boldsymbol{y}, \boldsymbol{x})$ defines $L''(x, y)$ where $L''(x, y) = L(y, x)$; and similarly for functions of more variables. *A function obtained from a definable function by substituting for a certain variable a definable function of other variables is definable (provided the domain of the replaced variable contains the domain of values of the substituted function).*

It is clear from the foregoing that every function recursive in the limited sense of Gödel (1931)* is definable, if we use $\lambda fx \cdot f(x)$, $S(\lambda fx \cdot f(x))$, $S(S(\lambda fx \cdot f(x))), \cdots$ as formulas for the numbers $0, 1, 2, \cdots$, resp. (thus going over from our theory of positive integers to a like theory of natural numbers), or if we replace natural numbers by positive integers in Gödel's theory. In either case Gödel's Theorems I-IV provide a convenient means

* Kurt Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I," *Monatshefte für Mathematik und Physik*, vol. 38 (1931), pp. 173-198. Cf. p. 179.

for showing that various functions, such as quotient, remainder, highest common factor, n-th prime number, are definable.*

It is also true that functions recursive in various more general senses may be defined formally.†

In some situations in which one of the above methods can be used a special device may be more expeditious.

Situations which do not come precisely within the scope of any one of the theorems of this and the following sections may often be dealt with by using several of them and by employing supplementary devices. As a general method of procedure, when it is not at once evident how to define a sequence $\boldsymbol{K}_1, \boldsymbol{K}_2, \cdots$, we attempt to find another sequence $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$ and a $\boldsymbol{J}$ such that $\boldsymbol{J}(\boldsymbol{K}'_1)$ conv $\boldsymbol{K}_1$, $\boldsymbol{J}(\boldsymbol{K}'_2)$ conv $\boldsymbol{K}_2, \cdots$ and to define $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$; or, more generally, to find and define two other sequences $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$ and $\boldsymbol{K}''_1, \boldsymbol{K}''_2, \cdots$ such that $\boldsymbol{K}''_1(\boldsymbol{K}'_1)$ conv $\boldsymbol{K}_1, \boldsymbol{K}''_2(\boldsymbol{K}'_2)$ conv $\boldsymbol{K}_2, \cdots$.

In case there is given a recursive situation like that in one of our theorems but with the function relating the members of the sequence in intuitive logic, the difficulty of finding a function $\boldsymbol{F}$ of the formal logic relating the members may often be evaded by the introduction into the terms of the sequence of an extra bound symbol on which a substitution can be made which transforms any member of the sequence $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$ thus obtained into the next member.

Given a positive integer n, let n_0 denote n^n, and n_{k+1} denote $(\cdots(n_k)_k\cdots)_k$ (n_k subscripts). n_n as a function of n is defined formally by $\mathfrak{Z}$ if $\mathfrak{Z} \rightarrow \lambda n \cdot [\lambda \rho m \cdot \rho^{\rho(m)}(m)]^n(\lambda r \cdot r^r, n)$. It is amazing that such a brief formula as $\mathfrak{Z}(3)$ should have so long a normal form (cf. § C5).

16. Finite sums and products. Let $\mathfrak{f} \rightarrow \lambda \pi \rho f m \cdot \rho(f, m) + f(m + \pi)$. By 15V, the sequence $1, \mathfrak{f}(1, 1), \mathfrak{f}(2, \mathfrak{f}(1, 1)), \cdots$ is definable by a formula $\mathfrak{S}$

* In the first case, it should be noted at the outset that sum, product, difference, etc., are definable in the resulting theory of natural numbers.

In the second case, the absence of 0 causes no difficulty in proving Gödel's I-IV (as modified in statement by the change from natural numbers to positive integers), since o may be used to multiply 1's and 2's as 0's and 1's, respectively (cf. 15Ij).

† As an example, given formulas $\boldsymbol{F}$ and $\boldsymbol{G}$ having the same free symbols, to obtain a formula $\boldsymbol{H}$ such that $\boldsymbol{H}(1, \boldsymbol{n})$ conv $\boldsymbol{F}(\boldsymbol{n})$, $\boldsymbol{H}(\boldsymbol{m}+1, 1)$ conv $\boldsymbol{G}(\boldsymbol{m})$, and $\boldsymbol{H}(\boldsymbol{m}+1, \boldsymbol{n}+1)$ conv $\boldsymbol{H}(\boldsymbol{m}, \boldsymbol{H}(\boldsymbol{m}+1, \boldsymbol{n}))$ $(m, n = 1, 2, \cdots)$, we may use 15III-15V, according to which formulas $\boldsymbol{L}$, $\mathfrak{R}$, and $\mathfrak{H}$ can be found such that $\boldsymbol{L}(1)$ conv $\boldsymbol{F}$, $\boldsymbol{L}(2)$ conv $\boldsymbol{G}$, $\mathfrak{R}(1)$ conv $\lambda hxyl \cdot h(1, Iy, I, l(2, x))$, $\mathfrak{R}(\boldsymbol{n}+1)$ conv $\lambda hxyl \cdot h(\mathfrak{R}(\boldsymbol{n}, h, x, y-1, l), l)$, $\mathfrak{H}(1)$ conv $\lambda yl \cdot l(1, y)$, and $\mathfrak{H}(\boldsymbol{m}+1)$ conv $\lambda y \cdot \mathfrak{R}(y, \mathfrak{H}(\boldsymbol{m}), \boldsymbol{m}, y)$, and let $\boldsymbol{H} \rightarrow \lambda pq \cdot \mathfrak{H}(p, q, \boldsymbol{L})$. By induction with respect to m, $\mathfrak{H}(\boldsymbol{m}, 1, I, I)$ conv I; using this fact, $\boldsymbol{H}$ will be found to have the desired properties.

such that $N(\boldsymbol{X}) \vdash' \mathfrak{S}(S(\boldsymbol{X})) =' \mathfrak{f}(\boldsymbol{X}, \mathfrak{S}(\boldsymbol{X}))$. Then $\mathfrak{S}(\boldsymbol{i}, \lambda \boldsymbol{x} \cdot \boldsymbol{F}(\boldsymbol{x}), \boldsymbol{m})$ conv $\boldsymbol{F}(\boldsymbol{m}) + \boldsymbol{F}(\boldsymbol{m} + 1) + \cdots + \boldsymbol{F}([\boldsymbol{m} + \boldsymbol{i}] - 1)$ $(m, i = 1, 2, \cdots)$. Let $\sum_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} [\boldsymbol{R}]$ be an abbreviation for $\mathfrak{S}([\boldsymbol{n} + 1] - \boldsymbol{m}, \lambda \boldsymbol{x} \cdot \boldsymbol{R}, \boldsymbol{m})$, and define $\prod_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} [\boldsymbol{R}]$ similarly, replacing the first occurrence of $+$ in $\mathfrak{f}$ by $\times$.

16I. *If m and n are positive integers and $m \leq n$, then* $\sum_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} \boldsymbol{F}(\boldsymbol{x})$ *conv* $\boldsymbol{F}(\boldsymbol{m}) + \boldsymbol{F}(\boldsymbol{m} + 1) + \cdots + \boldsymbol{F}(\boldsymbol{n})$ *and* $\prod_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} F(x)$ *conv* $\boldsymbol{F}(\boldsymbol{m}) \times \boldsymbol{F}(\boldsymbol{m} + 1) \times \cdots \times \boldsymbol{F}(\boldsymbol{n})$.

16.1: $\quad [N(\rho) \supset_\rho N(f(\rho))] \supset_f \cdot N(n) \supset_n \cdot N(\sum_{x=1}^{n} f(x))$.

16.2: $\quad [N(\rho) \supset_\rho N(f(\rho))] \supset_f \cdot N(n) \supset_n \cdot \sum_{x=1}^{S(n)} f(x) = \sum_{x=1}^{n} f(x) + f(S(n))$.

16.3: $\quad [N(\rho) \supset_\rho N(f(\rho))] \supset_f \cdot N(n) \supset_n \cdot S(\sum_{x=1}^{n} f(x)) > n$.

Proofs. Assume $N(\rho) \supset_\rho N(f(\rho))$. Then (1) $N(f(1))$, and by conversion, $N(\sum_{x=1}^{1} f(x))$. (2) Assume $N(n)$. Then $\sum_{x=1}^{S(n)} f(x)$ conv $\mathfrak{S}([S(n) + 1] - 1, \lambda x \cdot f(x), 1)$, $= \mathfrak{S}(S(n), \lambda x \cdot f(x), 1)$, $= \mathfrak{f}(n, \mathfrak{S}(n), \lambda x \cdot f(x), 1)$, conv $\mathfrak{S}(n, \lambda x \cdot f(x), 1) + f(1 + n)$, $= \mathfrak{S}([n + 1] - 1, \lambda x \cdot f(x), 1) + f(S(n))$, which is $\sum_{x=1}^{n} f(x) + f(S(n))$. (3) Assuming $N(n)$ and $N(\sum_{x=1}^{n} f(x))$, and using (2) and 5.2, $N(\sum_{x=1}^{S(n)} f(x))$. (4) From (1) and (3), by induction, $N(n) \supset_n \cdot N(\sum_{x=1}^{n} f(x))$. Hence $\vdash$ 16.1. (5) Assume $N(n)$. Using 16.1, $E(\sum_{x=1}^{S(n)} f(x))$. Hence, using (2) and § 2, $\sum_{x=1}^{S(n)} f(x) = \sum_{x=1}^{n} f(x) + f(S(n))$. Hence $\vdash$ 16.2. (6) By (1) and 12.8, $S(\sum_{x=1}^{1} f(x)) > 1$. Assume $N(n)$ and $S(\sum_{x=1}^{n} f(x)) > n$. Then $S^2(\sum_{x=1}^{S(n)} f(x)) = S^2(\sum_{x=1}^{n} f(x) + f(S(n)))$ (by (2)), $= S(\sum_{x=1}^{n} f(x)) + S(f(S(n)))$, $> S(\sum_{x=1}^{n} f(x)) + 1$ (12.8, 12.11), conv $S^2(\sum_{x=1}^{n} f(x))$, $> S(n)$ (by $S(\sum_{x=1}^{n} f(x)) > n$ and 12.11). Hence $S(\sum_{x=1}^{S(n)} f(x)) > S(n)$ (12.14). By induction, $N(n) \supset_n \cdot S(\sum_{x=1}^{n} f(x)) > n$.

16.4: $\quad N(k)[N(\rho) \supset_\rho \cdot f(\rho) = k] \supset_{fk} \cdot N(n) \supset_n \cdot \sum_{x=1}^{n} f(x) = nk$.

Proof. Assume $N(k)\cdot N(\rho)\supset_\rho\cdot f(\rho)=k$. Then $\sum_{x=1}^{1} f(x)$ conv $f(1)$, $=k$, $=1k$ (6.1); and, assuming $N(n)$ and $\sum_{x=1}^{n} f(x)=nk$, $\sum_{x=1}^{S(n)} f(x)=\sum_{x=1}^{n} f(x)$ $+f(S(n))$ (16.2), $=nk+k$, $=S(n)k$. By induction, $N(n)\supset_n\cdot\sum_{x=1}^{n} f(x)=nk$.

17. Formal definition: successions of finite sequences. By 15III, we can find a $\mathfrak{U}$ such that $\mathfrak{U}(1)$ conv $\lambda rpqm\cdot m(\delta^{r(p)}_{S(q)}, p, S(q))$ and $\mathfrak{U}(2)$ conv $\lambda rpqm\cdot I^q(m(\delta_1{}^{r(S(p))}, S(p), 1))$. Then, by 15IV, we can find a $\mathfrak{B}$ such that $\mathfrak{B}(1)$ conv $\lambda rm\cdot m(\delta_1{}^{r(1)}, 1, 1)$, $\mathfrak{B}(\boldsymbol{k}+1)$ conv $\lambda r\cdot\mathfrak{B}(\boldsymbol{k}, r, \lambda\pi\cdot\mathfrak{U}(\pi, r))$ $(k=1,2,\cdots)$, and $N(\boldsymbol{X})\vdash'\mathfrak{B}(S(\boldsymbol{X}))='\lambda\boldsymbol{r}\cdot\mathfrak{B}(\boldsymbol{X},\boldsymbol{r},\lambda\pi\cdot\mathfrak{U}(\pi,\boldsymbol{r}))$. Let $\mathfrak{Q}\rightarrow\lambda frn\cdot\mathfrak{B}(n, r, \lambda uvw\cdot I^u(f(v, w)))$.

17I. *If* $\boldsymbol{R}$ *defines the sequence* $r_1, r_2, \cdots$ *of positive integers, then* $\mathfrak{Q}(\boldsymbol{F},\boldsymbol{R})$ *defines the sequence* $\boldsymbol{F}(1,1)$, $\boldsymbol{F}(1,2),\cdots,\boldsymbol{F}(1,\boldsymbol{r}_1)$, $\boldsymbol{F}(2,1)$, $\boldsymbol{F}(2,2),\cdots,$ $\boldsymbol{F}(2,\boldsymbol{r}_2),\cdots$.

For, under the hypothesis, $\lambda n\cdot\mathfrak{B}(n,\boldsymbol{R})$ defines the sequence $\lambda m\cdot m(1,1,1)$, $\lambda m\cdot m(1,1,2),\cdots,\lambda m\cdot m(1,1,\boldsymbol{r}_1-1)$, $\lambda m\cdot m(2,1,\boldsymbol{r}_1)$, $\lambda m\cdot m(1,2,1)$, $\lambda m\cdot m(1,2,2),\cdots,\lambda m\cdot m(1,2,\boldsymbol{r}_2-1)$, $\lambda m\cdot m(2,2,\boldsymbol{r}_2),\cdots$, from which fact the conclusion follows.

$$17.1:\quad \begin{aligned}&[N(\xi)\supset_\xi N(r(\xi))]\supset_r\cdot N(p)\supset_p\\ &\quad\cdot[x<S(p)\supset_x\cdot y<S(r(x))\supset_y\cdot t(f(x,y))]\supset_{ft}\\ &\quad\cdot z<S(\sum_{i=1}^{p} r(i))\supset_z\cdot t(\mathfrak{Q}(f,r,z)).\end{aligned}$$

Proof. Note that $N(\boldsymbol{l})N(\boldsymbol{p})N(\boldsymbol{q})\vdash E(\lambda\boldsymbol{m}\cdot\boldsymbol{m}(\boldsymbol{l},\boldsymbol{p},\boldsymbol{q}))$. Assume $N(\xi)\supset_\xi N(r(\xi))$.

