$$\cos\frac{\pi}{5} = ?$$

Swipe For Solution $\Rightarrow$

Let's assign an arbitrary constant such as γ to be equal to $\frac{\pi}{5}$.

$$\begin{aligned} \gamma &= \frac{\pi}{5} \rightarrow 5\gamma \\ 3\gamma &= \pi - 2\gamma \end{aligned} \tag{1}$$

We will now apply $\sin$ on both sides of (1); we can apply cosine if we want but the addition formulas for *cos* give us squares which are somewhat difficult to deal with which is why we will start with *sin* functions and work our way towards and equation we can express in terms of $cos\gamma$

$$\sin(3\gamma) = \sin(\pi - 2\gamma)$$

We can write $\sin(\pi - 2\gamma)$ using its' addition formula which will give us:

$$\sin(3\gamma) = \sin(\pi)\cos(2\gamma) - \cos(\pi)\sin(2\gamma)$$

We know that $\sin(\pi) = 0$ and $\cos(\pi) =$ -1 so our equation simplifies to:

$$\begin{aligned} \sin(3\gamma) &= \sin(2\gamma) \\ \sin(\gamma + 2\gamma) &= \sin(2\gamma) \end{aligned}$$

which can be written as:

$$\sin(\gamma)cos(2\gamma) + \sin(2\gamma)\cos(\gamma) = 2\sin(\gamma)\cos(\gamma)$$

factoring out and cancelling a factor of $\sin(\gamma)$ gives us:

$$\cos(2\gamma) + \frac{\sin(2\gamma)\cos(\gamma)}{\sin(\gamma)} = 2\cos(\gamma)$$

Breaking down $\cos(2\gamma)$ and $\sin(2\gamma)$ by their addition formulas will take us closer to our goal of writing everything in terms of $\cos(\gamma)$.

$$\begin{aligned} \cos^2(\gamma) - \sin^2(\gamma) + \frac{2\sin(\gamma)\cos^2(\gamma)}{\sin(\gamma)} &= 2\cos(\gamma) \\ \cos^2(\gamma) - 1 + \cos^2(\gamma) + 2\cos^2(\gamma) &= 2\cos(\gamma) \end{aligned}$$

We can equate a variable such as λ to be equal to $\cos(\gamma)$ and under this our equation becomes:

$$4\lambda^2 - 2\lambda - 1 = 0$$

Solving this quadratic gives us 2 branches but we will consider the positive solution only as $\cos(\theta) \geq 0$ in the interval from $0 \to \frac{\pi}{2}$ and $\frac{\pi}{5}$ is in this interval.

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{4} = 0$$

$$\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16} = \frac{1}{4} + \frac{1}{16} \to \left(\lambda - \frac{1}{4}\right)^2 = \frac{5}{16}$$

$$\left(\lambda - \frac{1}{4}\right) = \frac{\sqrt{5}}{4} \to \lambda = \frac{\sqrt{5}}{4} + \frac{1}{4}$$

$$\lambda = \frac{1+\sqrt{5}}{4} \to \lambda = \frac{\phi}{2}$$

$\therefore \cos\left(\frac{\pi}{5}\right)$ equals $\frac{\phi}{2}$

$$\cos\left(\frac{\pi}{5}\right) = \frac{\phi}{2}$$