A THEORY OF POSITIVE INTEGERS IN FORMAL LOGIC.*
PART I.

By S. C. Kleene.

1. Introduction. In this paper we shall be concerned primarily with the development of the system of logic based on a set of postulates proposed by A. Church.† Our object is to demonstrate empirically that the system is adequate for the theory of positive integers, by exhibiting a construction of a significant portion of the theory within the system. By carrying out the construction on the basis of a certain subset of Church's formal axioms, we show that this portion at least of the theory of positive integers can be deduced from logic without the use of the notions of *negation*, *class*, and *description*.

Instead of limiting our discussion to the system of Church, we shall employ his rules of procedure in a generalized form, and present our results as valid for a logic based on these rules and any set of well-formed formal axioms which includes 1, 3-11, 14-16.‡ This program is explained in the introductory section of a previous article.§

We presuppose familiarity with the contents of the first article of Church, and of §§ 1-5 of our previous article.

As has been noted, significances can be assigned to the symbols of the logic in such a way that the formulas become assertions of logical truths. The mathematical interest arises from the fact that among the logical entities of the system it is possible to select certain ones which occur in relations of the same form as the relations between certain entities in mathematical theories. Hence if the mathematical entities are identified with, or defined to be, the logical entities, the propositions in which they occur will read as theorems of mathematics. It is in this sense that we are to develop or deduce the theory of positive integers within the logic, *i. e.* we are to define the numbers and

* Received October 9, 1933. Revised manuscript received June 18, 1934.

† A. Church, "A set of postulates for the foundation of logic," *Annals of Mathematics*, vol. 33 (1932), pp. 346-366, and a second paper under the same title, vol. 34 (1933), pp. 839-864. We shall refer to these articles by their dates.

‡ Church, 1933, p. 841, or 1932, p. 356.

§ S. C. Kleene, "Proof by cases in formal logic," *Annals of Mathematics*, vol. 35 (1934), no. 3. References to theorems or sections of this paper will be made by prefixing the letter C to the number of the theorem or section.

other notions employed in the theory as expressions of the logic, and prove formulas which assert, in the symbolism of the logic, that these expressions stand in the relations which the mathematical theory requires.

2. Equality. The definition

$$= \rightarrow \lambda\mu\nu \cdot \phi(\mu) \supset_{\phi} \phi(\nu),$$

$\{=\}(\boldsymbol{x}, \boldsymbol{y})$ abbreviated to $[\boldsymbol{x}] = [\boldsymbol{y}]$, and the theorems *

2.1: $\quad {}^{\epsilon} x \cdot x = x,$

2.2: $\quad [x = y] \supset_{xy} \cdot y = x,$

2.3: $\quad [x = y] \supset_{xy} \cdot [y = z] \supset_{z} \cdot x = z,$

show that we may carry out within the system certain familiar operations with equalities.

Specifically, it follows from the definition that we may pass from an expression $\boldsymbol{J}$ to an expression $\boldsymbol{J}'$ by the substitution for a free occurrence of $\boldsymbol{N}$ in $\boldsymbol{J}$ of an equal expression $\boldsymbol{N}'$, provided the resulting occurrence of $\boldsymbol{N}'$ in $\boldsymbol{J}'$ is also free, *i. e.* under these circumstances $\boldsymbol{J}, \boldsymbol{N} = \boldsymbol{N}' \vdash \boldsymbol{J}'$. For then, according to C5I, we may convert $\boldsymbol{J}$ into $\{\lambda\boldsymbol{x} \cdot \boldsymbol{K}\}(\boldsymbol{N})$ and $\boldsymbol{J}'$ into $\{\lambda\boldsymbol{x} \cdot \boldsymbol{K}\}(\boldsymbol{N}')$ for a suitably chosen $\boldsymbol{x}$ and $\boldsymbol{K}$, and we can pass from $\{\lambda\boldsymbol{x} \cdot \boldsymbol{K}\}(\boldsymbol{N})$ to $\{\lambda\boldsymbol{x} \cdot \boldsymbol{K}\}(\boldsymbol{N}')$ by means of Rule V using $\boldsymbol{N} = \boldsymbol{N}'$ as major premise. This argument also applies directly to the substitution of occurrences of $\boldsymbol{N}'$ for each of a set of occurrences of $\boldsymbol{N}$, provided the occurrences of $\boldsymbol{N}$ and of $\boldsymbol{N}'$ are free. (Cf. C2IX.) As a special case of this substitution rule we may pass from $\boldsymbol{A} = \boldsymbol{B}$ to $\boldsymbol{A} = \boldsymbol{C}$ where $\boldsymbol{C}$ is obtained by legitimate substitutions of $\boldsymbol{N}'$ for $\boldsymbol{N}$ within $\boldsymbol{B}$.

From 2.1 it follows that we may equate an expression $\boldsymbol{A}$ to itself, or to any expression $\boldsymbol{B}$ obtained from $\boldsymbol{A}$ by conversion, provided only that we can prove $E(\boldsymbol{A})$ or $E(\boldsymbol{B})$. For if we have $E(\boldsymbol{A})$ we infer $\boldsymbol{A} = \boldsymbol{A}$ by 2.1, and convert the $\boldsymbol{A}$ on the right into $\boldsymbol{B}$. If we have $E(\boldsymbol{B})$ we infer $\boldsymbol{B} = \boldsymbol{B}$ by 2.1, and convert the $\boldsymbol{B}$ on the left into $\boldsymbol{A}$. (Cf. § C5, paragraph 2.) C5II characterizes the situations in which we may prove $E(\boldsymbol{M})$.

2.2 enables us to pass from $\boldsymbol{A} = \boldsymbol{B}$ to $\boldsymbol{B} = \boldsymbol{A}$, and 2.3 from $\boldsymbol{A} = \boldsymbol{B}$ and $\boldsymbol{B} = \boldsymbol{C}$ to $\boldsymbol{A} = \boldsymbol{C}$.

The abbreviation $\boldsymbol{A}_1 = \boldsymbol{A}_2 = \boldsymbol{A}_3 = \cdots = \boldsymbol{A}_n$ will be used for $[\boldsymbol{A}_1 = \boldsymbol{A}_2]$

* Church, 1933, Theorems 1, 2, 3.

$[\boldsymbol{A}_2 = \boldsymbol{A}_3]\ [\boldsymbol{A}_3 = \boldsymbol{A}_4] \cdots [\boldsymbol{A}_{n-1} = \boldsymbol{A}_n]$. As a consequence of 2.2 and 2.3, each of the n^2 equalities $\boldsymbol{A}_i = \boldsymbol{A}_j$ $(i, j = 1, \cdots, n)$ is a consequence of the *chain of equalities* $\boldsymbol{A}_1 = \boldsymbol{A}_2 = \cdots = \boldsymbol{A}_n$. Frequently we are interested only in the equality $\boldsymbol{A}_1 = \boldsymbol{A}_n$.

We shall adopt the practice of starting with an expression, $\boldsymbol{A}_1$, and writing successively $= \boldsymbol{A}_2, = \boldsymbol{A}_3, \cdots, = \boldsymbol{A}_n$, whenever it can be shown that, for an arbitrary $\boldsymbol{F}$, $\boldsymbol{F}(\boldsymbol{A}_1) \vdash \boldsymbol{F}(\boldsymbol{A}_2)$ and $\boldsymbol{F}(\boldsymbol{A}_2) \vdash \boldsymbol{F}(\boldsymbol{A}_1)$, $\boldsymbol{F}(\boldsymbol{A}_2) \vdash \boldsymbol{F}(\boldsymbol{A}_3)$ and $\boldsymbol{F}(\boldsymbol{A}_3) \vdash \boldsymbol{F}(\boldsymbol{A}_2), \cdots, \boldsymbol{F}(\boldsymbol{A}_{n-1}) \vdash \boldsymbol{F}(\boldsymbol{A}_n)$ and $\boldsymbol{F}(\boldsymbol{A}_n) \vdash \boldsymbol{F}(\boldsymbol{A}_{n-1})$, respectively. Such a *chain of contingent equalities linking* $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n$ represents a chain of provable equalities if $E(\boldsymbol{A}_i)$ is provable for any one i $(1 \leq i \leq n)$. For then each of $E(\boldsymbol{A}_1)$, $E(\boldsymbol{A}_2), \cdots, E(\boldsymbol{A}_{n-1})$ follows from $E(\boldsymbol{A}_i)$ and the facts $\boldsymbol{F}(\boldsymbol{A}_1) \vdash \boldsymbol{F}(\boldsymbol{A}_2), \cdots, \boldsymbol{F}(\boldsymbol{A}_n) \vdash \boldsymbol{F}(\boldsymbol{A}_{n-1})$; and then each of the $n - 1$ factors of $\boldsymbol{A}_1 = \boldsymbol{A}_2 = \cdots = \boldsymbol{A}_n$ can be proved with the aid of Theorem I. A special case in which $E(\boldsymbol{A}_i)$ can be proved is that $\boldsymbol{A}_{i-1} = \boldsymbol{A}_i$ or $\boldsymbol{A}_i = \boldsymbol{A}_{i+1}$ is a previously established equality.

The circumstances which we most commonly use to justify linking $\boldsymbol{A}_{j+1}$ to $\boldsymbol{A}_j$ in forming a chain of contingent equalities, *i. e.* circumstances which imply that $\boldsymbol{F}(\boldsymbol{A}_j) \vdash \boldsymbol{F}(\boldsymbol{A}_{j+1})$ and $\boldsymbol{F}(\boldsymbol{A}_{j+1}) \vdash \boldsymbol{F}(\boldsymbol{A}_j)$, are the following: (1) $\boldsymbol{A}_j$ convertible into $\boldsymbol{A}_{j+1}$,* (2) $\boldsymbol{A}_j = \boldsymbol{A}_{j+1}$ or $\boldsymbol{A}_{j+1} = \boldsymbol{A}_j$ previously established, (3) $\boldsymbol{A}_{j+1}$ obtainable from $\boldsymbol{A}_j$ by a substitution of $\boldsymbol{N}'$ for $\boldsymbol{N}$ of the type described above ($\boldsymbol{N} = \boldsymbol{N}'$ or $\boldsymbol{N}' = \boldsymbol{N}$ being previously established).†

Ordinarily we introduce a chain of contingent equalities only as a preliminary to inferring their provability, or their provability from formulas which we have assumed. It is intended that it should be clear from the context when this is the case. It will suffice to point out in the course of building the chain or subsequently that one of the equalities is already known, or that one of the expressions exists. Even this formality may be omitted in situations of a type which has already occurred often, when we evidently make subsequent use of the chain of equalities as proved or as proved under our assumptions.