(i) Let $\mathfrak{C}_r\rightarrow\lambda p\sigma\cdot\mathfrak{B}([\sum_{i=1}^{p} r(i)+\min(\sigma, r(p))]-r(p), r)=\lambda m$ $\cdot\, m(\delta^{r(p)}_{\min(\sigma,r(p))}, p, \min(\sigma, r(p)))$. (1a) $\mathfrak{B}([\sum_{i=1}^{1} r(i)+\min(1, r(1))]$ $-r(1), r)$ conv $\mathfrak{B}([r(1)+\min(1, r(1))]-r(1), r)$, $=\mathfrak{B}(1, r)$, conv $\lambda m\cdot m(\delta_1{}^{r(1)}, 1, 1)$, $=\lambda m\cdot m(\delta^{r(1)}_{\min(1,r(1))}, 1, \min(1, r(1)))$. Thus $\mathfrak{C}_r(1,1)$. (b) Assume $N(\sigma)$ and $\mathfrak{C}_r(1,\sigma)$. Case 1: $\epsilon^{S(\sigma)}_{r(1)}=2$. Then $S(\sigma)>r(1)$; consequently $\min(\sigma, r(1))=r(1)$, $=\min(S(\sigma), r(1))$; and hence $\mathfrak{C}_r(1, S(\sigma))$ follows from $\mathfrak{C}_r(1,\sigma)$. Case 2: $\epsilon^{S(\sigma)}_{r(1)}=1$. Then $S(r(1))>S(\sigma)$, $r(1)>\sigma$, $\min(S(\sigma), r(1))=S(\sigma)$, and $\min(\sigma, r(1))=\sigma$. Hence $\mathfrak{B}([\sum_{i=1}^{1} r(i)+\min(S(\sigma), r(1))]-r(1), r)=\mathfrak{B}(\min(S(\sigma), r(1)), r)$,

$= \mathfrak{B}(S(\sigma), r)$, $= \mathfrak{B}(S(\min(\sigma, r(1))), r)$, $= \mathfrak{B}(\min(\sigma, r(1)), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$ (using the definition of $\mathfrak{B}$), $= \mathfrak{B}([\sum_{i=1}^{1} r(i) + \min(\sigma, r(1))] - r(1), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$, $= \{\lambda m \cdot m(\delta^{r(1)}_{\min(\sigma, r(1))}, 1, \min(\sigma, r(1)))\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$ (by $\mathfrak{C}_r(1, \sigma)$), $= \{\lambda m \cdot m(\delta_\sigma{}^{r(1)}, 1, \sigma)\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$, conv $\mathfrak{U}(\delta_\sigma{}^{r(1)}, r, 1, \sigma)$, $= \mathfrak{U}(1, r, 1, \sigma)$ (since $\sigma < r(1)$), conv $\lambda m \cdot m(\delta^{r(1)}_{S(\sigma)}, 1, S(\sigma))$ (using the def. of $\mathfrak{U}$), $= \lambda m \cdot m(\delta^{r(1)}_{\min(S(\sigma), r(1))}, 1, \min(S(\sigma), r(1)))$. Thus $\mathfrak{C}_r(1, S(\sigma))$. Hence, by cases (C9I), $\mathfrak{C}_r(1, S(\sigma))$. (c) From (a) and (b) by induction, $N(\sigma) \supset_\sigma \cdot \mathfrak{C}_r(1, \sigma)$. (2) Assume $N(p)$ and $N(\sigma) \supset_\sigma \cdot \mathfrak{C}_r(p, \sigma)$. (a) $\mathfrak{B}([\sum_{i=1}^{S(p)} r(i) + \min(1, r(S(p)))] - r(S(p)), r) = \mathfrak{B}(S(\sum_{i=1}^{n} r(i)), r)$ (16.2, 11.2, 5.4, 13.2, 12.8), $= \mathfrak{B}(\sum_{i=1}^{p} r(i), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$ (by the def. of $\mathfrak{B}$), $= \mathfrak{B}([\sum_{i=1}^{p} r(i) + r(p)] - r(p), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$, $= \mathfrak{B}([\sum_{i=1}^{p} r(i) + \min(r(p), r(p))] - r(p), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$, $= \{\lambda m \cdot m(\delta^{r(p)}_{\min(r(p), r(p))}, p, \min(r(p), r(p)))\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$ (by $N(\sigma) \supset_\sigma \mathfrak{C}_r(p, \sigma)$ and $N(r(p))$), $= \{\lambda m \cdot m(2, p, r(p))\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$, conv $\mathfrak{U}(2, r, p, r(p))$, conv $\lambda m \cdot I^{r(p)}(m(\delta_1{}^{r(S(p))}, S(p), 1))$ (using the def. of $\mathfrak{U}$), $= \lambda m \cdot m(\delta_1{}^{r(S(p))}, S(p), 1)$, $= \lambda m \cdot m(\delta^{r(S(p))}_{\min(1, r(S(p)))}, S(p), \min(1, r(S(p))))$. Thus $\mathfrak{C}_r(S(p), 1)$. (b) Assuming $N(\sigma)$ and $\mathfrak{C}_r(S(p), \sigma)$, $\mathfrak{C}_r(S(p), S(\sigma))$ follows by reasoning like that used in (1b) (in Case 2, 16.2 is used). (c) From (a) and (b) by induction, $N(\sigma) \supset_\sigma \mathfrak{C}_r(S(p), \sigma)$. (3) From (1) and (2) by induction, $N(p) \supset_p \cdot N(\sigma) \supset_\sigma \mathfrak{C}_r(p, \sigma)$. Thence we can infer $N(p) \supset_p \cdot \sigma < S(r(p)) \supset_\sigma \cdot \mathfrak{B}([\sum_{i=1}^{p} r(i) + \sigma] - r(p), r) = \lambda m \cdot m(\delta_\sigma{}^{r(p)}, p, \sigma)$.

(ii) Let $\mathfrak{L}_{rz} \to \lambda p \cdot \sum ab \cdot a < S(p) \cdot b < S(r(a)) \cdot z = [\sum_{i=1}^{a} r(i) + b] - r(a)$. (a) Assume $z < S(\sum_{i=1}^{1} r(i))$. Then $z = [r(1) + z] - r(1)$, conv $[\sum_{i=1}^{1} r(i) + z] - r(1)$; also $1 < S(1)$ and $z < S(\sum_{i=1}^{1} r(i))$, conv $S(r(1))$. Hence, using Axiom 14 and Rule IV, $\mathfrak{L}_{rz}(1)$. By Theorem I, $z < S(\sum_{i=1}^{1} r(i)) \supset_z \mathfrak{L}_{rz}(1)$. (b) Assume $N(p)$, $z < S(\sum_{i=1}^{p} r(i)) \supset_z \mathfrak{L}_{rz}(p)$, and $z < S(\sum_{i=1}^{S(p)} r(i))$. Case 1: $\epsilon(S(\sum_{i=1}^{p} r(i)), z) = 2$. Then $z < S(\sum_{i=1}^{n} r(i))$, and, using $z < S(\sum_{i=1}^{n} r(i)) \supset_z \mathfrak{L}_{rz}(p)$, we can prove $\mathfrak{L}_{rz}(S(p))$ by means of the second clause of Theorem I. Case 2: $\epsilon(S(\sum_{i=1}^{n} r(i)), z) = 1$. Then $z > \sum_{i=1}^{n} r(i)$, and hence $z = \sum_{i=1}^{p} r(i) + \cdot z - \sum_{i=1}^{p} r(i)$ (12.5), $= [\sum_{i=1}^{S(p)} r(i) + \cdot z - \sum_{i=1}^{p} r(i)] - r(S(p))$ (16.2, 11.2, 5.4). Case A: $\epsilon(z - \sum_{i=1}^{p} r(i), r(S(p))) = 1$. Then $z - \sum_{i=1}^{p} r(i) < S(r(S(p)))$. Case B: $\epsilon(z - \sum_{i=1}^{p} r(i), r(S(p))) = 2$. Then $z - \sum_{i=1}^{p} r(i) > r(S(p))$. Hence $\sum_{i=1}^{S(p)} r(i) = \sum_{i=1}^{p} r(i) + r(S(p))$ (16.2), $< \sum_{i=1}^{p} r(i) + \cdot z - \sum_{i=1}^{p} r(i)$ (12.11), $= z$. Hence $\epsilon(z, \sum_{i=1}^{S(p)} r(i)) = 2$. But $\epsilon(z, \sum_{i=1}^{S(p)} r(i)) = 1$ is a consequence of the assumption $z < S(\sum_{i=1}^{S(p)} r(i))$. Hence, by cases A and B and *reductio ad absurdum* (C10II), $z - \sum_{i=1}^{p} r(i) < S(r(S(p)))$. Also $S(p) < S^2(p)$. Hence, using Axiom 14 and Rule IV, $\mathfrak{L}_{rz}(S(p))$. Hence, by cases 1 and 2 (C9I), $\mathfrak{L}_{rz}(S(p))$. By Thm. I, $z < S(\sum_{i=1}^{S(p)} r(i)) \supset_z \mathfrak{L}_{rz}(S(p))$.

(c) From (a) and (b) by induction, $N(p) \supset_p \cdot z < S(\sum_{i=1}^{p} r(i)) \supset_z \mathfrak{L}_{rz}(p)$.

(iii) Assume $N(p)$, $x < S(p) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x, y))$, and $z < S(\sum_{i=1}^{p} r(i))$. Then, by (ii), $\sum ab \cdot a < S(p) \cdot b < S(r(a)) \cdot z = [\sum_{i=1}^{a} r(i) + b] - r(a)$. Assume $a < S(p) \cdot b < S(r(a)) \cdot z = [\sum_{i=1}^{a} r(i) + b] - r(a)$. Then $\mathfrak{Q}(f, r, z)$ conv $\mathfrak{B}(z, r, \lambda uvw \cdot I^u(f(v, w)))$, $= \mathfrak{B}([\sum_{i=1}^{a} r(i) + b] - r(a), r, \lambda uvw \cdot I^u(f(v, w)))$, $= \{\lambda m \cdot m(\delta_b{}^{r(a)}, a, b)\}(\lambda uvw \cdot I^u(f(v, w)))$ (by (i)), conv $\delta_b{}^{r(a)}(I, f(a, b))$, $= f(a, b)$ (7.2). Moreover $t(f(a, b))$ is provable from our assumptions. Hence $t(\mathfrak{Q}(f, r, z))$. By the second clause of Theorem I, $t(\mathfrak{Q}(f, r, z))$ is provable without the last assumption.

17.2: $[N(\xi) \supset_\xi N(r(\xi))] \supset_r \cdot [N(x) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x, y))] \supset_{ft} \cdot N(z) \supset_z \cdot t(\mathfrak{Q}(f, r, z))$.

Proof. Assuming $N(\xi) \supset_\xi N(r(\xi))$, $N(x) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x, y))$, and $N(z)$, we can prove $x < S(z) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x, y))$, and also, using 16.3, $z < S(\sum_{i=1}^{z} r(i))$. Hence, by 17.1, $t(\mathfrak{Q}(f, r, z))$.

Using 17I, the dyads (triads, $\cdots$) of positive integers can be *enumerated formally* (*i. e.*, there is an enumeration of them which is definable formally). As another application of $\mathfrak{Q}$, we establish the following theorem:

17II. *If* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$, $\boldsymbol{R}_1, \cdots, \boldsymbol{R}_{m+n}$ *contain no free symbols, then a* *formula* $\boldsymbol{H}$ *can be found such that* (1) $\boldsymbol{H}$ *enumerates formally* (*with repetitions*) *the formulas derivable from* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$ *by zero or more operations of passing from* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *to* $\boldsymbol{R}_1(\boldsymbol{A}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}), \boldsymbol{R}_{m+1}(\boldsymbol{A}, \boldsymbol{B}), \cdots$, *or* $\boldsymbol{R}_{m+n}(\boldsymbol{A}, \boldsymbol{B})$, *and* (2) $\boldsymbol{T}(\boldsymbol{A}_1), \cdots, \boldsymbol{T}(\boldsymbol{A}_l), \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_1(\boldsymbol{a})), \cdots, \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_m(\boldsymbol{a})), \boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+1}(\boldsymbol{a}, \boldsymbol{b})), \cdots, \boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+n}(\boldsymbol{a}, \boldsymbol{b})) \vdash \boldsymbol{N}(\boldsymbol{z}) \supset_{\boldsymbol{z}} \boldsymbol{T}(\boldsymbol{H}(\boldsymbol{z}))$.

Proof. Let $\boldsymbol{A}_{1i} \to \boldsymbol{A}_i$ $(i = 1, \cdots, l_1$, where $l_1 = l)$. Given $\boldsymbol{A}_{ki}$ $(i = 1, \cdots, l_k)$, let $\boldsymbol{A}_{k+1,1}, \cdots, \boldsymbol{A}_{k+1,l_{k+1}}$, where $l_{k+1} = (1 + m + n)l_k^2$, be the formulas $\boldsymbol{A}_{k1}, \cdots, \boldsymbol{A}_{kl_k}, \cdots, \boldsymbol{A}_{k1}, \cdots, \boldsymbol{A}_{kl_k}$; $\boldsymbol{R}_1(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_1(\boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_1(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_1(\boldsymbol{A}_{kl_k})$; $\cdots$; $\boldsymbol{R}_m(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}_{kl_k})$; $\boldsymbol{R}_{m+1}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+1}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_{m+1}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+1}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{kl_k})$; $\cdots$; $\boldsymbol{R}_{m+n}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+n}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_{m+n}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+n}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{kl_k})$ $(1 + m + n$ sets of l_k sets of l_k formulas each), respectively. Then the sequence of formulas $\boldsymbol{A}_{k1}, \cdots, \boldsymbol{A}_{kl_k}$ (defined by induction with respect to k) is an enumeration (with repetitions) of the formulas derivable from $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$ by not more than $k - 1$ applications of the operations under consideration.

