Math 252 Notes

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Contents

Chapter 1

Algebra and Geometry of Euclidean Space

1.1 Vectors in the Plane and Space

Brief review of points in the plane and space. Use $\mathbb{R}^n$ notation.

Talk about adding points together and scalar multiples. Zero and additive inverse / negative.

1. $(\alpha\beta)\big(a_1, a_2, a_3\big) \square \cdot = a\big(\beta\big(a_1, a_2, a_3\big)\big)$
2. Distributivity
3. Distributivity (other one)
4. zero for scalar multiplication
5. zero for scalar multiplication (other one)
6. scalar multiplication by 1

Introduce vectors as arrows. Vectors terminate at points.

Definition. If we insist vectors start at the origin, *bound vectors*. *Free vectors* or *vectors* can start anywhere.

1. Vector addition as parallelogram. Then as component-wise addition.
2. Scalar multiplication.

Definition.

$$\begin{aligned}
\hat{\mathbf{i}} &= (1, 0, 0) \\
\hat{\mathbf{j}} &= (0, 1, 0) \\
\hat{\mathbf{k}} &= (0, 0, 1)
\end{aligned}$$

If $\mathbf{a} = \big(a_1, a_2, a_3\big)$, then $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$.

Do some examples.

Do vector from P to P'.

Theorem. *The equation of line l through $P = (x_1, y_1, z_1)$ in the direction of $\mathbf{d} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is given by*

$$\begin{aligned} x =& x_1 + at, \\ y =& y_1 + bt, \\ z =& z_1 + ct. \end{aligned}$$

Modify this to go from one point to another. Do line segments.

Describe points in parallelogram and plane formed by two vectors.

Talk about dimension and xy-, xz-, and yz-planes.

1.2 The Inner Product and Distance

Definition. The *inner product* of $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{j}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties

1. $\mathbf{a} \cdot \mathbf{a} \geq 0$ and $\mathbf{a} \cdot \mathbf{a} = 0$ iff $\mathbf{a} = \mathbf{0}$.
2. Associativity with scalar multiplication.
3. Distributivity
4. Symmetry

Definition. The *length* or *norm* of a vector is $||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. Vectors with norm 1 are *unit vectors*. If $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a}/||\mathbf{a}||$ is a unit vector and we have *normalized* $\mathbf{a}$.

Normalize some vectors.

The distance from $\mathbf{a}$ to $\mathbf{b}$ is $||\mathbf{a} - \mathbf{b}||$.

Ex 1. Let $P_t = t(1, 1, 1)$. What is the distance from P_t to $(3, 0, 0)$? When is the distance the least?

Theorem. *Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors and let θ be the angle between them. Then*

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \, ||\mathbf{b}|| \cos\theta$$

Thus

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| \, ||\mathbf{b}||}\right)$$

Find some angles.

Definition. The inner product of two vectors is zero iff they are perpendicular. We say that perpendicular vectors are *orthogonal*.

Ex 2. Do some examples. Find a unit vector in the xy-plane orthogonal to $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

Theorem (Cauchy-Schwarz Inequality). *For any two vectors* $\mathbf{a}$ *and* $\mathbf{b}$*,*

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$$

Equality holds iff one vector is a scalar multiple of the other.

Do orthogonal projection. The projection of $\mathbf{v}$ onto $\mathbf{a}$ $\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{a}\|^2}\mathbf{a}$

Theorem (Triangle Inequality). *For vectors* $\mathbf{a}$ *and* $\mathbf{b}$*,*

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$

Definition. If P and Q are points, the vector $\mathbf{d} = \overrightarrow{PQ}$ is the *displacement vector*

Ex 3. Two ships can see a lighthouse, but not each other. What is their displacement vector?

Ex 4. Suppose a boat is traveling across a like at 10 km/h. Find the components of the displacement vector after 1 hour. This gives us the velocity vector.

Definition. If an object has a constant velocity vector $\mathbf{v}$, then in t units of time, the resulting displacement vector is $\mathbf{d} = t\mathbf{v}$.

Ex 5. Superman is flying in a straight line with velocity vector $10\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$ (in m/s). Suppose that (x, y) are his coordinates above the ground and z his height. If he starts at $(1, 2, 3)$, where is he after 1 minute? How many seconds does it take for him to climb 10 meters?

1.3 2x2 and 3x3 Matrices

Cover matrices, determinants, and transposes.

1.4 The Cross Product and Planes

1.4.1 Cross Product

Definition. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ be vectors in $\mathbb{R}^3$. The *cross product* of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \times \mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = \left(a_2b_3 - a_3b_2\right)\hat{\mathbf{i}} + \left(a_3b_1 - a_1b_3\right)\hat{\mathbf{j}} + \left(a_1b_2 - a_2b_1\right)\hat{\mathbf{k}}$$

Go over matrix calculations. Talk about quaternions. Do triple product $\left(\mathbf{a} \times \mathbf{b}\right) \cdot \mathbf{v}$ as determinant. Talk about orthogonality.

Do $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2|\sin\theta|$ algebraically. It's the area of the parallelogram.

2×2 determinant is area parallelogram. 3×3 is volume.

1.4.2 Planes

Derive formula for equation of a plane

Definition. The equation of a plane through $\left(x_0, y_0, z_0\right)$ that has a normal vector $\mathbf{n} = A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}$ is

$$A\left(x - x_0\right) + B\left(y - y_0\right) + C(z - z_0) = 0$$

Rewrite as $Ax + By + Cz + D = 0$

Do a direct example and then do one with three points (use cross product for normal)

Do distance from point to plane (construct vector and project onto $\mathbf{n}$)

Theorem. *The distance from (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is*

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

1.5 n-Dimensional Euclidean Space

Skip

1.6 Curves in the Plane and Space

Think of a curve as $f : [a, b] \to \mathbb{R}^n$. We call it a *path* and refer to it as $\mathbf{c}$. t is the most common variable, so $\mathbf{c}(t)$ is the position in time. $\mathbf{c}(t)$ parameterizes the curve. We can write $\mathbf{c}(t) = (x(t), y(t), z(t))$.

Ex 6. Do a line in space.

Ex 7. Parameterize a circle.

Ex 8. Parameterize a function $y = f(x)$.

Ex 9. Find the path $\mathbf{c}(t)$ of a point in the circle of radius R with initial velocity $v\hat{\mathbf{i}}$. The point starts r units directly below the center.

First find the path of the center: $\mathbf{C}(t) = vt\hat{\mathbf{i}} + R\hat{\mathbf{j}}$. The position of $\mathbf{c}(t)$ relative to the center is $\mathbf{d}(t) = \mathbf{c}(t) - \mathbf{C}(t)$. It has initial position $-r\hat{\mathbf{j}}$ and rotates clockwise.

