Boolean Algebra

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1 Boolean Function

Boolean algebra provides the operations and the rules for working with the set $\{0, 1\}$.
The three operations in Boolean algebra that we will use most are complementation, the Boolean sum, and the Boolean product.

- The **complement** of an element, denoted with a bar, is defined by:

$$\bar{0} = 1, \bar{1} = 0$$

- The Boolean **sum**, denoted by "+" or by OR, has the following values:

$$0 + 0 = 0, 1 + 0 = 1, 0 + 1 = 1, 1 + 1 = 1$$

- The Boolean **product**, denoted by "·" or by AND, has the following values:

$$0 \cdot 0 = 0, 1 \cdot 0 = 0, 0 \cdot 1 = 0, 1 \cdot 1 = 1$$

Definition
Boolean Expressions and Boolean Functions
Let $B = \{0, 1\}$. Then $B_n = \{(x_1, x_2, \cdots, x_n)|x_i \in B, 1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s. The variable x is called a Boolean variable if it assumes values only from B, that is, if its only possible values are 0 and 1. A function from B_n to B is called a Boolean function of degree n.

Example 1. *How many different Boolean functions are there of degree 7? Determine number of Boolean functions of degree n.*

Example 2. *Present the given Boolean function as a table*

$$F : B_2 \to B$$

$$F(x, y) = x\bar{y}$$

Example 3. *Find the values of the Boolean function as a table.*

$$F : B_3 \to B$$

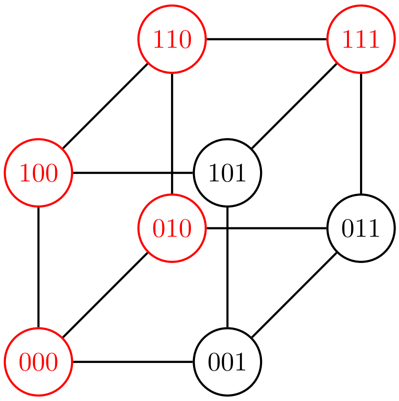
$$F(x, y, z) = xy + \bar{z}$$

Example 4. *Find the values of the Boolean function as a table.*

$$F : B_3 \to B$$

$$F(x, y, z) = x\bar{y} + \bar{x}\bar{z}$$

Example 5. *The function $f(x, y, z) = xy + \bar{z}$ can be represented by distinguish the vertices that have 1 output.*
Solution



Example 6. *Use 3-cube Q_3 to represent each of the Boolean functions.*

1. $f(x, y, z) = (x + \bar{y} + z)$
2. $f(x, y, z) = \bar{x} \cdot \bar{y} \cdot \bar{z}$
3. $f(x, y, z) = x\bar{y}z + \bar{y}z$
4. $f(x, y, z) = x(yz + \bar{y}\bar{z})$

Example 7. *Use 3-cube Q_3 to represent each of the Boolean functions.*

1. $f(x, y, z) = x(y + z)$
2. $f(x, y, z) = xy + xz$

2 Canonical and Standard Forms

Canonical Form

In Boolean algebra,Boolean function can be expressed as Canonical Disjunctive Normal Form known as minterm and some are expressed as Canonical Conjunctive Normal Form known as maxterm . In Minterm, we look for the functions where the output results in "1" while in Maxterm we look for function where the output results in "0".

- We perform Sum of minterm also known as Sum of products (SOP) .

  For example $xy + xz + yz$ or $\bar{x}y + yz + x\bar{z}$ are SOP expressions.

- We perform Product of maxterm also known as Product of sum (POS).

  For example $(x + y + z)(x + \bar{y} + \bar{z})$ or $(\bar{x} + \bar{y} + z)(x + y + \bar{z})$ are POS expressions.

  However, Boolean function like $(xy + yz)(\bar{x}\bar{y} + x\bar{z})$ is neither a sum of products form nor a product of sums form.

Boolean functions expressed as a sum of minterms or product of maxterms are said to be in canonical form.

Standard Form

A Boolean variable can be expressed in either true form or complemented form. In standard form Boolean function will contain all the variables in either true form or complemented form while in canonical number of variables depends on the output of SOP or POS.

