Calculus

Learn Calculus By Example

Armen Hayrapetian

Master of Mathematics

Department of Mathematical Science
Sharif University of Technology
June 2023

Contents

Chapter 1

Sequences

1.1 Limit of a Sequence

- We say that $\{a_n\}$ converges to a limit l, and we write $\lim_{n\to\infty} a_n = l$ or $a_n \to l$ if, for every $\epsilon > 0$, there is a number M such that $|a_n - l| < \epsilon$ for all $n > M$.
- We write $\lim_{n\to\infty} a_n = +\infty$ if for each $N > 0$ we can find $M > 0$ (depends on N) such that $a_n > N$ for all $n > M$.We say a_n diverges to $+\infty$
- We write $\lim_{n\to\infty} a_n = -\infty$ if for each $N > 0$ we can find $M > 0$ (depends on N) such that $a_n < -N$ for all $n > M$.We say a_n diverges to $-\infty$.
- Oscillated sequences such $\{(-1)^n\}$ are divergent.

Theorem 1 (Limit is Unique) *If a sequence $\{a_n\}$ has a limit, then this limit is unique.*

Example 1.1.1

Using definition of the limit of sequence (ϵ and $M(\epsilon)$ language) prove that:

1. $\lim_{n\to\infty} \frac{2n^2-1}{n^2+2} = 2$
2. $\lim_{n\to\infty} \frac{n+4\sqrt{n}+3}{\sqrt{n}+2} = +\infty$
3. $\lim_{n\to\infty} \frac{-n^2+6\sqrt{n}+3}{n+2} = -\infty$
4. $\lim_{n\to\infty} 3 + cos(n\pi) =$ does not exist.

Solution

1. We should prove, for all $\epsilon > 0$ there is $M \in \mathbf{N}$ such that $n > M \Rightarrow \left|\frac{2n^2-1}{n^2+2} - 2\right| < \epsilon$

$$\left|\frac{-5}{n^2+2}\right| < \epsilon \Rightarrow \frac{5}{n^2+2} < \epsilon \Rightarrow n^2 > \frac{5}{\epsilon} - 2 \Rightarrow n > \sqrt{\frac{5}{\epsilon} - 2}$$

Always ϵ presents a very small positive number, so, $M > \left[\sqrt{\frac{5}{\epsilon} - 2}\right] + 1$, is enough big to satisfy the definition.

2. We should prove, for all $N > 0$ there is $M \in \mathbf{N}$ such that $n > M \Rightarrow |a_n| > N$

$$\left|\frac{(\sqrt{n}+2)^2 - 1}{\sqrt{n}+2}\right| > N \Rightarrow \left|(\sqrt{n}+2) - \frac{1}{\sqrt{n+2}}\right| > N \Rightarrow \sqrt{n} > N \Rightarrow n > N^2$$

Now, select $M > \left[N^2\right] + 1$.This means the given sequence increase without bound and is divergent.

3. We should prove, for all $N > 0$ there is $M \in \mathbf{N}$ such that $n > M \Rightarrow a_n < -N$

$$\frac{-n^2+6\sqrt{n}+3}{n+2} < -N \Rightarrow \frac{n^2-6\sqrt{n}-3}{n+2} > N$$

$$\frac{n^2-6\sqrt{n}-3}{n+2} > \frac{n^2-6\sqrt{n}-3}{n+n} > \frac{n^2-6\sqrt{n}-\sqrt{n}}{2n} = \frac{n}{2} - \frac{7}{2\sqrt{n}} > \frac{n}{2} - \frac{7}{2} > N \Rightarrow n > 2N+7$$

Now, select $M > [2N+7]+1$.

4. Suppose $\lim_{n\to\infty} 3 + cos(n\pi) = l$ and select $\epsilon = 0.1$,by definition, there is $M \in \mathbf{N}$ such that $n > M \Rightarrow |a_n - l| < \epsilon$. Since $cos(n\pi) = (-1)^n$, $a_{2n} = 4$ and $a_{2n+1} = 2$. So, $|4-l| < \epsilon$ and $|2-l| < \epsilon$

$$2 = |(4-l)-(2-l)| \leq |(4-l)| + |(2-l)| \ < 2\epsilon \Rightarrow 2 < 2(0.1)$$

that is contradiction.This proves that, the sequence is divergent and there is no limit.

Example 1.1.2

Is the sequence $a_n = \frac{2n^2+3n-1}{n^2+6n+12}$ *convergent or divergent?*

Solution

$$\lim_{n\to\infty} \frac{2n^2+3n-1}{n^2+6n+12} = \lim_{n\to\infty} \frac{n^2(2+\frac{3}{n}-\frac{1}{n^2})}{n^2(1+\frac{6}{n}+\frac{12}{n^2})} = 2$$

So, $\{a_n\}$ is converge to 2.