The wheel makes one full rotation at $2\pi R$. This takes $2\pi R/v$ time units. The angular velocity is $d\theta/dt$ is v/R. Since the point is always r units from the center,

$$\mathbf{d}(t) = r\left(\cos\left(-\frac{v}{r}t + \theta\right)\hat{\mathbf{i}} + \sin\left(-\frac{v}{r}t + \theta\right)\hat{\mathbf{j}}\right)$$

where θ is the initial angle. Initial angle is $-\frac{\pi}{2}$. But $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ and $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

So

$$\mathbf{d}(t) = -r\sin\frac{vt}{r}\hat{\mathbf{i}} - r\cos\frac{vt}{r}\hat{\mathbf{j}}$$

Add to $(C)(t)$ to get answer.

Definition. The *velocity* of a path $\mathbf{c}(t)$ is $\mathbf{c}'(t)$. We normally draw $\mathbf{c}'(t)$ so that it starts at the point $\mathbf{c}(t)$. The *speed* is $s = \|\mathbf{c}'(t)\|$. This is done component-wise.

Ex 10. Do a couple examples.

Definition. If $\mathbf{c}(t)$ is a path, then its *tangent line* at the point $\mathbf{c}(t_0)$ is

$$\boldsymbol{\ell}(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0).$$

Ex 11. Do an example or two. Find where the particle is later.

Chapter 2

Differentiation

2.1 Graphs and Level Surfaces

2.1.1 Graphs

Definition. $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ Sometimes this is a *function of several variables*

Ex 12. Do some examples. Show why more than 3 dimensions can be useful.

Definition. If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a function of several variables (sometimes called a *real-valued function*), its *graph* consists of the set of points $\left(x_1, \ldots, x_n, f\left(x_1, \ldots, x_n\right)\right)$

Ex 13. $f(x, y) = x - y + 2$ and $f(x, y) = 3x$

2.1.2 Level Sets

The graph of a function in 3 variables is in $\mathbb{R}^4$

Definition. A *level set* is the set of solutions to $f(\mathbf{x}) = k$ where k is a constant. The level set of a 2-variable function is often called a *level curve* or *contour*. For a 3-variable function, it is a *level surface*.

Ex 14. Do previous ones. Then do $f(x, y) = x^2 + y^2$ and $z = x^2 + y^2 + 4x - 8y - 20$

Basic procedure:

1. Find any symmetries
2. See if any variables are missing.
3. For $z = f(x, y)$, do level curves. Also, do $x = 0$ and $y = 0$.
4. For $F(x, y, z) = k$, either do part 3 or do level curves for x.

Ex 15. $z = y^2$ and $x^2 + y^2 = 25$ and $f(x, y) = x^2 - y^2$

Definition. A *quadric surface* is defined by

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + m = 0$$

1. Parabaloid

2. Hyperbolic parabaloid
3. Hyperboloid of two sheets $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$
4. Hyperboloid of one sheet $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$
5. Ellipsoid
6. Cone

2.2 Partial Derivatives and Continuity

2.2.1 Limits

Definition. Suppose that f is defined on a set A such that every disk about (x_0, y_0) intersects A in at least one point other than (x_0, y_0). We write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ whenever $0 < d\big((x,y),(x_0,y_0)\big) < \delta$.

Theorem. *If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, then the two one-variable limits must be equal.*

Ex 16.

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

Ex 17.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$$

$y = x$ is the problem

Ex 18.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

$y = x^3$

Definition. Let f be defined for all points near (x_0, y_0) including (x_0, y_0). We say f is *continuous* if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

2.2.2 Partial Derivatives

Definition.

$$\frac{\partial f}{\partial x_i}(x_1,\ldots,x_n) = \lim_{h\to 0} \frac{f(x_1,\ldots,x_i+h,\ldots,x_n) - f(x_1,\ldots,x_n)}{h}$$

Immediately do more intuitive version. Do examples. Introduce $|_{x_0,y_0}$ notation.

2.3 Differentiability, the Derivative Matrix, and Tangent Planes

2.3.1 Tangent planes

Definition. The tangent plane of the graph $z = f(x, y)$ at the point (x_0, y_0) has the normal

$$\left(-\left.\frac{\partial f}{\partial x}\right|_{x_0, y_0}, -\left.\frac{\partial f}{\partial y}\right|_{x_0, y_0}, 1 \right)$$

and the equation is

Ex 19. Find some planes. Do top half of a sphere generically.

Differentiable means we have a tangent plane.

Definition. If $f : \mathbb{R}^2 \to \mathbb{R}$, then the *derivative matrix* is

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

So the tangent plane is

$$z = f\left(x_0, y_0\right) + Df\left(x_0, y_0\right) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Definition. The *derivative matrix* of f at $\mathbf{x}_0$ is the matrix $Df\left(\mathbf{x}_0\right)$ whose ijth entry is $\frac{\partial f_i}{\partial x_j}$ evaluated at $\mathbf{x}_0$. We say f is *differentiable* at $\mathbf{x}_0$ if all the partial derivatives are defined and

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\left\| f(\mathbf{x}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \right\|}{\left\| \mathbf{x} - \mathbf{x}_0 \right\|} = 0$$

Ex 20. Do some examples of matrices.

2.4 The Chain Rule

Theorem. *Let $z = f(x, y)$ have continuous partial derivatives and let x and y be differentiable functions of t. The*

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

If $u = f(x, y, z)$ then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Ex 21. Do some examples. Change the variable

Definition. A *path* in $\mathbb{R}^3$ is a mapping of an interval of real numbers to $\mathbb{R}^3$. It is written

$$\mathbf{c}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}} + k(t)\hat{\mathbf{k}}$$

The *tangent* vector to the curve at $\mathbf{c}(t_0)$ is

$$\mathbf{c}'\left(t_0\right) = g'\left(t_0\right)\hat{\mathbf{i}} + h'\left(t_0\right)\hat{\mathbf{j}} + k'\left(t_0\right)\hat{\mathbf{k}}$$

Ex 22. Do an example or two. Look at surfaces and parameterize by $(g(t), h(t), f(x, y))$

Theorem. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions. Let f be a function of u and v and write $g(x, y) = (u(x, y), v(x, y))$. Define h as*

$$h(x, y) = f(u(x, y), v(x, y))$$

Then

$$\begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Do some examples and extend.