A Boolean function can be expressed algebraically from a given truth table by forming a :

- minterm for each combination of the variables that produces a 1 in the function and then taking the OR of all those terms.
- maxterm for each combination of the variables that produces a 0 in the function and then taking the AND of all those terms. (see Fig1)

Example 8. *Show that if the number of variables is n, then the possible number of minterms is 2^n. Determine that in how many combinations of n input the value of minterm is 1 and in how many combinations of n input the value of minterm is 0.*

				Minterms		*Maxterms*
X	Y	Z		*Product Terms*		*Sum Terms*
0	0	0		$m_0 = \overline{X} \cdot \overline{Y} \cdot \overline{Z} = \min\left(\overline{X}, \overline{Y}, \overline{Z}\right)$		$M_0 = X + Y + Z = \max\left(X, Y, Z\right)$
0	0	1		$m_1 = \overline{X} \cdot \overline{Y} \cdot Z = \min\left(\overline{X}, \overline{Y}, Z\right)$		$M_1 = X + Y + \overline{Z} = \max\left(X, Y, \overline{Z}\right)$
0	1	0		$m_2 = \overline{X} \cdot Y \cdot \overline{Z} = \min\left(\overline{X}, Y, \overline{Z}\right)$		$M_2 = X + \overline{Y} + Z = \max\left(X, \overline{Y}, Z\right)$
0	1	1		$m_3 = \overline{X} \cdot Y \cdot Z = \min\left(\overline{X}, Y, Z\right)$		$M_3 = X + \overline{Y} + \overline{Z} = \max\left(X, \overline{Y}, \overline{Z}\right)$
1	0	0		$m_4 = X \cdot \overline{Y} \cdot \overline{Z} = \min\left(X, \overline{Y}, \overline{Z}\right)$		$M_4 = \overline{X} + Y + Z = \max\left(\overline{X}, Y, Z\right)$
1	0	1		$m_5 = X \cdot \overline{Y} \cdot Z = \min\left(X, \overline{Y}, Z\right)$		$M_5 = \overline{X} + Y + \overline{Z} = \max\left(\overline{X}, Y, \overline{Z}\right)$
1	1	0		$m_6 = X \cdot Y \cdot \overline{Z} = \min\left(X, Y, \overline{Z}\right)$		$M_6 = \overline{X} + \overline{Y} + Z = \max\left(\overline{X}, \overline{Y}, Z\right)$
1	1	1		$m_7 = X \cdot Y \cdot Z = \min\left(X, Y, Z\right)$		$M_7 = \overline{X} + \overline{Y} + \overline{Z} = \max\left(\overline{X}, \overline{Y}, \overline{Z}\right)$

Figure 1: Truth table representing minterm and maxterm.

2.1 Sum-of-Products expansion or the Disjunctive Normal Form of the Boolean function

The sum of minterms that represents the function is called the sum-of-products expansion or the disjunctive normal form of the Boolean function.

Example 9. *Find the SOP standard expansion for the function*

$$f(x,y) = x + y$$

Solution

$$f(x,y) = x + y = x \cdot 1 + y \cdot 1 = x(y+\bar{y}) + y(x+\bar{x}) = xy + x\bar{y} + xy + \bar{x}y$$
$$= xy + x\bar{y} + \bar{x}y.$$

Example 10. *Find the SOP standard expansion for the function*

$$f(x,y,z) = (x+y)\bar{z}$$

Solution

$$f(x,y,z) = (x+y)\bar{z} = x\bar{z} + y\bar{z} = x(y+\bar{y})\bar{z} + (x+\bar{x})y\bar{z}$$
$$= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}\bar{y}\bar{z} = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}.$$

Example 11. *Find the SOP standard expansion for the function*

$$f(x,y,z) = x + yz$$

Solution

$$f(x,y) = x + yz = x \cdot 1 \cdot 1 + 1 \cdot yz = x(y+\bar{y})(z+\bar{z}) + (x+\bar{x})yz$$
$$= (xy + x\bar{y})(z+\bar{z}) + xyz + \bar{x}yz = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + xyz + \bar{x}yz$$
$$= xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz.$$

Example 12. *Find the SOP expansion for the function*

$$f(x,y,z) = \sum(1,3,6) = m_1 + m_3 + m_6$$

Solution Each minterm is obtained by an AND operation. $m_1 = 001 = \bar{x}\bar{y}z$ and $m_3 = 011 = \bar{x}yz$ and $m_6 = 110 = xy\bar{z}$.