By 15III, there can be found a formula $\boldsymbol{F}_1$ such that $\boldsymbol{F}_1(\boldsymbol{i})$ conv $\boldsymbol{A}_{1i}$ $(i = 1, \cdots, l_1)$, and a formula $\boldsymbol{J}$ which defines the finite sequence $\lambda fji \cdot I^j(f(i))$, $\lambda fji \cdot \boldsymbol{R}_1(I^j(f(i))), \cdots, \lambda fji \cdot \boldsymbol{R}_m(I^j(f(i))), \lambda fji \cdot \boldsymbol{R}_{m+1}(f(j), f(i)), \cdots, \lambda fji \cdot \boldsymbol{R}_{m+n}(f(j), f(i))$. By 15IV, the sequence $l_1, l_2, l_3, \cdots$ can be defined by a formula $\boldsymbol{L}$ such that $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' [1 + \boldsymbol{m} + \boldsymbol{n}]\boldsymbol{L}(\boldsymbol{X})\boldsymbol{L}(\boldsymbol{X})$. If $\boldsymbol{F}_{k+1} \to \mathfrak{Q}\,(\lambda v \cdot \mathfrak{Q}\,(\boldsymbol{J}(v, \boldsymbol{F}_k), \lambda w \cdot I^w(\boldsymbol{L}(\boldsymbol{k}))), \lambda w \cdot I^w(\boldsymbol{L}(\boldsymbol{k})\boldsymbol{L}(\boldsymbol{k})))$ $(k = 1, 2, \cdots)$, then, by 15V, the sequence $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots$ is definable by a formula $\boldsymbol{F}$ such that $N(\boldsymbol{Y}) \vdash' \boldsymbol{F}(S(\boldsymbol{Y})) =' \mathfrak{Q}\,(\lambda \boldsymbol{v} \cdot \mathfrak{Q}\,(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{Y})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{Y}))), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{Y})\boldsymbol{L}(\boldsymbol{Y})))$. Let $\boldsymbol{H} \to \mathfrak{Q}(\boldsymbol{F}, \boldsymbol{L})$.

Assuming that $\boldsymbol{F}_k(\boldsymbol{i})$ conv $\boldsymbol{A}_{ki}$ $(i = 1, \cdots, l_k)$, it follows by 17I and the definitions of $\boldsymbol{F}_{k+1}$, $\boldsymbol{J}$ and $\boldsymbol{L}$ that $\boldsymbol{F}_{k+1}(\boldsymbol{i})$ conv $\boldsymbol{A}_{k+1,i}$ $(i = 1, \cdots, l_{k+1})$. By induction with respect to k, $\boldsymbol{F}_k(\boldsymbol{i})$ conv $\boldsymbol{A}_{ki}$ $(i = 1, \cdots, l_k;\ k = 1, 2, \cdots)$. Hence, by 17I and the definitions of $\boldsymbol{H}$, $\boldsymbol{F}$ and $\boldsymbol{L}$, $\boldsymbol{H}$ defines $\boldsymbol{A}_{11}, \cdots, \boldsymbol{A}_{1l_1}, \boldsymbol{A}_{21}, \cdots, \boldsymbol{A}_{2l_2}, \cdots$. Hence (1) is satisfied.

Assume $\boldsymbol{T}(\boldsymbol{A}_1), \cdots, \boldsymbol{T}(\boldsymbol{A}_l), \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_1(\boldsymbol{a})), \cdots, \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_m(\boldsymbol{a}))$, $\boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+1}(\boldsymbol{a}, \boldsymbol{b})), \cdots, \boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+n}(\boldsymbol{a}, \boldsymbol{b}))$. In the following we suppose $\boldsymbol{q}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ to represent variables distinct from each other and from the variables of $\boldsymbol{T}$. (1) $N(\xi) \supset_\xi N(\boldsymbol{L}(\xi))$ can be proved by induction. (2) Using $\boldsymbol{T}(\boldsymbol{A}_1), \cdots, \boldsymbol{T}(\boldsymbol{A}_l)$, we can prove $N(\boldsymbol{y}) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}_1(\min(\boldsymbol{y}, \boldsymbol{l})))$ by induction from an l-tuple basis, and thence infer $\boldsymbol{y} < S(\boldsymbol{l}) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}_1(\boldsymbol{y}))$ by use of Theorem I and 13.2. By conversion, $\boldsymbol{y} < S(\boldsymbol{L}(1)) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(1, \boldsymbol{y}))$.

(3) Assume $N(\boldsymbol{q})$ and $\boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$. (a) Assuming $\boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q}))$ and $\boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q}))$, we can infer $\boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{x}))$ and $\boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$; thence, using $\boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_c(\boldsymbol{a}))$, $\boldsymbol{T}(\boldsymbol{R}_c(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y})))$ $(c = 1, \cdots, m)$, and, using $\boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+d}(\boldsymbol{a}, \boldsymbol{b}))$, $\boldsymbol{T}(\boldsymbol{R}_{m+d}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{x}), \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y})))$ $(d = 1, \cdots, n)$; also, using the definition of $\boldsymbol{J}$, $\boldsymbol{J}(1, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y})$, $\boldsymbol{J}(1 + \boldsymbol{c}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{R}_c(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$, and $\boldsymbol{J}(1 + \boldsymbol{m} + \boldsymbol{d}, \boldsymbol{F}(\boldsymbol{q})\ \boldsymbol{x}, \boldsymbol{y}) \cdot = \boldsymbol{R}_{m+d}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{x}), \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$; hence $\boldsymbol{T}(\boldsymbol{J}(1, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$, $\boldsymbol{T}(\boldsymbol{J}(1 + \boldsymbol{c}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$, $\boldsymbol{T}(\boldsymbol{J}(1 + \boldsymbol{m} + \boldsymbol{d}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$. Thus, for $j = 1, \cdots, 1 + m + n$, $\boldsymbol{T}(\boldsymbol{J}(\boldsymbol{j}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$ is a consequence of $\boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q}))$, $\boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q}))$ and our other assumptions. Using these relations, we can prove by means of Thm. I and induction from a $1 + m + n$-tuple basis, $N(\boldsymbol{v}) \supset_{\boldsymbol{v}} \cdot \boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{x}} \cdot \boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{y}} \cdot \boldsymbol{T}(\boldsymbol{J}(\min(\boldsymbol{v}, 1 + \boldsymbol{m} + \boldsymbol{n}), \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$. (b) Assume $\boldsymbol{v} < S(1 + \boldsymbol{m} + \boldsymbol{n})$. Then from (a), by means of Theorem I, 13.2 and 7.2, we obtain $\boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{x}} \cdot \boldsymbol{y} < S(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{x})) \supset_{\boldsymbol{y}} \cdot \boldsymbol{T}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$. Using (1), $N(\boldsymbol{L}(\boldsymbol{q}))$; and hence, using 7.2 and Theorem I, $N(\boldsymbol{s}) \supset_{\boldsymbol{s}} N(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{s}))$. These results with 17.1 yield $\boldsymbol{z} < S(\sum_{\boldsymbol{u}=1}^{\boldsymbol{L}(\boldsymbol{q})} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u})) \supset_{\boldsymbol{z}} \boldsymbol{T}(\mathfrak{Q}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})), \boldsymbol{z}))$. Also, by using $N(\boldsymbol{L}(\boldsymbol{q}))$, 7.2 and Theorem I, $N(\boldsymbol{L}(\boldsymbol{q})) \cdot N(\boldsymbol{s}) \supset_{\boldsymbol{s}} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{s}) = \boldsymbol{L}(\boldsymbol{q})$; hence $\sum_{\boldsymbol{u}=1}^{\boldsymbol{L}(\boldsymbol{q})} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u}) = \boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})$ (by 16.4), $= \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})\}(\boldsymbol{v})$. Using this result with the preceding, and applying Theorem I, $\boldsymbol{v} < S(1 + \boldsymbol{m} + \boldsymbol{n}) \supset_{\boldsymbol{v}} \cdot \boldsymbol{z} < S(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{v})) \supset_{\boldsymbol{z}} \cdot \boldsymbol{T}(\{\lambda \boldsymbol{v} \cdot \mathfrak{Q}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})))\}(\boldsymbol{v}, \boldsymbol{z}))$. (c) By Theorem I, $N(\boldsymbol{s}) \supset_{\boldsymbol{s}} N(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{s}))$. Using the latter, $N(1 + \boldsymbol{m} + \boldsymbol{n})$, and the result of (b) with 17.1, $\boldsymbol{z} < S(\sum_{\boldsymbol{u}=1}^{1+\boldsymbol{m}+\boldsymbol{n}} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u})) \supset_{\boldsymbol{z}} \boldsymbol{T}(\mathfrak{Q}(\lambda \boldsymbol{v} \cdot \mathfrak{Q}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})), \boldsymbol{z}))$. Thence, using the definition of $\boldsymbol{F}$, Rule I, and the relation $\sum_{\boldsymbol{u}=1}^{1+\boldsymbol{m}+\boldsymbol{n}} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u}) = [1 + \boldsymbol{m} + \boldsymbol{n}]\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})$ (by 16.4), $= \boldsymbol{L}(S(\boldsymbol{q}))$ (by the def. of $\boldsymbol{L}$), we infer $\boldsymbol{y} < S(\boldsymbol{L}(S(\boldsymbol{q}))) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(S(\boldsymbol{q}), \boldsymbol{y}))$. (4) From (2) and (3) by induction, $N(\boldsymbol{q}) \supset_{\boldsymbol{q}} \cdot \boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$. This and (1) with 17.2 yield $N(\boldsymbol{z}) \supset_{\boldsymbol{z}} \boldsymbol{T}(\mathfrak{Q}(\boldsymbol{F}, \boldsymbol{L}, \boldsymbol{z}))$, or, by the definition of $\boldsymbol{H}$, $N(\boldsymbol{z}) \supset_{\boldsymbol{z}} \boldsymbol{T}(\boldsymbol{H}(\boldsymbol{z}))$.

18. The sequence of positive integers satisfying a given condition. By 15III, there can be found a formula $\mathfrak{J}$ such that

$$
\begin{aligned}
&\mathfrak{J}(1) \text{ conv } \lambda cdk \cdot c(1, d(k+1), c, d, k+1), \\
&\mathfrak{J}(2) \text{ conv } \lambda cdk \cdot c(2, I^{d(k)}, k),
\end{aligned}
\tag{1}
$$

and then a formula $\mathfrak{C}$ such that

(2) $\mathfrak{C}(1)$ conv $\mathfrak{Z}$, $\mathfrak{C}(2)$ conv I.

Let $\mathfrak{p} \rightarrow \lambda dk \cdot \mathfrak{Z}(d(k), \mathfrak{C}, d, k)$.

18I. *Given a positive integer k: If $\boldsymbol{D}(\boldsymbol{k})$ conv 2, $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\boldsymbol{k}$. If $\boldsymbol{D}(\boldsymbol{k})$ conv 1, $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k}+1)$.* Hence, *if $\boldsymbol{D}(\boldsymbol{k})$ conv $\boldsymbol{D}(\boldsymbol{k}+1)$ conv $\cdots$ conv $\boldsymbol{D}(\boldsymbol{l}-1)$ conv 1 and $\boldsymbol{D}(\boldsymbol{l})$ conv 2 ($l \geq k$), then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\boldsymbol{l}$.*

For if $\boldsymbol{D}(\boldsymbol{k})$ conv 2, then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{Z}(\boldsymbol{D}(\boldsymbol{k}), \mathfrak{C}, \boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{Z}(2, \mathfrak{C}, \boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{C}(2, I^{\boldsymbol{D}(\boldsymbol{k})}, \boldsymbol{k})$ conv $I(I^2, \boldsymbol{k})$ conv $\boldsymbol{k}$; and if $\boldsymbol{D}(\boldsymbol{k})$ conv 1, then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{Z}(1, \mathfrak{C}, \boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{C}(1, \boldsymbol{D}(\boldsymbol{k}+1), \mathfrak{C}, \boldsymbol{D}, \boldsymbol{k}+1)$ conv $\mathfrak{Z}(\boldsymbol{D}(\boldsymbol{k}+1), \mathfrak{C}, \boldsymbol{D}, \boldsymbol{k}+1)$ conv $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k}+1)$.

18II. *If $\boldsymbol{D}(\boldsymbol{i})$ conv 1 for every positive integer $i \geq$ the positive integer k, then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ has no normal form.**

Proof. A derivation of $\boldsymbol{B}$ from $\boldsymbol{A}$ by applications of I and II, including at least one of the latter, will be called a *reduction*. A conversion in which III is not used may be indicated by an accent. It will be shown in a forthcoming paper by A. Church and J. B. Rosser,† that if an expression $\boldsymbol{A}$ has a normal form, then every sequence $\boldsymbol{A}$ red $\boldsymbol{A}'$ red $\boldsymbol{A}''$ red $\cdots$ of reductions is finite; ‡ and that if $\boldsymbol{P}$ conv $\boldsymbol{Q}$, then there exists a conversion of $\boldsymbol{P}$ into $\boldsymbol{Q}$ in which all applications of III follow all applications of II.§ Hence if $\bar{\boldsymbol{A}}$ is a normal form of $\boldsymbol{A}$, $\boldsymbol{A}$ conv′ $\bar{\boldsymbol{A}}$. $\lambda cdk \cdot c(1, d(k+1), c, d, k+1)$ is a normal form of $\mathfrak{Z}(1)$. Consequently $\mathfrak{Z}$ has a normal form $\bar{\mathfrak{Z}}$, for otherwise there would exist an infinite sequence $\mathfrak{Z}$ red $\mathfrak{Z}'$ red $\mathfrak{Z}''$ red $\cdots$, and hence an infinite sequence $\mathfrak{Z}(1)$ red $\mathfrak{Z}'(1)$ red $\mathfrak{Z}''(1)$ red $\cdots$. (1) and (2) hold with $\mathfrak{Z}$ replaced by $\bar{\mathfrak{Z}}$, and *conv* by *conv′*. Moreover $\bar{\boldsymbol{i}}+1$ conv′ $\overline{\boldsymbol{i}+1}$, and from $\boldsymbol{D}(\boldsymbol{i})$ *conv* 1 follows $\boldsymbol{D}(\bar{\boldsymbol{i}})$ *conv′* 1. Then under the hypothesis, $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{i})$ red $\mathfrak{Z}(\boldsymbol{D}(\boldsymbol{i}), \mathfrak{C}, \boldsymbol{D}, \boldsymbol{i})$ conv′ $\bar{\mathfrak{Z}}(\boldsymbol{D}(\bar{\boldsymbol{i}}), \mathfrak{C}, \boldsymbol{D}, \bar{\boldsymbol{i}})$ conv′ $\bar{\mathfrak{Z}}(1, \mathfrak{C}, \boldsymbol{D}, \bar{\boldsymbol{i}})$ red $\mathfrak{C}(1, \boldsymbol{D}(\bar{\boldsymbol{i}}+1), \mathfrak{C}, \boldsymbol{D}, \bar{\boldsymbol{i}}+1)$ conv′ $\bar{\mathfrak{Z}}(\boldsymbol{D}(\bar{\boldsymbol{i}}+1), \mathfrak{C}, \boldsymbol{D}, \bar{\boldsymbol{i}}+1)$ conv′ $\bar{\mathfrak{Z}}(\boldsymbol{D}(\overline{\boldsymbol{i}+1}), \mathfrak{C}, \boldsymbol{D}, \overline{\boldsymbol{i}+1})$. Hence $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ red $\bar{\mathfrak{Z}}(\boldsymbol{D}(\bar{\boldsymbol{k}}), \mathfrak{C}, \boldsymbol{D}, \bar{\boldsymbol{k}})$ red $\bar{\mathfrak{Z}}(\boldsymbol{D}(\overline{\boldsymbol{k}+1}), \mathfrak{C}, \boldsymbol{D}, \overline{\boldsymbol{k}+1})$ red $\bar{\mathfrak{Z}}(\boldsymbol{D}(\overline{\boldsymbol{k}+2}), \mathfrak{C}, \boldsymbol{D}, \overline{\boldsymbol{k}+2})$ red $\cdots$ *ad infinitum*, which could not be if $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ had a normal form.