Theorem. *Let $g\left(x_1, \dots, x_n\right) = \left(g_1\left(x_1, \dots, x_n\right), \dots, g_m\left(x_1, \dots, x_n\right)\right)$ be m functions of n variables and assume that $f\left(u_1, \dots, u_m\right) = \left(f_1\left(u_1, \dots, u_m\right), \dots, f_p\left(u_1, \dots, u_m\right)\right)$. Assume that f and g are differentiable. Then $f \circ g$ is differentiable and*

$$D(f \circ g)(\mathbf{x}) = Df(g(\mathbf{x}))Dg(\mathbf{x})$$

Theorem.

1. $D(cu) = c\,Du$
2. $D(u + v) = Du + Dv$
3. $D(uv) = vDu + uDv$
4. $D\left(\frac{u}{v}\right) = \frac{vDu - uDv}{v^2}$

2.5 Gradients and Directional Derivatives

Definition. If $f(x, y, z)$ is a real-valued function of three variables, its *gradient*, which is denoted ∇f, is defined by

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

Do some examples

Definition. A rule $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y, z)$ to each point in space is a *vector field*.

Ex 23. $f(x, y) = x^2 + y^2$ Draw some vectors in the field.

Theorem. *Let f be a function of two or three variables, $\mathbf{c}(t)$ a curve, and $h(t) = f(\mathbf{c}(t))$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = h'(t) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$$

Do some examples.

Definition. If $\mathbf{v}$ is a unit vector, then $\mathbf{v}$ specifies a direction. We call $\nabla f(\mathbf{x}) \cdot \mathbf{v}$ the *directional derivative* of f in the direction of $\mathbf{v}$.

Prove gradient is the direction of greatest change.

Theorem. *The directional derivative at* $\mathbf{x}$ *is the greatest when*

$$\mathbf{v} = \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}$$

and least when its negative.

Do some examples.

Note that the gradient is thus orthogonal to $f(x, y, z) = k$ at every point. Use the gradient for the tangent plane

2.6 Implicit Differentiation

Look at an equation that implicitly defines y in terms of x. Get everything on one side to get

$$F(x, y) = 0$$

Use the chain rule

$$\begin{aligned}
\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\partial F / \partial x}{\partial F / \partial y}
\end{aligned}$$

Do some examples.

Chapter 3

Higher Derivatives and Extrema

3.1 Higher Order Partial Derivatives

Do some examples. Do some where order doesn't matter.

Theorem. *If $u = f\left(x_1, \ldots, x_n\right)$ has continuous second partial derivatives, then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

3.2 Taylor's Theorem

Theorem. *If $y = f(x)$ has a continuous derivative of order $k+1$, then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^k(a)}{k!}(x-a)^k + R_k$$

where

$$R_k = \int_a^x \frac{(x-t)^k}{k!} f^{k+1}(t)\, dt$$

Theorem (Second Order Taylor Formula)**.** *Let $z = f(x, y)$ have continuous partial derivatives up to second order. Then*

$$\begin{aligned} f(x,y) = f\left(x_0, y_0\right) &+ \left(x - x_0\right) \left.\frac{\partial f}{\partial x}\right|_{x_0, y_0} + (y - y_0) \left.\frac{\partial f}{\partial y}\right|_{x_0, y_0} \\ &+ \frac{1}{2}(x-x_0)^2 \left.\frac{\partial^2 f}{\partial x \partial x}\right|_{x_0, y_0} + \frac{1}{2}(y-y_0)^2 \left.\frac{\partial^2 f}{\partial y \partial y}\right|_{x_0, y_0} \\ &+ (x-x_0)(y-y_0) \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_0, y_0} + R_2 \end{aligned}$$

where $R_2/\|\mathbf{h}\|^2 \to 0$ as $\|\mathbf{h}\| \to 0$ where $\mathbf{h} = \left(x - x_0, y - x_0\right)$

Ex 24. $f(x, y) = e^{x+y}$ at $(0, 0)$

Ex 25. Approximate $(1.01)^2\left(1 - \sqrt{1.42}\right)$

3.3 Maxima and Minima

Define local maxima and minima. Define global.

Definition. A point is a *critical point* of $f(x,y)$ if

$$\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} = \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} = 0$$

Theorem. *If (x_0, y_0) is a local extremum of f and the partial derivatives of f exist at (x_0, y_0), then (x_0, y_0) is a critical point.*

Ex 26. $f(x,y) = x^2 - y^2 + xy + x - y$

Ex 27. $f(x,y) = e^{1+x^2-y^2}$

Talk about extreme value theorem and why it's more complicated (boundary)

3.4 Second Derivative Test

Talk about one variable:

If $f'(a) = 0$, then near $x = a$

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ &\approx f(a) + \frac{1}{2}f''(a)(x-a)^2 \end{aligned}$$

Now let's look at two variables.

If (x_0, y_0) is a critical point, then near (x_0, y_0)

$$\begin{aligned} f(x,y) \approx\, & f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y-y_0) \\ & + \frac{1}{2}(x-x_0)^2\left.\frac{\partial^2 f}{\partial x\partial x}\right|_{x_0,y_0} + \frac{1}{2}(y-y_0)^2\left.\frac{\partial^2 f}{\partial y\partial y}\right|_{x_0,y_0} \\ & + (x-x_0)(y-y_0)\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{x_0,y_0} \\ \approx\, & f(x_0,y_0) + \frac{1}{2}(x-x_0)^2\left.\frac{\partial^2 f}{\partial x\partial x}\right|_{x_0,y_0} + \frac{1}{2}(y-y_0)^2\left.\frac{\partial^2 f}{\partial y\partial y}\right|_{x_0,y_0} \\ & + (x-x_0)(y-y_0)\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{x_0,y_0} \end{aligned}$$

Let $A = \frac{1}{2}\left.\frac{\partial^2 f}{\partial x\partial x}\right|_{x_0,y_0}$, $B = \frac{1}{2}\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{x_0,y_0}$, and $C = \frac{1}{2}\left.\frac{\partial^2 f}{\partial y\partial y}\right|_{x_0,y_0}$

Then the previous approximation turns into

$$f(x,y) \approx f\left(x_0,y_0\right) + A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2$$

Let's look at $\left(x_0,y_0\right) = (0,0)$. We get the following approximation

$$f(0,0) + Ax^2 + 2Bxy + Cx^2$$

We don't care about the value of $f(0,0)$. We just need to know if $Ax^2+2Bxy+Cx^2$ is positive, negative, or neither near $(0,0)$. So define

$$z = g(x,y) = Ax^2 + 2Bxy + Cy^2$$

Show that $(0,0)$ is the only critical point if $AC - B^2 \neq 0$.

$$\begin{aligned} Ax_0 + By_0 =& 0 \\ Bx_0 + Cy_0 =& 0 \end{aligned}$$

Cancel out the y_0 to get $\left(AC - B^2\right)x_0 = 0$.

