POD_Fenics

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1 Proper Orthogonal Decomposition based reduced order modeling for the heat equation

1.1 By Julian Roth (2022)

In the following tutorial, we will learn how we can create a reduced order model of the heat equation by using the Proper Orthogonal Decomposition (POD). For the Finite Element Method (FEM) computations in this tutorial, we will use the FEniCS library.

1.1.1 Imports

The following code snippet installs FEniCS in the Google Colab environment.

[ ]:
```
try:
    import dolfin
except ImportError:
    !wget "https://fem-on-colab.github.io/releases/fenics-install.sh" -O "/tmp/
 ↪fenics-install.sh" && bash "/tmp/fenics-install.sh"
    import dolfin
```

```
--2022-03-30 12:13:16--  https://fem-on-colab.github.io/releases/fenics-
install.sh
Resolving fem-on-colab.github.io (fem-on-colab.github.io)… 185.199.108.153,
185.199.109.153, 185.199.110.153, …
Connecting to fem-on-colab.github.io (fem-on-
colab.github.io)|185.199.108.153|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 2831 (2.8K) [application/x-sh]
Saving to: ‘/tmp/fenics-install.sh’

/tmp/fenics-install 100%[===================>]   2.76K  --.-KB/s    in 0s

2022-03-30 12:13:16 (31.2 MB/s) - ‘/tmp/fenics-install.sh’ saved [2831/2831]

+ SHARE_PREFIX=/usr/local/share/fem-on-colab
+ FENICS_INSTALLED=/usr/local/share/fem-on-colab/fenics.installed
+ [[ ! -f /usr/local/share/fem-on-colab/fenics.installed ]]
+ PYBIND11_INSTALL_SCRIPT_PATH=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/e9e1ba9/releases/pybind11-install.sh
```

```
+ [[ https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/e9e1ba9/releases/pybind11-install.sh == http* ]]
+ PYBIND11_INSTALL_SCRIPT_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/e9e1ba9/releases/pybind11-install.sh
+ PYBIND11_INSTALL_SCRIPT_PATH=/tmp/pybind11-install.sh
+ [[ ! -f /tmp/pybind11-install.sh ]]
+ wget https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/e9e1ba9/releases/pybind11-install.sh -O
/tmp/pybind11-install.sh
--2022-03-30 12:13:16--  https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/e9e1ba9/releases/pybind11-install.sh
Resolving github.com (github.com)… 140.82.112.3
Connecting to github.com (github.com)|140.82.112.3|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://raw.githubusercontent.com/fem-on-colab/fem-on-colab.github.io/
e9e1ba91229f62d082b7118d533b60616b542dd8/releases/pybind11-install.sh
[following]
--2022-03-30 12:13:16--  https://raw.githubusercontent.com/fem-on-colab/fem-on-c
olab.github.io/e9e1ba91229f62d082b7118d533b60616b542dd8/releases/pybind11-instal
l.sh
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.110.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 1487 (1.5K) [text/plain]
Saving to: ‘/tmp/pybind11-install.sh’

/tmp/pybind11-insta 100%[===================>]   1.45K  --.-KB/s    in 0s

2022-03-30 12:13:16 (9.82 MB/s) - ‘/tmp/pybind11-install.sh’ saved [1487/1487]

+ source /tmp/pybind11-install.sh
++ set -e
++ set -x
++ SHARE_PREFIX=/usr/local/share/fem-on-colab
++ PYBIND11_INSTALLED=/usr/local/share/fem-on-colab/pybind11.installed
++ [[ ! -f /usr/local/share/fem-on-colab/pybind11.installed ]]
++ MPI4PY_INSTALL_SCRIPT_PATH=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/534f2c3/releases/mpi4py-install.sh
++ [[ https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/534f2c3/releases/mpi4py-install.sh == http* ]]
++ MPI4PY_INSTALL_SCRIPT_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/534f2c3/releases/mpi4py-install.sh
++ MPI4PY_INSTALL_SCRIPT_PATH=/tmp/mpi4py-install.sh
++ [[ ! -f /tmp/mpi4py-install.sh ]]
++ wget https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/534f2c3/releases/mpi4py-install.sh -O /tmp/mpi4py-install.sh
```

```
--2022-03-30 12:13:16--  https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/534f2c3/releases/mpi4py-install.sh
Resolving github.com (github.com)… 140.82.113.4
Connecting to github.com (github.com)|140.82.113.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/534f2c3456e2d14b51256b962a88e26f16341798/releases/mpi4py-
install.sh [following]
--2022-03-30 12:13:16--  https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/534f2c3456e2d14b51256b962a88e26f16341798/releases/mpi4py-
install.sh
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.111.133, 185.199.110.133, 185.199.109.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.111.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 1796 (1.8K) [text/plain]
Saving to: ‘/tmp/mpi4py-install.sh’

/tmp/mpi4py-install 100%[===================>]   1.75K  --.-KB/s    in 0s

2022-03-30 12:13:17 (28.2 MB/s) - ‘/tmp/mpi4py-install.sh’ saved [1796/1796]

++ source /tmp/mpi4py-install.sh
+++ set -e
+++ set -x
+++ SHARE_PREFIX=/usr/local/share/fem-on-colab
+++ MPI4PY_INSTALLED=/usr/local/share/fem-on-colab/mpi4py.installed
+++ [[ ! -f /usr/local/share/fem-on-colab/mpi4py.installed ]]
+++ GCC_INSTALL_SCRIPT_PATH=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/fabd340/releases/gcc-install.sh
+++ [[ https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/fabd340/releases/gcc-install.sh == http* ]]
+++ GCC_INSTALL_SCRIPT_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/fabd340/releases/gcc-install.sh
+++ GCC_INSTALL_SCRIPT_PATH=/tmp/gcc-install.sh
+++ [[ ! -f /tmp/gcc-install.sh ]]
+++ wget https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/fabd340/releases/gcc-install.sh -O /tmp/gcc-install.sh
--2022-03-30 12:13:17--  https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/fabd340/releases/gcc-install.sh
Resolving github.com (github.com)… 140.82.112.3
Connecting to github.com (github.com)|140.82.112.3|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/fabd340ffcab5dbd49b73fb20e83209027e06cf3/releases/gcc-install.sh
[following]
--2022-03-30 12:13:17--  https://raw.githubusercontent.com/fem-on-colab/fem-on-
```

```
colab.github.io/fabd340ffcab5dbd49b73fb20e83209027e06cf3/releases/gcc-install.sh
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.110.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 3734 (3.6K) [text/plain]
Saving to: ‘/tmp/gcc-install.sh’

/tmp/gcc-install.sh 100%[===================>]   3.65K  --.-KB/s    in 0s

2022-03-30 12:13:17 (50.7 MB/s) - ‘/tmp/gcc-install.sh’ saved [3734/3734]

+++ source /tmp/gcc-install.sh
++++ set -e
++++ set -x
++++ SHARE_PREFIX=/usr/local/share/fem-on-colab
++++ GCC_INSTALLED=/usr/local/share/fem-on-colab/gcc.installed
++++ [[ ! -f /usr/local/share/fem-on-colab/gcc.installed ]]
++++ add-apt-repository -y ppa:ubuntu-toolchain-r/test
Get:1 https://cloud.r-project.org/bin/linux/ubuntu bionic-cran40/ InRelease
[3,626 B]
Ign:2 https://developer.download.nvidia.com/compute/cuda/repos/ubuntu1804/x86_64
InRelease
Ign:3 https://developer.download.nvidia.com/compute/machine-
learning/repos/ubuntu1804/x86_64  InRelease
Get:4 https://developer.download.nvidia.com/compute/cuda/repos/ubuntu1804/x86_64
Release [696 B]
Hit:5 https://developer.download.nvidia.com/compute/machine-
learning/repos/ubuntu1804/x86_64  Release
Get:6 https://developer.download.nvidia.com/compute/cuda/repos/ubuntu1804/x86_64
Release.gpg [836 B]
Get:7 http://security.ubuntu.com/ubuntu bionic-security InRelease [88.7 kB]
Hit:8 http://archive.ubuntu.com/ubuntu bionic InRelease
Get:9 http://ppa.launchpad.net/c2d4u.team/c2d4u4.0+/ubuntu bionic InRelease
[15.9 kB]
Get:10 http://archive.ubuntu.com/ubuntu bionic-updates InRelease [88.7 kB]
Get:11 https://cloud.r-project.org/bin/linux/ubuntu bionic-cran40/ Packages
[80.8 kB]
Get:13 http://archive.ubuntu.com/ubuntu bionic-backports InRelease [74.6 kB]
Hit:14 http://ppa.launchpad.net/cran/libgit2/ubuntu bionic InRelease
Get:15
https://developer.download.nvidia.com/compute/cuda/repos/ubuntu1804/x86_64
Packages [946 kB]
Get:16 http://ppa.launchpad.net/deadsnakes/ppa/ubuntu bionic InRelease [15.9 kB]
Get:17 http://ppa.launchpad.net/graphics-drivers/ppa/ubuntu bionic InRelease
[21.3 kB]
Get:18 http://security.ubuntu.com/ubuntu bionic-security/restricted amd64
```

```
Packages [869 kB]
Get:19 http://ppa.launchpad.net/ubuntu-toolchain-r/test/ubuntu bionic InRelease
[20.8 kB]
Get:20 http://security.ubuntu.com/ubuntu bionic-security/universe amd64 Packages
[1,486 kB]
Get:21 http://security.ubuntu.com/ubuntu bionic-security/main amd64 Packages
[2,670 kB]
Get:22 http://ppa.launchpad.net/c2d4u.team/c2d4u4.0+/ubuntu bionic/main Sources
[1,830 kB]
Get:23 http://archive.ubuntu.com/ubuntu bionic-updates/universe amd64 Packages
[2,264 kB]
Get:24 http://archive.ubuntu.com/ubuntu bionic-updates/restricted amd64 Packages
[911 kB]
Get:25 http://archive.ubuntu.com/ubuntu bionic-updates/main amd64 Packages
[3,115 kB]
Get:26 http://archive.ubuntu.com/ubuntu bionic-updates/multiverse amd64 Packages
[29.8 kB]
Get:27 http://ppa.launchpad.net/c2d4u.team/c2d4u4.0+/ubuntu bionic/main amd64
Packages [937 kB]
Get:28 http://archive.ubuntu.com/ubuntu bionic-backports/universe amd64 Packages
[12.9 kB]
Get:29 http://archive.ubuntu.com/ubuntu bionic-backports/main amd64 Packages
[12.2 kB]
Get:30 http://ppa.launchpad.net/deadsnakes/ppa/ubuntu bionic/main amd64 Packages
[45.3 kB]
Get:31 http://ppa.launchpad.net/graphics-drivers/ppa/ubuntu bionic/main amd64
Packages [44.3 kB]
Get:32 http://ppa.launchpad.net/ubuntu-toolchain-r/test/ubuntu bionic/main amd64
Packages [50.4 kB]
Fetched 15.6 MB in 7s (2,179 kB/s)
Reading package lists… Done
++++ apt update
Hit:1 https://cloud.r-project.org/bin/linux/ubuntu bionic-cran40/ InRelease
Ign:2 https://developer.download.nvidia.com/compute/cuda/repos/ubuntu1804/x86_64
InRelease
Ign:3 https://developer.download.nvidia.com/compute/machine-
learning/repos/ubuntu1804/x86_64  InRelease
Hit:4 https://developer.download.nvidia.com/compute/cuda/repos/ubuntu1804/x86_64
Release
Hit:5 https://developer.download.nvidia.com/compute/machine-
learning/repos/ubuntu1804/x86_64  Release
Hit:6 http://security.ubuntu.com/ubuntu bionic-security InRelease
Hit:7 http://ppa.launchpad.net/c2d4u.team/c2d4u4.0+/ubuntu bionic InRelease
Hit:8 http://archive.ubuntu.com/ubuntu bionic InRelease
Hit:9 http://archive.ubuntu.com/ubuntu bionic-updates InRelease
Hit:10 http://archive.ubuntu.com/ubuntu bionic-backports InRelease
Hit:11 http://ppa.launchpad.net/cran/libgit2/ubuntu bionic InRelease
Hit:12 http://ppa.launchpad.net/deadsnakes/ppa/ubuntu bionic InRelease
```

```
Hit:14 http://ppa.launchpad.net/graphics-drivers/ppa/ubuntu bionic InRelease
Hit:16 http://ppa.launchpad.net/ubuntu-toolchain-r/test/ubuntu bionic InRelease
Reading package lists… Done
Building dependency tree
Reading state information… Done
91 packages can be upgraded. Run 'apt list --upgradable' to see them.
++++ apt install -y -qq gcc-11 gfortran-11 g++-11 libgcc1
The following additional packages will be installed:
  cpp-11 gcc-11-base libasan6 libatomic1 libcc1-0 libgcc-11-dev libgcc-s1
  libgfortran-11-dev libgfortran5 libgomp1 libitm1 liblsan0 libquadmath0
  libstdc++-11-dev libstdc++6 libtsan0 libubsan1
Suggested packages:
  gcc-11-locales g++-11-multilib gcc-11-doc libstdc++6-11-dbg gcc-11-multilib
  libgcc-s1-dbg libgomp1-dbg libitm1-dbg libatomic1-dbg libasan6-dbg
  liblsan0-dbg libtsan0-dbg libubsan1-dbg libhwasan0-dbg libquadmath0-dbg
  gfortran-11-multilib gfortran-11-doc libgfortran5-dbg libcoarrays-dev
  libstdc++-11-doc
The following NEW packages will be installed:
  cpp-11 g++-11 gcc-11 gcc-11-base gfortran-11 libasan6 libgcc-11-dev
  libgcc-s1 libgfortran-11-dev libgfortran5 libstdc++-11-dev libubsan1
The following packages will be upgraded:
  libatomic1 libcc1-0 libgcc1 libgomp1 libitm1 liblsan0 libquadmath0
  libstdc++6 libtsan0
9 upgraded, 12 newly installed, 0 to remove and 82 not upgraded.
Need to get 53.0 MB of archives.
After this operation, 196 MB of additional disk space will be used.
Selecting previously unselected package gcc-11-base:amd64.
(Reading database … 156210 files and directories currently installed.)
Preparing to unpack …/gcc-11-base_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking gcc-11-base:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up gcc-11-base:amd64 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package libgcc-s1:amd64.
(Reading database … 156215 files and directories currently installed.)
Preparing to unpack …/libgcc-s1_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libgcc-s1:amd64 (11.1.0-1ubuntu1~18.04.1) …
Replacing files in old package libgcc1:amd64 (1:8.4.0-1ubuntu1~18.04) …
Setting up libgcc-s1:amd64 (11.1.0-1ubuntu1~18.04.1) …
(Reading database … 156217 files and directories currently installed.)
Preparing to unpack …/libgcc1_1%3a11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libgcc1 (1:11.1.0-1ubuntu1~18.04.1) over (1:8.4.0-1ubuntu1~18.04) …
Setting up libgcc1 (1:11.1.0-1ubuntu1~18.04.1) …
(Reading database … 156218 files and directories currently installed.)
Preparing to unpack …/libstdc++6_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libstdc++6:amd64 (11.1.0-1ubuntu1~18.04.1) over (8.4.0-1ubuntu1~18.04)
…
Setting up libstdc++6:amd64 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package cpp-11.
(Reading database … 156218 files and directories currently installed.)
```

```
Preparing to unpack …/00-cpp-11_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking cpp-11 (11.1.0-1ubuntu1~18.04.1) …
Preparing to unpack …/01-libcc1-0_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libcc1-0:amd64 (11.1.0-1ubuntu1~18.04.1) over (8.4.0-1ubuntu1~18.04)
…
Preparing to unpack …/02-libgomp1_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libgomp1:amd64 (11.1.0-1ubuntu1~18.04.1) over (8.4.0-1ubuntu1~18.04)
…
Preparing to unpack …/03-libitm1_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libitm1:amd64 (11.1.0-1ubuntu1~18.04.1) over (8.4.0-1ubuntu1~18.04)
…
Preparing to unpack …/04-libatomic1_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libatomic1:amd64 (11.1.0-1ubuntu1~18.04.1) over (8.4.0-1ubuntu1~18.04)
…
Selecting previously unselected package libasan6:amd64.
Preparing to unpack …/05-libasan6_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libasan6:amd64 (11.1.0-1ubuntu1~18.04.1) …
Preparing to unpack …/06-liblsan0_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking liblsan0:amd64 (11.1.0-1ubuntu1~18.04.1) over (8.4.0-1ubuntu1~18.04)
…
Preparing to unpack …/07-libtsan0_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libtsan0:amd64 (11.1.0-1ubuntu1~18.04.1) over (8.4.0-1ubuntu1~18.04)
…
Selecting previously unselected package libubsan1:amd64.
Preparing to unpack …/08-libubsan1_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libubsan1:amd64 (11.1.0-1ubuntu1~18.04.1) …
Preparing to unpack …/09-libquadmath0_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libquadmath0:amd64 (11.1.0-1ubuntu1~18.04.1) over
(8.4.0-1ubuntu1~18.04) …
Selecting previously unselected package libgcc-11-dev:amd64.
Preparing to unpack …/10-libgcc-11-dev_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libgcc-11-dev:amd64 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package gcc-11.
Preparing to unpack …/11-gcc-11_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking gcc-11 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package libstdc++-11-dev:amd64.
Preparing to unpack …/12-libstdc++-11-dev_11.1.0-1ubuntu1~18.04.1_amd64.deb
…
Unpacking libstdc++-11-dev:amd64 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package g++-11.
Preparing to unpack …/13-g++-11_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking g++-11 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package libgfortran5:amd64.
Preparing to unpack …/14-libgfortran5_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking libgfortran5:amd64 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package libgfortran-11-dev:amd64.
Preparing to unpack …/15-libgfortran-11-dev_11.1.0-1ubuntu1~18.04.1_amd64.deb
…
```

```
Unpacking libgfortran-11-dev:amd64 (11.1.0-1ubuntu1~18.04.1) …
Selecting previously unselected package gfortran-11.
Preparing to unpack …/16-gfortran-11_11.1.0-1ubuntu1~18.04.1_amd64.deb …
Unpacking gfortran-11 (11.1.0-1ubuntu1~18.04.1) …
Setting up libquadmath0:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libgomp1:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libatomic1:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libcc1-0:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libtsan0:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up liblsan0:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up cpp-11 (11.1.0-1ubuntu1~18.04.1) …
Setting up libasan6:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libgfortran5:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libitm1:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libubsan1:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libgcc-11-dev:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up libgfortran-11-dev:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up gcc-11 (11.1.0-1ubuntu1~18.04.1) …
Setting up libstdc++-11-dev:amd64 (11.1.0-1ubuntu1~18.04.1) …
Setting up g++-11 (11.1.0-1ubuntu1~18.04.1) …
Setting up gfortran-11 (11.1.0-1ubuntu1~18.04.1) …
Processing triggers for man-db (2.8.3-2ubuntu0.1) …
Processing triggers for libc-bin (2.27-3ubuntu1.3) …
/sbin/ldconfig.real: /usr/local/lib/python3.7/dist-
packages/ideep4py/lib/libmkldnn.so.0 is not a symbolic link

++++ update-alternatives --install /usr/bin/g++ g++ /usr/bin/g++-7 7
update-alternatives: using /usr/bin/g++-7 to provide /usr/bin/g++ (g++) in auto
mode
++++ update-alternatives --install /usr/bin/gcc gcc /usr/bin/gcc-7 7
update-alternatives: using /usr/bin/gcc-7 to provide /usr/bin/gcc (gcc) in auto
mode
++++ update-alternatives --install /usr/bin/gcc-ar gcc-ar /usr/bin/gcc-ar-7 7
update-alternatives: using /usr/bin/gcc-ar-7 to provide /usr/bin/gcc-ar (gcc-ar)
in auto mode
++++ update-alternatives --install /usr/bin/gcc-nm gcc-nm /usr/bin/gcc-nm-7 7
update-alternatives: using /usr/bin/gcc-nm-7 to provide /usr/bin/gcc-nm (gcc-nm)
in auto mode
++++ update-alternatives --install /usr/bin/gcc-ranlib gcc-ranlib /usr/bin/gcc-
ranlib-7 7
update-alternatives: using /usr/bin/gcc-ranlib-7 to provide /usr/bin/gcc-ranlib
(gcc-ranlib) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-g++ x86_64-linux-
gnu-g++ /usr/bin/x86_64-linux-gnu-g++-7 7
update-alternatives: using /usr/bin/x86_64-linux-gnu-g++-7 to provide
/usr/bin/x86_64-linux-gnu-g++ (x86_64-linux-gnu-g++) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc x86_64-linux-
gnu-gcc /usr/bin/x86_64-linux-gnu-gcc-7 7
```

```
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-7 to provide
/usr/bin/x86_64-linux-gnu-gcc (x86_64-linux-gnu-gcc) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc-ar
x86_64-linux-gnu-gcc-ar /usr/bin/x86_64-linux-gnu-gcc-ar-7 7
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-ar-7 to provide
/usr/bin/x86_64-linux-gnu-gcc-ar (x86_64-linux-gnu-gcc-ar) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc-nm
x86_64-linux-gnu-gcc-nm /usr/bin/x86_64-linux-gnu-gcc-nm-7 7
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-nm-7 to provide
/usr/bin/x86_64-linux-gnu-gcc-nm (x86_64-linux-gnu-gcc-nm) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc-ranlib
x86_64-linux-gnu-gcc-ranlib /usr/bin/x86_64-linux-gnu-gcc-ranlib-7 7
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-ranlib-7 to provide
/usr/bin/x86_64-linux-gnu-gcc-ranlib (x86_64-linux-gnu-gcc-ranlib) in auto mode
++++ update-alternatives --install /usr/bin/g++ g++ /usr/bin/g++-11 11
update-alternatives: using /usr/bin/g++-11 to provide /usr/bin/g++ (g++) in auto
mode
++++ update-alternatives --install /usr/bin/gcc gcc /usr/bin/gcc-11 11
update-alternatives: using /usr/bin/gcc-11 to provide /usr/bin/gcc (gcc) in auto
mode
++++ update-alternatives --install /usr/bin/gcc-ar gcc-ar /usr/bin/gcc-ar-11 11
update-alternatives: using /usr/bin/gcc-ar-11 to provide /usr/bin/gcc-ar (gcc-
ar) in auto mode
++++ update-alternatives --install /usr/bin/gcc-nm gcc-nm /usr/bin/gcc-nm-11 11
update-alternatives: using /usr/bin/gcc-nm-11 to provide /usr/bin/gcc-nm (gcc-
nm) in auto mode
++++ update-alternatives --install /usr/bin/gcc-ranlib gcc-ranlib /usr/bin/gcc-
ranlib-11 11
update-alternatives: using /usr/bin/gcc-ranlib-11 to provide /usr/bin/gcc-ranlib
(gcc-ranlib) in auto mode
++++ update-alternatives --install /usr/bin/gfortran gfortran
/usr/bin/gfortran-11 11
update-alternatives: using /usr/bin/gfortran-11 to provide /usr/bin/gfortran
(gfortran) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-g++ x86_64-linux-
gnu-g++ /usr/bin/x86_64-linux-gnu-g++-11 11
update-alternatives: using /usr/bin/x86_64-linux-gnu-g++-11 to provide
/usr/bin/x86_64-linux-gnu-g++ (x86_64-linux-gnu-g++) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc x86_64-linux-
gnu-gcc /usr/bin/x86_64-linux-gnu-gcc-11 11
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-11 to provide
/usr/bin/x86_64-linux-gnu-gcc (x86_64-linux-gnu-gcc) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc-ar
x86_64-linux-gnu-gcc-ar /usr/bin/x86_64-linux-gnu-gcc-ar-11 11
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-ar-11 to provide
/usr/bin/x86_64-linux-gnu-gcc-ar (x86_64-linux-gnu-gcc-ar) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc-nm
x86_64-linux-gnu-gcc-nm /usr/bin/x86_64-linux-gnu-gcc-nm-11 11
```

```
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-nm-11 to provide
/usr/bin/x86_64-linux-gnu-gcc-nm (x86_64-linux-gnu-gcc-nm) in auto mode
++++ update-alternatives --install /usr/bin/x86_64-linux-gnu-gcc-ranlib
x86_64-linux-gnu-gcc-ranlib /usr/bin/x86_64-linux-gnu-gcc-ranlib-11 11
update-alternatives: using /usr/bin/x86_64-linux-gnu-gcc-ranlib-11 to provide
/usr/bin/x86_64-linux-gnu-gcc-ranlib (x86_64-linux-gnu-gcc-ranlib) in auto mode
++++ update-alternatives --set g++ /usr/bin/g++-11
++++ update-alternatives --set gcc /usr/bin/gcc-11
++++ update-alternatives --set gcc-ar /usr/bin/gcc-ar-11
++++ update-alternatives --set gcc-nm /usr/bin/gcc-nm-11
++++ update-alternatives --set gcc-ranlib /usr/bin/gcc-ranlib-11
++++ update-alternatives --set gfortran /usr/bin/gfortran-11
++++ update-alternatives --set x86_64-linux-gnu-g++ /usr/bin/x86_64-linux-
gnu-g++-11
++++ update-alternatives --set x86_64-linux-gnu-gcc /usr/bin/x86_64-linux-gnu-
gcc-11
++++ update-alternatives --set x86_64-linux-gnu-gcc-ar /usr/bin/x86_64-linux-
gnu-gcc-ar-11
++++ update-alternatives --set x86_64-linux-gnu-gcc-nm /usr/bin/x86_64-linux-
gnu-gcc-nm-11
++++ update-alternatives --set x86_64-linux-gnu-gcc-ranlib
/usr/bin/x86_64-linux-gnu-gcc-ranlib-11
++++ cp -f /usr/lib/gcc/x86_64-linux-gnu/11/libstdc++.so
/usr/lib/gcc/x86_64-linux-gnu/11/libstdc++.so.6
++++ mkdir -p /usr/local/share/fem-on-colab
++++ touch /usr/local/share/fem-on-colab/gcc.installed
+++ MPI4PY_ARCHIVE_PATH=https://github.com/fem-on-colab/fem-on-
colab/releases/download/mpi4py-20220216-073713-8f2b82c/mpi4py-install.tar.gz
+++ [[ https://github.com/fem-on-colab/fem-on-
colab/releases/download/mpi4py-20220216-073713-8f2b82c/mpi4py-install.tar.gz ==
http* ]]
+++ MPI4PY_ARCHIVE_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab/releases/download/mpi4py-20220216-073713-8f2b82c/mpi4py-install.tar.gz
+++ MPI4PY_ARCHIVE_PATH=/tmp/mpi4py-install.tar.gz
+++ wget https://github.com/fem-on-colab/fem-on-
colab/releases/download/mpi4py-20220216-073713-8f2b82c/mpi4py-install.tar.gz -O
/tmp/mpi4py-install.tar.gz
--2022-03-30 12:14:15--  https://github.com/fem-on-colab/fem-on-
colab/releases/download/mpi4py-20220216-073713-8f2b82c/mpi4py-install.tar.gz
Resolving github.com (github.com)… 140.82.112.4
Connecting to github.com (github.com)|140.82.112.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://objects.githubusercontent.com/github-production-release-
asset-2e65be/370599515/98fe2932-05b2-4477-8561-6e3b3cce812e?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121415Z&X-Amz-Expires=300&X-Amz-S
ignature=a70d46fee1bafb44dc6eb0fc2b5e27cf3c7bd24176fe9b725e7838724d19d532&X-Amz-
```

```
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dmpi4py-install.tar.gz&response-content-
type=application%2Foctet-stream [following]
--2022-03-30 12:14:15--  https://objects.githubusercontent.com/github-
production-release-
asset-2e65be/370599515/98fe2932-05b2-4477-8561-6e3b3cce812e?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121415Z&X-Amz-Expires=300&X-Amz-S
ignature=a70d46fee1bafb44dc6eb0fc2b5e27cf3c7bd24176fe9b725e7838724d19d532&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dmpi4py-install.tar.gz&response-content-
type=application%2Foctet-stream
Resolving objects.githubusercontent.com (objects.githubusercontent.com)…
185.199.109.133, 185.199.111.133, 185.199.110.133, …
Connecting to objects.githubusercontent.com
(objects.githubusercontent.com)|185.199.109.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 7763326 (7.4M) [application/octet-stream]
Saving to: ‘/tmp/mpi4py-install.tar.gz’

/tmp/mpi4py-install 100%[===================>]   7.40M  --.-KB/s    in 0.1s

2022-03-30 12:14:15 (56.3 MB/s) - ‘/tmp/mpi4py-install.tar.gz’ saved
[7763326/7763326]

+++ [[ /tmp/mpi4py-install.tar.gz != skip ]]
+++ tar -xzf /tmp/mpi4py-install.tar.gz --strip-components=2
--directory=/usr/local
+++ [[ /tmp/mpi4py-install.tar.gz != skip ]]
+++ ln -fs /usr/local/lib/libmca_common_dstore.so
/usr/local/lib/libmca_common_dstore.so.1
/usr/local/lib/libmca_common_dstore.so.1.0.2
/usr/local/lib/libmca_common_monitoring.so
/usr/local/lib/libmca_common_monitoring.so.50
/usr/local/lib/libmca_common_monitoring.so.50.20.0
/usr/local/lib/libmca_common_ompio.so /usr/local/lib/libmca_common_ompio.so.41
/usr/local/lib/libmca_common_ompio.so.41.29.2 /usr/local/lib/libmca_common_sm.so
/usr/local/lib/libmca_common_sm.so.40 /usr/local/lib/libmca_common_sm.so.40.30.0
/usr/lib
+++ ln -fs /usr/local/lib/libmpi_cxx.so /usr/local/lib/libmpi_cxx.so.40
/usr/local/lib/libmpi_cxx.so.40.30.1 /usr/local/lib/libmpi_mpifh.so
/usr/local/lib/libmpi_mpifh.so.40 /usr/local/lib/libmpi_mpifh.so.40.30.0
/usr/local/lib/libmpi.so /usr/local/lib/libmpi.so.40
/usr/local/lib/libmpi.so.40.30.2 /usr/local/lib/libmpi_usempif08.so
/usr/local/lib/libmpi_usempif08.so.40 /usr/local/lib/libmpi_usempif08.so.40.30.0
/usr/local/lib/libmpi_usempi_ignore_tkr.so
/usr/local/lib/libmpi_usempi_ignore_tkr.so.40
```

```
/usr/local/lib/libmpi_usempi_ignore_tkr.so.40.30.0 /usr/lib
+++ ln -fs /usr/local/lib/libopen-pal.so /usr/local/lib/libopen-pal.so.40
/usr/local/lib/libopen-pal.so.40.30.2 /usr/local/lib/libopen-rte.so
/usr/local/lib/libopen-rte.so.40 /usr/local/lib/libopen-rte.so.40.30.2 /usr/lib
+++ ln -fs /usr/local/lib/ompi_monitoring_prof.so /usr/lib
+++ mkdir -p /usr/local/share/fem-on-colab
+++ touch /usr/local/share/fem-on-colab/mpi4py.installed
++ PYBIND11_ARCHIVE_PATH=https://github.com/fem-on-colab/fem-on-
colab/releases/download/pybind11-20220216-075631-8f2b82c/pybind11-install.tar.gz
++ [[ https://github.com/fem-on-colab/fem-on-
colab/releases/download/pybind11-20220216-075631-8f2b82c/pybind11-install.tar.gz
== http* ]]
++ PYBIND11_ARCHIVE_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab/releases/download/pybind11-20220216-075631-8f2b82c/pybind11-install.tar.gz
++ PYBIND11_ARCHIVE_PATH=/tmp/pybind11-install.tar.gz
++ wget https://github.com/fem-on-colab/fem-on-
colab/releases/download/pybind11-20220216-075631-8f2b82c/pybind11-install.tar.gz
-O /tmp/pybind11-install.tar.gz
--2022-03-30 12:14:16--  https://github.com/fem-on-colab/fem-on-
colab/releases/download/pybind11-20220216-075631-8f2b82c/pybind11-install.tar.gz
Resolving github.com (github.com)… 140.82.112.4
Connecting to github.com (github.com)|140.82.112.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://objects.githubusercontent.com/github-production-release-
asset-2e65be/370599515/d96a9b7b-6cb1-4644-b88d-b44d68e3ed53?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121416Z&X-Amz-Expires=300&X-Amz-S
ignature=d29754a94ce3df5ebece7f52861fa3b8c850c78f8bc6e26f076e452697404d5c&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dpybind11-install.tar.gz&response-content-
type=application%2Foctet-stream [following]
--2022-03-30 12:14:16--  https://objects.githubusercontent.com/github-
production-release-
asset-2e65be/370599515/d96a9b7b-6cb1-4644-b88d-b44d68e3ed53?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121416Z&X-Amz-Expires=300&X-Amz-S
ignature=d29754a94ce3df5ebece7f52861fa3b8c850c78f8bc6e26f076e452697404d5c&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dpybind11-install.tar.gz&response-content-
type=application%2Foctet-stream
Resolving objects.githubusercontent.com (objects.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.111.133, …
Connecting to objects.githubusercontent.com
(objects.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 366678 (358K) [application/octet-stream]
```

```
Saving to: ‘/tmp/pybind11-install.tar.gz’

/tmp/pybind11-insta 100%[===================>] 358.08K  --.-KB/s    in 0.02s

2022-03-30 12:14:17 (18.6 MB/s) - ‘/tmp/pybind11-install.tar.gz’ saved
[366678/366678]

++ [[ /tmp/pybind11-install.tar.gz != skip ]]
++ tar -xzf /tmp/pybind11-install.tar.gz --strip-components=2
--directory=/usr/local
++ mkdir -p /usr/local/share/fem-on-colab
++ touch /usr/local/share/fem-on-colab/pybind11.installed
+ BOOST_INSTALL_SCRIPT_PATH=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/c88365b/releases/boost-install.sh
+ [[ https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/c88365b/releases/boost-install.sh == http* ]]
+ BOOST_INSTALL_SCRIPT_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/c88365b/releases/boost-install.sh
+ BOOST_INSTALL_SCRIPT_PATH=/tmp/boost-install.sh
+ [[ ! -f /tmp/boost-install.sh ]]
+ wget https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/c88365b/releases/boost-install.sh -O /tmp/boost-install.sh
--2022-03-30 12:14:17--  https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/c88365b/releases/boost-install.sh
Resolving github.com (github.com)… 140.82.112.4
Connecting to github.com (github.com)|140.82.112.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/c88365b41d0f8b9f5230bf8431d1c5e13f2c6497/releases/boost-
install.sh [following]
--2022-03-30 12:14:17--  https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/c88365b41d0f8b9f5230bf8431d1c5e13f2c6497/releases/boost-
install.sh
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.108.133, 185.199.110.133, 185.199.111.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 1681 (1.6K) [text/plain]
Saving to: ‘/tmp/boost-install.sh’

/tmp/boost-install. 100%[===================>]   1.64K  --.-KB/s    in 0s

2022-03-30 12:14:17 (13.8 MB/s) - ‘/tmp/boost-install.sh’ saved [1681/1681]

+ source /tmp/boost-install.sh
++ set -e
++ set -x
```

```
++ SHARE_PREFIX=/usr/local/share/fem-on-colab
++ BOOST_INSTALLED=/usr/local/share/fem-on-colab/boost.installed
++ [[ ! -f /usr/local/share/fem-on-colab/boost.installed ]]
++ GCC_INSTALL_SCRIPT_PATH=/tmp/gcc-install.sh
++ [[ /tmp/gcc-install.sh == http* ]]
++ source /tmp/gcc-install.sh
+++ set -e
+++ set -x
+++ SHARE_PREFIX=/usr/local/share/fem-on-colab
+++ GCC_INSTALLED=/usr/local/share/fem-on-colab/gcc.installed
+++ [[ ! -f /usr/local/share/fem-on-colab/gcc.installed ]]
++ apt install -y -qq zlib1g-dev
zlib1g-dev is already the newest version (1:1.2.11.dfsg-0ubuntu2).
zlib1g-dev set to manually installed.
0 upgraded, 0 newly installed, 0 to remove and 82 not upgraded.
++ BOOST_ARCHIVE_PATH=https://github.com/fem-on-colab/fem-on-
colab/releases/download/boost-20220216-073720-8f2b82c/boost-install.tar.gz
++ [[ https://github.com/fem-on-colab/fem-on-
colab/releases/download/boost-20220216-073720-8f2b82c/boost-install.tar.gz ==
http* ]]
++ BOOST_ARCHIVE_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab/releases/download/boost-20220216-073720-8f2b82c/boost-install.tar.gz
++ BOOST_ARCHIVE_PATH=/tmp/boost-install.tar.gz
++ wget https://github.com/fem-on-colab/fem-on-
colab/releases/download/boost-20220216-073720-8f2b82c/boost-install.tar.gz -O
/tmp/boost-install.tar.gz
--2022-03-30 12:14:19--  https://github.com/fem-on-colab/fem-on-
colab/releases/download/boost-20220216-073720-8f2b82c/boost-install.tar.gz
Resolving github.com (github.com)… 140.82.112.4
Connecting to github.com (github.com)|140.82.112.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://objects.githubusercontent.com/github-production-release-
asset-2e65be/370599515/03ac9222-0aad-48a5-8b01-62ed555f52ea?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121419Z&X-Amz-Expires=300&X-Amz-S
ignature=b41ae489e0d3842685d9058ad4a70d20f57edc951cd8b1c7dce154e7c0690bc9&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dboost-install.tar.gz&response-content-
type=application%2Foctet-stream [following]
--2022-03-30 12:14:19--  https://objects.githubusercontent.com/github-
production-release-
asset-2e65be/370599515/03ac9222-0aad-48a5-8b01-62ed555f52ea?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121419Z&X-Amz-Expires=300&X-Amz-S
ignature=b41ae489e0d3842685d9058ad4a70d20f57edc951cd8b1c7dce154e7c0690bc9&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
```

```
disposition=attachment%3B%20filename%3Dboost-install.tar.gz&response-content-
type=application%2Foctet-stream
Resolving objects.githubusercontent.com (objects.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.111.133, …
Connecting to objects.githubusercontent.com
(objects.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 33816088 (32M) [application/octet-stream]
Saving to: ‘/tmp/boost-install.tar.gz’

/tmp/boost-install. 100%[===================>]  32.25M  26.1MB/s    in 1.2s

2022-03-30 12:14:21 (26.1 MB/s) - ‘/tmp/boost-install.tar.gz’ saved
[33816088/33816088]

++ [[ /tmp/boost-install.tar.gz != skip ]]
++ tar -xzf /tmp/boost-install.tar.gz --strip-components=2
--directory=/usr/local
++ [[ /tmp/boost-install.tar.gz != skip ]]
++ ln -fs /usr/local/lib/libboost_atomic.so
/usr/local/lib/libboost_atomic.so.1.78.0 /usr/local/lib/libboost_chrono.so
/usr/local/lib/libboost_chrono.so.1.78.0 /usr/local/lib/libboost_container.so
/usr/local/lib/libboost_container.so.1.78.0 /usr/local/lib/libboost_context.so
/usr/local/lib/libboost_context.so.1.78.0 /usr/local/lib/libboost_contract.so
/usr/local/lib/libboost_contract.so.1.78.0 /usr/local/lib/libboost_coroutine.so
/usr/local/lib/libboost_coroutine.so.1.78.0 /usr/local/lib/libboost_date_time.so
/usr/local/lib/libboost_date_time.so.1.78.0
/usr/local/lib/libboost_filesystem.so
/usr/local/lib/libboost_filesystem.so.1.78.0 /usr/local/lib/libboost_graph.so
/usr/local/lib/libboost_graph.so.1.78.0 /usr/local/lib/libboost_iostreams.so
/usr/local/lib/libboost_iostreams.so.1.78.0 /usr/local/lib/libboost_json.a
/usr/local/lib/libboost_json.so /usr/local/lib/libboost_json.so.1.78.0
/usr/local/lib/libboost_locale.so /usr/local/lib/libboost_locale.so.1.78.0
/usr/local/lib/libboost_log_setup.so /usr/local/lib/libboost_log_setup.so.1.78.0
/usr/local/lib/libboost_log.so /usr/local/lib/libboost_log.so.1.78.0
/usr/local/lib/libboost_math_c99f.so /usr/local/lib/libboost_math_c99f.so.1.78.0
/usr/local/lib/libboost_math_c99l.so /usr/local/lib/libboost_math_c99l.so.1.78.0
/usr/local/lib/libboost_math_c99.so /usr/local/lib/libboost_math_c99.so.1.78.0
/usr/local/lib/libboost_math_tr1f.so /usr/local/lib/libboost_math_tr1f.so.1.78.0
/usr/local/lib/libboost_math_tr1l.so /usr/local/lib/libboost_math_tr1l.so.1.78.0
/usr/local/lib/libboost_math_tr1.so /usr/local/lib/libboost_math_tr1.so.1.78.0
/usr/local/lib/libboost_nowide.so /usr/local/lib/libboost_nowide.so.1.78.0
/usr/local/lib/libboost_prg_exec_monitor.so
/usr/local/lib/libboost_prg_exec_monitor.so.1.78.0
/usr/local/lib/libboost_program_options.so
/usr/local/lib/libboost_program_options.so.1.78.0
/usr/local/lib/libboost_random.so /usr/local/lib/libboost_random.so.1.78.0
/usr/local/lib/libboost_regex.so /usr/local/lib/libboost_regex.so.1.78.0
```

```
/usr/local/lib/libboost_serialization.so
/usr/local/lib/libboost_serialization.so.1.78.0
/usr/local/lib/libboost_system.so /usr/local/lib/libboost_system.so.1.78.0
/usr/local/lib/libboost_thread.so /usr/local/lib/libboost_thread.so.1.78.0
/usr/local/lib/libboost_timer.so /usr/local/lib/libboost_timer.so.1.78.0
/usr/local/lib/libboost_type_erasure.so
/usr/local/lib/libboost_type_erasure.so.1.78.0
/usr/local/lib/libboost_unit_test_framework.so
/usr/local/lib/libboost_unit_test_framework.so.1.78.0
/usr/local/lib/libboost_wave.so /usr/local/lib/libboost_wave.so.1.78.0
/usr/local/lib/libboost_wserialization.so
/usr/local/lib/libboost_wserialization.so.1.78.0 /usr/lib
++ mkdir -p /usr/local/share/fem-on-colab
++ touch /usr/local/share/fem-on-colab/boost.installed
+ SLEPC4PY_INSTALL_SCRIPT_PATH=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/9ad8c8d/releases/slepc4py-install-real.sh
+ [[ https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/9ad8c8d/releases/slepc4py-install-real.sh == http* ]]
+ SLEPC4PY_INSTALL_SCRIPT_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/9ad8c8d/releases/slepc4py-install-real.sh
+ SLEPC4PY_INSTALL_SCRIPT_PATH=/tmp/slepc4py-install.sh
+ [[ ! -f /tmp/slepc4py-install.sh ]]
+ wget https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/9ad8c8d/releases/slepc4py-install-real.sh -O /tmp/slepc4py-
install.sh
--2022-03-30 12:14:23--  https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/9ad8c8d/releases/slepc4py-install-real.sh
Resolving github.com (github.com)… 140.82.114.4
Connecting to github.com (github.com)|140.82.114.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/9ad8c8d3ba902933816362120e7b7f0b353c6a3f/releases/slepc4py-
install-real.sh [following]
--2022-03-30 12:14:24--  https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/9ad8c8d3ba902933816362120e7b7f0b353c6a3f/releases/slepc4py-
install-real.sh
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.111.133, 185.199.110.133, 185.199.109.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.111.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 1546 (1.5K) [text/plain]
Saving to: ‘/tmp/slepc4py-install.sh’

/tmp/slepc4py-insta 100%[==================>]   1.51K  --.-KB/s    in 0s

2022-03-30 12:14:24 (16.8 MB/s) - ‘/tmp/slepc4py-install.sh’ saved [1546/1546]
```

```
+ source /tmp/slepc4py-install.sh
++ set -e
++ set -x
++ SHARE_PREFIX=/usr/local/share/fem-on-colab
++ SLEPC4PY_INSTALLED=/usr/local/share/fem-on-colab/slepc4py.installed
++ [[ ! -f /usr/local/share/fem-on-colab/slepc4py.installed ]]
++ PETSC4PY_INSTALL_SCRIPT_PATH=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/305465a/releases/petsc4py-install-real.sh
++ [[ https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/305465a/releases/petsc4py-install-real.sh == http* ]]
++ PETSC4PY_INSTALL_SCRIPT_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/305465a/releases/petsc4py-install-real.sh
++ PETSC4PY_INSTALL_SCRIPT_PATH=/tmp/petsc4py-install.sh
++ [[ ! -f /tmp/petsc4py-install.sh ]]
++ wget https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/305465a/releases/petsc4py-install-real.sh -O /tmp/petsc4py-
install.sh
--2022-03-30 12:14:24--  https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/305465a/releases/petsc4py-install-real.sh
Resolving github.com (github.com)… 140.82.114.4
Connecting to github.com (github.com)|140.82.114.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/305465a7c079fde491f51dca11a245166fd44086/releases/petsc4py-
install-real.sh [following]
--2022-03-30 12:14:24--  https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/305465a7c079fde491f51dca11a245166fd44086/releases/petsc4py-
install-real.sh
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.109.133, 185.199.111.133, 185.199.108.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.109.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 1598 (1.6K) [text/plain]
Saving to: ‘/tmp/petsc4py-install.sh’

/tmp/petsc4py-insta 100%[===================>]   1.56K  --.-KB/s    in 0s

2022-03-30 12:14:25 (22.0 MB/s) - ‘/tmp/petsc4py-install.sh’ saved [1598/1598]

++ source /tmp/petsc4py-install.sh
+++ set -e
+++ set -x
+++ SHARE_PREFIX=/usr/local/share/fem-on-colab
+++ PETSC4PY_INSTALLED=/usr/local/share/fem-on-colab/petsc4py.installed
+++ [[ ! -f /usr/local/share/fem-on-colab/petsc4py.installed ]]
+++ H5PY_INSTALL_SCRIPT_PATH=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/eb9a171/releases/h5py-install.sh
```

```
+++ [[ https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/eb9a171/releases/h5py-install.sh == http* ]]
+++ H5PY_INSTALL_SCRIPT_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/eb9a171/releases/h5py-install.sh
+++ H5PY_INSTALL_SCRIPT_PATH=/tmp/h5py-install.sh
+++ [[ ! -f /tmp/h5py-install.sh ]]
+++ wget https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/eb9a171/releases/h5py-install.sh -O /tmp/h5py-install.sh
--2022-03-30 12:14:25--  https://github.com/fem-on-colab/fem-on-
colab.github.io/raw/eb9a171/releases/h5py-install.sh
Resolving github.com (github.com)… 140.82.114.4
Connecting to github.com (github.com)|140.82.114.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/eb9a1712c4dbd4b8c4a00d297140f75968c58d8d/releases/h5py-
install.sh [following]
--2022-03-30 12:14:25--  https://raw.githubusercontent.com/fem-on-colab/fem-on-
colab.github.io/eb9a1712c4dbd4b8c4a00d297140f75968c58d8d/releases/h5py-
install.sh
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.110.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 1533 (1.5K) [text/plain]
Saving to: ‘/tmp/h5py-install.sh’

/tmp/h5py-install.s 100%[==================>]   1.50K  --.-KB/s    in 0s

2022-03-30 12:14:25 (12.8 MB/s) - ‘/tmp/h5py-install.sh’ saved [1533/1533]

+++ source /tmp/h5py-install.sh
++++ set -e
++++ set -x
++++ SHARE_PREFIX=/usr/local/share/fem-on-colab
++++ H5PY_INSTALLED=/usr/local/share/fem-on-colab/h5py.installed
++++ [[ ! -f /usr/local/share/fem-on-colab/h5py.installed ]]
++++ MPI4PY_INSTALL_SCRIPT_PATH=/tmp/mpi4py-install.sh
++++ [[ /tmp/mpi4py-install.sh == http* ]]
++++ source /tmp/mpi4py-install.sh
+++++ set -e
+++++ set -x
+++++ SHARE_PREFIX=/usr/local/share/fem-on-colab
+++++ MPI4PY_INSTALLED=/usr/local/share/fem-on-colab/mpi4py.installed
+++++ [[ ! -f /usr/local/share/fem-on-colab/mpi4py.installed ]]
++++ apt install -y -qq zlib1g-dev
zlib1g-dev is already the newest version (1:1.2.11.dfsg-0ubuntu2).
0 upgraded, 0 newly installed, 0 to remove and 82 not upgraded.
```

```
++++ H5PY_ARCHIVE_PATH=https://github.com/fem-on-colab/fem-on-
colab/releases/download/h5py-20220216-075632-8f2b82c/h5py-install.tar.gz
++++ [[ https://github.com/fem-on-colab/fem-on-
colab/releases/download/h5py-20220216-075632-8f2b82c/h5py-install.tar.gz ==
http* ]]
++++ H5PY_ARCHIVE_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab/releases/download/h5py-20220216-075632-8f2b82c/h5py-install.tar.gz
++++ H5PY_ARCHIVE_PATH=/tmp/h5py-install.tar.gz
++++ wget https://github.com/fem-on-colab/fem-on-
colab/releases/download/h5py-20220216-075632-8f2b82c/h5py-install.tar.gz -O
/tmp/h5py-install.tar.gz
--2022-03-30 12:14:27--  https://github.com/fem-on-colab/fem-on-
colab/releases/download/h5py-20220216-075632-8f2b82c/h5py-install.tar.gz
Resolving github.com (github.com)… 140.82.114.4
Connecting to github.com (github.com)|140.82.114.4|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://objects.githubusercontent.com/github-production-release-
asset-2e65be/370599515/058d40f6-0168-4b26-b68d-22239394e67d?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121428Z&X-Amz-Expires=300&X-Amz-S
ignature=f8e73c7106e82f249f87b7e3d3a5a08bedd2453c1a74ab9ed57f723d65213d73&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dh5py-install.tar.gz&response-content-
type=application%2Foctet-stream [following]
--2022-03-30 12:14:28--  https://objects.githubusercontent.com/github-
production-release-
asset-2e65be/370599515/058d40f6-0168-4b26-b68d-22239394e67d?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121428Z&X-Amz-Expires=300&X-Amz-S
ignature=f8e73c7106e82f249f87b7e3d3a5a08bedd2453c1a74ab9ed57f723d65213d73&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dh5py-install.tar.gz&response-content-
type=application%2Foctet-stream
Resolving objects.githubusercontent.com (objects.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.111.133, …
Connecting to objects.githubusercontent.com
(objects.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 11735509 (11M) [application/octet-stream]
Saving to: ‘/tmp/h5py-install.tar.gz’

/tmp/h5py-install.t 100%[==================>]  11.19M  47.8MB/s    in 0.2s

2022-03-30 12:14:28 (47.8 MB/s) - ‘/tmp/h5py-install.tar.gz’ saved
[11735509/11735509]
```

```
++++ [[ /tmp/h5py-install.tar.gz != skip ]]
++++ rm -rf /usr/local/lib/python3.7/dist-packages/h5py
/usr/local/lib/python3.7/dist-packages/h5py-3.1.0.dist-info
/usr/local/lib/python3.7/dist-packages/h5py.libs
++++ tar -xzf /tmp/h5py-install.tar.gz --strip-components=2
--directory=/usr/local
++++ mkdir -p /usr/local/share/fem-on-colab
++++ touch /usr/local/share/fem-on-colab/h5py.installed
+++ apt install -y -qq libblas-dev liblapack-dev
libblas-dev is already the newest version (3.7.1-4ubuntu1).
libblas-dev set to manually installed.
liblapack-dev is already the newest version (3.7.1-4ubuntu1).
0 upgraded, 0 newly installed, 0 to remove and 82 not upgraded.
+++ PETSC4PY_ARCHIVE_PATH=https://github.com/fem-on-colab/fem-on-
colab/releases/download/petsc4py-20220216-081056-8f2b82c-real/petsc4py-
install.tar.gz
+++ [[ https://github.com/fem-on-colab/fem-on-
colab/releases/download/petsc4py-20220216-081056-8f2b82c-real/petsc4py-
install.tar.gz == http* ]]
+++ PETSC4PY_ARCHIVE_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab/releases/download/petsc4py-20220216-081056-8f2b82c-real/petsc4py-
install.tar.gz
+++ PETSC4PY_ARCHIVE_PATH=/tmp/petsc4py-install.tar.gz
+++ wget https://github.com/fem-on-colab/fem-on-
colab/releases/download/petsc4py-20220216-081056-8f2b82c-real/petsc4py-
install.tar.gz -O /tmp/petsc4py-install.tar.gz
--2022-03-30 12:14:31--  https://github.com/fem-on-colab/fem-on-
colab/releases/download/petsc4py-20220216-081056-8f2b82c-real/petsc4py-
install.tar.gz
Resolving github.com (github.com)… 140.82.114.3
Connecting to github.com (github.com)|140.82.114.3|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://objects.githubusercontent.com/github-production-release-
asset-2e65be/370599515/6427a65a-2ba7-40c2-bf92-f2a9b8804008?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121431Z&X-Amz-Expires=300&X-Amz-S
ignature=2be237b063c186091ade9f599643a446ac1a457df24c7e2eed355baf71c365d6&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dpetsc4py-install.tar.gz&response-content-
type=application%2Foctet-stream [following]
--2022-03-30 12:14:31--  https://objects.githubusercontent.com/github-
production-release-
asset-2e65be/370599515/6427a65a-2ba7-40c2-bf92-f2a9b8804008?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121431Z&X-Amz-Expires=300&X-Amz-S
ignature=2be237b063c186091ade9f599643a446ac1a457df24c7e2eed355baf71c365d6&X-Amz-
```

```
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dpetsc4py-install.tar.gz&response-content-
type=application%2Foctet-stream
Resolving objects.githubusercontent.com (objects.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.111.133, …
Connecting to objects.githubusercontent.com
(objects.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 153425745 (146M) [application/octet-stream]
Saving to: ‘/tmp/petsc4py-install.tar.gz’

/tmp/petsc4py-insta 100%[===================>] 146.32M  37.6MB/s    in 3.9s

2022-03-30 12:14:35 (37.6 MB/s) - ‘/tmp/petsc4py-install.tar.gz’ saved
[153425745/153425745]

+++ [[ /tmp/petsc4py-install.tar.gz != skip ]]
+++ tar -xzf /tmp/petsc4py-install.tar.gz --strip-components=2
--directory=/usr/local
+++ mkdir -p /usr/local/share/fem-on-colab
+++ touch /usr/local/share/fem-on-colab/petsc4py.installed
++ SLEPC4PY_ARCHIVE_PATH=https://github.com/fem-on-colab/fem-on-
colab/releases/download/slepc4py-20220216-085519-8f2b82c-real/slepc4py-
install.tar.gz
++ [[ https://github.com/fem-on-colab/fem-on-
colab/releases/download/slepc4py-20220216-085519-8f2b82c-real/slepc4py-
install.tar.gz == http* ]]
++ SLEPC4PY_ARCHIVE_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab/releases/download/slepc4py-20220216-085519-8f2b82c-real/slepc4py-
install.tar.gz
++ SLEPC4PY_ARCHIVE_PATH=/tmp/slepc4py-install.tar.gz
++ wget https://github.com/fem-on-colab/fem-on-
colab/releases/download/slepc4py-20220216-085519-8f2b82c-real/slepc4py-
install.tar.gz -O /tmp/slepc4py-install.tar.gz
--2022-03-30 12:14:42--  https://github.com/fem-on-colab/fem-on-
colab/releases/download/slepc4py-20220216-085519-8f2b82c-real/slepc4py-
install.tar.gz
Resolving github.com (github.com)… 140.82.114.3
Connecting to github.com (github.com)|140.82.114.3|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://objects.githubusercontent.com/github-production-release-
asset-2e65be/370599515/87d6f807-5f78-401e-b82f-64fd3b16e00d?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121442Z&X-Amz-Expires=300&X-Amz-S
ignature=4f0c82c48ea3fac38052131e2826effe0b2c975c6180319c59fcc8be06e36c27&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dslepc4py-install.tar.gz&response-content-
```

```
type=application%2Foctet-stream [following]
--2022-03-30 12:14:42--  https://objects.githubusercontent.com/github-
production-release-
asset-2e65be/370599515/87d6f807-5f78-401e-b82f-64fd3b16e00d?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121442Z&X-Amz-Expires=300&X-Amz-S
ignature=4f0c82c48ea3fac38052131e2826effe0b2c975c6180319c59fcc8be06e36c27&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dslepc4py-install.tar.gz&response-content-
type=application%2Foctet-stream
Resolving objects.githubusercontent.com (objects.githubusercontent.com)…
185.199.108.133, 185.199.110.133, 185.199.111.133, …
Connecting to objects.githubusercontent.com
(objects.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 12638215 (12M) [application/octet-stream]
Saving to: ‘/tmp/slepc4py-install.tar.gz’

/tmp/slepc4py-insta 100%[===================>]  12.05M  47.2MB/s    in 0.3s

2022-03-30 12:14:43 (47.2 MB/s) - ‘/tmp/slepc4py-install.tar.gz’ saved
[12638215/12638215]

++ [[ /tmp/slepc4py-install.tar.gz != skip ]]
++ tar -xzf /tmp/slepc4py-install.tar.gz --strip-components=2
--directory=/usr/local
++ mkdir -p /usr/local/share/fem-on-colab
++ touch /usr/local/share/fem-on-colab/slepc4py.installed
+ FENICS_ARCHIVE_PATH=https://github.com/fem-on-colab/fem-on-
colab/releases/download/fenics-20220216-155654-c357fcf/fenics-install.tar.gz
+ [[ https://github.com/fem-on-colab/fem-on-
colab/releases/download/fenics-20220216-155654-c357fcf/fenics-install.tar.gz ==
http* ]]
+ FENICS_ARCHIVE_DOWNLOAD=https://github.com/fem-on-colab/fem-on-
colab/releases/download/fenics-20220216-155654-c357fcf/fenics-install.tar.gz
+ FENICS_ARCHIVE_PATH=/tmp/fenics-install.tar.gz
+ wget https://github.com/fem-on-colab/fem-on-
colab/releases/download/fenics-20220216-155654-c357fcf/fenics-install.tar.gz -O
/tmp/fenics-install.tar.gz
--2022-03-30 12:14:43--  https://github.com/fem-on-colab/fem-on-
colab/releases/download/fenics-20220216-155654-c357fcf/fenics-install.tar.gz
Resolving github.com (github.com)… 140.82.114.3
Connecting to github.com (github.com)|140.82.114.3|:443… connected.
HTTP request sent, awaiting response… 302 Found
Location: https://objects.githubusercontent.com/github-production-release-
asset-2e65be/370599515/192bc12b-a792-4d92-b4c8-5fbcfce4252e?X-Amz-
Algorithm=AWS4-HMAC-SHA256&X-Amz-
```

```
Credential=AKIAIWNJYAX4CSVEH53A%2F20220330%2Fus-
east-1%2Fs3%2Faws4_request&X-Amz-Date=20220330T121443Z&X-Amz-Expires=300&X-Amz-S
ignature=5198684b264c5e1c2a1ba9332718933342baf039fd8292fb3b4e5d3dc1bfa97f&X-Amz-
SignedHeaders=host&actor_id=0&key_id=0&repo_id=370599515&response-content-
disposition=attachment%3B%20filename%3Dfenics-install.tar.gz&response-content-
type=application%2Foctet-stream [following]
--2022-03-30 12:14:43--  https://objects.githubusercontent.com/github-
production-release-
asset-2e65be/370599515/192bc12b-a792-4d92-b4c8-5fbcfce4252e?X-Amz-
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disposition=attachment%3B%20filename%3Dfenics-install.tar.gz&response-content-
type=application%2Foctet-stream
Resolving objects.githubusercontent.com (objects.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.110.133, …
Connecting to objects.githubusercontent.com
(objects.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 86526374 (83M) [application/octet-stream]
Saving to: '/tmp/fenics-install.tar.gz'

/tmp/fenics-install 100%[===================>]  82.52M  37.6MB/s    in 2.2s

2022-03-30 12:14:46 (37.6 MB/s) - '/tmp/fenics-install.tar.gz' saved
[86526374/86526374]

+ [[ /tmp/fenics-install.tar.gz != skip ]]
+ tar -xzf /tmp/fenics-install.tar.gz --strip-components=2
--directory=/usr/local
+ [[ /tmp/fenics-install.tar.gz != skip ]]
+ ln -fs /usr/local/lib/libdolfin.so /usr/local/lib/libdolfin.so.2019.2
/usr/local/lib/libdolfin.so.2019.2.0.dev0 /usr/lib
+ ln -fs /usr/local/lib/libmshr.so /usr/local/lib/libmshr.so.2019.2
/usr/local/lib/libmshr.so.2019.2.0.dev0 /usr/lib
+ mkdir -p /usr/local/share/fem-on-colab
+ touch /usr/local/share/fem-on-colab/fenics.installed
```

