SIMON FRASER UNIVERSITY
ENGAGING THE WORLD

BIJECTIVE ASPECTS OF MAP ENUMERATION

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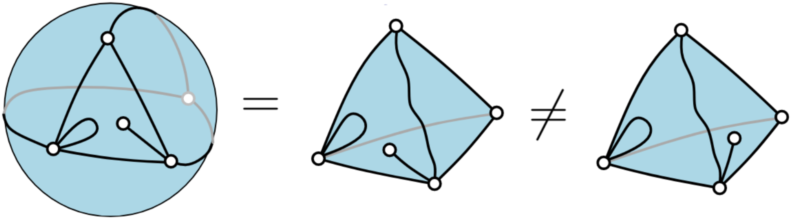
INTRODUCTION TO MAPS

What is a map?

Definition

A **map** is a connected graph embedded on an orientable surface, such that all edges are non-intersecting. A **planar map** is a map embedded on the sphere.



(a) (b) (c)

Map (b) has a face of degree 6, whereas map (c) does not. Thus they are not equivalent maps. [Fusy '15]

Parameters to consider

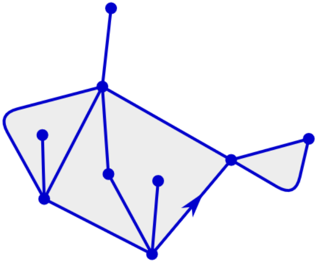
How many edges?

How many vertices? What is the distribution of vertex valences?

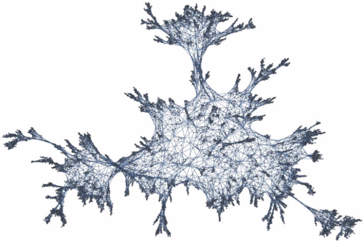
How many faces? What is the degree of each face?

Is the map rooted? i.e. having a marked corner

Are there any marked faces?



[Bouttier '10]



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Why study maps?

Motivations for study:

Physics

Probability distributions over simple classes of maps give discretizations of 2D quantum gravity.

Applications to the Ising model (ferromagnetism)

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3D mesh, computer graphics

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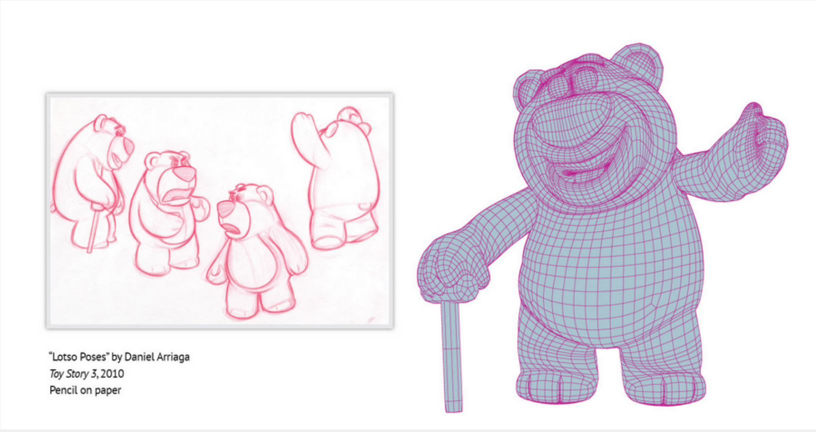


3D mesh for a character from Pixar's Toy Story 3

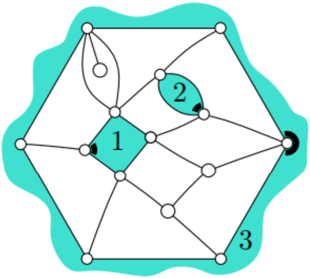
<http://sciencebehindpixar.org/pipeline/modeling>

Map classes: Examples

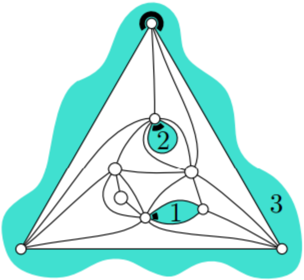
Definition

A **p-angulation** is a planar map such that every face is of degree p. **e.g.** a quadrangulation will have faces of degree 4.

A **boundary** is a marked face, and can be inner/outer.