$\mathbf{AC - B^2 > 0}$:

Since $AC - B^2 > 0$, $A, B \neq 0$. Complete the square

$$\begin{aligned} g(x,y) =& A\left(x^2 + \frac{2B}{A}xy + \frac{C}{A}y^2\right) \\ =& A\left(x^2 + \frac{2B}{A}xy + \frac{B^2}{A^2}y^2 - \frac{B^2}{A^2}y^2 + \frac{C}{A}y^2\right) \\ =& A\left(\left(x + \frac{B}{A}y\right)^2 + \frac{AC - B^2}{A^2}y^2\right) \\ =& A\left(x + \frac{B}{A}y\right)^2 + \frac{AC-B^2}{A}y^2 \end{aligned}$$

If $A > 0$, then $C > 0$. Thus $g(x,y) \geq 0$ and is equal to zero only when both terms are zero. But if that's the case, then $y = 0$ and $A\left(x + \frac{B}{A}\right)^2 = Ax^2 = 0$. Thus the only point is where it's zero is $(0,0)$. So everywhere else, $g(x,y) > 0$ and so $(0,0)$ is a minimum.

If $A < 0$, the same argument follows and $(0,0)$ is a maximum.

But recall we're dealing with $f(x,y) \approx f(0,0) + g(x,y)$.

$\mathbf{AC - B^2 < 0}$:

If $A \neq 0$, then we're still good with

$$g(x,y) = A\left(x + \frac{B}{A}y\right)^2 + \frac{AC-B^2}{A}y^2$$

If $A < 0$ then $A\left(x + \frac{B}{A}y\right)^2 \leq 0$ and $\frac{AC-B^2}{A}y^2 \geq 0$

But we have $g(x,0) = Ax^2 < 0$ and $g\left(-\frac{B}{A}y, y\right) = \frac{AC-B^2}{A}y^2 \geq 0$. Thus $g(x,y)$ can be both positive and negative near $(0,0)$.

If $A = 0$, then $g(x,y) = 2Bxy + Cy^2 = y(2B + Cy)$. This can also be positive or negative around $(0,0)$.

What does this mean for other points? Shift.

Theorem. *Let $z = f(x,y)$ have continuous partial derivatives up to the third order and suppose that (x_0, y_0) is a critical point of f. Consider the expression*

$$D = \left(\left.\frac{\partial^2 f}{\partial x \partial x}\right|_{(x_0,y_0)}\right)\left(\left.\frac{\partial^2 f}{\partial y \partial y}\right|_{(x_0,y_0)}\right) - \left(f\Big|xy_{(x_0,y_0)}\right)^2$$

called the discriminant*.*

1. *If $D > 0$ and $\left.\frac{\partial^2 f}{\partial x \partial x}\right|_{(x_0,y_0)} > 0$, it's a local min.*
2. *If $D > 0$ and $\left.\frac{\partial^2 f}{\partial x \partial x}\right|_{(x_0,y_0)} < 0$, it's a local max.*
3. *If $D < 0$, it's a saddle point.*
4. *If $D = 0$, the test fails.*

Ex 28. $f(x,y) = x^3 - 3xy + y^3$

Ex 29. $f(x,y) = x^2y - y^2x - x^2 - y^2, (0,0), \left(-\frac{2}{3}, \frac{2}{3}\right), \left(1+\sqrt{5}, -1+\sqrt{5}\right), \left(1-\sqrt{5}, -1-\sqrt{5}\right)$

3.5 Constrained Extrema

Look at level curves and paths.

Ex 30. Assume there is a 3D printing company whose revenue is $R(h,m) = h^{\frac{2}{3}}m^{\frac{1}{3}}$. Each hour of labor h costs the company \$30. Each unit of material m costs the company \$5. If their budget is \$60000, What should their production level be to maximize revenue?

Theorem. *Let f and g be functions of two variables with continuous partial derivatives. Suppose that the function f, when restricted to the level curve C defined by $g(x,y) = c$, has an extremum at (x_0, y_0) and that $\nabla g(x_0, y_0) \neq \mathbf{0}$. Then there is a number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Ex 31. $f(x,y) = x - y, x^2 - y^2 = 2$

Ex 32. $f(x,y) = xy, x + y = 1$

Ex 33. $f(x,y) = \cos^2 x + \cos^2 y, x + y = \frac{\pi}{4}$

Ex 34. Assume there is a 3D printing company whose revenue is $R(h,m) = h^{\frac{2}{3}}m^{\frac{1}{3}}$. Each hour of labor h costs the company \$30. Each unit of material m costs the company \$5. If their budget is \$60000, What should their production level be to maximize revenue?

$\left(\frac{4000}{3}, 40000\right)$. $\lambda = \frac{1}{15\sqrt[3]{9}}$

Ex 35. $f(x, y) = 2x^2 + 3y^2, x^2 + y^2 \leq 1$

Ex 36. $f(x, y) = 3x + 2y, 2x^2 + 3y^2 \leq 3$

Chapter 4

Vector-valued Functions

4.1 Acceleration

Definition. If $\mathbf{c}$ is a path in $\mathbb{R}^n$, its *tangent* or *velocity vector* is $\mathbf{v} = \mathbf{c}'(t)$ and its *speed* is $s = \|\mathbf{v}\|$.

Sum Rule: $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{b}(t) + \mathbf{c}(t)\right] = \mathbf{b}'(t) + \mathbf{c}'(t)$

Scalar Multiplication Rule: $\frac{\mathrm{d}}{\mathrm{d}t}[p(t)\mathbf{c}(t)] = p'(t)\mathbf{c}(t) + p(t)\mathbf{c}'(t)$

Dot Product Rule: $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{b}(t) \cdot \mathbf{c}(t) = \mathbf{b}'(t) \cdot \mathbf{c}(t) + \mathbf{b}(t) \cdot \mathbf{c}'(t)$

Cross Product Rule: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{b}(t) \times \mathbf{c}(t)\right) = \mathbf{b}'(t) \times \mathbf{c}(t) + \mathbf{b}(t) \times \mathbf{c}'(t)$

Chain Rule: $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}(p(t)) = \mathbf{c}'(p(t))p'(t)$

Ex 37. If $\|\mathbf{c}(t)\|$ is constant, then $\mathbf{c}'(t)$ is orthogonal to $\mathbf{c}(t)$ for all t.

