Boolean Algebra

Armen Hayrapetian (UFAR)

1 Boolean Function

Boolean algebra provides the operations and the rules for working with the set $\{0, 1\}$.
The three operations in Boolean algebra that we will use most are **complementation**, the **Boolean sum**, and the **Boolean product**.

- The **complement** of an element, denoted with a bar, is defined by:

$$\bar{0} = 1, \bar{1} = 0$$

- The Boolean **sum**, denoted by "+" or by OR, has the following values:

$$0 + 0 = 0, 1 + 0 = 1, 0 + 1 = 1, 1 + 1 = 1$$

- The Boolean **product**, denoted by "·" or by AND, has the following values:

$$0 \cdot 0 = 0, 1 \cdot 0 = 0, 0 \cdot 1 = 0, 1 \cdot 1 = 1$$

Example 1. *Translate logic propositional into an identity in Boolean Algebra.*

1. $T \wedge \sim (F \vee F) \wedge T \vee F$
2. $T \wedge (T \vee F) \vee \sim T \wedge F$
3. $T \wedge (T \vee F)$
4. $\sim (T \wedge F)$

Example 2. *By using translating logic propositional into an identity in Boolean Algebra, prove De Morgan's laws in logic.*

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

Example 3. *By translating logic propositional into Boolean Algebra, prove given law in logic.*

$$p \wedge (p \vee q) \equiv p$$

Definition

Boolean Expressions and Boolean Functions
Let $B = \{0, 1\}$. Then $B_n = \{(x_1, x_2, \cdots, x_n) | x_i \in B, 1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s. The variable x is called a Boolean variable if it assumes values only from B, that is, if its only possible values are 0 and 1. A function from B_n to B is called a Boolean function of degree n.

Example 4. *How many different Boolean functions are there of degree 7? Determine number of Boolean functions of degree n.*

Example 5. *Present the given Boolean function as a table.*

$$F : B_2 \to B$$

$$F(x, y) = x\bar{y}$$

Example 6. *Find the values of the Boolean function as a table.*

$$F : B_3 \to B$$

$$F(x, y, z) = xy + \bar{z}$$

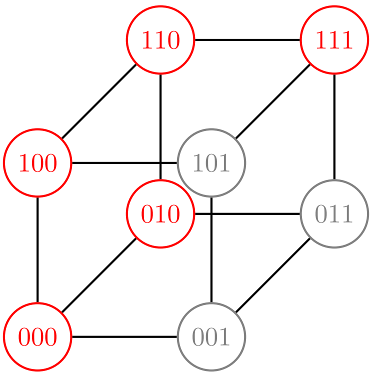
Example 7. *Find the values of the Boolean function as a table.*

$$F : B_3 \to B$$

$$F(x, y, z) = x\bar{y} + \bar{x}\bar{z}$$

Example 8. *The function $f(x, y, z) = xy + \bar{z}$ can be represented by distinguish the vertices that have 1 output.*

Solution



Example 9. *Use 3-cube Q_3 to represent each of the Boolean functions.*

1. $f(x, y, z) = (x + \bar{y} + z)$
2. $f(x, y, z) = \bar{x} \cdot \bar{y} \cdot \bar{z}$
3. $f(x, y, z) = x\bar{y}z + \bar{y}z$
4. $f(x, y, z) = x(yz + \bar{y}\bar{z})$

Example 10. *Use 3-cube Q_3 to represent each of the Boolean functions.*

1. $f(x, y, z) = x(y + z)$
2. $f(x, y, z) = xy + xz$

Example 11. *Use 2-cube Q_2 to represent the functions and get the equivalence $x + xy = x$.*

Example 12. *Use 2-cube Q_2 to represent the functions and get the equivalence $x(x + y) = x$.*

Example 13. *. Show that $F(x, y, z) = xy + xz + yz$ has the value 1 if and only if at least two of the variables x, y, and z have the value 1.*

Create a new Function

Let F and G be Boolean functions of degree n. The Boolean sum $F + G$ and the Boolean product FG are defined by

$$(F + G)(x_1, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n) + G(x_1, x_2, \cdots, x_n)$$

$$(FG)(x_1, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)G(x_1, x_2, \cdots, x_n)$$

Example 14. *For the given functions write $F+G$ and FG functions.*

$$F(x_1, x_2, x_3) = (x_1 + x_2)\bar{x_3}$$

$$G(x_1, x_2, x_3) = \bar{x_1} + x_2\bar{x_3}$$

Example 15. A Boolean function F is called **self-dual** if and only if

$$F(x_1, \cdots, x_n) = \overline{F(\bar{x_1}, \cdots, \bar{x_n})}.$$

Which of these functions are self-dual?