However chains of contingent equalities are of interest in themselves since a chain of contingent equalities linking $\boldsymbol{N}$ and $\boldsymbol{N}'$ can be used in place of a proved $\boldsymbol{N} = \boldsymbol{N}'$ in passing from $\boldsymbol{J}$ to $\boldsymbol{J}'$ under the conditions above, *i. e.* $\boldsymbol{J} \vdash \boldsymbol{J}'$ can be inferred by means of the former. In particular, in the construction of one chain of contingent equalities we may use, (2′), instead of $\boldsymbol{A}_i = \boldsymbol{A}_{i+1}$ or $\boldsymbol{A}_{i+1} = \boldsymbol{A}_i$ under (2), a previously obtained chain of contingent equalities

* In this case we may write *conv*, instead of $=$, before $\boldsymbol{A}_{j+1}$ in constructing the chain.

† (3) includes (2). The transitive property, $\boldsymbol{A} = \boldsymbol{B}, \boldsymbol{B} = \boldsymbol{C} \vdash \boldsymbol{A} = \boldsymbol{C}$, of equality is a special case of the substitution rule, $\boldsymbol{J}, \boldsymbol{N} = \boldsymbol{N}' \vdash \boldsymbol{J}'$.

linking $\boldsymbol{A}_i$ and $\boldsymbol{A}_{i+1}$, and, (3′), instead of $\boldsymbol{N} = \boldsymbol{N}'$ or $\boldsymbol{N}' = \boldsymbol{N}$ under (3), a previously obtained chain linking $\boldsymbol{N}$ and $\boldsymbol{N}'$. Chains of contingent equalities are derived in § 15, under hypotheses which do not imply their provability, with a view to their subsequent use in this manner.

If one chain of contingent equalities is used in obtaining another in the fashion just described, and the latter established as true, then the provability of the former follows. For $E(\boldsymbol{A}_i)$ or $E(\boldsymbol{N})$ can then be proved according to C5II. Thus starting with conversions and known equalities we can build a hierarchy of contingent equalities. If one of the expressions linked in the chain at the top occurs in a provable formula as a free expression, then all the equalities of the hierarchy are provable.

As with formulas, contingent equalities $\boldsymbol{M} = \boldsymbol{N}$ will often occur below when it is shown merely that they hold as a consequence of "assumptions" $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \cdots$, *i. e.* when $\boldsymbol{F}(\boldsymbol{M}), \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \cdots \vdash \boldsymbol{F}(\boldsymbol{N})$ and $\boldsymbol{F}(\boldsymbol{N}), \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \cdots \vdash \boldsymbol{F}(\boldsymbol{M})$.

When there is ambiguity, we may use an accent to distinguish a contingent equality $\boldsymbol{M} =' \boldsymbol{N}$ from an equality $\boldsymbol{M} = \boldsymbol{N}$. The assertion that $\boldsymbol{M} =' \boldsymbol{N}$ would hold if $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \cdots$ were added to the list of axioms may be written $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \cdots \vdash' \boldsymbol{M} =' \boldsymbol{N}$.

3. Definition of the positive integers. Peano's axioms. A construction of the theory of positive integers in the logic has been begun by Church,* who selected formulas to represent the three undefined terms, one, successor, and the notion of being a number, of Peano,† and adopted Peano's definitions of $2, 3, \cdots$ by means of the first two of them, thus:

$$
\begin{aligned}
&1 \rightarrow \lambda f x \cdot f(x). \\
&S \rightarrow \lambda \rho f x \cdot f(\rho(f, x)). \\
&N \rightarrow \lambda \mu \cdot [\phi(1) \cdot \phi(x) \supset_x \phi(S(x))] \supset_\phi \phi(\mu). \\
&2 \rightarrow S(1), \qquad 3 \rightarrow S(2), \qquad 4 \rightarrow S(3), \cdots.
\end{aligned}
$$

We follow Church in these definitions, but not in the definition of addition, because J. B. Rosser has proposed one which leads to simpler formal proofs, nor in the definitions of multiplication and subtraction, because it is our program to avoid the use of the function ι.

Our first objective will, of course, be the formulation and proof in the logic of Peano's axioms, except the fourth which involves negation. Formula-

* Church, 1933, § 9.

† G. Peano, *Rivista di Matematica,* vol. 1 (1891), pp. 87-102. Peano called the "first" number "zero."

tions have been given by Church, which we adopt except in the case of the fifth. The fifth we express by 3. 3, which may be a better rendering of Peano's axiom, and which we need in the course of our development of the theory.

The first two, 3. 1 and 3. 2, may be proved thus:

$\mathfrak{A}_1$: $\Sigma(\Sigma)$—by two applications of IV to any axiom.

$\mathfrak{A}_2$: $\{\lambda\rho \cdot \rho(I, \Sigma(\Sigma))\}(1)$—by conversion from $\mathfrak{A}_1$. $(I \rightarrow \lambda x \cdot x.)$

$\mathfrak{A}_3$: $\Sigma y \cdot \{\lambda\rho \cdot \rho(I, \Sigma(\Sigma))\}(y)$—IV, (III, $\mathfrak{A}_2$).

$\mathfrak{D}_y$: $\{\lambda\rho \cdot \rho(I, \Sigma(\Sigma))\}(y)$—assumed, in preparation for an application of Theorem I.

$\mathfrak{C}_{y1}$: $\{\lambda\rho \cdot \rho(I, \Sigma(\Sigma))\}(S(y))$—by conversion from $\mathfrak{D}_y$.

$\mathfrak{A}_4$: $\{\lambda\rho \cdot \rho(I, \Sigma(\Sigma))\}(y) \supset_y \{\lambda\rho \cdot \rho(I, \Sigma(\Sigma))\}(S(y))$—provable, according to Theorem I, using $\mathfrak{A}_3$ and the proof of $\mathfrak{C}_{y1}$ from $\mathfrak{D}_y$.

$\mathfrak{A}_5$: $\Sigma\phi \cdot \phi(1) \cdot \phi(y) \supset_y \phi(S(y))$—IV, III, (14, $\mathfrak{A}_2$, $\mathfrak{A}_4$).

$\mathfrak{D}_\phi$: $\phi(1) \cdot \phi(y) \supset_y \phi(S(y))$—assumed.

$\mathfrak{C}_{\phi 2}$: $\phi(1)$—15, $\mathfrak{D}_\phi$.

$\mathfrak{A}_6$: $[\phi(1) \cdot \phi(y) \supset_y \phi(S(y))] \supset_\phi \phi(1)$—Theorem I, $\mathfrak{A}_5$, $\mathfrak{C}_{\phi 2}$.

3. 1: $N(1)$—III, $\mathfrak{A}_6$.

$\mathfrak{A}_7$: $\Sigma x \cdot N(x)$—IV, (III, 3. 1).

$\mathfrak{D}_x$: $N(x)$—assumed.

$\mathfrak{C}_{\phi x3}$: $\phi(x)$—V, $\mathfrak{D}_x$, $\mathfrak{D}_\phi$, conversion.

$\mathfrak{C}_{\phi 4}$: $\phi(y) \supset_y \phi(S(y))$—16, $\mathfrak{D}_\phi$.

$\mathfrak{C}_{\phi x5}$: $\phi(S(x))$—V, $\mathfrak{C}_{\phi 4}$, $\mathfrak{C}_{\phi x3}$, conversion.

$\mathfrak{C}_{x6}$: $[\phi(1) \cdot \phi(y) \supset_y \phi(S(y))] \supset_\phi \phi(S(x))$—Theorem I, $\mathfrak{A}_5$, $\mathfrak{C}_{\phi x5}$.

$\mathfrak{C}_{x7}$: $N(S(x))$—III, $\mathfrak{C}_{x6}$.

3. 2: $N(x) \supset_x N(S(x))$—Theorem I, $\mathfrak{A}_7$, $\mathfrak{C}_{x7}$.

The fifth Peano axiom we establish as follows:

$\mathfrak{A}_1$: $N(y)N(y) \supset_y N(S(y))$—provable by Theorem I, using 3. 1, 3. 2, 14, 15.

$\mathfrak{D}_\phi$: $\phi(1) \cdot N(y)\phi(y) \supset_y \phi(S(y))$—assumed; $\Sigma\phi \cdot \mathfrak{D}_\phi$ is provable from 14, 3. 1, $\mathfrak{A}_1$.

$\mathfrak{C}_{\phi 1}$: $N(1)\phi(1)$—14, 3. 1, 15($\mathfrak{D}_\phi$).

$\mathfrak{D}_{\phi,y}$: $N(y)\phi(y)$—assumed; $\Sigma y \cdot \mathfrak{D}_{\phi,y}$ is provable by means of III and IV from $\mathfrak{C}_{\phi 1}$.

$\mathfrak{C}_{\phi y2}$: $\phi(S(y))$—provable by means of 16($\mathfrak{D}_\phi$) and $\mathfrak{D}_{\phi,y}$.

$\mathfrak{C}_{y3}$: $N(S(y))$—provable by means of 3. 2, 15($\mathfrak{D}_{\phi,y}$).

$\mathfrak{C}_{\phi 4}$: $N(y)\phi(y) \supset_y \cdot N(S(y))\phi(S(y))$—Theorem I, 14($\mathfrak{C}_{\phi y2}$, $\mathfrak{C}_{y3}$).

$\mathfrak{C}_{\phi 5}$: $N(1)\phi(1) \cdot N(y)\phi(y) \supset_y N(S(y))\phi(S(y))$—14, $\mathfrak{C}_{\phi 1}$, $\mathfrak{C}_{\phi 4}$.

$\mathfrak{D}_x$: $N(x)$—assumed; $\Sigma x \cdot N(x)$ is provable by means of 3. 1.

$\mathfrak{C}_{\phi x 6}$: $N(x)\phi(x)$—V, $\mathfrak{D}_x$, $\mathfrak{C}_{\phi 5}$, conversion.
$\mathfrak{C}_{\phi x 7}$: $\phi(x)$—16, $\mathfrak{C}_{\phi x 6}$.
3.3: $[\phi(1) \cdot N(y)\phi(y) \supset_y \phi(S(y))] \supset_\phi \cdot N(x) \supset_x \phi(x)$ — Theorem I, (Theorem I, $\mathfrak{C}_{\phi x 7}$).