Definition. The *acceleration* of $\mathbf{c}(t)$ is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{c}''(t)$

Ex 38. If a particle moves so that its acceleration is constant $-\hat{\mathbf{k}}$ and its position and velocity at $t = 0$ are $(0, 0, 1)$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, when does it fall below $z = 0$?

4.2 Arc Length

Theorem. *The arc length of* $\mathbf{c}(t)$ *on the interval* $\left[t_0, t_1\right]$ *is*

$$\ell = \int_{t_0}^{t_1} \left\|\mathbf{c}'(t)\right\| dt$$

Definition.

$$d\mathbf{s} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{k}}\right)dt$$

So

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y^2}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, dt$$

We'll often say

$$\ell = \int_{t_0}^{t_1} ds$$

Ex 39. $\left(2\cos t, 2\sin t, t\right), 0 \leq t \leq 2\pi$

Ex 40. $\left(t, t, t^2\right), 1 \leq t \leq 2$

Let $\mathbf{c}(t)$ be a given path, $a \leq t \leq b$. Let $s = \alpha(t)$ be a new variable, where α is a strictly increasing continuous function given on $[a, b]$. Thus α is one-to-one. Let $\mathbf{d}(s) = \mathbf{c}(t)$.

- The images of $\mathbf{c}$ and $\mathbf{d}$ are the same.
- They have the same arc length.
- Let $s = \alpha(t) = \int_a^t \left\|\mathbf{c}'(\tau)\right\|, d\tau$. Let $\mathbf{d}(s) = \mathbf{c}(t)$. Then

$$\left\|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{d}(s)\right\| = 1$$

This is *arc length reparametrization*

Ex 41.

$$\begin{aligned}\mathbf{c}(t) =& \left(2, 6\cos(2t), -6\sin(2t)\right)\\ \left\|\mathbf{c}'(t)\right\| =& 2\sqrt{10}\end{aligned}$$

So

$$s(t) = \int_0^t 2\sqrt{10}\, d\tau = 2\sqrt{10}t$$

This means

$$t = \frac{s}{2\sqrt{10}}$$

And so

$$\mathbf{c}(t(s)) = \left(2, 6\cos\left(\frac{s}{\sqrt{10}}\right), -6\sin\left(\frac{s}{\sqrt{10}}\right)\right)$$

Chapter 5

Multiple Integrals

5.1 Volume and Cavalieri's Principle

Recall definite integral in single variable. Talk about volume now.

$$\iint_D f(x,y)\,dx\,dy \text{ or } \iint_D f(x,y)dA$$

A rectangle is determined by two closed intervals $[a,b]$ and $[c,d]$. R is the *Cartesian product* of them and $R = [a,b] \times [c,d]$.

Ex 42. Integrate a constant over a rectangle.

Ex 43. $\iint_R (1-x)\,dxdy$ (Wedge, so it's half the volume)

Theorem (Cavalieri's Principle)**.** *Let S be a solid and P_x be a family of parallel planes such that*

1. *S lies between P_a and P_b and*
2. *the area of the slice of S cut by P_x is $A(x)$.*

Then the volume of S is

$$\int_a^b A(x)\,dx$$

So

$$V = \iint_R f(x,y)\,dxdy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

where $\int_c^d f(x,y)\,dy$ is found by integrating with respect to y and treating x as a constant.

Ex 44. $\int_{-1}^1 \int_0^1 \left(x^4 y + y^2\right) dydx$

Ex 45. $\int_0^1 \int_1^2 \left(-x \ln y\right) dydx$

5.2 Double Integral Over a Rectangle

Talk about rectangles.

Theorem. 1. *Every continuous function is integrable*

2. *If a rectangle R is divided into two rectangles R_1 and R_2 by a line segment,*

$$\iint_R f\,dxdy = \iint_{R_1} f\,dxdy + \iint_{R_2} f\,dxdy$$

3. *If f_1 and f_2 are integrable on R and $f_1(x,y) \le f_2(x,y)$ for all (x,y) in R, then*

$$\iint_R f_1\,dxdy \le \iint_R f_2\,dxdy$$

4. $\iint_R k\,dxdy = k(\text{area of } R)$

Ex 46.

$$\int_0^2 \int_{-1}^1 (yx)^2\,dydx$$

Ex 47.

$$\int_{-1}^1 \int_0^3 y^5 e^{xy^3}\,dxdy$$

Ex 48.

$$\iint_R \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\,dA$$

Show order matters

Ex 49. Find the volume of the solid below $z = x^2 + y$ and lying above $R = [0,1] \times [1,2]$

Ex 50. The state of Colorado occupies the region between 37° and 41° latitude and 102° and 109° longitude. A degree of latitude is above 110 km and a degree of longitude is about 83 km. The intensity of solar radiation at time t on day T at latitude l is (in units of watts per square km)

$$I = \cos l \sqrt{1 - \sin^2\alpha \cos^2\left(\frac{2\pi T}{365}\right)} \cos\left(\frac{2\pi t}{24}\right) + \sin l \sin\alpha \cos\left(\frac{2\pi T}{365}\right)$$

1. What is the integrated intensity of solar energy over Colorado at time t on day T?
2. Suppose we integrate that with respect to t from t_1 to t_2. What does that represent?

5.3 The Double Integral Over Regions

Ex 51.

$$\int_0^1 \int_{x^2}^x xy^2\,dydx$$

Ex 52.

$$\iint_R xy\, dA$$

where R is the region between x^2 and $\sqrt{x}$

Ex 53.

$$\iint_R x^2y^2\, dA$$

where R is the region between x^3 and $\sqrt[3]{x}$.

Ex 54.

$$\iint_R (3x-2y)\, dA$$

where R is the region enclosed by $x^2+y^2=1$.

Ex 55.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy dx$$

Ex 56.

$$\int_0^1 \int_0^y e^{x^2}\, dx dy$$

Theorem (Mean Value Theorem)**.** *Suppose that $f : D \to \mathbb{R}$ is continuous and D is an elementary region. Then for some point (x_0, y_0) in D, we have*

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x,y)\, dA$$

5.4 Triple Integrals

$$\iiint_B f\, dV$$

Theorem. *Let f be integrable on $B = [a,b]\times[c,d]\times[p,q]$. Then any iterated integral exists and is*

$$\begin{aligned}\iiint_B f, dV &= \int_a^b \int_c^d \int_p^q f(x,y,z)\, dz dy dx\\ &= \int_a^b \int_p^q \int_c^d f(x,y,z)\, dy dz dx\\ &= \dots\end{aligned}$$

Ex 57.

$$\iiint_B (2x+3y+z)\, dx dy dz\,,\ B=[0,2]\times[-1,1]\times[0,1]$$

Ex 58.

$$\int_0^1 \int_0^x \int_0^y (y+xz)\, dz dy dx$$

Ex 59. Find the volume bounded by $x = y, z = 0, y = 0, x = 1$ and $x + y + z = 0$

Ex 60. Find the volume bounded by $z = x^2 + y^2$ and $z = 10 - x^2 - 2y^2$.

$$\frac{50\pi}{\sqrt{3}}$$

5.5 Change of Variables, Cylindrical and Spherical Coordinates

Ex 61. $\iint_D 2xy, dA$ where D is the region between the circles of radii 2 and 5 centered at the origin.