Example 1.1.3

Is the sequence $a_n = \frac{n}{\sqrt{n+2}}$ *convergent or divergent?*

Solution

$$\lim_{n\to\infty} \frac{n}{\sqrt{n+2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n(1+\frac{2}{n})}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+\frac{2}{n}}} = \frac{\infty}{\sqrt{1+0}} = \infty$$

So, $\{a_n\}$ is divergent.

Theorem 2 (Bounded Sequence) *Every convergent sequence is a bounded sequence. There are two numbers m(lower bound) and M(upper bound) such that* $m \leq \{a_n\} \leq M$.

Rate of Grow

A sequence $\{b_n\}$ **grows faster than** $\{a_n\}$ if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \infty \ or \ \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

In this case we write $\{a_n\} \ll \{b_n\}$. We can write

$$\frac{slow}{fast} \to 0 \ and \ \frac{fast}{slow} \to \infty.$$

Sequence $\{a_n\}$ and $\{b_n\}$ **grow at the same rate** if for some $L, 0 < L < \infty$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

We call $\{a_n\}$ and $\{b_n\}$ **Equivalent Sequences** if $L = 1$.

Theorem 3 (Orders of Growth)

$$lnlnn \ll lnn \ll \cdots \ll \sqrt[3]{n} \ll \sqrt{n} \ll n \ll n^2 \ll \cdots \ll 2^n \ll e^n \ll n! \ll n^n \ll e^{n^2}$$

Example 1.1.4

*By using **Rate of Grow**, evaluate limit of sequences.*

1. $\lim_{n\to\infty} \frac{3+4lnn}{3n+5}$
2. $\lim_{n\to\infty} \frac{\sqrt{n+2}+1}{lnn+5}$
3. $\lim_{n\to\infty} \frac{2^n+3^n}{3^{n+1}+2^{n+5}}$
4. $\lim_{n\to\infty} \frac{3\cdot 4^n}{(n+3)!}$

Solution

1. $\lim_{n\to\infty} \frac{3+4lnn}{3n+5} = \lim_{n\to\infty} \frac{4lnn}{3n} = \frac{4}{3}\lim_{n\to\infty} \frac{lnn}{n} = 0$
2. $\lim_{n\to\infty} \frac{\sqrt{n+2}+1}{lnn+5} = \lim_{n\to\infty} \frac{\sqrt{n}}{lnn} = +\infty$
3. $\lim_{n\to\infty} \frac{2^n+3^n}{3^{n+1}+2^{n+5}} = \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3}$
4. $\lim_{n\to\infty} \frac{3\cdot 4^n}{(n+3)!} = 3\lim_{n\to\infty} \frac{4^n}{(n+3)!} = 0$

Theorem 4 (Squeeze Theorem) *If $a_n \le b_n \le c_n$ for $n \ge m$ and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l,$$

then $\lim_{n\to\infty} b_n = l$.

Example 1.1.5

*By using **Squeeze Theorem**, evaluate limit of sequences.*

1. $\lim_{n\to\infty} \frac{n!}{n^n}$
2. $\lim_{n\to\infty} \sqrt[n]{2^n + 3^n + 4^n}$
3. $\lim_{n\to\infty} \frac{3}{n}cos^3(n!)$
4. $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$

Solution

1. $a_n = \frac{n!}{n^n} = \frac{1}{n}(\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n})$
$$0 < a_n \leq \frac{1}{n}$$
We know that $\frac{1}{n} \to 0$. Therefore $a_n \to 0$ by the Squeeze Theorem.

2. $a_n = \sqrt[n]{2^n + 3^n + 4^n}$
$$4^n \leq 2^n + 3^n + 4^n \leq 4^n + 4^n + 4^n = 3 \cdot 4^n$$
$$\lim_{n\to\infty} \sqrt[n]{4^n} = 4,\ \lim_{n\to\infty} \sqrt[n]{3\cdot 4^n} = 4 \lim_{n\to\infty} \sqrt[n]{3} = 4 \cdot 1 = 4$$
Therefore $a_n \to 4$ by the Squeeze Theorem.

3. $a_n = \frac{3}{n} cos^3(n!)$
$$-\frac{3}{n} \leq \frac{3}{n} cos^3(n!) \leq \frac{3}{n}$$
Since $-\frac{3}{n} \to 0$ and $\frac{3}{n} \to 0$, $a_n \to 0$, by the Squeeze Theorem.

4. $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$
$$\frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}}$$
$$n\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq n\frac{1}{\sqrt{n^2+1}}$$
$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1,\ \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$$
Therefore $a_n \to 1$ by the Squeeze Theorem.

Theorem 5 (Sub-Sequences) *If a sequence converges then all sub-sequences converge and all convergent sub-sequences converge to the same limit.*
If $\{a_n\}$is a sequence that either has a sub-sequence that diverges or two convergent sub-sequences with different limits then $\{a_n\}$ is divergent.