[ ]:
```
from fenics import *
import numpy as np
import matplotlib.pyplot as plt
import math
```

1.2 The UFL / FEniCS way of solving the heat equation

1.2.1 Strong formulation of heat equation

First, we briefly recall how one generally goes about solving the heat equation using FEniCS. A core feature of FEniCS is its support of the Unified Form Language (UFL) which enables its users to solve partial differential equations (PDEs) by specifying the function space of the FEM solution and the variational formulation of the PDE.

The strong formulation of the heat equation reads: \ Find $u : [0,T] \times \bar{\Omega} \to \mathbb{R}$ such that

$$\partial_t u - \Delta_x u = f \quad \text{in} \quad (0,T)\times\Omega, \qquad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \qquad u = u_0 \quad \text{on} \quad \{0\}\times\Omega,$$

where $\Omega \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$ is an open domain and the Laplacian of u, denoted by $\Delta_x u$, is defined as

$$\Delta_x u = \sum_{i=1}^{d} \partial_{x_i} u.$$

1.2.2 Temporal discretization: One-Step-θ scheme

To solve this time dependent PDE, we first discretize it in time with finite differences and then use finite elements in space. For the temporal discretization, we use the One-Step-θ scheme, which for the model problem

$$\partial_t u = g(u)$$

reads

$$\frac{u(t_{n+1}) - u(t_n)}{t_{n+1} - t_n} = \theta \cdot g(u(t_{n+1})) + (1-\theta)\cdot g(u(t_n)).$$

In this tutorial, we denote the constant time step size as $\Delta t := t_{n+1} - t_n$ and use the short hand notation $u^{n+1} := u(t_{n+1})$ and $u^n := u(t_n)$. For the heat equation, we have $g(u) := \Delta_x u + f$ and thus its time-discrete strong formulation reads

$$\frac{u^{n+1} - u^n}{\Delta t} = \theta \cdot (\Delta_x u^{n+1} + f^{n+1}) + (1-\theta)\cdot(\Delta_x u^n + f^n).$$

We now reorder this formulation such that after u^n has been computed all implicit terms are on the left side and all the explicit terms are on the right side:

$$u^{n+1} - \theta\cdot\Delta t\cdot\Delta_x u^{n+1} = u^n + (1-\theta)\cdot\Delta t\cdot\Delta_x u^n + \theta\cdot\Delta t\cdot f^{n+1} + (1-\theta)\cdot\Delta t\cdot f^n$$

1.2.3 Variational formulation of heat equation

Now that the heat equation has already been discretized in time, we discretize it in space with finite elements. For this we need the theory of Sobolev spaces. The time discrete formulation doesn't need to be fulfilled in the strong sense anymore, but only in a variational sense. This means that we multiply the strong formulation with a test function and integrate it over the domain Ω. For this we introduce the space $V := H_0^1(\Omega)$ as the space of functions that vanish at $\partial\Omega$ whose weakly derivattives are $L^2(\Omega)$-integrable. After testing with a function $v \in V$, we get

$$\int_\Omega (u^{n+1} - \theta\cdot\Delta t\cdot\Delta_x u^{n+1})\cdot v \,\mathrm{d}(x,y) = \int_\Omega (u^n + (1-\theta)\cdot\Delta t\cdot\Delta_x u^n + \theta\cdot\Delta t\cdot f^{n+1} + (1-\theta)\cdot\Delta t\cdot f^n)\cdot v\,\mathrm{d}(x,y).$$

We can apply the integration by parts formula

$$\int_\Omega -\Delta_x u \cdot v \, \mathrm{d}(x,y) = \int_\Omega \nabla_x u \cdot \nabla_x v \, \mathrm{d}(x,y) - \int_{\partial\Omega} \nabla_x u \cdot v \cdot \vec{n} \, \mathrm{d}s,$$

where the integral over $\partial\Omega$ equals 0, since $v = 0$ on $\partial\Omega$ due to $v \in V = H_0^1(\Omega)$.

The variational formulation of the heat equation thus reads: \ Find $u^{n+1} \in V$ such that

$$a(u^{n+1}, v) = L(v) \qquad \forall v \in V,$$

where

$$a(u,v) := \int_\Omega u \cdot v \, \mathrm{d}(x,y) + \theta \cdot \Delta t \cdot \int_\Omega \nabla_x u \cdot \nabla_x v \, \mathrm{d}(x,y), \tag{1}$$

$$L(v) := \int_\Omega u^n \cdot v \, \mathrm{d}(x,y) - (1-\theta) \cdot \Delta t \cdot \int_\Omega \nabla_x u^n \cdot \nabla_x v \, \mathrm{d}(x,y) + \theta \cdot \Delta t \cdot \int_\Omega f^{n+1} \cdot v \, \mathrm{d}(x,y) + (1-\theta) \cdot \Delta t \cdot \int_\Omega f^n \cdot v \, \mathrm{d}(x,y). \tag{2}$$

1.2.4 Model problem for the heat equation

We choose a model problem of the heat equation that has been proposed in the deal.II tutorials. \
Find $u : [0,T] \times [0,1]^2 \to \mathbb{R}$ such that

$$\partial_t u - \Delta_x u = 0 \quad \text{in} \quad (0,T) \times (0,1)^2, \qquad u = 0 \quad \text{on} \quad (0,T) \times [0,1]^2 \, (0,1)^2, \qquad u(x,y) = \text{si}$$

For this PDE the solution is given by

$$u(t,x,y) = \exp(-2\pi^2 t) \sin(\pi x) \sin(\pi y).$$

1.2.5 Code

First, we specify the parameters of the time discretization:

- start time: t
- end time: T
- time step size: Δt
- θ

Famous choices are

- $\theta = 0$: forward Euler scheme
- $\theta = 1$: backward Euler scheme
- $\theta = \frac{1}{2}$: Crank-Nicholson scheme

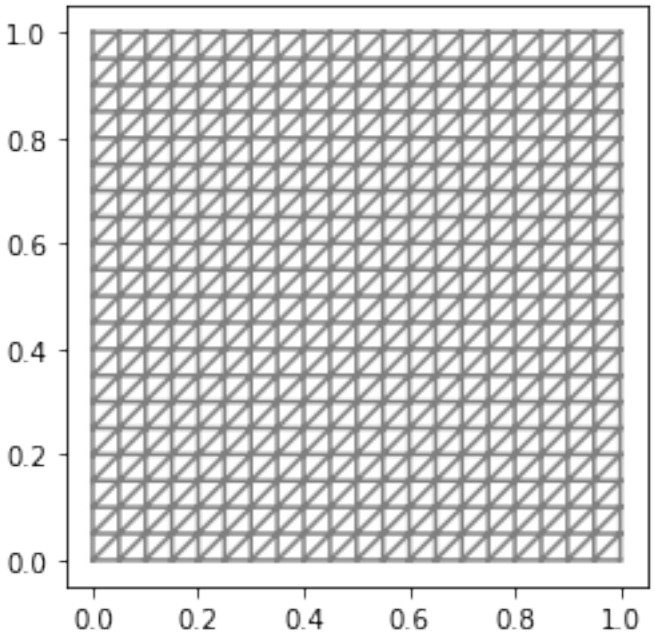
and we choose the Crank-Nicholson scheme, i.e. $\theta = 0.5$.

[ ]:
```
# start time
t = 0.
# end time
T = 0.5
# time step size
Δt = 0.01
# one-step theta
```

```
 = 0.5
```

Next, we create a triangulation of the domain $\Omega = (0,1) \times (0,1)$.

[ ]:
```
nx = ny = 20
mesh = UnitSquareMesh(nx, ny)
# visualize the triangulation
plot(mesh)
plt.show()
```



Then, we define on this mesh the space of bilinear finite elements and create homogeneous Dirichlet boundary conditions on the entire boundary.

[ ]:
```
V = FunctionSpace(mesh, 'P', 1)
bc = DirichletBC(V, Constant(0.), lambda _, on_boundary: on_boundary)
```

```
Calling FFC just-in-time (JIT) compiler, this may take some time.
```

We now define the expression for the initial condition as u_0 and interpolate it in the function space V. Here we will use the variable u_n as the solution from the last timestep.

[ ]:
```
# initial condition
u_0 = Expression('sin(pi*x[0])*sin(pi*x[1])', degree=2, pi=math.pi)
# u_n: solution from last time step
u_n = interpolate(u_0, V)
```

With all these preliminaries out of the way, we can now define the variational problem (1) and (2) using UFL.

```
[ ]: # variational problem
u = TrialFunction(V)
v = TestFunction(V)

a = u*v*dx + *Δt*dot(grad(u), grad(v))*dx
L = u_n*v*dx - (1.- )*Δt*dot(grad(u_n), grad(v))*dx
```

Finally, we come to the time-stepping loop itself. For this we create the function u, which represents the solution at the current time step, i.e. $u := u_{n+1}$. In the loop itself, we than compute t_{n+1} as $t_n + \Delta t$. Afterwards we solve the variational problem $a(u_{n+1}, v) = L(v)$. Having computed the solution, we can verify its correctness, since we know that the analytical solution is given as

$$u(t, x, y) = \exp(-2\pi^2 t) \sin(\pi x) \sin(\pi y).$$

A common choice is to ensure that the $L^2((0, T), L^2(\Omega))$ norm of $u - u_h$ converges to 0 and the convergence rate should depend on the type of spatio-temporal discretization. However, in this tutorial, we go a different route and want to verify that $J(u(t, \cdot, \cdot)) - J(u_h(t, \cdot, \cdot))$ converges to 0 for $\Delta t, h \to 0$, where $J(\cdot)$ is a quantity of interest that we are monitoring. For our model problem, we decide that we are interested in evaluating the solution at the center of the domain, i.e.

$$J(\tilde{u}(t, \cdot, \cdot)) := \tilde{u}\left(t, \frac{1}{2}, \frac{1}{2}\right)$$

and we have for the analytical solution u that

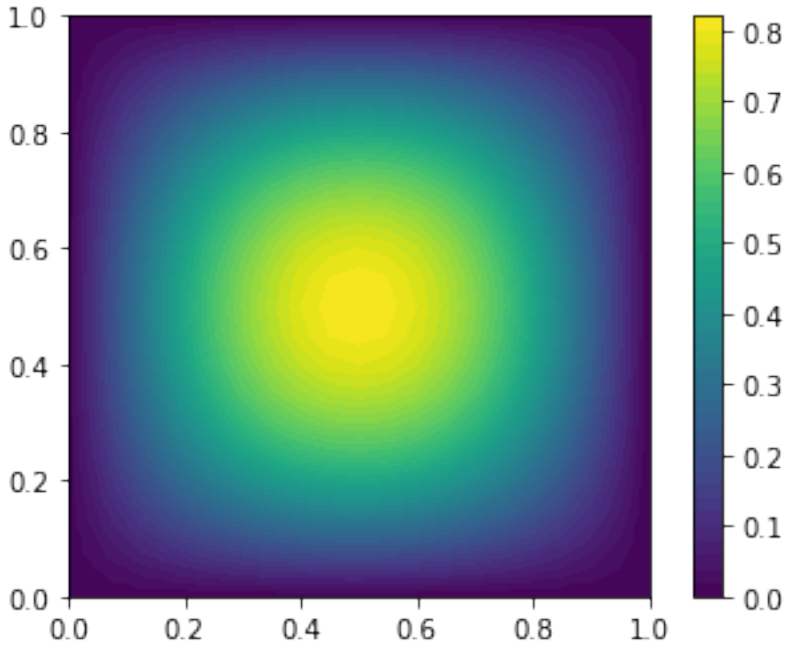
$$J(u(t, \cdot, \cdot)) = u\left(t, \frac{1}{2}, \frac{1}{2}\right) = \exp(-2\pi^2 t) \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) = \exp(-2\pi^2 t).$$

Having compared the difference between the analytical solution u and its FEM approximation u_h in the goal functional $J(\cdot)$, we plot the FEM solution and finally update u_n for the computations in the next time step.

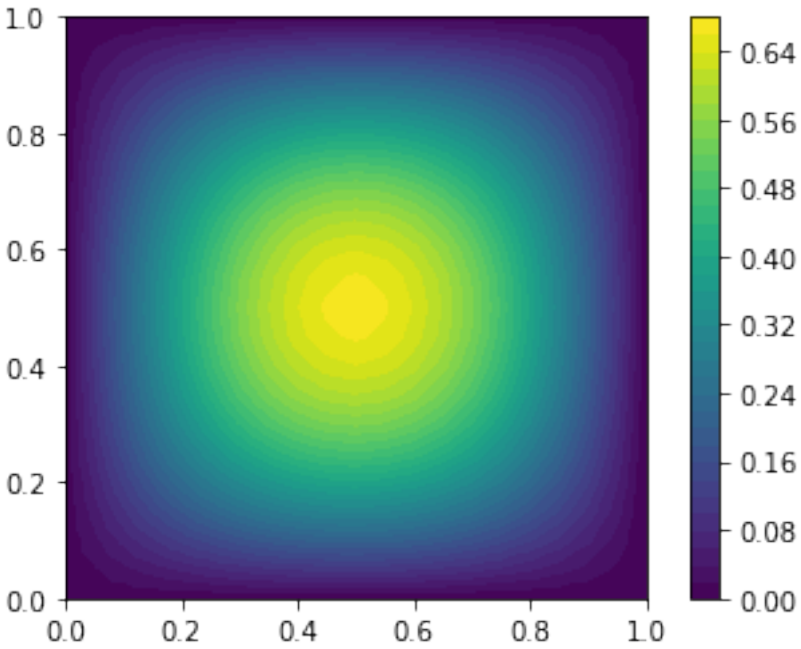
```
[ ]: # Time-stepping
u = Function(V) # u = u_{n+1}: current solution

while(t+Δt <= T+1e-8):
    # Update current time
    t += Δt

    # Compute solution
    solve(a == L, u, bc)

    # Print results for quantity of interest J(u) := u(0.5,0.5)
    print(f"t = {round(t,4)}:")
    print(f"  u_h(0.5,0.5) = {round(u(Point(0.5, 0.5)),6)}")
    print(f"  u(0.5,0.5)   = {round(math.exp(-2 * math.pi**2 * t),6)}")

    # Plot solution
    c = plot(u)
    plt.colorbar(c)
```

```
    plt.show()

    # Update previous solution
    u_n.assign(u)
```

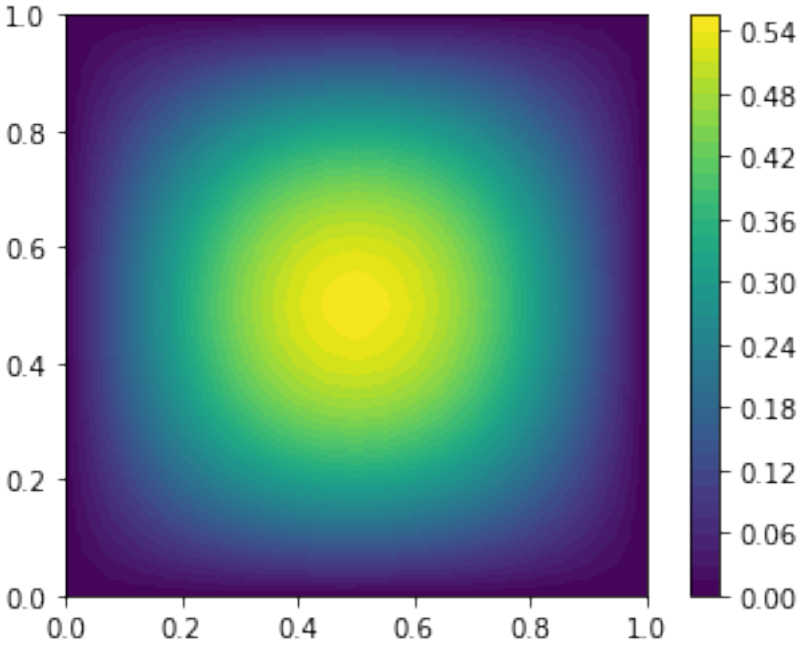
```
Calling FFC just-in-time (JIT) compiler, this may take some time.
Calling FFC just-in-time (JIT) compiler, this may take some time.
Calling FFC just-in-time (JIT) compiler, this may take some time.
Calling FFC just-in-time (JIT) compiler, this may take some time.
t = 0.01:
  u_h(0.5,0.5) = 0.819327
  u(0.5,0.5)   = 0.820869
```



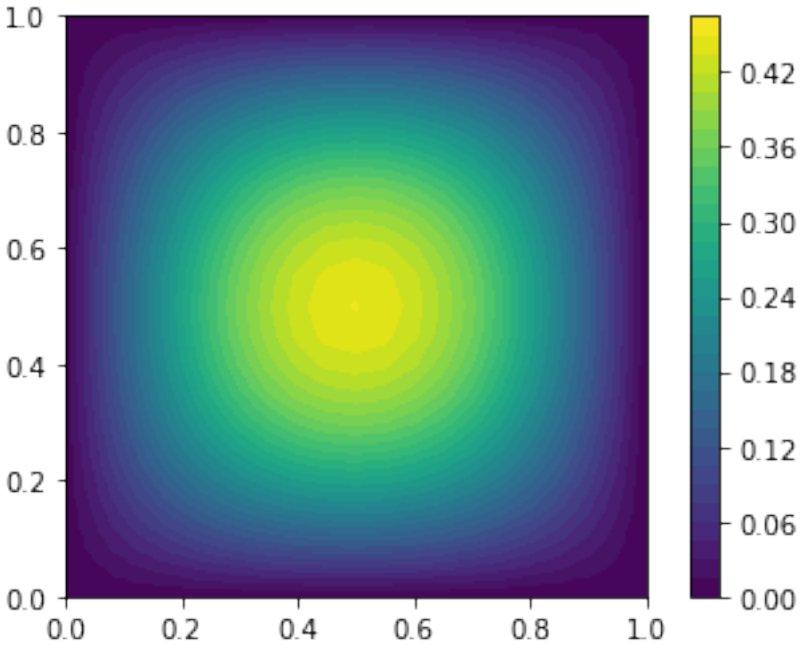

```
t = 0.02:
  u_h(0.5,0.5) = 0.671299
  u(0.5,0.5)   = 0.673825
```



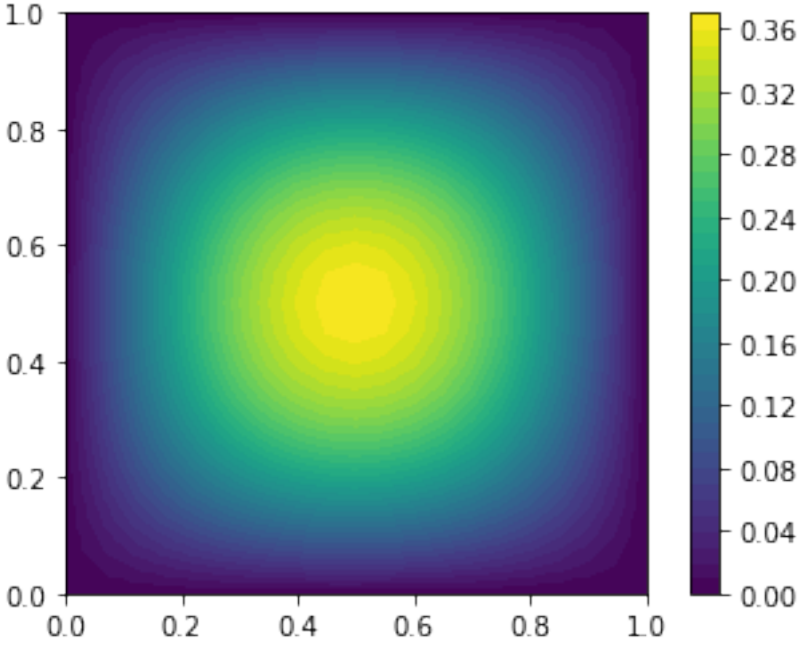

```
t = 0.03:
  u_h(0.5,0.5) = 0.550015
  u(0.5,0.5)   = 0.553122
```



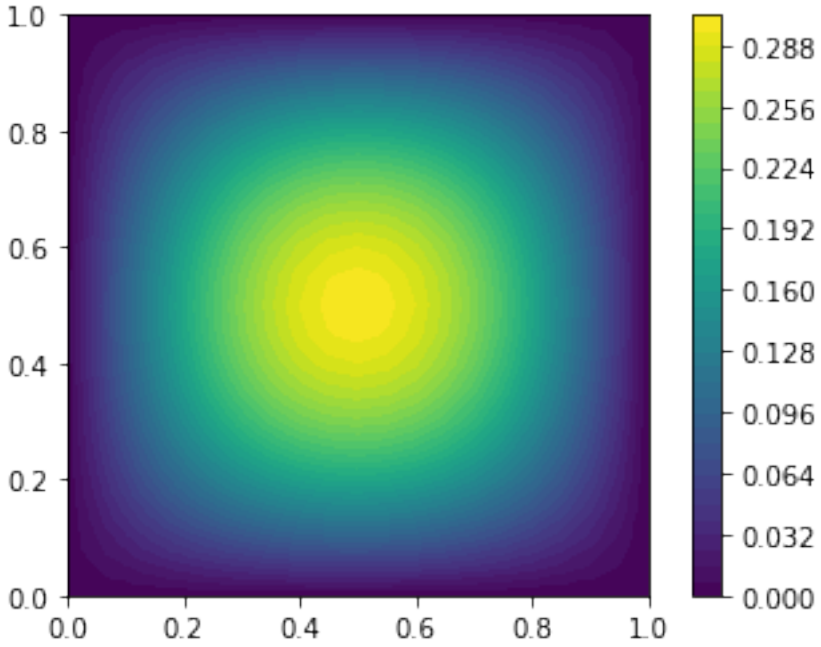
```
t = 0.04:
  u_h(0.5,0.5) = 0.450645
  u(0.5,0.5)   = 0.454041
```



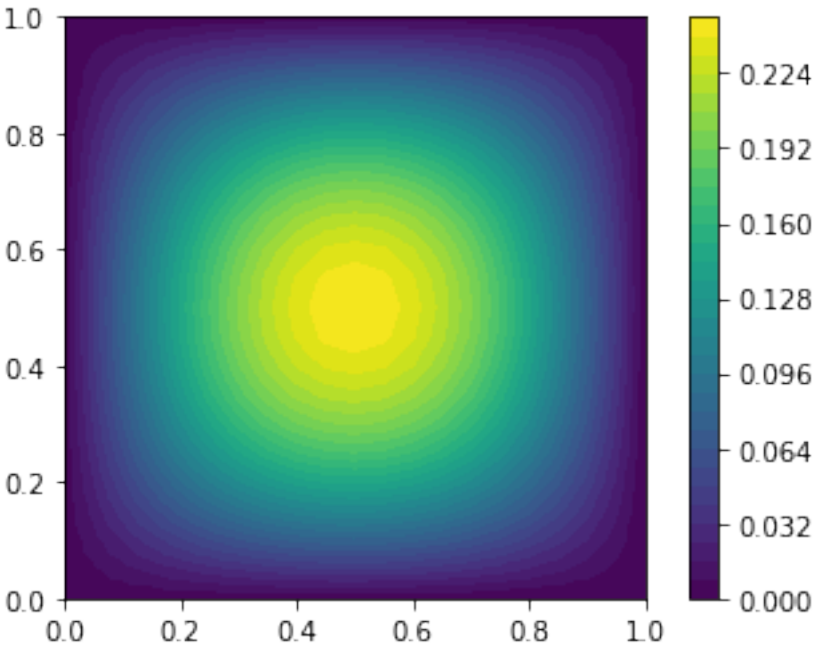

```
t = 0.05:
  u_h(0.5,0.5) = 0.369228
  u(0.5,0.5)   = 0.372708
```



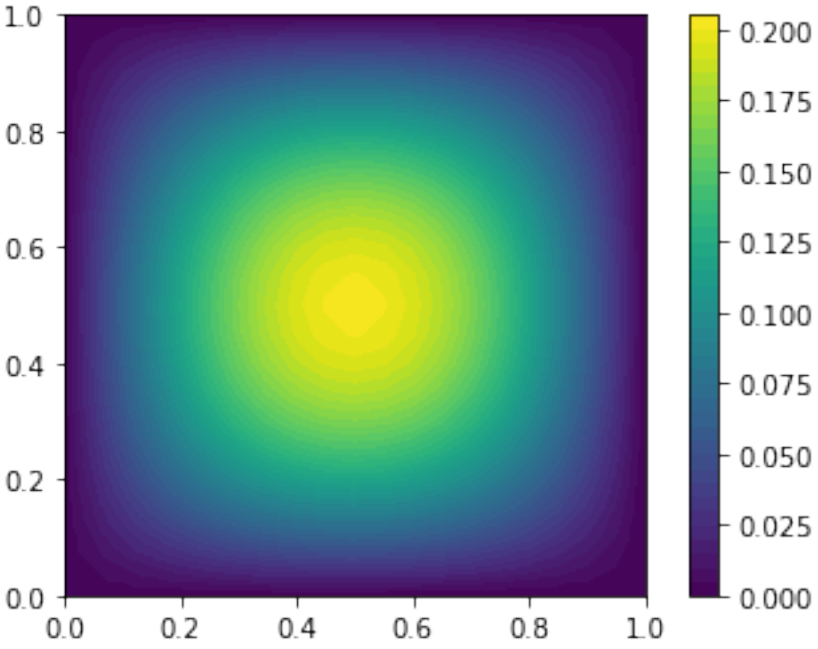
```
t = 0.06:
  u_h(0.5,0.5) = 0.302519
  u(0.5,0.5)   = 0.305944
```



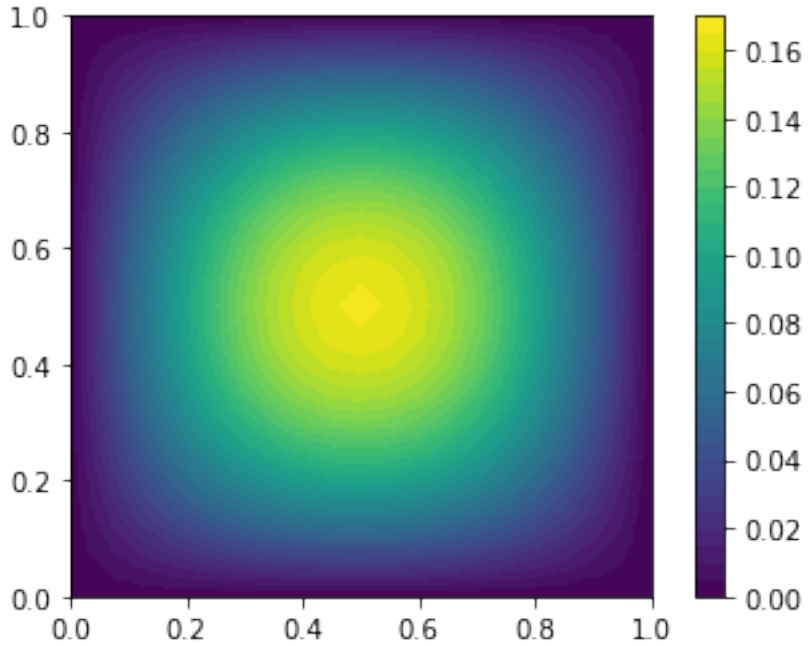

```
t = 0.07:
  u_h(0.5,0.5) = 0.247864
  u(0.5,0.5)   = 0.25114
```



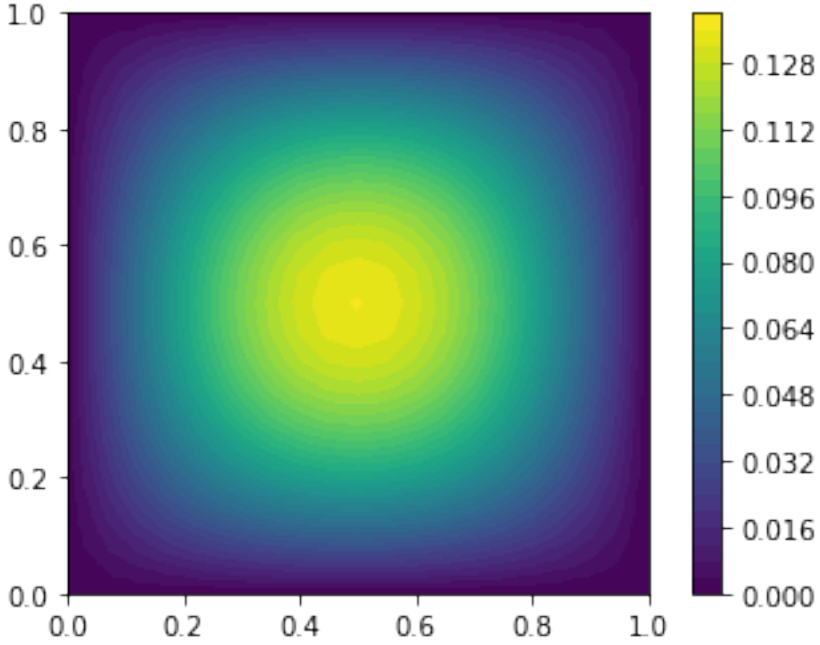

```
t = 0.08:
  u_h(0.5,0.5) = 0.203082
  u(0.5,0.5)   = 0.206153
```



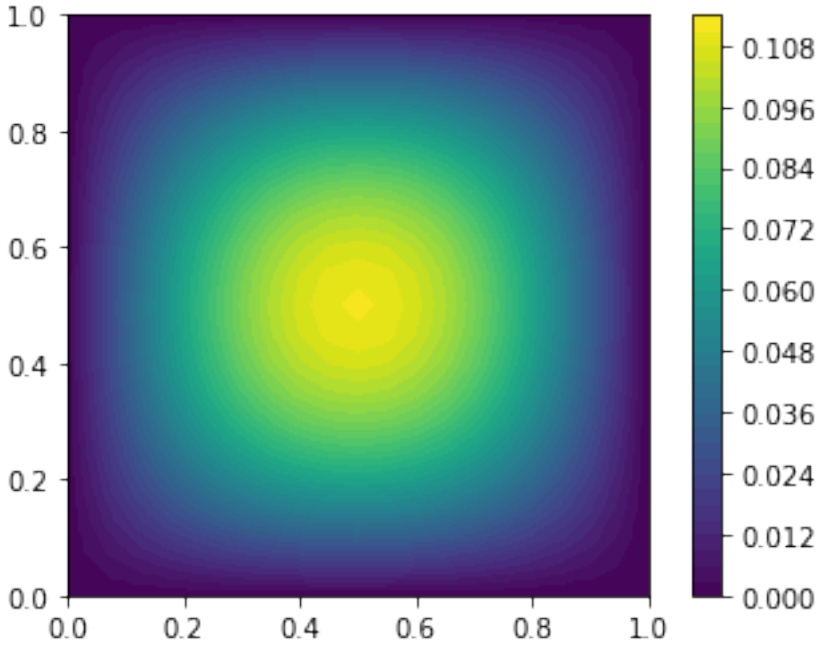
```
t = 0.09:
  u_h(0.5,0.5) = 0.166391
  u(0.5,0.5)   = 0.169225
```



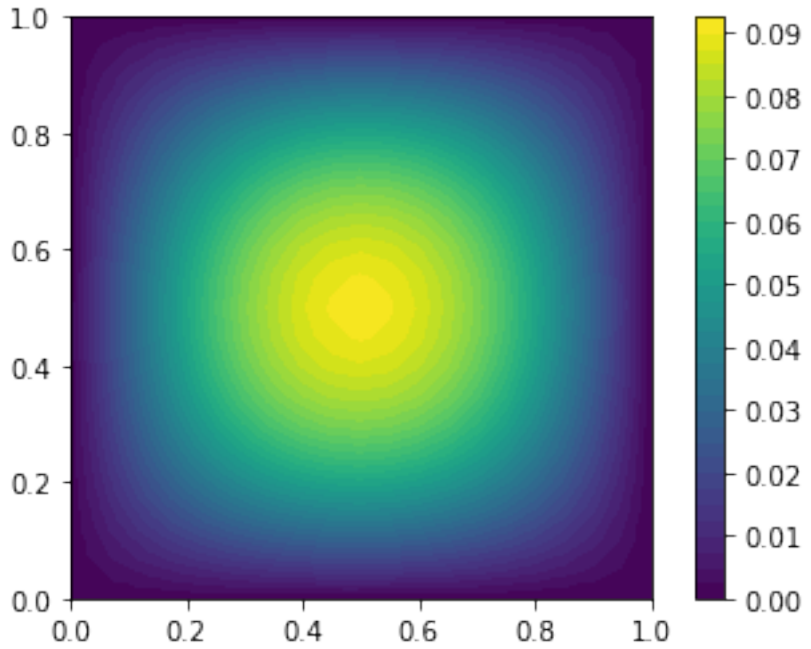

```
t = 0.1:
  u_h(0.5,0.5) = 0.13633
  u(0.5,0.5)   = 0.138911
```



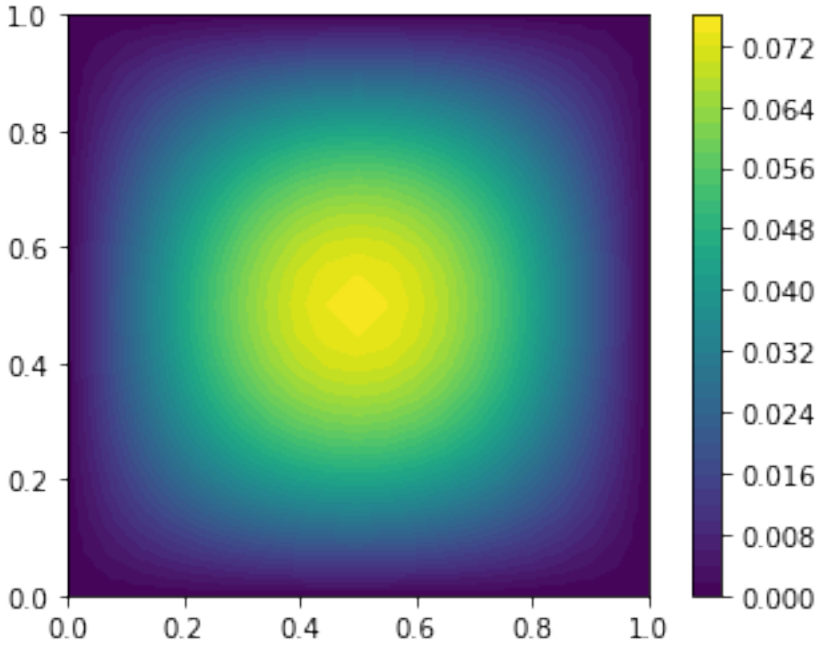
```
t = 0.11:
  u_h(0.5,0.5) = 0.111699
  u(0.5,0.5)   = 0.114028
```