The third Peano axiom will be proved as Theorem 10.1 below.

4. Proof by induction. In the development of the theory of positive integers by means of intuitive logic, the object of Peano's fifth axiom is to justify proofs by induction. By following Frege in making it the definition of x being a positive integer that propositions involving p for which a mathematical induction with respect to p can be carried out should hold when p is taken to be x, we are enabled to express the fifth axiom as a provable formula of the logic. If the logic is adequate, it should then be possible to carry out proofs by induction within the logic. This we show to be the case.

4I. *If* $\boldsymbol{x}$ *is a variable which does not occur in* $\boldsymbol{F}$ *as a free variable, if* $\vdash \boldsymbol{F}(1)$, *and if* $N(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x}) \vdash \boldsymbol{F}(S(\boldsymbol{x}))$, *then* $\vdash N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x})$.

For by means of 14, 3.1, and $\boldsymbol{F}(1)$, we can prove $N(1)\boldsymbol{F}(1)$, and thence, since $\boldsymbol{x}$ does not occur in $\boldsymbol{F}$ as a free symbol and, being a variable, is distinct from the free symbols, Π, &, of N, we can obtain $\{\lambda \boldsymbol{x} \cdot N(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})\}(1)$ by conversion. An application of IV gives $\Sigma \boldsymbol{x} \cdot N(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})$, and, assuming $N(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})$, we can prove $N(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{x})$ by 15 and 16, and thence, according to the third hypothesis of the Theorem, $\boldsymbol{F}(S(\boldsymbol{x}))$. Then by Theorem I: $N(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(S(\boldsymbol{x}))$. Combining this with $\boldsymbol{F}(1)$ by 14, we obtain a formula convertible into $\{\lambda\phi \cdot \phi(1) \cdot N(y)\phi(y) \supset_y \phi(S(y))\}(\boldsymbol{F})$ since $\boldsymbol{F}, N$, &, S do not contain $\boldsymbol{x}$ as a free symbol. Using this formula as minor premise with 3.3 as major premise to an application of Rule V, we obtain $\{\lambda\phi \cdot N(x) \supset_x \phi(x)\}(\boldsymbol{F})$, and by conversion, since $\boldsymbol{F}$ does not contain $\boldsymbol{x}$ as a free variable, $N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x})$. Under these circumstances we may say that $N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x})$ is proved by *induction with respect to* $\boldsymbol{x}$ from the *basis* $\boldsymbol{F}(1)$; and we may call $\boldsymbol{F}(\boldsymbol{x})$ the *hypothesis of the induction.* This terminology, with appropriate modifications, will also be used in connection with the generalizations of this theorem.

Certain generalizations of the simple inductive procedure described in 4I are permissible, because they can be reduced to one or more simple inductions. Although the nature of these reductions is quite evident intuitively, we shall give them explicitly to ensure that they can be carried out wholly within the confines of the system.

4II. *If* $\boldsymbol{x}$ *is a variable which does not occur in* $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_n$ *as a free variable, if* $\vdash \boldsymbol{F}_1(1)$, $\vdash \boldsymbol{F}_2(1), \cdots, \vdash \boldsymbol{F}_n(1)$, *and if* $N(\boldsymbol{x}), \boldsymbol{F}_1(\boldsymbol{x}), \boldsymbol{F}_2(\boldsymbol{x}), \cdots, \boldsymbol{F}_n(\boldsymbol{x}) \vdash \boldsymbol{F}_i(S(\boldsymbol{x}))$ $(i = 1, \cdots, n)$, *then* $\vdash N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}_1(\boldsymbol{x})$, $\vdash N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}_2(\boldsymbol{x}), \cdots, \vdash N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}_n(\boldsymbol{x})$.

For then we can take $\lambda \boldsymbol{x} \cdot \boldsymbol{F}_1(\boldsymbol{x}) \boldsymbol{F}_2(\boldsymbol{x}) \cdots \boldsymbol{F}_n(\boldsymbol{x})$ as the $\boldsymbol{F}$ of the simple case, and prove $N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \cdot \boldsymbol{F}_1(\boldsymbol{x}) \boldsymbol{F}_2(\boldsymbol{x}) \cdots \boldsymbol{F}_n(\boldsymbol{x})$.* With the aid of this result we can prove each of the theorems $N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}_1(\boldsymbol{x}), \cdots, N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}_n(\boldsymbol{x})$ by means of Theorem I. The restrictions on $\boldsymbol{x}$ have been used tacitly in this argument. A set of theorems which are inferred to be provable by an application of this theorem may be said to be proved by a *simultaneous induction*.

4III. *If* $\boldsymbol{x}$ *is a variable which does not occur in* $\boldsymbol{F}$ *as a free variable, if* $\vdash \boldsymbol{F}(1)$, $\vdash \boldsymbol{F}(2), \cdots, \vdash \boldsymbol{F}(\boldsymbol{n})$, *and if* $N(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x}), \boldsymbol{F}(S(\boldsymbol{x})), \cdots, \boldsymbol{F}(S(\cdots n-1 \text{ times} \cdots (S(\boldsymbol{x})) \cdots)) \vdash \boldsymbol{F}(S(\cdots n \text{ times} \cdots (S(\boldsymbol{x})) \cdots))$, *then* $\vdash N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x})$.†

This theorem we establish by an intuitive induction with respect to n. It has been proved as 4I for the case that n is 1. We assume it for a value p of n, and apply it under the hypotheses with n taken as $p+1$ to the function $\lambda \boldsymbol{x} \cdot \boldsymbol{F}(\boldsymbol{x}) \boldsymbol{F}(S(\boldsymbol{x}))$, obtaining $N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}) \boldsymbol{F}(S(\boldsymbol{x}))$. Thence by Theorem I and Axiom 15 we can prove $N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x})$. Under the circumstances of this theorem we may say that $N(\boldsymbol{x}) \supset_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x})$ is proved by induction with respect to $\boldsymbol{x}$ from the n-tuple basis $\boldsymbol{F}(1), \boldsymbol{F}(2), \cdots, \boldsymbol{F}(\boldsymbol{n})$.

4IV. *If* $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ *are distinct variables which do not occur as free variables in* $\boldsymbol{F}$, *and if* $\vdash \boldsymbol{F}(1, 1, \cdots, 1)$; $N(\boldsymbol{x}_1), \boldsymbol{F}(\boldsymbol{x}_1, 1, \cdots, 1) \vdash \boldsymbol{F}(S(\boldsymbol{x}_1), 1, \cdots, 1)$; $N(\boldsymbol{x}_1), N(\boldsymbol{x}_2), \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, 1) \vdash \boldsymbol{F}(\boldsymbol{x}_1, S(\boldsymbol{x}_2), \cdots, 1)$; $\cdots$; $N(\boldsymbol{x}_1), N(\boldsymbol{x}_2), \cdots, N(\boldsymbol{x}_n), \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n) \vdash \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, S(\boldsymbol{x}_n))$, *then* $\vdash N(\boldsymbol{x}_1) N(\boldsymbol{x}_2) \cdots N(\boldsymbol{x}_n) \supset_{\boldsymbol{x}_1 \boldsymbol{x}_2 \ldots \boldsymbol{x}_n} \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$.

This theorem concerning *induction with respect to n variables* reduces to 4I in case n is 1. If the theorem be assumed for the case that n is p, and applied under the hypotheses with n taken as $p+1$ to the function $\lambda \boldsymbol{x}_1 \boldsymbol{x}_2 \cdots \boldsymbol{x}_{n-1} \cdot \boldsymbol{F}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{n-1}, 1)$, we obtain $N(\boldsymbol{x}_1) \cdots N(\boldsymbol{x}_{n-1}) \supset_{\boldsymbol{x}_1 \ldots \boldsymbol{x}_{n-1}}$

* Henceforth we often omit mention of applications of Axioms 14-16.

† When n is being used to represent a given positive integer of intuitive logic, we may use $\boldsymbol{n}$ to represent the corresponding positive integer $S(\cdots n-1 \text{ times} \cdots (S(1)) \cdots)$ of the formal logic, and *vice versa*. We sometimes employ symbols of the formal theory in a familiar intuitive sense, the context being supposed to indicate when this is being done. For example, 1, n, $+$, $-$ in x_1, x_n, x_{n+1}, $n-1$ times, and n-th.

$\cdot \boldsymbol{F}(\boldsymbol{x}_1 \cdots, \boldsymbol{x}_{n-1}, 1)$. Also by means of the last hypothesis and a corollary of Theorem I we obtain that $N(\boldsymbol{x}_n)$, $N(\boldsymbol{x}_1) \cdots N(\boldsymbol{x}_{n-1}) \supset_{\boldsymbol{x}_1 \ldots \boldsymbol{x}_{n-1}} \cdot \boldsymbol{F}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{n-1}, \boldsymbol{x}_n) \vdash N(\boldsymbol{x}_1) \cdots N(\boldsymbol{x}_{n-1}) \supset_{\boldsymbol{x}_1 \ldots \boldsymbol{x}_{n-1}} \boldsymbol{F}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{n-1}, S(\boldsymbol{x}_n))$. Hence, applying 4I to the function $\lambda \boldsymbol{x}_n \cdot N(\boldsymbol{x}_1) \cdots N(\boldsymbol{x}_{n-1}) \supset_{\boldsymbol{x}_1 \ldots \boldsymbol{x}_{n-1}} \cdot \boldsymbol{F}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$, we can prove $N(\boldsymbol{x}_n) \supset_{\boldsymbol{x}_n} \cdot N(\boldsymbol{x}_1) \cdots N(\boldsymbol{x}_{n-1}) \supset_{\boldsymbol{x}_1 \ldots \boldsymbol{x}_{n-1}} \cdot \boldsymbol{F}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$, and thence, by a corollary of Theorem I, $N(\boldsymbol{x}_1) \cdots N(\boldsymbol{x}_n) \supset_{\boldsymbol{x}_1 \ldots \boldsymbol{x}_n} \boldsymbol{F}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$.

Finally we may vary the simple inductive process of 4I in a combination of the three directions represented by 4II, 4III, and 4IV, and justify the procedure by a corresponding combination of the devices used in establishing 4II, 4III, and 4IV.