Example 1.1.6

Use Theorem 4,to proof divergence of sequences.

1. $a_n = tan(4n+1)\frac{\pi}{3}$
2. $a_n = \frac{ncos(n\pi)+2}{2n+3}$
3. $a_n = \frac{n}{2n+3} + cos\frac{2n\pi}{3}$
4. $a_n = \left[2 + \frac{sinn}{n}\right]$

Solution

1. Select two sub-sequences with different limits. Select a_{3n} and a_{3n+1},
$$a_{3n} = tan(4(3n)+1)\frac{\pi}{3} = tan(4n\pi + \frac{\pi}{3}) = tan(\frac{\pi}{3}) = \sqrt{3} \to \sqrt{3}$$
$$a_{3n+1} = tan(4(3n+1)+1)\frac{\pi}{3} = tan(4n\pi + \frac{5\pi}{3}) = -\sqrt{3} \to -\sqrt{3}$$
Therefore,$\{a_n\}$is a sequence that has two convergent sub-sequences with different limits, so $\{a_n\}$ is divergent.

2. Select two sub-sequences with different limits. Select a_{2n} and a_{2n+1}. We know that $cos(n\pi) = (-1)^n$.
$$a_{2n} = \frac{2ncos(2n\pi)+2}{4n+3} = \frac{2n+2}{4n+3} = \frac{1}{2} \to \frac{1}{2}$$
$$a_{2n+1} = \frac{(2n+1)cos((2n+1)\pi)+2}{4n+5} = \frac{-2n+1}{4n+5} = -\frac{1}{2} \to -\frac{1}{2}$$
Therefore,$\{a_n\}$is a sequence that has two convergent sub-sequences with different limits, so $\{a_n\}$ is divergent.

3. Select two sub-sequences with different limits. Select a_{3n} and a_{3n+1}.
$$a_{3n} = \frac{3n}{6n+3} + cos\frac{2(3n)\pi}{3} = \frac{3n}{6n+3} + cos2n\pi \to \frac{1}{2} + 1 = \frac{3}{2}$$
$$a_{3n+1} = \frac{3n+1}{6n+5} + cos\frac{2(3n+1)\pi}{3} = \frac{3n+1}{6n+5} + cos(2n\pi + \frac{2\pi}{3}) \to \frac{1}{2} - \frac{1}{2} = 0$$
Therefore,$\{a_n\}$is a sequence that has two convergent sub-sequences with different limits, so $\{a_n\}$ is divergent.

4. By Squeeze Theorem we know:
$$-1 \le sinn \le 1 \Rightarrow \frac{sinn}{n} \to 0 \Rightarrow 2 + \frac{sinn}{n} \to 2$$
Select two types of a_n terms, where $sinn > 0$ and where $sinn < 0$

If $sinn > 0$ we have $2 + \frac{sinn}{n} > 2$, so $a_n \to 2$ and if $sinn < 0$ we have $2 + \frac{sinn}{n} < 2$, so $a_n \to 1$.

Therefore,$\{a_n\}$is divergent.

Example 1.1.7

Prove the divergence of the sequence. $1, \frac{1}{2}, 3, 1, \frac{1}{4}, 6, 1, \frac{1}{8}, 9, 1, \frac{1}{16}, 27, \cdots$.
Solution Here we can see 3 sub-sequences $a_{3n+1} \to 1, a_{3n+2} \to 0$ and $a_{3n} \to +\infty$, so the sequence is unbounded and therefore divergence.

Asymptotic Sequences

We say $\{a_n\}$ is asymptotic to $\{b_n\}$ and write $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$.
Consider k, m, r are constants and $r > 0$. We can write:
$$a_n = \sqrt{n+k} - \sqrt{n+m} \sim \frac{k-m}{2\sqrt{n}} \to 0 \tag{1.1}$$
$$a_n = \sqrt[r]{n+k} - \sqrt[r]{n+m} \sim \frac{k-m}{r\sqrt[r]{n^{r-1}}} \to 0 \tag{1.2}$$

Example 1.1.8

Find limit of the sequence.