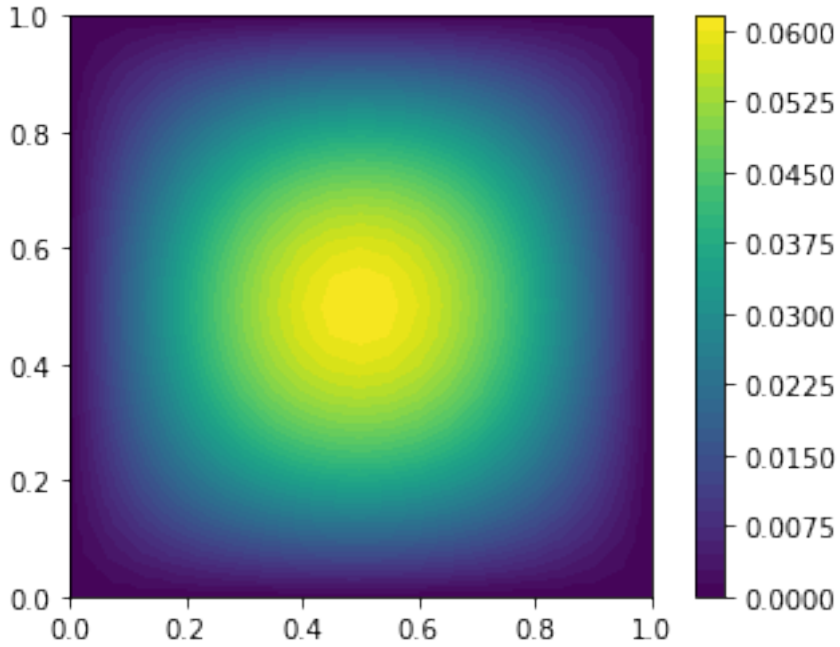

```
t = 0.12:
  u_h(0.5,0.5) = 0.091518
  u(0.5,0.5)   = 0.093602
```



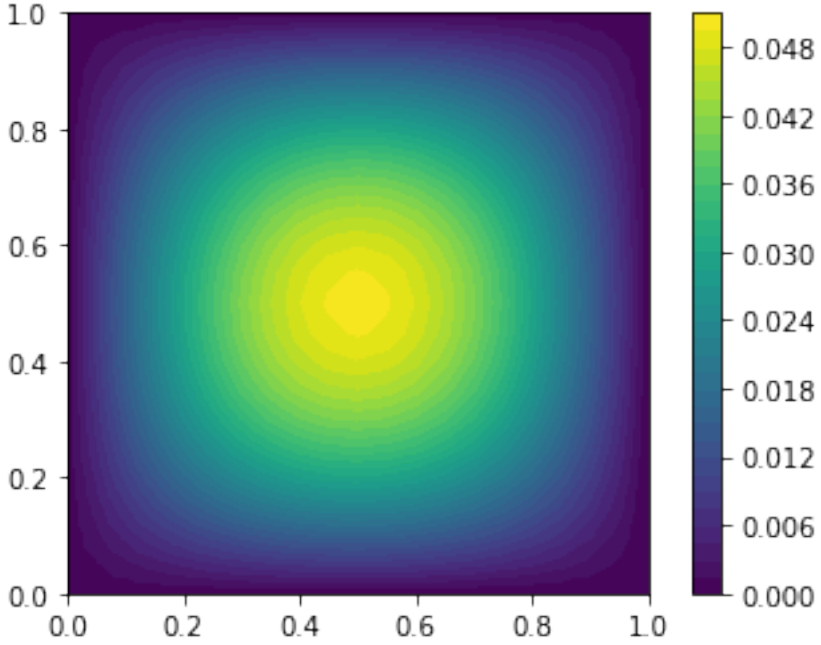

```
t = 0.13:
  u_h(0.5,0.5) = 0.074984
  u(0.5,0.5)   = 0.076835
```



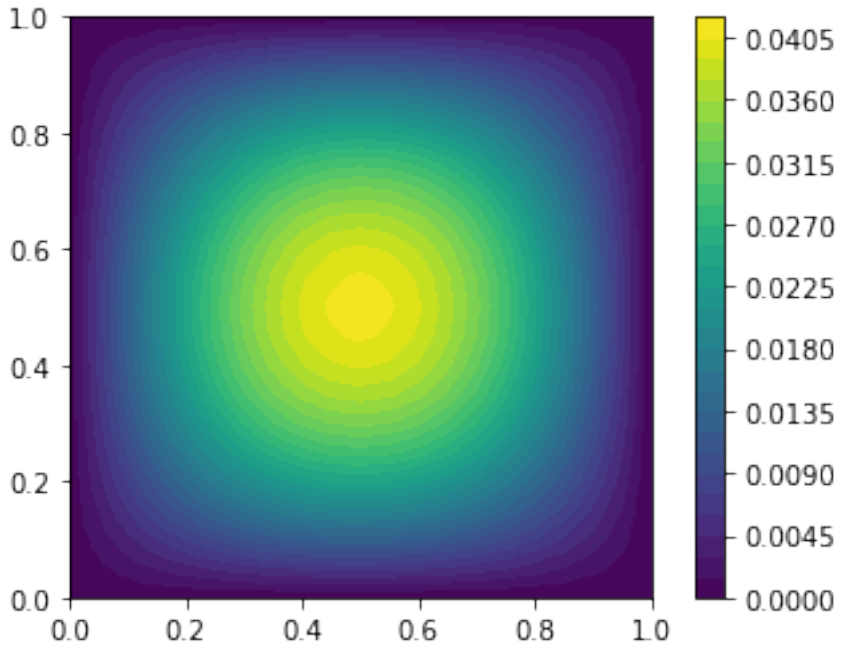
```
t = 0.14:
  u_h(0.5,0.5) = 0.061436
  u(0.5,0.5)   = 0.063071
```



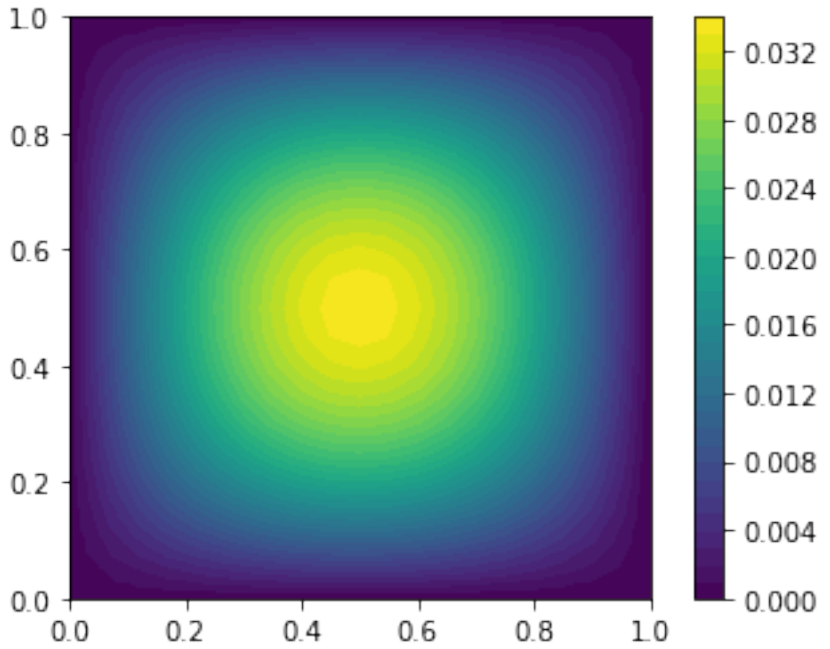

```
t = 0.15:
  u_h(0.5,0.5) = 0.050337
  u(0.5,0.5)   = 0.051773
```



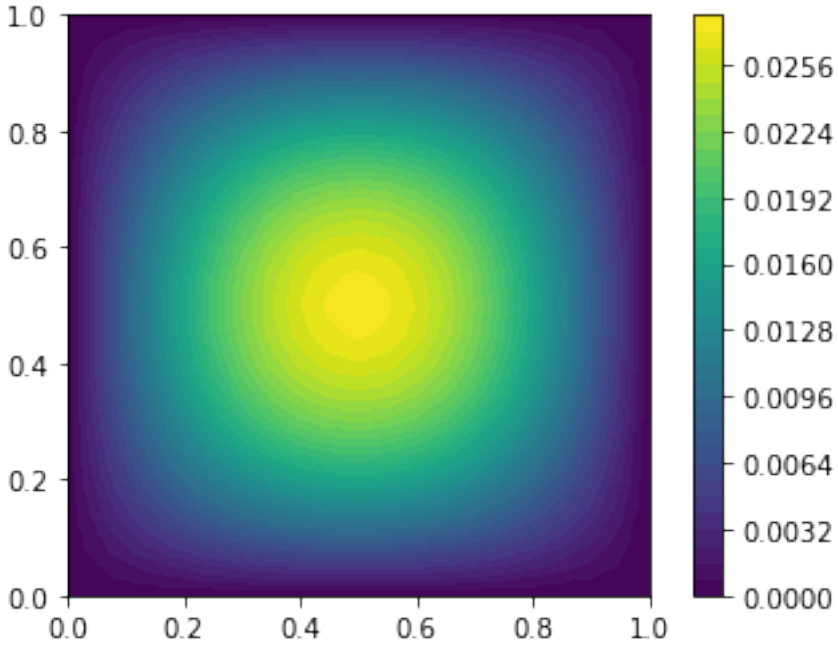
```
t = 0.16:
  u_h(0.5,0.5) = 0.041242
  u(0.5,0.5)   = 0.042499
```



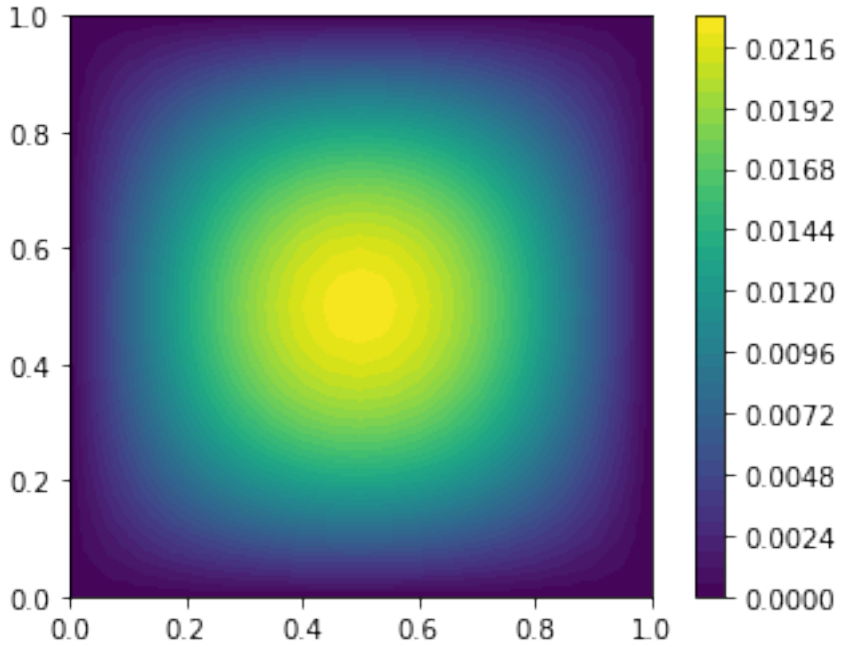

```
t = 0.17:
  u_h(0.5,0.5) = 0.033791
  u(0.5,0.5)   = 0.034886
```



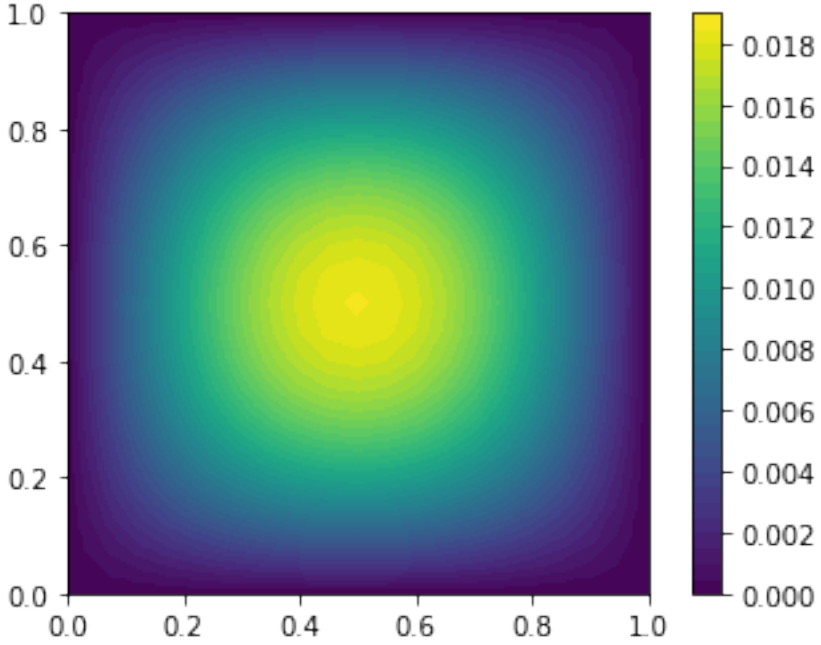

```
t = 0.18:
  u_h(0.5,0.5) = 0.027686
  u(0.5,0.5)   = 0.028637
```



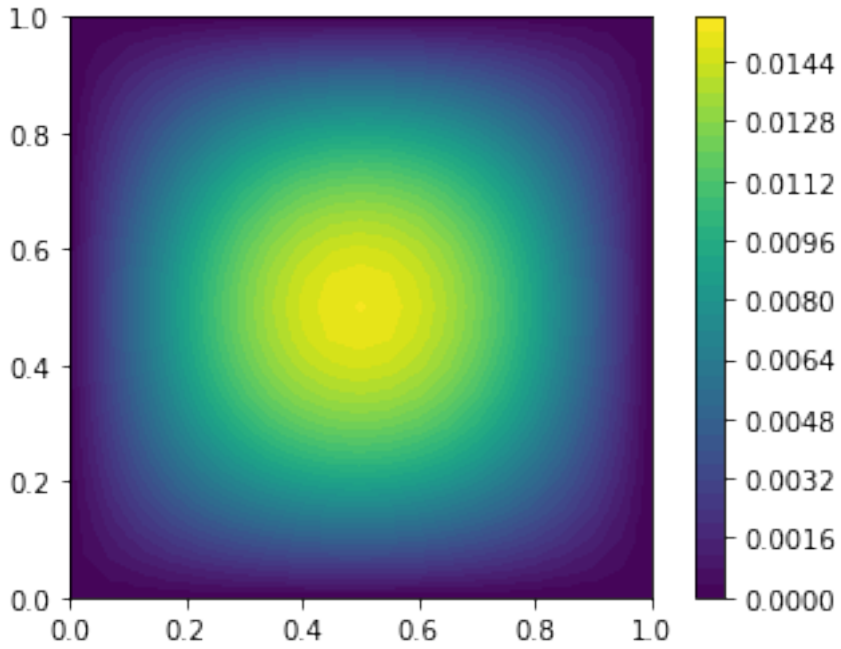
```
t = 0.19:
  u_h(0.5,0.5) = 0.022684
  u(0.5,0.5)   = 0.023507
```



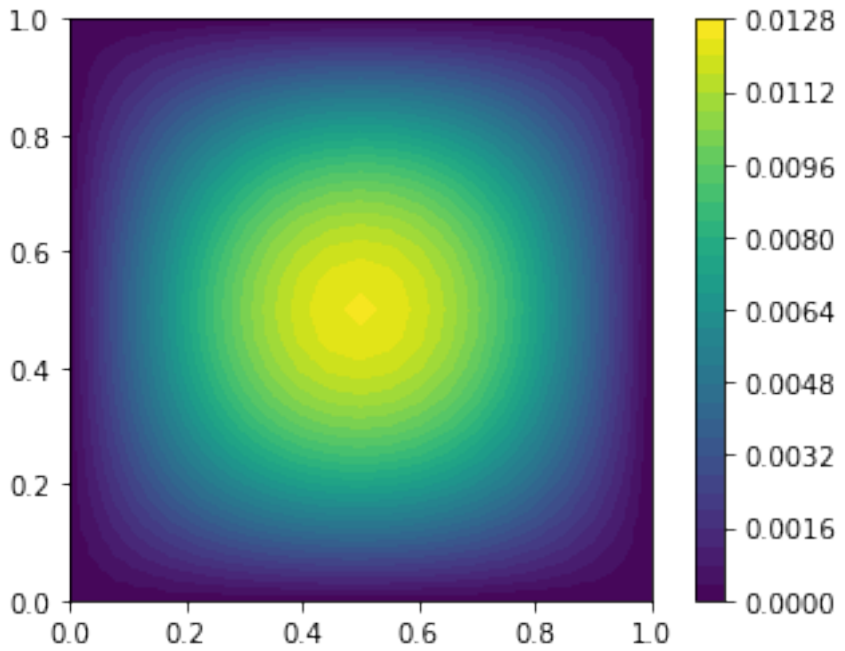

```
t = 0.2:
  u_h(0.5,0.5) = 0.018586
  u(0.5,0.5)   = 0.019296
```



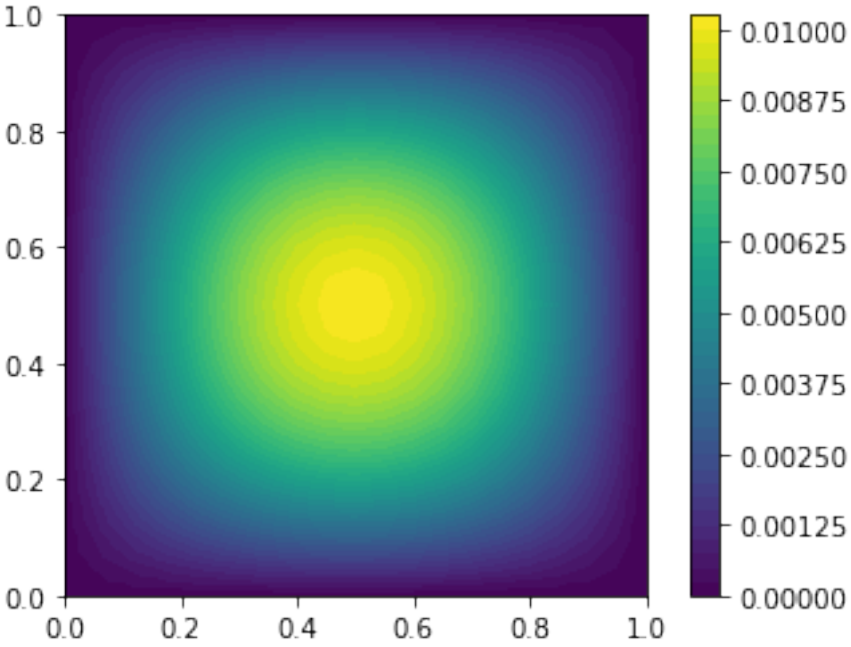
```
t = 0.21:
  u_h(0.5,0.5) = 0.015228
  u(0.5,0.5)   = 0.01584
```



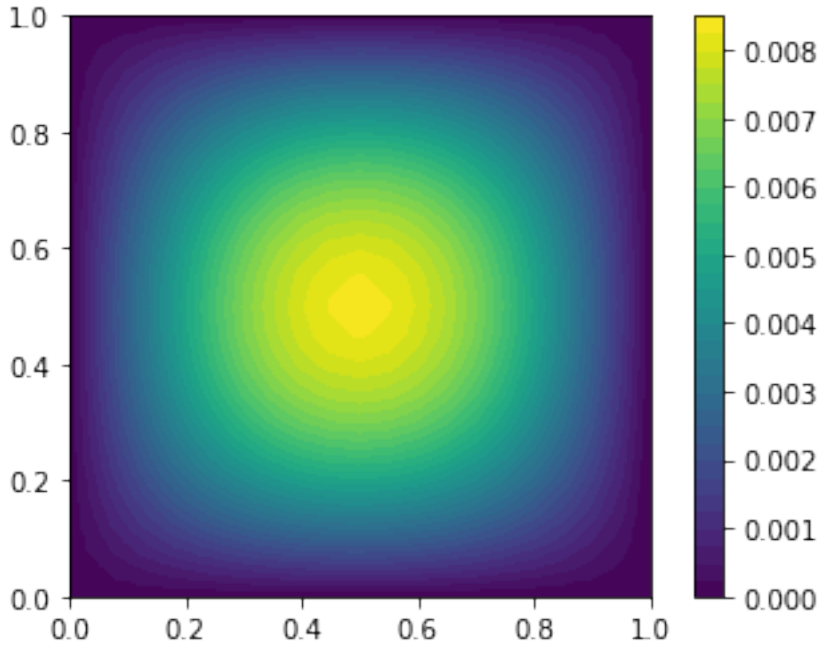

```
t = 0.22:
  u_h(0.5,0.5) = 0.012477
  u(0.5,0.5)   = 0.013002
```



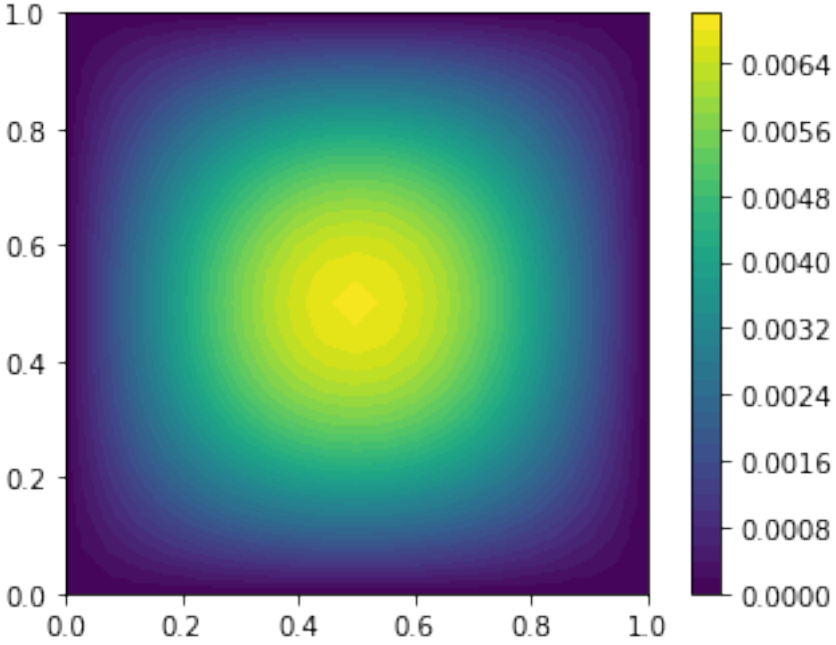

```
t = 0.23:
  u_h(0.5,0.5) = 0.010223
  u(0.5,0.5)   = 0.010673
```



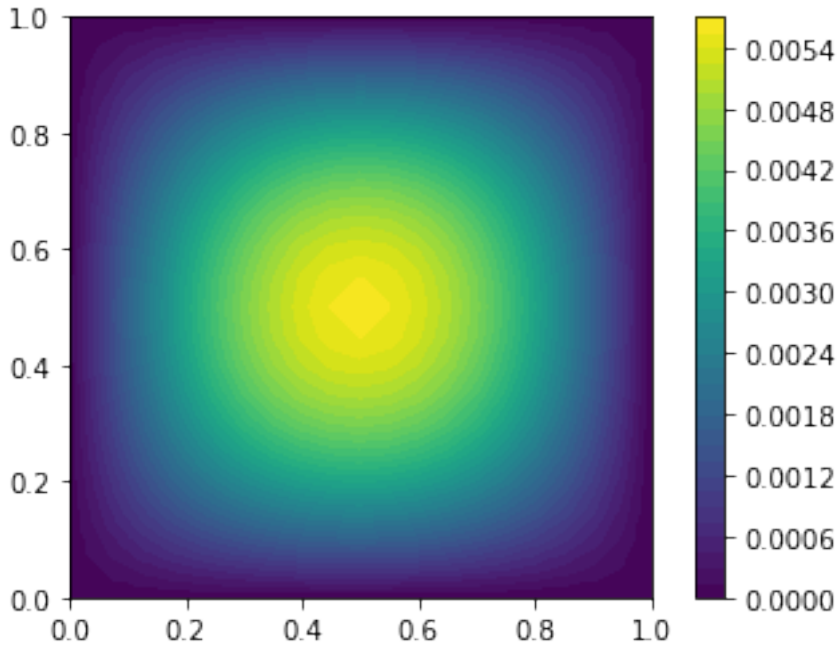
```
t = 0.24:
  u_h(0.5,0.5) = 0.008376
  u(0.5,0.5)   = 0.008761
```



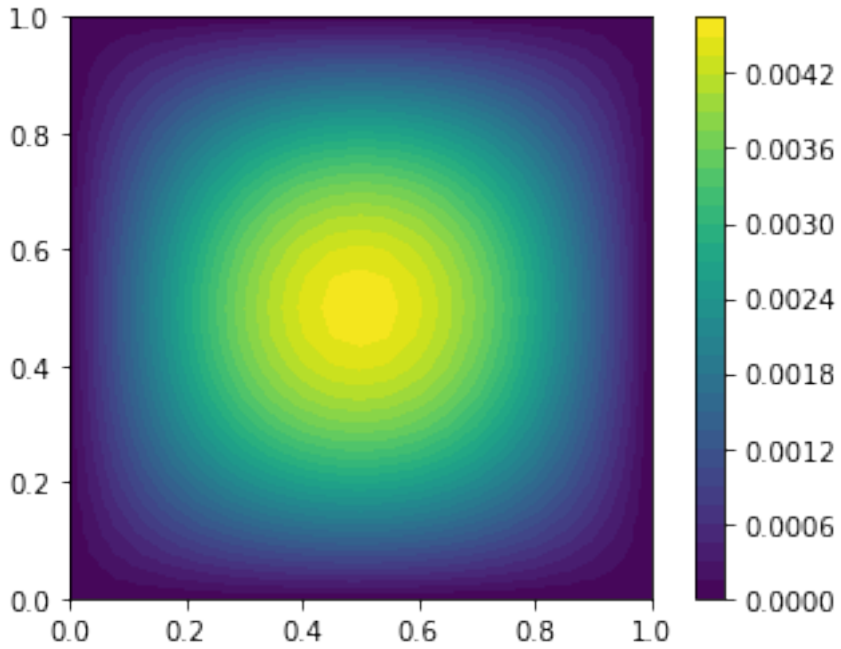

```
t = 0.25:
  u_h(0.5,0.5) = 0.006862
  u(0.5,0.5)   = 0.007192
```



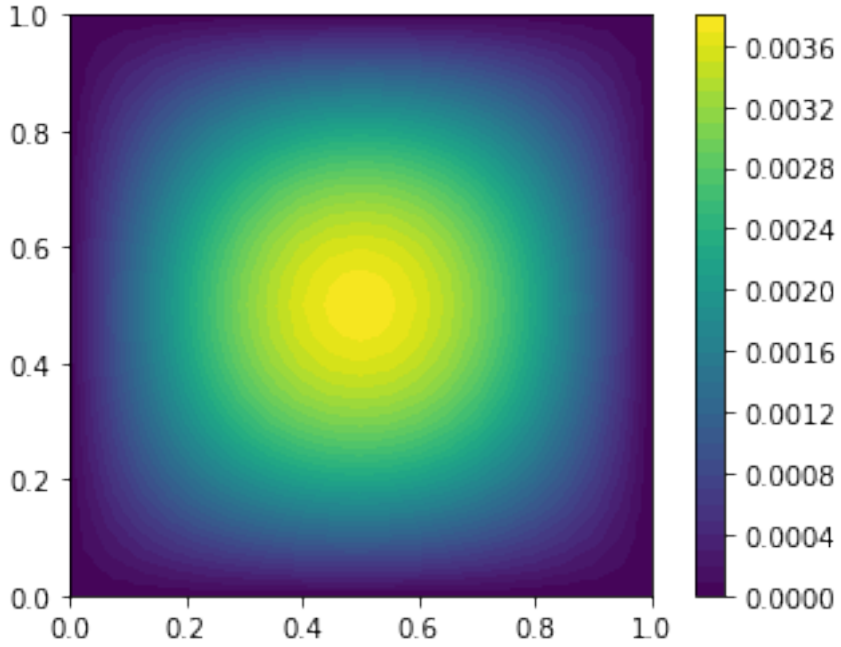
```
t = 0.26:
  u_h(0.5,0.5) = 0.005623
  u(0.5,0.5)   = 0.005904
```



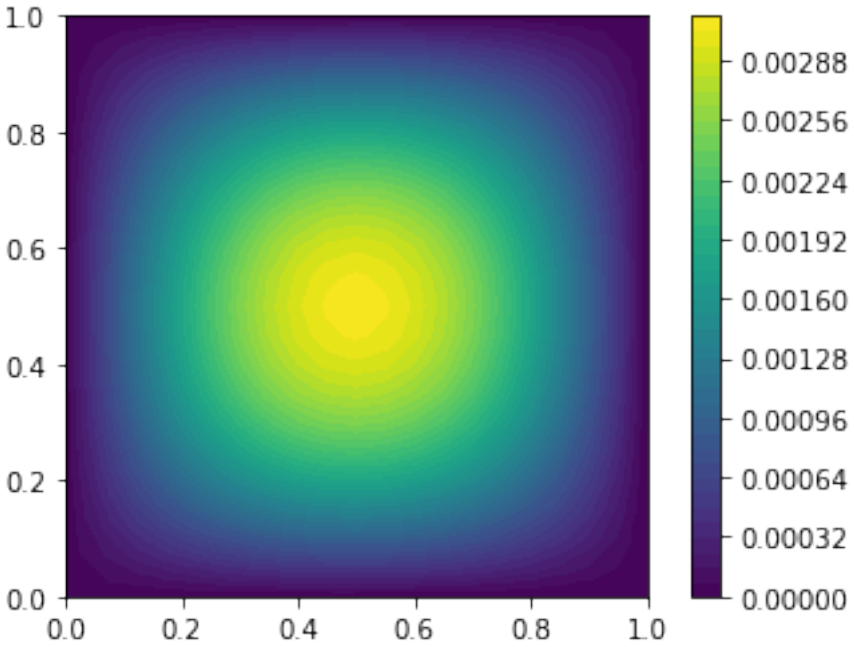

```
t = 0.27:
  u_h(0.5,0.5) = 0.004607
  u(0.5,0.5)   = 0.004846
```



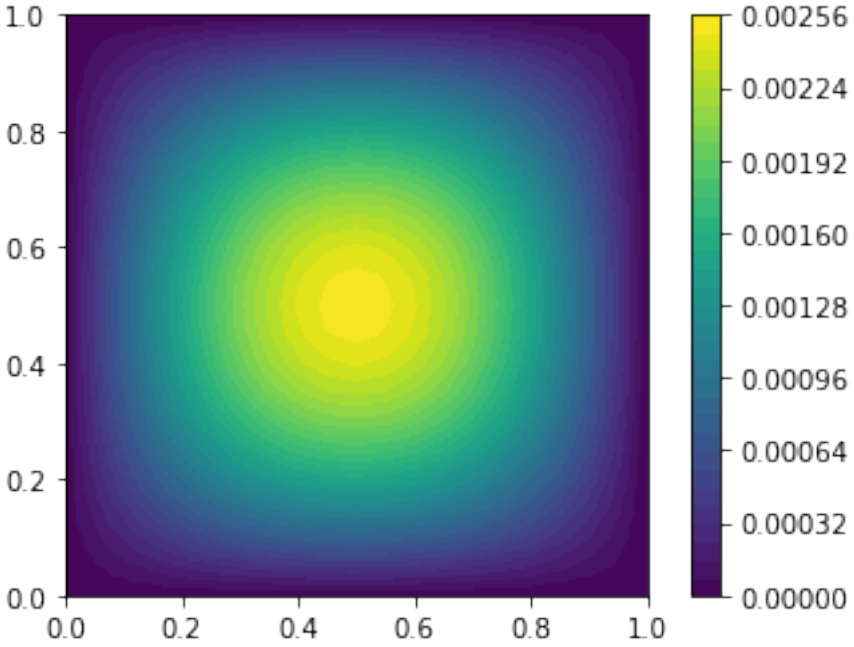

```
t = 0.28:
  u_h(0.5,0.5) = 0.003774
  u(0.5,0.5)   = 0.003978
```



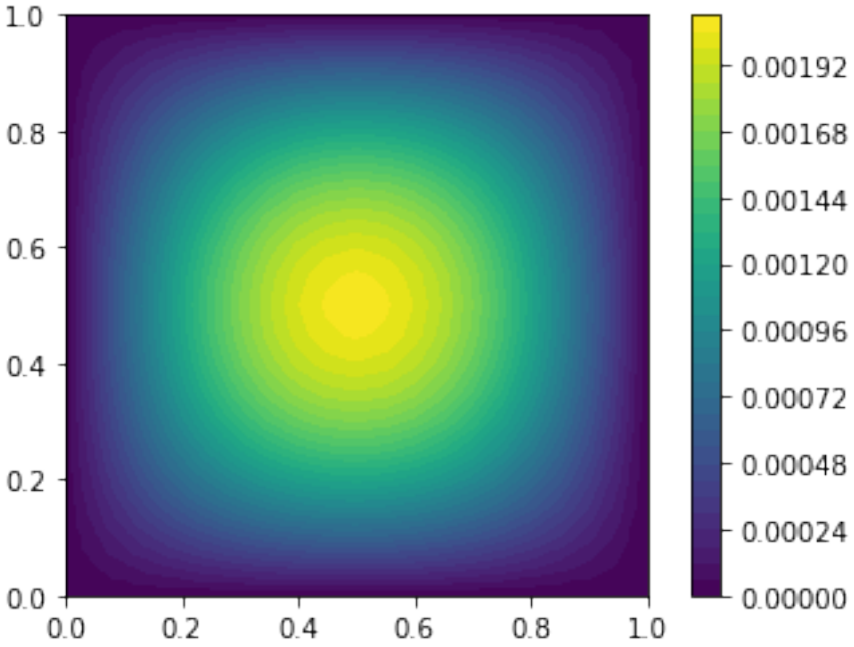
```
t = 0.29:
  u_h(0.5,0.5) = 0.003093
  u(0.5,0.5)   = 0.003265
```



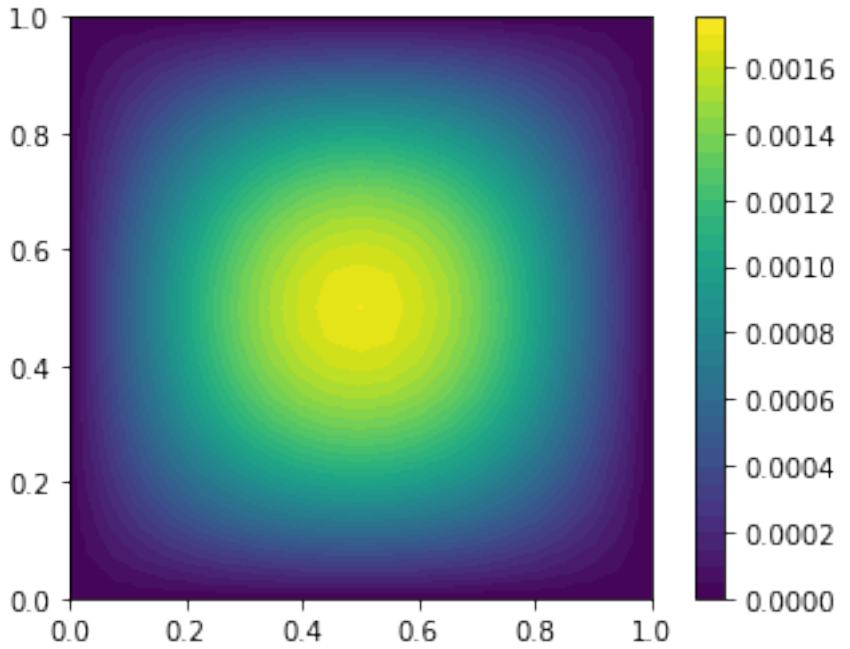

```
t = 0.3:
  u_h(0.5,0.5) = 0.002534
  u(0.5,0.5)   = 0.00268
```



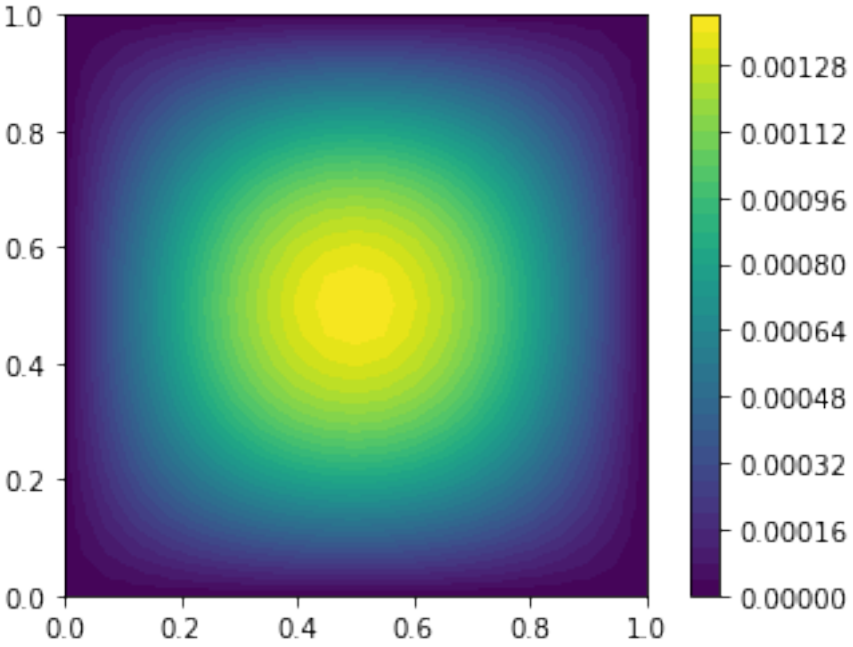
```
t = 0.31:
  u_h(0.5,0.5) = 0.002076
  u(0.5,0.5)   = 0.0022
```



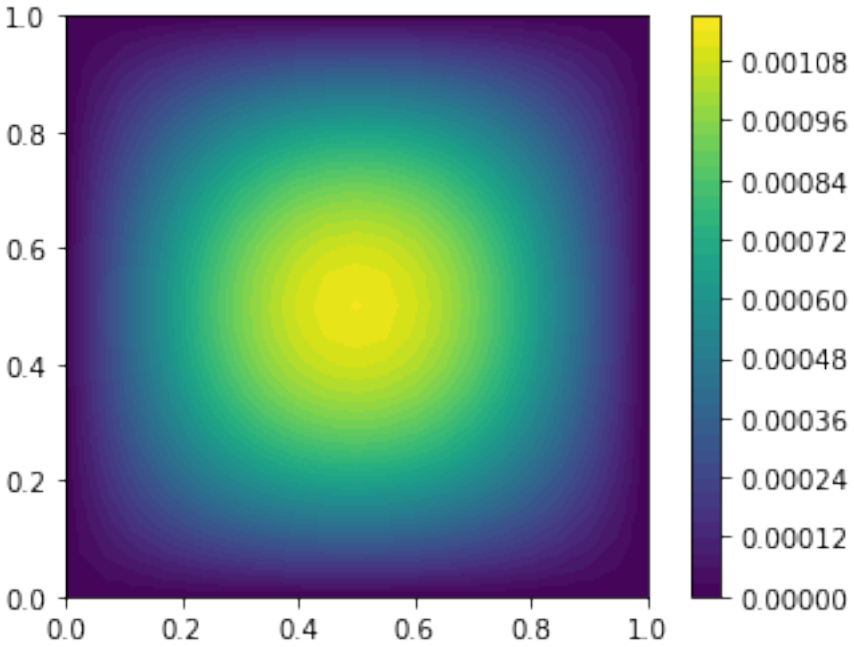

```
t = 0.32:
  u_h(0.5,0.5) = 0.001701
  u(0.5,0.5)   = 0.001806
```



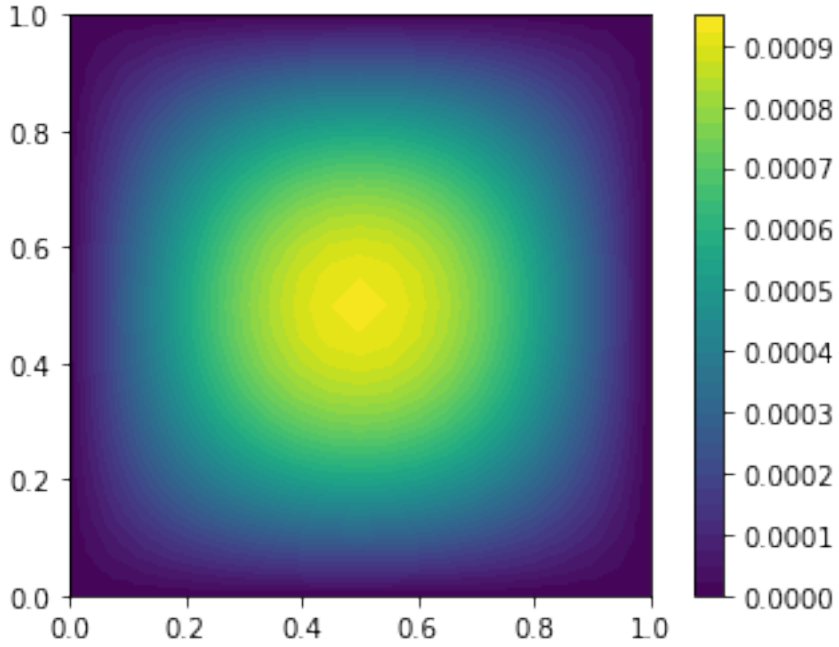

```
t = 0.33:
  u_h(0.5,0.5) = 0.001394
  u(0.5,0.5)   = 0.001483
```



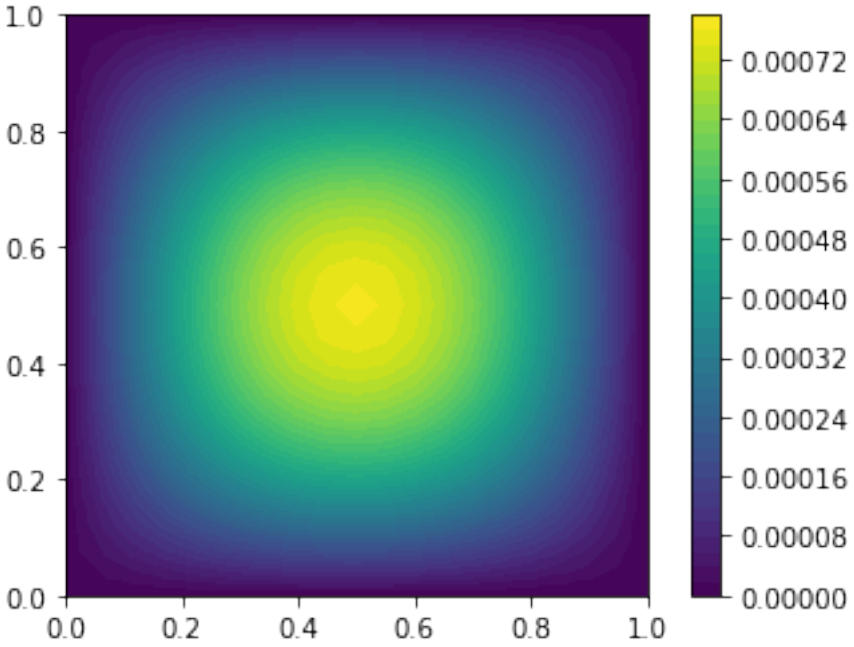
```
t = 0.34:
  u_h(0.5,0.5) = 0.001142
  u(0.5,0.5)   = 0.001217
```



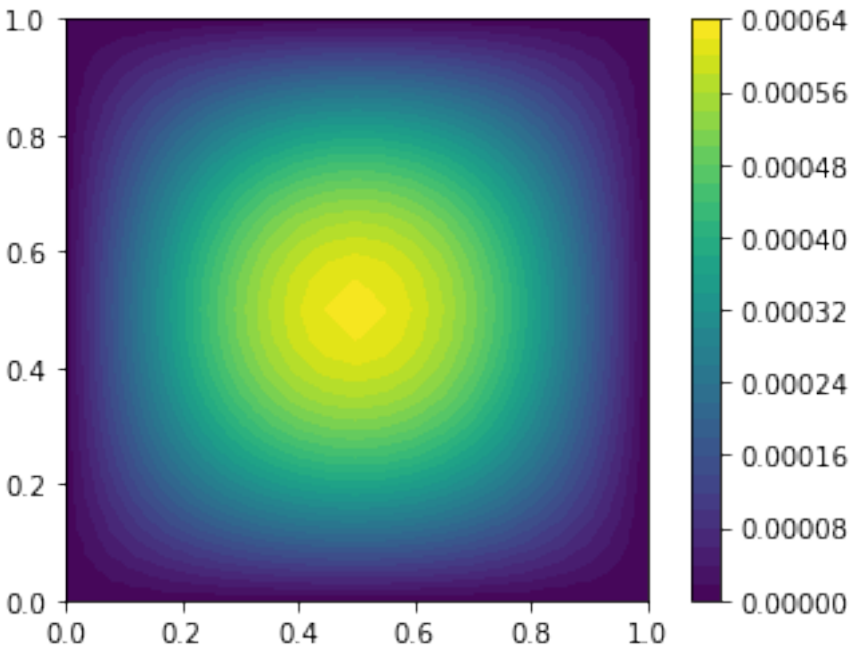

```
t = 0.35:
  u_h(0.5,0.5) = 0.000936
  u(0.5,0.5)   = 0.000999
```



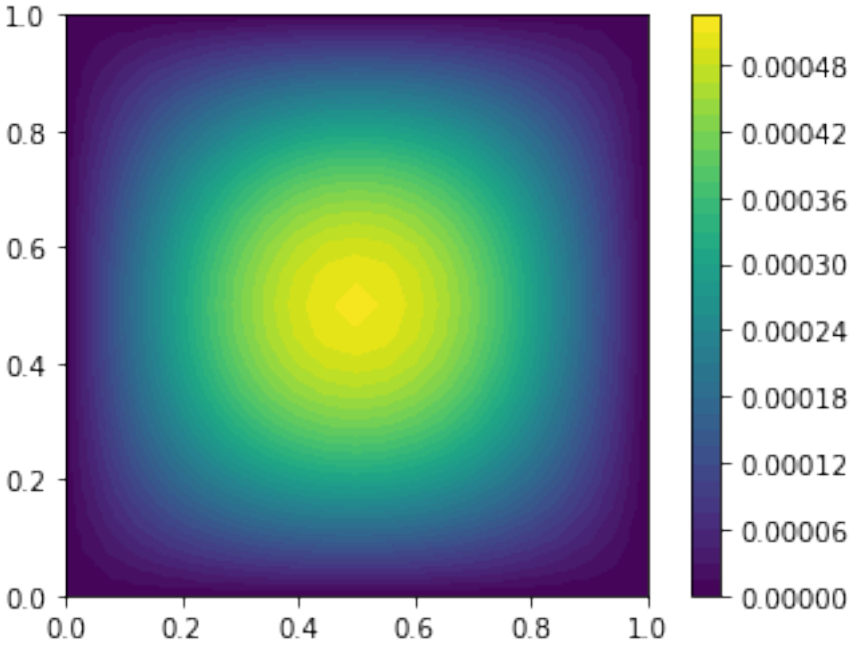
```
t = 0.36:
  u_h(0.5,0.5) = 0.000767
  u(0.5,0.5)   = 0.00082
```