We shall apply the theorems of this section, and similar theorems below, when the situation exhibited is not exactly that described by the theorem, but one that could be made so by evident conversions. (Cf. § C5.)

5. Addition. We adopt the definition

$$+ \rightarrow \lambda \rho \sigma f x \cdot \rho(f, \sigma(f, x)),$$

due to J. B. Rosser, and abbreviate $\{+\}(\boldsymbol{x}, \boldsymbol{y})$ to $[\boldsymbol{x}] + [\boldsymbol{y}]$.

5I. $\{+\}(1)$ *conv* S. $[\boldsymbol{x} + \boldsymbol{y}] + \boldsymbol{z}$ *conv* $\boldsymbol{x} + [\boldsymbol{y} + \boldsymbol{z}]$. $S(\boldsymbol{x} + \boldsymbol{y})$ *conv* $S(\boldsymbol{x}) + \boldsymbol{y}$.

The last conversion follows from the first two thus: $S(\boldsymbol{x} + \boldsymbol{y})$ conv $1 + \cdot \boldsymbol{x} + \boldsymbol{y}$ conv $[1 + \boldsymbol{x}] + \boldsymbol{y}$ conv $S(\boldsymbol{x}) + \boldsymbol{y}$. It is often more convenient to employ this theorem and like theorems of §§ 6, 7 (often tacitly) than to refer to the formal theorems proved by means of them.

Since $E(S)$ can be proved (*e. g.* from 3. 2), the first conversion of 5I leads, by means of 2. 1,* to

$$5.1: \qquad \{+\}(1) = S.$$

Assume $N(y)$. Then, (1) we can prove $N(S(y))$ by means of 3. 2, and thence, by conversion, $N(1 + y)$, and (2) if we assume $N(x + y)$, we can prove $N(S(x + y))$ by 3. 2, and thence, by conversion, $N(S(x) + y)$. Having (1) and (2), we can prove $N(x) \supset_x N(x + y)$ by induction.† This was done on the assumption $N(y)$, and $\Sigma y \cdot N(y)$ is provable from 3. 1.

* Cf. the remarks on 2. 1 in § 2.

† Cf. 4I. In this case only the second of the assumptions $N(\boldsymbol{x})$, $\boldsymbol{F}(\boldsymbol{x})$ is used.

By Theorem I we obtain $N(y) \supset_y \cdot N(x) \supset_x N(x+y)$, and thence, by a corollary of Theorem I,*

5.2: $$N(x)N(y) \supset_{xy} N(x+y).$$

5.2 with 15 and 16 enable us to prove $E([x+y]+z)$ as a consequence of $N(x)N(y)N(z)$, and hence the second conversion of 5I leads to

5.3: $$N(x)N(y)N(z) \supset_{xyz} \cdot [x+y]+z = x+[y+z].$$

We shall use $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}$ as an abbreviation for $[\boldsymbol{x}+\boldsymbol{y}]+\boldsymbol{z}$.

(1) Since $E(2)$ is provable (*e. g.* from 3.1, 3.2) and 2 is convertible into $1+1$, we can prove $1+1=1+1$. (2) Assume $N(x)$ and $x+1=1+x$. Now $S(x)+1$ conv $S(x+1)$, $=S(1+x)$ (by means of the assumption $x+1=1+x$), conv $S(S(x))$ (and $E(S(S(x)))$ is provable from $N(x)$ with the aid of 3.2), conv $1+S(x)$. Hence, by § 2, $S(x)+1=1+S(x)$ is provable from our assumptions. (3) Using (1) and (2) we can prove, by induction

$\mathfrak{A}_1$: $$N(x) \supset_x \cdot x+1 = 1+x.$$

(4) Assume $N(x)$, $N(y)$, and $x+y=y+x$. Then $x+S(y)$ conv $x+\cdot 1+y$, $=x+\cdot y+1$ (since $y+1=1+y$ is provable from $N(y)$ and $\mathfrak{A}_1$), conv $[x+y]+1$, $=1+[x+y]$ (by means of $\mathfrak{A}_1$, $N(x)$, $N(y)$, 5.2—this equality is proved from our assumptions), $=1+[y+x]$ (by means of the hypothesis $x+y=y+x$), conv $S(y)+x$. This establishes $x+S(y)=S(y)+x$ from our assumptions, according to § 2. Having (1), (2), and (4), we infer by induction (cf. 4IV)

5.4: $$N(x)N(y) \supset_{xy} \cdot x+y = y+x.$$

Another fundamental theorem on addition will be obtained as 11.4 below.

6. Multiplication. We adopt the definition,

$$\times \rightarrow \lambda\rho\sigma x \cdot \rho(\sigma(x)),$$

of J. B. Rosser, and abbreviate $\{\times\}(\boldsymbol{x}, \boldsymbol{y})$ to $[\boldsymbol{x}] \times [\boldsymbol{y}]$ or $[\boldsymbol{x}][\boldsymbol{y}]$.†

---

* In making applications of Theorem I and its corollaries we may omit mention of the formulas $\Sigma \boldsymbol{x} \cdot \boldsymbol{M}$, $\Sigma \boldsymbol{xy} \cdot \boldsymbol{M}, \cdots$, when their proof is obvious, as here from 3.1 and 14. (Cf. the statements of Theorem I and its corollaries, Church, 1932, pp. 358, 366.)

† The abbreviation $[\boldsymbol{x}][\boldsymbol{y}]$ for the product of two positive integers, $\boldsymbol{x}$ and $\boldsymbol{y}$, employed in the presence of $N(\boldsymbol{x})$ and $N(\boldsymbol{y})$, should not lead to confusion with the

6I. $[\boldsymbol{xy}]\boldsymbol{z}$ *conv* $\boldsymbol{x}[\boldsymbol{yz}]$. $[\boldsymbol{x}+\boldsymbol{y}]\boldsymbol{z}$ *conv* $\boldsymbol{xz}+\boldsymbol{yz}$.

(1) 3.1 $\vdash E(1)$. $E(1)$, 2.1 $\vdash 1=1$ By conversion from $1=1$, $[1.1]=1$. (2) Assume $N(x)$. Then $E(S(x))$ is provable by means of 3.2. $1S(x)$ conv $S(x)$. Hence $[1S(x)]=S(x)$. From (1) and (2) by induction: *

$$6.1: \qquad N(x)\supset_x \cdot 1x = x.$$

Assume $N(y)$. Then (1) by means of 6.1 we have $N(1y)$, and (2) assuming $N(x)$ and $N(xy)$, we can prove $N(1y+xy)$ by means of 5.2, $N(xy)$, and (1), and pass by conversion to $N(S(x)y)$. Having (1) and (2) we infer by induction that $N(y)\vdash N(x)\supset_x N(xy)$. With this result we can prove, by Theorem I Corollary,

$$6.2: \qquad N(x)N(y)\supset_{xy} N(xy).$$

Assuming $N(x)N(y)N(z)$, we can prove $E([xy]z)$ by means of 6.2, and $E([x+y]z)$ by means of 5.2 and 6.2. Hence 6I leads to the pair of formal theorems

$$6.3: \qquad N(x)N(y)N(z)\supset_{xyz} \cdot [xy]z = x[yz],$$

$$6.4: \qquad N(x)N(y)N(z)\supset_{xyz} \cdot [x+y]z = xz+yz.$$

Assume $N(k)$. Then: (1) $1[1+k]=[1\cdot 1]+1k$, by 3.2, 2.1, conversion and 6.1. (2) Assume also $N(l)$ and $l[1+k]=l1+lk$. Then $S(l)[1+k]$ conv $1[1+k]+l[1+k]$, $=1+k+l[1+k]$ (by 6.1, 3.1, 5.2), $=1+k+\cdot\, l1+lk$ (by our assumption $l[1+k]=l1+lk$), conv $1+[k+l1]+lk$, $=1+[l1+k]+lk$ (by 3.1, 6.2, 5.4), conv $[1\cdot 1]+l1+\cdot\, k+lk$, $=[1\cdot 1]+l1+\cdot\, 1k+lk$ (by 6.1), conv $S(l)1+S(l)k$. We can prove the existence of any one of these expressions from our assumptions by means of 3.1, 5.2, 6.2. Hence, by § 2, $S(l)[1+k]=S(l)1+S(l)k$. (3) By induction, using (1) and (2), $N(l)\supset_l \cdot l[1+k]=l1+lk$. According to Theorem I, this argument enables us to infer the provability of

$$\mathfrak{A}_1: \qquad N(k)\supset_k \cdot N(l)\supset_l \cdot l[1+k]=l1+lk.$$

---

abbreviation for the logical product of two propositions. The abbreviation $[\boldsymbol{x}][\boldsymbol{y}][\boldsymbol{z}]$ for $[[\boldsymbol{x}][\boldsymbol{y}]][\boldsymbol{z}]$ will be employed with the arithmetical product as with the logical product.

* Cf. 4I. In this case only the first of the assumptions $N(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{x})$ is used.

(1) By conversion from $1 = 1$: $[1 \cdot 1] = 1 \cdot 1$. (2) Assume $N(x)$ and $x1 = 1x$. Then $S(x)1$ conv $1 + x1$, $= 1 + 1x$ (by the second assumption), $= 1 + x$ (by 6.1 and $N(x)$), conv $S(x)$ ($E(S(x))$ is provable from $N(x)$, using 3.2), conv $1S(x)$. Hence by § 2: $S(x)1 = 1S(x)$. (3) By induction, using (1) and (2),

$$\mathfrak{A}_2: \qquad N(x) \supset_x \cdot x1 = 1x.$$

(4) Assume $N(x)$, $N(y)$, $xy = yx$. Then $xS(y)$ conv $x[1 + y]$, $= x1 + xy$ (by $\mathfrak{A}_1$, $N(x)$, $N(y)$), $= x1 + yx$ (by the hypothesis $xy = yx$), $= 1x + yx$ (by $\mathfrak{A}_2$, $N(x)$), conv $S(y)x$. Hence $xS(y) = S(y)x$. Having (1), (2), and (4), we can prove, by induction (cf. 4IV),

$$6.5: \qquad N(x)N(y) \supset_{xy} \cdot xy = yx.$$

7. Exponentiation. $1(\boldsymbol{F}, \boldsymbol{A})$ conv $\boldsymbol{F}(\boldsymbol{A})$, and $S(\boldsymbol{x}, \boldsymbol{F}, \boldsymbol{A})$ conv $\boldsymbol{F}(\boldsymbol{x}(\boldsymbol{F}, \boldsymbol{A}))$. By an intuitive induction, utilizing the definitions of 2, 3, 4, $\cdots$ from 1, we infer that for any given positive integer z, $\boldsymbol{z}(\boldsymbol{F}, \boldsymbol{A})$ is convertible into $\boldsymbol{F}(\cdots z \text{ times} \cdots (\boldsymbol{F}(\boldsymbol{A}))\cdots)$. Church's definitions of the positive integers were framed with a view to providing this formal means of representing the z-th power of a function $\boldsymbol{F}$ of an argument $\boldsymbol{A}$. We recognize it by introducing the abbreviation $[\boldsymbol{x}]^{[\boldsymbol{p}]}$ for $\{\boldsymbol{p}\}(\boldsymbol{x})$, so that $\boldsymbol{z}(\boldsymbol{F}, \boldsymbol{A})$ may be written $\boldsymbol{F}^{\boldsymbol{z}}(\boldsymbol{A})$.