Show that since we're taking a chunk of a circle, we get $d!A = r\,dr\,d\theta$

Ex 62. $\iint_D e^{x^2+y^2}\,dA$ where D is the unit circle.

Ex 63. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin\left(x^2 + y^2\right)\,dy\,dx$

Cylindrical Coordinates

Ex 64. $\iiint_W ze^{x^2+y^2}\,dA$ where W is the cylinder given by $x^2 + y^2 \leq 4$ and $2 \leq z \leq 3$.

Spherical Coordinates

Ex 65. $\iiint_W \left(x^2 + y^2 + z^2\right)\,dA$ where W is the unit ball.

Ex 66. $\iiint_W \left(x^2 + y^2 + z^2\right)^{3/2}\,dV$ where W is the solid bounded by $x^2 + y^2 + z^2 = \alpha^2$ and $x^2 + y^2 + z^2 = \beta^2$ where $\alpha > \beta > 0$.

Chapter 6

Integrals Over Curves and Surfaces

6.1 Line Integrals

Definition. Let $\mathbf{F}$ be a vector field and $\mathbf{c}$ be a continuously differentiable path defined on $[a, b]$. The *line integral* of $\mathbf{F}$ over $\mathbf{c}$ is

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t))\mathbf{c}'(t)\, dt.$$

Ex 67. $\mathbf{F}(x, y, z) = 8x^2yz\hat{\mathbf{i}} + 5z\hat{\mathbf{j}} - 4xy\hat{\mathbf{k}}, \mathbf{c}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}, 0 \le t \le 1$

Definition. Let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ and $\mathbf{c}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ on $a \le t \le b$. Then

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F} \cdot dsv &= \int_{\mathbf{c}} P\, dx + Q\, dy + R\, dz \\ &= \int_a^b \left(Px'(t) + Qy'(t) + Rz'(t)\right) dt\end{aligned}$$

Ex 68. $\int_C x^2\, dx + xy\, dy + dz$ where C is parameterized by $\mathbf{c}(t) = \left(t, t^2, 1\right)$.

Theorem. *If $f(x, y, z)$ is a given real-valued function and $\mathbf{c}(t)$ is a path joining $\mathbf{x}_0$ and $\mathbf{x}_1$, then*

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f\left(\mathbf{x}_1\right) - f\left(\mathbf{x}_0\right)$$

Proof. Recall from calc 1 that $\int_a^b g'(t)\, dt = g(b) - g(a)$. Let $\mathbf{c}(a) = \mathbf{x}_0$ and $\mathbf{c}(b) = \mathbf{x}_1$ and $g(t) = f(\mathbf{c}(t))$. Then from the chain rule we get

$$g(b) - g(a) = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

But $g(b) = f\left(\mathbf{x}_1\right)$ and $g(a) = f\left(\mathbf{x}_0\right)$. So

$$f\left(\mathbf{x}_1\right) - f\left(\mathbf{x}_0\right) = \int_C \nabla f \cdot d\mathbf{s}$$

□

Ex 69. $\mathbf{F} = \left(z^3 + 2xy\right)\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 3xz^2\hat{\mathbf{k}}$ around the square with vertices $(\pm 1, \pm 1, 0)$.

Ex 70. $\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}}$ along

$$\begin{aligned}
\mathbf{c}_1(t) =& (\cos t)\hat{\mathbf{i}} + \left(\sin v\right)\hat{\mathbf{j}} + \frac{t}{2\pi}\hat{\mathbf{k}} && 0 \le t \le 2\pi \\
\mathbf{c}_2(t) =& \left(\cos\left(t^3\right)\right)\hat{\mathbf{i}} + \left(\sin\left(t^3\right)\right)\hat{\mathbf{j}} + \frac{t^3}{2\pi}\hat{\mathbf{k}} && 0 \le t \le \sqrt[3]{2\pi} \\
\mathbf{c}_2(t) =& (\cos t)\hat{\mathbf{i}} - \left(\sin t\right)\hat{\mathbf{j}} + \frac{t}{2\pi}\hat{\mathbf{k}} && 0 \le t \le 2\pi
\end{aligned}$$

Theorem. *If the path C has two parametrizations $\mathbf{c}$ and $\mathbf{b}$, then for any vector field $\mathbf{F}$*

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{b}} \mathbf{F} \cdot d\mathbf{s}$$

Ex 71. $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}(x, y) = xy^2\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ and C is the unit square oriented counterclockwise.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar function and $\mathbf{c} : [a, b] \to \mathbb{R}^n$ a continuously differentiable path. The *integral* of f along $\mathbf{c}$ is

$$\int_{\mathbf{c}} f\, ds = \int_a^b f(\mathbf{c}(t))\left\|\mathbf{c}'(t)\right\|\, dt.$$

Ex 72. $\int_C (x \cos z)\, ds$ where $\mathbf{c}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$.

6.2 Parametrized Surfaces

Lots of surfaces are not the graphs of functions.

Definition. A *parametrized surface* is a vector-valued function $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ where D is some domain in $\mathbb{R}^2$. The *geometric surface S* corresponding to the function Φ is its image: $S = \Phi(D)$

Ex 73. The graph of a real-valued function $f(x, y)$ is a parametrized surface.

What about tangent planes?

Let $\mathbf{c}(v) = \Phi\left(u_0, v\right)$. The tangent vector is $\Phi_v = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}} + \frac{\partial z}{\partial v}\hat{\mathbf{k}}$. There's a similar one for Φ_u.

Theorem. *Let Φ be a parametrized surface. To compute the tangent plane at $\Phi\left(u_0, v_0\right) = \left(x_0, y_0, z_0\right)$, we get the normal vector*

$$\mathbf{n} = \Phi_u \times \Phi_v$$

evaluated at $\left(u_0, x_0\right)$.

Ex 74. $x = 2u, y = u^2 + v, z = v^2$ at $u = 0, v = 1$.