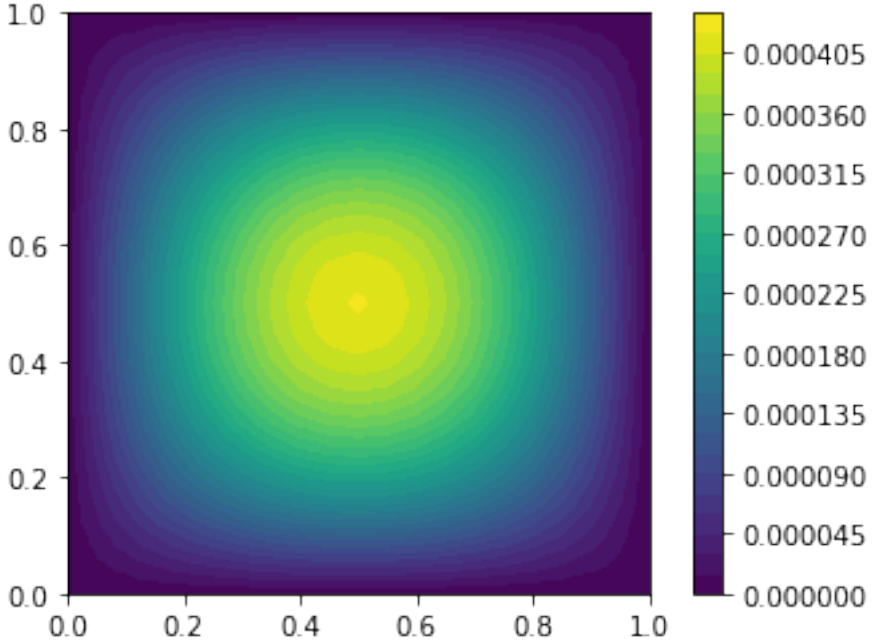

```
t = 0.37:
  u_h(0.5,0.5) = 0.000628
  u(0.5,0.5)   = 0.000673
```



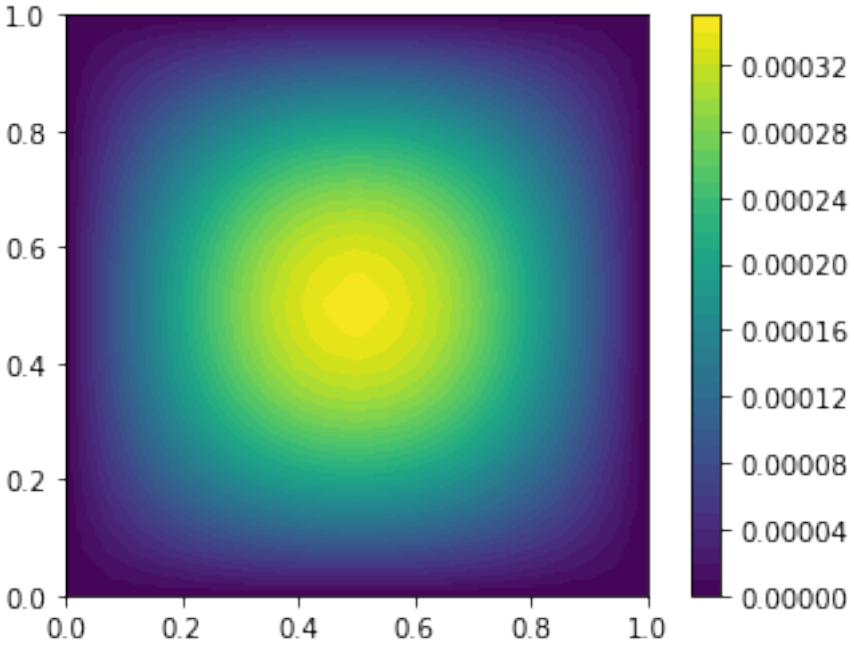

```
t = 0.38:
  u_h(0.5,0.5) = 0.000515
  u(0.5,0.5)   = 0.000553
```



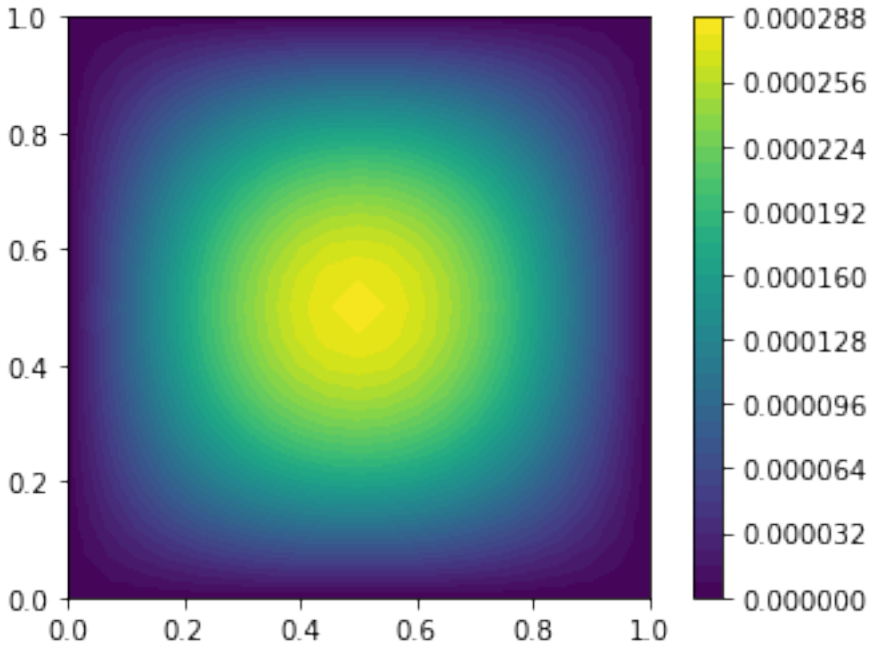
```
t = 0.39:
  u_h(0.5,0.5) = 0.000422
  u(0.5,0.5)   = 0.000454
```



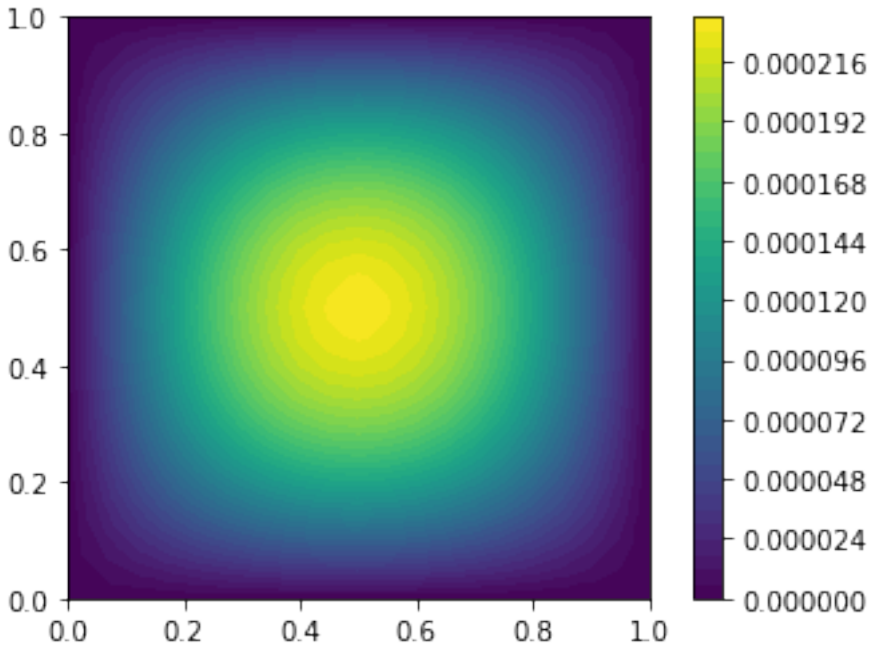

```
t = 0.4:
  u_h(0.5,0.5) = 0.000345
  u(0.5,0.5)   = 0.000372
```



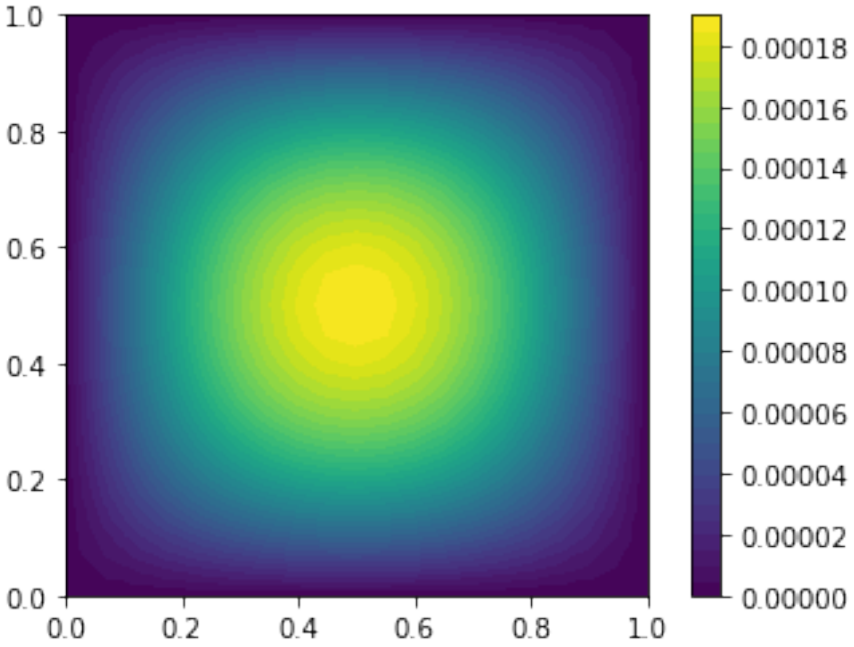
```
t = 0.41:
  u_h(0.5,0.5) = 0.000283
  u(0.5,0.5)   = 0.000306
```



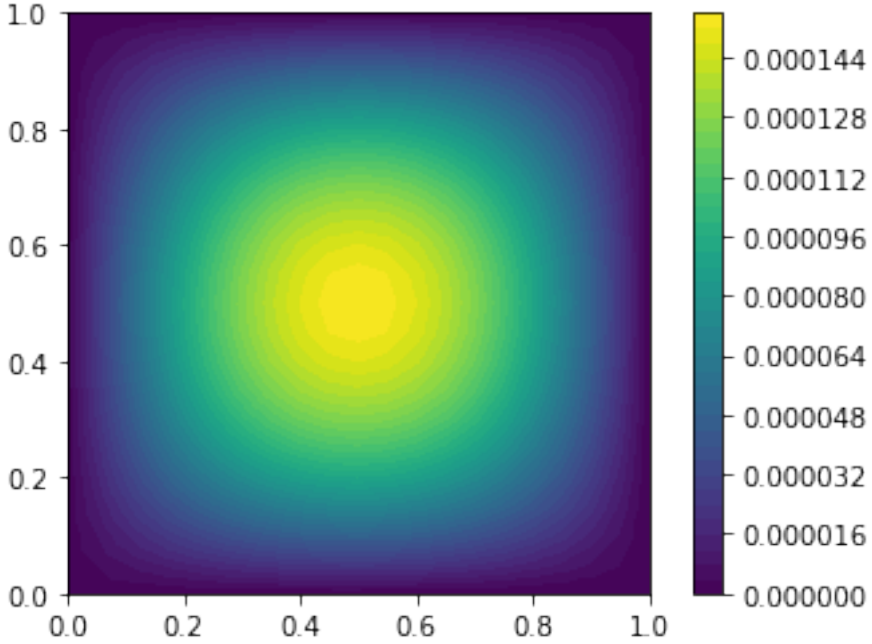

```
t = 0.42:
  u_h(0.5,0.5) = 0.000232
  u(0.5,0.5)   = 0.000251
```



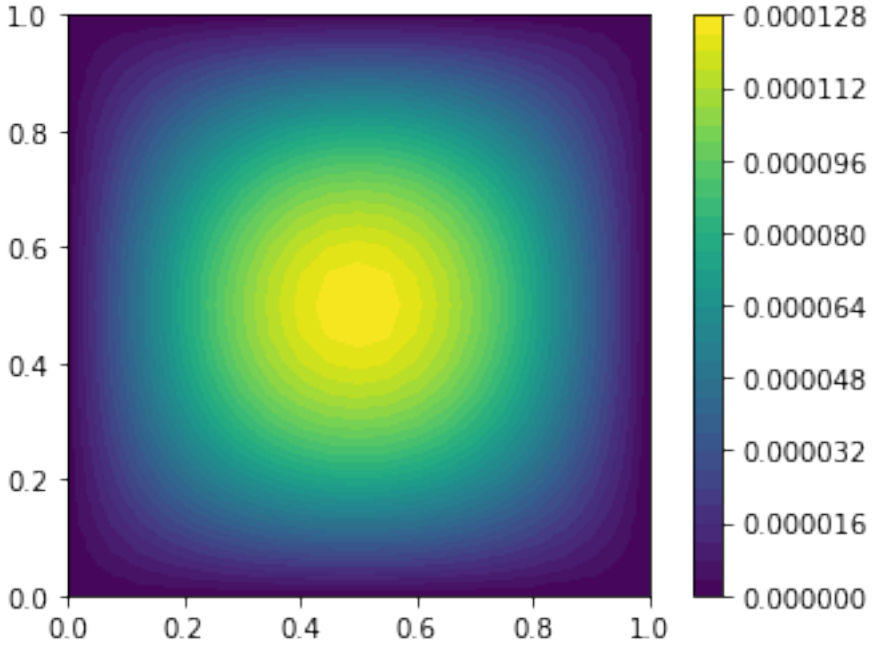

```
t = 0.43:
  u_h(0.5,0.5) = 0.00019
  u(0.5,0.5)   = 0.000206
```



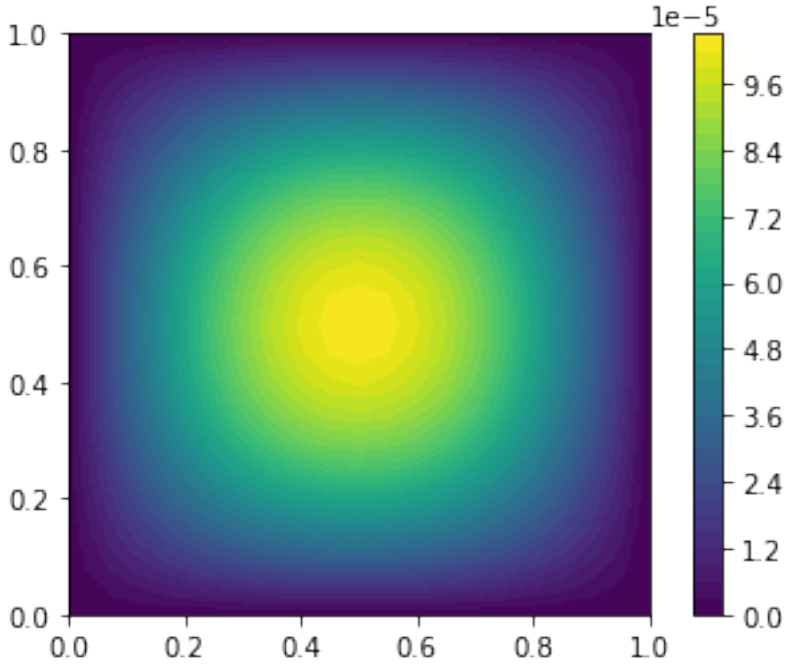
```
t = 0.44:
  u_h(0.5,0.5) = 0.000156
  u(0.5,0.5)   = 0.000169
```



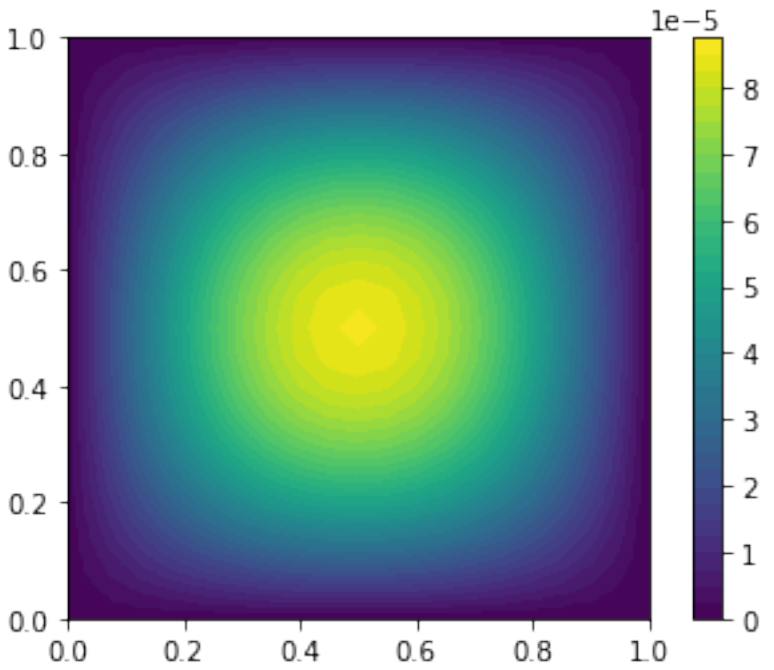

```
t = 0.45:
  u_h(0.5,0.5) = 0.000128
  u(0.5,0.5)   = 0.000139
```



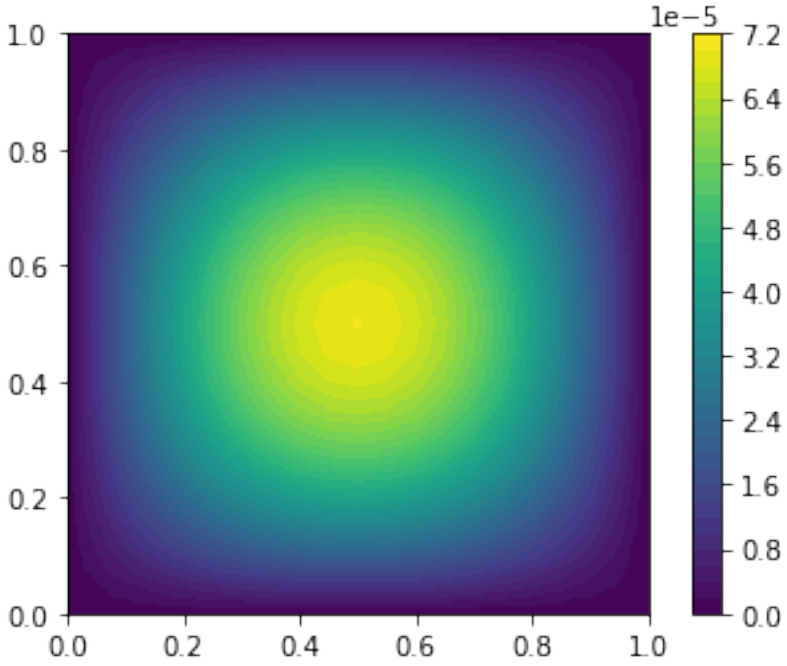
```
t = 0.46:
  u_h(0.5,0.5) = 0.000105
  u(0.5,0.5)   = 0.000114
```



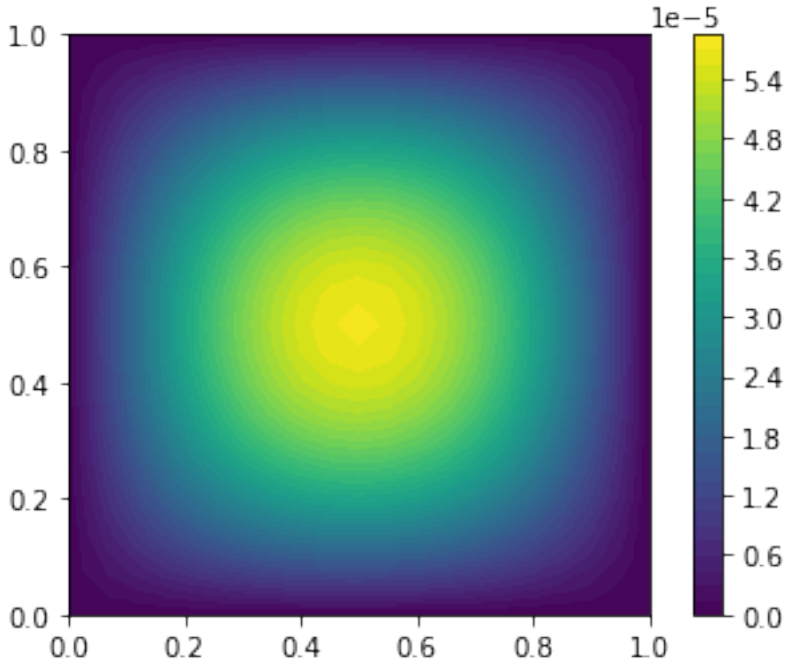

```
t = 0.47:
  u_h(0.5,0.5) = 8.6e-05
  u(0.5,0.5)   = 9.4e-05
```



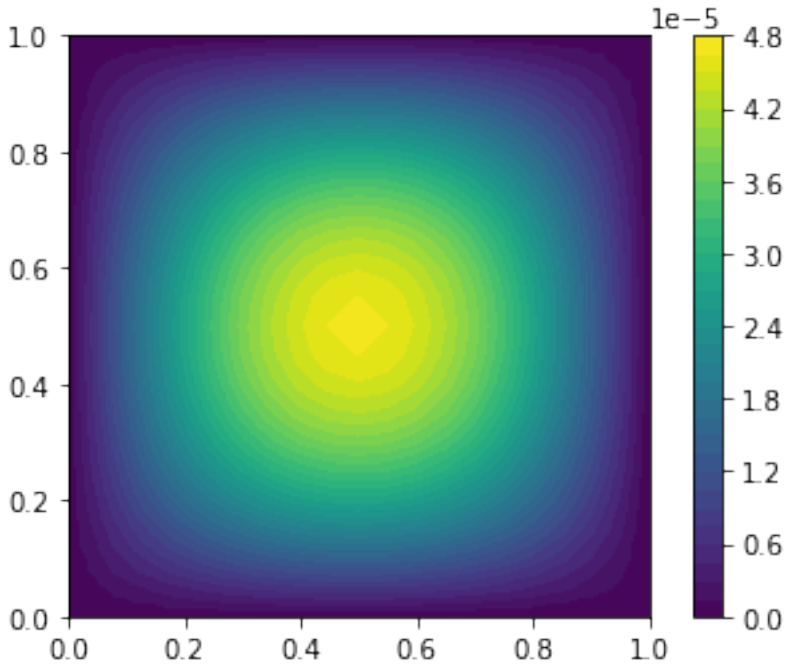

```
t = 0.48:
  u_h(0.5,0.5) = 7e-05
  u(0.5,0.5)   = 7.7e-05
```



```
t = 0.49:
  u_h(0.5,0.5) = 5.7e-05
  u(0.5,0.5)   = 6.3e-05
```




```
t = 0.5:
  u_h(0.5,0.5) = 4.7e-05
  u(0.5,0.5)   = 5.2e-05
```



We observe that our FEM solution resembles a peak centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ whose amplitude decreases over time. This is what we expected, knowing that the analytical solution reads

$$u(t, x, y) = \exp(-2\pi^2 t)\sin(\pi x)\sin(\pi y).$$

Morever, we see that evaluating the analytical solution and its FEM approximation in the quantity of interest $J(\cdot)$ yields similar results, even though the relative error is at times close to 10%, since we use a coarse spatial grid and large time step size.

1.2.6 Exercises

1. *Verify that* $J(u) - J(u_h) \to 0$ *for* $\Delta t, h \to 0$*:* Try different values for Δt and n_x, n_y. What is the smallest relative error in $J(\cdot)$ that you can get experimentally?
2. *Try out different values for* θ*:* How does the quality of your FEM results depend on the choice of θ? Try out $\theta = 0, \theta = 1$ and $\theta = \frac{1}{2}$. Interpret your results and compare your observations with the literature on numerical methods for Ordinary Differential Equations, in particular the literature on the forward/backward Euler scheme and the Crank-Nicholson scheme.

1.3 The matrix assembly / deal.II way of solving the heat equation

In this section, we will make the first step necessary for Proper Orthogonality Decomposition based Reduced Order Modeling. We will tackle the same model problem as before, but instead of solving the variational formulation in UFL form, we will assembly the linear equation system explicitly since this will also be required for creating the reduced order model. Implementing the assembly of the linear system is the cornerstone of many FEM programs, e.g. the deal.II library . Additionally, we store each of the snapshots of the FEM computation in the snapshot matrix Y, which we will need for finding the POD basis that will replace the large dimensional FEM basis.

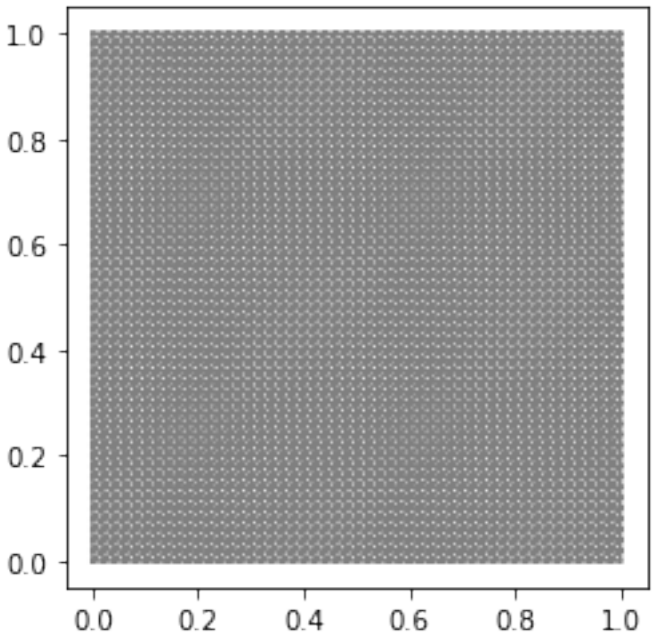
1.3.1 Code

First, we specify the parameters of the time discretization:

[ ]:
```
# start time
t = 0.
# end time
T = 0.5
# time step size
Δt = 0.01
# one-step theta
  = 0.5
```

Next, we create a triangulation of the domain $\Omega = (0, 1) \times (0, 1)$.

[ ]:
```
nx = ny = 50
mesh = UnitSquareMesh(nx, ny)
# visualize the triangulation
plot(mesh)
plt.show()
```



Then, we define on this mesh the space of bilinear finite elements and create homogeneous Dirichlet boundary conditions on the entire boundary.

[ ]:
```
V = FunctionSpace(mesh, 'P', 1)
bc = DirichletBC(V, Constant(0.), lambda _, on_boundary: on_boundary)
```

We now define the expression for the initial condition as u_0 and interpolate it in the function space

V. Here we will use the variable u_n as the solution from the last timestep.

[ ]:
```
# initial condition
u_0 = Expression('sin(pi*x[0])*sin(pi*x[1])', degree=2, pi=math.pi)
# u_n: solution from last time step
u_n = interpolate(u_0, V)
```

Now, we are at the point where things get a little different than in the previous section, where we used UFL to define the variational formulation (1) and (2). In the following, we will use the corresponding linear equation system. For a more detailed explanation of how to get from the variational formulation to a linear equation system, please take a look at my tutorial on the finite element method .

Mainly to get the linear system, we need to use the fact that we know the finite dimensional basis $\{\varphi_i\}_{i=1}^m$ of V. Since u_{n+1} and u_n are functions from V, they can be expressed as a linear combination of this basis, i.e.

$$u_{n+1}(x,y) := \sum_{i=1}^{m} U_i^{n+1} \cdot \varphi_i(x,y), u_n(x,y) := \sum_{i=1}^{m} U_i^n \cdot \varphi_i(x,y).$$

Plugging this into the bilinear form (1) and the right hand side (2) of the variational formulation, we only need to be able to assemble the mass matrix $M \in \mathbb{R}^{m \times m}$, which is defined by

$$M_{ij} = \int_\Omega \varphi_i \cdot \varphi_j \, \mathrm{d}(x,y),$$

and we need to assemble the Laplacian/stiffness matrix $K \in \mathbb{R}^{m \times m}$, which is defined by

$$K_{ij} = \int_\Omega \nabla_x \varphi_i \cdot \nabla_x \varphi_j \, \mathrm{d}(x,y).$$

Using these two matrices, the linear equation system reads: \ Find $\vec{U}^{n+1} \in \mathbb{R}^m$ such that

$$[M + \theta \cdot \Delta t \cdot K]\vec{U}^{n+1} = [M - (1-\theta) \cdot \Delta t \cdot K]\vec{U}^n.$$

Thus, we start off by assembling the mass matrix M and the Laplace matrix K.

[ ]:
```
# Define variational problem
u = TrialFunction(V)
v = TestFunction(V)

# mass matrix
M = assemble(u*v*dx)

# Laplace matrix
K = assemble(dot(grad(u), grad(v))*dx)
```

```
Calling FFC just-in-time (JIT) compiler, this may take some time.
Calling FFC just-in-time (JIT) compiler, this may take some time.
Calling FFC just-in-time (JIT) compiler, this may take some time.
Calling FFC just-in-time (JIT) compiler, this may take some time.
Calling FFC just-in-time (JIT) compiler, this may take some time.
```

With the help of the mass matrix M and the Laplace matrix K, we can now define the system matrix

$$A^{n+1} := M + \theta \cdot \Delta t \cdot K,$$

and the right hand side matrix

$$A^n := M - (1-\theta) \cdot \Delta t \cdot K.$$

Later, we will then only need to solve the linear system

$$A^{n+1}\vec{U}^{n+1} = A^n\vec{U}^n.$$

[ ]:
```
# system matrix
system_matrix = M +   * Δt * K

# right hand side matrix
rhs_matrix = M - (1. -  ) * Δt * K
```

Next, we create a snapshot matrix $Y \in \mathbb{R}^{m \times n}$, where m is the size of the FEM basis and n is the number of snapshots of the FEM solution. We add the initial condition as the first column of this matrix and afterwards in the time-stepping scheme fill the remaining columns.

[ ]:
```
# number of snapshots
n = int(math.ceil((T-t)/Δt))+1

# number of degrees of freedom
m = V.dim()

# snapshot matrix
Y = np.zeros((m, n))

# store initial condition in snapshot matrix
Y[:, 0] = u_n.vector()
```

Finally, we come to the time-stepping loop itself. For this we create the function u, which represents the solution at the current time step, i.e. $u := u_{n+1}$. In the loop itself, we then compute t_{n+1} as $t_n + \Delta t$. Afterwards we solve the linear system

$$A^{n+1}\vec{U}^{n+1} = A^n\vec{U}^n,$$

which corresponds to the variational problem $a(u_{n+1}, v) = L(v)$. Having computed the solution, we can verify its correctness, since we know that the analytical solution is given as

$$u(t, x, y) = \exp(-2\pi^2 t)\sin(\pi x)\sin(\pi y).$$

We want to verify that $J(u(t,\cdot,\cdot)) - J(u_h(t,\cdot,\cdot))$ converges to 0 for $\Delta t, h \to 0$, where $J(\cdot)$ is a quantity of interest that we are monitoring. For our model problem, we decide that we are interested in evaluating the solution at the center of the domain, i.e.

$$J(\tilde{u}(t,\cdot,\cdot)) := \tilde{u}\left(t, \frac{1}{2}, \frac{1}{2}\right)$$

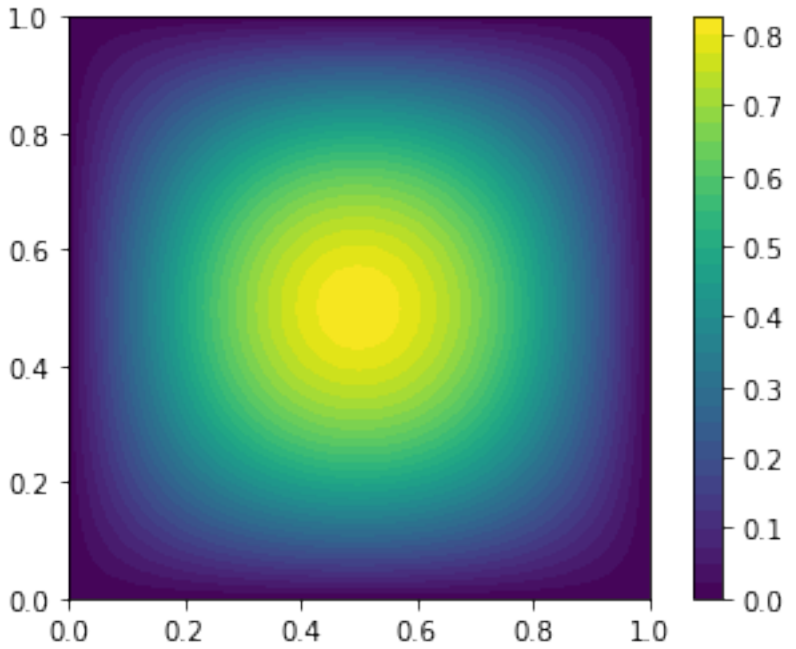
and we have for the analytical solution u that

$$J(u(t,\cdot,\cdot)) = \exp(-2\pi^2 t).$$

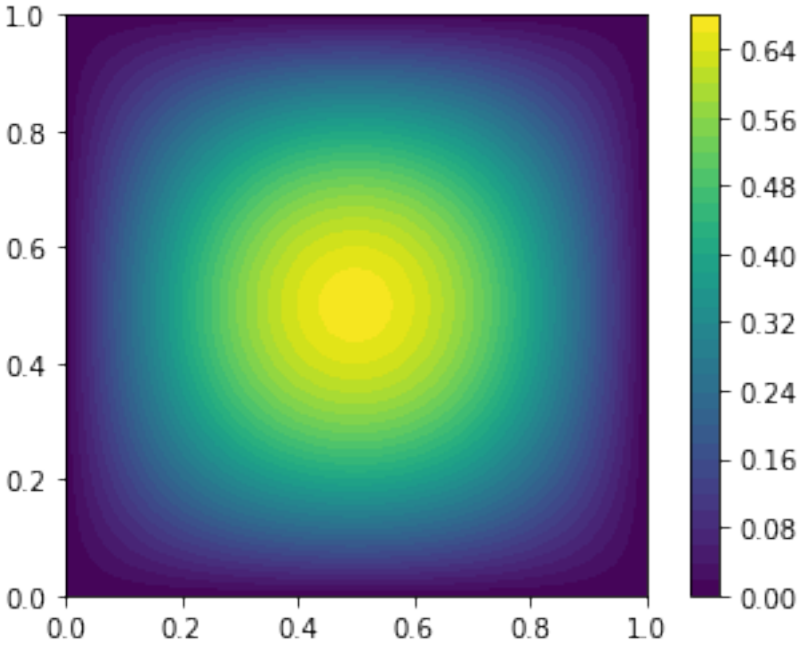
Having compared the difference between the analytical solution u and its FEM approximation u_h in the goal functional $J(\cdot)$, we plot the FEM solution, store the solution in the snapshot matrix and finally update u_n for the computations in the next time step.

[ ]:
```
# Time-stepping
u = Function(V) # u_{n+1}: current solution

n = 1
while(t+Δt <= T+1e-8):
    # Update current time
    t += Δt

    # Compute solution
    A = system_matrix
    # Compute RHS
    b = u_n.vector().copy()
    rhs_matrix.mult(u_n.vector(), b)
    # Apply boundary conditions
    bc.apply(A,b)
    # Solve linear system
    solve(A, u.vector(), b)

    # Print results
    print(f"t = {round(t,4)}:")
    print(f"  u_h(0.5,0.5) = {round(u(Point(0.5, 0.5)),6)}")
    print(f"  u(0.5,0.5)   = {round(math.exp(-2 * math.pi**2 * t),6)}")

    # Plot solution
    c = plot(u)
    plt.colorbar(c)
    plt.show()

    # Store solution in snapshot matrix
    Y[:, n] = u.vector()
    n += 1

    # Update previous solution
    u_n.assign(u)
```

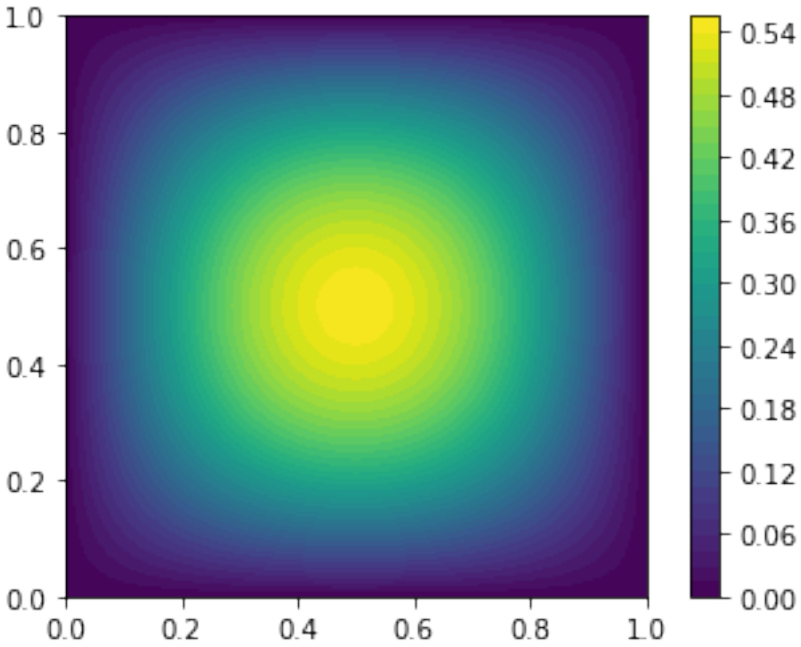
```
t = 0.01:
  u_h(0.5,0.5) = 0.820178
  u(0.5,0.5)   = 0.820869
```



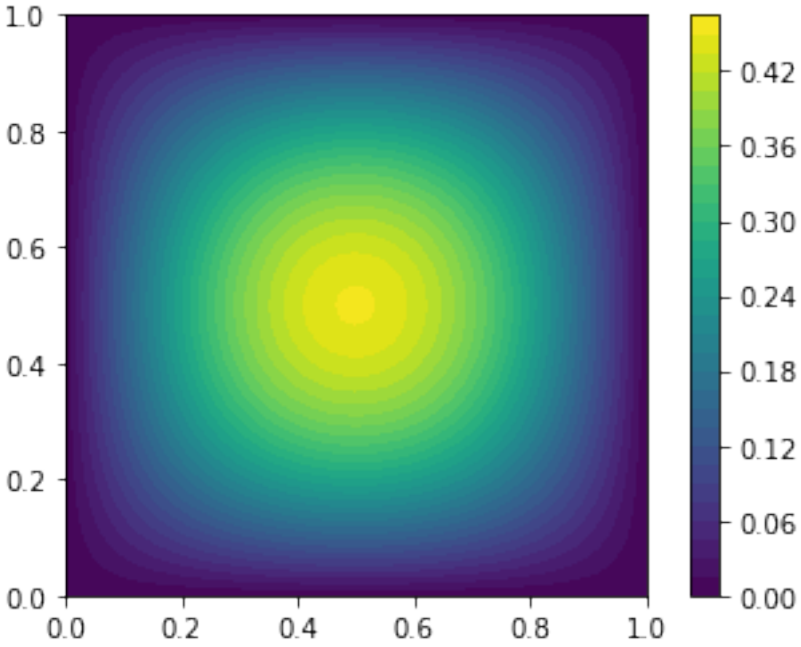

```
t = 0.02:
  u_h(0.5,0.5) = 0.672692
  u(0.5,0.5)   = 0.673825
```



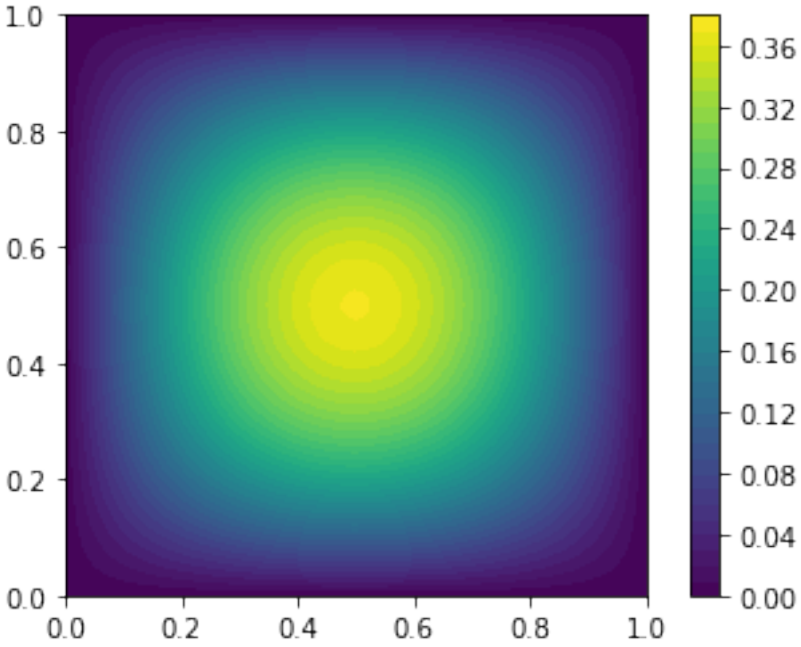
```
t = 0.03:
  u_h(0.5,0.5) = 0.551728
  u(0.5,0.5)   = 0.553122
```



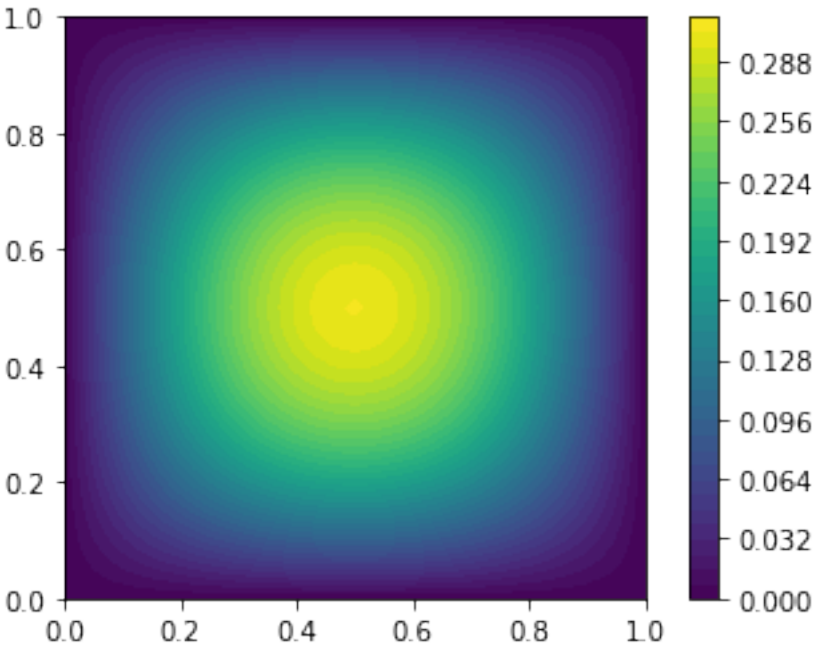

```
t = 0.04:
  u_h(0.5,0.5) = 0.452515
  u(0.5,0.5)   = 0.454041
```



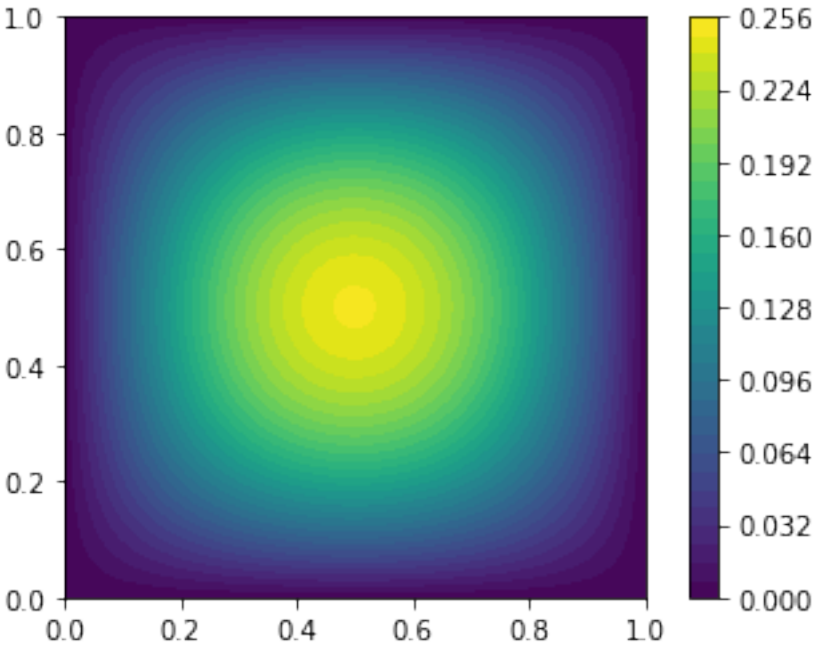
```
t = 0.05:
  u_h(0.5,0.5) = 0.371143
  u(0.5,0.5)   = 0.372708
```



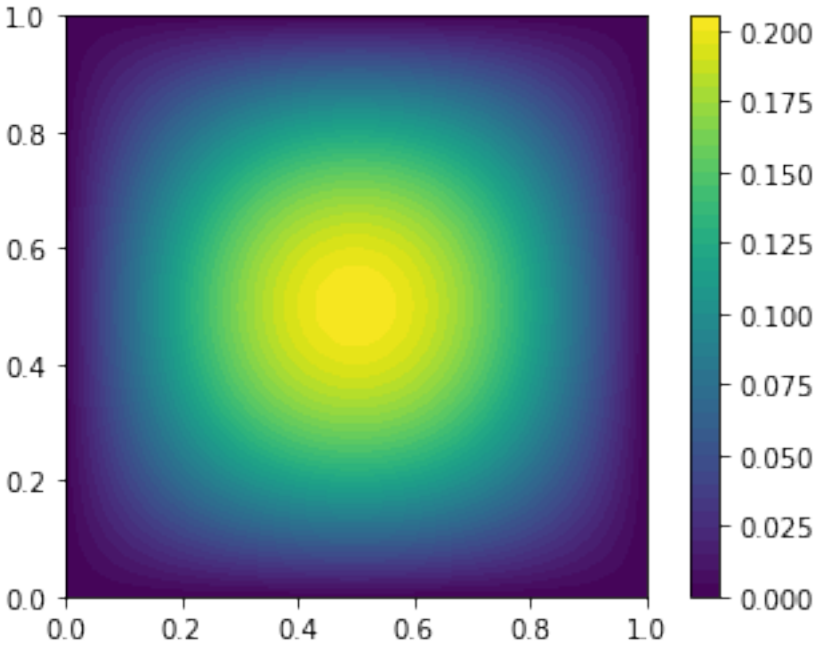

```
t = 0.06:
  u_h(0.5,0.5) = 0.304403
  u(0.5,0.5)   = 0.305944
```



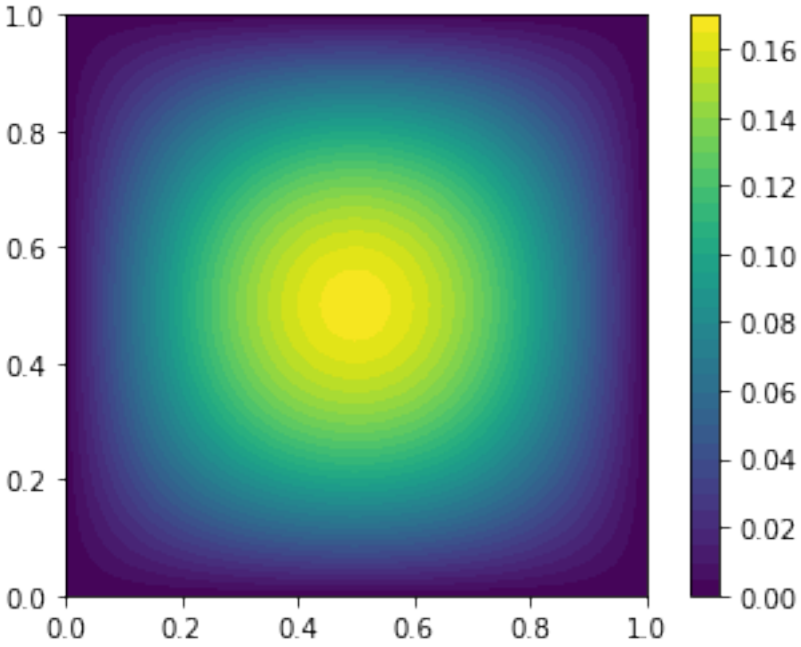

```
t = 0.07:
  u_h(0.5,0.5) = 0.249665
  u(0.5,0.5)   = 0.25114
```



