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- **Quasi-Bipartite map** := A planar map such that none or only two of its faces has odd degree.
- **p-constellation**:= A planar map whose faces have degrees multiples of p. (p=2 corresponds to bipartite)

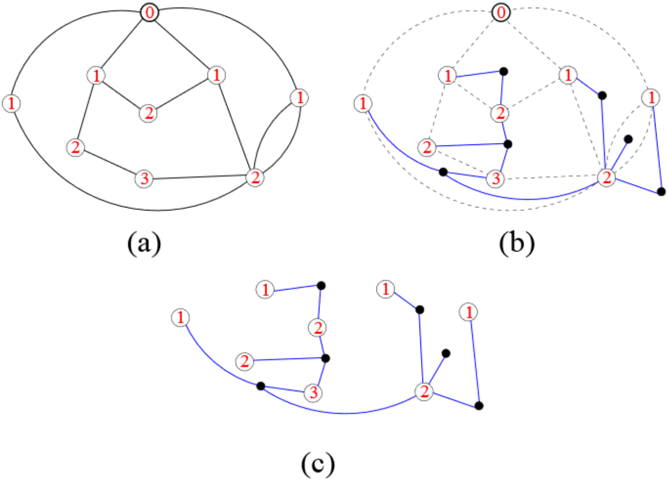
Planar Maps as Labelled Mobiles

Bouttier, Di Francesco, and Guitter [1] describe a process for finding a labelled mobile given a bipartite map M.

1. Designate a vertex as the origin, and label all other vertices by their geodesic distance.
2. Within each face of M, place an unlabelled vertex.
3. For a face of degree $2k$, add an edge from the k labelled vertices immediately followed clockwise by a smaller label. Do so for every face of M, for the outer face the opposite convention holds.
4. Remove all edges from the original map.

The result is a plane tree with two types of vertices, labelled and unlabelled ones.

Planar Maps as Labelled Mobiles



Planar Maps as Labelled Mobiles

The bijection by Bouttier, Di Francesco, and Guitter [1] yields the following useful relationship:

- Define $R = R(t) = R(t; x_1, x_2, ...)$ as

$$R = t + \sum_{i \geq 1} x_i \binom{2i-1}{i} R^i$$

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  - t denotes the number of vertices.
  - x_i denotes the number of non-boundary faces of degree $2i$.
  - Then, $M'(t) = 2R(t)$

Bipartite and Quasi-Bipartite maps with marked faces

In his book, Eynard [2] describes a formula for computing the generating functions of maps with two boundaries of prescribed length l_1, l_2. A boundary is also called a marked face with degree l_i.
For quasi-bipartite maps these boundaries will both be odd. Every other face has degree $2k$ for some $k \geq 1$.

$$G_{l_1,l_2} = \gamma^{l_1+l_2} \sum_{j=0}^{\lfloor l_2/2 \rfloor} (l_2 - 2j) \frac{l_1! l_2!}{j! (\frac{l_1 - l_2}{2} - 2)! (\frac{l_1+l_2}{2} - j)! (l_2 - j)!}$$

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- $l_1 \geq l_2$ and $l_1 + l_2$ must be even.

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- $l_1 \geq l_2$ and $l_1 + l_2$ must be even.
- γ^2 = $R(t)$.

Implementation: R, M, and G_{l_1,l_2}

- R will be useful in finding generating functions with a prescribed number of marked faces, but we can also find the generating function M, for the number of rooted bipartite maps.
- R is defined as a recursive function, and will become more accurate over greater iterations.
- On the next slide is the maple code for this, note that we are only looking at $j = 1..2$ which means faces of degree 2 and 4, as well as keeping only 16 terms of the series in each iteration to allow the code to finish running!

Implementation: R, M, and G_{l_1,l_2}

```
R[0]:= 0:

for i from 1 to 50 do
     R[i]:= convert(series(t + expand(add(x[2*j]*binomial(2*j-1, j-1)*R[i-1]^j, j=1..2)), t, 16),polynom);
od:
```

M(t;x1,x2,x3,x4,...) encodes the number of rooted planar bipartite maps.
R(t;x1,x2,x3,x4,...) is constructed with the help of the bijection using mobiles. (Bouttier, Di Francesco, Guitter)
x[2*i] := number of non-boundary faces of degree 2*i

> **R[i]** is more accurate as i increases
Display up to t^5

```
series(R[40],t):
```

> Fusy et Collet 2012, defines the **relationship between M and R** to be
M'(t) = 2R(t)

Issue *** Taking the integral of R gives rational coefficients for M *** ------> Improves with greater iterations

```
M:= series(int(2*R[50], t),t, 4);   M_1:= series(int(2*R[49], t),t, 4):   M_0:= series(int(2*R[48], t),t, 4):
```

Check stability of t^3 coeff

```
coeff(M, t^3):   coeff(M_1, t^3):   coeff(M_0, t^3):
```