* *Normal form* is defined in § C5.

† A. Church and J. B. Rosser, "Some properties of conversion."

‡ In other words, given any well-formed expression $\boldsymbol{P}$, either all or none of the sequences $\boldsymbol{P}$ red $\boldsymbol{P}'$ red $\boldsymbol{P}''$ red $\cdots$ can be continued *ad infinitum*.

§ Consequently, if $\boldsymbol{A}$ has a normal form, all normal forms of $\boldsymbol{A}$ are derivable from a given one by applications of I.

4

By 15IV, a formula $\mathfrak{A}$ such that $\mathfrak{A}(1)$ conv $\lambda d \cdot \mathfrak{p}(d, 1)$ and $\mathfrak{A}(\boldsymbol{n}+1)$ conv $\lambda d \cdot \mathfrak{p}(d, \mathfrak{A}(\boldsymbol{n}, d)+1)$ $(n=1,2,\cdots)$ can be found. Let $\mathfrak{P} \rightarrow \lambda dn \cdot \mathfrak{A}(n, d)$.

18III. *If* $\boldsymbol{D}$ *defines the infinite sequence* $d_1, d_2, d_3, \cdots$ *of* 1's *and* 2's, *and* $d_{n_1}, d_{n_2}, d_{n_3}, \cdots$ *is the subsequence which are* 2's, *then* $\mathfrak{P}(\boldsymbol{D})$ *defines the sequence* $n_1, n_2, n_3, \cdots$. *If the latter is a finite sequence* $n_1, \cdots, n_k$ $(k \geq 0)$, *then, for* $i > k$, $\mathfrak{P}(\boldsymbol{D}, \boldsymbol{i})$ *has no normal form.*

This result, together with 15Ij, l and above results concerning the formal definability and enumerability of n-tuples of positive integers, leads to the following:

18IV. *Given functions* $F_i(x_1, \cdots, x_n)$ *and* $G_i(x_1, \cdots, x_n)$ $(i=1, \cdots, m)$ *which are defined for all n-tuples of positive integers* $(x_1, \cdots, x_n)$ *and whose values are positive integers, if* F_i *and* G_i *are definable formally, then there can be found a formula* $\boldsymbol{L}$ *such that* (a) *if solutions of the system of equations*

$$F_i(x_1, \cdots, x_n)=G_i(x_1, \cdots, x_n) \qquad (i=1, \cdots, m) \tag{3}$$

exist, $\boldsymbol{L}$ *enumerates them formally,** *and* (b) *if less than k different solutions exist,* $\boldsymbol{L}(\boldsymbol{k})$ *does not have a normal form.*

For example, a formula $\mathfrak{F}$ can be found such that (a) $\mathfrak{F}$ enumerates the solutions of $x^t+y^t=z^t$ $(t>2)$ in positive integers, if such solutions exist, and (b) the Fermat problem is equivalent to the problem of whether $\mathfrak{F}(1)$ has a normal form.

We have noted that a theory of formal definition of functions of natural numbers, similar to our theory for functions of positive integers, can be constructed. It is also easy to construct a like theory for integers, if the integer x is represented by the formula $[\boldsymbol{x}_1, \boldsymbol{x}_2]$, where x_1, x_2 are the least positive integers such that $x_1 - x_2 = x$; and a like theory for rational numbers, if the rational number x is represented by the formula $[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3]$ where x_1, x_2, x_3 are the least positive integers such that $(x_1 - x_2)/x_3 = x$. In particular, theorems corresponding to 18IV can be proved for each of these theories.

Given any formula T in the notation of *Principia Mathematica*, there can be found a well-formed expression $\boldsymbol{K}$ such that the problem whether T is provable in the system of *Principia* is equivalent to the problem, whether $\boldsymbol{K}$ has a normal form. Indeed, suppose we have given any formula T and any system of formal logic F, for which the condition is satisfied that there is a

* That is, there is an enumeration of the solutions as $(x_{j1}, \cdots, x_{jn})$ $(j=1,2, \cdots)$ such that $L(\boldsymbol{j})$ conv $[\boldsymbol{x}_{j1}, \cdots, \boldsymbol{x}_{jn}]$, in the notation of § 8.

class M of formulas such that (a) all provable formulas of F belong to M, (b) T belongs to M, and (c) there exists a one-to-one correspondence of M to a class of positive integers such that the numbers corresponding to provable formulas are enumerable formally in the sense of the present theory (let t correspond to T, and $\boldsymbol{L}$ enumerate the numbers ordered to provable formulas). Then the problem whether T is provable in F is equivalent to the problem whether $\mathcal{P}(\lambda n \cdot \delta_{\boldsymbol{t}}^{\boldsymbol{L}(n)}, 1)$ has a normal form.

19. A representation of the logic C_1 within itself. Let C_1 denote the logic whose formal axioms are 1, 3-11, 14-16, and whose rules of procedure are I-V.

The objective of this section is its last theorem, to establish which we utilize a representation of the logic C_1 within itself in the fashion of Gödel.* Our particular choice of a representation serves to simplify the formal proofs. Instead of setting it up directly, we first set up a representation of the combinations without free symbols by formulas which will be called "metads," and then avail ourselves of a relation suggested by Rosser between C_1 and a certain system of combinations without free symbols.

Let $\mathfrak{r}$ be an expression such that $\mathfrak{r}(1)$ conv $\lambda m \cdot m(\lambda pq \cdot I^q(p))$ and $\mathfrak{r}(S(\boldsymbol{k}))$ conv $\lambda m \cdot m(\lambda pqr \cdot I^{\mathfrak{r}(\boldsymbol{k},q)}(I^{\mathfrak{r}(\boldsymbol{k},r)}(p)))$, and $\mathfrak{h}$ an expression such that $\mathfrak{h}(1)$ conv $\lambda p \cdot \mathfrak{r}(1, \lambda m \cdot m(1, p))$ and $\mathfrak{h}(S(\boldsymbol{k}))$ conv $\lambda pq \cdot \mathfrak{r}(S(\boldsymbol{k}), \lambda m \cdot m(S(\boldsymbol{k}), p, q))$ $(k = 1, 2, 3, \cdots)$.† Abbreviate $\boldsymbol{a}(\mathfrak{h})$ to $|\boldsymbol{a}|$, $\{\lambda xm \cdot m(1, x)\}(\boldsymbol{x})$ to $[\boldsymbol{x}]$, $\{\lambda abm \cdot m(S(|a|), a, b)\}(\boldsymbol{a}, \boldsymbol{b})$ to $[\boldsymbol{a}, \boldsymbol{b}]$, and $[[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}], [\boldsymbol{x}_{2^{r-1}+1}, \cdots, \boldsymbol{x}_{2^r}]]$ to $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^r}]$. A formula $\boldsymbol{a}$ shall be called a *metad* (of *rank* r) if $\boldsymbol{a}$ conv $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$, where $x_1, \cdots, x_{2^{r-1}}$ is a set of 1's and 2's.

19I. *If $\boldsymbol{a}$ is a metad of rank r, then $|\boldsymbol{a}|$ conv $\boldsymbol{r}$.*

For, by induction with respect to r, if $\boldsymbol{a}$ is a metad of rank r, then $\mathfrak{r}(\boldsymbol{r}, \boldsymbol{a})$ conv $\boldsymbol{r}$ and $|\boldsymbol{a}|$ conv $\mathfrak{r}(\boldsymbol{r}, \boldsymbol{a})$ (cf. the proof of 19.1).

Let ad $\rightarrow \lambda\mu \cdot [\phi([1])\phi([2]) \cdot [\phi(a)\phi(b) \; |a| = |b|] \supset_{ab} \phi([a, b])] \supset_\phi \phi(\mu)$.

19.1(x): $\quad$ ad$([\boldsymbol{x}])$ $\quad$ $(x = 1, 2)$.

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* *Loc. cit.*

† Henceforth the introduction of expressions in accordance with the Theorems 15III-15V will be made in an abbreviated manner, as here where $\mathfrak{r}$ is supposed to satisfy not only the stated relations but also the relation $N(\boldsymbol{K}) \vdash' \mathfrak{r}(S(\boldsymbol{K})) =' \lambda m \cdot \boldsymbol{m}(\lambda \boldsymbol{pqr} \cdot \mathfrak{r}(\boldsymbol{K}, \boldsymbol{q}, \boldsymbol{I}, \mathfrak{r}(\boldsymbol{K}, \boldsymbol{r}, \boldsymbol{I}, \boldsymbol{p})))$ (cf. 15IV), and $\mathfrak{h}$ is supposed to satisfy not only the stated relations but also the relation $N(\boldsymbol{K}) \vdash' \mathfrak{h}(S(\boldsymbol{K})) =' \lambda \boldsymbol{pq} \cdot \mathfrak{r}(S(\boldsymbol{K}), \lambda m \cdot \boldsymbol{m}(S(\boldsymbol{K}), \boldsymbol{p}, \boldsymbol{q}))$ (cf. 15III).

19.2: $$\mathrm{ad}(a) \supset_a N(|\,a\,|).$$

Proofs. (1) $N(|[\boldsymbol{x}]|) \cdot |[\boldsymbol{x}]| = \mathfrak{r}(|[\boldsymbol{x}]|, [\boldsymbol{x}])$ $(x = 1, 2)$ is provable by conversion from $N(1) \cdot 1 = 1$. (2) Assume $N(|\,a\,|)$, $|\,a\,| = \mathfrak{r}(|\,a\,|, a)$, $N(|\,b\,|)$, $|\,b\,| = \mathfrak{r}(|\,b\,|, b)$, $|\,a\,| = |\,b\,|$. Then $|[a, b]|$ conv $\mathfrak{h}(S(|\,a\,|), a, b)$, $= \{\lambda pq \cdot \mathfrak{r}(S(|\,a\,|), \lambda m \cdot m(S(|\,a\,|), p, q))\}(a, b)$ (using the assumption $N(|\,a\,|)$ and the last property of $\mathfrak{h}$ as selected in accordance with 15III), conv $\mathfrak{r}(S(|\,a\,|), [a, b])$, $= \{\lambda m \cdot m(\lambda pqr \cdot I^{\mathfrak{r}(|a|,q)}(I^{\mathfrak{r}(|a|,r)}(p)))\}([a, b])$ (using $N(|\,a\,|)$ and the last property of $\mathfrak{r}$ as selected in accordance with 15IV), conv $I^{\mathfrak{r}(|a|,a)}(I^{\mathfrak{r}(|a|,b)}(S(|\,a\,|)))$, $= I^{\mathfrak{r}(|a|,a)}(I^{\mathfrak{r}(|b|,b)}(S(|\,a\,|)))$ (by the assumption $|\,a\,| = |\,b\,|$), $= I^{|a|}(I^{|b|}(S(|\,a\,|))$ (by the assumptions $|\,a\,| = \mathfrak{r}(|\,a\,|, a)$ and $|\,b\,| = \mathfrak{r}(|\,b\,|, b)$), $= I(I(S(|\,a\,|)))$ $(N(|\,a\,|), N(|\,b\,|), 7.2)$, conv $S(|\,a\,|)$. Hence $|[a, b]| = \mathfrak{r}(|[a, b]|, [a, b])$ (note the occurrence of $\mathfrak{r}(S(|\,a\,|), [a, b])$ in the foregoing chain of equalities) and $N(|[a, b]|)$ (using $N(|\,a\,|)$ and 3.2). (3) Hence, if $\mathfrak{D}_\phi \rightarrow \phi([1])\phi([2]) \cdot [\phi(a)\phi(b) \mid |\,a\,| = |\,b\,|] \supset_{ab} \phi([a, b])$, $\{\lambda\phi \cdot \mathfrak{D}_\phi\}(\lambda\alpha \cdot N(|\,\alpha\,|) \cdot |\,\alpha\,| = \mathfrak{r}(|\,\alpha\,|, \alpha))$ is provable. Hence $\Sigma\phi \cdot \mathfrak{D}_\phi$. $\mathfrak{D}_\phi \vdash \phi([\boldsymbol{x}])$ $(x = 1, 2)$. By Theorem I, $\mathrm{ad}([\boldsymbol{x}])$. (4) Now $\Sigma a \cdot \mathrm{ad}(a)$. Assume $\mathrm{ad}(a)$. From $\mathrm{ad}(a)$ and $\{\lambda\phi \cdot \mathfrak{D}_\phi\}(\lambda\alpha \cdot N(|\,\alpha\,|) \cdot |\,\alpha\,| = \mathfrak{r}(|\,\alpha\,|, \alpha))$ by Rule V, $\{\lambda\phi \cdot \phi(a)\}(\lambda\alpha \cdot N(|\,\alpha\,|) \cdot |\,\alpha\,| = \mathfrak{r}(|\,\alpha\,|, \alpha))$. Thence, $N(|\,a\,|)$.

