De Casteljau algoritm proof

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Cubic Bezier curve:

$$\begin{gathered} n \in \mathbb{N}, \\ p_i \in \mathbb{R}^n, \quad i \in \{0,1,2,3\} \\ f : [0,1] \to \mathbb{R}^n \end{gathered}$$

$$f(u) = (1-u)^3 p_0 + 3(1-u)^2 u p_1 + 3(1-u)u^2 p_2 + u^3 p_3$$

De Casteljau algorithm:

$$\begin{gathered} n \in \mathbb{N}, \\ p_i \in \mathbb{R}^n, \quad i \in \{0,1,2,3\} \end{gathered}$$

$$\begin{aligned} &r_i : [0,1] \to \mathbb{R}^n, i \in \{0,1,2\} \\ &s_i : [0,1] \to \mathbb{R}^n, i \in \{0,1\} \\ &t_0 : [0,1] \to \mathbb{R}^n \end{aligned}$$

$$\begin{aligned} r_i(u) &= (1-u)p_i + up_{i+1} \\ s_i(u) &= (1-u)r_i(u) + ur_{i+1}(u) \\ t_0(u) &= (1-u)s_0(u) + us_1(u) \end{aligned}$$

Theorem:

$$\begin{gathered} n \in \mathbb{N}, \\ \forall p_i \in \mathbb{R}^n, i \in 0,1,2,3, \text{ and } u \in [0,1], \\ f(u) = t_0(u) \end{gathered}$$

Proof:

$$\begin{aligned} f(u) &= (1-u)^3 p_0 + 3(1-u)^2 u p_1 + 3(1-u)u^2 p_2 + u^3 p_3 \\ t_0(u) &= (1-u)s_0(u) + us_1(u) = \\ &= (1-u)((1-u)r_0(u) + ur_1(u)) + u((1-u)r_1(u) + ur_2(u)) = \\ &= (1-u)((1-u)((1-u)p_0 + up_1) + u((1-u)p_1 + up_2)) + \\ &+ u((1-u)((1-u)p_1 + up_2) + u((1-u)p_2 + up_3)) = \\ &= (1-u)((1-u)^2 p_0 + 2(1-u)up_1 + u^2 p_2) + u((1-u)^2 p_1 + 2(1-u)up_2 + u^2 p_3) = \\ &= (1-u)^3 p_0 + 2(1-u)^2 up_1 + (1-u)u^2 p_2 + (1-u)^2 up_1 + 2(1-u)u^2 p_2 + u^3 p_3 = \\ &= (1-u)^3 p_0 + 3(1-u)^2 up_1 + 3(1-u)u^2 p_2 + u^3 p_3 = \\ &= f(u) \end{aligned}$$

$$\therefore f(u) = t_0(u)$$