Ex 75. Find a parametrization of the hyperboloid $x^2 + y^2 - z^2 = 25$. Find an expression for the unit normal. Find an equation of the tangent plane at $(3, 4, 0)$ where $x_0^2 + y_0^2 = 25$.

$$\begin{aligned}
x =& 5\cosh v \cos u \\
y =& 5\cosh v \sin u \\
z =& 5\sinh v
\end{aligned}$$

6.3 Area of a Surface

Recall the surface area of revolution:

$$2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2}\, dx$$

For a general surface $z = f(x, y)$, divide it up into parallelograms by dividing xy into a grid. The area is the cross product

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ dx & 0 & \frac{\partial}{\partial x} f(x, y)\, dx \\ 0 & dy & \frac{\partial}{\partial y} f(x, y)\, dy \end{vmatrix}$$

Thus $dS = \sqrt{1 + f_x^2 + f_y^2}$

Theorem.

$$\begin{aligned} \textit{Area} &= \iint_D dS \\ &= \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx dy \end{aligned}$$

Ex 76. Find the area of $z = xy$ that lies above $x^2 + y^2 = 1$.

Ex 77. Find the area of $x^2 + z^2 = 1$ cut out by $x^2 + y^2 = 1$.

Theorem. *For a geometric surface S that is the image of the parametrization Φ, we have*

$$d\mathbf{S} = \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) du dv$$

and

$$dS = \|d\mathbf{S}\| = \left\|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right\| du dv$$

so that

$$\mathbf{n} = d\mathbf{S}/dS \text{ and } d\mathbf{S} = \mathbf{n}\, dS$$

where $\mathbf{n}$ is a unit normal vector to the surface. The surface area is

$$A = \iint_D dS = \iint_D \sqrt{\left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2}\, du dv$$

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Ex 78. Find the surface area of a sphere of radius R

Ex 79. Find the surface area of a sphere of radius 4 inside $x^2 + y^2 = 12$ above the xy-plane.

Definition. Let S be parametrized by $\Phi : D \to S \subset \mathbb{R}^3$ with $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$. If $f(x,y,z)$ is a real-valued continuous function defined on S, we define the *integral of f over S* to be

$$\iint_S f\, dS = \iint_D f(\Phi(u,v)) \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| du dv$$

Ex 80. $\iint_D z\, dS$ where S where S is the upper half of a sphere of radius 2.

6.4 Surface Integrals

Definition. The *surface integral* of a vector field $\mathbf{F}$ over a parametrized surface is the *number*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F}(\Phi(u,v)) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right) du dv$$

where $\mathbf{n}$ is a unit normal vector

Ex 81. $\mathbf{F}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + z\left(x^2+y^2\right)^2 \hat{\mathbf{k}}$ over the cylinder $x^2+y^2 \le 1, 0 \le z \le 1$.

Ex 82. $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where S is the surface $x^2+y^2+3z^2 = 1, z \le 0$ and $\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + zx^3y^2\hat{\mathbf{k}}$. Answer -2π.

Definition. An *oriented surface* is a two-sided surface with one side specified as the *outside* or *positive side* and the other as the *inside* or *negative side*. We will use the *outward pointing unit normal* $\mathbf{n}$.

Ex 83. Consider $x^2+y^2+z^2 = 1$ and $\mathbf{n}(x,y,z) = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

Ex 84. Consider $z = f(x,y)$.

Definition. If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ is a vector field in space and S is the surface $z = f(x,y)$ where $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, the *surface integral* of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R \right) dxdy$$

Ex 85. $\mathbf{F} = y\hat{\mathbf{j}} - z\hat{\mathbf{k}}$ and S is $y = x^2+y^2, 0 \le y \le 1$. Answer: $-\pi$.

Chapter 7

The Integral Theorems of Vector Analysis

7.1 Green's Theorem

Definition. If D is a planar region with boundary curve C, the *positive orientation* of C is given by the vector $\hat{\mathbf{k}} \times \mathbf{v}_{\text{out}}$ where $\mathbf{v}_{\text{out}}$ is the outward pointing normal vector.

As you walk along the boundary in the positive orientation, the region is on your left.

Theorem (Green's Theorem)**.** *If D is a region in $\mathbb{R}^2$, ∂D as its boundary, oriented in the positive sense, and P and Q are differentiable functions of x and y, then*

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Ex 86. $P(x, y) = x^2 y$, $Q(x, y) = x$, D is the region bounded by $y = 0$, $x = 1$, and $y = 2x$ oriented positively.

Ex 87. $\oint \left(x^2 - y^2\right) dx + xdy, x^2 + y^2 = 9$

7.2 Stokes' Theorem

Ex 88. $\oint_{\partial S} 2z\,dx + 3x\,dy + 5y\,dy$ where S is $z = 4 - x^2 - y^2$ with $z \geq 0$. Integrate $\nabla \times \mathbf{F}$ over S.

Ex 89. $\mathbf{F} = 2x\hat{\mathbf{i}} - y\hat{\mathbf{j}} + (x + z)\hat{\mathbf{k}}$, triangle with vertices $(1, 0, 1)$, $(0, 1, 0)$, $(0, 0, 1)$. Curl is $-\hat{\mathbf{j}}$.

Plane is $0(x - 1) + (y - 0) + (z - 1) = 0$ or $y + z = 1$ or $z = 1 - y$.

$\iint_S (\nabla \times F) \cdot\, d\mathbf{S}$ becomes $\iint_D \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial x} + R\right) dA$ The triangle is $(0, 0)$, $(0, 1)$, $(1, 0)$

$\int_0^1 \int_0^{1-x} (-1)\,dydx$

Ex 90. $\iint_S (\nabla \times \mathbf{F}) \cdot\, d\mathbf{S}$ where $\mathbf{F} = z^2\hat{\mathbf{i}} - 3xy\hat{\mathbf{j}} + x^3y^3\hat{\mathbf{k}}$ and S is $z = 5 - x^2 - y^2$ above $z = 1$. Answer is 0.

Ex 91. $\oint_C \mathbf{F} \cdot\, d\mathbf{s}$ where $\mathbf{F} = z^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ and S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Answer is $-\frac{1}{6}$.

Ex 92. $\mathbf{F} = (x - y)\hat{\mathbf{i}} + (y - z)\hat{\mathbf{j}} + (z - x)\hat{\mathbf{k}}$ and C is $x^2 + y^2 = 16$.

Ex 93. $\mathbf{F} = ax\hat{\mathbf{i}} + by\hat{\mathbf{j}} + cz\hat{\mathbf{k}}$ and C is in a plane with normal $\mathbf{n}$ enclosing an area A. Find an expression for $\int_C \mathbf{F} \cdot d\mathbf{s}$.