1. $\lim_{n\to\infty} n(\sqrt{n^2+1}-n)$
2. $\lim_{n\to\infty} n^2(\sqrt[3]{n^3+1}-n)$
3. $\lim_{n\to\infty} \sqrt{n}(\sqrt[3]{n+12}-\sqrt[3]{n+4})$
4. $\lim_{n\to\infty} 2\sqrt{n+1}-\sqrt{n+3}-\sqrt{n}$

Solution

1. Use (1.1) equivalency, substitute n^2 to n

$$\lim_{n\to\infty} n(\sqrt{n^2+1}-n) = \lim_{n\to\infty} n(\sqrt{n^2+1}-\sqrt{n^2}) = \lim_{n\to\infty} n(\frac{1}{2\sqrt{n^2}}) = \frac{1}{2}$$

2. Use (1.2) equivalency, substitute n^3 to n

$$\lim_{n\to\infty} n^2(\sqrt[3]{n^3+1}-n) = \lim_{n\to\infty} n^2(\sqrt[3]{n^3+1}-\sqrt[3]{n^3}) = \lim_{n\to\infty} n^2(\frac{1}{3\sqrt[3]{n^6}}) = \frac{1}{3}$$

3. Use (1.2) equivalency,

$$\lim_{n\to\infty} \sqrt{n}(\sqrt[3]{n+12}-\sqrt[3]{n+4}) = \lim_{n\to\infty} \sqrt{n}(\frac{12-4}{3\sqrt[3]{n}^2}) = 0$$

4. By dividing into subtraction, we will use (1.1) equivalency, two times.

$$\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n+3}+\sqrt{n+1}-\sqrt{n} = \lim_{n\to\infty} \frac{1-3}{2\sqrt{n}} + \frac{1}{2\sqrt{n}} = 0$$

Example 1.1.9

Find limit of the sequence

$$\lim_{n\to\infty} \sqrt{n^2+6n+1}-n$$

Solution

$$\lim_{n\to\infty} \sqrt{n^2+6n+1}-n = \lim_{n\to\infty} \sqrt{n^2+6n+9-8}-n =$$

$$\lim_{n\to\infty} \sqrt{(n+3)^2-8}-n = \lim_{n\to\infty} (n+3)-n = 3$$

Asymptotic Sequences

Consider a, b, c, k are constants and $a > 0, k > 0$. We can write:

$$\sqrt{an^2+bn+c} \sim \sqrt{a}(n+\frac{b}{2a}) \tag{1.3}$$

$$\sqrt[k]{an^k+bn^{k-1}+c+\cdots} \sim \sqrt[k]{a}(n+\frac{b}{ka}) \tag{1.4}$$

Example 1.1.10

Find the limit of the sequence or show that it diverges.

1. $\lim_{n\to\infty} \sqrt{4n^2+6n+1}-\sqrt{4n^2+2}$
2. $\lim_{n\to\infty} \sqrt[3]{2n^3+18n^2+n+1}-\sqrt[3]{2n^3+24n^2-5}$

3. $\lim_{n\to\infty} \sqrt{2n^2+12n-1} - \sqrt{2n^2-8n+11}$

Solution Use (1.3) and (1.4) equations.

1. $\lim_{n\to\infty} \sqrt{4n^2+6n+1} - \sqrt{4n^2+2} = \lim_{n\to\infty} 2(n+\frac{6}{8}) - 2n = \frac{3}{2}$
2. $\lim_{n\to\infty} \sqrt[3]{2n^3+18n^2+n+1} - \sqrt[3]{2n^3+24n^2-5} = \lim_{n\to\infty} \sqrt[3]{2}((n+\frac{18}{6}) - (n+\frac{24}{6})) = -\sqrt[3]{2}$
3. $\lim_{n\to\infty} \sqrt{2n^2+12n-1} - \sqrt{2n^2-8n+11} = \lim_{n\to\infty} \sqrt{2}((n+\frac{12}{4}) - (n-\frac{8}{4})) = 5\sqrt{2}$

1.2 Monotonic Sequences

We say that $\{x_n\}$ is **increasing** if $x_n \leq x_{n+1}$ for all n and strictly increasing if $x_n < x_{n+1}$ for all n. Similarly, We say that $\{x_n\}$ is **decreasing** if $x_{n+1} \leq x_n$ for all n and strictly decreasing if $x_{n+1} < x_n$ for all n.Sequences which are either increasing or decreasing are called **monotone**.

Theorem 6 (Monotonic Sequence Theorem) *Every bounded, monotonic sequence is convergent.*

1. *Suppose $\{x_n\}$ is a bounded and increasing sequence. Then the $sup(x_n)$ is the limit of $\{x_n\}$.*
2. *Suppose $\{x_n\}$ is a bounded and decreasing sequence. Then the $inf(x_n)$ is the limit of $\{x_n\}$.*

Example 1.2.11

Prove that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded. Find $inf(a_n)$ and $sup(a_n)$.
Solution

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}).\frac{(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} =$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