1. $F(x, y) = x + y$
2. $F(x, y) = xy + \bar{x}\bar{y}$
3. $F(x, y) = xy + \bar{x}y$
4. $F(x, y, z) = \bar{x}\bar{y}z + \bar{x}y\bar{z}$
5. $F(x, y, z) = xyz + x\bar{y}z + x\bar{y}\bar{z}$
6. $F(x, y, z) = xyz + x\bar{y}z + x\bar{y}\bar{z}$
7. $F(x, y, z) = \bar{x}\bar{y}\bar{z} + \bar{x}y\bar{z} + xy\bar{z} + \bar{x}yz$

Example 16. *The Boolean operator $\bigoplus$, called the "**XOR**" operator, is defined by*

$$1 \bigoplus 1 = 0, 1 \bigoplus 0 = 1, 0 \bigoplus 1 = 1, 0 \bigoplus 0 = 0$$

Also, we can see that $\bigoplus$ is the sum in mod 2 system. Simplify these expressions:

1. $x \bigoplus 0$
2. $x \bigoplus 1$
3. $x \bigoplus x$
4. $x \bigoplus \bar{x}$

Show that these identities hold.

1. $x \bigoplus y = (x + y)(\overline{xy})$
2. $x \bigoplus y = (x\bar{y}) + (\bar{x}y)$

2 Functional Completeness

Because every Boolean function can be represented using $\{\cdot, +, -\}$ operators we say that the set $\{\cdot, +, -\}$ is **functionally complete**. Can we find a smaller set of functionally complete operators? Since by De Morgan's law we can represent sum as a product $x + y = \overline{\bar{x}\bar{y}}$ the set $\{\cdot, -\}$ is functionally complete and similarly, product can be present as sum by De Morgan's law $xy = \overline{\bar{x} + \bar{y}}$, so the set $\{+, -\}$ is also functionally complete. We have found sets containing two operators that are functionally complete. Can we find a smaller set of functionally complete operators, a set containing just one operator? Such sets exist.

Define $|$ or "**NAND**" operator as below:

$$1|0 = 0|1 = 0|0 = 1, 1|1 = 0$$

The set $\{|\}$ is functionally complete, because

$$\bar{x} = x|x$$

$$x \cdot y = (x|y)|(x|y)$$

Also, define $\downarrow$ or "**NOR**" operator as below:

$$1 \downarrow 0 = 0 \downarrow 1 = 1 \downarrow 1 = 0, 0 \downarrow 0 = 1$$

The set $\{\downarrow\}$ is also functionally complete, because

$$\bar{x} = x \downarrow x$$

$$x \cdot y = (x \downarrow x) \downarrow (y \downarrow y)$$

Example 17. *Show that*

1. $\bar{x} = x|x$
2. $x \cdot y = (x|y)|(x|y)$
3. $x + y = (x|x)|(y|y)$

Example 18. *Show that*

1. $\bar{x} = x \downarrow x$
2. $x \cdot y = (x \downarrow x) \downarrow (y \downarrow y)$
3. $x + y = (x \downarrow y) \downarrow (x \downarrow y)$

3 Canonical and Standard Forms

Canonical Form
In Boolean algebra, Boolean function can be expressed as *Canonical Disjunctive Normal Form* known as **minterm** and some are expressed as *Canonical Conjunctive Normal Form* known as **maxterm**. In minterm, we look for the functions where the output results in "1" while in maxterm we look for function where the output results in "0".

- We perform Sum of minterm also known as Sum of products (SOP) .