3.1, 3.2, 14 $\vdash \Sigma\phi a f \cdot \phi(a) \cdot \phi(\rho) \supset_\rho \phi(f(\rho))$. Assuming $\phi(a) \cdot \phi(\rho) \supset_\rho \phi(f(\rho))$, we have, by the use of the first factor with the second, $\phi(f(a))$, which is convertible into $\phi(f^1(a))$. Assuming also $\phi(f^x(a))$, the second factor gives $\phi(f(f^x(a)))$, which is convertible into $\phi(f^{S(x)}(a))$. Hence, by induction and the corollary of Theorem I,

$$7.1: \qquad [\phi(a) \cdot \phi(\rho) \supset_\rho \phi(f(\rho))] \supset_{\phi a f} \cdot N(x) \supset_x \phi(f^x(a)).$$

(1) $I^1 = I$ is provable from $I = I$ by conversion.* (2) Assuming $I^x = I$, we have $I^{S(x)}$ conv $\lambda y \cdot I^x(y)$, $= \lambda y \cdot I(y)$ (by means of our assumption), conv I. $\vdash E(I)$. Hence, by § 2, $I^{S(x)} = I$. From (1) and (2) by induction,

$$7.2: \qquad N(x) \supset_x \cdot I^x = I.$$

It follows, since $I(\boldsymbol{A})$ conv $\boldsymbol{A}$, that we can express formally a function of positive integers having any given constant value $\boldsymbol{A}$. Indeed, if

* $I = I$ is provable by 2.1 since $\vdash E(I)$ (cf. §§ C6, C8).

$$\mathfrak{C} \rightarrow \lambda\pi \cdot [\lambda\rho\sigma \cdot I^{\sigma}(\rho)]^{\pi},$$

then $\mathfrak{C}(1, \boldsymbol{A})$, $\mathfrak{C}(2, \boldsymbol{A})$, $\cdots$ are functions whose values for any one, two, $\cdots$ members of the sequence $1, 2, \cdots$ (*i. e.* "known" positive integers), respectively, are convertible into $\boldsymbol{A}$.

The formula 6.1 is convertible into

7.3: $$N(x) \supset_x \cdot [\lambda fa \cdot f^{x}(a)] = x.$$

7I. $\boldsymbol{F}^{\boldsymbol{x}}(\boldsymbol{F}^{\boldsymbol{y}}(\boldsymbol{A}))$ *conv* $\boldsymbol{F}^{\boldsymbol{x}+\boldsymbol{y}}(\boldsymbol{A})$.

Let $\boldsymbol{f}$ and $\boldsymbol{a}$ be proper symbols not occurring in $\boldsymbol{x}$ and $\boldsymbol{y}$ as free symbols. Then the formula $\boldsymbol{x} + \boldsymbol{y}$ is convertible into $\lambda \boldsymbol{fa} \cdot \boldsymbol{f}^{\boldsymbol{x}}(\boldsymbol{f}^{\boldsymbol{y}}(\boldsymbol{a}))$ and $\lambda \boldsymbol{fa} \cdot \boldsymbol{f}^{\boldsymbol{x}+\boldsymbol{y}}(\boldsymbol{a})$. This leads evidently to the conversion 7I, and also shows from a new point of view why the definition given in § 5 for the addition of positive integers is suitable.

The observation is due to J. B. Rosser that if $\boldsymbol{p}$ and $\boldsymbol{q}$ are positive integers the formula $\boldsymbol{x}^{\boldsymbol{p}}$ is actually the p-th power of $\boldsymbol{x}$ in the arithmetic sense.

7II. $\boldsymbol{x}^{\boldsymbol{p}}\boldsymbol{x}^{\boldsymbol{q}}$ *conv* $\boldsymbol{x}^{\boldsymbol{p}+\boldsymbol{q}}$. $[\boldsymbol{x}^{\boldsymbol{p}}]^{\boldsymbol{q}}$ *conv* $\boldsymbol{x}^{\boldsymbol{qp}}$.

8. Number dyads and triads. A finite ordered set of expressions can be defined intuitively by enumerating its members, $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_k$. In order to carry out an argument in the logic about such a set whose members vary according to given laws, we may require a formula $\boldsymbol{A}$ such that (1) $\boldsymbol{A}$ is a function of the k members $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$, and (2) $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$ are k functions of $\boldsymbol{A}$. If there exist expressions $\boldsymbol{H}_1, \cdots, \boldsymbol{H}_k$ such that $\boldsymbol{H}_1(\boldsymbol{A}_1) = I, \cdots, \boldsymbol{H}_k(\boldsymbol{A}_k) = I$, we can take as $\boldsymbol{A}$ the expression $\lambda\boldsymbol{x} \cdot \boldsymbol{x}(\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k)$, where $\boldsymbol{x}$ is any proper symbol not occurring in $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$ as a free symbol. In the important case that $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$ are positive integers there are the simpler constructions which follow.

To represent the ordered pair of numbers $\boldsymbol{x}$ and $\boldsymbol{y}$, we employ the formula $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{y})$, abbreviated $[\boldsymbol{x}, \boldsymbol{y}]$, where

$$\boldsymbol{D} \rightarrow \lambda\rho\sigma fga \cdot f^{\rho}(g^{\sigma}(a)).$$

Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are the functions $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$,

$$\boldsymbol{D}_1 \rightarrow \lambda\rho f \cdot \rho(f, I) \quad \text{and} \quad \boldsymbol{D}_2 \rightarrow \lambda\rho \cdot \rho(I),$$

respectively, of $[\boldsymbol{x}, \boldsymbol{y}]$, as is established in the theorems

8.1: $$N(x)N(y) \supset_{xy} \cdot \boldsymbol{D}_1([x, y]) = x,$$

8.2: $$N(x)N(y) \supset_{xy} \cdot \boldsymbol{D}_2([x, y]) = y.$$

The first is provable, according to § 2 and the corollary of Theorem I, since assuming $N(x)N(y)$ we have $\boldsymbol{D}_1([x, y])$ conv $\lambda fa \cdot f^x(I^y(a)), = \lambda fa \cdot f^x(I(a))$ (by 7.2, $N(y)$), conv $\lambda fa \cdot f^x(a)$, $= x$ (by 7.3, $N(x)$); the second, since $\boldsymbol{D}_2([x, y])$ conv $\lambda fa \cdot I^x(f^y(a))$, from which we can pass similarly to y.

Let

$$\boldsymbol{T} \rightarrow \lambda\rho\sigma\tau fgha \cdot f^\rho(g^\sigma(h^\tau(a))),$$

and abbreviate $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ to $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$. If

$$\boldsymbol{T}_1 \rightarrow \lambda\rho f \cdot \rho(f, I, I), \quad \boldsymbol{T}_2 \rightarrow \lambda\rho f \cdot \rho(I, f, I), \quad \boldsymbol{T}_3 \rightarrow \lambda\rho \cdot \rho(I, I),$$

then

8.3: $$N(x)N(y)N(z) \supset_{xyz} \cdot \boldsymbol{T}_1([x, y, z]) = x,$$

8.4: $$N(x)N(y)N(z) \supset_{xyz} \cdot \boldsymbol{T}_2([x, y, z]) = y,$$

8.5: $$N(x)N(y)N(z) \supset_{xyz} \cdot \boldsymbol{T}_3([x, y, z]) = z.$$

Similarly for tetrads, etc.

9. Predecessor. Given the third Peano axiom, as formulated in 10.1, a function for the predecessor of a positive integer can be obtained by means of the description, ι. Conversely, given a predecessor function, 10.1 can be proved. By obtaining a predecessor function first, and making it the basis of the proof of 10.1, and of further definitions, we are able to limit our formal assumptions, avoiding, particularly, the use of the description.

The predecessor function, P, which we shall define has, besides the property that *if* $\boldsymbol{x}$ is one of the expressions $1, 2, \cdots$, then $P(S(\boldsymbol{x}))$ conv $\boldsymbol{x}$, also the property $P(1)$ conv 1. The arithmetical functions, such as subtraction and division, defined by means of it, are likewise given a value within the class of positive integers for every set of positive integral values of the arguments, even in cases when this is not ordinarily done. The resulting special properties of the functions are useful in the development of the theory.