So, $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$,

$$a_{n+1} < a_n$$

$$inf = 0 < a_n \ \leq sup = \sqrt{2} - 1$$

Example 1.2.12

Prove that $a_n = \frac{2n+1}{n+5}cos(\frac{\pi}{2n+2})$ is monotone and find limit.
Solution

$$lim_{n\to\infty} \frac{2n+1}{n+5} cos(\frac{\pi}{2n+2}) = 2 \cdot cos(0) = 2$$

We know that $(\frac{ax+b}{cx+d})' = \frac{ad-bc}{(cx+d)^2}$, therefore if $f(x) = \frac{2x+1}{x+5}$ then $f'(x) > 0$. It shows that $x_n = \frac{2n+1}{n+5}$ is increasing. Also by growing n, the $arc = \frac{\pi}{2n+2}$ is variate from $\frac{\pi}{4}$ to zero, therefore $y_n = cos(\frac{\pi}{2n+2}) > 0$ and increasing between $\frac{\sqrt{2}}{2}$ to 1. So, $a_n = x_n \cdot y_n$ is multiply of two positive and increasing sequences, therefore a_n is increasing and converges to 2 and bounded $a_1 = \frac{\sqrt{2}}{4} \leq a_n < 2$.

Example 1.2.13

Prove that $a_n = (1+\frac{1}{n})^n$ is increasing and bounded.
SolutionWe can use binomial theorem to expand the expression

$$a_n = (1+\frac{1}{n})^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2!}(\frac{1}{n})^2 + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}(\frac{1}{n})^n$$

$$a_n = 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n\cdot n} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n\cdot n\cdot n} + \cdots + \frac{1}{n!}\frac{n(n-1)(n-2)\cdots 1}{n\cdot n\cdot n\cdots n}$$

$$a_n = 2 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$

Similarly for a_{n+1} we have

$$a_{n+1} = 2+\frac{1}{2!}(1-\frac{1}{n+1})+\frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})+\cdots+\frac{1}{(n+1)!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n}{n+1})$$

Because $(1-\frac{k}{n+1}) - (1-\frac{k}{n}) = \frac{k}{n(n+1)} \geq 0$ we have $a_{n+1} - a_n \geq 0$, therefore it is increasing sequence.
Next, we need to show it is bounded.

$$a_n = 2 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$
$$a_n \leq 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} < 3$$

So, a_n is increasing and bounded.

Theorem 7 (Leonhard Euler "e" number) *The sequence $a_n = (1+\frac{1}{n})^n$ is increasing and bounded, and so has a limit which denote by e. The value of $e = 2.718281828459045\cdots$.*

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

Remark Suppose $a_n \to 0$ and $b_n \to +\infty$, then $\lim_{n\to\infty}(1+a_n)^{b_n} = exp(\lim_{n\to\infty} a_n \cdot b_n)$

Example 1.2.14

Evaluate following limits.

1. $\lim_{n\to\infty}(1+\frac{5n+3}{n^3+n+2})^{n^2}$
2. $\lim_{n\to\infty}(\frac{4n+3}{4n+1})^{3n+2}$
3. $\lim_{n\to\infty}(cos\frac{1}{3n})^{n^2}$

Solution

1. $\lim_{n\to\infty}(1+\frac{n+3}{5n^3+n+2})^{n^2} = exp(\lim_{n\to\infty}(\frac{5n+3}{n^3+n+2})\cdot n^2) == exp(\lim_{n\to\infty}\frac{5n^3+3n^2}{n^3+n+2}) = e^5$

2. $\lim_{n\to\infty}(\frac{4n+3}{4n+1})^{3n+2} = \lim_{n\to\infty}(1+(\frac{4n+3}{4n+1}-1))^{3n+2} = \lim_{n\to\infty}(1+\frac{2}{4n+1})^{3n+2} =$

$$exp(\lim_{n\to\infty}(\frac{2}{4n+1})\cdot 3n+2) = exp(\lim_{n\to\infty}\frac{6n+4}{4n+1}) = e^{\frac{6}{4}}$$

3. $\lim_{n\to\infty}(cos\frac{1}{3n})^{n^2} = \lim_{n\to\infty}(1+(cos\frac{1}{3n}-1))^{n^2} =$

$$exp(\lim_{n\to\infty}(cos\frac{1}{3n}-1)\cdot n^2) = exp(\lim_{n\to\infty}(-\frac{1}{2}\cdot\frac{1}{9n^2})\cdot n^2) = e^{-\frac{1}{18}}$$

1.3 Cauchy Sequences

A sequence a_n called Cauchy sequence if

$$\forall\epsilon > 0 \exists n_0 \in N(\forall n, m > n_0 : |a_n - a_m| < \epsilon)$$

Theorem 8 (Cauchy's general principle of convergence) *Every convergent sequence is a Cauchy sequence. Every Cauchy sequence of real numbers converges.*

In fact, often we use Cauchy's general principle of convergence theorem to show divergence of a sequence, as shown in below example.