Check stability of t^2 coeff

```
coeff(M, t^2):   coeff(M_1, t^2):   coeff(M_0, t^2):
```

$$M := \left(x_2^{49} + x_2^{48} + x_2^{47} + x_2^{46} + x_2^{45} + x_2^{44} + x_2^{43} + x_2^{42} + x_2^{41} + x_2^{40} + x_2^{39} + x_2^{38} + x_2^{37} + x_2^{36} + x_2^{35} + x_2^{34} + x_2^{33} + x_2^{32} + x_2^{31} + x_2^{30} + x_2^{29} + x_2^{28} + x_2^{27} + x_2^{26} + x_2^{25} + x_2^{24} + x_2^{23} + x_2^{22} + x_2^{21} + x_2^{20} + x_2^{19} + x_2^{18} + x_2^{17} + x_2^{16} + x_2^{15}\right.$$

$$\left. x_2^7 + x_2^6 + x_2^5 + x_2^4 + x_2^3 + x_2^2 + x_2 + 1\right) t^2 + \left(2 x_2^{96} x_4 + 6 x_2^{95} x_4 + 12 x_2^{94} x_4 + 20 x_2^{93} x_4 + 30 x_2^{92} x_4 + 42 x_2^{91} x_4 + 56 x_2^{90} x_4 + 72 x_2^{89} x_4 + 90 x_2^{88} x_4 + 110 x_2^{87} x_4 + 132 x_2^{86} x_4 + 156 x_2^{85} x_4 + 182 x_2^{84} x_4 + 210 x_2^{83} x_4 + 240 x\right.$$

$$+ 380 x_2^{78} x_4 + 420 x_2^{77} x_4 + 462 x_2^{76} x_4 + 506 x_2^{75} x_4 + 552 x_2^{74} x_4 + 600 x_2^{73} x_4 + 650 x_2^{72} x_4 + 702 x_2^{71} x_4 + 756 x_2^{70} x_4 + 812 x_2^{69} x_4 + 870 x_2^{68} x_4 + 930 x_2^{67} x_4 + 992 x_2^{66} x_4 + 1056 x_2^{65} x_4 + 1122 x_2^{64} x_4 + 1190 x_2^{63} x_4 + 1260$$

$$x_2^{59} x_4 + 1560 x_2^{58} x_4 + 1640 x_2^{57} x_4 + 1722 x_2^{56} x_4 + 1806 x_2^{55} x_4 + 1892 x_2^{54} x_4 + 1980 x_2^{53} x_4 + 2070 x_2^{52} x_4 + 2162 x_2^{51} x_4 + 2256 x_2^{50} x_4 + 2352 x_2^{49} x_4 + 2450 x_2^{48} x_4 + 2352 x_2^{47} x_4 + 2256 x_2^{46} x_4 + 2162 x_2^{45} x_4 + 2070 x_2^{44} x_4$$

$$+ 1722 x_2^{40} x_4 + 1640 x_2^{39} x_4 + 1560 x_2^{38} x_4 + 1482 x_2^{37} x_4 + 1406 x_2^{36} x_4 + 1332 x_2^{35} x_4 + 1260 x_2^{34} x_4 + 1190 x_2^{33} x_4 + 1122 x_2^{32} x_4 + 1056 x_2^{31} x_4 + 992 x_2^{30} x_4 + 930 x_2^{29} x_4 + 870 x_2^{28} x_4 + 812 x_2^{27} x_4 + 756 x_2^{26} x_4 + 702 x_2^{25} x$$

$$x_2^{21} x_4 + 462 x_2^{20} x_4 + 420 x_2^{19} x_4 + 380 x_2^{18} x_4 + 342 x_2^{17} x_4 + 306 x_2^{16} x_4 + 272 x_2^{15} x_4 + 240 x_2^{14} x_4 + 210 x_2^{13} x_4 + 182 x_2^{12} x_4 + 156 x_2^{11} x_4 + 132 x_2^{10} x_4 + 110 x_2^{9} x_4 + 90 x_2^{8} x_4 + 72 x_2^{7} x_4 + 56 x_2^{6} x_4 + 42 x_2^{5} x_4 + 30 x_2^{4} x_4 + 20$$

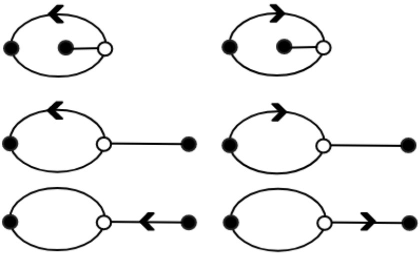
Implementation: R, M, and G_{l_1,l_2}

- Each term of M encodes information:
  - t^j, j denotes the number of vertices.
  - x_i^k denotes a map has k faces of degree i, i is even.
  - The coefficient counts the number of maps with these properties.

Implementation: R,M, and G_{l_1,l_2}

Example: (stable)

- $6x_2x_4t^3$ tells us that there are 6 distinct rooted maps that have 3 vertices with 1 face of degree 2 and 1 face of degree 4.



