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Bijective Aspects of Map Enumeration

Megan Monkman Marni Mishna

Simon Fraser University, Department of Mathematics

Background: Planar maps

A **map** is a connected graph embedded on an orientable surface, such that all edges are non-intersecting. A **planar map** is a map embedded on the sphere.

Planar maps provide many useful discretizations of physical problems in areas such as physics and computing.

Map Enumeration

Elegant counting formulas exist for many map classes. For example, the number of rooted planar maps with n edges, $n \geq 1$:

$$M_n = \frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$$

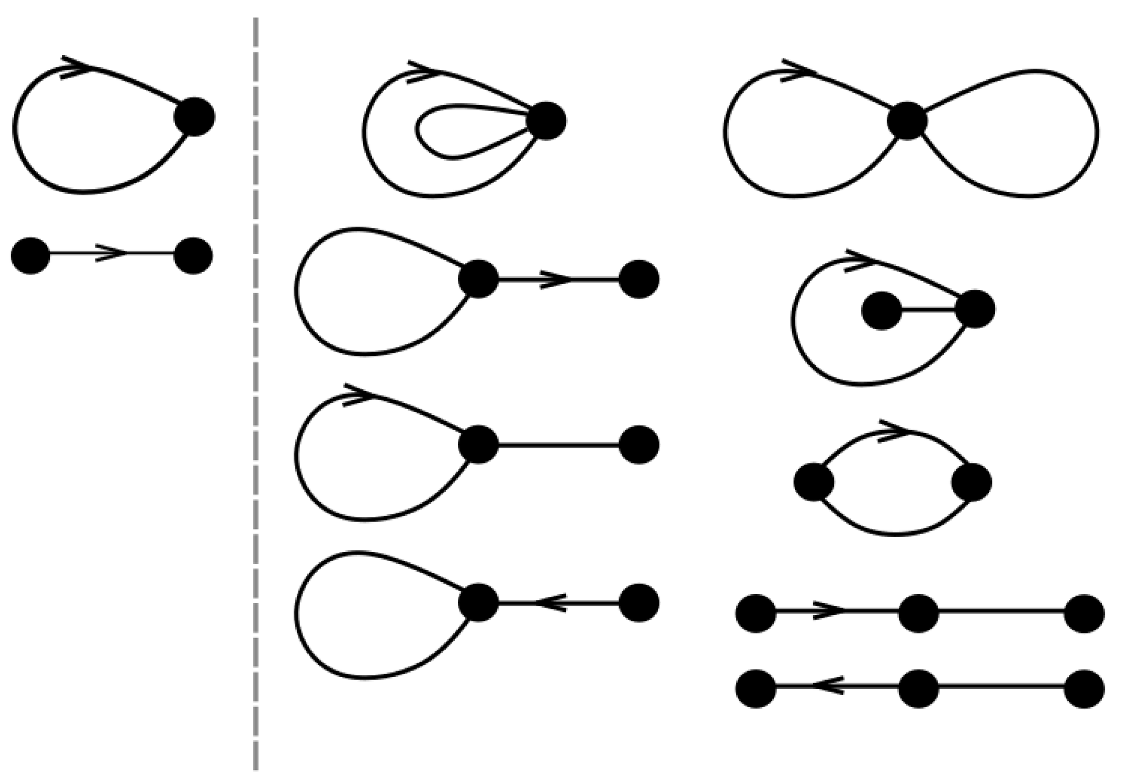

Figure 1: Rooted maps of size n=1 and n=2. M_1=2, $M_2 = 9$.

One method to explain these formulas is to consider a bijection between maps and other combinatorial objects that are more easily enumerated.

Objectives

- Enumerate bipartite planar maps using generating functions.
- Understand the bijective relationship between maps and objects known as mobiles.
- Randomly generate large trees from the generating function of mobiles.