Example 1.3.15

Show that $\{a_n\}$ is divergent.

$$a_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

Solution We will show that $\{a_n\}$ is not a Cauchy sequence.

$$a_{2n} = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}$$

$$a_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$|a_{2n}-a_n| = \frac{1}{n+1}+\cdots+\frac{1}{2n} > \frac{1}{2n}+\cdots+\frac{1}{2n} = n\frac{1}{2n} = \frac{1}{2}$$

This implies that $\{a_n\}$ is not a Cauchy sequence. So $\{a_n\}$ is divergent.

Example 1.3.16

Show that $\{a_n\}$ is divergent.

$$a_n = \frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}.$$

Solution We will show that $\{a_n\}$ is not a Cauchy sequence.

$$a_{2n} = \frac{1}{1}+\frac{1}{3}+\cdots+\frac{1}{2n-1}+\frac{1}{2n+1}+\cdots+\frac{1}{4n-1}$$

$$a_n = \frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}$$

$$|a_{2n}-a_n| = \frac{1}{2n+1}+\cdots+\frac{1}{4n-1} > \frac{1}{4n-1}+\cdots+\frac{1}{4n-1} = n\frac{1}{4n-1} = \frac{1}{4}$$

This implies that $\{a_n\}$ is not a Cauchy sequence. So $\{a_n\}$ is divergent.

Theorem 9 (Cauchy's first theorem on limits) *If* $\lim_{n\to\infty} a_n = l$ *then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty}\left(\frac{a_1+a_2+a_3+\cdots+a_n}{n}\right) = l.$$

Example 1.3.17

Evaluate

1. $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n}\frac{1}{k}$
2. $\lim_{n\to\infty}\sum_{k=1}^{k=n}\frac{1}{\sqrt{n^2+k}}$

Solution

1. $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n}\frac{1}{k} = \lim_{n\to\infty}\frac{1}{n}(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n})$, let $a_n=\frac{1}{n}$, then by Cauchy's first theorem on limits, we have $\lim_{n\to\infty}\frac{1}{n}(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}) = \lim_{n\to\infty} a_n = 0$
2. $\lim_{n\to\infty}\sum_{k=1}^{k=n}\frac{1}{\sqrt{n^2+k}} = \lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}) =$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)\left(\frac{n}{\sqrt{n^2+1}}+\frac{n}{\sqrt{n^2+2}}+\cdots+\frac{n}{\sqrt{n^2+n}}\right) = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}} = 1.$$

Example 1.3.18

Cauchy Formula on Limits
If $a_n > 0$ for all value of n, then

$$\lim_{n\to\infty}(a_{n+1})^{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}.$$

Proof

$$(a_{n+1})^{\frac{1}{n+1}} = \left(\frac{a_1}{1}\frac{a_2}{a_1}\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}}\frac{a_{n+1}}{a_n}\right)^{\frac{1}{n+1}}$$

Take logarithm of either side to obtain

$$log(a_{n+1})^{\frac{1}{n+1}} = \left(\frac{1}{n+1}\right)\left(log\left(\frac{a_1}{1}\right)+log\left(\frac{a_2}{a_1}\right)+log\left(\frac{a_3}{a_2}\right)+\cdots+log\left(\frac{a_{n+1}}{a_n}\right)\right)$$

Let $A_n = log(\frac{a_n}{a_{n-1}}), a_0 = 1$, we have

$$log(a_{n+1})^{\frac{1}{n+1}} = \frac{1}{n+1}(A_0+A_1+\cdots+A_{n+1})$$

Using Cauchy's first theorem on limits we get

$$\lim_{n\to\infty} log(a_{n+1})^{\frac{1}{n+1}} = \lim_{n\to\infty} A_{n+1} = \lim_{n\to\infty}\left(log\left(\frac{a_{n+1}}{a_n}\right)\right)$$

$$log \lim_{n\to\infty}(a_{n+1})^{\frac{1}{n+1}} = log \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

$$\lim_{n\to\infty}(a_{n+1})^{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}.$$

Example 1.3.19

Evaluate

1. $\lim_{n\to\infty}(n)^{\frac{1}{n}}$.
2. $\lim_{n\to\infty}(\frac{n^n}{n!})^{\frac{1}{n}}$.

Solution

1. Let $a_n = n$, then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n+1}{n} = 1$$

Using Cauchy formula on limits, we have,

$$\lim_{n\to\infty}(n)^{\frac{1}{n}} = \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{a_n}{a_{n-1}} = 1.$$

2. Let $a_n = \frac{n^n}{n!}$, then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{n!(n+1)^n(n+1)}{n!(n+1)n^n = (\frac{n+1}{n})^n}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}(\frac{n+1}{n})^n = e$$

Using Cauchy formula on limits, we have,

$$\lim_{n\to\infty}(\frac{n^n}{n!})^{\frac{1}{n}} = \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{a_n}{a_{n-1}} = e.$$

1.4 Recurrent Sequences

Example 1.4.20

Consider $\{x_n\}$ defined by recurrent formula :

$$x_1 = 10, x_{n+1} = \sqrt{21+4x_n}$$

Prove $\{x_n\}$ is convergent and find its limit.