  For example $xy + xz + yz$ or $\bar{x}y + yz + x\bar{z}$ are SOP expressions.

- We perform Product of maxterm also known as Product of sum (POS).

  For example $(x + y + z)(x + \bar{y} + \bar{z})$ or $(\bar{x} + \bar{y} + z)(x + y + \bar{z})$ are POS expressions.

  However, Boolean function like $(xy + yz)(\bar{x}\bar{y} + x\bar{z})$ is neither a sum of products form nor a product of sums form.

Boolean functions expressed as a sum of minterms or product of maxterms are said to be in canonical form.

Standard Form
A Boolean variable can be expressed in either true form or complemented form. In standard form Boolean function will contain all the variables in either true form or complemented form while in canonical number of variables depends on the output of SOP or POS.

A Boolean function can be expressed algebraically from a given truth table by forming a :

- minterm for each combination of the variables that produces a 1 in the function and then taking the OR of all those terms.
- maxterm for each combination of the variables that produces a 0 in the function and then taking the AND of all those terms. (see Fig1)

Example 19. *Show that if the number of variables is n, then the possible number of minterms is 2^n. Determine that in how many combinations of n input the value of minterm is 1 and in how many combinations of n input the value of minterm is 0.*

				Minterms		*Maxterms*
X	Y	Z		*Product Terms*		*Sum Terms*
0	0	0		$m_0 = \overline{X}\cdot\overline{Y}\cdot\overline{Z} = \min\left(\overline{X},\overline{Y},\overline{Z}\right)$		$M_0 = X + Y + Z = \max\left(X,Y,Z\right)$
0	0	1		$m_1 = \overline{X}\cdot\overline{Y}\cdot Z = \min\left(\overline{X},\overline{Y},Z\right)$		$M_1 = X + Y + \overline{Z} = \max\left(X,Y,\overline{Z}\right)$
0	1	0		$m_2 = \overline{X}\cdot Y\cdot\overline{Z} = \min\left(\overline{X},Y,\overline{Z}\right)$		$M_2 = X + \overline{Y} + Z = \max\left(X,\overline{Y},Z\right)$
0	1	1		$m_3 = \overline{X}\cdot Y\cdot Z = \min\left(\overline{X},Y,Z\right)$		$M_3 = X + \overline{Y} + \overline{Z} = \max\left(X,\overline{Y},\overline{Z}\right)$
1	0	0		$m_4 = X\cdot\overline{Y}\cdot\overline{Z} = \min\left(X,\overline{Y},\overline{Z}\right)$		$M_4 = \overline{X} + Y + Z = \max\left(\overline{X},Y,Z\right)$
1	0	1		$m_5 = X\cdot\overline{Y}\cdot Z = \min\left(X,\overline{Y},Z\right)$		$M_5 = \overline{X} + Y + \overline{Z} = \max\left(\overline{X},Y,\overline{Z}\right)$
1	1	0		$m_6 = X\cdot Y\cdot\overline{Z} = \min\left(X,Y,\overline{Z}\right)$		$M_6 = \overline{X} + \overline{Y} + Z = \max\left(\overline{X},\overline{Y},Z\right)$
1	1	1		$m_7 = X\cdot Y\cdot Z = \min\left(X,Y,Z\right)$		$M_7 = \overline{X} + \overline{Y} + \overline{Z} = \max\left(\overline{X},\overline{Y},\overline{Z}\right)$

Figure 1: Truth table representing minterm and maxterm.

3.1 Sum-of-Products expansion or the Disjunctive Normal Form of the Boolean function

The sum of minterms that represents the function is called the sum-of-products expansion or the disjunctive normal form of the Boolean function.