A **quadrangulation** with 3 boundary faces of degrees 4, 2, and 6



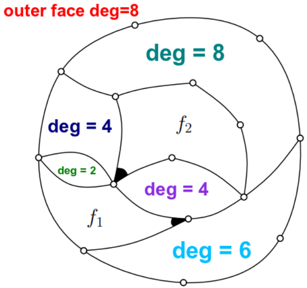
A **triangulation** with 3 boundary faces of degrees 1, 2, and 3

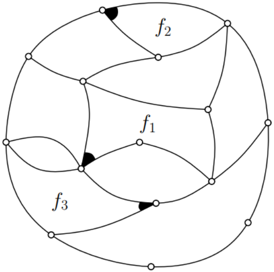
[Bernardi, Fusy '15]

Definitions

A **bipartite map** has faces of even degree.

A **quasi-bipartite map** is a bipartite map that has exactly 2 marked boundary faces that are not even.





L: A bipartite map with two marked faces f_1 and f_2 **R**: A quasi-bipartite map with 3 marked faces of degrees 5, 3, and 4 [Collet, Fusy '12]

MAP ENUMERATION

Map Enumeration

How many maps are there in each class? (given a size parameter)

In the 1960's Tutte found many elegant counting formulas, e.g. the number of rooted planar maps with n edges, $n \geq 1$: **(Tutte)**

$$M_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$$

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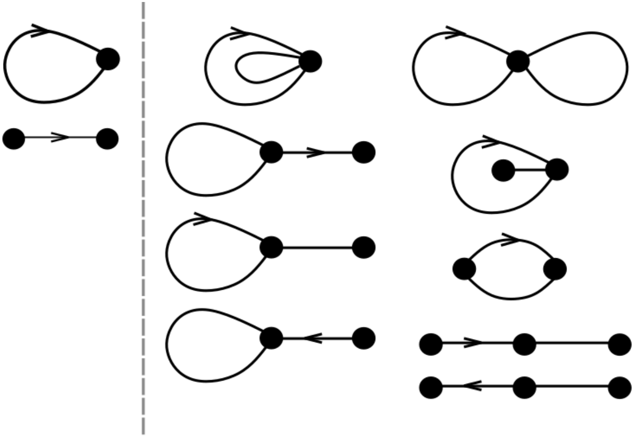

How many maps are there in each class? (that satisfy some parameter)

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Many possible counting methods (recursive decomposition, matrix integrals, ...)

One method that explains the elegant formulas is to consider a **bijection** between maps and other combinatorial objects.

Let us consider an example!

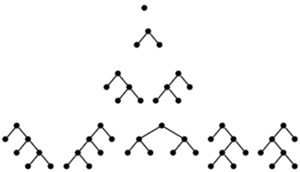
First, a brief description of generating functions.

A generating function **encodes a counting sequence as a series.**

Example: binary trees

The number of binary trees with n internal nodes is encoded in the generating function $C(z)$ by c_n.

$$C(z) = 1 + z + 2z^2 + 5z^3 + ... + c_n z^n$$



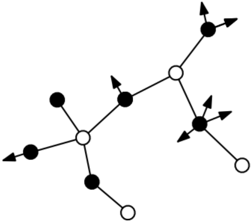
[wikipedia]

BIJECTION: PLANAR MAPS TO MOBILES

Planar maps $\longleftrightarrow$ mobiles

Definition

A **mobile** is a rooted plane tree with buds, and alternating white and black vertices.

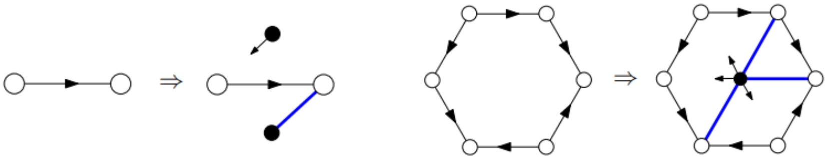


[Fusy '15]

We will consider bipartite maps with a pointed vertex and **balanced orientation***.