Let

$$\mathfrak{F} \rightarrow \lambda\rho \cdot [\boldsymbol{T}_2(\rho), \boldsymbol{T}_3(\rho), S(\boldsymbol{T}_3(\rho))] \quad \text{and} \quad \mathfrak{J} \rightarrow [1, 1, 1].$$

Then (1) $\mathfrak{F}^{S(1)}(\mathfrak{J})$ conv $[1, 2, 3]$, and $E([1, 2, 3])$ is provable since

$[1, 2, 3]$ occurs in the proposition $\boldsymbol{T}_1([1, 2, 3]) = 1$ which is provable from 8.3, 3.1, and 3.2. Hence, by 2.1, $\mathfrak{F}^{S(1)}(\mathfrak{J}) = [1, S(1), S(S(1))]$. (2) Assume $N(x)$ and $\mathfrak{F}^{S(x)}(\mathfrak{J}) = [x, S(x), S(S(x))]$. From $N(x)$ we infer $N(S(x))$, $N(S(S(x)))$, and $N(S(S(S(x))))$ by 3.2. Now $\mathfrak{F}^{S(S(x))}(\mathfrak{J})$ conv $[\boldsymbol{T}_2(\mathfrak{F}^{S(x)}(\mathfrak{J})), \boldsymbol{T}_3(\mathfrak{F}^{S(x)}(\mathfrak{J})), S(\boldsymbol{T}_3(\mathfrak{F}^{S(x)}(\mathfrak{J})))]$, $= [\boldsymbol{T}_2([x, S(x), S(S(x))]), \boldsymbol{T}_3([x, S(x), S(S(x))]), S(\boldsymbol{T}_3([x, S(x), S(S(x))]))]$ (by the hypothesis of the induction), $= [S(x), S(S(x)), S(S(S(x)))]$ (by means of the equalities obtained by using $N(x)$, $N(S(x))$, and $N(S(S(x)))$ in 8.4 and 8.5). $E([S(x), S(S(x)), S(S(S(x)))])$ is given by 8.3, $N(S(x))$, $N(S(S(x)))$, and $N(S(S(S(x))))$. Hence, by § 2, $\mathfrak{F}^{S(S(x))}(\mathfrak{J}) = [S(x), S(S(x)), S(S(S(x)))]$. Having (1) and (2), we can prove by induction

$$\mathfrak{A}_1: \qquad N(x) \supset_x \cdot \mathfrak{F}^{S(x)}(\mathfrak{J}) = [x, S(x), S(S(x))].$$

Let

$$P \to \lambda\rho \cdot \boldsymbol{T}_1(\mathfrak{F}^{\rho}(\mathfrak{J})).$$

Assume $N(x)$. By 3.2, $N(S(x))$ and $N(S(S(x)))$. Then $P(S(x))$ conv $\boldsymbol{T}_1(\mathfrak{F}^{S(x)}(\mathfrak{J}))$, $= \boldsymbol{T}_1([x, S(x), S(S(x))])$ (by $N(x)$ and $\mathfrak{A}_1$), $= x$ (by means of 8.3, $N(x)$, $N(S(x))$, and $N(S(S(x)))$). Hence, by § 2 and Theorem I,

$$9.1: \qquad N(x) \supset_x \cdot P(S(x)) = x.$$

The theorem

$$9.2: \qquad P(1) = 1$$

is provable by conversion from $1 = 1$; and the theorem

$$9.3: \qquad N(x) \supset_x N(P(x))$$

is provable by induction, since (1) $N(P(1))$ follows by conversion from 3.1, and (2) assuming $N(x)$, we can obtain $P(S(x)) = x$ from 9.1, and then $N(P(S(x)))$ from $N(x)$.

10. Peano's third axiom. $N(1)N(1) \cdot S(1) = S(1)$ is provable by means of 3.1, 3.2, 2.1, and 14; and thence $\Sigma xy \cdot N(x)N(y) \cdot S(x) = S(y)$ can be proved. Assume $N(x)N(y) \cdot S(x) = S(y)$. Then $x = P(S(x))$ (by 9.1 and $N(x)$), $= P(S(y))$ (by $S(x) = S(y)$), $= y$ (by 9.1 and $N(y)$). Hence, by § 2 and Theorem I Cor.,

$$10.1: \qquad [N(x)N(y) \cdot S(x) = S(y)] \supset_{xy} \cdot x = y.$$

11. Subtraction. Let

$$- \rightarrow \lambda\mu\nu \cdot P^{\nu}(\mu),$$

and abbreviate $\{-\}(\boldsymbol{x}, \boldsymbol{y})$ to $[\boldsymbol{x}] - [\boldsymbol{y}]$. If x and y are positive integers, then $\boldsymbol{x} - \boldsymbol{y}$ has the usual significance if x is greater than y, and $\boldsymbol{x} - \boldsymbol{y}$ is 1 if x is less than or equal to y.

$N(x)$, 9. 3, 7. 1 $\vdash N(y) \supset_y N(x - y)$. Hence, by Theorem I Cor.,

11. 1: $$N(x)N(y) \supset_{xy} N(x - y).$$

(1) Assuming $N(x)$, we have $[x + 1] - 1 = [1 + x] - 1$ (by 5. 4, $N(x)$, and 3. 1), conv $P(S(x))$, $= x$ (by 9. 1 and $N(x)$). Hence, by § 2 and Theorem I, $N(x) \supset_x \cdot [x + 1] - 1 = x$. (2) Assume $N(y)$ and $N(x) \supset_x \cdot [x + y] - y = x$. Assume also $N(x)$. Then $[x + S(y)] - S(y)$ $= [S(y) + x] - S(y)$ (by 5. 4, $N(y)$, 3. 2, $N(x)$), conv $P([1 + y + x] - y)$, $= P([y + 1 + x] - y)$ (by 5. 4, $N(y)$, 3. 1), conv $P([y + S(x)] - y)$, $= P([S(x) + y] - y)$ (by 5. 4, $N(y)$, $N(x)$, 3. 2), $= P(S(x))$ (by the second assumption, $N(x)$, 3. 2), $= x$ (by 9. 1, $N(x)$). Hence, by § 2 and Theorem I, $N(x) \supset_x \cdot [x + S(y)] - S(y) = x$. From (1) and (2) by induction, $N(y) \supset_y \cdot N(x) \supset_x \cdot [x + y] - y = x$; whence, by Theorem I Cor.,

11. 2: $$N(x)N(y) \supset_{xy} \cdot [x + y] - y = x.$$

(1) Assume $N(x)N(y)$. Then $[x - y] - 1$ conv $P(y(P, x))$, conv $S(y, P, x)$, conv $x - S(y)$, conv $x - \cdot 1 + y$, $= x - \cdot y + 1$ (by 5. 4, 3. 1, $N(y)$). $N(x)$, $N(y)$, 3. 1, 5. 2, 11. 1 $\vdash N(x - \cdot y + 1)$, from which $E(x - \cdot y + 1)$ is provable. Hence, by § 2 and Theorem I Cor., $N(x)N(y) \supset_{xy}$ $\cdot [x - y] - 1 = x - \cdot y + 1$. (2) Assume $N(z)$ and $N(x)N(y) \supset_{xy}$ $\cdot [x - y] - z = x - \cdot y + z$. Assume also $N(x)N(y)$. Then $[x - y] - S(z)$ conv $[x - y] - \cdot 1 + z$, $= [[x - y] - 1] - z$ (by the hypothesis of the induction, $N(x)$, $N(y)$, 11. 1, 3. 1), $= [x - \cdot y + 1] - z$ (by (1), $N(x)$, $N(y)$), $= x - \cdot y + 1 + z$ (by the hypothesis of the induction, $N(x)$, $N(y)$, 3. 1, 5. 2), conv $x - \cdot y + S(z)$. We can prove $E(x - \cdot y + S(z))$ by means of $N(x)$, $N(y)$, $N(z)$, 3. 2, 5. 2, 11. 1. Hence, by § 2 and Theorem I Cor., $N(x)N(y) \supset_{xy} \cdot [x - y] - S(z) = x - \cdot y + S(z)$. Having (1) and (2), we infer by induction $N(z) \supset_z \cdot N(x)N(y) \supset_{xy} \cdot [x - y] - z = x - \cdot y + z$, and thence

11. 3: $$N(x)N(y)N(z) \supset_{xyz} \cdot [x - y] - z = x - [y + z].$$

Assume $N(x)N(y)N(z) \cdot x+y=x+z$. Then $y=[x+y]-x$ (by 11.2, 5.4, $N(y)$, $N(x)$), $=[x+z]-x$ (by $x+y=x+z$), $=z$ (by 11.2, 5.4, $N(x)$, $N(z)$). Hence

11.4: $$[N(x)N(y)N(z) \cdot x+y=x+z] \supset_{xyz} \cdot y=z.$$

12. Order. Let $< \rightarrow \lambda\mu\nu \cdot N(\mu) \cdot [\phi(S(\mu)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))] \supset_\phi \phi(\nu)$, and abbreviate $\{<\}(\boldsymbol{x}, \boldsymbol{y})$ to $\boldsymbol{x} < \boldsymbol{y}$ or $\boldsymbol{y} > \boldsymbol{x}$.

12.1: $$N(x) \supset_x \cdot x < S(x).$$

Proof. Assume $N(x)$. Then, using 3.2, $\Sigma\phi \cdot \phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi)) \cdot \phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi)) \vdash \phi(S(x))$. By Theorem I, $[\phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))] \supset_\phi \phi(S(x))$, which with $N(x)$ yields $x < S(x)$.*

12.2: $$x < y \supset_{xy} \cdot N(x)N(y).$$

Proof. 12.1 $\vdash \Sigma xy \cdot x < y$. $x < y \vdash N(x) \cdot [\phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))] \supset_\phi \phi(y)$. $N(x)$, 3.2 $\vdash N(S(x))$. $N(S(x))$, 3.2, $[\phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))] \supset_\phi \phi(y) \vdash N(y)$.

12.3: $$[x < y \cdot y < z] \supset_{xyz} \cdot x < z.$$

Proof. Assume $x < y \cdot y < z$ and $\phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))$. The latter with $x < y$ yields $\phi(y)$. $\phi(y)$, $\phi(\xi) \supset_\xi \phi(S(\xi)) \vdash \phi(S(y)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))$, which with $y < z$ yields $\phi(z)$. Hence $[\phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))] \supset_\phi \phi(z)$, and $x < z$.

When each a_i $(i=1, \cdots, n)$ is either $=$ or $<$ ($=$ or $>$), we abbreviate $[\boldsymbol{x}_1 a_1 \boldsymbol{x}_2][\boldsymbol{x}_2 a_2 \boldsymbol{x}_3] \cdots [\boldsymbol{x}_n a_n \boldsymbol{x}_{n+1}]$ to $\boldsymbol{x}_1 a_1 \boldsymbol{x}_2 a_2 \boldsymbol{x}_3 \cdots \boldsymbol{x}_n a_n \boldsymbol{x}_{n+1}$. By § 2 and 12.3, we can make formally the usual inferences concerning the order relation of two expressions linked in such a chain of inequalities.†

---

* Henceforth our proofs will be abbreviated by omissions of such details as applications of Theorem I and Corollaries, applications of § 2 and like principles for inequalities, and references to formal theorems under circumstances in which it is clear what theorems are being used. In particular, required formulas of the form $N(\boldsymbol{A})$ will not be mentioned, when they are obtainable from the hypotheses and such theorems as 3.1, 3.2, 5.2, 12.2. In case several theorems are used at a given step in the argument, those playing a subordinate rôle may not be cited.