Example 20. *Find the SOP standard expansion for the function*

$$f(x,y) = x + y$$

Solution

$$\begin{aligned} f(x,y) = x + y = x\cdot 1 + y\cdot 1 = x(y+\bar{y}) + y(x+\bar{x}) = xy + x\bar{y} + xy + \bar{x}y \\ = xy + x\bar{y} + \bar{x}y. \end{aligned}$$

Example 21. *Find the SOP standard expansion for the function*

$$f(x,y,z) = (x+y)\bar{z}$$

Solution

$$\begin{aligned} f(x,y,z) = (x+y)\bar{z} = x\bar{z} + y\bar{z} = x(y+\bar{y})\bar{z} + (x+\bar{x})y\bar{z} \\ = xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z} = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}. \end{aligned}$$

Example 22. *Find the SOP standard expansion for the function*

$$f(x,y,z) = x + yz$$

Solution

$$\begin{aligned} f(x,y) = x + yz = x\cdot 1\cdot 1 + 1\cdot yz = x(y+\bar{y})(z+\bar{z}) + (x+\bar{x})yz \\ = (xy + x\bar{y})(z+\bar{z}) + xyz + \bar{x}yz = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + xyz + \bar{x}yz \\ = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz. \end{aligned}$$

Example 23. *Find the SOP expansion for the function*

$$f(x,y,z) = \sum(1,3,6) = m_1 + m_3 + m_6$$

Solution Each minterm is obtained by an AND operation. $m_1 = 001 = \bar{x}\bar{y}z$ and $m_3 = 011 = \bar{x}yz$ and $m_6 = 110 = xy\bar{z}$.

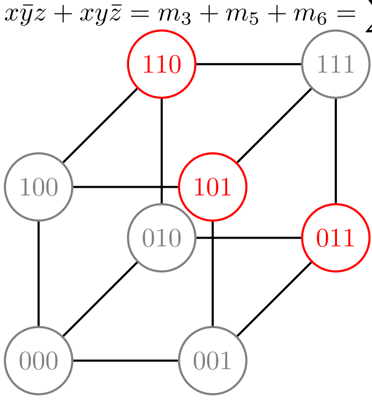
$$f(x,y,z) = \sum(1,3,6) = m_1 + m_3 + m_6 = \bar{x}\bar{y}z + \bar{x}yz + xy\bar{z}$$

Example 24. *Represent the given Boolean function as the sum of the decimal codes. Use 3-cube Q_3 to represent the Boolean function.*

$$f(x,y,z) = \bar{x}yz + x\bar{y}z + xy\bar{z}$$

Solution The expression $\bar{x}yz$ representation as a binary code is 011 that is 3 in decimal code. Next terms are 101, 110 or in decimal codes are 5, 6.

$$f(x,y,z) = \bar{x}yz + x\bar{y}z + xy\bar{z} = m_3 + m_5 + m_6 = \sum m(3,5,6)$$



3.2 Product-of-Sums or the Conjunctive Normal form of the function.

If the number of variables is n, then the possible number of maxterms is 2^n. The main property of a maxterm is that it posssess the value of 0 for only one combination of n input variables and the rest of the $2^n - 1$ combinations have the logic value of 1.

Example 25. *Find the POS expansion for the function*

$$f(x,y,z) = \Pi(0,2,5) = M_0 M_2 M_5$$

Solution Each maxterm is obtained by an OR operation. $M_0 = 000 = x + y + z$ and $M_2 = 010 = x\bar{y} + z$ and $M_5 = 101 = \bar{x} + y + \bar{z}$.

$$f(x,y,z) = \Pi(0,2,5) = M_0 M_2 M_5 = (x + y + z)(x + \bar{y} + z)(\bar{x} + y + \bar{z}).$$

Example 26. *Obtain the standard canonical POS form of the function. Write as product of maxterms.*

$$f(x,y,z) = (x + \bar{y})(y + z)(x + \bar{z})$$

Solution $f(x,y,z) = (x + \bar{y})(y + z)(x + \bar{z}) = (x + \bar{y} + 0)(0 + y + z)(x + 0 + \bar{z})$

- z is missing in the first term. Therefore $z\bar{z}$ is to be added with the first term
- x is missing in the second term. Add $x\bar{x}$.
- y is missing in the third term. Add $y\bar{y}$.

$$f(x,y,z) = (x+\bar{y}+z\bar{z})(x\bar{x}+y+z)(x+y\bar{y}+\bar{z})$$

Using the distributive property, $a+bc=(a+b)(a+c)$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(\bar{x}+y+z)(x+y+\bar{z})(x+\bar{y}+\bar{z})$$