Map Enumeration: Bijection with mobiles

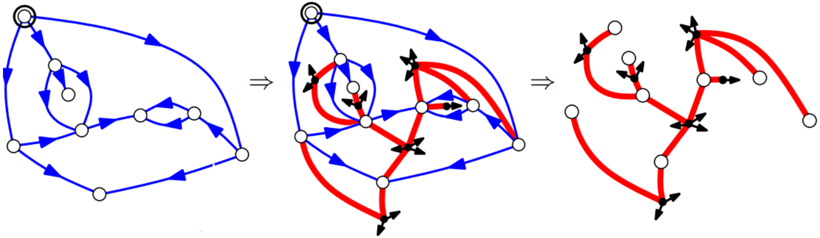
Within each face, place a single black vertex.


(a) (b)

[Fusy '15]

Follow the local rule!

Example:



[Bernardi, Fusy '15]

Erase the original edges of the map, as well as the pointed vertex.

Bijection with mobiles

Proposition: (Fusy 2015)

Vertex-pointed bipartite maps are in bijection (via the master bijection) with balanced mobiles.

For M a vertex-pointed bipartite map and R the associated balanced mobile, each non-pointed vertex of M corresponds to a white vertex of R, and each face in M corresponds to a black vertex of same (even) degree in R.

The bijection with mobiles yields the following generating function:

$$R = R(t) = R(t; x_1, x_2, ...) = t + \sum_{i \geq 1} x_i \binom{2i-1}{i} R^i$$

Denote $M = M(t) = M(t; x_1, x_2, ...)$ as the generating function of rooted bipartite maps.

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t^k, k denotes the number of vertices.

Map Enumeration: Bijection with mobiles

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t^k, k denotes the number of vertices.

x_i^l, l denotes the number of faces of degree $2i$.

Map enumeration: Bijection with mobiles

This GF relationship encodes information about rooted bipartite maps.

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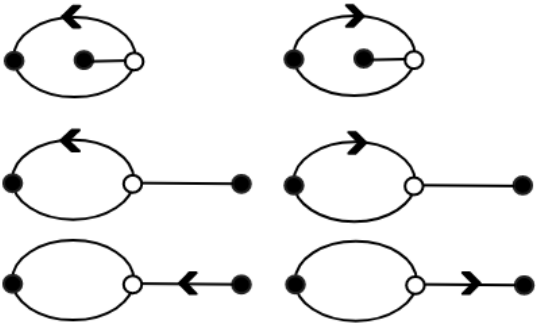
For example, the term $6x_1x_2t^3$ in M tells us that there are 6 bipartite rooted maps with 1 face of degree 2, 1 face of degree 4, and 3 vertices.

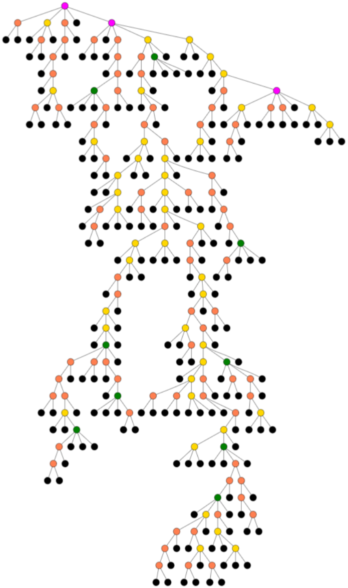


R also describes trees with x_i nodes (node that has i children) and leaves.

$$R = R(t) = R(t; x_1, x_2, ...) = t + \sum_{i \geq 1} x_i \binom{2i-1}{i} R^i$$

By random generation (Boltzmann), we can generate large trees in large numbers and record parameters such as subtree sum size, node distribution, number of leaves, etc.

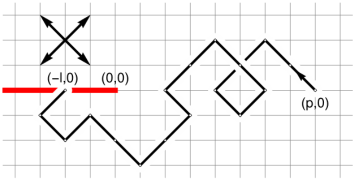
Randomly Generated R trees



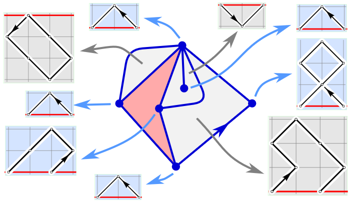
FINAL REMARKS

End goal

The generating function for bipartite maps with marked vertices appears in the GF for lattice walks that avoid the negative x-axis.



(a) Example of a lattice path walk, where p corresponds to the degree of the root face of the accompanying bipartite map. **[Budd '17]**



(b) Example assigning excursions to the faces and vertices of a bipartite map, which can then be combined as a lattice walk. **[Budd '17]**

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Thank you!