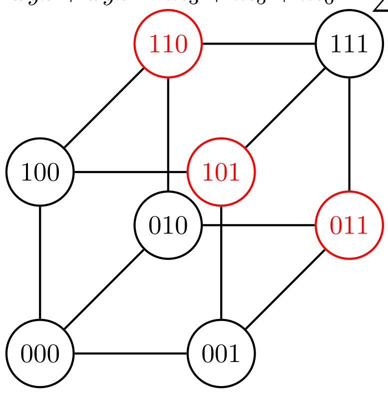
$$f(x,y,z) = \sum(1,3,6) = m_1 + m_3 + m_6 = \bar{x}\bar{y}z + \bar{x}yz + xy\bar{z}$$

Example 13. *Represent the given Boolean function as the sum of the decimal codes.Use 3-cube Q_3 to represent the Boolean function.*

$$f(x,y,z) = \bar{x}yz + x\bar{y}z + xy\bar{z}$$

Solution The expression $\bar{x}yz$ representation as a binary code is 011 that is 3 in decimal code. Next terms are 101, 110 or in decimal codes are 5, 6.

$$f(x,y,z) = \bar{x}yz + x\bar{y}z + xy\bar{z} = m_3 + m_5 + m_6 = \sum m(3,5,6)$$



2.2 Product-of-Sums or the Conjunctive Normal form of the function.

If the number of variables is n, then the possible number of maxterms is 2^n. The main property of a maxterm is that it posssess the value of 0for only one combination of n input variables and the rest of the $2^n - 1$ combinations have the logic value of 1.

Example 14. *Find the POS expansion for the function*

$$f(x,y,z) = \Pi(0,2,5) = M_0 M_2 M_5$$

Solution Each maxterm is obtained by an OR operation. $M_0 = 000 = x + y + z$ and $M_2 = 010 = x\bar{y} + z$ and $M_5 = 101 = \bar{x} + y + \bar{z}$.

$$f(x,y,z) = \Pi(0,2,5) = M_0 M_2 M_5 = (x+y+z)(x+\bar{y}+z)(\bar{x}+y+\bar{z}).$$

Example 15. *Obtain the standard canonical POS form of the function. Write as product of maxterms.*

$$f(x,y,z) = (x+\bar{y})(y+z)(x+\bar{z})$$

Solution $f(x,y,z) = (x+\bar{y})(y+z)(x+\bar{z}) = (x+\bar{y}+0)(0+y+z)(x+0+\bar{z})$

- z is missing in the first term. Therefore $z\bar{z}$ is to be added with the first term
- x is missing in the second term. Add $x\bar{x}$.
- y is missing in the third term. Add $y\bar{y}$.

$$f(x,y,z) = (x+\bar{y}+z\bar{z})(x\bar{x}+y+z)(x+y\bar{y}+\bar{z})$$

Using the distributive property, $a+bc=(a+b)(a+c)$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(\bar{x}+y+z)(x+y+\bar{z})(x+\bar{y}+\bar{z})$$

Since $(x+\bar{y}+\bar{z})(x+\bar{y}+\bar{z}) = x+\bar{y}+\bar{z}$ we have

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(\bar{x}+y+z)(x+y+\bar{z})$$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(\bar{x}+y+z)(x+y+\bar{z}) = M_2M_3M_0M_4M_1 = \Pi(0,1,3,4).$$

Example 16. *Obtain the standard canonical POS form of the function. Write as product of maxterms.*

$$f(x,y,z) = x+\bar{y}z$$

Solution The function is given at sum of product (SOP) form. First, the function needs to be changed to POS form by using distribution law $a+bc=(a+b)(a+c)$