Solution By induction we will show that x_n is bounded and $7 \leq x_n \leq 10$ and is a decreasing sequence, then by Monotonic Sequence Theorem x_n is convergent.This is true for initial steps $x_1 = 10 \geq 7, x_2 = \sqrt{61} \geq 7$, assume $x_n \geq 7$, then

$$x_n \geq 7 \Rightarrow 21 + 4x_n \geq 49 \Rightarrow x_{n+1} \geq 7.$$

Now, we will prove that x_n is decreasing

$$\frac{x_{n+1}}{x_n} = \sqrt{\frac{21+4x_n}{x_n^2}} = \sqrt{\frac{21}{x_n^2}+\frac{4}{x_n}} \leq \sqrt{\frac{21}{49}+\frac{4}{7}} = 1 \Rightarrow \frac{x_{n+1}}{x_n} \leq 1 \Rightarrow x_{n+1} \leq x_n$$

So, x_n is decreasing and has lower bound, assume the limit is equal to l

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \sqrt{21+4x_n} = \lim_{n\to\infty} x_n$$

$$\sqrt{21+4l} = l \Rightarrow l = 7$$

Example 1.4.21

Consider $\{x_n\}$ *defined by recurrent formula :*

$$x_1 = 1, x_{n+1} = \sqrt{10+3x_n}$$

Prove $\{x_n\}$ *is convergent and find its limit.*
Solution By induction we will show that x_n is bounded and $1 \le x_n \le 5$ and is a increasing sequence, then by Monotonic Sequence Theorem x_n is convergent.This is true for initial steps $x_1 = 1 \le 5, x_2 = \sqrt{13} \le 5$, assume $x_n \le 5$, then

$$x_n \le 5 \Rightarrow 10 + 3x_n \le 25 \Rightarrow x_{n+1} \le 5.$$

Now, we will prove that x_n is increasing

$$\frac{x_{n+1}}{x_n} = \sqrt{\frac{10+3x_n}{x_n^2}} = \sqrt{\frac{10}{x_n^2} + \frac{3}{x_n}} \ge \sqrt{\frac{10}{25} + \frac{3}{5}} = 1 \Rightarrow \frac{x_{n+1}}{x_n} \ge 1 \Rightarrow x_{n+1} \ge x_n$$

So, x_n is increasing and has upper bound, assume the limit is equal to l

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \sqrt{10+3x_n} = \lim_{n\to\infty} x_n$$

$$\sqrt{10+3l} = l \Rightarrow l = 5$$

Example 1.4.22

Newton Sequence to Approximate $\sqrt{a}$
Let $x_1 = 2$ *and* $x_{n+1} = \frac{1}{2}(x_n + \frac{5}{x_n})$ *for* $n \ge 2$.
1.Find x_2, x_3.
2.Prove that x_n *is convergent.*
3.Find the limit of sequence.
Solution $x_1 = 2, x_2 = \frac{161}{72} = 2.2361\cdots, x_3 = \frac{51841}{23184} = 2.236067977$
By induction we will show that x_n is bounded and $1 \le x_n \le \sqrt{5}$ and is a increasing sequence, then by Monotonic Sequence Theorem x_n is convergent.This is true for initial steps $x_1 = 1 \le \sqrt{5}, x_2 \le \sqrt{5}$. Now, we will prove that x_n is increasing

$$x_{n+1} - x_n = \frac{5 - x_n^2}{2x_n} \ge 0 \Rightarrow x_{n+1} \ge x_n$$

So, x_n is increasing and has upper bound, assume the limit is equal to l

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \frac{1}{2}(l + \frac{5}{l}) = l \Rightarrow \frac{1}{2}(l + \frac{5}{l}) = l \Rightarrow l = \sqrt{5}$$

Example 1.4.23

Let $a_n \ge 0$ *such that*

$$a_0 = 0, a_1 = 1, a_n = \frac{1}{2}(a_{n-1} + a_{n-2}), n \ge 2$$

Show that $\{a_n\}$ is convergent and find its limit.
Solution Rewrite the recursive relation:

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \Rightarrow 2a_n = a_{n-1} + a_{n-2}$$

$$2a_n - 2a_{n-1} = a_{n-1} + a_{n-2} - 2a_{n-1} = a_{n-2} - a_{n-1}$$

$$a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2})$$

$$a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2}) = -\frac{1}{2}(-\frac{1}{2}(a_{n-2} - a_{n-3})) = (-\frac{1}{2})^3(a_{n-3} - a_{n-4}) = \cdots$$

$$a_n - a_{n-1} = (-\frac{1}{2})^{n-1}(a_1 - a_0) = (-\frac{1}{2})^{n-1}(1 - 0) = (-\frac{1}{2})^{n-1}$$