Since $(x+\bar{y}+\bar{z})(x+\bar{y}+\bar{z}) = x+\bar{y}+\bar{z}$ we have

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(\bar{x}+y+z)(x+y+\bar{z})$$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(\bar{x}+y+z)(x+y+\bar{z}) = M_2M_3M_0M_4M_1 = \Pi(0,1,2,3,4).$$

Example 27. *Obtain the standard canonical POS form of the function. Write as product of maxterms.*

$$f(x,y,z) = x+\bar{y}z$$

Solution The function is given at sum of product (SOP) form. First, the function needs to be changed to POS form by using distribution law $a+bc=(a+b)(a+c)$

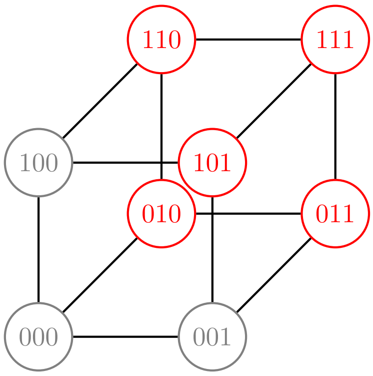
$$f(x,y,z) = x+\bar{y}z = (x+\bar{y})(x+z) = (x+\bar{y}+z\bar{z})(x+y\bar{y}+z)$$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(x+\bar{y}+z)$$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)$$

$$f(x,y,z) = M_2M_3M_0 = \Pi(0,2,3)$$

Example 28. *Here is given Q_3 representation of a Boolean function. Write the function as POS and SOP forms.*



Solution For SOP form $010 = m_2, 011 = m_3, 101 = m_5, 110 = m_6, 111 = m_7$

$$f(x,y,z) = m_2+m_3+m_5+m_6+m_7 = \sum(2,3,5,6,7)$$

$$f(x,y,z) = \bar{x}y\bar{z}+\bar{x}yz+x\bar{y}z+xy\bar{z}+xyz.$$

For POS form, since in maxterm the output for an input such 010 for $M_2 = x+\bar{y}+z$ is 0, we should select other vertices (gray vertices), $000 = M_0, 001 = M_1, 100 = M_4$

$$f(x,y,z) = M_0 \cdot M_1 \cdot M_4 = \Pi(0,1,4)$$

$$f(x,y,z) = (x+y+z)(x+y+\bar{z})(\bar{x}+y+z).$$

Example 29. *Write given functions as POS and SOP forms.*

1. $f(x,y,z) = \bar{x}+\bar{y}z$
2. $f(x,y,z) = \bar{x}+\bar{y}+\bar{z}$
3. $f(x,y,z) = \overline{x\bar{y}+\bar{z}}$
4. $f(x,y,z) = \overline{\bar{x}\bar{y}z+x\bar{z}}$

3.3 Simplifying the SOP form Karnaugh map method

To reduce the number of terms in a Boolean expression representing a circuit, it is necessary to find terms to combine. There is a graphical method, called a **Karnaugh map** or **K-map**, for finding terms to combine for Boolean functions involving a relatively small number of variables. K-maps give us a visual method for simplifying sum-of-products expansions.

We will first illustrate how K-maps are used to simplify expansions of Boolean functions in two variables. There are four possible minterms in the sum-of-products expansion of a Boolean function in the two variables x and y. A K-map for a Boolean function in these two variables consists of four cells, where a 1 is placed in the cell representing a minterm if this minterm is present in the expansion. Cells are said to be adjacent if the minterms that they represent differ in exactly one literal. For instance, the cell representing $\bar{x}y$ is **adjacent** to the cells representing $\bar{x}\bar{y}$ and xy.

K-map in two variables		
	y	$\bar{y}$
x	xy	$x\bar{y}$
$\bar{x}$	$\bar{x}y$	$\bar{x}\bar{y}$

Example 30. *Find the K-maps for $xy + \bar{x}y$.*
Solution We include a 1 in a cell when the minterm represented by this cell is present in the sum-of-products expansion.