19.3: $[\mathrm{ad}(a)\mathrm{ad}(b) \mid |\,a\,| = |\,\mathrm{b}\,|] \supset_{ab} \mathrm{ad}([a, b])$.

19.4: $[\phi([1])\phi([2])$
$\quad \cdot [\mathrm{ad}(a)\phi(a)\mathrm{ad}(b)\phi(b) \mid |\,a\,| = |\,b\,|] \supset_{ab} \phi([a, b])] \supset_\phi \cdot \mathrm{ad}(c) \supset_c \phi(c)$.

These theorems follow from 19.1 and the formula $\Sigma\phi \cdot \mathfrak{D}_\phi$ occurring in the proof of 19.1 in the same manner as 3.2 and 3.3 from 3.1 and $\mathfrak{A}_5$ of the proof of 3.1. The inference of an expression of the form $\mathrm{ad}(\boldsymbol{c}) \supset_{\boldsymbol{c}} \boldsymbol{F}(\boldsymbol{c})$ by means of 19.4 will be said to be by *induction* (*with respect to* $\boldsymbol{c}$).

Choose $\mathfrak{m}_j$ so that $\mathfrak{m}_j(1)$ conv I, $\mathfrak{m}_1(S(\boldsymbol{k}))$ conv $\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q)))$, $\mathfrak{m}_2(S(\boldsymbol{k}))$ conv $\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|q|}(r)))$, and let $\mathfrak{M}_j \rightarrow \lambda\rho \cdot \mathfrak{m}_j(|\,\rho\,|, \rho)$ $(j = 1, 2;\ k = 1, 2, 3, \cdots)$.* We abbreviate $\mathfrak{M}_j(\boldsymbol{a})$ to $\boldsymbol{a}_j$, $\mathfrak{M}_i(\mathfrak{M}_j(\boldsymbol{a}))$ to $\boldsymbol{a}_{ji}$, etc., when $\boldsymbol{a}$ is a metad or represents a metad in the formal argument.

19II. *If* $\boldsymbol{a}$ *conv* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$, *where* $x_1, \cdots, x_{2^{r-1}}$ *are* 1's *and* 2's *and* $r > 1$, *then* $\boldsymbol{a}_1$ *conv* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-2}}]$ *and* $\boldsymbol{a}_2$ *conv* $[\boldsymbol{x}_{2^{r-2}+1}, \cdots, \boldsymbol{x}_{2^{r-1}}]$.

19.5: $[\mathrm{ad}(a)\mathrm{ad}(b) \mid |a| = |b|] \supset_{ab} \cdot |[a, b]| = S(|a|) \cdot [a, b]_1 = a \cdot [a, b]_2 = b$.

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* We know that there exists an expression $\mathfrak{m}_1$ having the specified properties, and the property $N(\boldsymbol{K}) \vdash' \mathfrak{m}_1(S(\boldsymbol{K})) =' \lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q)))$, by use of 15III in conjunction with 15Ie and 7.2, taking for $\boldsymbol{F}$ the expression $\lambda x \cdot I^x(\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q))))$. The introduction of $\mathfrak{m}_2$ is justified in the same manner.

Proof. Assume $\mathrm{ad}(a)\ \mathrm{ad}(b) \mid a \mid = \mid b \mid$. By (4) of the proof of 19.1 and 19.2, we can infer $N(\mid a \mid)$, $\mid a \mid = \mathfrak{r}(\mid a \mid, a)$, $N(\mid b \mid)$, $\mid b \mid = \mathfrak{r}(\mid b \mid, b)$, and hence, by (2) of the same proof, $|[a, b]| = S(\mid a \mid)$. Then also $[a, b]_1$ conv $\mathfrak{m}_1(|[a, b]|, [a, b])$, $= \mathfrak{m}_1(S(\mid a \mid), [a, b])$, $= \{\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q)))\}([a, b])$ (by a supposition concerning $\mathfrak{m}_1$), conv $I^{S(|a|)}(I^{|b|}(a))$, $= I(I(a))$ (19.2, 7.2), conv a. $\mathrm{ad}(a) \vdash E(a)$. Hence, by § 2, $[a, b]_1 = a$. Similarly $[a, b]_2 = b$.

Let $\mathfrak{c} \rightarrow \lambda m \cdot m(\lambda p \cdot I^p)$.

19.6: $[\mathrm{ad}(a) \mid a \mid = 1] \supset_a \cdot \mathfrak{c}(a) < 3 \cdot a = [\mathfrak{c}(a)]$.

19.7: $[\mathrm{ad}(a) | a | > 1] \supset_a \cdot a = [a_1, a_2] \cdot ad(a_1) \cdot ad(a_2) \cdot |a_1| = |a_2| = |a| - 1$.

Proofs. Choose an expression $\mathfrak{B}$ such that $\mathfrak{B}(1)$ conv $\lambda a \cdot \mathfrak{c}(a) < 3 \cdot a = [\mathfrak{c}(a)] \cdot E(I)$ and $\mathfrak{B}(2)$ conv $\lambda a \cdot a = [a_1, a_2] \cdot \mathrm{ad}(a_1) \cdot \mathrm{ad}(a_2) \cdot \mid a_1 \mid = \mid a_2 \mid = \mid a \mid - 1$. Then, using 19.5, the lemma $\mathrm{ad}(a) \supset_a \mathfrak{B}(\epsilon_1{}^{|a|}, a)$ can be proved by induction. 19.6 and 19.7 follow.

19.8: $$\mathrm{ad}(a) \supset_a \cdot \mathrm{ad}(a_1) ad(a_2).$$

Proof. Assume $\mathrm{ad}(a)$. Case 1: $\epsilon_1{}^{|a|} = 1$. Then $\mid a \mid = 1$; and, using the definitions of $\mathfrak{M}_j$ and $\mathfrak{m}_j$, $a = a_j$ $(j = 1, 2)$. Using the latter, $\mathrm{ad}(a_j)$ follows from $\mathrm{ad}(a)$. Case 2: $\epsilon_1{}^{|a|} = 2$. Then $\mid a \mid > 1$; and $\mathrm{ad}(a_1)\mathrm{ad}(a_2)$ can be proved by means of 19.7.

In the remainder of this paper, we shall mean by a *combination* a combination, in the sense of § C6, which contains no free symbols; in other words, a combination whose terms are I's and J's. If $\boldsymbol{T}$ is the only term of a combination, the *rank* of $\boldsymbol{T}$ shall be 1; if $\boldsymbol{T}$ is a term of $\boldsymbol{M}$ of rank r, then the rank of $\boldsymbol{T}$ as a term of $\{\boldsymbol{M}\}(\boldsymbol{N})$ or $\{\boldsymbol{N}\}(\boldsymbol{M})$ shall be $r + 1$. The *rank* of a combination shall be the rank of its term of highest rank. A combination shall be *uniform* if all its terms have the same rank. (A uniform combination $\boldsymbol{A}'$ of rank r has 2^{r-1} terms, and they occur in $\boldsymbol{A}'$ in a linear series—cf. C6III). A uniform combination $\boldsymbol{A}'$ shall *represent* a combination $\boldsymbol{A}$, if $\boldsymbol{A}'$ is derivable from $\boldsymbol{A}$ by zero or more substitutions of $I(\boldsymbol{T})$ for $\boldsymbol{T}$, where $\boldsymbol{T}$ is a term. Given the correspondence $\binom{I\ J}{1\ 2}$, $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ shall *correspond* to a uniform combination $\boldsymbol{A}'$ if $x_1, \cdots, x_{2^{r-1}}$ is the series of the numbers which correspond to the respective terms of $\boldsymbol{A}'$. If $\boldsymbol{A}$ is a combination and $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ corresponds to a uniform combination $\boldsymbol{A}'$ which represents $\boldsymbol{A}$, we write "$[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$." A metad $\boldsymbol{a}$ shall *represent* a combination $\boldsymbol{A}$ if $\boldsymbol{a}$ conv $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ where $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$.

19III. *Suppose that* x_i, y_i $(i = 1, \cdots, 2^{r-1})$ *are* 1's *and* 2's. a. *Given a combination* $\boldsymbol{A}$, *a representing metad* $\boldsymbol{a}$ *can be found.* b. *If the metad* $\boldsymbol{a}$ *represents the combination* $\boldsymbol{A}$, *then* $\boldsymbol{a}$ *is of rank* $\geq$ *the rank of* $\boldsymbol{A}$. c. *If* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$ *and* $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2_{r-1}}] \sim \boldsymbol{B}$, *then* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \{\boldsymbol{A}\}(\boldsymbol{B})$. d. *If* $[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$, *then* $[1, \boldsymbol{x}_1, 1, \boldsymbol{x}_2, \cdots, 1, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$. e. *If both* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$ and $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \boldsymbol{A}$, *then* $x_i = y_i$. f. *If the metad* $\boldsymbol{a}$ *represents the combination* $\{\boldsymbol{F}\}(\boldsymbol{P})$, *then* $\boldsymbol{a}_1$ *represents* $\boldsymbol{F}$, *and* $\boldsymbol{a}_2$ *represents* $\boldsymbol{P}$.

Let $\mathfrak{e}$ be an expression such that $\mathfrak{e}(1)$ conv $\lambda p \cdot [[1], p]$ and $\mathfrak{e}(S(\boldsymbol{k}))$ conv $\lambda p \cdot [\mathfrak{e}(\boldsymbol{k}, p_1), \mathfrak{e}(\boldsymbol{k}, p_2)]$ $(k = 1, 2, \cdots)$. Let $\mathfrak{E} \rightarrow \lambda\rho \cdot \mathfrak{e}(|\,\rho\,|, \rho)$.

19IV. *If* $\boldsymbol{a}$ *conv* $[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{2^{r-1}}]$ $(x_1, \cdots, x_{2^{r-1}}$ *being* 1's *and* 2's), *then* $\mathfrak{E}(\boldsymbol{a})$ *conv* $[1, \boldsymbol{x}_1, 1, \boldsymbol{x}_2, \cdots, 1, \boldsymbol{x}_{2^{r-1}}]$.

The proof is by induction with respect to r (using 19I and 19II).

19.9: $$N(\rho) \supset_\rho \cdot \mathrm{ad}(a) \supset_a \cdot \mathrm{ad}(\mathfrak{E}^\rho(a)) \cdot |\,\mathfrak{E}^\rho(a)\,| = \rho + |\,a\,|.$$

Proof. $\mathrm{ad}(a) \supset_a \cdot \mathrm{ad}(\mathfrak{E}(a)) \cdot |\,\mathfrak{E}(a)\,| = S(|\,a\,|)$ is provable by induction with respect to a (using 19.1-19.3, 19.5), and 19.9 follows by induction with respect to ρ.

19.10: $$[N(\rho)\,\mathrm{ad}(a)\,\mathrm{ad}(b) \cdot \mathfrak{E}^\rho(a) = \mathfrak{E}^\rho(b)] \supset_{\rho a b} \cdot a = b.$$

Proof. Let $\mathfrak{e}'$ be an expression such that $\mathfrak{e}'(1)$ conv $\lambda p \cdot p_2$ and $\mathfrak{e}'(S(\boldsymbol{k}))$ conv $\lambda p \cdot [\mathfrak{e}'(\boldsymbol{k}, p_1), \mathfrak{e}'(\boldsymbol{k}, p_2)]$ $(k = 1, 2, \cdots)$. Let $\mathfrak{E}' \rightarrow \lambda\rho \cdot \mathfrak{e}'(|\,\rho\,| - 1, \rho)$. Then $\mathrm{ad}(a) \supset_a \cdot \mathfrak{E}'(\mathfrak{E}(a)) = a$ is provable by induction with respect to a, and $N(\rho) \supset_\rho \cdot \mathrm{ad}(a) \supset_a \cdot \mathfrak{E}'^\rho(\mathfrak{E}^\rho(a)) = a$ follows by induction with respect to ρ. 19.10 follows from the latter in the same manner as 11.4 from 11.2.

Let $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \rightarrow [\mathfrak{E}^{|\boldsymbol{b}|}(\boldsymbol{a}), \mathfrak{E}^{|\boldsymbol{a}|}(\boldsymbol{b})]$.

19V. *If the metads* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *represent the combinations* $\boldsymbol{A}$ *and* $\boldsymbol{B}$, *respectively, then* $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ *is a metad which represents* $\{\boldsymbol{A}\}(\boldsymbol{B})$.

This follows from 19I, 15Id, 19IV, 19IIId, c.

19.11: $$\mathrm{ad}(a)\,\mathrm{ad}(b) \supset_{ab} \cdot \mathrm{ad}(\langle a, b \rangle).$$

Let $\mathfrak{D}$ be an expression such that $\mathfrak{D}(1)$ conv $\lambda pq \cdot \delta(p(\lambda n \cdot \boldsymbol{I}^n), q(\lambda n \cdot \boldsymbol{I}^n))$ and $\mathfrak{D}(S(\boldsymbol{k}))$ conv $\lambda pq \cdot \mathfrak{D}(\boldsymbol{k}, p_1, q_1) \circ \mathfrak{D}(\boldsymbol{k}, p_2, q_2)$ $(k = 1, 2, \cdots)$. Let $\Delta \rightarrow \lambda pq \cdot \mathfrak{D}(|\,p\,| + |\,q\,|, \mathfrak{E}^{|q|}(p), \mathfrak{E}^{|p|}(q))$, and abbreviate $\Delta(\boldsymbol{a}, \boldsymbol{b})$ to $\Delta^{\boldsymbol{a}}_{\boldsymbol{b}}$.

19VI. *If the metads* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *both represent the combination* $\boldsymbol{A}$, *then* $\Delta^{\boldsymbol{a}}_{\boldsymbol{b}}$ *conv* 2.