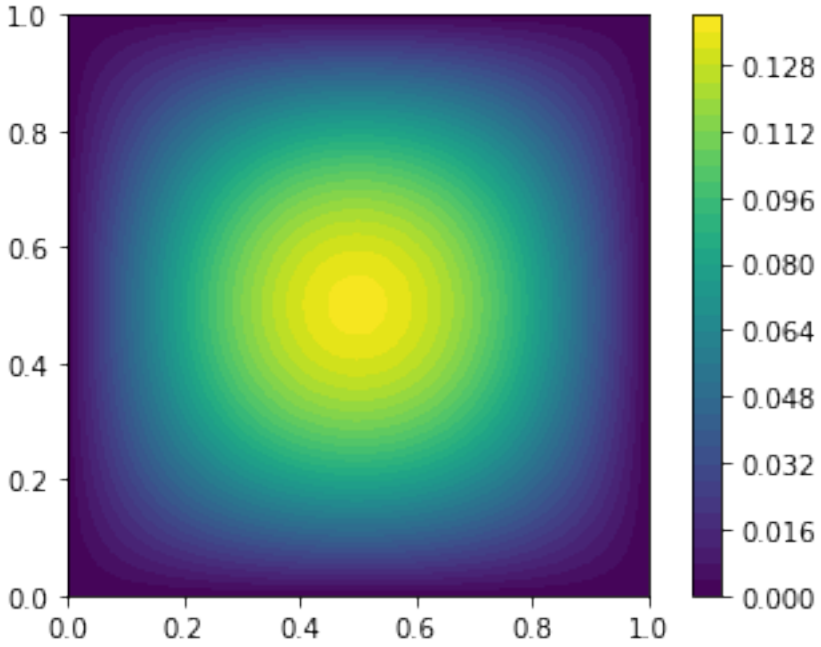
```
t = 0.08:
  u_h(0.5,0.5) = 0.20477
  u(0.5,0.5)   = 0.206153
```



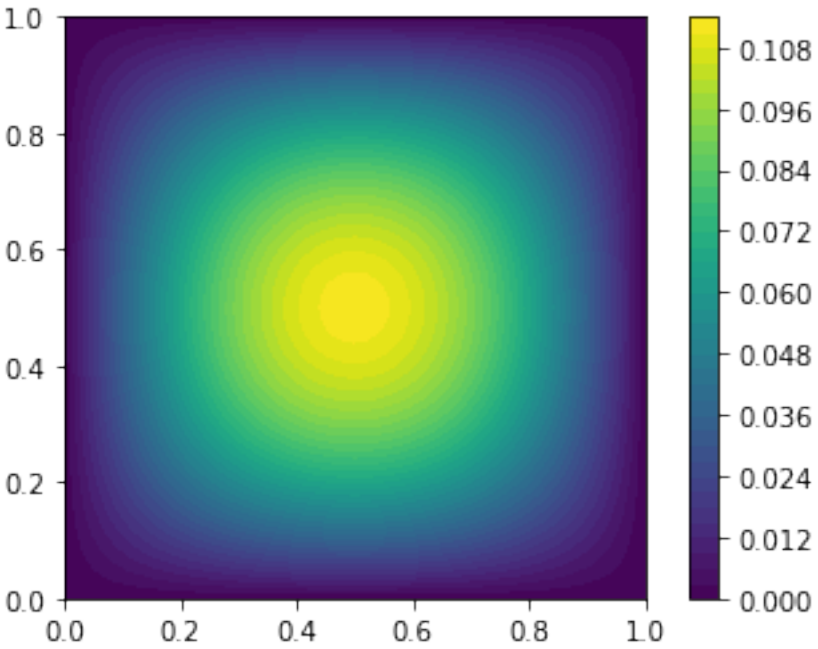

```
t = 0.09:
  u_h(0.5,0.5) = 0.167948
  u(0.5,0.5)   = 0.169225
```



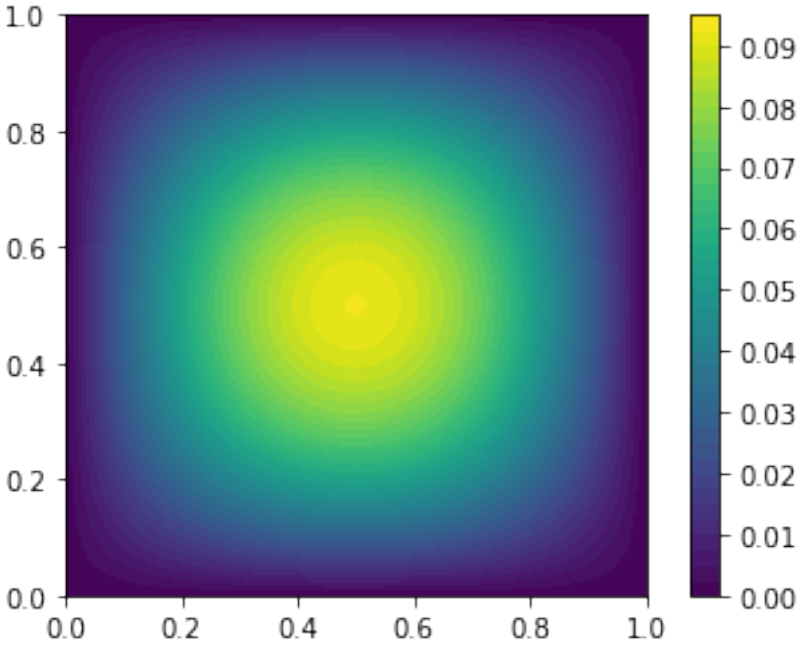
```
t = 0.1:
  u_h(0.5,0.5) = 0.137747
  u(0.5,0.5)   = 0.138911
```



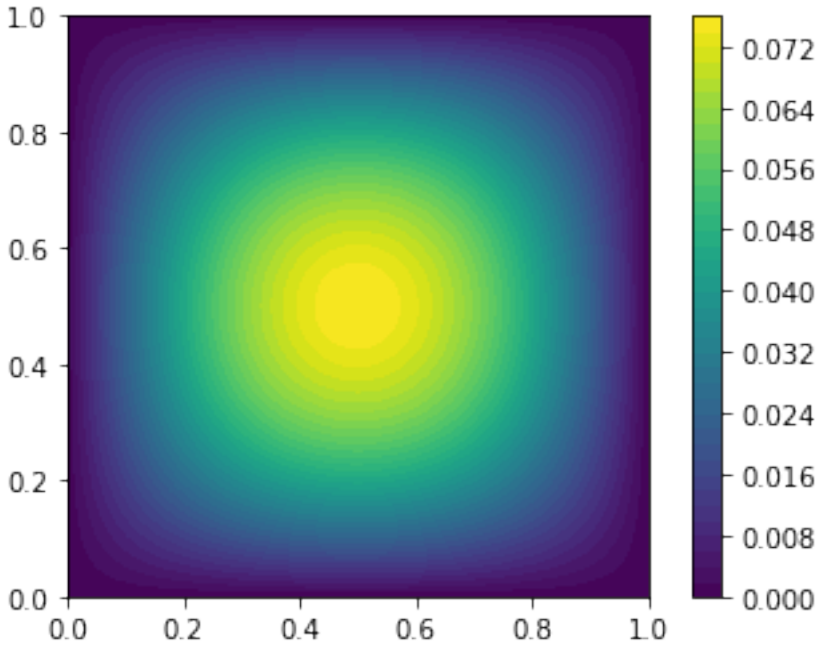

```
t = 0.11:
  u_h(0.5,0.5) = 0.112977
  u(0.5,0.5)   = 0.114028
```



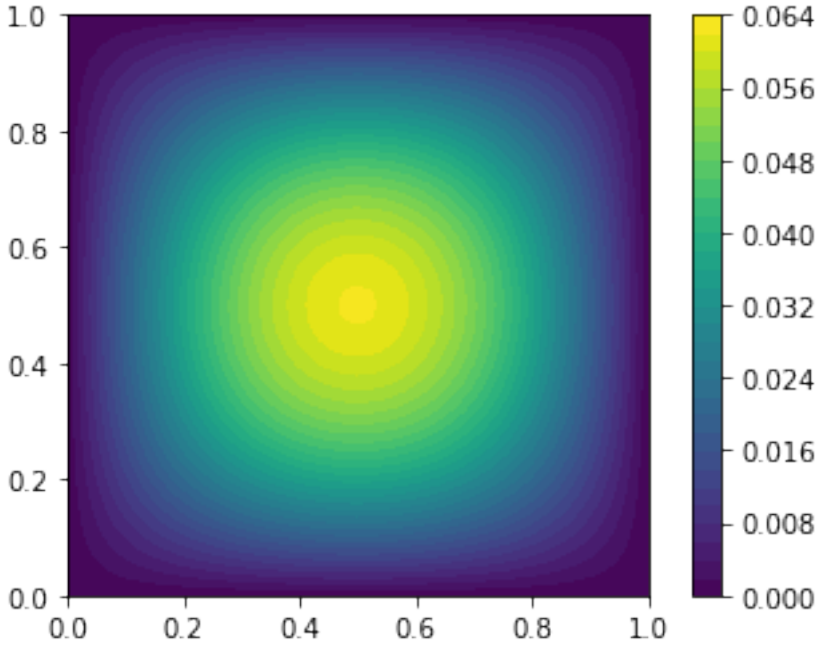

```
t = 0.12:
  u_h(0.5,0.5) = 0.092661
  u(0.5,0.5)   = 0.093602
```



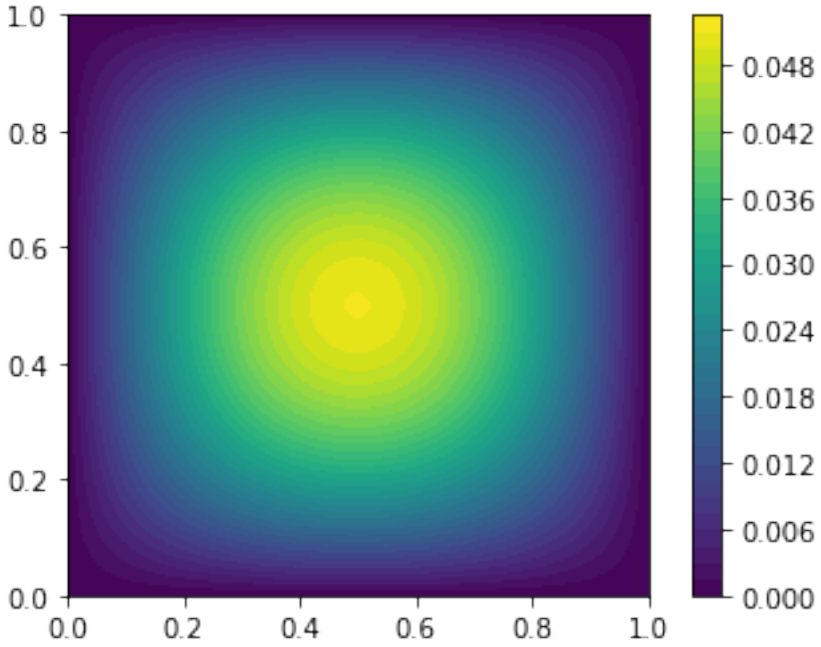
```
t = 0.13:
  u_h(0.5,0.5) = 0.075999
  u(0.5,0.5)   = 0.076835
```



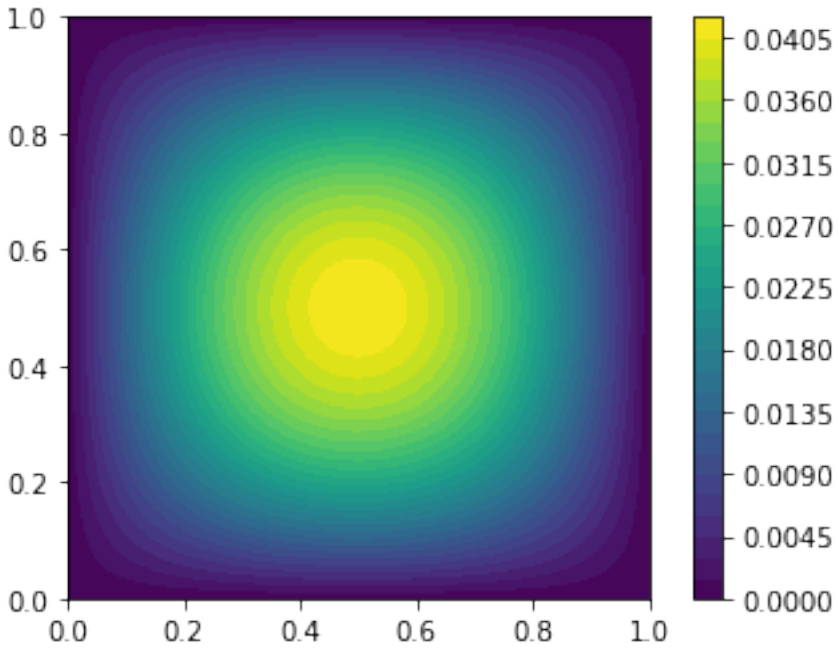

```
t = 0.14:
  u_h(0.5,0.5) = 0.062333
  u(0.5,0.5)   = 0.063071
```



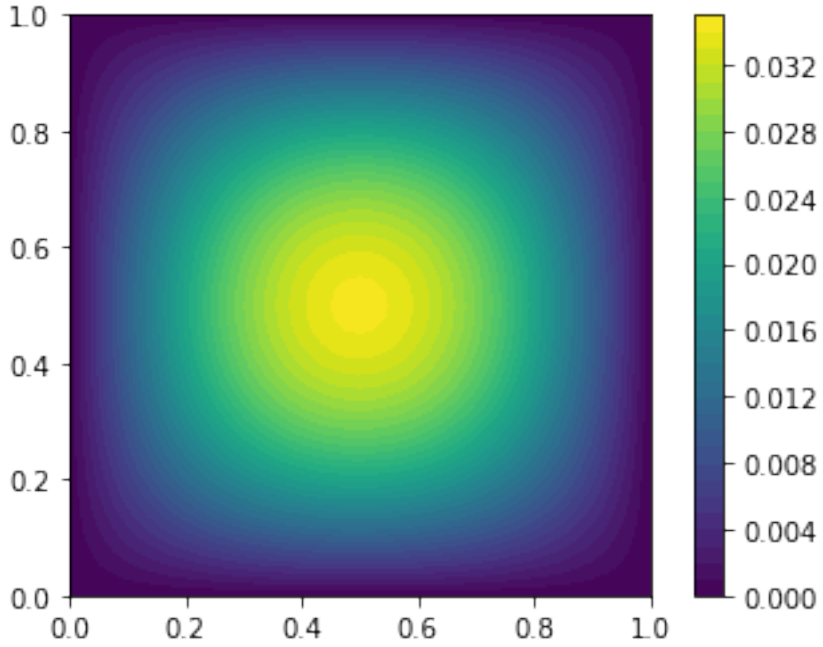
```
t = 0.15:
  u_h(0.5,0.5) = 0.051124
  u(0.5,0.5)   = 0.051773
```



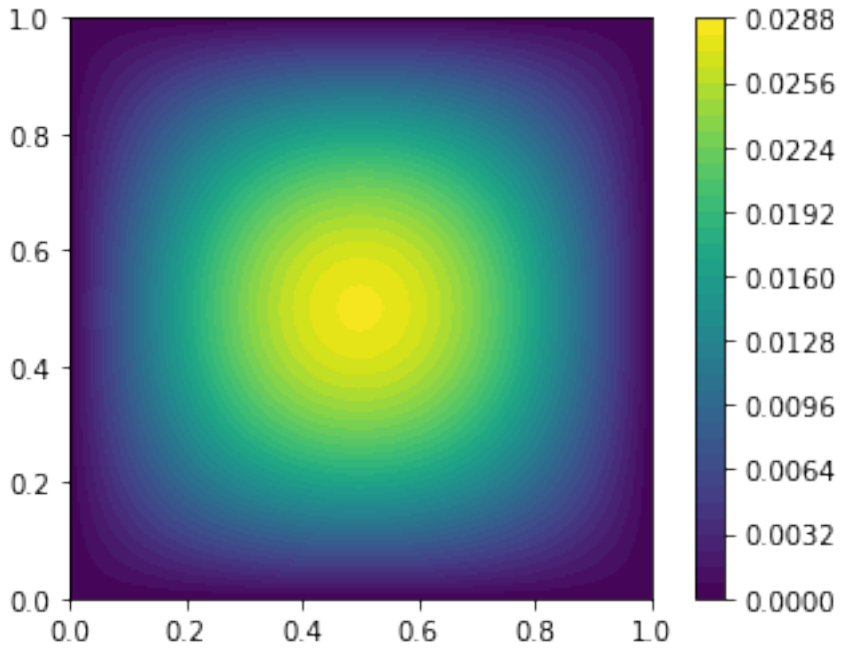

```
t = 0.16:
  u_h(0.5,0.5) = 0.041931
  u(0.5,0.5)   = 0.042499
```



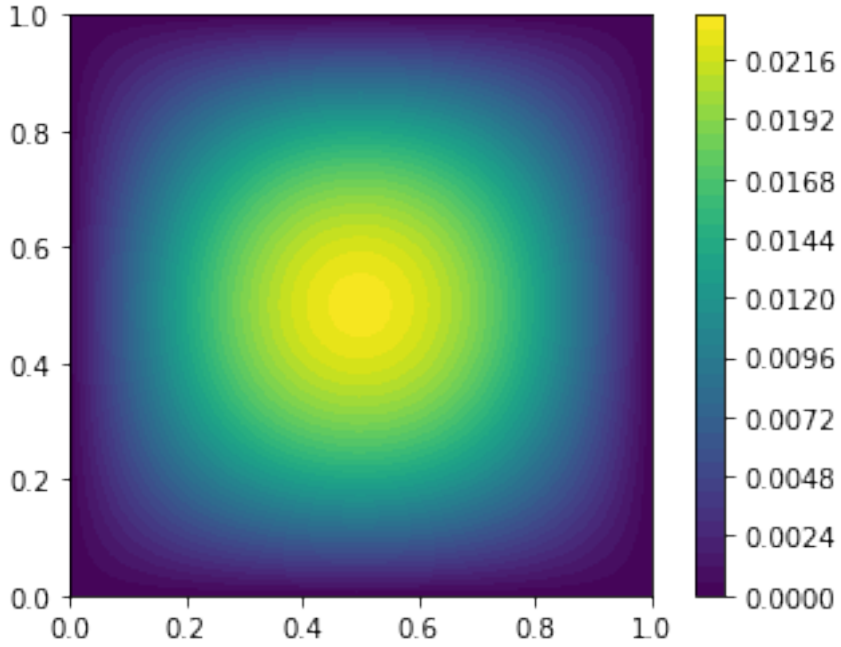

```
t = 0.17:
  u_h(0.5,0.5) = 0.034391
  u(0.5,0.5)   = 0.034886
```



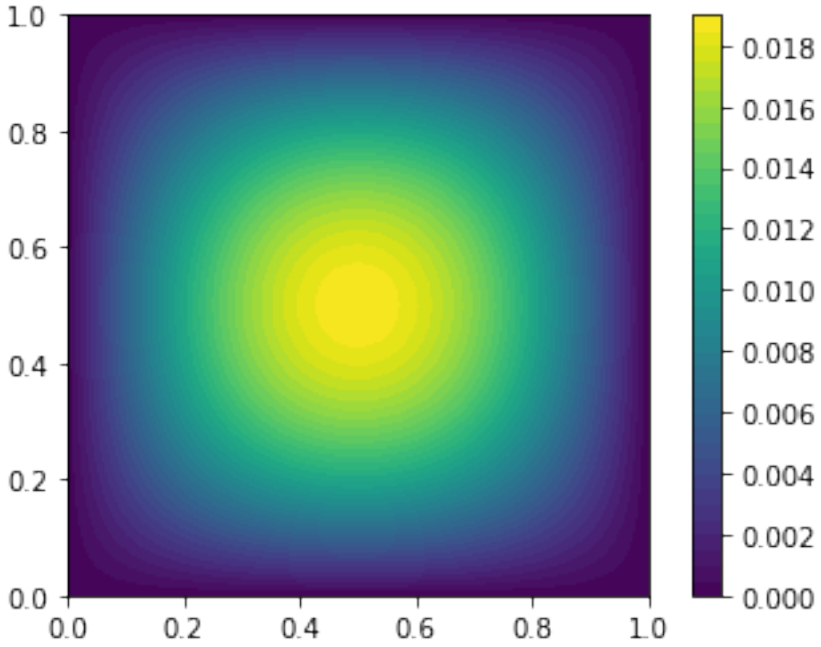
```
t = 0.18:
  u_h(0.5,0.5) = 0.028206
  u(0.5,0.5)   = 0.028637
```



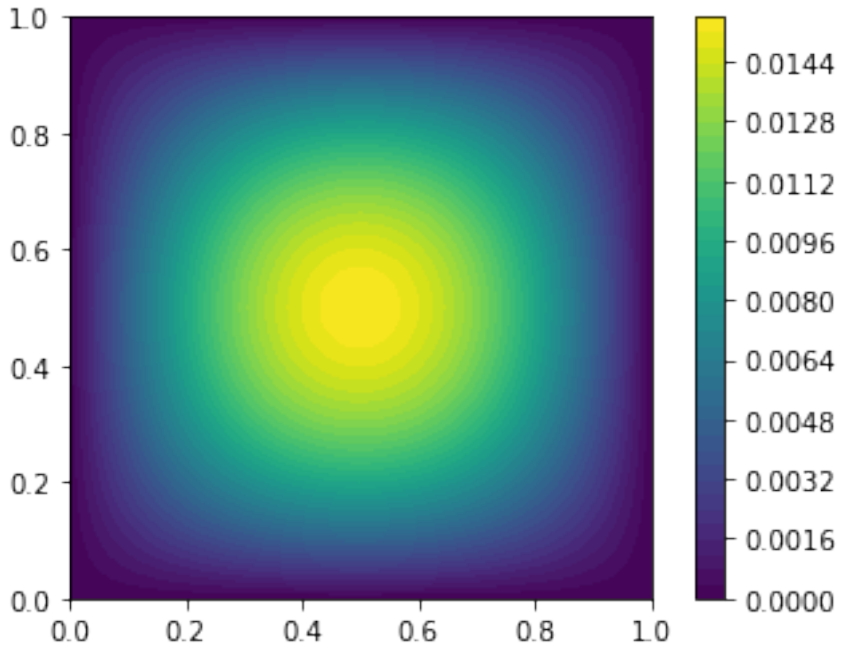

```
t = 0.19:
  u_h(0.5,0.5) = 0.023134
  u(0.5,0.5)   = 0.023507
```



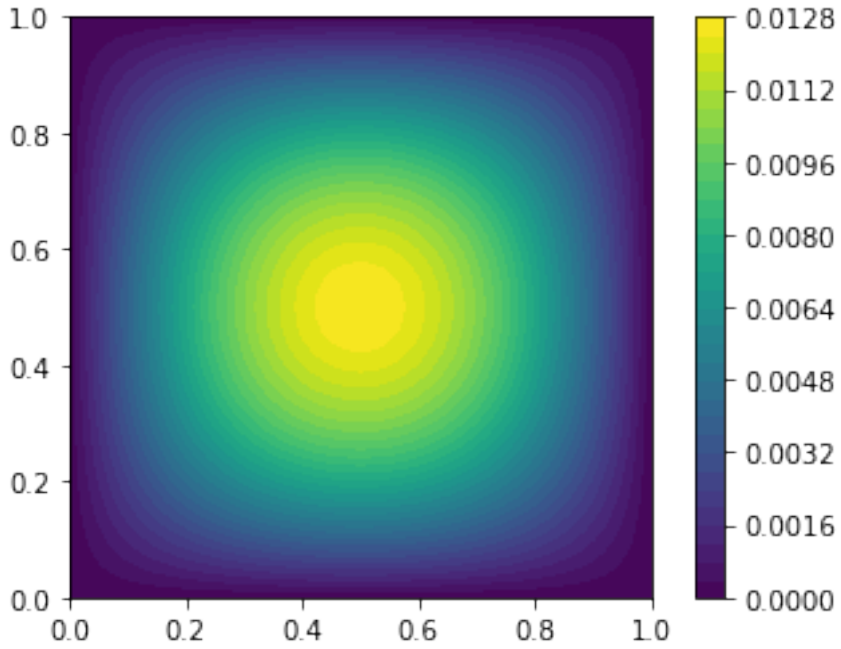
```
t = 0.2:
  u_h(0.5,0.5) = 0.018974
  u(0.5,0.5)   = 0.019296
```



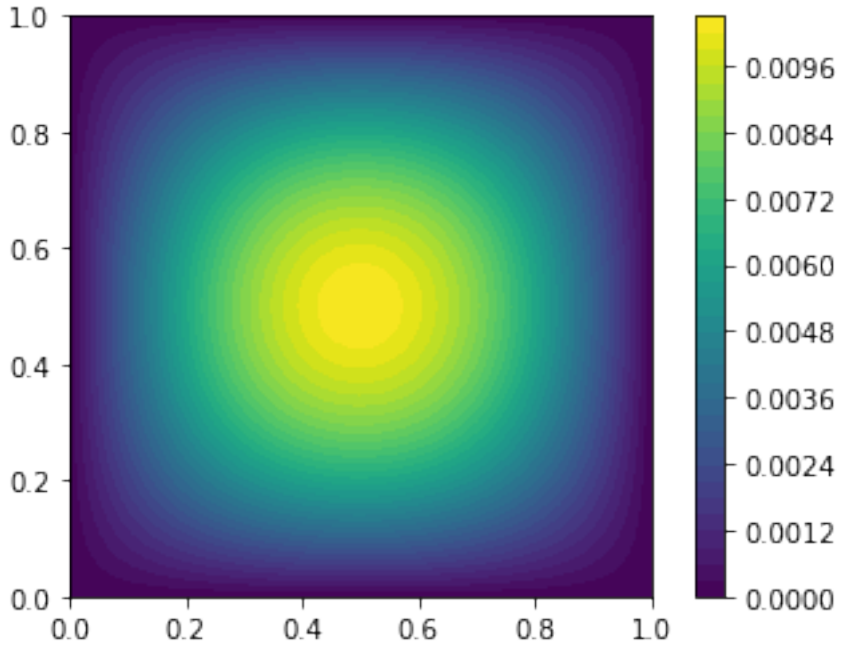

```
t = 0.21:
  u_h(0.5,0.5) = 0.015562
  u(0.5,0.5)   = 0.01584
```



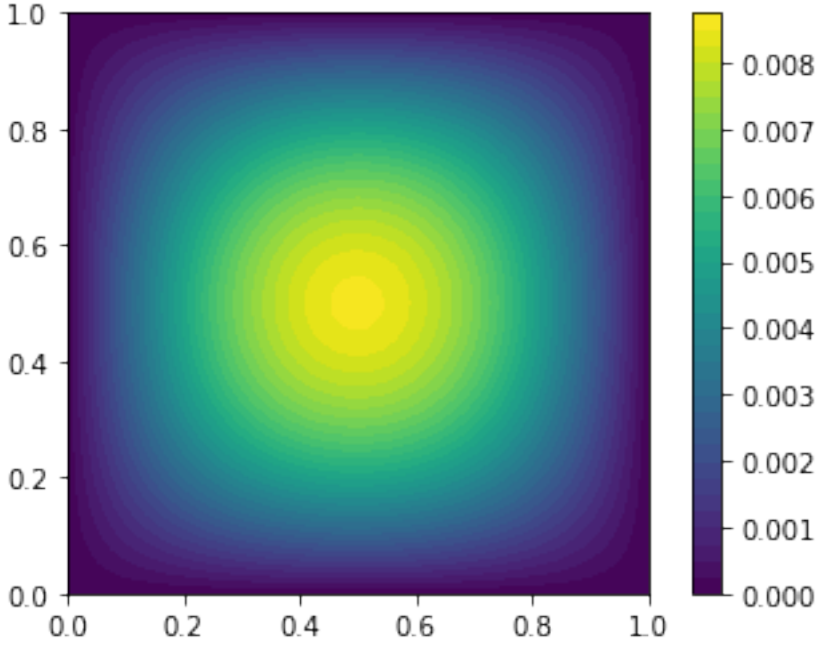

```
t = 0.22:
  u_h(0.5,0.5) = 0.012764
  u(0.5,0.5)   = 0.013002
```



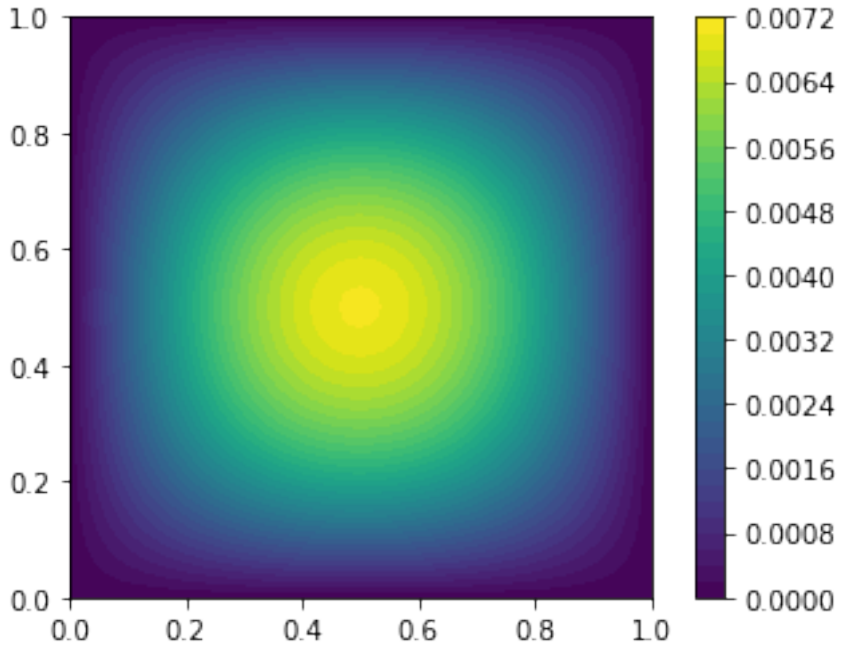
```
t = 0.23:
  u_h(0.5,0.5) = 0.010469
  u(0.5,0.5)   = 0.010673
```



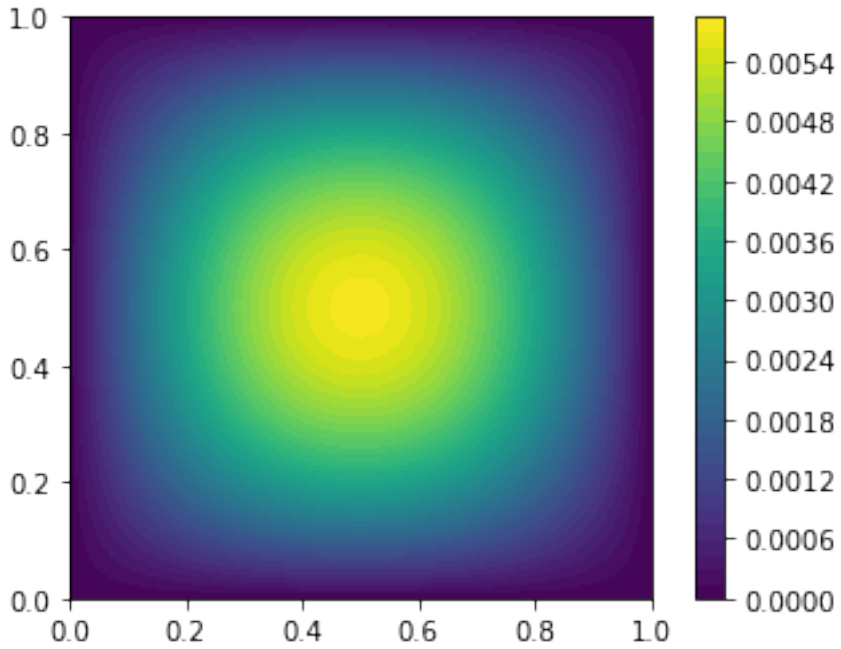

```
t = 0.24:
  u_h(0.5,0.5) = 0.008586
  u(0.5,0.5)   = 0.008761
```



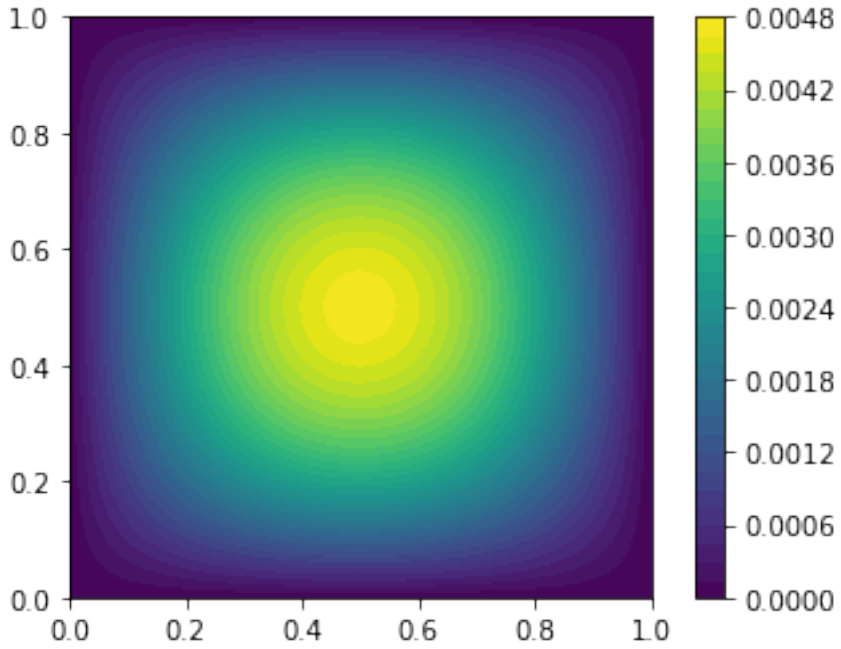
```
t = 0.25:
  u_h(0.5,0.5) = 0.007042
  u(0.5,0.5)   = 0.007192
```



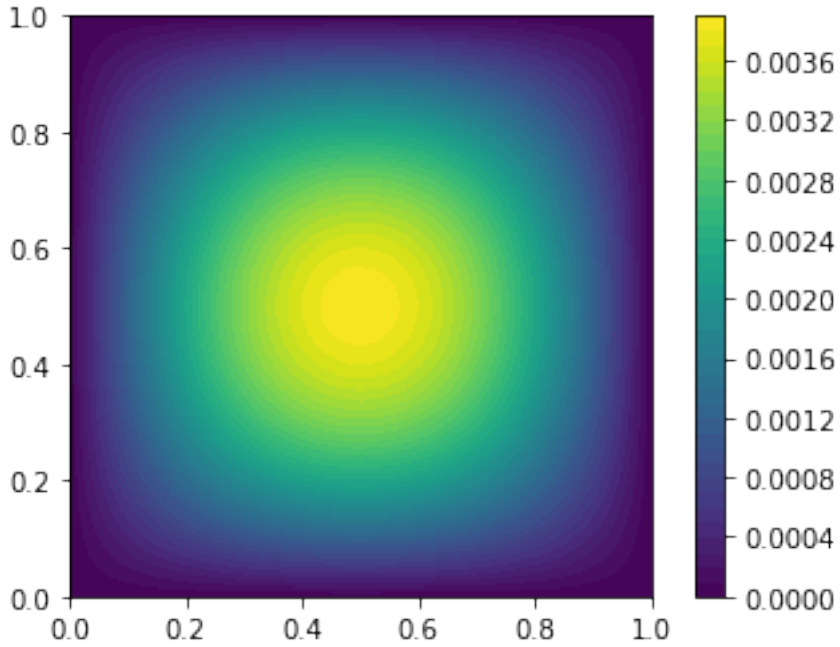

```
t = 0.26:
  u_h(0.5,0.5) = 0.005776
  u(0.5,0.5)   = 0.005904
```



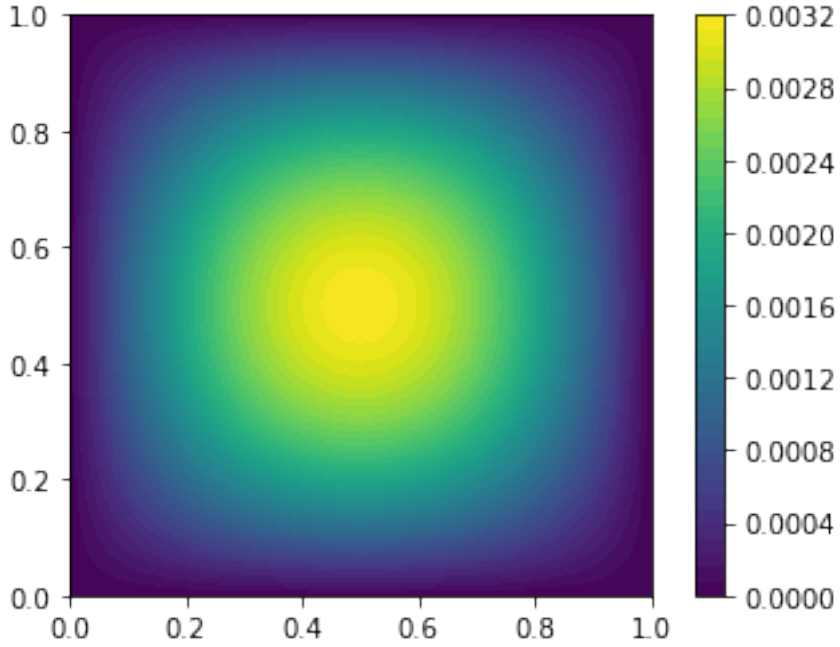

```
t = 0.27:
  u_h(0.5,0.5) = 0.004737
  u(0.5,0.5)   = 0.004846
```



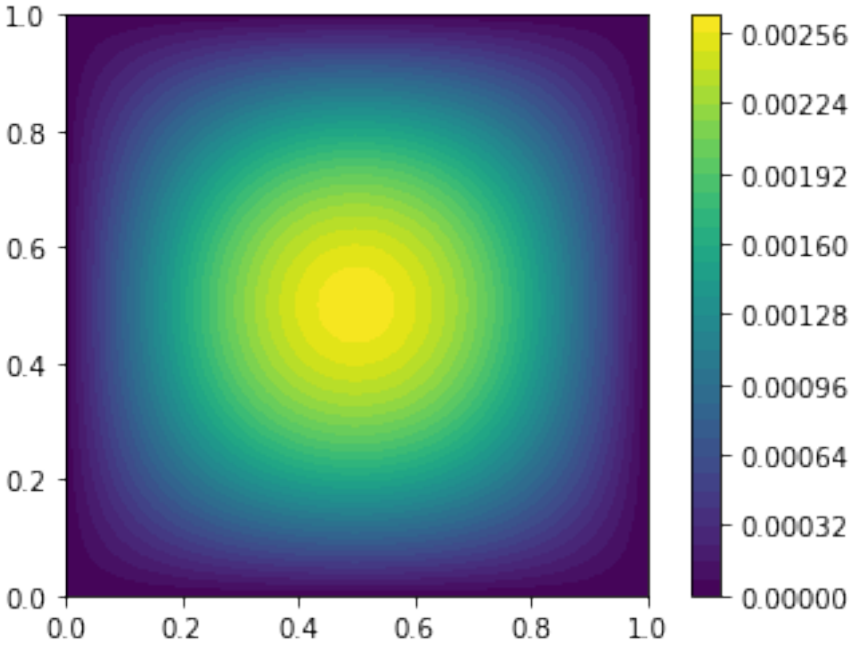
```
t = 0.28:
  u_h(0.5,0.5) = 0.003885
  u(0.5,0.5)   = 0.003978
```



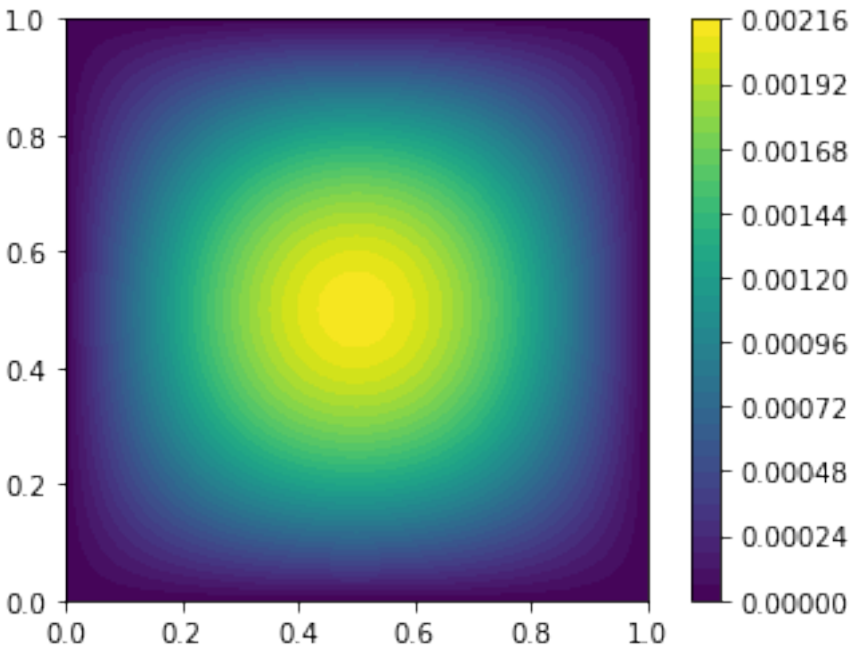

```
t = 0.29:
  u_h(0.5,0.5) = 0.003187
  u(0.5,0.5)   = 0.003265
```



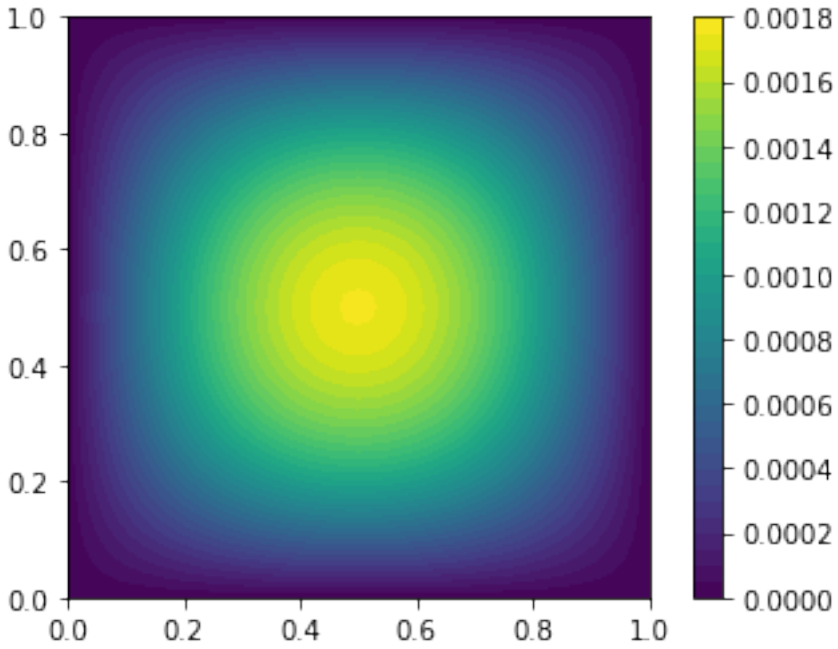
```
t = 0.3:
  u_h(0.5,0.5) = 0.002614
  u(0.5,0.5)   = 0.00268
```



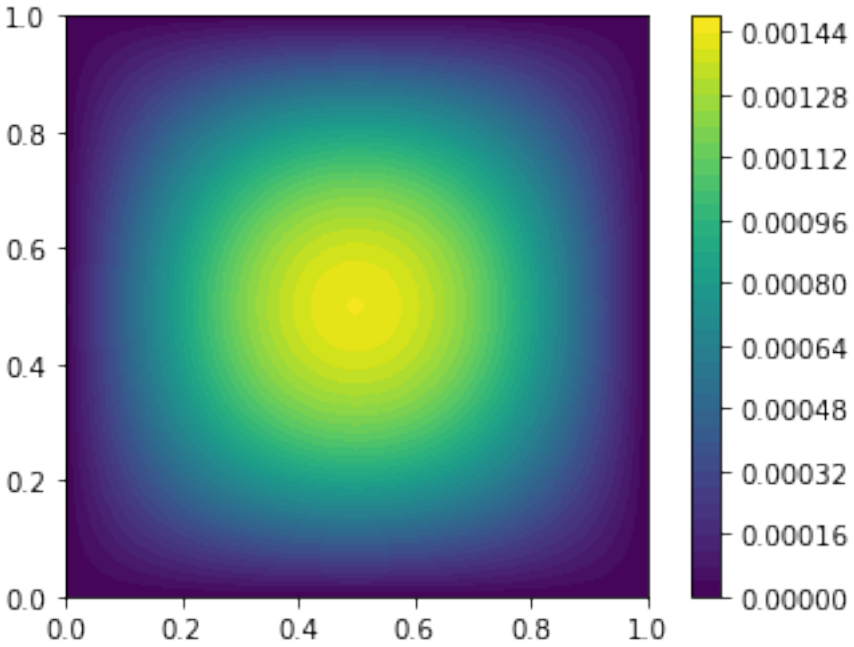

```
t = 0.31:
  u_h(0.5,0.5) = 0.002144
  u(0.5,0.5)   = 0.0022
```



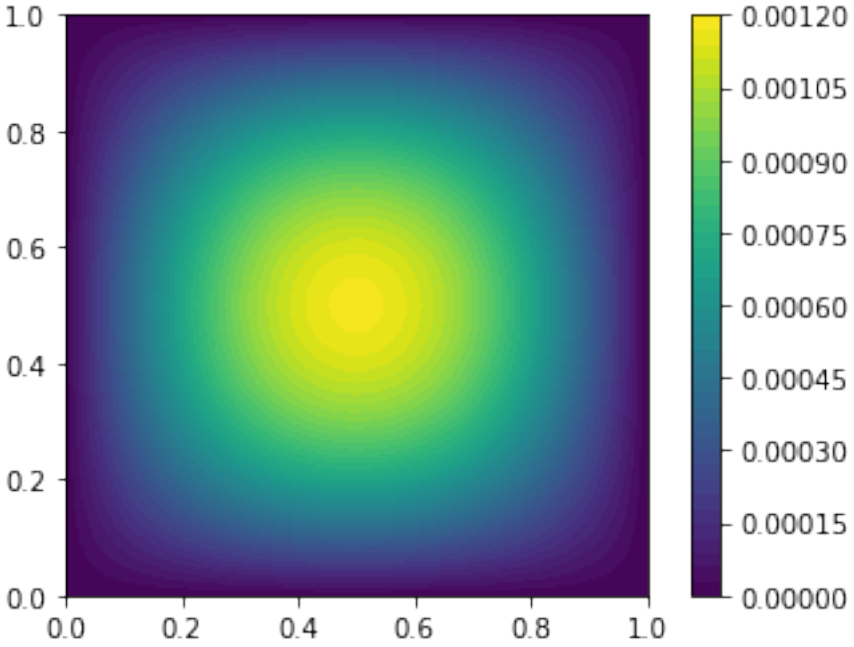

```
t = 0.32:
  u_h(0.5,0.5) = 0.001758
  u(0.5,0.5)   = 0.001806
```