K-map for $xy + \bar{x}y$		
	y	$\bar{y}$
x	1	
$\bar{x}$	1	

Example 31. *Find the K-maps for $xy + \bar{x}\bar{y}$.*
Solution We include a 1 in a cell when the minterm represented by this cell is present in the sum-of-products expansion.

K-map for $xy + \bar{x}\bar{y}$		
	y	$\bar{y}$
x	1	
$\bar{x}$		1

A K-map in three variables is a rectangle divided into eight cells. The cells represent the eight possible minterms in three variables. Two cells are said to be adjacent if the minterms that they represent differ in exactly one literal.

K-map in three variables				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

K-map for $x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z} = \bar{y}\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x			1	
$\bar{x}$			1	

K-map for $\bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z = \bar{x}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x				
$\bar{x}$	1	1	1	1

Example 32. *Use K-maps to minimize these sum-of-products expansions.*

1. $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$
2. $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$
3. $xyz + \bar{x}yz + xy\bar{z} + \bar{x}y\bar{z}$

Solution

1.

K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$	1		1	

$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z} = x\bar{z} + \bar{y}\bar{z} + \bar{x}yz.$

2.

K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$			1	1

$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} = x\bar{z} + \bar{x}\bar{y}.$

3.

K-map for $xyz + \bar{x}yz + xy\bar{z} + \bar{x}y\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$	1	1		

$xyz + \bar{x}yz + xy\bar{z} + \bar{x}y\bar{z} = y.$

Example 33. *Use K-maps to minimize these sum-of-products expansions.*

1. $\bar{x}yz + \bar{x}\bar{y}z$
2. $xyz + xy\bar{z} + \bar{x}yz$
3. $xyz + x\bar{y}z$
4. $xy\bar{z} + x\bar{y}z + \bar{x}y\bar{z}$

3.4 Simplifying the SOP form Quine–McCluskey method

There is a need for a procedure for simplifying sum-of-products expansions that can be mechanized. The Quine–McCluskey method is such a procedure. It can be used for Boolean functions in any number of variables.

Minterms that can be combined are those that differ in exactly one literal. Hence, two terms that can be combined differ by exactly one in the number of 1s in the bit strings that represent them. When two minterms are combined into a product, this product contains two literals. A product in two literals is represented using a dash to denote the variable that does not occur. For example two strings 101 and 001 can be combined to one string, and in the next time of combining we will simplify strings that have one $-$, for example -01 and -11 will be combined to $--1$ string. By doing this procedure we get simplify form.

Example 34. *We will show how the Quine–McCluskey method can be used to find a minimal expansion equivalent to*

$$xyz + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}.$$

Solution We will represent the minterms in this expansion by bit strings.

$$xyz = 111, x\bar{y}z = 101, \bar{x}yz = 011, \bar{x}\bar{y}z = 001, \bar{x}\bar{y}\bar{z} = 000$$

Quine–McCluskey method					
1	xyz	111	(1,2) $1-1$	((1,2),(3,4)) $--1$	$--1$
2	$x\bar{y}z$	101	(1,3) -11	((1,3),(2,4)) $--1$	$00-$
3	$\bar{x}yz$	011	(2,4) -01	(4,5) $00-$	
4	$\bar{x}\bar{y}z$	001	(3,4) $0-1$		
5	$\bar{x}\bar{y}\bar{z}$	000	(4,5) $00-$		

So the function is equivalent to $z + \bar{x}\bar{y}$.