† We arrange the introduction and proof of chains of inequalities in the same manner as that of chains of equalities. Any link may be a conversion or contingent equality.

12. 4: $$N(x) \supset_x \cdot N(y) \supset_y \cdot x + y > y.$$

Proof. Assume $N(x)$, $N(y) \supset_y \cdot x + y > y$, and $N(y)$. Then $S(x) + y$ conv $S(x + y)$, $> x + y$ (12. 1), $> y$ (by means of the hypotheses). Hence, by Theorem I, $N(x), N(y) \supset_y \cdot x + y > y \vdash N(y) \supset_y \cdot S(x) + y > y$. Thus 12. 4 is provable by induction from 12. 1 as basis.

12. 5: $$x < y \supset_{xy} \cdot y = [y - x] + x.$$

Proof. Assume $x < y$. (1) $S(x)$ conv $1 + x$, $= [[1 + x] - x] + x$ (11. 2), conv $[S(x) - x] + x$. Hence $N(S(x)) \cdot S(x) = [S(x) - x] + x$. (2) Assume $N(\xi) \cdot \xi = [\xi - x] + x$. Then $S(\xi) = S([\xi - x] + x)$ (by the hyp.), conv $S(\xi - x) + x$, $= [[S(\xi - x) + x] - x] + x$ (11. 2), conv $[S([\xi - x] + x) - x] + x$, $= [S(\xi) - x] + x$ (by the hyp.). Hence $N(S(\xi)) \cdot S(\xi) = [S(\xi) - x] + x$. By Theorem I, $[N(\xi) \cdot \xi = [\xi - x] + x] \supset_\xi \cdot N(S(\xi)) \cdot S(\xi) = [S(\xi) - x] + x$. (3) $x < y \vdash [\phi(S(x)) \cdot \phi(\xi) \supset_\xi \phi(S(\xi))] \supset_\phi \phi(y)$, which with (1) and (2) yields $N(y) \cdot y = [y - x] + x$.

12. 6: $$x < y \supset_{xy} \cdot x = y - [y - x].$$

12. 7: $$[N(x) \cdot y > z] \supset_{xyz} \cdot [x + y] - z = x + [y - z].$$

Proofs. Assuming $x < y$ and letting $\boldsymbol{p} \rightarrow y - x$, $x = [\boldsymbol{p} + x] - \boldsymbol{p}$ (11. 2), $= [\boldsymbol{p} + x] - \cdot [\boldsymbol{p} + x] - x$ (11. 2), $= y - \cdot y - x$ (12. 5). Assuming $N(x) \cdot y > z$ and letting $\boldsymbol{p} \rightarrow y - z$, $[x + y] - z = [x + \boldsymbol{p} + z] - z$ (12. 5), $= x + \boldsymbol{p}$ (11. 2), $= x + \cdot [\boldsymbol{p} + z] - z$ (11. 2), $= x + \cdot y - z$ (12. 5).

12. 8: $$N(x) \supset_x \cdot S(x) > 1.$$

12. 9: $$N(y) \supset_y \cdot [x > y] \supset_x \cdot x > 1.$$

12. 10: $$N(x) N(y) \supset_{xy} \cdot x + y > 1.$$

Proofs. 12. 4 $\vdash$ 12. 8, and 12. 9 is provable by induction, since $[x > 1] \supset_x \cdot x > 1$ and $N(y) \vdash [x > S(y)] \supset_x \cdot x > 1$ (12. 4). 12. 10 follows from 12. 9 by 12. 4.

12. 11: $$[N(x) \cdot y < z] \supset_{xyz} \cdot x + y < x + z.$$

12. 12: $$[N(x) N(y) N(z) \cdot x + y < x + z] \supset_{xyz} \cdot y < z.$$

Proofs. Assuming $N(x) \cdot y < z$, then $x + z = x + y + \cdot z - y$ (12. 5),

$> x + y$ (12.4). Assuming $N(x)N(y)N(z) \cdot x + y < x + z$, then $z = [x+z] - x$ (11.2), $= [\boldsymbol{p} + x + y] - x$ (by 12.5, if $\boldsymbol{p} \rightarrow [x+z] - \cdot x + y$), $= \boldsymbol{p} + \cdot [x+y] - x$ (12.7, 12.4), $= \boldsymbol{p} + y$ (11.2), $> y$ (12.4).

12.13: $$[N(z) \cdot x < S(y) \cdot y < S(z)] \supset_{xyz} \cdot x < S(z).$$

12.14: $$[N(z) \cdot x < y \cdot y < S(z)] \supset_{xyz} \cdot x < z.$$

12.15: $$[x < S(y) \cdot y < z] \supset_{xyz} \cdot x < z.$$

Proofs. Assuming $N(z) \cdot x < S(y) \cdot y < S(z)$, we have $1 + S(z) = 1 + \cdot [S(z) - y] + y$ (12.5), $= [S(z) - y] + S(y)$, $= [S(z) - y] + [S(y) - x] + x$ (12.5), $> 1 + x$ (12.11, 12.10), and hence, by 12.12, $S(z) > x$. Similarly for 12.14 and 12.15.

We let $\boldsymbol{x} \leq \boldsymbol{y} \rightarrow \boldsymbol{x} < S(\boldsymbol{y})$,* and employ the relation $\leq$ as well as $<$ and $=$ in our chains of inequalities. 12.13-12.15 together with previously noted facts show that we can make formally the usual inferences concerning the order relation of two expressions linked in such a chain.

12.16: $$[N(y) \cdot x < S^2(y)] \supset_{xy} \cdot x - y = 1.$$

Proof. Assume $N(y) \cdot x < S^2(y)$. By 11.2, $[S^2(y) - 1] - y = 1$. Assuming $N(p)$ and $[S^2(y) - p] - y = 1$, $[S^2(y) - S(p)] - y = [[S^2(y) - p] - y] - 1$ (11.3), $= 1 - 1$ (by the hyp.), conv 1. Hence, by induction, $N(p) \supset_p \cdot [S^2(y) - p] - y = 1$. This with $N(S^2(y) - x)$ yields $[S^2(y) - \cdot S^2(y) - x] - y = 1$, and, by 12.6, $x - y = 1$.

12.17: $$[x < S(y) \cdot y < S(x)] \supset_{xy} \cdot x = y.$$

Proof. Assume $x < S(y) \cdot y < S(x)$. $x < S(y)$, 12.11 $\vdash S(x) < S^2(y)$. (1) $2 + y$ conv $S^2(y)$, $= [S^2(y) - S(x)] + S(x)$ (12.5, $S(x) < S^2(y)$), $= [S^2(y) - S(x)] + [S(x) - y] + y$ (12.5, $y < S(x)$). Hence, by 11.4, (2) $2 = [S^2(y) - S(x)] + \cdot S(x) - y$. (3) $S^2(y) - S(x) = [[S^2(y) - S(x)] + \cdot S(x) - y] - \cdot S(x) - y$ (11.2), $= 2 - \cdot S(x) - y$ (by (2)), $= 2 - 1$ (12.16, $S(x) < S^2(y)$), conv 1. (4) $2 + x$ conv $1 + S(x)$, $= [S^2(y) - S(x)] + S(x)$ (by (3)), $= 2 + y$ (as in (1)). Hence, by 11.4, $x = y$.

12.18: $$[x > 1] \supset_x \cdot N(y) \supset_y \cdot x - y < x.$$

* Or let $\leq \rightarrow \lambda xy \cdot x < S(y)$ and abbreviate $\{\leq\}(\boldsymbol{x}, \boldsymbol{y})$ to $\boldsymbol{x} \leq \boldsymbol{y}$. Similarly below.

Proof. Assume $x > 1$. (1) $1 - 1$ conv 1, $< S(1)$. Assuming $N(p)$, $S(p) - 1 = p$, $< S^2(p)$. Hence, by induction, $N(p) \supset_p \cdot p - 1 < S(p)$. (2) $x - 1 < S(x - 1)$, $= x$ (12.5). (3) Assuming $N(y)$ and $x - y < x$, $x - S(y) = [x - y] - 1$ (11.3), $\leq x - y$ (by means of (1)), $< x$ (hyp. induction). From (2) and (3), by induction, $N(y) \supset_y \cdot x - y < x$.

12.19: $N(k) \supset_k \cdot [\rho < S(k) \supset_\rho \phi(\rho)]\, [\rho > k \supset_\rho \phi(\rho)] \supset_\phi \cdot N(x) \supset_x \phi(x)$.

Proof. Let $\mathfrak{D}_\phi(\boldsymbol{a}) \rightarrow [\rho < S(\boldsymbol{a}) \supset_\rho \phi(\rho)][\rho > \boldsymbol{a} \supset_\rho \phi(\rho)]$. (1) $\vdash \Sigma\phi \cdot \mathfrak{D}_\phi(1)$. By induction and Theorem I, $\mathfrak{D}_\phi\, (1) \supset_\phi \cdot N(x) \supset_x \phi(x)$. (2) Assume $N(k)$ and $\mathfrak{D}_\phi(k) \supset_\phi \cdot N(x) \supset_x \phi(x)$. Then $\Sigma\phi \cdot \mathfrak{D}_\phi(S(k))$. Assume $\mathfrak{D}_\phi(S(k))$. (a) $\rho < S(k) \vdash \rho < S^2(k)$; and hence, by the first factor of $\mathfrak{D}_\phi(S(k))$, and Theorem I, $\rho < S(k) \supset_\rho \phi(\rho)$. (b) $N(k) \vdash 1 + k < S^2(k)$, whence, by the first factor of $\mathfrak{D}_\phi(S(k))$, $\phi(1 + k)$. Assuming $N(p)$, we have $S(p) + k > S(k)$, whence, by the second factor of $\mathfrak{D}_\phi(S(k))$, $\phi(S(p) + k)$. By induction, $N(p) \supset_p \phi(p + k)$. Thence, assuming $\rho > k$, we obtain $\phi([\rho - k] + k)$, and, by 12.5, $\phi(\rho)$. By Theorem I, $\rho > k \supset_\rho \phi(\rho)$. (c) $\mathfrak{D}_\phi(k) \supset_\phi \cdot N(x) \supset_x \phi(x)$ with (a) and (b) yields $N(x) \supset_x \phi(x)$. By Theorem I, $\mathfrak{D}_\phi(S(k)) \supset_\phi \cdot N(x) \supset_x \phi(x)$. (3) 12.19 follows from (1) and (2) by induction.