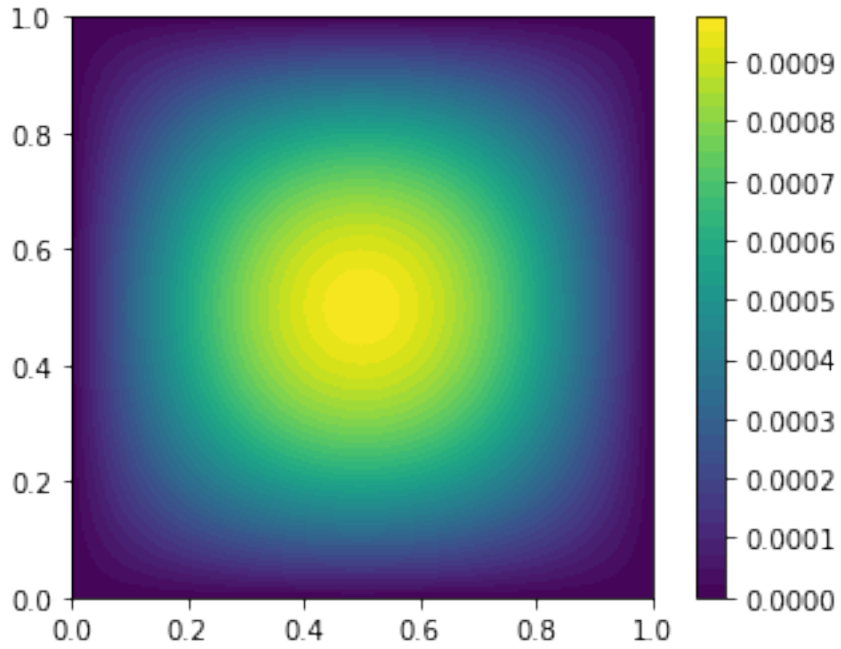
```
t = 0.33:
  u_h(0.5,0.5) = 0.001442
  u(0.5,0.5)   = 0.001483
```



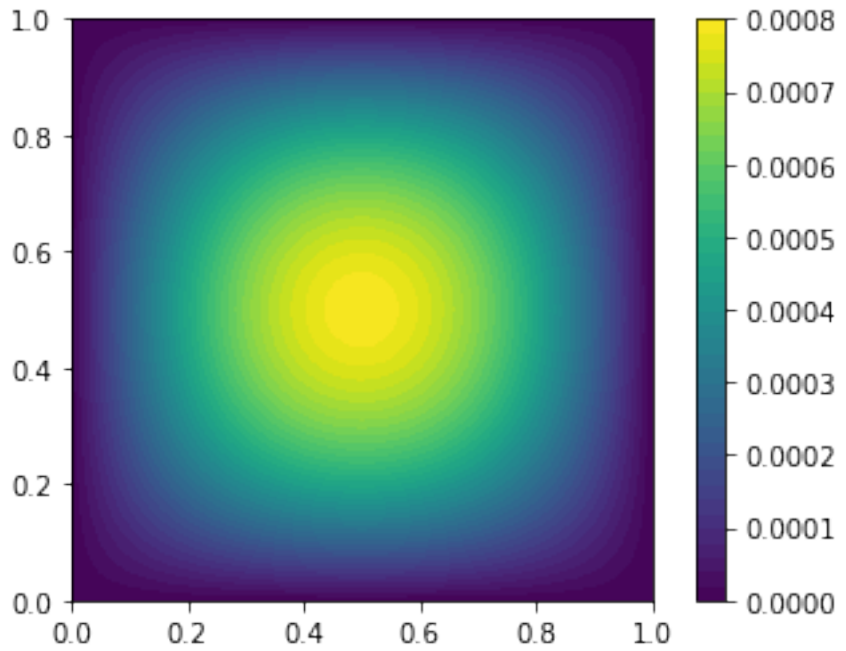

```
t = 0.34:
  u_h(0.5,0.5) = 0.001183
  u(0.5,0.5)   = 0.001217
```



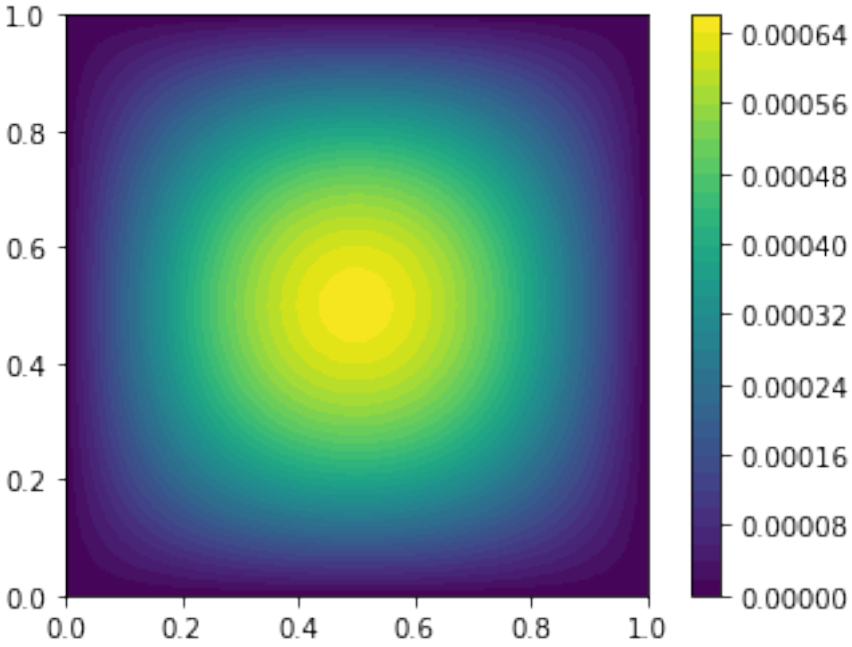
```
t = 0.35:
  u_h(0.5,0.5) = 0.00097
  u(0.5,0.5)   = 0.000999
```



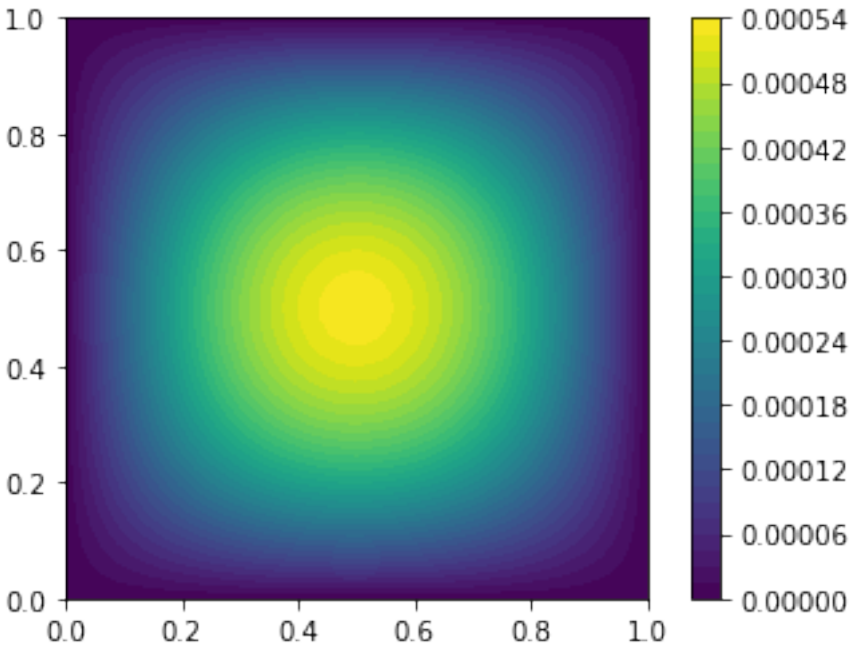

```
t = 0.36:
  u_h(0.5,0.5) = 0.000796
  u(0.5,0.5)   = 0.00082
```



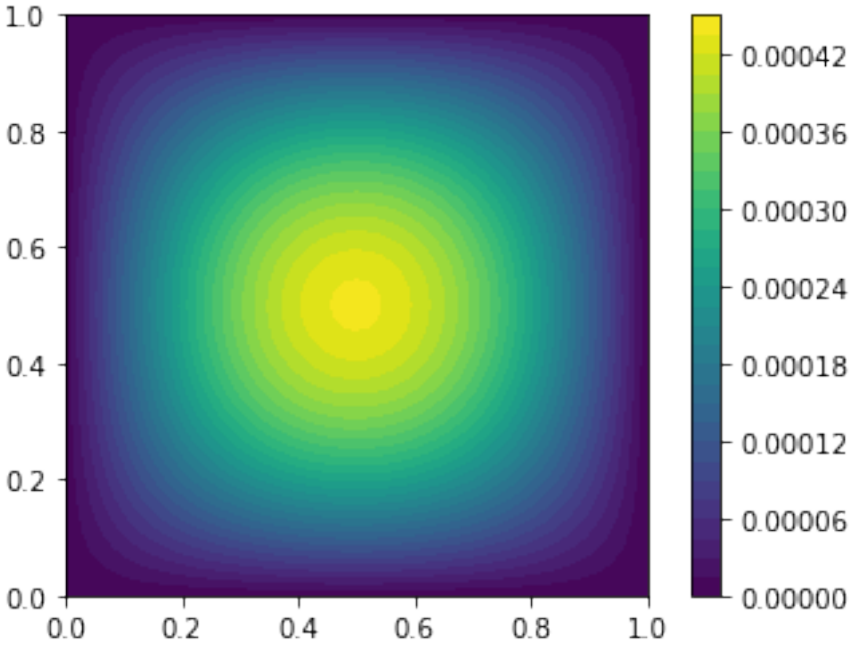

```
t = 0.37:
  u_h(0.5,0.5) = 0.000653
  u(0.5,0.5)   = 0.000673
```



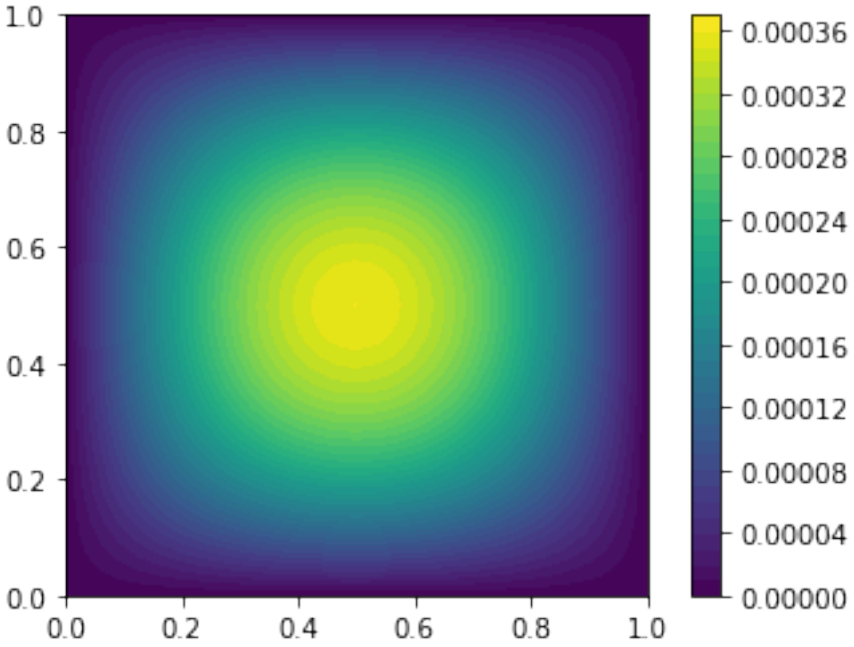
```
t = 0.38:
  u_h(0.5,0.5) = 0.000535
  u(0.5,0.5)   = 0.000553
```



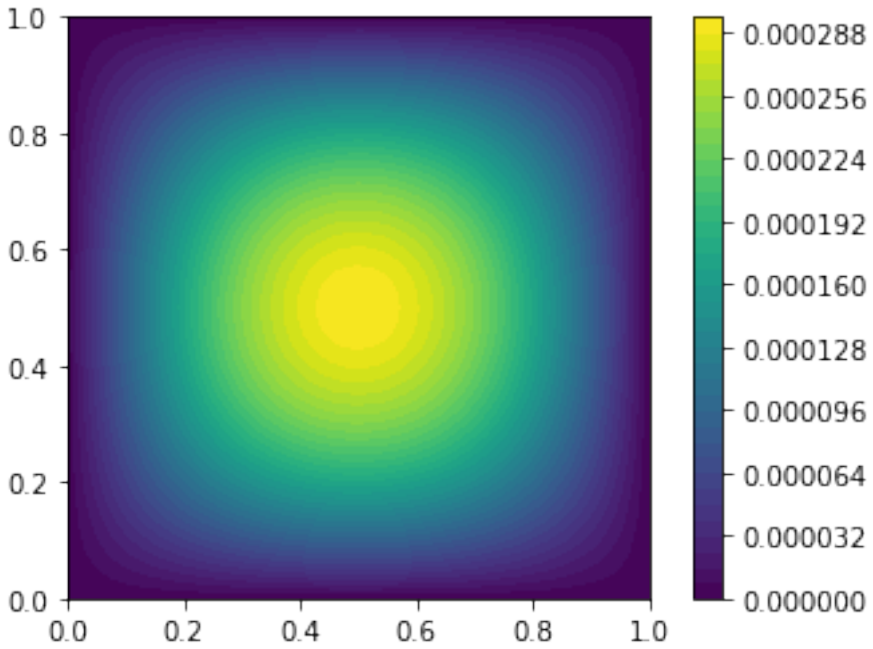

```
t = 0.39:
  u_h(0.5,0.5) = 0.000439
  u(0.5,0.5)   = 0.000454
```



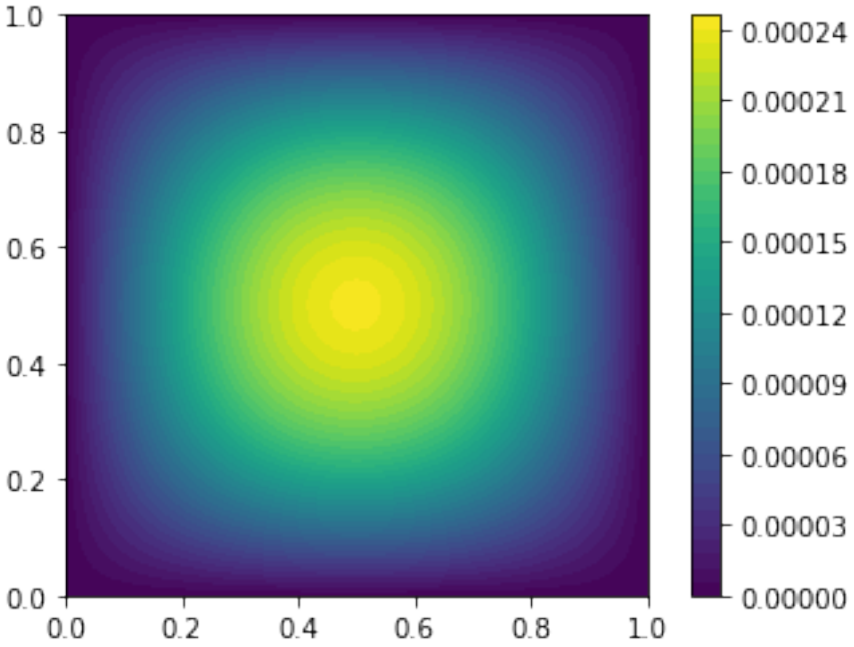
```
t = 0.4:
  u_h(0.5,0.5) = 0.00036
  u(0.5,0.5)   = 0.000372
```



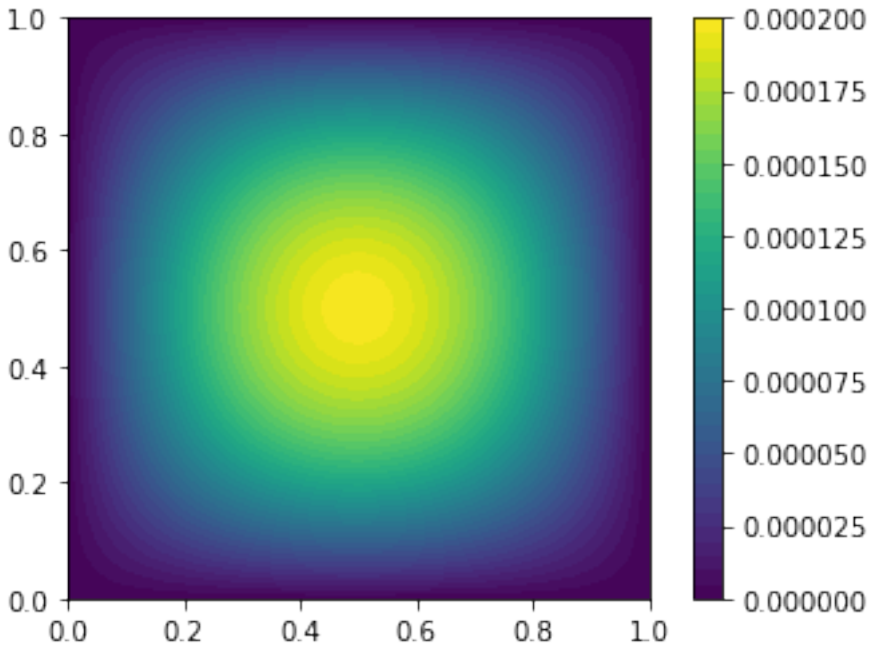

```
t = 0.41:
  u_h(0.5,0.5) = 0.000295
  u(0.5,0.5)   = 0.000306
```



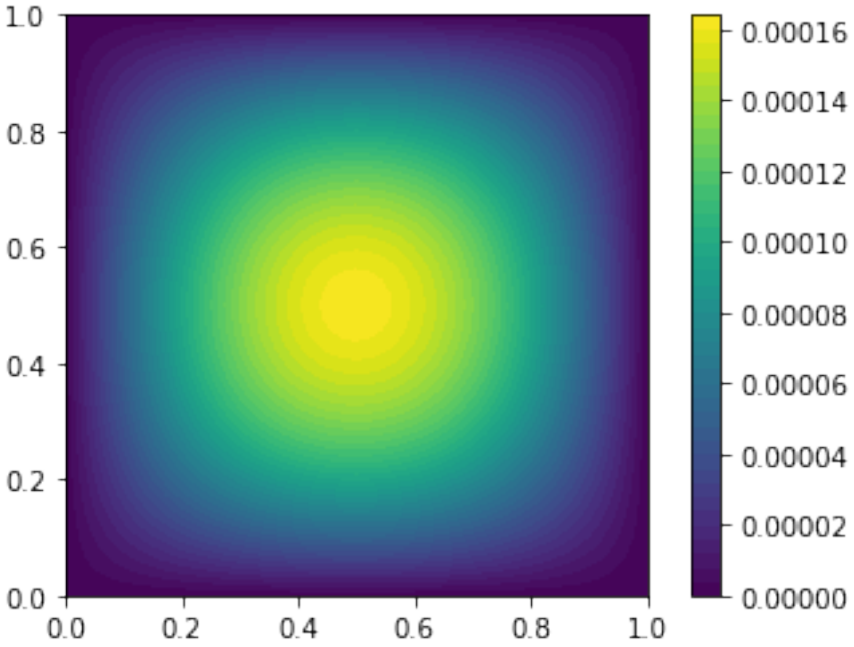

```
t = 0.42:
  u_h(0.5,0.5) = 0.000242
  u(0.5,0.5)   = 0.000251
```



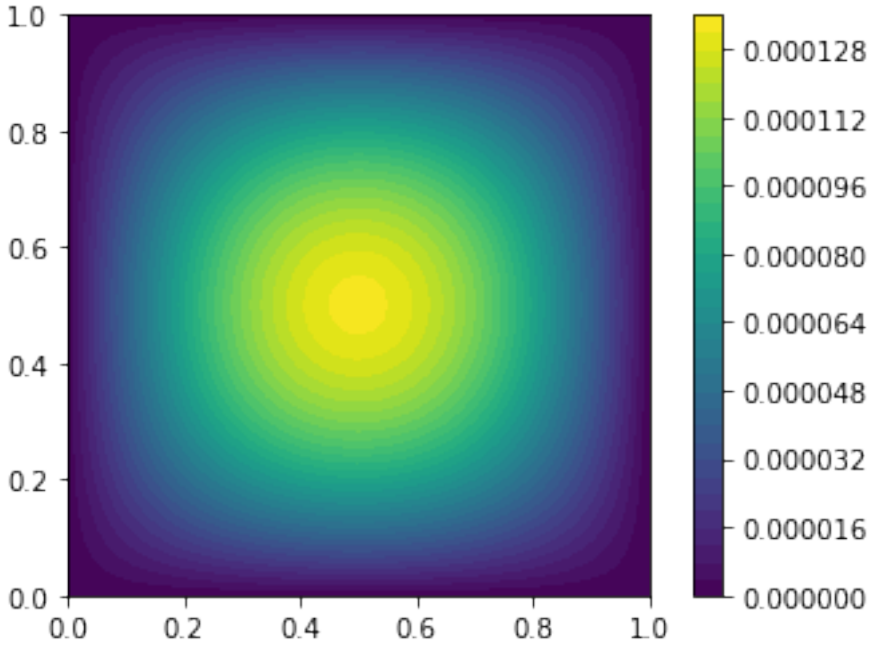
```
t = 0.43:
  u_h(0.5,0.5) = 0.000199
  u(0.5,0.5)   = 0.000206
```



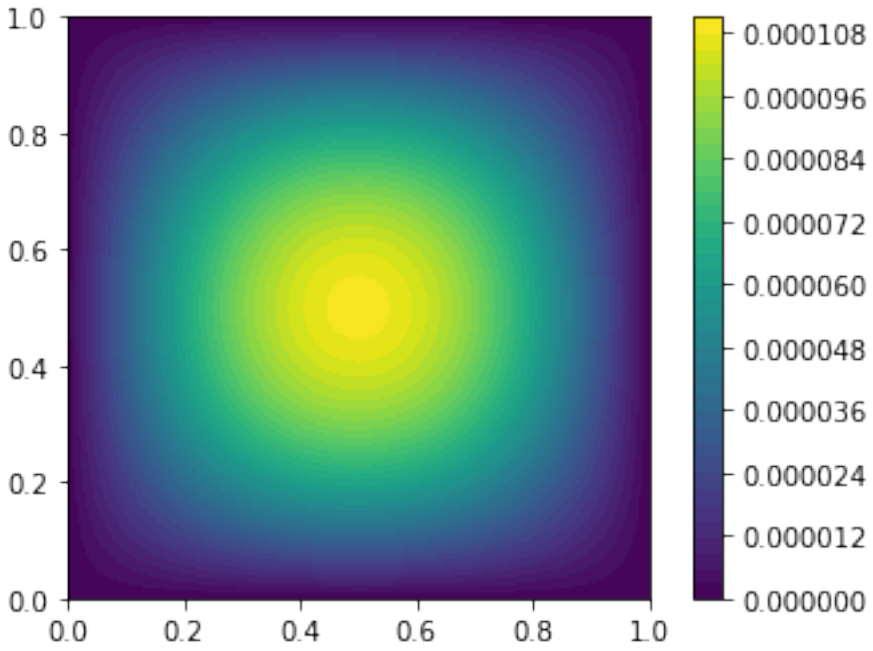

```
t = 0.44:
  u_h(0.5,0.5) = 0.000163
  u(0.5,0.5)   = 0.000169
```



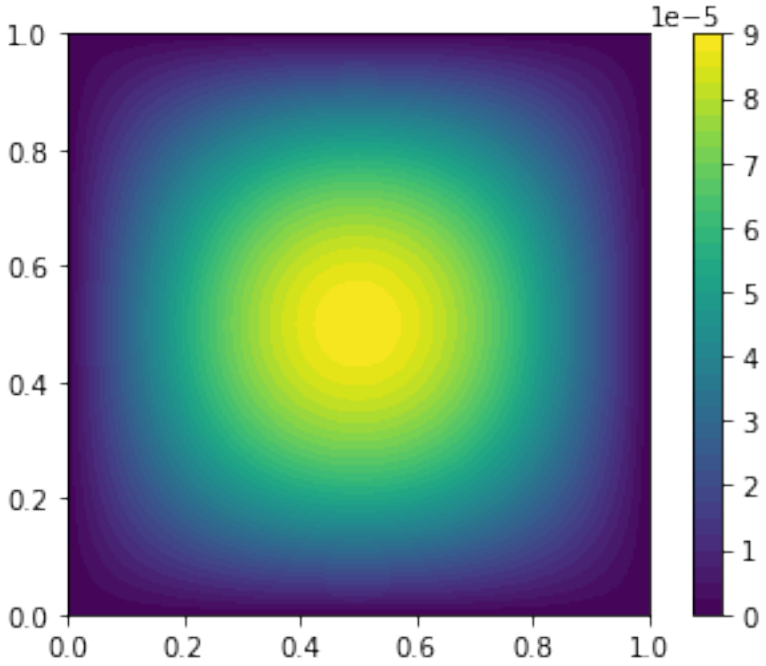
```
t = 0.45:
  u_h(0.5,0.5) = 0.000134
  u(0.5,0.5)   = 0.000139
```



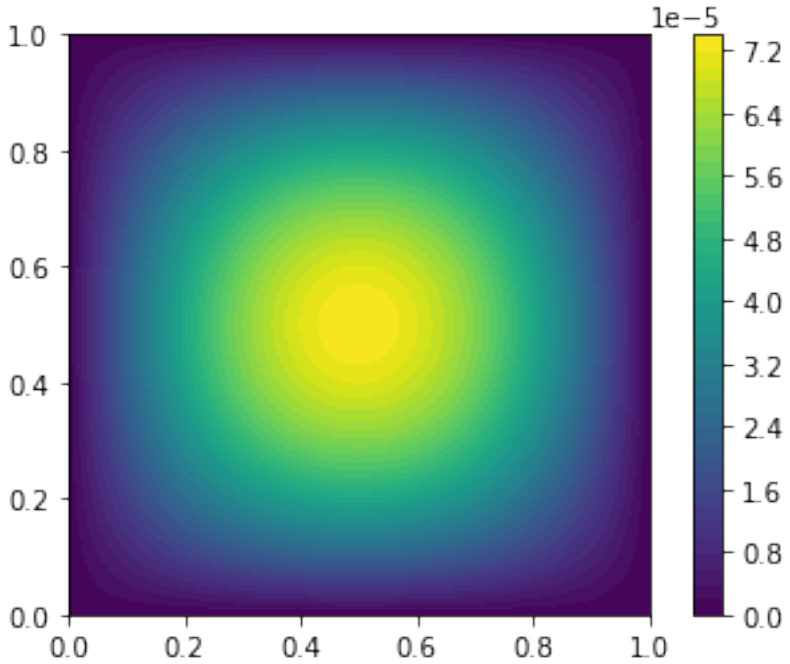

```
t = 0.46:
  u_h(0.5,0.5) = 0.00011
  u(0.5,0.5)   = 0.000114
```



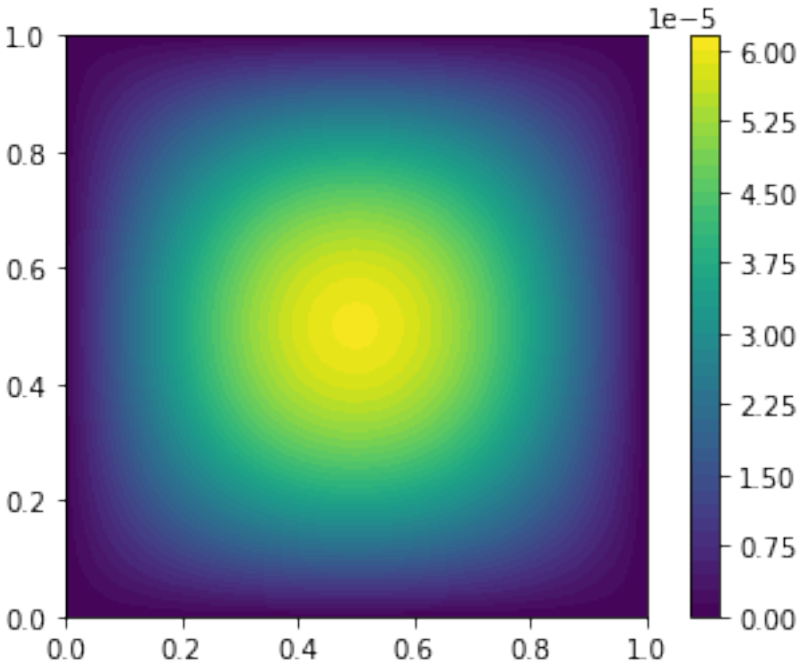

```
t = 0.47:
  u_h(0.5,0.5) = 9e-05
  u(0.5,0.5)   = 9.4e-05
```



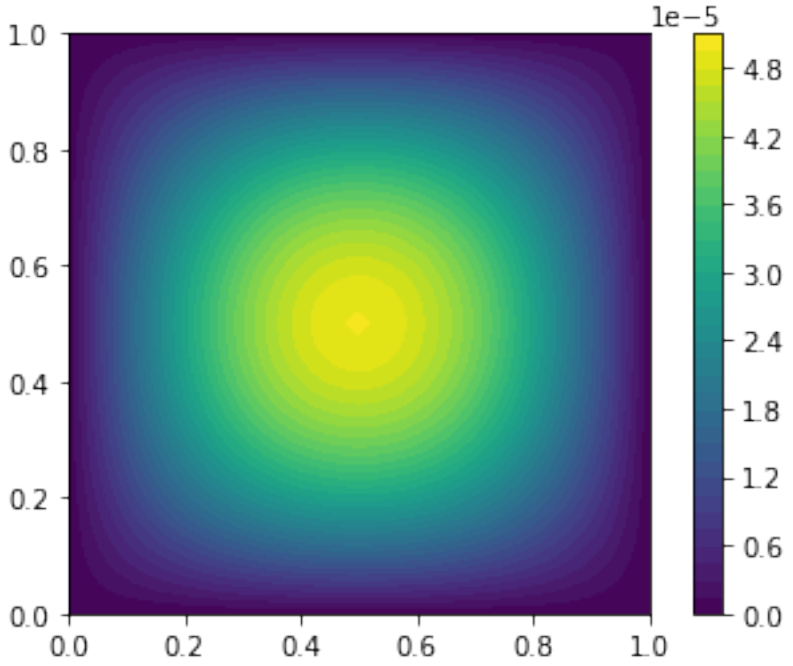
```
t = 0.48:
  u_h(0.5,0.5) = 7.4e-05
  u(0.5,0.5)   = 7.7e-05
```




```
t = 0.49:
  u_h(0.5,0.5) = 6e-05
  u(0.5,0.5)   = 6.3e-05
```




```
t = 0.5:
  u_h(0.5,0.5) = 5e-05
  u(0.5,0.5)   = 5.2e-05
```



1.4 Reduced Order Modeling

We saw in the previous section that to solve the heat equation with finite elements we need to solve a linear equation system of size m, where m denotes the number of degrees of freedom. For 3D problems, complex geometries or fine meshes the number of degrees of freedom can quickly become very large, which makes this linear system computationally expensive to solve. However, for many applications in industry it might be required that these simulations are (almost) real-time. Hence, in such scenarios reduced order models of size $r \ll m$ are being used. A popular approach for this is Proper Orthogonal Decomposition (POD) based reduced order modeling (ROM), where one tries to find a new basis from the snapshots of the high fidelity simulation and then carries out all remaining simulations in this low dimensional function space.

In the following, we will discuss the main aspects of POD-ROM, but for more details please check out my master's thesis or Prof. Volkwein's lecture notes.

1.4.1 Compute Proper Orthogonal Decomposition (POD) basis

In the first part of reduced order modeling, we first need to find a new basis of size $r \ll m$, which preserves the majority of the information of the high fidelity simulation. To find this POD basis, we need to follow the steps:

1. assemble the correlation matrix
2. compute the eigenvalues/-vectors of the correlation matrix
3. determine the POD basis size r
4. caclulate the POD basis vectors from the eigenvalues/-vectors of the correlation matrix

1. Assemble the correlation matrix \ The correlation matrix is defined as

$$C := D^{\frac{1}{2}} Y^T W Y D^{\frac{1}{2}},$$

where Y is the snapshot matrix, W is the weight matrix, which is usually either the mass matrix, i.e. $W = M$, or the Laplace matrix, i.e. $W = K$, and D is a diagonal matrix containing the time quadrature weights. For our computations, we choose $W = M$ and due to a uniform time step size, we get with the trapezoidal rule for the integration of the time integral that $D \approx \frac{1}{\Delta t} I$. Here, I stands for the identity matrix in $\mathbb{R}^{m \times m}$. With these simplifications, we get

$$C := \frac{1}{\Delta t} \cdot Y^T M Y.$$

[ ]:
```
# assemble correlation matrix
correlation_matrix = (1. / Δt) * np.dot(np.dot(Y.T, M.array()),Y)
```

2. Compute the eigenvalues/-vectors of the correlation matrix \ Next, we compute the eigenvalues and eigenvectors of this correlation matrix. Since M is symmetric, the correlation matrix is a real, symmetric matrix and thus only has real eigenvalues and eigenvectors. Additionally, we sort the eigenavalues in descending order, which we will need to determine the size of the POD basis.

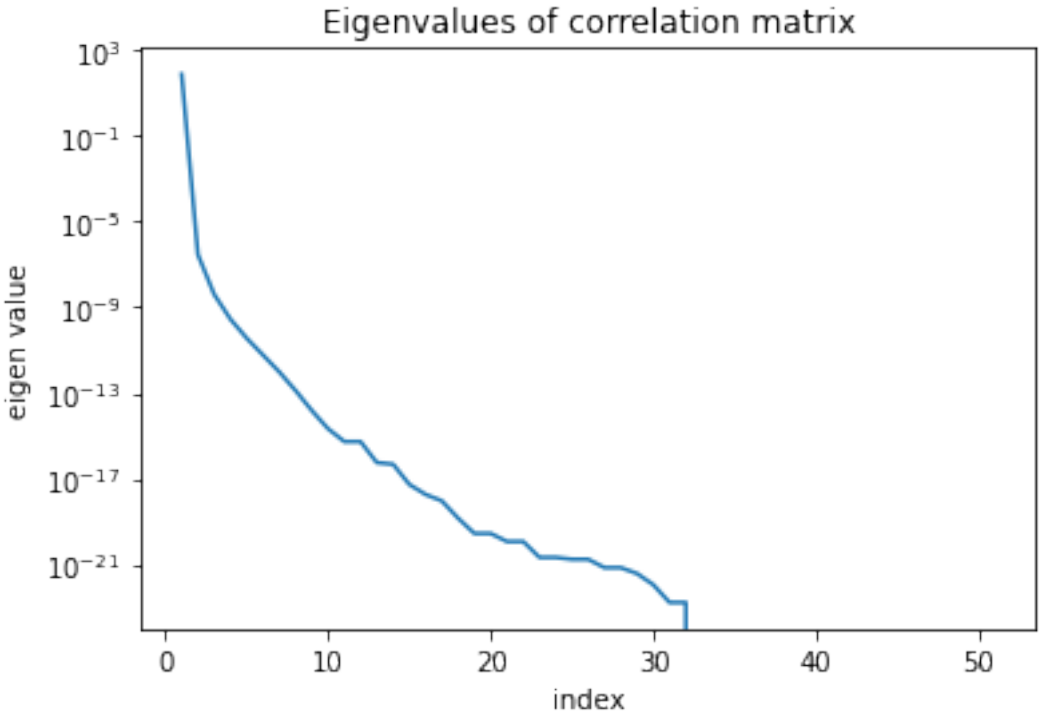
[ ]:
```
# compute eigenvalues of correlation matrix
eigen_values, eigen_vectors = np.linalg.eig(correlation_matrix)
```

```
# ignore the complex part of the eigen values and eigen vectors
eigen_values, eigen_vectors = eigen_values.astype('float64'), eigen_vectors.
 ↪astype('float64')

# sort the eigenvalues in descending order
eigen_values, eigen_vectors = eigen_values[eigen_values.argsort()[::-1]],␣
 ↪eigen_vectors[eigen_values.argsort()[::-1]]
```

```
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:5: ComplexWarning:
Casting complex values to real discards the imaginary part
  """
```

To verify whether using a POD based reduced order model actually makes sense, we need to check whether the the eigenvalues of the correlation matrix decay sufficiently fast. If this is the case, then only a small number of POD modes will be sufficient to build a reduced order model. There are also PDEs for which this assumption is not satisfied, e.g. one-dimensional linear transport. For more information on this, please check out the book from Quarteroni and Rozza "Reduced Order Methods for Modeling and Computation".

[ ]:
```
plt.plot(range(1,min(n,m)+1), eigen_values)
plt.yscale("log")
plt.xlabel("index")
plt.ylabel("eigen value")
plt.title("Eigenvalues of correlation matrix")
plt.show()
```



We observe that the eigenvalues decay very rapidly and hence we expect to get a reduced order model with only a few POD basis vectors.

3. Determine the POD basis size r \

There is no formula for the optimal size for the POD basis. Instead, most of the POD literature uses a heuristic, which tries to preserve a certain amount of the energy of the snapshots. The energy of the high fidelity solution is

$$\sum_{i=1}^{n} \lambda_i,$$

where $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of the correlation matrix. If we take only the r largest eigenvalues/-vectors of the correlation matrix, it can be shown that they have the energy

$$\sum_{i=1}^{r} \lambda_i.$$

The heuristic from the literature is then to find the smallest $r \in \mathbb{N}$ such that the energy ratio

$$\frac{\sum_{i=1}^{r} \lambda_i}{\sum_{i=1}^{n} \lambda_i}$$

is larger than a user-defined threshold, e.g. 0.99, 0.999, ...

[ ]:
```
# desired energy ratio
ENERGY_RATIO_THRESHOLD = 0.999

# determine number of POD basis vectors needed to preserve certain ratio of␣
 ↪energy
r = np.sum([(eigen_values[:i].sum() / eigen_values.sum()) <␣
 ↪ENERGY_RATIO_THRESHOLD for i in range(n)])
print(f"To preserve {ENERGY_RATIO_THRESHOLD} of information we need {r} POD␣
 ↪vector(s). (result: {eigen_values[:r].sum() / eigen_values.sum()}␣
 ↪information).")
```

```
To preserve 0.999 of information we need 1 POD vector(s). (result:
0.9999999961606815 information).
```

4. Caclulate the POD basis vectors from the eigenvalues/-vectors of the correlation matrix \ Finally, we find that the POD basis is given by

$$\vec{\psi}_i := \frac{Y D^{\frac{1}{2}} \vec{w}_i}{\sqrt{\lambda_i}} \qquad \forall 1 \leq i \leq r,$$

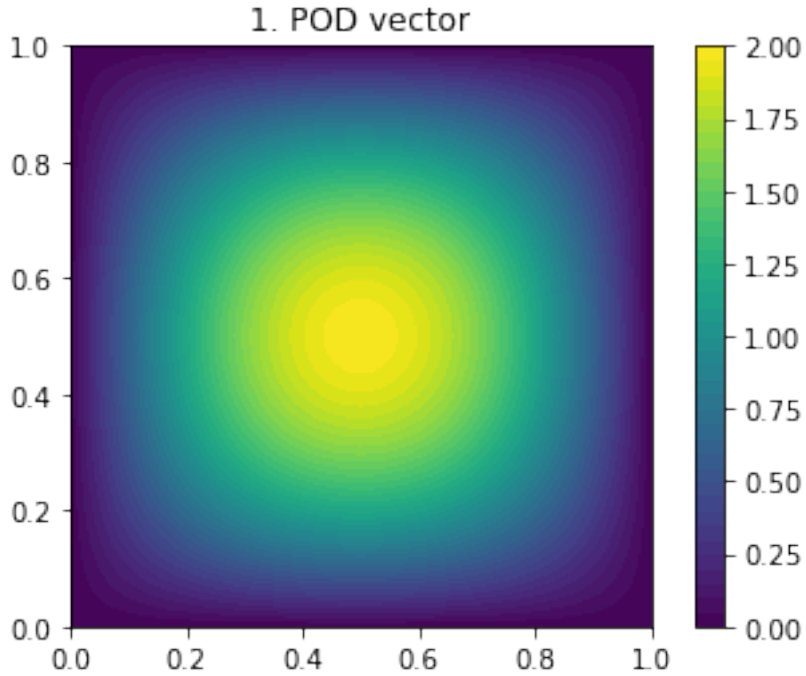
where $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues and $\{\vec{w}_i\}_{i=1}^n$ denote the eigenvectors of the correlation matrix. Using $D \approx \frac{1}{\Delta t} I$, we then get that

$$\vec{\psi}_i := \sqrt{\frac{1}{\Delta t}} \frac{Y \vec{w}_i}{\sqrt{\lambda_i}} \qquad \forall 1 \leq i \leq r.$$

[ ]:
```
# compute POD basis vectors
pod_basis = np.sqrt(1. / Δt) * np.dot(Y, eigen_vectors[:,:r])  / np.diag(np.
  ↪sqrt(eigen_values[:r]))
```

Having computed our POD basis, we can now visualize the individual POD vectors.

[ ]:
```
# Plot solution
w = Function(V)
for i in range(r):
  w.vector()[:] = pod_basis[:, i]
  c = plot(w)
  plt.colorbar(c)
  plt.title(f"{i+1}. POD vector")
  plt.show()
```



Since $r = 1$, we have only one POD vector. This was also to be expected, since the analytical solution is given by

$$u(t, x, y) = \exp(-2\pi^2 t) \sin(\pi x) \sin(\pi y).$$

and thus we only expected one POD mode in the first place. Furthermore, we see that the POD mode resembles a scaled version of

$$\sin(\pi x) \sin(\pi y),$$

which is what we hoped for.

1.4.2 Compute reduced matrices

For the reduced order modeling, we now need to transform all our matrices and vectors from the FEM space to the ROM space. For this we write all our POD basis vectors in a matrix

$$\Psi := [\vec{\psi}_1, \cdots, \vec{\psi}_r] \in \mathbb{R}^{m \times r}.$$

Let us first discuss how we can transform a general FEM matrix $A_h \in \mathbb{R}^{m \times m}$, where the subscript h indicates that this is a FEM matrix. To get its reduced counterpart, we need to perform a change of basis by

$$A_r := \underbrace{\Psi^T}_{\in \mathbb{R}^{r \times m}} \underbrace{A_h}_{\in \mathbb{R}^{m \times m}} \underbrace{\Psi}_{\in \mathbb{R}^{m \times r}} \in \mathbb{R}^{r \times r}.$$

Here the subscript r indicates that this is a ROM matrix. \ Similarly, for a general FEM vector $\vec{v}_h \in \mathbb{R}^m$, we get its reduced counterpart by the change of basis

$$\vec{v}_r := \underbrace{\Psi^T}_{\in \mathbb{R}^{r \times m}} \underbrace{\vec{v}_h}_{\in \mathbb{R}^m} \in \mathbb{R}^r.$$

We now apply this change of basis to the system matrix $A^{n+1} \in \mathbb{R}^{m \times m}$ and the right hand side matrix $A^n \in \mathbb{R}^{m \times m}$. We now need to solve the reduced linear equation system: \ Find $\vec{U}_r^{n+1} \in \mathbb{R}^r$ such that

$$\underbrace{A_r^{n+1}}_{\in \mathbb{R}^{r \times r}} \vec{U}_r^{n+1} = \underbrace{A_r^n}_{\in \mathbb{R}^{r \times r}} \vec{U}_r^n.$$

[ ]:
```
# change from the FOM to the POD basis
reduced_system_matrix = np.dot(np.dot(pod_basis.T, system_matrix.array()),␣
 ↪pod_basis)
reduced_rhs_matrix    = np.dot(np.dot(pod_basis.T,    rhs_matrix.array()),␣
 ↪pod_basis)
```

1.4.3 Solve reduced order model

Now that all the groundwork has been laid, we will now actually solve the recuced order model. \ First, we specify the parameters of the time discretization:

[ ]:
```
# start time
t = 0.
# end time
T = 0.5
# time step size
Δt = 0.01
# one-step theta
 = 0.5
```

We now define the expression for the initial condition as u_0 and interpolate it in the function space V. Here we will use the variable u_n as the solution from the last timestep. We need to briefly discuss how the reduced initial condition can be computed because in general it is not sufficient to simply restrict the initial condition from the FEM to the POD space, i.e.

$$\vec{u}_r^0 \neq \Psi^T \vec{u}_h^0.$$

Instead we need to include the weight matrix W and it holds in general

$$\vec{u}_r^0 = \Psi^T W \vec{u}_h^0$$

and for our particular case where we chose $W = M$, we thus get

$$\vec{u}_r^0 = \Psi^T M \vec{u}_h^0.$$

To understand why we need to use the weighting matrix when computing the reduced intial condition, we need to take into account that we get the initial condition through the projection

$$(u_r(0) - u_h(0), \psi_r^j) = 0 \quad \forall 1 \leq i \leq r,$$

where ψ_r^j is the j.th POD basis function which corresponds to the basis vector ψ_j and $(\cdot, \cdot)$ stands for the inner product that we are using, e.g. in the case $W = M$ this is the $L^2(\Omega)$ inner product. It is left as an exercise to the reader that by expressing $u_r(0)$ and $u_h(0)$ as a linear combination of the POD or FEM basis, we arrive at the formula

$$\vec{u}_r^0 = \Psi^T W \vec{u}_h^0.$$

[ ]:
```
# initial condition
u_0 = Expression('sin(pi*x[0])*sin(pi*x[1])', degree=2, pi=math.pi)

# u_n: solution from last time step
u_n = np.dot(np.dot(pod_basis.T, M.array()), np.array(interpolate(u_0, V).
 ↪vector()))
```

Next, we create a matrix to store the reduced snapshots $Y_r \in \mathbb{R}^{r \times n}$, where r is the size of the POD basis and n is the number of snapshots of the POD solution. We add the initial condition as the first column of this matrix and afterwards in the time-stepping scheme fill the remaining columns.

[ ]:
```
# reduced snapshot matrix
reduced_snapshots = np.zeros((r, n))

# store initial condition in reduced snapshot matrix
reduced_snapshots[:, 0] = u_n
```

Finally, we come to the time-stepping loop itself. For this we create the function w, which will only be used for visualizing the current ROM solution. In the loop itself, we then compute t_{n+1} as $t_n + \Delta t$. Afterwards we solve the linear system

$$A_r^{n+1} \vec{U}_r^{n+1} = A_r^n \vec{U}_r^n.$$

A major difference to the FEM codes from before is that in our ROM, we don't need to enforce the boundary conditions. But why? We know that not including boundary conditions in the finite element model means that we are dealing with homogeneous Neumann boundary conditions and that the solution is usually not unique. However, the POD basis is a linear combination of the snapshots of the FEM solution. Therefore, the POD basis satisfies the homogeneous Dirichlet boundary conditions by construction and the same holds for the ROM solution, which is a linear combination of the POD basis. Although dealing with the boundary conditions was easy for our

problem, this is not always this way. For inhomogeneous, time-independent boundary conditions one can use a centering approach, where the time-average is being substracted from the snapshots such that these centered snapshots again satisfy homogeneous boundary conditions. For time-dependent boundary conditions the treatment of the boundary conditions can be more involved and can require for example additional penalization terms in the reduced order model. For more information on this, please check out the book from Quarteroni and Rozza "Reduced Order Methods for Modeling and Computation".