Proof. By induction with respect to r (using 15Ie, 19II), if $x_1, \cdots, y_{2^{r-1}}$ are 1's and 2's, $\mathfrak{D}(\boldsymbol{r}, [\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}], [\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}])$ conv $\delta^{x_1}_{y_1} \circ \cdots \circ \delta^{x_{2^{r-1}}}_{y_{2^{r-1}}}$. Hence, by 15Il, j and 19IIIe, if $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$ and $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \boldsymbol{A}$, then $\mathfrak{D}(\boldsymbol{r}, [\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}], [\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}])$ conv 2. Moreover, by 19IV, 19I, 15Id and 19IIId, if $\boldsymbol{a}$ conv $[\boldsymbol{x}'_1, \cdots, \boldsymbol{x}'_{2^{m-1}}]$, $[\boldsymbol{x}'_1, \cdots, \boldsymbol{x}'_{2^{m-1}}] \sim \boldsymbol{A}$, $\boldsymbol{b}$ conv $[\boldsymbol{y}'_1, \cdots, \boldsymbol{y}'_{2^{n-1}}]$, $[\boldsymbol{y}'_1, \cdots, \boldsymbol{y}'_{2^{n-1}}] \sim \boldsymbol{A}$, then there are $x_1, \cdots, x_{2^{r-1}}$, $y_1, \cdots, y_{2^{r-1}}$ $(r = m + n)$ such that $\mathfrak{E}^{|\boldsymbol{b}|}(\boldsymbol{a})$ conv $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$, $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$, $\mathfrak{E}^{|\boldsymbol{a}|}(\boldsymbol{b})$ conv $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}]$, $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \boldsymbol{A}$.

19.12: $\quad \mathrm{ad}(a)\,\mathrm{ad}(b) \supset_{ab} \cdot M(\Delta_b{}^a)$.

19.13: $\quad \mathrm{ad}(a)\,\mathrm{ad}(b) \supset_{ab} \cdot \Delta_b{}^a = \Delta_a{}^b$.

19.14: $\quad \mathrm{ad}(a)\,\mathrm{ad}(b) \supset_{ab} \cdot N(\rho) \supset_\rho \cdot \Delta(a, \mathfrak{E}^\rho(b)) = \Delta_b{}^a$.

Proofs. If $\mathfrak{B} \rightarrow \lambda a \cdot [\mathrm{ad}(b) \mid a \mid = \mid b \mid] \supset_b \cdot M(\mathfrak{D}(\mid a \mid, a, b)) \cdot \mathfrak{D}(\mid a \mid, a, b) = \mathfrak{D}(\mid a \mid, b, a) \cdot \mathfrak{D}(S(\mid a \mid), \mathfrak{E}(a), \mathfrak{E}(b)) = \mathfrak{D}(\mid a \mid, a, b)$, the lemma $\mathrm{ad}(a) \supset_a \mathfrak{B}(a)$ can be proved by induction, using first 19.6, 14.10, 14.11, 14.5, and then the relation $\mathrm{ad}(l)$, $\mathrm{ad}(m)$, $\mid l \mid = \mid m \mid$, $\mathrm{ad}(b)$, $\mid [l, m] \mid = \mid b \mid \vdash \mathfrak{D}(\mid [l, m] \mid, [l, m], b) = \mathfrak{D}(\mid l \mid, l, b_1) \circ \mathfrak{D}(\mid m \mid, m, b_2) \cdot \mathfrak{D}(\mid [l, m] \mid, b, [l, m]) = \mathfrak{D}(\mid l \mid, b_1, l) \circ \mathfrak{D}(\mid m \mid, b_2, m) \cdot \mathfrak{D}(S(\mid [l, m] \mid), \mathfrak{E}([l, m]), \mathfrak{E}(b)) = \mathfrak{D}(S(\mid l \mid), \mathfrak{E}(l), \mathfrak{E}(b_1)) \circ \mathfrak{D}(S(\mid m \mid), \mathfrak{E}(m), \mathfrak{E}(b_2))$ (which follows from 19.2, 19.5, 19.9, 19.7), and 19.2, 19.3, 19.5, 19.7, 14.2. 19.12-19.14 follow from the lemma, 19.2, 19.9, and the relation $\mathrm{ad}(a)\,\mathrm{ad}(b) \vdash \Delta(a, \mathfrak{E}(b)) = \mathfrak{D}(S(\mid \mathfrak{E}^{|b|}(a) \mid), \mathfrak{E}(\mathfrak{E}^{|b|}(a)), \mathfrak{E}(\mathfrak{E}^{|a|}(b)))$.

19.15: $\quad [\mathrm{ad}(a)\,\mathrm{ad}(b) \cdot \mid a \mid = \mid b \mid \cdot \Delta_b{}^a = 2] \supset_{ab} \cdot a = b$.

19.16: $\quad \mathrm{ad}(a) \supset_a \cdot \Delta_a{}^a = 2$.

Proofs. If $\mathfrak{C} \rightarrow \lambda a \cdot [\mathfrak{D}(\mid a \mid, a, a) = 2] \cdot [\mathrm{ad}(b) \cdot \mid a \mid = \mid b \mid \cdot \mathfrak{D}(\mid a \mid, a, b) = 2] \supset_b \cdot a = b$, then $\mathrm{ad}(a) \supset_a \mathfrak{C}(a)$ is provable by induction, using first 19.6, 14.14, and then $\mathrm{ad}(a) \supset_a \mathfrak{B}(a)$, 19.2, 19.3, 19.5, 19.7, 14.6 and the relation $\mathrm{ad}(l)$, $\mathrm{ad}(m)$, $\mid l \mid = \mid m \mid$, $\mathrm{ad}(b)$, $\mid [l, m] \mid = \mid b \mid \vdash \mathfrak{D}(\mid [l, m] \mid, [l, m], b) = \mathfrak{D}(\mid l \mid, l, b_1) \circ \mathfrak{D}(\mid m \mid, m, b_2)$. 19.15 and 19.16 follow, using 19.10.

Let $\mathfrak{J}$ be an expression such that $\mathfrak{J}(1)$ conv I and $\mathfrak{J}(2)$ conv J, and $\mathfrak{g}$ an expression such that $\mathfrak{g}(1)$ conv $\lambda a \cdot a(\lambda pq \cdot I^p(\mathfrak{J}(q)))$ and $\mathfrak{g}(S(\boldsymbol{k}))$ conv $\lambda a \cdot \mathfrak{g}(\boldsymbol{k}, a_1, \mathfrak{g}(\boldsymbol{k}, a_2))$ $(k = 1, 2, \cdots)$. Let $\mathfrak{G} \rightarrow \lambda a \cdot \mathfrak{g}(\mid a \mid, a)$.

19VII. *If the metad* $\boldsymbol{a}$ *represents the combination* $\boldsymbol{A}$, *then* $\mathfrak{G}(\boldsymbol{a})$ *conv* $\boldsymbol{A}$.

For, by induction with respect to r, if $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ corresponds to a uniform combination $\boldsymbol{A}'$, $\mathfrak{G}([\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}])$ conv $\boldsymbol{A}'$. If $\boldsymbol{A}'$ represents $\boldsymbol{A}$, $\boldsymbol{A}'$ conv $\boldsymbol{A}$.

Let $\mathfrak{i}$ be an expression such that $\mathfrak{i}(1)$ conv $[1]$ and $\mathfrak{i}(S(\boldsymbol{k}))$ conv $[\mathfrak{i}(\boldsymbol{k}), \mathfrak{i}(\boldsymbol{k})]$ $(k = 1, 2, \cdots)$.

19.17: $N(r) \supset_r \cdot \mathrm{ad}(\mathfrak{i}(r)) \cdot |\, \mathfrak{i}(r) \,| = r \cdot \mathfrak{G}(\mathfrak{i}(r)) = I.$

19.18: $[\mathrm{ad}(a) \cdot |\, a \,| > 1 \cdot E(\mathfrak{G}(a))] \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(a_1, \mathfrak{G}(a_2)).$

Proofs. 19.17 is provable by induction with respect to r. 19.17 $\vdash \Sigma a \cdot \mathrm{ad}(a) \cdot |\, a \,| > 1 \cdot E(\mathfrak{G}(a))$; and, assuming $\mathrm{ad}(a) \cdot |\, a \,| > 1 \cdot E(\mathfrak{G}(a))$, $\mathfrak{G}(a) = \mathfrak{G}(a_1, \mathfrak{G}(a_2))$ (by 19.7, 12.5, § 2). Hence, by Theorem I, $\vdash$ 19.18.

19.19:
$$N(\rho) \supset_\rho \cdot \mathrm{ad}(a) E(\mathfrak{G}(a)) \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{E}^\rho(a)).$$
$$N(\rho) \supset_\rho \cdot \mathrm{ad}(a) E(\mathfrak{G}(\mathfrak{E}^\rho(a))) \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{E}^\rho(a)).$$

Proof. Note that $N(\boldsymbol{n})$, 19.17 $\vdash \Sigma \boldsymbol{a} \cdot \mathrm{ad}(\boldsymbol{a}) \cdot |\, \boldsymbol{a} \,| = \boldsymbol{n} \cdot E(\mathfrak{G}(\boldsymbol{a}))$. Using this relation, 19.1, 19.9, 19.5, 19.7, 11.2, §2, and Theorem I, we can prove $N(r) \supset_r \cdot [\mathrm{ad}(a) \cdot |\, a \,| = r \cdot E(\mathfrak{G}(a))] \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{E}(a))$ by induction with respect to r. Thence, using 19.17, 19.2 and Theorem I, $\mathrm{ad}(a) E(\mathfrak{G}(a)) \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{E}(a))$. The first of the formulas 19.19 follows by induction with respect to ρ; and the second is proved similarly.

19.20: $\mathrm{ad}(a)\mathrm{ad}(b) E(\mathfrak{G}(a, \mathfrak{G}(b))) \supset_{ab} \cdot \mathfrak{G}(a, \mathfrak{G}(b)) = \mathfrak{G}(\langle a, b \rangle).$

Proof. 19.17 $\vdash \Sigma ab \cdot \mathrm{ad}(a)\mathrm{ad}(b) E(\mathfrak{G}(a, \mathfrak{G}(b)))$. Assume $\mathrm{ad}(a)\mathrm{ad}(b) E(\mathfrak{G}(a, \mathfrak{G}(b)))$. Then $\mathfrak{G}(a, \mathfrak{G}(b)) = \mathfrak{G}(\mathfrak{E}^{|b|}(a), \mathfrak{G}(\mathfrak{E}^{|a|}(b)))$ (19.19, 19.2), $= \mathfrak{G}(\langle a, b \rangle)$ (19.2, 19.9, 19.5, def. of $\mathfrak{G}$).

19.21: $[\mathrm{ad}(a)\mathrm{ad}(b) E(\mathfrak{G}(a)) \cdot \Delta_b{}^a = 2] \supset_{ab} \cdot \mathfrak{G}(a) = \mathfrak{G}(b).$

Proof. 19.17, 19.16 $\vdash \Sigma ab \cdot \mathrm{ad}(a)\mathrm{ad}(b) E(\mathfrak{G}(a)) \cdot \Delta_b{}^a = 2$. Assuming $\mathrm{ad}(a)\mathrm{ad}(b) E(\mathfrak{G}(a)) \cdot \Delta_b{}^a = 2$, then $\mathfrak{G}(a) = \mathfrak{G}(\mathfrak{E}^{|b|}(a))$ (19.19, 19.2), $= \mathfrak{G}(\mathfrak{E}^{|a|}(b))$ (19.15, 19.14, 19.13, 19.2, 19.9), $= \mathfrak{G}(b)$.

A combination $\bar{\boldsymbol{A}}$ shall be said to be *representative* of a formula $\boldsymbol{A}$, if $\bar{\boldsymbol{A}}$ conv $\lambda \Pi \Sigma \& \cdot \boldsymbol{A} \cdot E(\Pi)$.

19VIII. *Given a formula* $\boldsymbol{A}$ *having no free symbols other than* Π, Σ *and* &, *a representative combination* $\bar{\boldsymbol{A}}$ *can be found.*

Proof. By C6V, there is a combination $\bar{\boldsymbol{A}}$ (in the sense of § C6) such that $\bar{\boldsymbol{A}}$ conv $\lambda \Pi \Sigma \& \cdot \boldsymbol{A} \cdot E(\Pi)$. Under the hypothesis, $\lambda \Pi \Sigma \& \cdot \boldsymbol{A} \cdot E(\Pi)$ contains no free symbols, and hence, by C5VI, $\bar{\boldsymbol{A}}$ is a combination in the present sense.

Let the subsequences (including the null sequence) of the sequence $\Pi, \Sigma, \&$ be $X_{i1}, \cdots, X_{i\alpha_i}$ $(i = 1, \cdots, 2^3)$. By C6V and C5VI, there are combinations $\mathfrak{S}_{ij}$, $\mathfrak{T}_i$ and $\mathfrak{U}_{ij}$ convertible into $\lambda fp\Pi\Sigma\& \cdot f(X_{i1}, \cdots, X_{i\alpha_i}, p(X_{j1}, \cdots, X_{j\alpha_j})) \cdot E(\Pi)$, $\lambda f\Pi\Sigma\& \cdot \Sigma(f(X_{i1}, \cdots, X_{i\alpha_i})) \cdot E(\Pi)$ and $\lambda fg\Pi\Sigma\& \cdot \Pi(f(X_{i1}, \cdots, X_{i\alpha_i}), g(X_{j1}, \cdots, X_{j\alpha_j})) \cdot E(\Pi)$, respectively $(i, j = 1, \cdots, 2^3)$.*

We denote the rules of procedure of Rosser, *loc. cit.*, Section H,† by $\mathrm{R}_1, \cdots, \mathrm{R}_{38}$, and list the rules R_{ik}, "*If* $\mathfrak{S}_{ik}(\boldsymbol{f}, \boldsymbol{p})$, *then* $\mathfrak{T}_i(\boldsymbol{f})$," $(i, k = 1, \cdots, 2^3)$, as R_{39}-R_{102}, and the rules R_{ijk}, "*If* $\mathfrak{U}_{ij}(\boldsymbol{f}, \boldsymbol{g})$ *and* $\mathfrak{S}_{ik}(\boldsymbol{f}, \boldsymbol{p})$, *then* $\mathfrak{S}_{jk}(\boldsymbol{g}, \boldsymbol{p})$," $(i, j, k = 1, \cdots, 2^3)$ as R_{103}-R_{614}.

19IX(t). *If* $\boldsymbol{C}$ *is derivable from* $\boldsymbol{A}$ *(*$\boldsymbol{A}$ *and* $\boldsymbol{B}$*) by an application of* R_t, *then* $\boldsymbol{A}(\Pi, \Sigma, \&)$ $(\boldsymbol{A}(\Pi, \Sigma, \&), \boldsymbol{B}(\Pi, \Sigma, \&)) \vdash \boldsymbol{C}(\Pi, \Sigma, \&)$, $(t = 1, \cdots, 614)$.