13. The lesser and greater of two positive integers. Let $\min \rightarrow \lambda xy \cdot S(y) - \cdot S(y) - x$ and $\max \rightarrow \lambda xy \cdot [x + y] - \min(x, y)$.

13.1: $N(x)N(y) \supset_{xy} N(\min(x, y))$.
13.2: $[N(y) \cdot x < S(y)] \supset_{xy} \cdot \min(x, y) = x$.
13.3: $N(x) \supset_x \cdot N(y) \supset_y \cdot \min(x, y) = \min(y, x)$.
13.4: $N(x)N(y) \supset_{xy} \cdot \min(x, y) < S(y)$.
13.5: $N(x)N(y)N(z) \supset_{xyz} \cdot \min(z + x, z + y) = z + \min(x, y)$.

Proofs. 3.2, 11.1 $\vdash$ 13.1. 12.6 $\vdash$ 13.2. 13.3 may be established by an application of 12.19, since, assuming $N(x)$, (1) assuming $y < S(x)$, $\min(x, y)$ conv $S(y) - \cdot S(y) - x$, $= S(y) - 1$ (by 12.16, since $y < S(x)$, 12.11 $\vdash S(y) < S^2(x)$), $= y$, $= \min(y, x)$ (13.2), and (2) assuming $y > x$, a like series of steps takes us from $\min(y, x)$ to $\min(x, y)$. 12.18, 12.8 $\vdash$ 13.4. $\vdash$ 13.5, since, assuming $N(x)N(y)N(z)$, $\min(z + x, z + y)$ conv $S(z + y) - \cdot S(z + y) - \cdot z + x$, $= [z + S(y)] - \cdot S(y) - x$ (11.2, 11.3), $= z + \cdot S(y) - \cdot S(y) - x$ (12.7, 12.18, 12.8), conv $z + \min(x, y)$.

14. Proof by cases. We now establish theorems which connect the present theory with that of §§ C7, C9, C10. $M \rightarrow \lambda\mu \cdot \phi(1)\phi(2) \supset_\phi \phi(\mu)$, and $[\boldsymbol{P}] \equiv_{x_1 \ldots x_n} \cdot \boldsymbol{Q}$ will be used as an abbreviation for $[[\boldsymbol{P}] \supset_{x_1 \ldots x_n} \cdot \boldsymbol{Q}]$ $[[\boldsymbol{Q}] \supset_{x_1 \ldots x_n} \cdot \boldsymbol{P}]$.*

14.1: $$M(x) \equiv_x \cdot x < 3.$$

Proof. C7.1 $\vdash \Sigma x \cdot M(x)$. $1 < 3$, $2 < 3$, $M(x) \vdash x < 3$, by Rule V. Hence $M(x) \supset_x \cdot x < 3$. Conversely, assume $x < 3$ and $\phi(1)\phi(2)$. Then $\phi(\min(1, 2))$ ($\phi(1)$, 13.2); and assuming $N(p)$, $\phi(\min(S(p), 2))$ ($\phi(2)$, 13.2, 13.3, 12.4). Hence, by induction, $N(p) \supset_p \phi(\min(p, 2))$. Thence, since $x < 3 \vdash N(x)$, we obtain $\phi(\min(x, 2))$, and, by 13.2, $\phi(x)$.

Note that $M(\boldsymbol{x}) \vdash N(\boldsymbol{x})$ (by 14.1, 15).

Let $[\boldsymbol{x}] \circ [\boldsymbol{y}] \rightarrow \min(\boldsymbol{x}, \boldsymbol{y})$. o multiplies 1's and 2's as 0's and 1's resp.

14.2: $M(x)M(y) \supset_{xy} M(x \circ y)$.
14.3: $M(x)M(y) \supset_{xy} \cdot x \circ y = y \circ x$.
14.4: $M(y) \supset_y \cdot 1 \circ y = 1$.
14.5: $M(y) \supset_y \cdot 2 \circ y = y$.
14.6: $[M(x)M(y) \cdot x \circ y = 2] \equiv_{xy} \cdot [x = 2]\,[y = 2]$.

Proofs. 14.5 follows from $2 \circ 1 = 1$ and $2 \circ 2 = 2$ by the definition of M. For 14.6a (i.e. the first factor of 14.6), assume $M(x)M(y) \cdot x \circ y = 2$. Then, by 13.4, $x \circ y < S(y)$. Hence $2 < S(y)$. Also, by 14.1, $y < S(2)$. Hence, by 12.17, $y = 2$. Likewise $x = 2$ (cf. 14.3).

Let $\epsilon \rightarrow \lambda xy \cdot \min(2, S(x) - y)$, and abbreviate $\epsilon(\boldsymbol{x}, \boldsymbol{y})$ to $\epsilon_{\boldsymbol{y}}^{\boldsymbol{x}}$.

14.7: $N(x)N(y) \supset_{xy} M(\epsilon_y{}^x)$.
14.8: $[N(y) \cdot x < S(y)] \equiv_{xy} \cdot N(x)N(y) \cdot \epsilon_y{}^x = 1$.
14.9: $x > y \equiv_{xy} \cdot N(x)N(y) \cdot \epsilon_y{}^x = 2$.

Proofs. 13.3, 13.4, 14.1 $\vdash$ 14.7. 12.16, 12.11 $\vdash$ 14.8a; and for 14.9a we have, assuming $x > y$, $\epsilon_y{}^x = \min(2, S(x) - y)$, $= \min(2, S(x - y))$ (12.7), $= 2$ (by 13.2, since 12.4 $\vdash S^2(x - y) > 2$). 14.8b and 14.9b we prove as follows: By C7I, there exists a formula $\mathfrak{B}$ such that $\mathfrak{B}(1)$ conv $\lambda ab \cdot a < S(b)$ and $\mathfrak{B}(2)$ conv $\lambda ab \cdot a > b$. Assuming $N(y)$, then (1) assuming $x < S(y)$ we infer $\epsilon_y{}^x = 1$ by 14.8a, and hence $\mathfrak{B}(\epsilon_y{}^x, x, y)$, and (2) assuming $x > y$, we infer $\epsilon_y{}^x = 2$ by 14.9a, and hence $\mathfrak{B}(\epsilon_y{}^x, x, y)$. By 12.19 and Theorem I, $N(y) \supset_y \cdot N(x) \supset_x \mathfrak{B}(\epsilon_y{}^x, x, y)$. This lemma enables us to infer $x < S(y)$ from $N(x)N(y) \cdot \epsilon_y{}^x = 1$ and $x > y$ from $N(x)N(y) \cdot \epsilon_y{}^x = 2$.

* Cf. Church, 1932, p. 355.

Let $\delta \rightarrow \lambda xy \cdot 4 \,\dot{-}\, \epsilon_y{}^x + \epsilon_x{}^y$, and abbreviate $\delta(\boldsymbol{x}, \boldsymbol{y})$ to $\delta^{\boldsymbol{x}}_{\boldsymbol{y}}$.

14.10: $N(x)N(y) \supset_{xy} M(\delta_y{}^x)$.

14.11: $N(x)N(y) \supset_{xy} \cdot \delta_y{}^x = \delta_x{}^y$.

14.12: $x < y \supset_{xy} \cdot \delta_y{}^x = 1$.

14.13: $[N(y) \cdot x < S(y) \cdot \delta_y{}^x = 1] \supset_{xy} \cdot x < y$.

14.14: $[N(x)N(y) \cdot x = y] \equiv_{xy} \cdot N(x)N(y) \cdot \delta_y{}^x = 2$.

Proofs. Assuming $N(x)N(y)$, $\delta_y{}^x = 4 \,\dot{-}\, \epsilon_y{}^x + \epsilon_x{}^y$, $= [4 - \epsilon_y{}^x] - \epsilon_x{}^y$ (11.3), $< 4 - \epsilon_y{}^x$ (by 12.18, since $4 - \epsilon_y{}^x = 1 + \cdot 3 - \epsilon_y{}^x$ (12.7, 14.1, 14.7), > 1 (12.10)), ≤ 3 (12.18), and hence, by 14.1, $M(\delta_y{}^x)$. 5.4 $\vdash$ 14.11. Assuming $x < y$, $\delta_y{}^x = 4 \,\dot{-}\, \epsilon_y{}^x + 2$ (14.9), $= [4 - 2] - \epsilon_y{}^x$ (11.3), conv $2 - \epsilon_y{}^x$, $= 1$ (12.16). Assuming $N(y) \cdot x < S(y) \cdot \delta_y{}^x = 1$, $\epsilon_x{}^y = 3 \,\dot{-}\, 3 - \epsilon_x{}^y$ (12.6, 14.1, 14.7), $= 3 \,\dot{-}\, 4 \,\dot{-}\, 1 + \epsilon_x{}^y$ (11.2, 11.3), $= 3 \,\dot{-}\, 4 \,\dot{-}\, \epsilon_y{}^x + \epsilon_x{}^y$ (14.8), $= 3 - \delta_y{}^x$ (def.), $= 3 - 1$ (hyp.), conv 2; and hence, by 14.9, $x < y$. Assuming $N(x)N(y) \cdot x = y$, then $x < S(y)$ and, by 14.8, $\epsilon_y{}^x = 1$; also $\epsilon_x{}^y = 1$; hence $\delta_y{}^x = 4 \,\dot{-}\, 1 + 1$, conv 2. It remains to establish 14.14b, which we do as follows. Assume $N(x)N(y) \cdot \delta_y{}^x = 2$. By 14.7, $M(\epsilon_y{}^x)$. *(1) Assuming $\epsilon_y{}^x = 1$, then $x < S(y)$ (14.8). (2) Assuming $\epsilon_y{}^x = 2$, then $x > y$ (14.9); hence $\delta_y{}^x = 1$ (14.11, 14.12). Hence, by C10II, $x < S(y)$. Similarly, $y < S(x)$. Hence, by 12.17, $x = y$.*

Henceforth we use tacitly results of this section in conjunction with C9I and C10II. The headings "Case 1," "Subcase a," etc., *will indicate applica*tions of C9I or C10II.

PRINCETON UNIVERSITY.