Implementation: R,M, and G_{l_1,l_2}

```
>
  Formula for R(t), consider only quadrangulations (j=2)

  R[0]:= 0:

  for i from 1 to 50 do
      R[i]:= convert(series(t + expand(add(x[2*j]*binomial(2*j-1, j-1)*R[i-1]^j, j=2..2)), t, 16),polynom);
  od:

> Display R[50];
  series(R[50],t,4);
```

$$t+3x_4t^2+18x_4^2t^3+O(t^4)$$

```
> M'(t) = R(t)
  M:= series(int(2*R[50], t),t, 9);
```

$$M:=t^2+2x_4t^3+9x_4^2t^4+54x_4^3t^5+378x_4^4t^6+2916x_4^5t^7+24057x_4^6t^8+O(t^9)$$

```
>

  Procedure to compute the generating function of bipartite/quasi-bipartite maps with 2 marked faces.
  G_two:= proc(x::integer,y::integer)
  local a,i,B;

  B:= add( (y-2*i)*x!*y!/(i!*((x-y)/2+i)!*((x+y)/2-i)!*(y-i)!), i=0..floor(y/2));
  expand(R[50]^((x+y)/2)*B);
  end proc;
```

G_two := **proc**(x::$integer$, y::$integer$)

local a, i, B;

$B := add((y-2*i)*\mathrm{factorial}(x)*\mathrm{factorial}(y)/(\mathrm{factorial}(i)*\mathrm{factorial}(1/2*x-1/2*y+i)*\mathrm{factorial}(1/2*x+1/2*y-i)*\mathrm{factorial}(y-i)), i=0..\mathrm{floor}(1/2*y)); \; expand(R[50]\hat{}(1/2*x+1/2*y)*B)$

end proc

```
> Given in Eynard  for gamma^2
  gamma_sqrd_quad:= (1/(6*x[4]))*(1-sqrt(1-12*t*x[4]));
>

> series(G_two(4,4),t,9);
```

$$36t^4+432x_4t^5+4536x_4^2t^6+46656x_4^3t^7+481140x_4^4t^8+O(t^9)$$

```
> Values given in Eynard, 36*gamma^8

  series(36*gamma_sqrd_quad^4,t,9);
```

$$36t^4+432x_4t^5+4536x_4^2t^6+46656x_4^3t^7+481140x_4^4t^8+O(t^9)$$

Implementation: R,M, and G_{l_1,l_2}

To check the accuracy of G_{l_1,l_2}, compare directly to the example given by Eynard for the number of quadrangulations.He notes that for quadrangulations $\gamma^2 = \dfrac{1-\sqrt{1-12tx_4}}{6tx_4}$.

The last line of the previous slide uses this definition of γ, whereas the line above uses R to find γ.

Both calculations result in the same generating function.

$$36t^4 + 432x_4t^5 + 4536x_4^2t^6 + ...$$

Implementation: R,M, and G_{l_1,l_2}

$$36t^4 + 432x_4t^5 + 4536x_4^2t^6 + ...$$

The information in this generating function can be read similiarly to M with one exception. x_4^k denotes the k unmarked faces of degree 4. This does not included the 2 marked faces we chose when finding $G_{l1,l2}$.

For example,

- $36t^4$ says there are 36 distinct quadrangulations that have 4 vertices, 2 marked faces of degree 4, and 0 unmarked faces of degree 4.
- $432x_4t^5$ says there are 432 distinct quadrangulations that have 5 vertices, 2 marked faces of degree 4, and 1 unmarked face of degree 4.

More than two marked faces

Eynard also gives a formula for 3 marked faces. Collet et Fusy generalize these formulae for any number of marked faces.

- $l_1, l_2, ..., l_r$ denote the degrees of r marked faces, $r \geq 1$.
- Holds only when none or exactly two of l_i are odd.

$$G_{l_1,l_2,...,l_r} = \prod_{i=1}^{r} \alpha(l_i) \cdot \frac{1}{s} \cdot \frac{d^{r-2}}{dt^{r-2}} R^s$$

- with $\alpha(l) = \dfrac{l!}{\lfloor \frac{l}{2} \rfloor ! \lfloor \frac{l-1}{2} \rfloor !}$
- and $s = \dfrac{l_1 + l_2 + ... + l_r}{2}$ and R

More than two marked faces

- Implementing this formula for G, with a list of two marked faces of degree 4 also returns the expected values for quadrangulations. The maple code allows the user to input a list of any size with prescribed marked face valences.
- Adding an error for lists that do not have exactly 0 or 2 odd valued l_i would improve the code.
- Next step: Collet and Fusy further generalize this work to p-constellations

References

1. Bouttier, J., Di Francesco, P., and Guitter, E. (2004). Planar maps as labeled mobiles. Elec. Jour. of Combinatorics Vol 11 (2004)
2. Eynard, B. (2016). Counting Surfaces (Vol. 70, Progress in Mathematical Physics). Basel: Springer Basel.
3. Collet, G., and Fusy, E. (2012). A simple formula for the series of constellations and quasi-constellations with boundaries.