Proof. If $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ by an application of one of R_1-R_{38}, then $\boldsymbol{A}$ conv $\boldsymbol{C}$. If $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ ($\boldsymbol{A}$ and $\boldsymbol{B}$) by an application of one of the rules $\mathrm{R}_{ik}(\mathrm{R}_{ijk})$, then $\boldsymbol{C}(\Pi, \Sigma, \&)$ is derivable from $\boldsymbol{A}(\Pi, \Sigma, \&)$ ($\boldsymbol{A}(\Pi, \Sigma, \&)$ and $\boldsymbol{B}(\Pi, \Sigma, \&)$) by conversion, Rule IV (V) and the relations $\boldsymbol{PQ} \vdash \boldsymbol{P}$, $\boldsymbol{PQ} \vdash \boldsymbol{Q}$ and $\boldsymbol{P}, \boldsymbol{Q} \vdash \boldsymbol{PQ}$.

Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ be combinations representative of Axioms 1, 3-11, 14-16, respectively, and let $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$ be metads representing $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$, respectively (cf. 19VIII, 19IIIa).

19X. *If the combination* $\bar{\boldsymbol{D}}$ *is representative of a formula* $\boldsymbol{D}$ *provable in* C_1, *then* $\bar{\boldsymbol{D}}$ *is derivable from* $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ *by means of Rules* R_1-R_{614}.

Proof. Under the hypothesis, $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by means of conversion and the two rules

IV′. *If* $\lambda\Pi\Sigma\& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$, *then* $\lambda\Pi\Sigma\& \cdot \Sigma(\boldsymbol{F}) \cdot E(\Pi)$.

V′. *If* $\lambda\Pi\Sigma\& \cdot \Pi(\boldsymbol{F}, \boldsymbol{G}) \cdot E(\Pi)$ *and* $\lambda\Pi\Sigma\& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$, *then* $\lambda\Pi\Sigma\& \cdot \boldsymbol{G}(\boldsymbol{P}) \cdot E(\Pi)$.

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* More explicitly, let $\alpha_1 = 3$, $\alpha_2 = \alpha_3 = \alpha_4 = 2$, $\alpha_5 = \alpha_6 = \alpha_7 = 1$, $\alpha_8 = 0$; and let X_{11}, X_{12}, X_{13}; X_{21}, X_{22}; X_{31}, X_{32}; X_{41}, X_{42}; X_{51}; X_{61}; X_{71} stand for $\Pi, \Sigma, \&$; Π, Σ; $\Pi, \&$; $\Sigma, \&$; Π; Σ; $\&$, respectively. Then $\mathfrak{S}_{13}$ shall be a combination convertible into $\lambda fp\Pi\Sigma\& \cdot f(\Pi, \Sigma, \&, p(\Pi, \&)) \cdot E(\Pi)$, $\mathfrak{S}_{85}$ a combination convertible into $\lambda fp\Pi\Sigma\& \cdot f(p(\Pi)) \cdot E(\Pi)$, etc.

† See the footnote of § C6 (*Annals of Mathematics*, vol. 35, p. 537, ([12])).

If $\lambda\Pi\Sigma\& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$ contains no free symbols, and if $X_{i1}, \cdots, X_{i a_i}$ and $X_{k1}, \cdots, X_{k a_k}$ are the sets of the symbols Π, Σ, & which occur in $\boldsymbol{F}$ and $\boldsymbol{P}$, respectively, as free symbols, then $\lambda X_{i1} \cdots X_{i a_i} \cdot \boldsymbol{F}$ and $\lambda X_{k1} \cdots X_{k a_k} \cdot \boldsymbol{P}$ contain no free symbols, and are hence convertible into combinations $\boldsymbol{F}'$ and $\boldsymbol{P}'$, respectively (C6V, C5VI). Then $\lambda\Pi\Sigma\& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$ conv $\mathfrak{S}_{ik}(\boldsymbol{F}', \boldsymbol{P}')$ and $\lambda\Pi\Sigma\& \cdot \Sigma(\boldsymbol{F}) \cdot E(\Pi)$ conv $\mathfrak{T}_i(\boldsymbol{F}')$. Hence, if $\boldsymbol{A}$ (containing no free symbols) yields $\boldsymbol{C}$ by an application of IV$'$, then $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ by conversion and an application of one of the rules R_{ik} in which the premise and conclusion are combinations. Similarly, if $\boldsymbol{A}$ and $\boldsymbol{B}$ (containing no free symbols) yield $\boldsymbol{C}$ by an application of V$'$, then $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ and $\boldsymbol{B}$ by conversion and an application of one of the rules R_{ijk} in which the premise and conclusion are combinations. The formulas derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by conversion, IV$'$ and V$'$ contain no free symbols (cf. C5V Cor.). Hence $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by conversion and applications of R_{ik} and R_{ijk} in which the premises and conclusions are combinations. Now R_1-R_{38} have the property that if $\boldsymbol{A}$ and $\boldsymbol{C}$ are combinations, and $\boldsymbol{A}$ conv $\boldsymbol{C}$, then $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ by R_1-R_{38}. Hence $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by R_1-R_{38}, R_{ik}, R_{ijk}, *i. e.* by R_1-R_{614}.

We now define expressions $\mathfrak{R}_t$ corresponding to the rules R_t $(t = 1, \cdots, 614)$.

For typical rules of the set R_1-R_{38}, the definition of $\mathfrak{R}_t$ follows ($\mathfrak{r}_t$ standing for an expression satisfying the condition $\mathfrak{r}_t(1)$ conv I and the condition given below):

R_1. *If $I(\boldsymbol{p})$, then $\boldsymbol{p}$.*

$$\mathfrak{r}_1(2) \text{ conv } \lambda a \cdot a_2, \qquad \mathfrak{R}_1 \to \lambda a \cdot \mathfrak{r}_1(\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{[1]}, a).$$

R_2. *If $\boldsymbol{p}$, then $I(\boldsymbol{p})$.*

$$\mathfrak{R}_2 \to \lambda a \cdot \langle [1], a \rangle.$$

R_3. *If $\boldsymbol{f}(I(\boldsymbol{p}, \boldsymbol{q}))$, then $\boldsymbol{f}(\boldsymbol{p}(\boldsymbol{q}))$.*

$$\mathfrak{r}_3(2) \text{ conv } \lambda a \cdot \langle a_1, \langle a_{212}, a_{22} \rangle \rangle, \qquad \mathfrak{R}_3 \to \lambda a \cdot \mathfrak{r}_3(\epsilon_3{}^{|a|} \circ \Delta^{a_{211}}_{[1]}, a).$$

R_6. *If $\boldsymbol{f}(\boldsymbol{p}(\boldsymbol{q}, \boldsymbol{p}(\boldsymbol{s}, \boldsymbol{r})))$, then $\boldsymbol{f}(J(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}))$.*

$$\mathfrak{r}_6(2) \text{ conv } \lambda a \cdot \langle a_1, \langle\langle\langle\langle [2], a_{211} \rangle, a_{212} \rangle, a_{222} \rangle, a_{2212} \rangle\rangle,$$
$$\mathfrak{R}_6 \to \lambda a \cdot \mathfrak{r}_6(\epsilon_4{}^{|a|} \circ \Delta^{a_{211}}_{a_{2211}}, a).*$$

If R_t is the rule R_{ik} (for a certain i and k), then $\mathfrak{R}_t$ shall be the expres-

* The considerations governing the choice of the $\mathfrak{R}_t$ will appear in the proofs of 19XI(t) and 19.23(t). a_2, a_{211}, $\cdots$ are our abbreviations for $\mathfrak{M}_2(a)$, $\mathfrak{M}_1(\mathfrak{M}_1(\mathfrak{M}_2(a))), \cdots$.

sion $\mathfrak{R}_{ik}$ defined thus: Let $\mathfrak{s}_{ik}$ and $\mathfrak{t}_i$ be metads which represent the combinations $\mathfrak{S}_{ik}$ and $\mathfrak{T}_i$, respectively (cf. 19IIIa), and let $\mathfrak{r}_{ik}$ be an expression such that $\mathfrak{r}_{ik}(1)$ conv I and $\mathfrak{r}_{ik}(2)$ conv $\lambda a \cdot \langle \mathfrak{t}_i, a_{12} \rangle$. Let $\mathfrak{R}_{ik} \to \lambda a \cdot \mathfrak{r}_{ik}(\epsilon_2{}^{|a|} \circ \Delta(a_{11}, \mathfrak{s}_{ik}), a)$.

If R_t is the rule R_{ijk} (for a certain i, j and k), then $\mathfrak{R}_t$ shall be the expression $\mathfrak{R}_{ijk}$ defined thus: Let $\mathfrak{u}_{ij}$ be a metad which represents $\mathfrak{U}_{ij}$, and $\mathfrak{r}_{ijk}$ an expression such that $\mathfrak{r}_{ijk}(1)$ conv $\lambda pq \cdot I^{|p|}(q)$ and $\mathfrak{r}_{ijk}(2)$ conv $\lambda ab \cdot \langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle$. Let $\mathfrak{R}_{ijk} \to \lambda ab \cdot \mathfrak{r}_{ijk}(\epsilon_2{}^{|a|} \circ \epsilon_2{}^{|b|} \circ \Delta(a_{11}, \mathfrak{u}_{ij}) \circ \Delta(b_{11}, \mathfrak{s}_{ik}) \circ \Delta^{a_{12}}_{b_{12}}, a, b)$.

19XI(t). *If the metad* $\boldsymbol{a}$ *represents* (*the metads* $\boldsymbol{a}$, $\boldsymbol{b}$ *represent*) *a combination* $\boldsymbol{A}$ (*combinations* $\boldsymbol{A}$, $\boldsymbol{B}$) *such that* R_t *is applicable to* $\boldsymbol{A}$ (*to the pair* $\boldsymbol{A}$, $\boldsymbol{B}$), *then* $\mathfrak{R}_t(\boldsymbol{a})$ ($\mathfrak{R}_t(\boldsymbol{a}, \boldsymbol{b})$) *is a metad which represents the combination resulting from the application.* ($t = 1, \cdots, 614$).

As illustrative of the arguments for the several values of t, we take the case of a $t > 102$ (≤ 614). Then R_t is the rule R_{ijk}, for a certain i, j and k; and $\boldsymbol{A}$ and $\boldsymbol{B}$ are of the forms $\mathfrak{U}_{ij}(\boldsymbol{f}, \boldsymbol{g})$ and $\mathfrak{S}_{ik}(\boldsymbol{f}, \boldsymbol{p})$, respectively ($\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{p}$ being combinations, by C6II). Then the ranks of $\boldsymbol{A}$ and $\boldsymbol{B}$ are both at least 3. Hence, by 15Ik, 19IIIb and 19I, $\epsilon_2{}^{|\boldsymbol{a}|}$ conv $\epsilon_2{}^{|\boldsymbol{b}|}$ conv 2. $\mathfrak{u}_{ij}$, $\mathfrak{s}_{ik}$ and $\mathfrak{s}_{jk}$ are, by definition, metads which represent the combinations $\mathfrak{U}_{ij}$, $\mathfrak{S}_{ik}$, and $\mathfrak{S}_{jk}$, respectively. Also, by 19IIIf, the metads $\boldsymbol{a}_{11}$, $\boldsymbol{a}_{12}$, $\boldsymbol{a}_2$, $\boldsymbol{b}_{11}$, $\boldsymbol{b}_{12}$, $\boldsymbol{b}_2$ represent the combinations $\mathfrak{U}_{ij}$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\mathfrak{S}_{ik}$, $\boldsymbol{f}$, $\boldsymbol{p}$, respectively. Hence, by 19VI, $\Delta(\boldsymbol{a}_{11}, \mathfrak{u}_{ij})$ conv $\Delta(\boldsymbol{b}_{11}, \mathfrak{s}_{ik})$ conv $\Delta^{\boldsymbol{a}_{12}}_{\boldsymbol{b}_{12}}$ conv 2. Then, by 15Ij, $\epsilon_2{}^{|\boldsymbol{a}|} \circ \epsilon_2{}^{|\boldsymbol{b}|} \circ \Delta(\boldsymbol{a}_{11}, \mathfrak{u}_{ij}) \circ \Delta(\boldsymbol{b}_{11}, \mathfrak{s}_{ik}) \circ \Delta^{\boldsymbol{a}_{12}}_{\boldsymbol{b}_{12}}$ conv 2. Consequently $\mathfrak{R}_{ijk}(\boldsymbol{a}, \boldsymbol{b})$ conv $\mathfrak{r}_{ijk}(2, \boldsymbol{a}, \boldsymbol{b})$, conv $\langle\langle \mathfrak{s}_{jk}, \boldsymbol{a}_2\rangle, \boldsymbol{b}_2\rangle$. By 19V, the latter is a metad which represents $\mathfrak{S}_{jk}(\boldsymbol{g}, \boldsymbol{p})$, which is the formula resulting from the application of R_{ijk} to $\boldsymbol{A}$, $\boldsymbol{B}$.

COROLLARY. *If the combination* $\bar{\boldsymbol{D}}$ *is derivable from* $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ *by* R_1-R_{614}, *the set of formulas derivable from* $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$ *by zero or more operations of passing from* $\boldsymbol{a}$, $\boldsymbol{b}$ *to* $\mathfrak{R}_1(\boldsymbol{a}), \cdots, \mathfrak{R}_{102}(\boldsymbol{a})$, $\mathfrak{R}_{103}(\boldsymbol{a}, \boldsymbol{b}), \cdots$, *or* $\mathfrak{R}_{614}(\boldsymbol{a}, \boldsymbol{b})$ *contains a metad which represents* $\bar{\boldsymbol{D}}$.

This follows from the Theorems 19XI(t) by the definition of $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$ as metads representing the combinations $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$, respectively.

Now let $\mathfrak{H}$ be an expression which has the properties (1) and (2) of $\boldsymbol{H}$ in 17II when $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$, $\boldsymbol{R}_1, \cdots, \boldsymbol{R}_{m+n}$, m, n are taken to be $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$, $\mathfrak{R}_1, \cdots, \mathfrak{R}_{614}$, 102, 512, respectively.