Example 35. *Use the Quine–McCluskey method to find a minimal for given function.*

$$wx\bar{y}\bar{z} + w\bar{x}y\bar{z} + w\bar{x}yz + w\bar{x}\bar{y}\bar{z} + \bar{w}x\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z}$$

Solution

Quine–McCluskey method				
1	$w\bar{x}yz$	1011	$(1,3)$ $101-$	$(1,3)$ $101-$
2	$wx\bar{y}\bar{z}$	1100	$(2,4)$ $1-00$	$((2,5),(4,7))$ $--00$
3	$w\bar{x}y\bar{z}$	1010	$(2,5)$ -100	$((3,4),(6,7))$ $-0-0$
4	$w\bar{x}\bar{y}\bar{z}$	1000	$(3,4)$ $10-0$	$((2,4),(5,7))$ $--00$
5	$\bar{w}x\bar{y}\bar{z}$	0100	$(3,6)$ -010	$((3,6),(4,7))$ $-0-0$
6	$\bar{w}\bar{x}y\bar{z}$	0010	$(4,7)$ -000	
7	$\bar{w}\bar{x}\bar{y}\bar{z}$	0000	$(5,7)$ $0-00$	
8			$(6,7)$ $00-0$	

So the function is equivalent to $w\bar{x}y + \bar{y}\bar{z} + \bar{x}\bar{z}$.

Example 36. *Use the Quine–McCluskey method to find a minimal form of given function.*

1. $xy\bar{z} + x\bar{y}z + \bar{x}\bar{y}z$
2. $wxyz + \bar{w}\bar{x}yz + \bar{w}x\bar{y}\bar{z}$

4 Metric Properties in Boolean Algebra

Let $B_n = \{\hat{x} = (x_1, x_2, \cdots, x_n) | x_i \in \{0,1\}, 1 \le i \le n\}$, then every member $\hat{x}$ in B_n is like a vector with length n with k ones and $n-k$ zeroes. The **number of ones** in $\hat{x}$ vector called **weight** of $\hat{x} = (x_1, x_2, \cdots, x_n)$ and defined by

$$\|\hat{x}\| = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n. \tag{1}$$

If $\|\hat{x}\| = k$, then we say $\hat{x}$ is in *the k-th layer*. For $k = 0, 1, 2, \cdots, n$ layers are defined by B_n^k set, this is the set of all vectors in B_n with k weight:

$$B_n^k = \{\hat{x} | \hat{x} \in B_n, \|\hat{x}\| = k\} \tag{2}$$

For example in B_4,

$$B_4^0 = \{\hat{x} | \hat{x} \in B_4, \|\hat{x}\| = 0\} = \{(0,0,0,0)\}$$

$$B_4^1 = \{\hat{x} | \hat{x} \in B_4, \|\hat{x}\| = 1\} = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

$$B_4^2 = \{\hat{x} | \hat{x} \in B_4, \|\hat{x}\| = 2\} = \{(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1)\}.$$

$$B_4^3 = \{\hat{x} | \hat{x} \in B_4, \|\hat{x}\| = 3\} = \{(0,1,1,1), (1,0,1,1), (1,1,0,1), (1,1,1,0)\}$$

$$B_4^3 = \{\hat{x} | \hat{x} \in B_4, \|\hat{x}\| = 4\} = \{(1,1,1,1)\}$$

Example 37. *Write the layers of B_5.*

Example 38. *Find the number of elements in every layer of B_n.*

Example 39. *In B_4 write neighbor vertices to vertex $\hat{\alpha} = (0,1,0,1)$. Determine layers of neighbor vertices.*

Example 40. *Explain that for every $\hat{\alpha} \in B_n^k$, where $k > 0$, there are k neighbors in $k-1$ layer and $n-k$ neighbors in $k+1$ layer.*

Example 41. *Suppose,in B_n there is a path from $\hat{\alpha}$ to $\hat{\beta}$ with length l. Explain that $||\hat{\alpha}|| + ||\hat{\beta}|| + l$ is an even number.*

4.1 Hamming distance

What is the Hamming distance?
The Hamming distance is a **metric** used in **information theory** to measure how much **two messages with the same length differ**.
The Hamming distance indicates the **number of different digits/letters in a pair of messages**. Take the words "Hamming" and "Humming": what is the Hamming distance?

Let's see. "H**a**mming" and "H**u**mming", there is one letter of difference, so the **Hamming distance is 1**. Increase the Hamming distance to two or three: *farming, camping. fasting, hosting, hammock or Hamburg.*
Where do we use the Hamming distance?