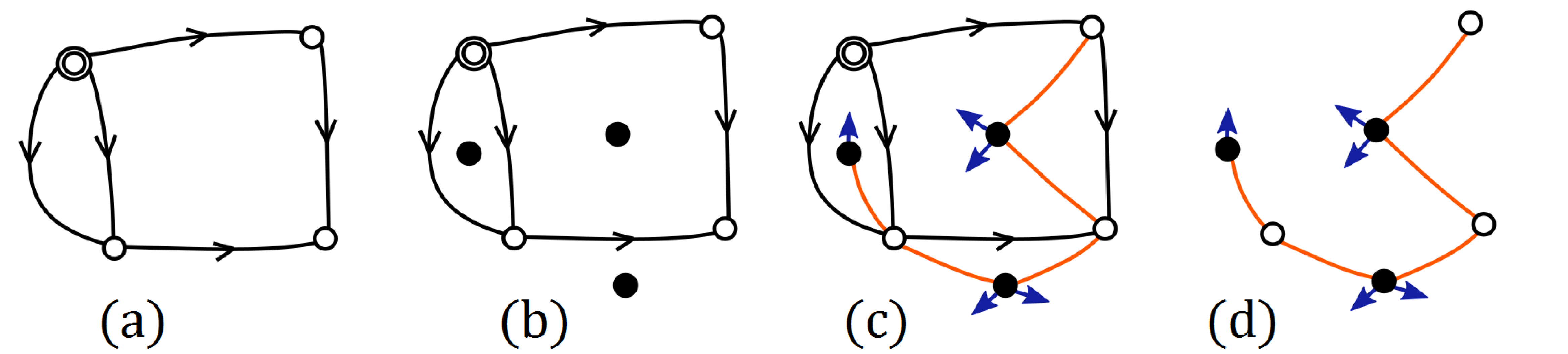

Figure 2: (a) Bipartite map M with balanced orientation and a pointed vertex. (b) A black vertex is placed within each face of M. (c) For each face of degree $2k$, k buds and k edges are added according to orientation. (d) Pointed vertex and original edges of M are erased, resulting in a balanced mobile.

Proposition: Maps as Mobiles [Fusy '15]

Vertex-pointed bipartite maps are in bijection with balanced mobiles.

For M a vertex-pointed bipartite map and R the associated balanced mobile, each non-pointed vertex of M corresponds to a white vertex of R, and each face in M corresponds to a black vertex of same (even) degree in R.

A **mobile** is a rooted plane tree with buds, and alternating black and white vertices.

Initial terms of $R(t; x_1, x_2, ...)$:

$$R(t) = t(1+x_1+x_1^2+...)+t^2(3x_2+9x_1x_2+18x_1^2x_2 + 30x_1^3x_2+...)+t^3(18x_2^2+90x_1x_2^2+...)+...$$

Initial terms of $M(t; x_1, x_2, ...)$:

$$M(t) = t^2(x_1+x_1^2+x_1^3+...)+t^3(2x_2+ 6x_1x_2+12x_1^2x_2+20x_1^3x_2+...)+...$$

Generating function for mobiles

The bijection with mobiles yields the following generating function:

$$R = R(t; x_1, x_2, ...) = t + \sum_{i \geq 1} x_i \binom{2i-1}{i} R^i$$

Denote $M = M(t; x_1, x_2, ...)$ as the generating function of rooted bipartite maps. Then,

$$M'(t) = 2R(t)$$

M encodes counting information for bipartite maps, for each term:

- t^k, k denotes the number of vertices.
- x_i^l, l denotes the number of faces of degree $2i$.

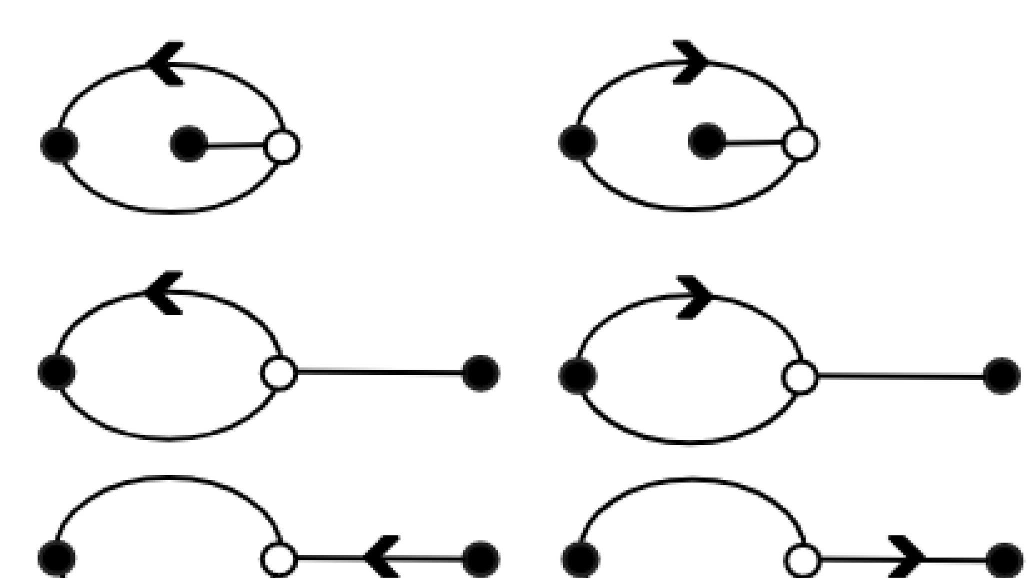

Figure 3: Example of the rooted maps encoded by the term $6x_1x_2t^3$ in M.

R is useful in enumerating bipartite and quasi-bipartite maps with r marked faces of degree $l_1, l_2, ..., l_r$ where none or exactly two of l_i are odd.

Random Generation using R

Figure 4: R object randomly generated using Boltzmann sampling. In R, x_i denotes nodes with i children.

Further Work

- Connection to lattice walks
  - The generating function for bipartite maps with marked vertices appears in the generating function for lattice walks that avoid the negative x-axis. Excursions are assigned to the faces and vertices of a bipartite map, and can be combined as a lattice walk.
- Coding the bijection.
- Generalization to non-bipartite oriented maps.

References

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