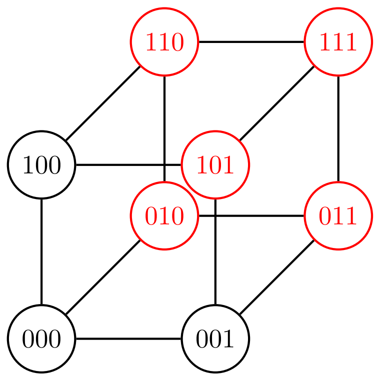
$$f(x,y,z) = x+\bar{y}z = (x+\bar{y})(x+z) = (x+\bar{y}+z\bar{z})(x+y\bar{y}+z)$$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)(x+\bar{y}+z)$$

$$f(x,y,z) = (x+\bar{y}+z)(x+\bar{y}+\bar{z})(x+y+z)$$

$$f(x,y,z) = M_2M_3M_0 = \Pi(0,2,3)$$

Example 17. *Here is given Q_3 representation of a Boolean function. Write the function as POS and SOP forms.*



Solution For SOP form $010 = m_2, 011 = m_3, 101 = m_5, 110 = m_6, 111 = m_7$

$$f(x,y,z) = m_2+m_3+m_5+m_6+m_7 = \sum(2,3,5,6,7)$$

$$f(x,y,z) = \bar{x}y\bar{z}+\bar{x}yz+x\bar{y}z+xy\bar{z}+xyz.$$

For POS form $010 = M_5, 011 = M_4, 101 = M_2, 110 = M_1, 111 = M_0$

$$f(x,y,z) = M_0\cdot M_1\cdot M_2\cdot M_4\cdot M_5 = \Pi(0,1,2,4,5)$$

$$f(x,y,z) = (x+y+z)(x+y+\bar{z})(x+\bar{y}+z)(\bar{x}+y+z)(\bar{x}+y+\bar{z}).$$

Example 18. *Write given functions as POS and SOP forms.*

1. $f(x,y,z) = \bar{x}+\bar{y}z$
2. $f(x,y,z) = \bar{x}+\bar{y}+\bar{z}$
3. $f(x,y,z) = \overline{x\bar{y}+\bar{z}}$
4. $f(x,y,z) = \overline{\bar{x}\bar{y}z+x\bar{z}}$

2.3 Simplifying the SOP form Karnaugh map method

To reduce the number of terms in a Boolean expression representing a circuit, it is necessary to find terms to combine. There is a graphical method, called a **Karnaugh map** or **K-map**, for finding terms to combine for Boolean functions involving a relatively small number of variables. K-maps give us a visual method for simplifying sum-of-products expansions.

We will first illustrate how K-maps are used to simplify expansions of Boolean functions in two variables. There are four possible minterms in the sum-of-products expansion of a Boolean function in the two variables x and y. A K-map for a Boolean function in these two variables consists of four cells, where a 1 is placed in the cell representing a minterm if this minterm is present in the expansion. Cells are said to be adjacent if the minterms that they represent differ in exactly one literal. For instance, the cell representing $\bar{x}y$ is **adjacent** to the cells representing $\bar{x}\bar{y}$ and xy.

K-map in three variables		
	y	$\bar{y}$
x	xy	$x\bar{y}$
$\bar{x}$	$\bar{x}y$	$\bar{x}\bar{y}$

Example 19. *Find the K-maps for $xy + \bar{x}y$.*
Solution We include a 1 in a cell when the minterm represented by this cell is present in the sum-of-products expansion.

K-map for $xy + \bar{x}y$		
	y	$\bar{y}$
x	1	
$\bar{x}$	1	

Example 20. *Find the K-maps for $xy + \bar{x}\bar{y}$.*
Solution We include a 1 in a cell when the minterm represented by this cell is present in the sum-of-products expansion.

K-map for $xy + \bar{x}\bar{y}$		
	y	$\bar{y}$
x	1	
$\bar{x}$		1

A K-map in three variables is a rectangle divided into eight cells. The cells represent the eight possible minterms in three variables. Two cells are said to be adjacent if the minterms that they represent differ in exactly one literal.

K-map in three variables				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

K-map for $x\bar{y}\bar{z} + x\bar{y}\bar{z} = \bar{y}\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x			1	
$\bar{x}$			1	

K-map for $\bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z = \bar{x}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x				
$\bar{x}$	1	1	1	1

Example 21. *Use K-maps to minimize these sum-of-products expansions.*

1. $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$
2. $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$
3. $xyz + \bar{x}yz + xy\bar{z} + \bar{x}y\bar{z}$

Solution

1.