The Hamming distance is a fundamental concept in the field of **error detection and correction**. An error in a message is quantifiable by measuring the number of different bits between the corrupted message and the original one.
How to calculate the Hamming distance? Take two binary messages with equal length, and write them down:

$$a = 1110000101$$

$$b = 1100001100$$

Now compare the messages "bitwise", and mark on one of them where the values of the bit in a given position differ between the messages, then count the number of bits you found: that is the Hamming distance.

$$a = 1110000101$$

$$b = 11\color{red}{0}0001\color{red}{1}0\color{red}{0}$$

$$Hamming(a,b) = \rho(a,b) = 3.$$

Hamming distance in B_n

Let $\hat{a}, \hat{b} \in B_n$, for counting the difference between two vectors, we can define

$$\rho(\hat{a}, \hat{b}) = \sum_{i=1}^{n} |a_i - b_i|$$

or

$$\rho(\hat{a}, \hat{b}) = \sum_{i=1}^{n} (a_i \bigoplus b_i) = ||\hat{a} \bigoplus \hat{b}||.$$

Example 42. *Prove that Hamming distance in B_n is a metric measure, that is*

1. $\rho(\hat{a}, \hat{b}) \geq 0$
2. $\rho(\hat{a}, \hat{b}) = \rho(\hat{b}, \hat{a})$
3. $\rho(\hat{a}, \hat{b}) = 0 \Leftrightarrow \hat{a} = \hat{b}$
4. $\rho(\hat{a}, \hat{b}) \leq \rho(\hat{a}, \hat{c}) + \rho(\hat{c}, \hat{b})$

Ball and Sphere in B_n

For $r \geq 0$, sphere and ball with center $\hat{\alpha} \in B_n$ with radius r is defined by

$$S(\hat{\alpha}, r) = \{\hat{\beta} | \hat{\beta} \in B_n, \rho(\hat{\alpha}, \hat{\beta}) \leq r\},$$

$$C(\hat{\alpha}, r) = \{\hat{\beta} | \hat{\beta} \in B_n, \rho(\hat{\alpha}, \hat{\beta}) = r\}.$$

Example 43. *Determine members of the sets:*

1. $C((0, 0, 0, 0), 2)$
2. $C((0, 0, 0, 1, 0), 3)$
3. $S((0, 1, 1, 0), 3)$
4. $S((1, 1, 1, 0, 0), 2)$

Example 44. *Is there intersection between $C((0, 0, 0, 0), 3)$ and $C((1, 1, 1, 1), 2)$? Compare this with circles intersection in Euclidean geometry.*

Example 45. *Show that*

1. *For every $\hat{a} \in B_n$, $|C(\hat{a}, r)| = C(n, r)$.*
2. *For every $\hat{a} \in B_n$, $|S(\hat{a}, r)| = C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n, r)$.*

Example 46. *Is there, $X \in B_7^3$ and $Y \in B_7^4$, such that $\rho(X, Y) = 7$?*

Example 47. Let $\hat{a} = 1000010010 \in B_{10}$. Find $|C(\hat{a}, r = 3)|$.

Example 48. *Write a path from $\hat{a} = 111110000 \in B_9^5$ to $\hat{b} = 000111111 \in B_9^6$, and find $\rho(\hat{a}, \hat{b})$.*

Example 49. *Write shortest and longest path from 7-th vertex in B_5 to 17-th vertex in B_5, and find Hamming distance between these vertices.*

Example 50. *Suppose f is a Boolean function on B_{10}, such that maps every item that contains 6 or less than ones to one and others to zero. Find the 25-th and 65-th rows output of function f.*

Example 51. *Let $\hat{a}, \hat{b} \in B_n$, such that $\rho(\hat{a}, \hat{b}) = R$. Find the number of vertices $\hat{c}$, that satisfy*

$$\rho(\hat{a}, \hat{b}) = \rho(\hat{a}, \hat{c}) + \rho(\hat{c}, \hat{b}).$$

Example 52. *We call function* ***sum*** *on B_n* to *$\{0, 1, 2, \cdots, n\}$, the output is sum of the digits of the string in B_n, for example $sum(10101) = 3$. Determine number of strings map to 2.*