To visualize the reduced solution, we now project it into the FEM space by computing

$$\Psi \vec{U}_r^{n+1} \in \mathbb{R}^m.$$

With this we can verify the solution's correctness, since we know that the analytical solution is given as

$$u(t, x, y) = \exp(-2\pi^2 t) \sin(\pi x) \sin(\pi y).$$

For our model problem, we decide that we are interested in evaluating the solution at the center of the domain, i.e.

$$J(\tilde{u}(t, \cdot, \cdot)) := \tilde{u}\left(t, \frac{1}{2}, \frac{1}{2}\right)$$

and we have for the analytical solution u that

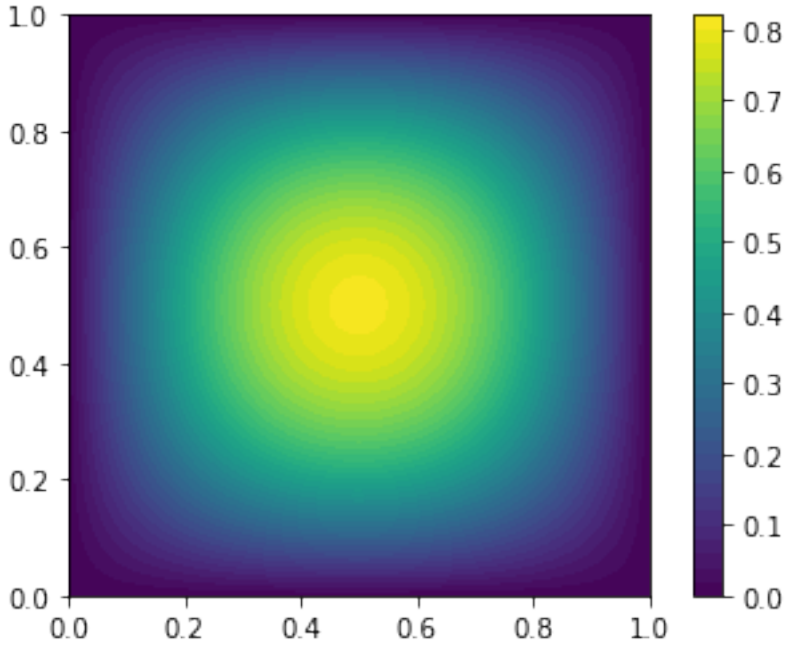
$$J(u(t, \cdot, \cdot)) = \exp(-2\pi^2 t).$$

Having compared the difference between the analytical solution u and its POD-ROM approximation u_r in the goal functional $J(\cdot)$, we plot the POD-ROM solution and store it in the reduced snapshot matrix.

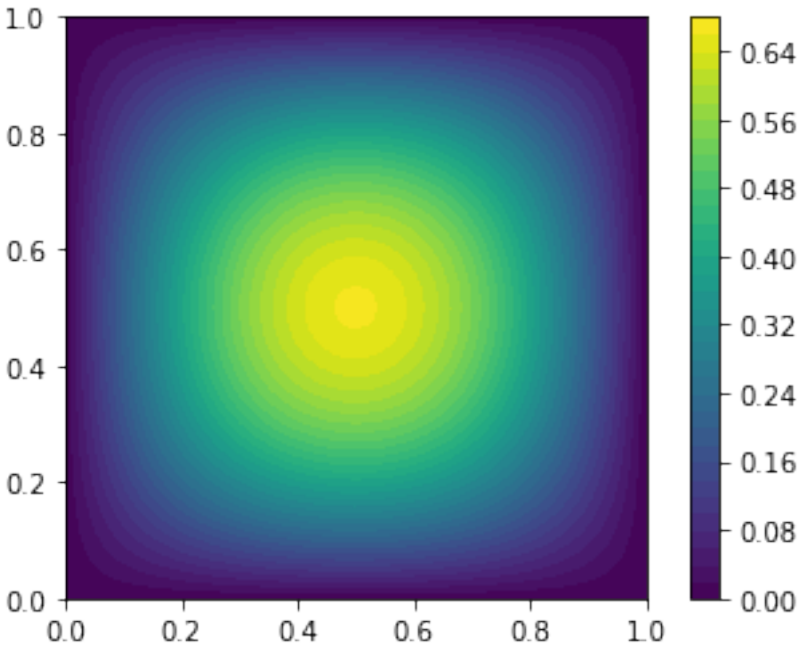
[ ]:
```
# Time-stepping
n = 1
w = Function(V)

while(t+Δt <= T+1e-8):
    # Update current time
    t += Δt

    # Solve reduced linear system
    u_n = np.linalg.solve(reduced_system_matrix, np.dot(reduced_rhs_matrix,␣
 ↪u_n))

    # Prolongate the reduced solution into the FOM space
    w.vector()[:] = np.dot(pod_basis, u_n)

    # Print results
    print(f"t = {round(t,4)}:")
    print(f"  u_r(0.5,0.5) = {round(w(Point(0.5, 0.5)),6)}")
    print(f"  u(0.5,0.5)   = {round(math.exp(-2 * math.pi**2 * t),6)}")

    # Plot solution
    c = plot(w)
```

```
    plt.colorbar(c)
    plt.show()

    # Store solution in snapshot matrix
    reduced_snapshots[:, n] = u_n
    n += 1
```

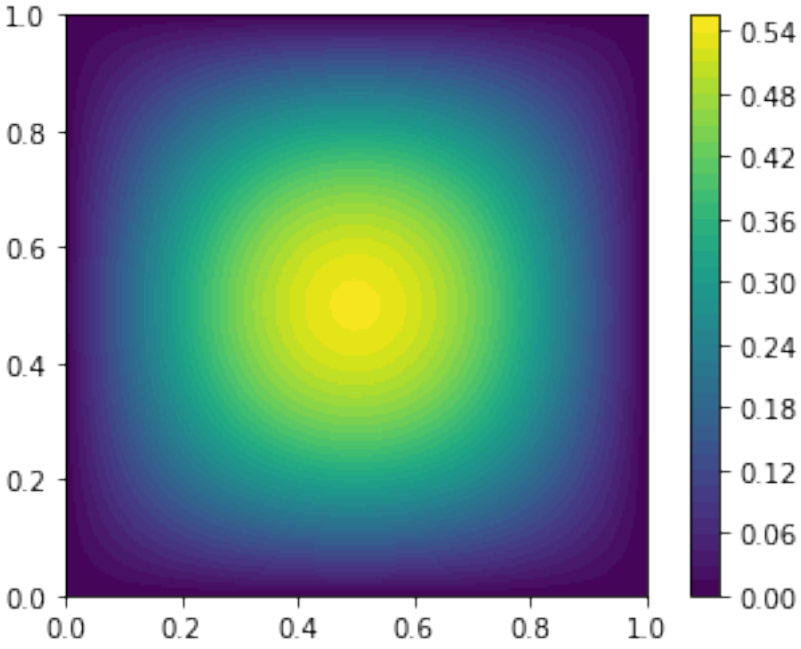
```
t = 0.01:
  u_r(0.5,0.5) = 0.809922
  u(0.5,0.5)   = 0.820869
```



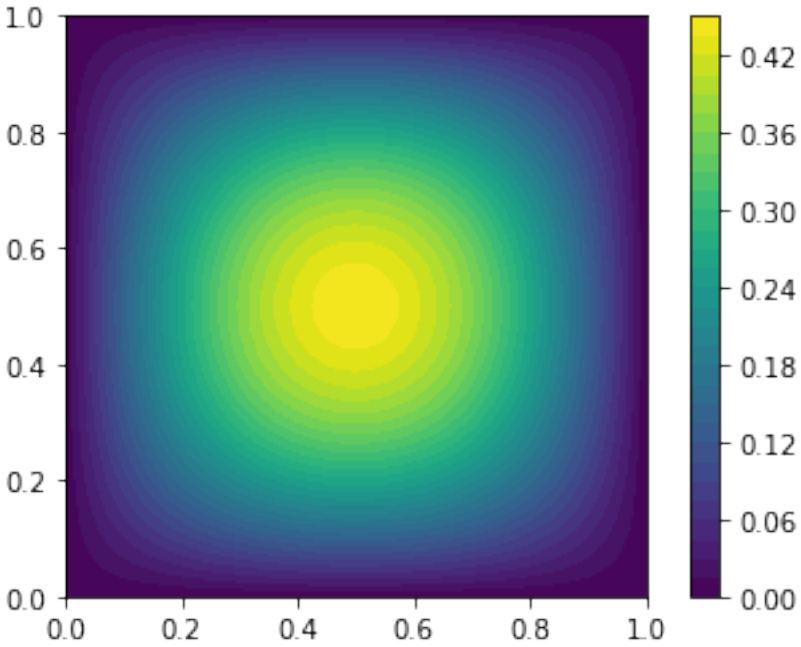

```
t = 0.02:
  u_r(0.5,0.5) = 0.66428
  u(0.5,0.5)   = 0.673825
```



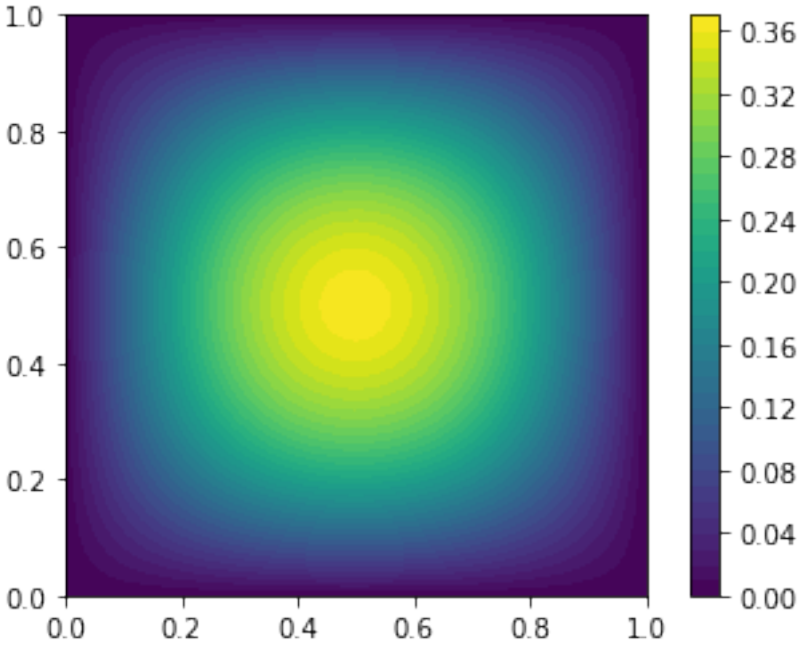

```
t = 0.03:
  u_r(0.5,0.5) = 0.544828
  u(0.5,0.5)   = 0.553122
```



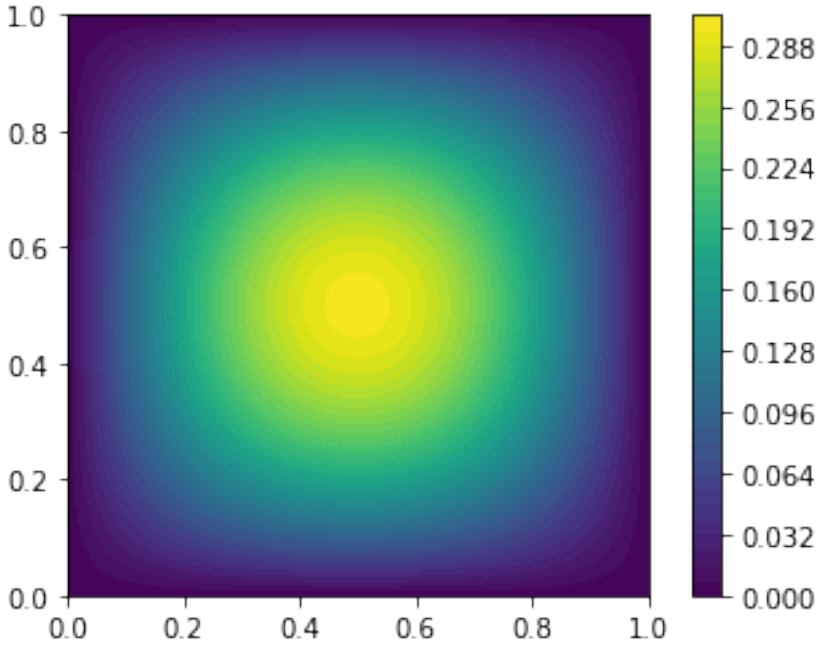
```
t = 0.04:
  u_r(0.5,0.5) = 0.446856
  u(0.5,0.5)   = 0.454041
```



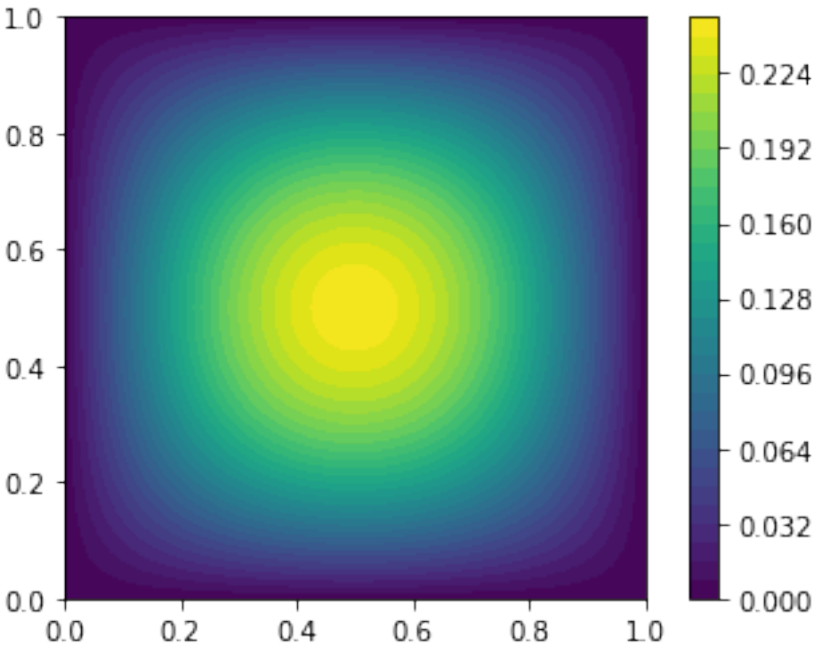

```
t = 0.05:
  u_r(0.5,0.5) = 0.366502
  u(0.5,0.5)   = 0.372708
```



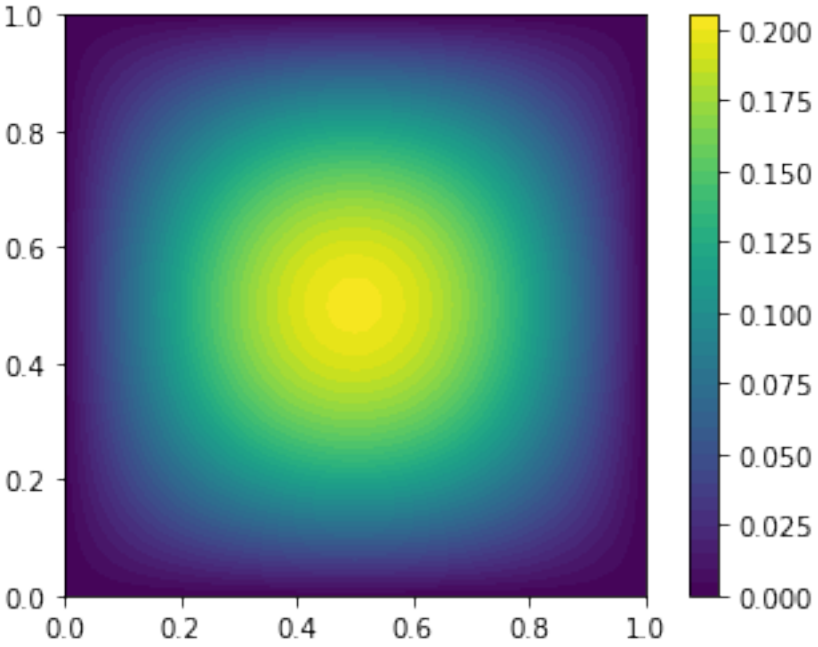
```
t = 0.06:
  u_r(0.5,0.5) = 0.300597
  u(0.5,0.5)   = 0.305944
```



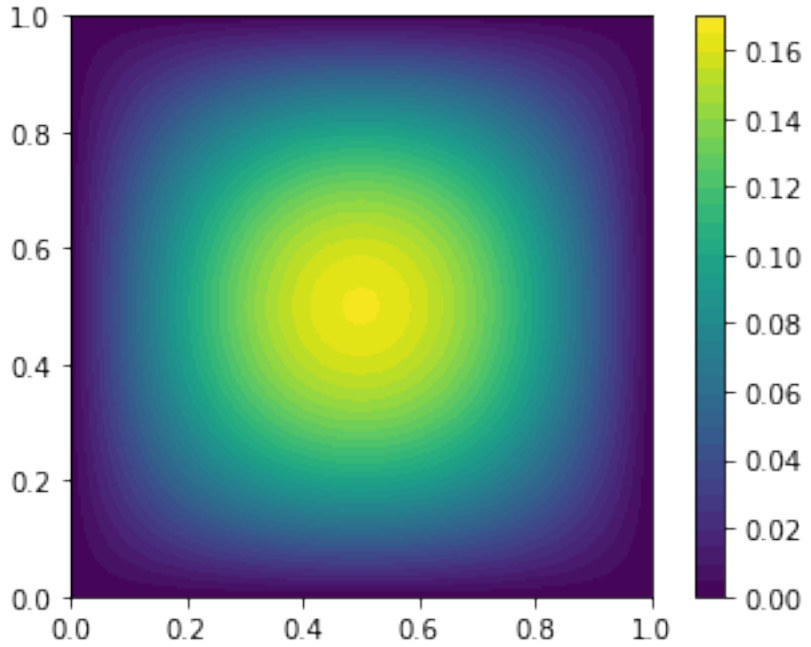

```
t = 0.07:
  u_r(0.5,0.5) = 0.246543
  u(0.5,0.5)   = 0.25114
```



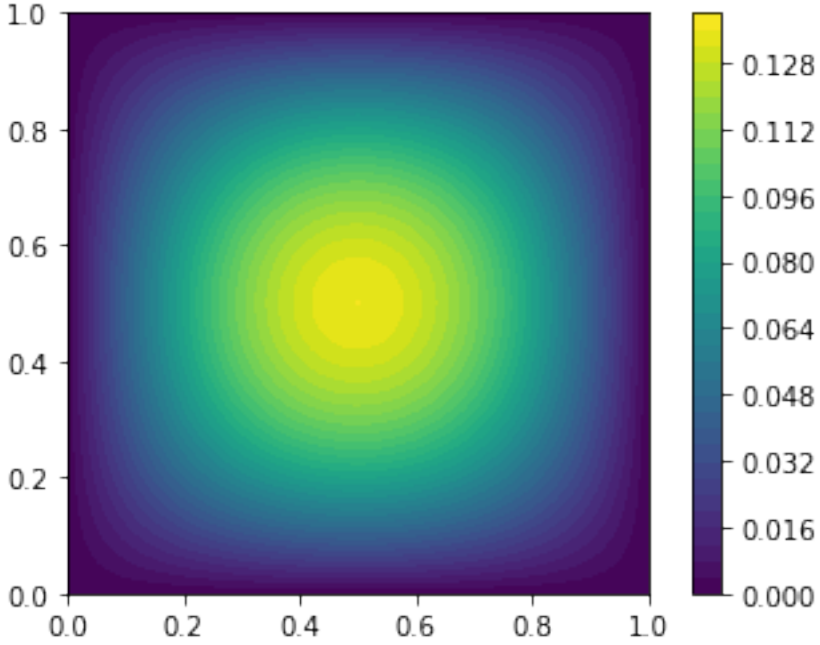

```
t = 0.08:
  u_r(0.5,0.5) = 0.202209
  u(0.5,0.5)   = 0.206153
```



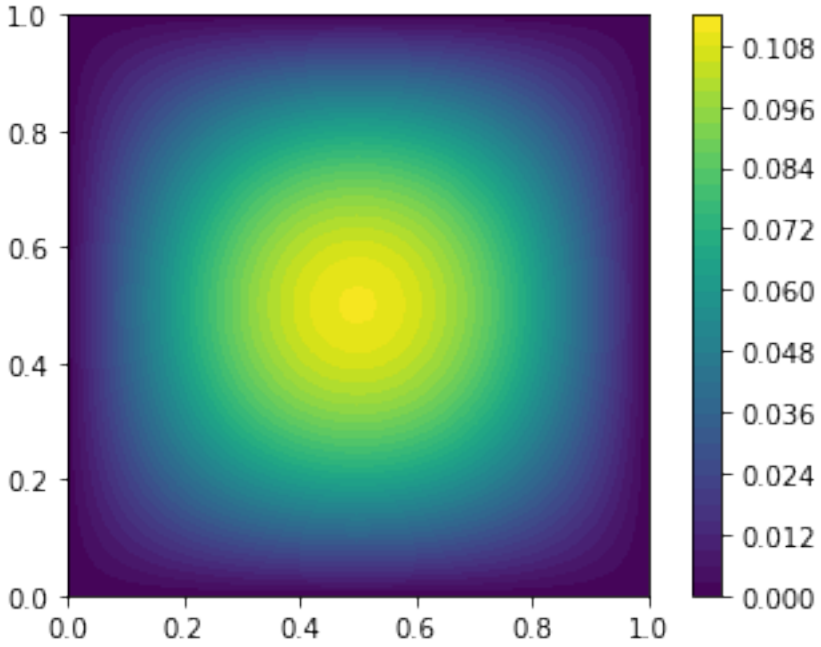
```
t = 0.09:
  u_r(0.5,0.5) = 0.165848
  u(0.5,0.5)   = 0.169225
```



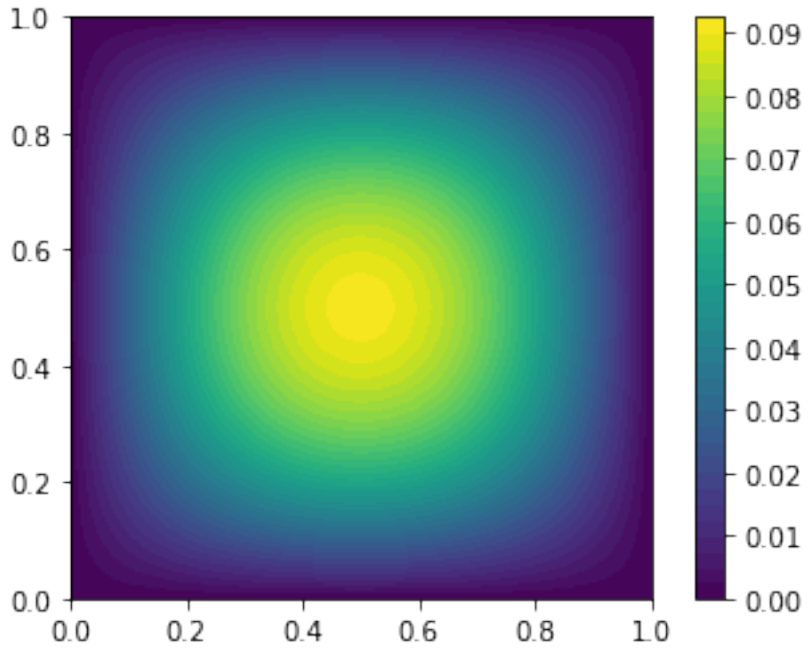

```
t = 0.1:
  u_r(0.5,0.5) = 0.136025
  u(0.5,0.5)   = 0.138911
```



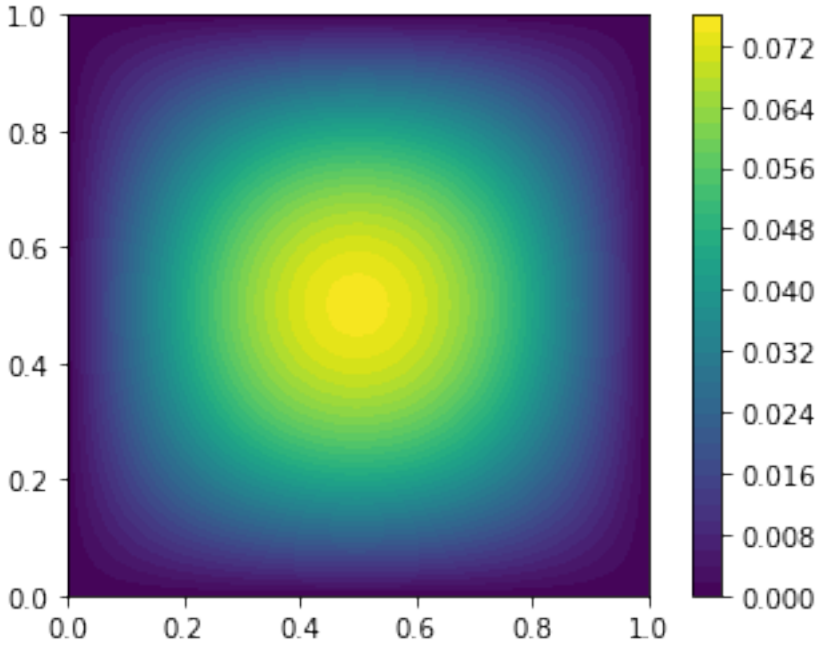
```
t = 0.11:
  u_r(0.5,0.5) = 0.111564
  u(0.5,0.5)   = 0.114028
```



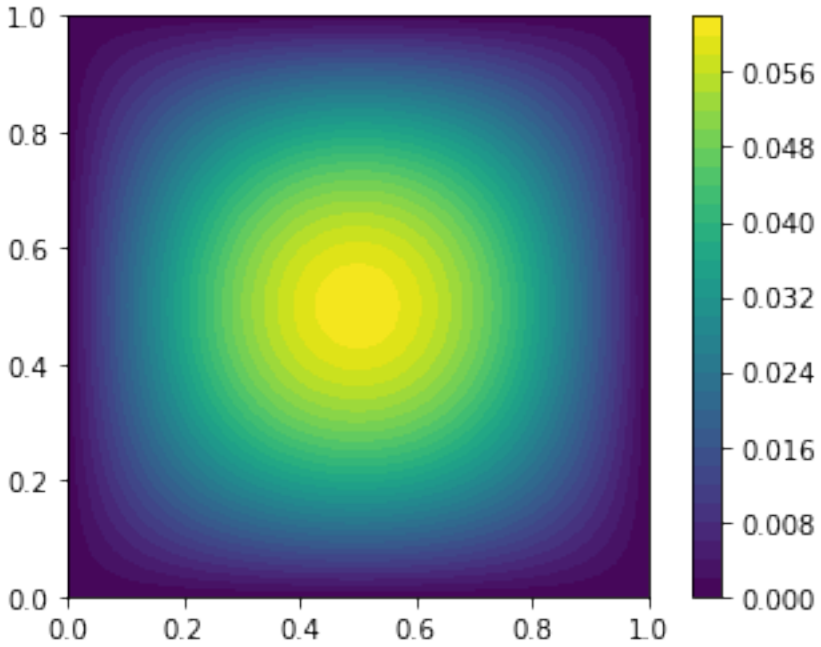

```
t = 0.12:
  u_r(0.5,0.5) = 0.091503
  u(0.5,0.5)   = 0.093602
```



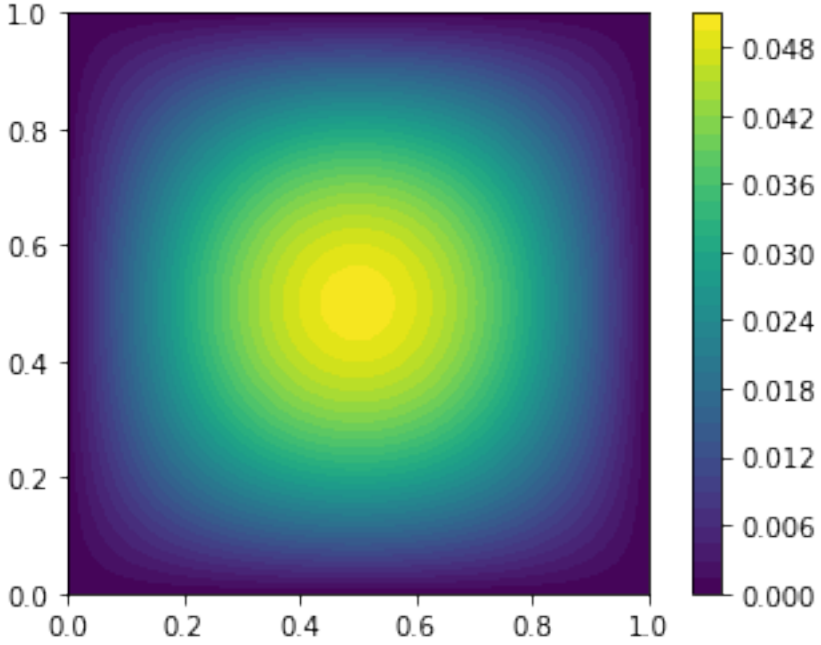

```
t = 0.13:
  u_r(0.5,0.5) = 0.075049
  u(0.5,0.5)   = 0.076835
```



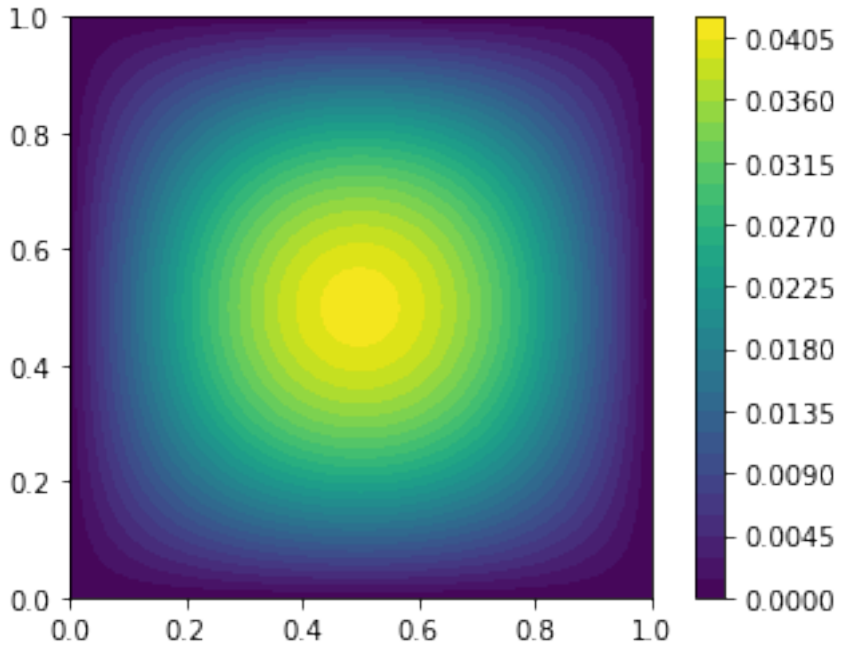
```
t = 0.14:
  u_r(0.5,0.5) = 0.061553
  u(0.5,0.5)   = 0.063071
```



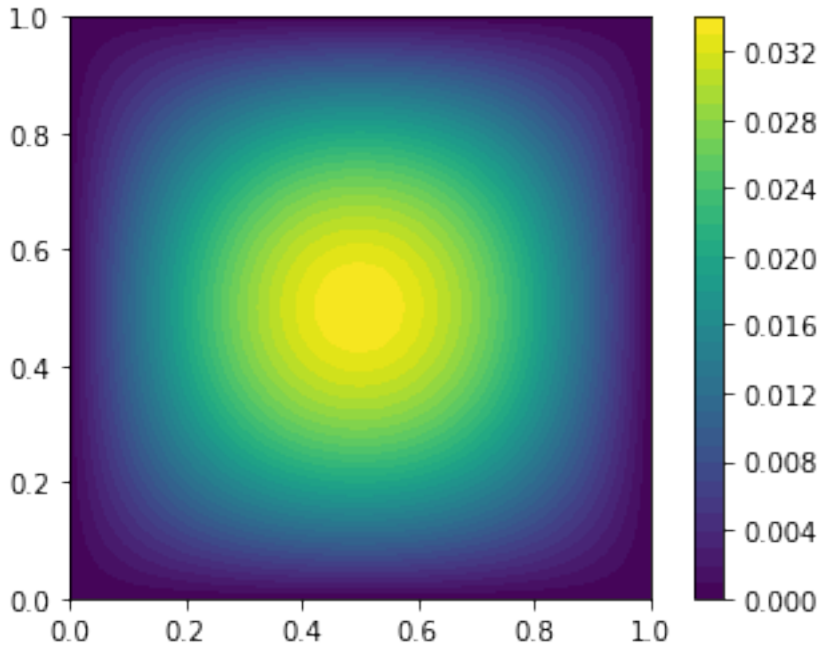

```
t = 0.15:
  u_r(0.5,0.5) = 0.050485
  u(0.5,0.5)   = 0.051773
```



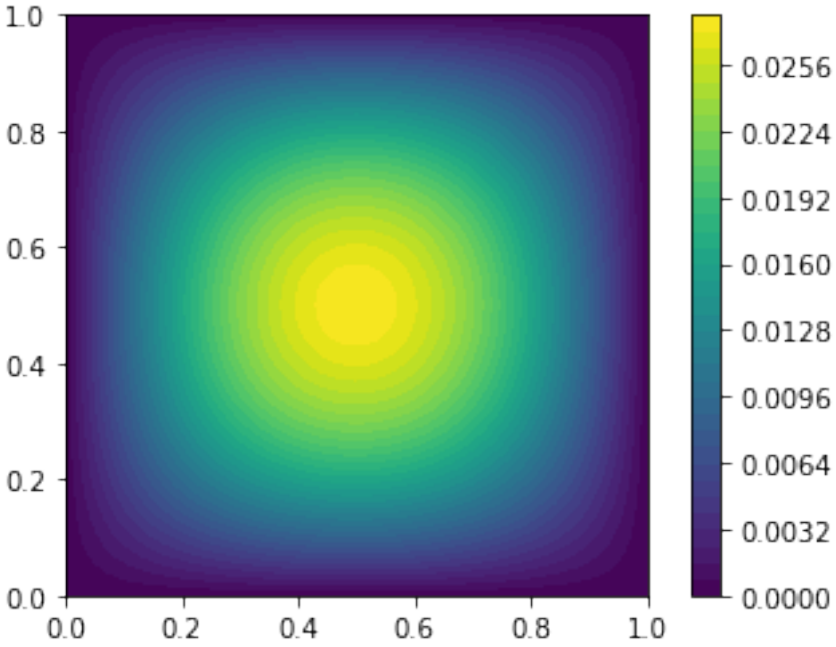
```
t = 0.16:
  u_r(0.5,0.5) = 0.041406
  u(0.5,0.5)   = 0.042499
```



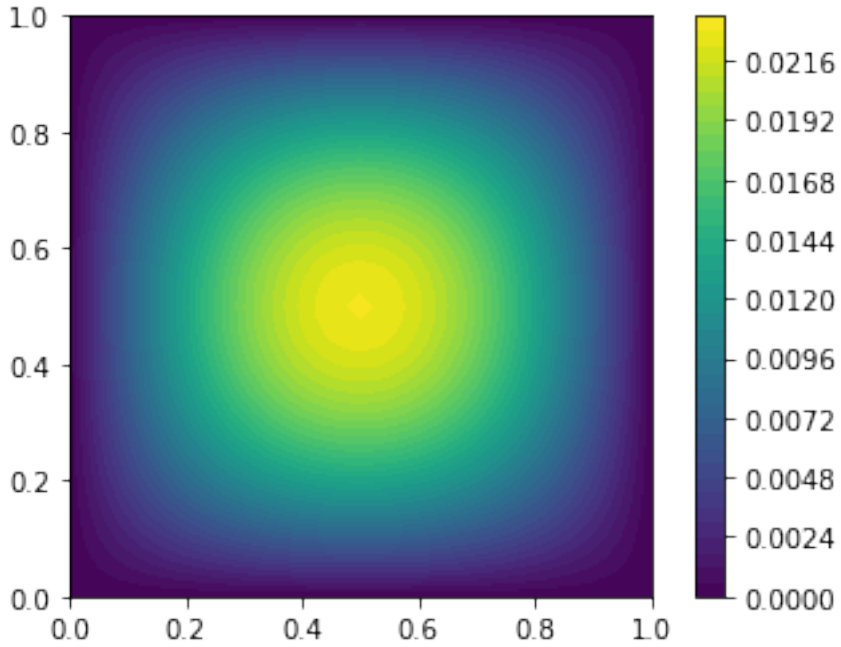

```
t = 0.17:
  u_r(0.5,0.5) = 0.033961
  u(0.5,0.5)   = 0.034886
```



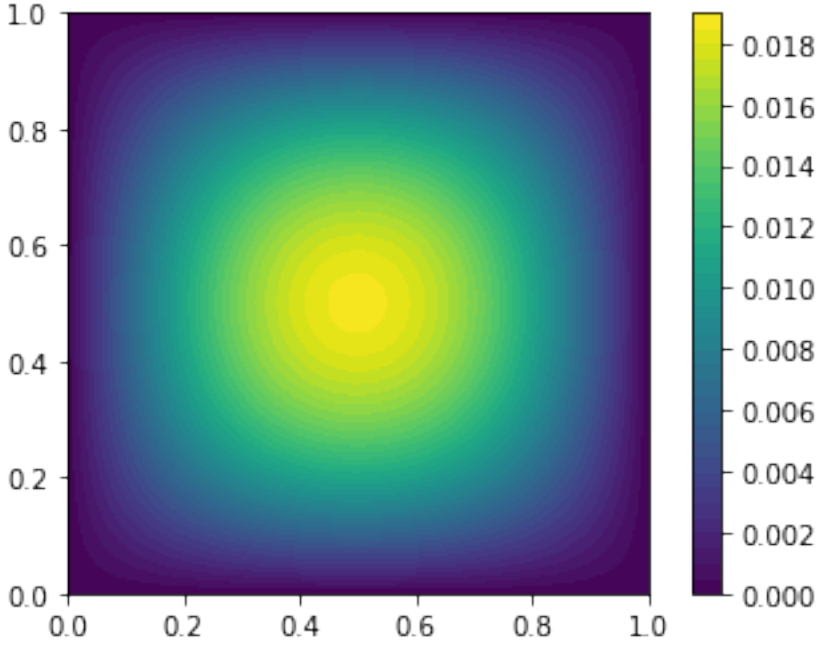

```
t = 0.18:
  u_r(0.5,0.5) = 0.027854
  u(0.5,0.5)   = 0.028637
```



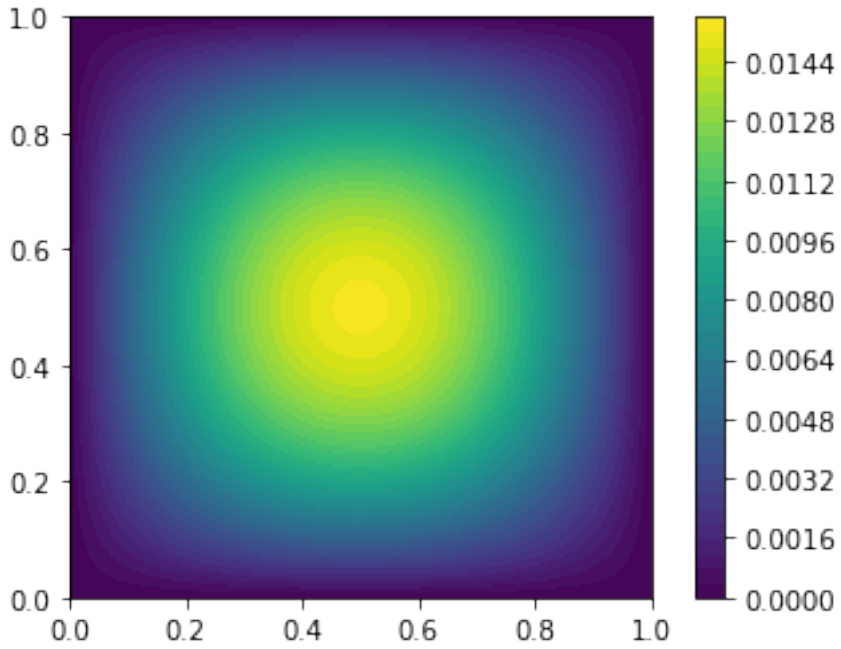
```
t = 0.19:
  u_r(0.5,0.5) = 0.022845
  u(0.5,0.5)   = 0.023507
```



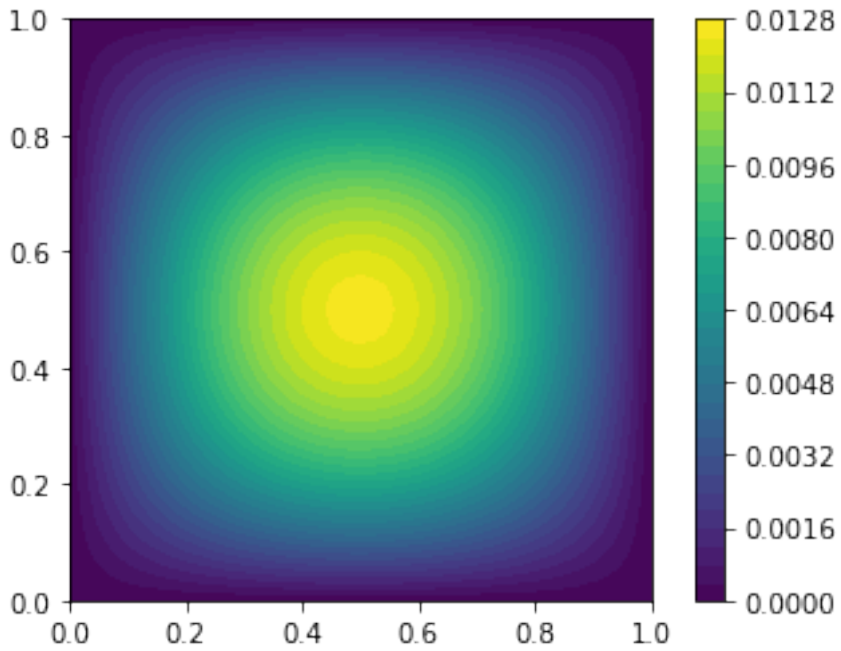

```
t = 0.2:
  u_r(0.5,0.5) = 0.018737
  u(0.5,0.5)   = 0.019296
```



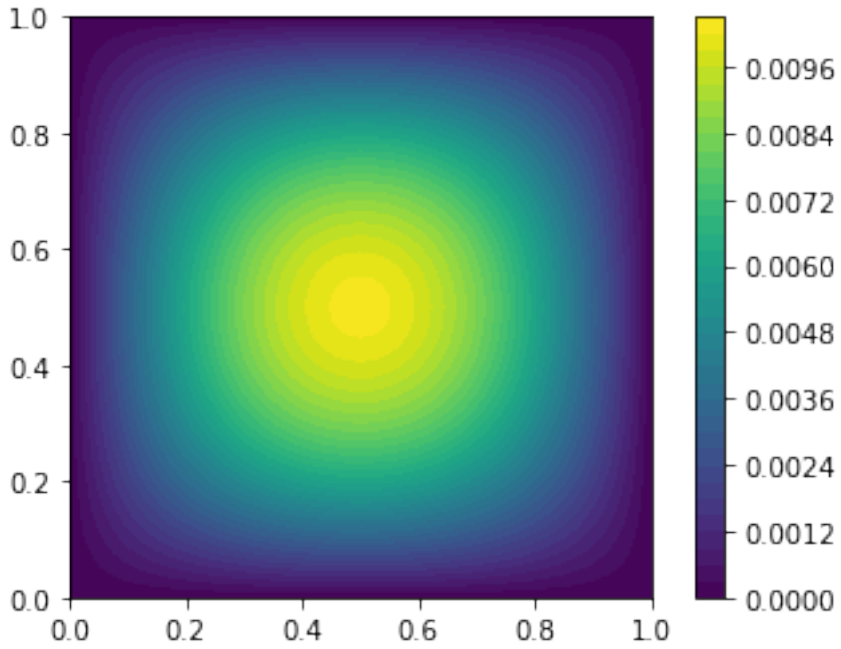
```
t = 0.21:
  u_r(0.5,0.5) = 0.015368
  u(0.5,0.5)   = 0.01584
```



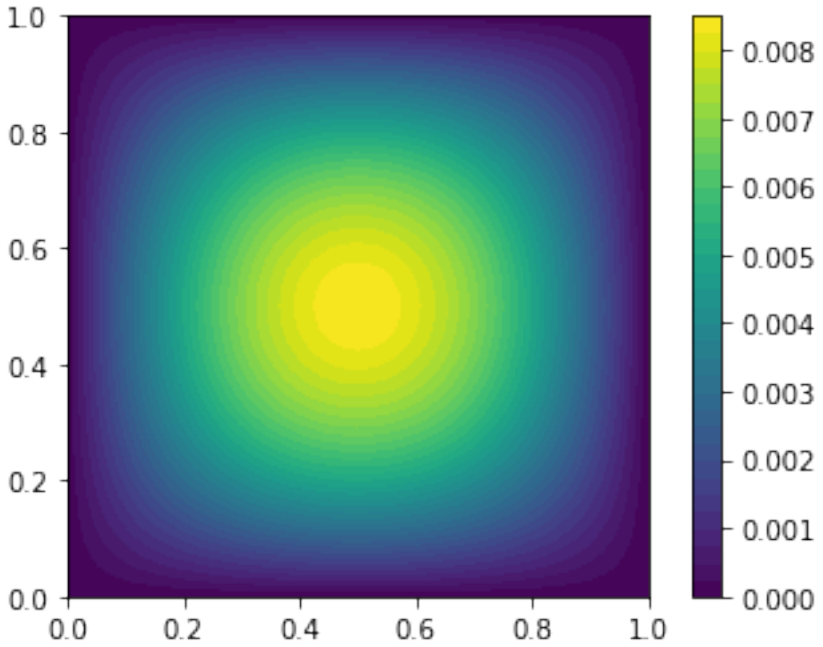

```
t = 0.22:
  u_r(0.5,0.5) = 0.012604
  u(0.5,0.5)   = 0.013002
```



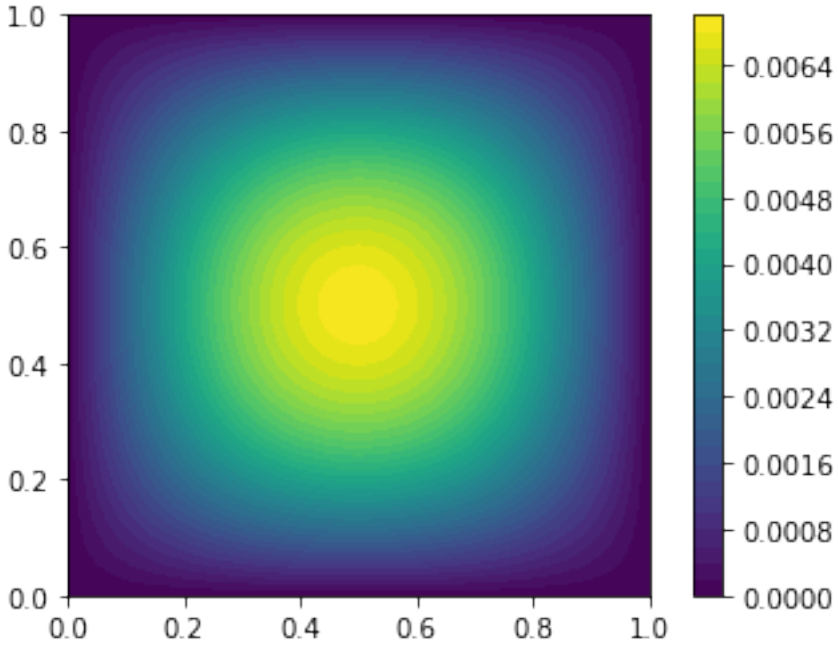

```
t = 0.23:
  u_r(0.5,0.5) = 0.010338
  u(0.5,0.5)   = 0.010673
```



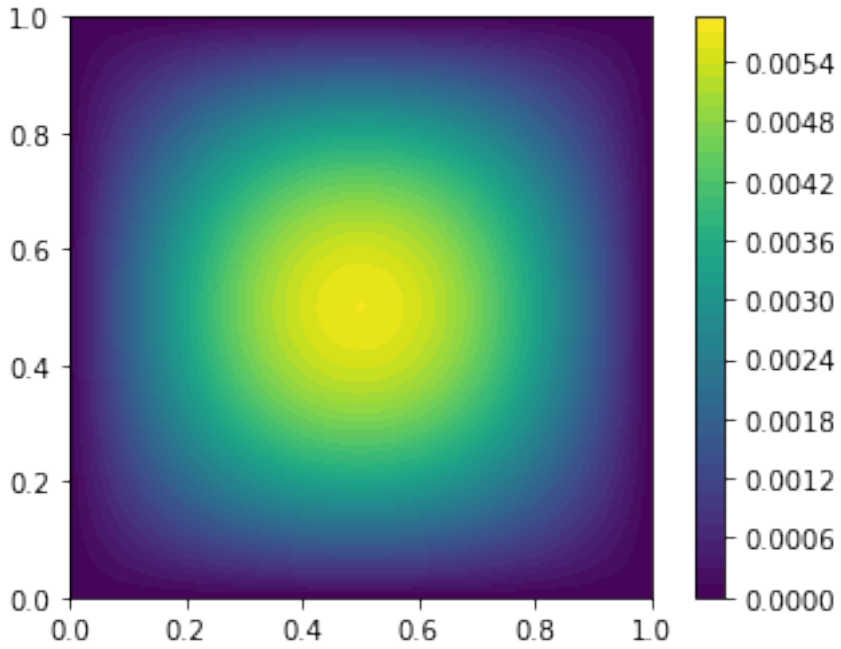
```
t = 0.24:
  u_r(0.5,0.5) = 0.008479
  u(0.5,0.5)   = 0.008761
```