19XII. *If the combination* $\bar{\boldsymbol{D}}$ *is representative of a formula* $\boldsymbol{D}$ *provable* *in C_1, then there is a positive integer n such that $\mathfrak{H}(\boldsymbol{n})$ is a metad which represents $\bar{\boldsymbol{D}}$.*

Proof. By 19X, $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by R_1-R_{614}. The conclusion follows by 19XI Cor. and 17II(1) (under our definition of $\mathfrak{H}$).

Let $G \rightarrow \lambda a \cdot \mathfrak{G}(a, \Pi, \Sigma, \&)$.

$$19.22(s) \qquad \mathrm{ad}(\mathfrak{a}_s) G(\mathfrak{a}_s) \qquad (s = 1, \cdots, 13).$$

Proof. Since $\mathfrak{a}_s$ is a given metad, $\mathrm{ad}(\mathfrak{a}_s)$ is provable from the formulas 19.1 by a succession of applications of 19.3. Since $\mathfrak{a}_s$ represents the combination $\mathfrak{A}_s$, which is representative of an axiom $\boldsymbol{A}_s$, $G(\mathfrak{a}_s)$ conv $\mathfrak{G}(\mathfrak{a}_s, \Pi, \Sigma, \&)$, conv $\mathfrak{A}_s(\Pi, \Sigma, \&)$ (19VII), conv $\{\lambda \Pi\Sigma\& \cdot \boldsymbol{A}_s \cdot E(\Pi)\}(\Pi, \Sigma, \&)$, conv $\boldsymbol{A}_s \cdot E(\Pi)$, which is a provable formula.

$$19.23(t): \quad \begin{array}{l} \mathrm{ad}(a) G(a) \supset_a \mathrm{ad}(\mathfrak{R}_t(a)) G(\mathfrak{R}_t(a)) \qquad (t = 1, \cdots, 102). \\ {[\mathrm{ad}(a) G(a) \cdot \mathrm{ad}(b) G(b)]} \supset_{ab} \mathrm{ad}(\mathfrak{R}_t(a, b)) G(\mathfrak{R}_t(a, b)) \\ \qquad\qquad\qquad\qquad\qquad\qquad (t = 103, \cdots, 614). \end{array}$$

Proof. We take as typical the case of a $t > 102$. Then $\mathfrak{R}_t$ is one of the expressions $\mathfrak{R}_{ijk}$ for a certain i, j and k. 19.22 $\vdash \Sigma ab \cdot \mathrm{ad}(a) G(a) \cdot \mathrm{ad}(b) G(b)$. Assume $\mathrm{ad}(a) G(a) \cdot \mathrm{ad}(b) G(b)$. Since $\mathfrak{u}_{ij}$, $\mathfrak{s}_{ik}$ and $\mathfrak{s}_{jk}$ are given metads, $\mathrm{ad}(\mathfrak{u}_{ij})$, $\mathrm{ad}(\mathfrak{s}_{ik})$ and $\mathrm{ad}(\mathfrak{s}_{jk})$ are provable. Using 14.2, 14.7, 19.2, 19.8, and 19.12, $M(\epsilon_2{}^{|a|} \circ \epsilon_2{}^{|b|} \circ \Delta(a_{11}, \mathfrak{u}_{ij}) \circ \Delta(b_{11}, \mathfrak{s}_{ik}) \circ \Delta^{a_{12}}_{b_{12}})$. Case 1: $\epsilon_2{}^{|a|} \circ \epsilon_2{}^{|b|} \circ \Delta(a_{11}, \mathfrak{u}_{ij}) \circ \Delta(b_{11}, \mathfrak{s}_{ik}) \circ \Delta^{a_{12}}_{b_{12}} = 1$. Then $\mathfrak{R}_{ijk}(a, b) = \mathfrak{r}_{ijk}(1, a, b)$, conv $I^{|a|}(b)$, $= b$ (19.2, 7.2), and $\mathrm{ad}(\mathfrak{R}_{ijk}(a, b)) G(\mathfrak{R}_{ijk}(a, b))$ follows from $\mathrm{ad}(b) G(b)$. Case 2: $\epsilon_2{}^{|a|} \circ \epsilon_2{}^{|b|} \circ \Delta(a_{11}, \mathfrak{u}_{ij}) \circ \Delta(b_{11}, \mathfrak{s}_{ik}) \circ \Delta^{a_{12}}_{b_{12}} = 2$. Then $\mathfrak{R}_{ijk}(a, b) = \mathfrak{r}_{ijk}(2, a, b)$, conv $\langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle$, and $\mathrm{ad}(\mathfrak{R}_{ijk}(a, b))$ follows from $\mathrm{ad}(\mathfrak{s}_{jk})$, $\mathrm{ad}(a)$ and $\mathrm{ad}(b)$ by means of 19.8 and 19.11. Also $|a| > 2$, $|b| > 2$, $\Delta(a_{11}, \mathfrak{u}_{ij}) = 2$, $\Delta(b_{11}, \mathfrak{s}_{ik}) = 2$, $\Delta^{a_{12}}_{b_{12}} = 2$ (14.2, 14.6, 14.7, 14.9, 19.2, 19.8, 19.12). Now, from $G(a)$ by conversion, $\mathfrak{G}(a, \Pi, \Sigma, \&)$; thence, by 19.18, $\mathfrak{G}(a_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; by another application of 19.18, $\mathfrak{G}(a_{11}, \mathfrak{G}(a_{12}), \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; by two applications of 19.21, $\mathfrak{G}(\mathfrak{u}_{ij}, \mathfrak{G}(b_{12}), \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; and, by conversion (cf. 19VII), $\mathfrak{A}(\Pi, \Sigma, \&)$, where $\mathfrak{A} \rightarrow \mathfrak{U}_{ij}(\mathfrak{G}(b_{12}), \mathfrak{G}(a_2))$. Similarly, from $G(b)$ we infer $\mathfrak{B}(\Pi, \Sigma, \&)$, where $\mathfrak{B} \rightarrow \mathfrak{S}_{ik}(\mathfrak{G}(b_{12}), \mathfrak{G}(b_2))$. If $\mathfrak{C} \rightarrow \mathfrak{S}_{jk}(\mathfrak{G}(a_2), \mathfrak{G}(b_2))$, then $\mathfrak{C}$ is derivable from $\mathfrak{A}$ and $\mathfrak{B}$ by an application of R_{ijk}. Hence, by 19IX, we can infer $\mathfrak{C}(\Pi, \Sigma, \&)$ from $\mathfrak{A}(\Pi, \Sigma, \&)$ and $\mathfrak{B}(\Pi, \Sigma, \&)$. From $\mathfrak{C}(\Pi, \Sigma, \&)$, by conversion, $\mathfrak{G}(\mathfrak{s}_{jk}, \mathfrak{G}(a_2), \mathfrak{G}(b_2), \Pi, \Sigma, \&)$ (19VII); by applications of 19.20,

$\mathfrak{G}(\langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle, \Pi, \Sigma, \&)$; by conversion, $G(\langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle)$; and thence $G(\mathfrak{R}_{ijk}(a,b))$. Using Axiom 14, $\mathrm{ad}(\mathfrak{R}_{ijk}(a,b)) \cdot G(\mathfrak{R}_{ijk}(a,b))$. By cases (C9I), $\mathrm{ad}(\mathfrak{R}_{ijk}(a,b))G(\mathfrak{R}_{ijk}(a,b))$.

19.24: $$N(n)\supset_n \mathrm{ad}(\mathfrak{H}(n))G(\mathfrak{H}(n)).$$

This formula follows from the formulas 19.22(s) and 19.23(t) by 17II(2) and our definition of $\mathfrak{H}$.

19XIII. *If $\boldsymbol{F}(\boldsymbol{P})$ is provable in C_1, and $\boldsymbol{P}$ contains no free symbols, then a formula $\boldsymbol{U}$ (containing no free symbols) can be found such that* (1) *if $\boldsymbol{F}(\boldsymbol{Q})$ is provable in C_1, and $\boldsymbol{Q}$ contains no free symbols, then there is a positive integer q such that $\boldsymbol{U}(\boldsymbol{q})$ conv $\boldsymbol{Q}$, and* (2) *$N(n)\supset_n \boldsymbol{F}(\boldsymbol{U}(n))$ is provable.*

Proof. Assume the hypothesis. Let $\boldsymbol{F}'$ and $\boldsymbol{P}'$ be combinations such that $\boldsymbol{F}'$ conv $\lambda p\Pi\Sigma\& \cdot \boldsymbol{F}(p) \cdot E(\Pi)$, and $\boldsymbol{P}'$ conv $\boldsymbol{P}$ (C6V, C5VI, C5V Cor.). Let $\boldsymbol{c}$ be a metad representing $\boldsymbol{F}'(\boldsymbol{P}')$ (19IIIa). Let $\boldsymbol{K}$ be an expression such that $\boldsymbol{K}(1)$ conv $\lambda a \cdot I^{|a|}(\boldsymbol{c})$ and $\boldsymbol{K}(2)$ conv I. Let $\boldsymbol{L} \rightarrow \lambda a \cdot \boldsymbol{K}(\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{\boldsymbol{c}_1}, a)$.

(1) *If the metad $\boldsymbol{a}$ represents a combination of the form $\boldsymbol{F}'(\boldsymbol{Q}')$, then $\boldsymbol{L}(\boldsymbol{a})$ conv $\boldsymbol{a}$* (15Ij, k, 19I, 19IIIb, f, 19VI).

(2) $\vdash \mathrm{ad}(a)G(a)\supset_a \cdot \mathrm{ad}(\boldsymbol{L}(a))G(\boldsymbol{L}(a)) \cdot \epsilon(|\boldsymbol{L}(a)|,1) \circ \Delta(\boldsymbol{L}(a)_1, \boldsymbol{c}_1) = 2.$

Proof. Assume $\mathrm{ad}(a)G(a)$. Case 1: $\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{\boldsymbol{c}_1} = 1$. Then $\boldsymbol{L}(a) = \boldsymbol{K}(1,a)$, $= \boldsymbol{c}$ (19.2, 7.2). 19.1, 19.3 $\vdash \mathrm{ad}(\boldsymbol{c})$; $G(\boldsymbol{c})$ is provable by conversion from $\boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$; and $\epsilon_1{}^{|\boldsymbol{c}|} \circ \Delta^{\boldsymbol{c}_1}_{\boldsymbol{c}_1}$ conv 2. Case 2: $\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{\boldsymbol{c}_1} = 2$. Then $\boldsymbol{L}(a) = \boldsymbol{K}(2,a)$, conv a. In both cases $\mathrm{ad}(\boldsymbol{L}(a))G(\boldsymbol{L}(a)) \cdot \epsilon(|\boldsymbol{L}(a)|,1) \circ \Delta(\boldsymbol{L}(a)_1, \boldsymbol{c}_1) = 2$ is provable from the assumptions; and hence, by applications of C9I and Theorem I, (2) holds.

Let $\mathfrak{W} \rightarrow \lambda a \cdot \mathfrak{G}(a_2)$.

(3) *If the metad $\boldsymbol{a}$ represents a combination of the form $\boldsymbol{F}'(\boldsymbol{Q}')$, then $\mathfrak{W}(\boldsymbol{a})$ conv $\boldsymbol{Q}'$* (19IIIf, 19VII).

(4) $\vdash [\mathrm{ad}(a)G(a) \cdot \epsilon_1{}^{|a|} \circ \Delta^{a_1}_{\boldsymbol{c}_1} = 2]\supset_a F(\mathfrak{W}(a)).$

Proof. By 19.22 and (2), $\Sigma a \cdot \mathrm{ad}(a)G(a) \cdot \epsilon_1{}^{|a|} \circ \Delta^{a_1}_{\boldsymbol{c}_1} = 2$. Assume $\mathrm{ad}(a)G(a) \cdot \epsilon_1{}^{|a|} \circ \Delta^{a_1}_{\boldsymbol{c}_1} = 2$. Then $|a| > 1$ and $\Delta^{a_1}_{\boldsymbol{c}_1} = 2$ (14.6, 14.7, 14.9, 19.2, 19.8, 19.12). Now $G(a)$ conv $\mathfrak{G}(a, \Pi, \Sigma, \&)$; thence, by 19.18, $\mathfrak{G}(a_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; by 19.21, $\mathfrak{G}(\boldsymbol{c}_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$. $\mathfrak{G}(\boldsymbol{c}_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$

conv $\boldsymbol{F}'(\mathfrak{G}(a_2), \Pi, \Sigma, \&)$ (19IIIf, 19VII), conv $\boldsymbol{F}(\mathfrak{G}(a_2)) \cdot E(\Pi)$ (def. of $\boldsymbol{F}'$), conv $\boldsymbol{F}(\mathfrak{W}(a)) \cdot E(\Pi)$, whence, by Axiom 15, $\boldsymbol{F}(\mathfrak{W}(a))$.

Let $\boldsymbol{U} \rightarrow \lambda n \cdot \mathfrak{W}(\boldsymbol{L}(\mathfrak{H}(n)))$.

(5) Suppose that $\boldsymbol{F}(\boldsymbol{Q})$ is provable in C_1, and that $\boldsymbol{Q}$ contains no free symbols. Let $\boldsymbol{Q}'$ be a combination such that $\boldsymbol{Q}'$ conv $\boldsymbol{Q}$. Then the combination $\boldsymbol{F}'(\boldsymbol{Q}')$ is representative of $\boldsymbol{F}(\boldsymbol{Q})$. Hence, by 19XII, there is a positive integer q such that $\mathfrak{H}(\boldsymbol{q})$ represents $\boldsymbol{F}'(\boldsymbol{Q}')$. Now $\boldsymbol{U}(\boldsymbol{q})$ conv $\mathfrak{W}(\boldsymbol{L}(\mathfrak{H}(\boldsymbol{q})))$, conv $\mathfrak{W}(\mathfrak{H}(\boldsymbol{q}))$ (by (1)), conv $\boldsymbol{Q}'$ (by (3)), conv $\boldsymbol{Q}$.

(6) Assume $N(n)$. By 19.24, $\mathrm{ad}(\mathfrak{H}(n)) \cdot G(\mathfrak{H}(n))$. Thence, using (2) and (4), $\boldsymbol{F}(\mathfrak{W}(\boldsymbol{L}(\mathfrak{H}(n))))$, and, by conversion, $\boldsymbol{F}(\boldsymbol{U}(n))$. By Theorem I, $N(n) \supset_n \boldsymbol{F}(\boldsymbol{U}(n))$.

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