K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$	1		1	

$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z} = x\bar{z} + \bar{y}\bar{z} + \bar{x}yz.$

2.

K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$			1	1

$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} = x\bar{z} + \bar{x}\bar{y}.$

3.

K-map for $xyz + \bar{x}yz + xy\bar{z} + \bar{x}y\bar{z}$				
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$	1	1		

$xyz + \bar{x}yz + xy\bar{z} + \bar{x}y\bar{z} = y.$

Example 22. *Use K-maps to minimize these sum-of-products expansions.*

1. $\bar{x}yz + \bar{x}\bar{y}z$
2. $xyz + xy\bar{z} + \bar{x}yz$
3. $xyz + x\bar{y}z$
4. $xy\bar{z} + x\bar{y}z + \bar{x}y\bar{z}$

2.4 Simplifying the SOP form Quine–McCluskey method

There is a need for a procedure for simplifying sum-of-products expansions that can be mechanized. The Quine–McCluskey method is such a procedure. It can be used for Boolean functions in any number of variables.

Minterms that can be combined are those that differ in exactly one literal. Hence, two terms that can be combined differ by exactly one in the number of 1s in the bit strings that represent them. When two minterms are combined into a product, this product contains two literals. A product in two literals is represented using a dash to denote the variable that does not occur. For example two strings 101 and 001 can be combined to one string, and in the next time of combining we will simplify strings that have one $-$, for example -01 and -11 will be combined to $--1$ string. By doing this procedure we get simplify form.

Example 23. *We will show how the Quine–McCluskey method can be used to find a minimal expansion equivalent to*

$$xyz + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}.$$

Solution We will represent the minterms in this expansion by bit strings.

$$xyz = 111, x\bar{y}z = 101, \bar{x}yz = 011, \bar{x}\bar{y}z = 001, \bar{x}\bar{y}\bar{z} = 000$$

Quine–McCluskey method					
1	xyz	111	(1,2) $1-1$	((1,2),(3,4)) $--1$	$--1$
2	$x\bar{y}z$	101	(1,3) -11	((1,3),(2,4)) $--1$	$00-$
3	$\bar{x}yz$	011	(2,4) -01	(4,5) $00-$	
4	$\bar{x}\bar{y}z$	001	(3,4) $0-1$		
5	$\bar{x}\bar{y}\bar{z}$	000	(4,5) $00-$		

So the function is equivalent to $z + \bar{x}\bar{y}$.

Example 24. *Use the Quine–McCluskey method to find a minimal for given function.*

$$wx\bar{y}\bar{z} + w\bar{x}y\bar{z} + w\bar{x}yz + w\bar{x}\bar{y}\bar{z} + \bar{w}x\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z}$$

Solution

Quine–McCluskey method				
1	$w\bar{x}yz$	1011	$(1,3)$ $101-$	$(1,3)$ $101-$
2	$wx\bar{y}\bar{z}$	1100	$(2,4)$ $1-00$	$((2,5),(4,7))$ $--00$
3	$w\bar{x}y\bar{z}$	1010	$(2,5)$ -100	$((3,4),(6,7))$ $-0-0$
4	$w\bar{x}\bar{y}\bar{z}$	1000	$(3,4)$ $10-0$	$((2,4),(5,7))$ $--00$
5	$\bar{w}x\bar{y}\bar{z}$	0100	$(3,6)$ -010	$((3,6),(4,7))$ $-0-0$
6	$\bar{w}\bar{x}y\bar{z}$	0010	$(4,7)$ -000	
7	$\bar{w}\bar{x}\bar{y}\bar{z}$	0000	$(5,7)$ $0-00$	
8			$(6,7)$ $00-0$	

So the function is equivalent to $w\bar{x}y + \bar{y}\bar{z} + \bar{x}\bar{z}$.

Example 25. *Use the Quine–McCluskey method to find a minimal for given function.*

1. $xy\bar{z} + x\bar{y}z + \bar{x}\bar{y}z$
2. $wxyz + \bar{w}\bar{x}yz + \bar{w}x\bar{y}\bar{z}$