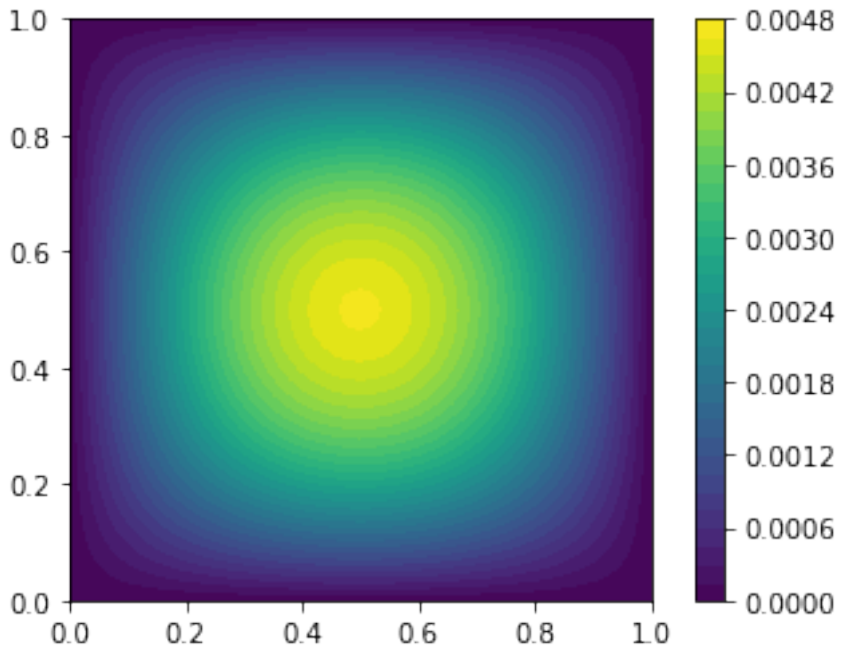

```
t = 0.25:
  u_r(0.5,0.5) = 0.006954
  u(0.5,0.5)   = 0.007192
```



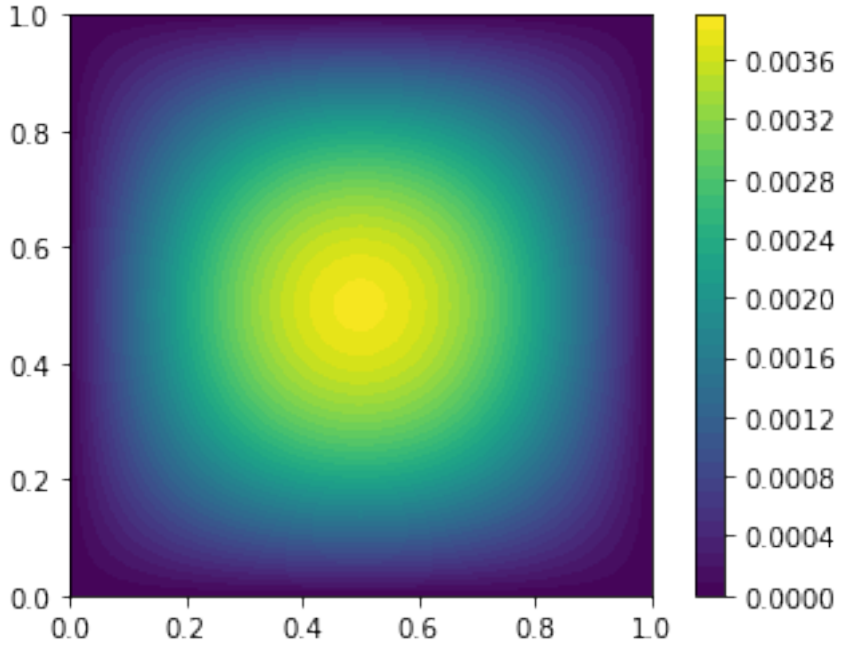
```
t = 0.26:
  u_r(0.5,0.5) = 0.005704
  u(0.5,0.5)   = 0.005904
```



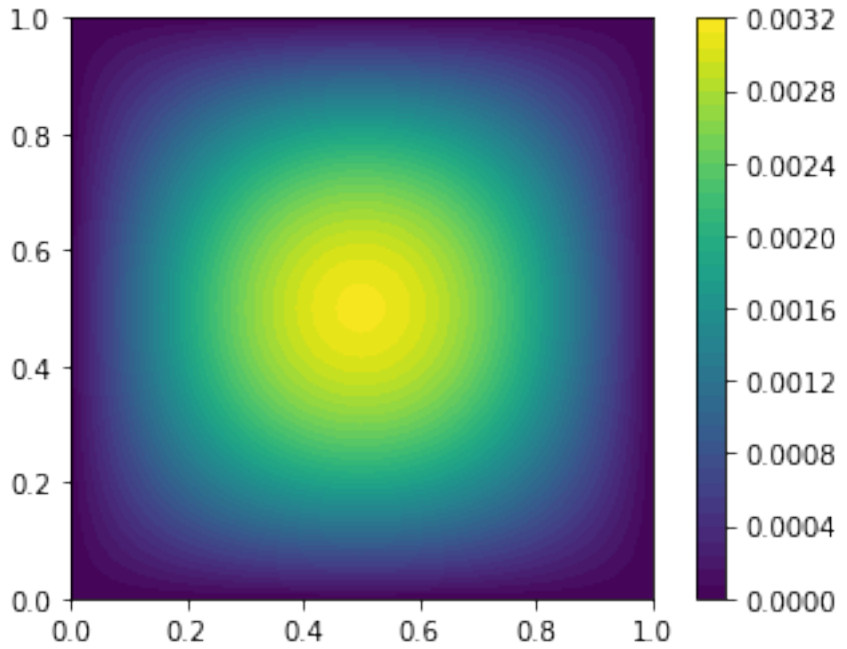

```
t = 0.27:
  u_r(0.5,0.5) = 0.004678
  u(0.5,0.5)   = 0.004846
```



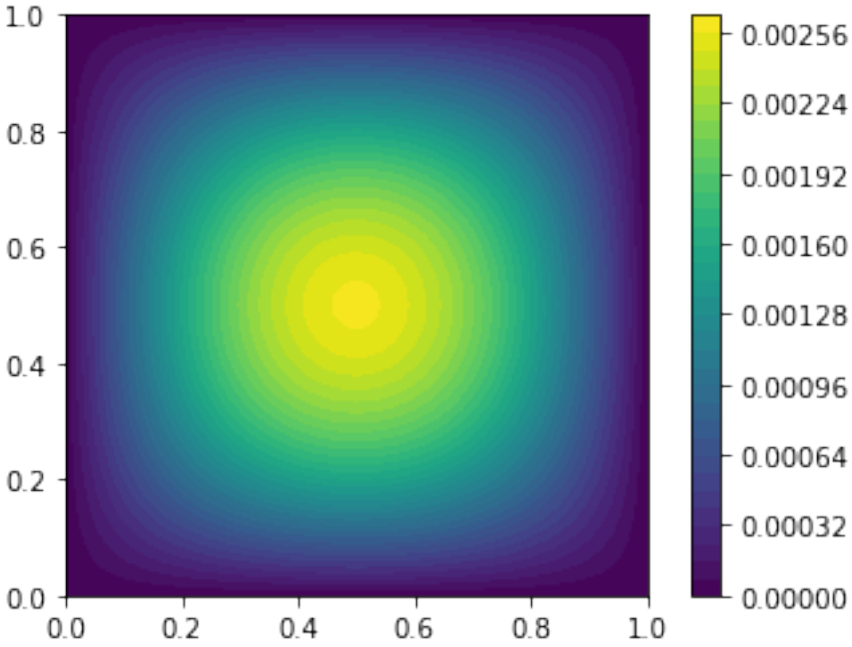

```
t = 0.28:
  u_r(0.5,0.5) = 0.003837
  u(0.5,0.5)   = 0.003978
```



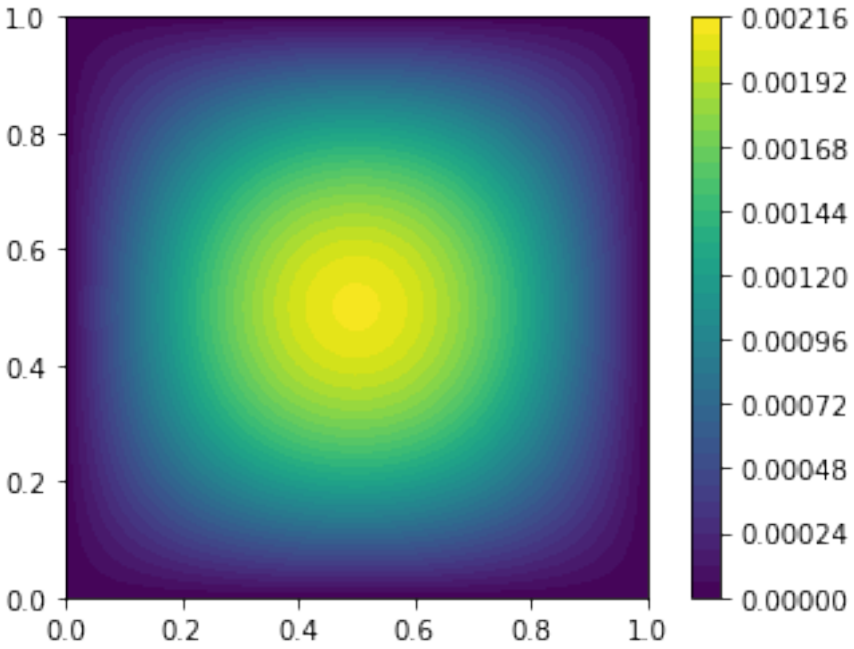
```
t = 0.29:
  u_r(0.5,0.5) = 0.003147
  u(0.5,0.5)   = 0.003265
```



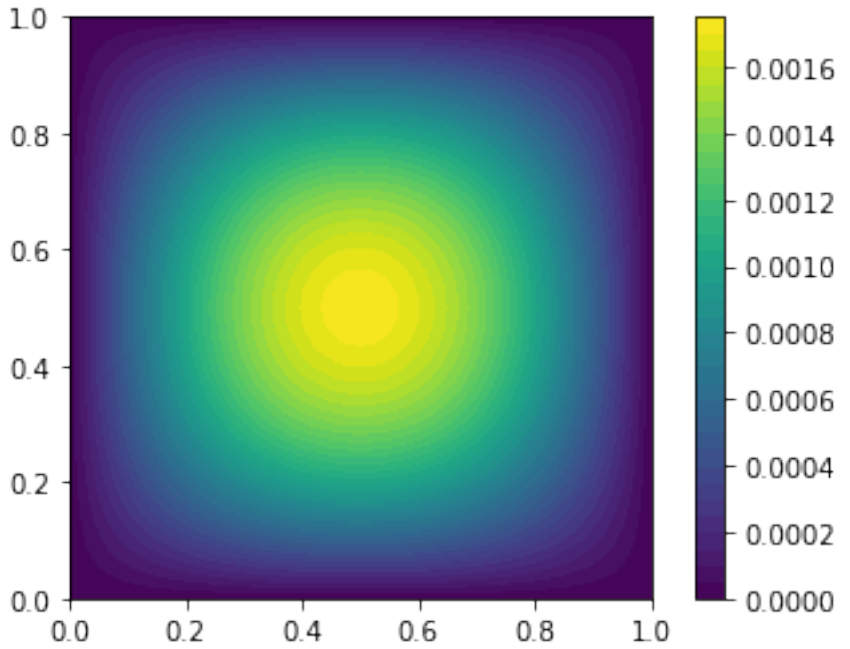

```
t = 0.3:
  u_r(0.5,0.5) = 0.002581
  u(0.5,0.5)   = 0.00268
```



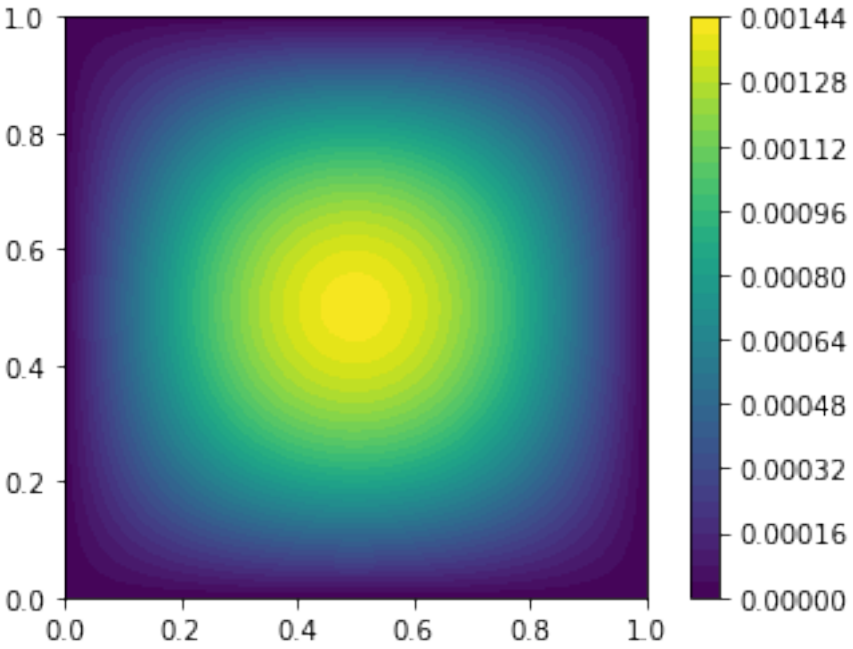
```
t = 0.31:
  u_r(0.5,0.5) = 0.002117
  u(0.5,0.5)   = 0.0022
```



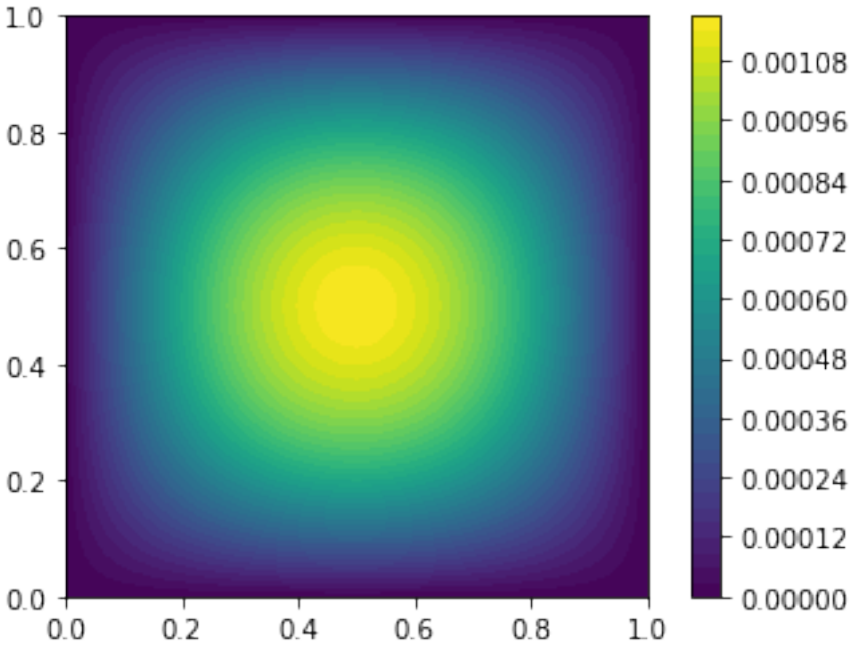

```
t = 0.32:
  u_r(0.5,0.5) = 0.001736
  u(0.5,0.5)   = 0.001806
```



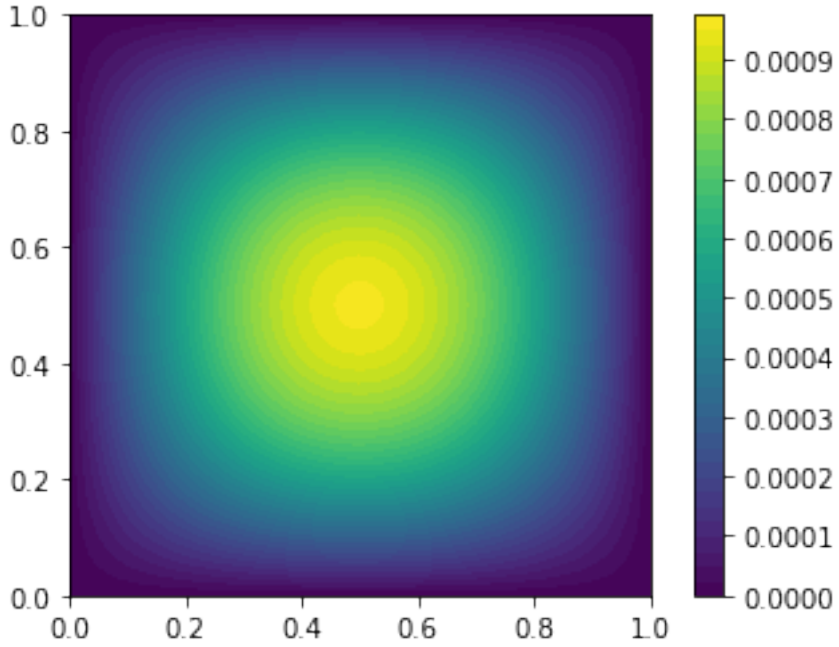

```
t = 0.33:
  u_r(0.5,0.5) = 0.001424
  u(0.5,0.5)   = 0.001483
```



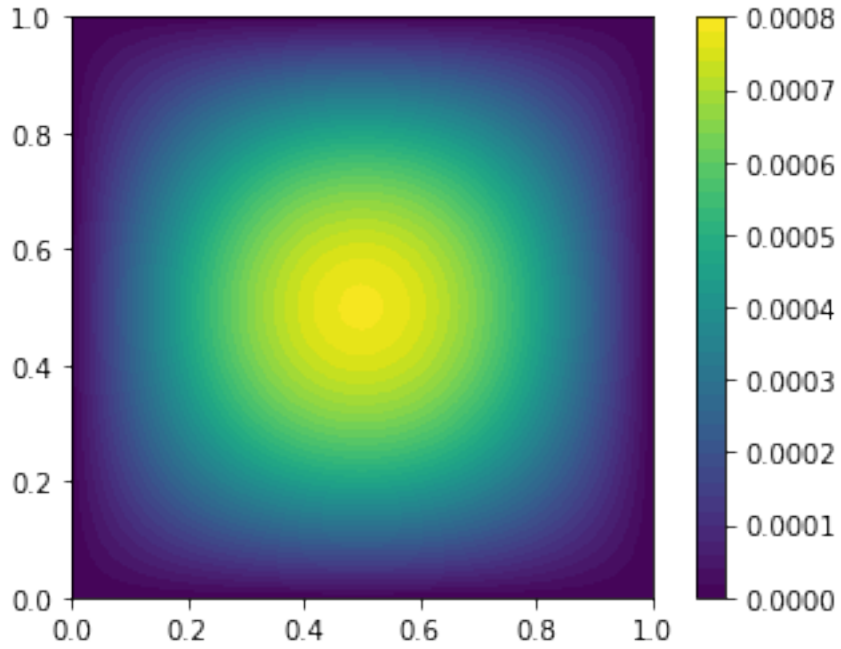
```
t = 0.34:
  u_r(0.5,0.5) = 0.001168
  u(0.5,0.5)   = 0.001217
```



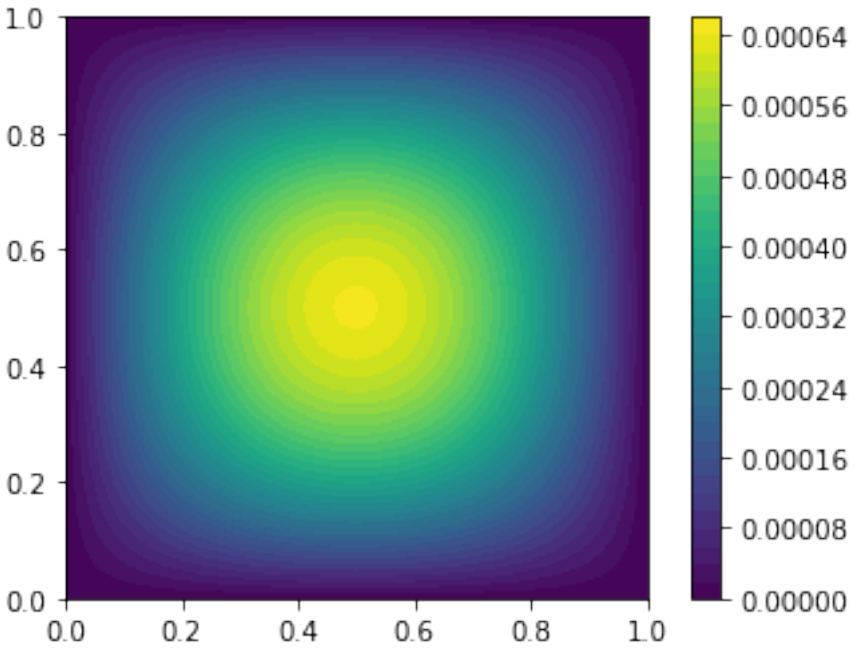

```
t = 0.35:
  u_r(0.5,0.5) = 0.000958
  u(0.5,0.5)   = 0.000999
```



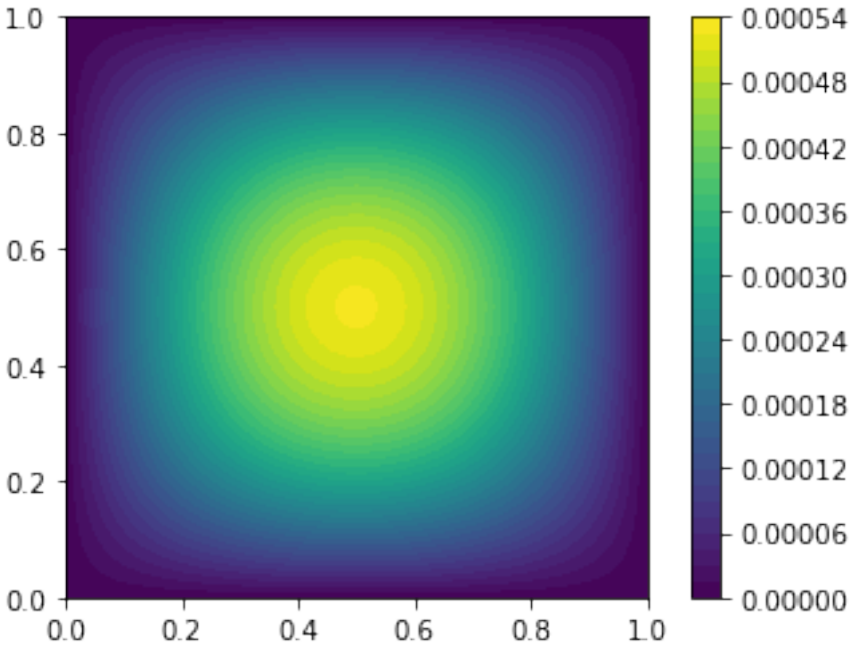
```
t = 0.36:
  u_r(0.5,0.5) = 0.000786
  u(0.5,0.5)   = 0.00082
```



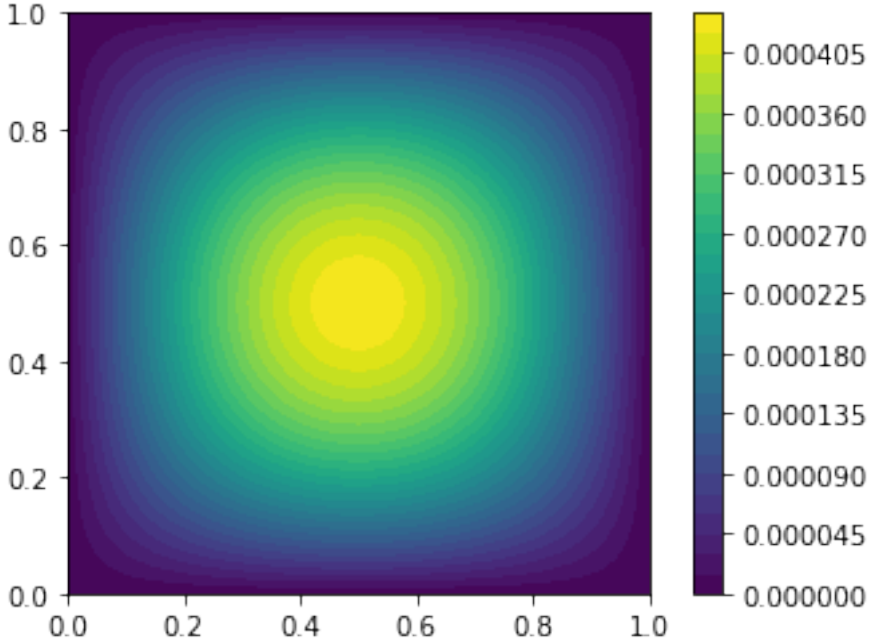

```
t = 0.37:
  u_r(0.5,0.5) = 0.000644
  u(0.5,0.5)   = 0.000673
```



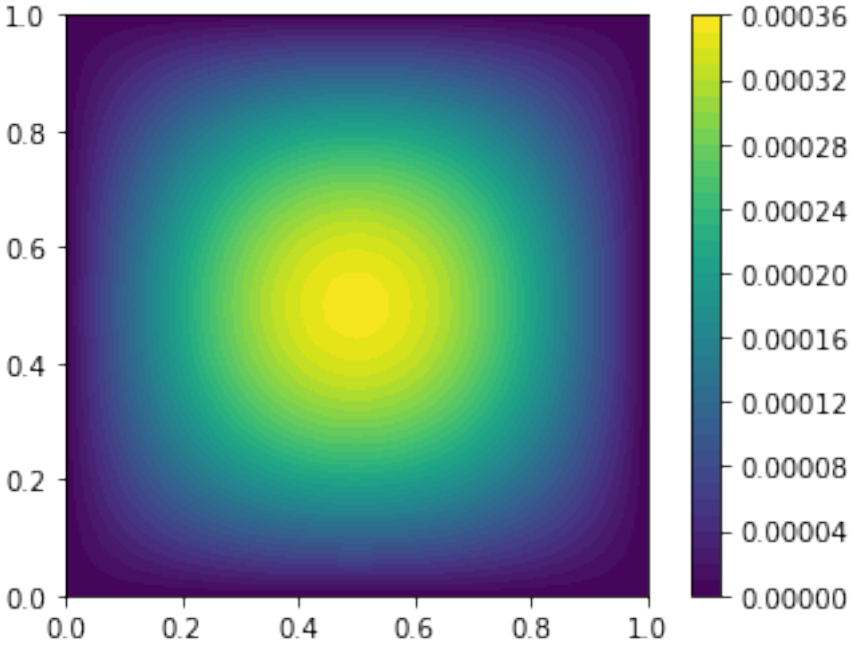

```
t = 0.38:
  u_r(0.5,0.5) = 0.000529
  u(0.5,0.5)   = 0.000553
```



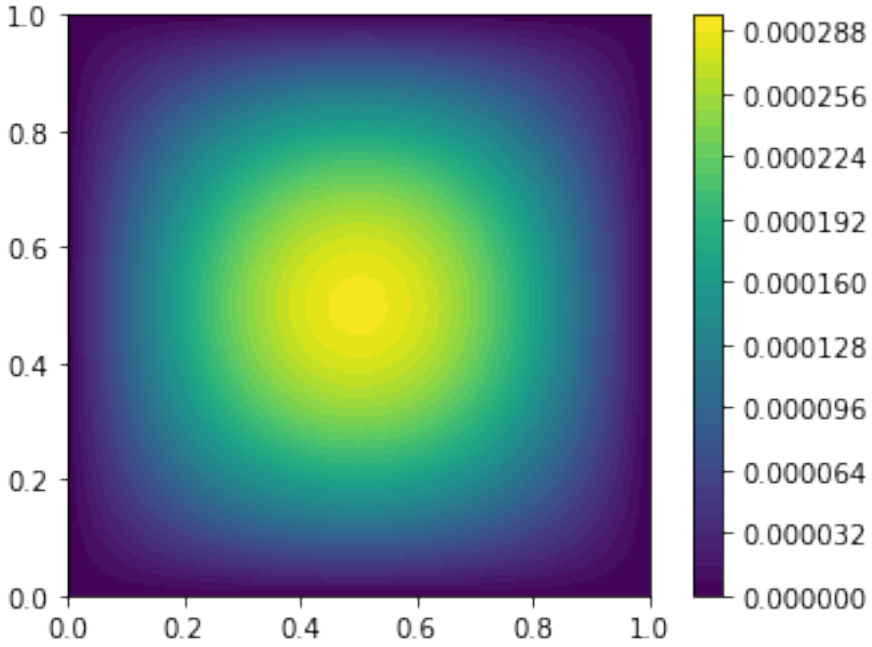
```
t = 0.39:
  u_r(0.5,0.5) = 0.000433
  u(0.5,0.5)   = 0.000454
```



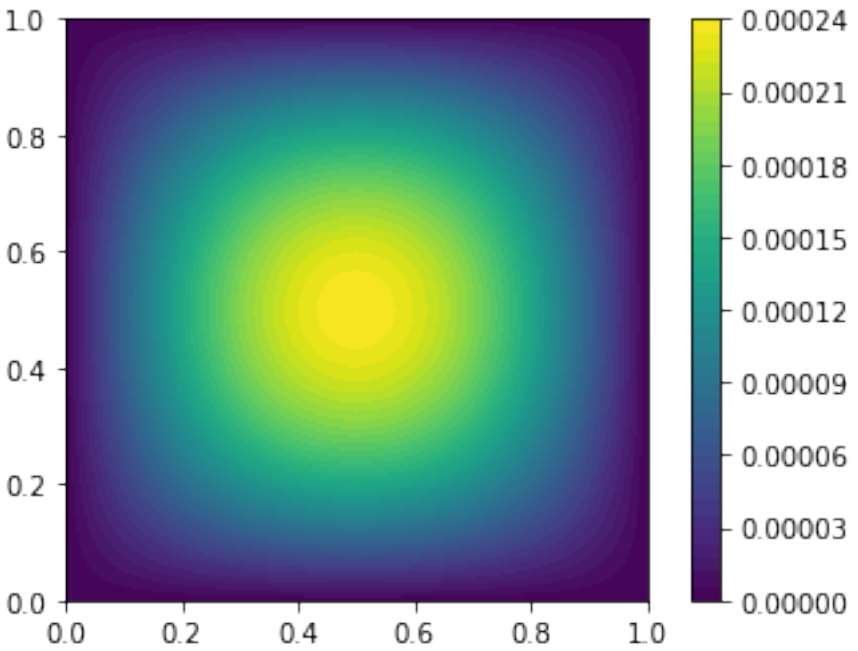

```
t = 0.4:
  u_r(0.5,0.5) = 0.000356
  u(0.5,0.5)   = 0.000372
```



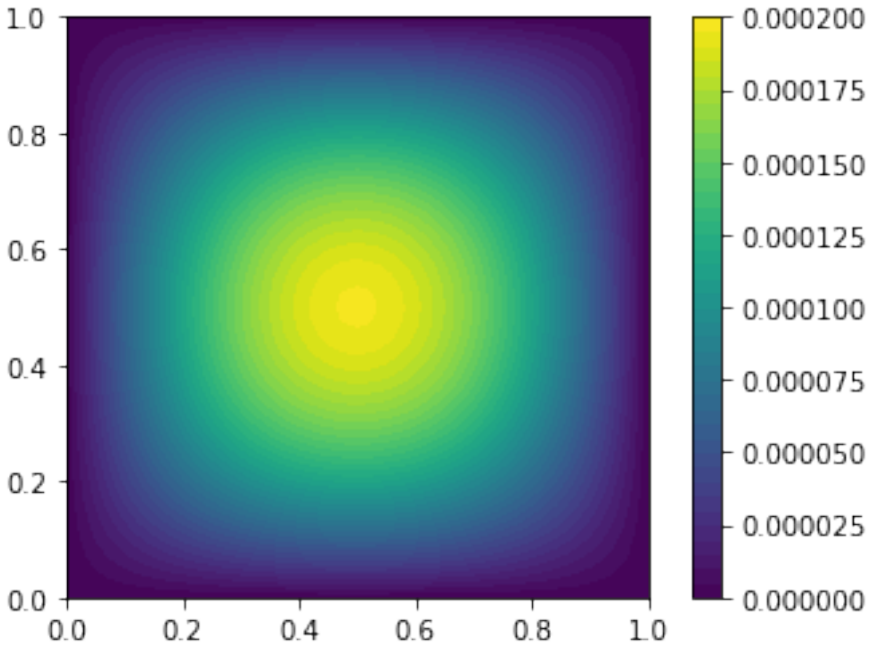
```
t = 0.41:
  u_r(0.5,0.5) = 0.000292
  u(0.5,0.5)   = 0.000306
```



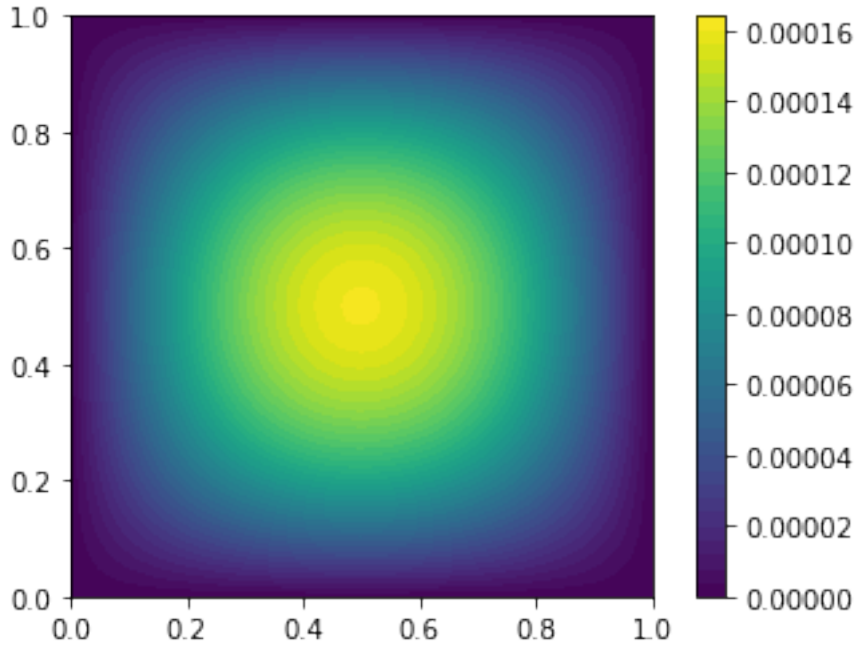

```
t = 0.42:
  u_r(0.5,0.5) = 0.000239
  u(0.5,0.5)   = 0.000251
```



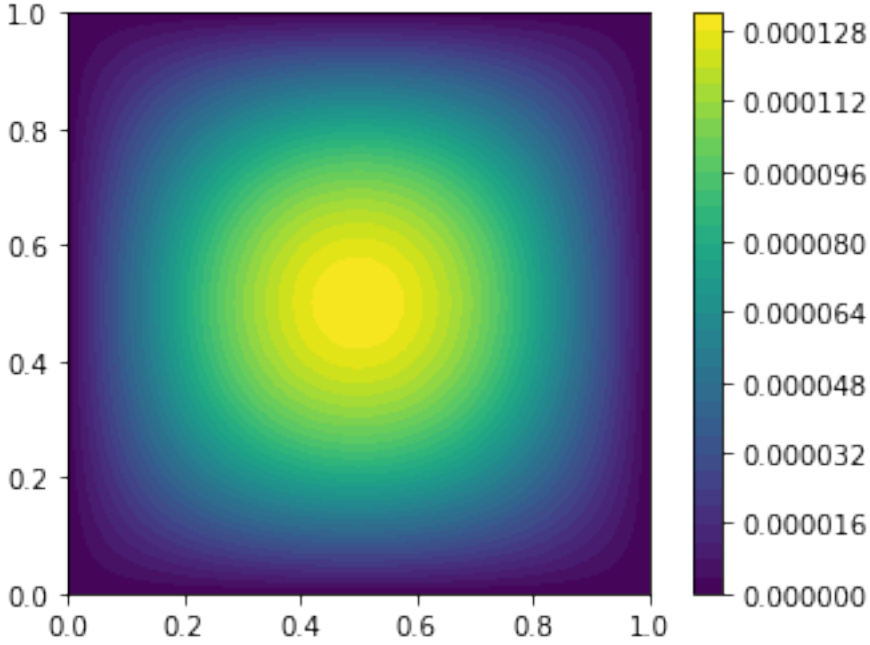

```
t = 0.43:
  u_r(0.5,0.5) = 0.000196
  u(0.5,0.5)   = 0.000206
```



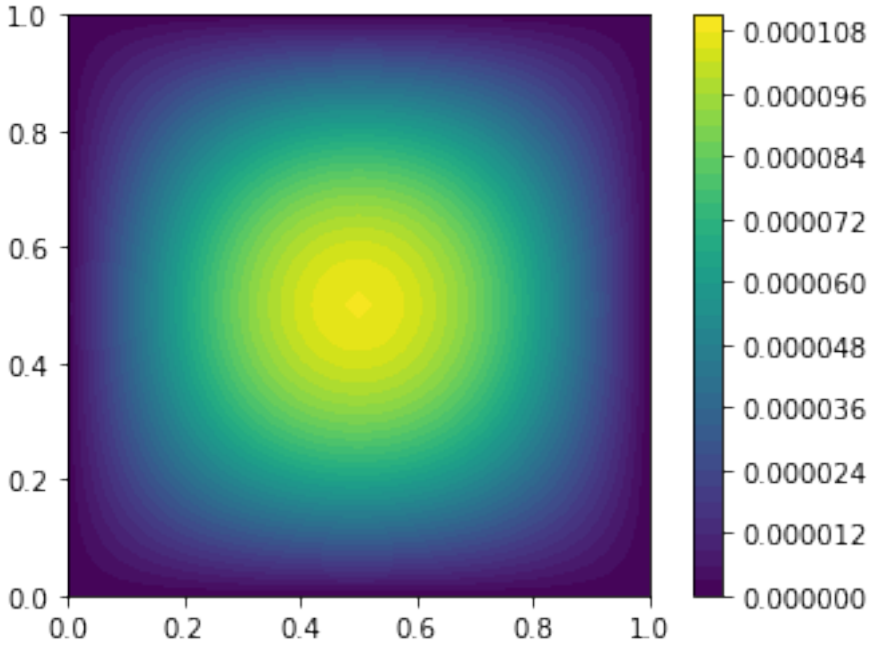
```
t = 0.44:
  u_r(0.5,0.5) = 0.000161
  u(0.5,0.5)   = 0.000169
```



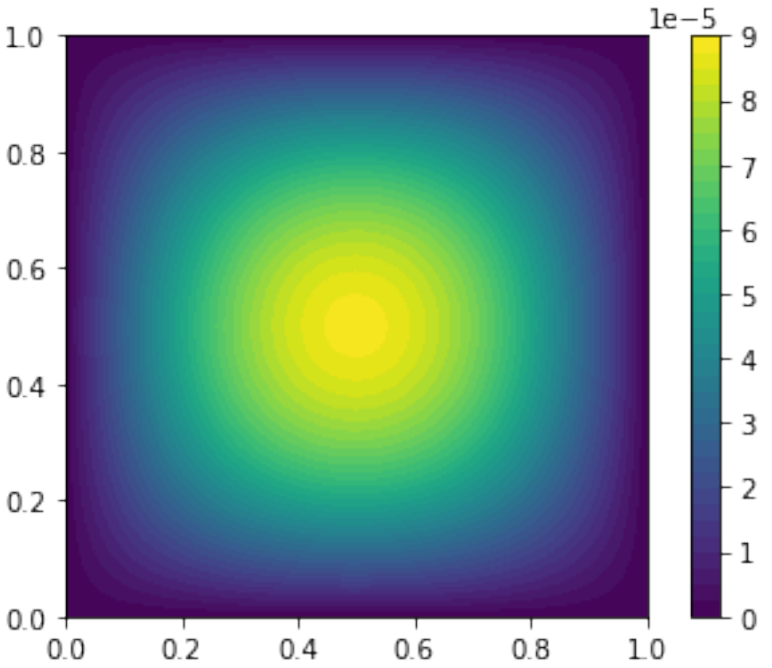

```
t = 0.45:
  u_r(0.5,0.5) = 0.000132
  u(0.5,0.5)   = 0.000139
```



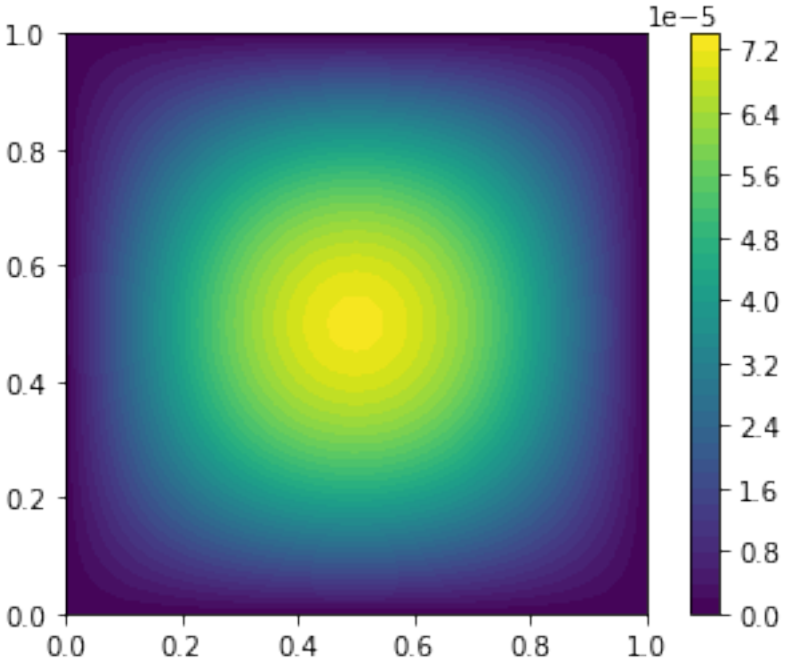
```
t = 0.46:
  u_r(0.5,0.5) = 0.000108
  u(0.5,0.5)   = 0.000114
```



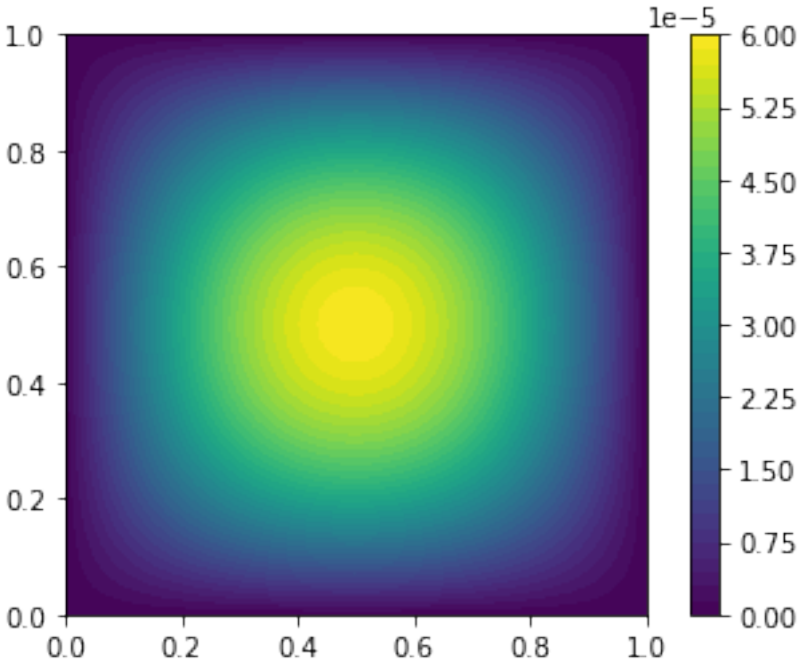

```
t = 0.47:
  u_r(0.5,0.5) = 8.9e-05
  u(0.5,0.5)   = 9.4e-05
```



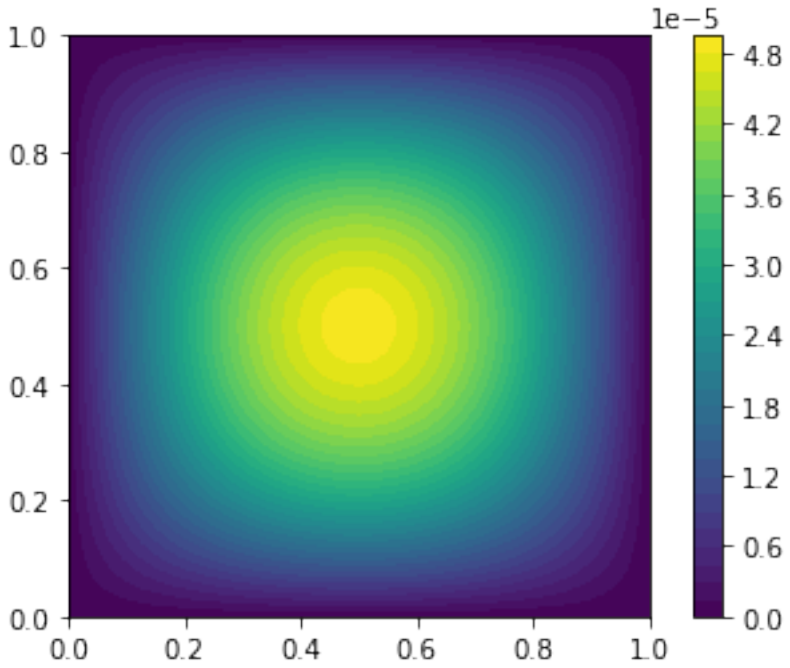

```
t = 0.48:
  u_r(0.5,0.5) = 7.3e-05
  u(0.5,0.5)   = 7.7e-05
```



```
t = 0.49:
  u_r(0.5,0.5) = 6e-05
  u(0.5,0.5)   = 6.3e-05
```




```
t = 0.5:
  u_r(0.5,0.5) = 4.9e-05
  u(0.5,0.5)   = 5.2e-05
```



1.4.4 Exercises

3. *Try out another model problem:* In this tutorial, we dealt with a heat equation with a unimodal analytical solution and right hand side $f = 0$. Try a different problem, e.g. the heat equation from step 26 of the deal.II tutorials with a time-dependent right hand side f. If you need a little help with this, please take a look at Chapter 4 of my master's thesis which solves exactly this problem.

1.4.5 Further resources on Reduced Order Modeling

- For a rigorous mathematical analysis of POD-ROM, please check out Volkwein's tutorials on this topic, e.g. Chapter 2 from the book "Snapshot-Based Methods and Algorithms" or his lecture notes .
- For the application of ROMs to problems from fluid mechanics, I can recommend Chapter 9 on "Model Order Reduction in Fluid Dynamics: Challenges and Perspectives" from Quarteroni's and Rozza's book "Reduced Order Methods for Modeling and Computation".
- Finally, feel free to take a closer look at my master's thesis, where I also demonstrate how proper orthogonal decomposition based reduced order models can be created for the heat equation and the laminar flow around a cylinder, which is a benchmark problem for the Navier-Stokes equations.