We will write Cauchy sequence condition

$$|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{m+1} - a_m)|$$

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{m+1} - a_m|$$

$$|a_n - a_m| \leq (\frac{1}{2})^{n-1} + (\frac{1}{2})^{n-2} + \cdots + (\frac{1}{2})^m = (\frac{1}{2})^m((\frac{1}{2})^{n-m-1} + \ \cdots + 1)$$

$$|a_n - a_m| \leq (\frac{1}{2})^{m-1} \to 0$$

Hence a_n is a Cauchy sequence and is convergent to l.

$$2a_n + a_{n-1} = 2a_{n-1} + a_{n-2} = 2a_{n-2} + a_{n-3} = \cdots = 2a_1 + a_0$$

$$\lim_{n\to\infty} 2a_n + a_{n-1} = \lim_{n\to\infty} 2a_1 + a_0 = 2$$

$$2l + l = 2 \Rightarrow l = \frac{2}{3}.$$

1.5 Sequential Characterization of Limits of Functions

Theorem 10 *$lim_{x\to a}f(x) = l$ if and only if $lim_{n\to\infty}f(x_n) = l$ for every sequence $x_n \neq a$, and $x_n \to a$.*
Conclusion *If there are two sequences $\{a_n\}$ and $\{b_n\}$, such that $a_n \neq a$ and $b_n \neq a$ and $lim_{n\to\infty}f(a_n) \neq lim_{n\to\infty}f(b_n)$ then $lim_{x\to a}f(x) =$does not exist.*

Example 1.5.24

By using sequences, show that $lim_{x\to 0}sin(\frac{1}{x}) =$does not exist.
Solution Let $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{2n\pi+\frac{\pi}{2}}$

$$f(a_n) = sin(\frac{1}{a_n}) = sin(2n\pi) = 0 \to 0$$

$$f(b_n) = sin(\frac{1}{b_n}) = sin(2n\pi + \frac{\pi}{2}) = 1 \to 1$$

So, $lim_{x\to 0}sin(\frac{1}{x}) =$does not exist.

Example 1.5.25

Show that $lim_{x\to 0}\left[x\left[\frac{1}{x}\right]\right]$ *=does not exist.*
Solution Let $a_n=\frac{1}{n}$ and $b_n=\frac{1}{n+0.5}$, $a_n\to 0$ and $b_n\to 0$.

$$f(a_n)=\left[\frac{1}{n}\left[n\right]\right]=\left[\frac{n}{n}\right]=1\to 1$$

$$f(b_n)=\left[\frac{1}{n+0.5}\left[n+0.5\right]\right]=\left[\frac{n}{n+0.5}\right]=0\to 0$$

So, $lim_{x\to 0}\left[x\left[\frac{1}{x}\right]\right]$ =does not exist.

Example 1.5.26

For given function f *and* $\{a_n\}$ *sequence.Find limit of* $\{f(a_n)\}$ *sequence.*

1. $f(x)=\frac{3}{x^2-1}\left|cos(\frac{\pi}{2}x)\right|$ and $a_n=\sqrt{n^2+6n+10}-n-2$.
2. $f(x)=\frac{3x^2-4\left[x^2\right]}{x-2}$ and $a_n=\frac{2n+1}{n+3}$.

Solution

1. Use asymptotic sequences to evaluate limit of a_n ,

$$a_n=\sqrt{n^2+6n+10}-n-2\sim(n+3)-n-2\to 1 \text{ and}$$
$$a_n=\sqrt{(n+3)^2+1}-n-2>(n+3)-n-2=1$$

$$lim_{n\to\infty}f(a_n)=lim_{x\to 1^+}f(x)=-lim_{x\to 1^+}\frac{3}{x^2-1}cos(\frac{\pi}{2}x)=-\frac{3}{2}lim_{x\to 1^+}\frac{cos(\frac{\pi}{2}x)}{x-1}=$$
$$\frac{3}{2}lim_{x\to 1^+}\frac{sin(\frac{\pi}{2}x-\frac{\pi}{2})}{x-1}=\frac{3}{2}lim_{x\to 1^+}\frac{sin(\frac{\pi}{2}(x-1))}{x-1}=\frac{3}{2}lim_{x\to 1^+}\frac{\frac{\pi}{2}(x-1)}{x-1}=\frac{3\pi}{4}$$

2. $a_n\to 2$ and $a_n<2$.

$$lim_{n\to\infty}f(a_n)=lim_{x\to 2^-}f(x)=lim_{x\to 2^-}\frac{3x^2-4\left[x^2\right]}{x-2}=$$

$$lim_{x\to 2^-}\frac{3x^2-12}{x-2}=lim_{x\to 2^-}\frac{3(x-2)(x+2)}{x-2}=12$$