Theorem. *Let C_ϵ be the circle of radius ϵ centered at a point P_0 and lying in the plane through P_0 with unit normal $\mathbf{n}$. Let $\mathbf{F}$ be a smooth vector field defined in a region containing P_0. Then*

$$\left((\nabla \times \mathbf{F})\left(P_0\right)\right) \cdot \mathbf{n} = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}$$

$\left((\nabla \times \mathbf{F})\left(P_0\right)\right) \cdot \mathbf{n}$ *is the circulation per unit area in a plane.*

7.3 Divergence Theorem

Theorem. *Let $S = \partial W$ be a closed surface, oriented by the outward normal to the three dimensional region W of which it is the boundary. If $\mathbf{F}$ is a continuously differentiable vector field defined on W, then*

$$\iiint_W (\nabla \cdot \mathbf{F})\, dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$$

Ex 94. $\mathbf{F} = 2x\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ over $x^2 + y^2 + z^2 = 1$

Ex 95. $\mathbf{F} = 2x\hat{\mathbf{i}} + 3y\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ across unit cube.

Ex 96. $\mathbf{F} = x^3\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ across the cylinder $x^2 + y^2 = 9$, $z = 0$, and $z = 2$

Ex 97. $\mathbf{F} = 2xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ where S is the region bounded by $z = \sqrt{z^2 - x^2 - y^2}$ and $z = 0$.

We are concerned with incompressible fluids. At a fixed point, the velocity of the fluid doesn't change with respect to time. This flow is in a *steady state*. The *flux* (Φ) is the net volume of fluid that passes through the surface per unit of time.

Theorem. *Let W_r be a solid ball of radius r centered at a point P in space and $S_r = \partial W_r$ be the bounding sphere. For a vector field $\mathbf{F}$ in space, $\nabla \cdot \mathbf{F}(P)$ is the outward flux per unit volume at P.*

$$\nabla \cdot \mathbf{F}(P) = \lim_{r \to 0} \frac{1}{\mathit{vol}\left(W_r\right)} \iint_{S_r} \mathbf{F} \cdot d\mathbf{S}$$

Proof. Let $\Phi\left(W_r\right)$ be the flux across ∂W_r. Since $\mathbf{F}$ is continuous and *r* is small, the value of $\nabla \times \mathbf{F}$ won't vary much from the value at the center. Thus we can approximate $\nabla \times \mathbf{F}$ by $\nabla \times \mathbf{F}(P)$ in W_r.

So by the divergence theorem

$$\begin{aligned}
\Phi\left(W_r\right) &= \iint_{\partial W_R} \mathbf{F} \cdot d\mathbf{S} \\
&= \iiint_{W_r} (\nabla \cdot \mathbf{F})\, dV \\
&\approx \iiint_{W_r} \nabla \cdot \mathbf{F}(P)\, dV \\
&= \nabla \cdot \mathbf{F}(P) \iiint_{W_r} dV \\
&= \nabla \cdot \mathbf{F}(P) \cdot \mathrm{vol}\left(W_r\right)
\end{aligned}$$

And so

$$\nabla \cdot \mathbf{F}(P) \approx \frac{\Phi\left(W_r\right)}{\mathrm{vol}\left(W_r\right)}$$

As we let the radius decrease, the approximation gets better so any error will go to zero. $\square$

This is the *outward flux density of* $\mathbf{F}$ *at* P. Positive divergence at point means it is a *source*. Negative divergence means it is a *sink*.

7.4 Path Independence

Write FTC as the integral on a boundary. Show how Stokes' and divergence theorems are the similar.

Theorem. *A vector field* $\mathbf{F}$ *on a region in* $\mathbb{R}^n$ *is the gradient of some function if and only if, for any two paths* C_1 *and* C_2 *with the same endpoints,*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

Proof. Given $\mathbf{F}$ defined on some domain D, choose a reference point $\mathbf{x}_0$ in D. Assuming path independence, *define* f by

$$f(\mathbf{x}) = \int_{C_x} \mathbf{F} \cdot d\mathbf{s}$$

. Suppose that C is a path from $\mathbf{x}$ to $\mathbf{y}$. Let C_x be the path from $\mathbf{x}_0$ to $\mathbf{x}$ and let C_y be the path from $\mathbf{x}_0$ to $\mathbf{y}$. So we have $C_y = C_x + C$.

By path independence, we have

$$\int_{C_x} \mathbf{F} \cdot d\mathbf{s} + \int_{C} \mathbf{F} \cdot d\mathbf{s} = \int_{C_y} \mathbf{F} \cdot d\mathbf{s}$$

and so

$$\int_{C} \mathbf{F} \cdot d\mathbf{s} = \int_{C_y} \mathbf{F} \cdot d\mathbf{s} - \int_{C_x} \mathbf{F} \cdot d\mathbf{s} = f\left(\mathbf{y}\right) - f(\mathbf{x})$$

by definition of f. But $\int_C \nabla f \cdot d\mathbf{s}$ is also $f\left(\mathbf{y}\right) - f(\mathbf{x})$.

Thus $\mathbf{F}$ and ∇f have the same line integral over *every* path C. What if $\mathbf{F} \neq \nabla f$? Well, there is some point $\mathbf{z}$ where some component of ∇f is greater than the corresponding component of $\mathbf{F}$. Choose a short path near $\mathbf{z}$ in the direction of this component. Then $\int_C \nabla f \cdot d\mathbf{s} > \int_C \mathbf{F} \cdot d\mathbf{s}$. But that can't happen, so they must be equal. $\square$

Definition. A vector field $\mathbf{F}$ on a region in $\mathbb{R}^n$ is called *conservative* if it is the gradient of a function.

Theorem. *A vector field* $\mathbf{F}$ *defined on all* $\mathbb{R}^3$ *is conservative iff* $\nabla \times \mathbf{F} = \mathbf{0}$.

Proof. Show by computation that $\nabla \times (\nabla f) = \mathbf{0}$.

Now assume that $\nabla \times \mathbf{F} = \mathbf{0}$. Let C_1 and C_2 be different paths from $\mathbf{x}$ to $\mathbf{y}$. Then $C = C_1 + \left(-C_2\right)$ is a closed curve. Use Stokes' theorem.

$$
\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s} &= \int_{C} \mathbf{F} \cdot d\mathbf{s} \\
&= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= 0 \\
\int_{C_1} \mathbf{F} \cdot d\mathbf{s} &= \int_{C_2} \mathbf{F} \cdot d\mathbf{s}
\end{aligned}
$$

Since this is true for any two paths, it must be the gradient of a function. □