Calculus

Learn Calculus By Example

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Contents

1 Limit of a Sequence

- We say that $\{a_n\}$ converges to a limit l, and we write $\lim_{n\to\infty} a_n = l$ or $a_n \to l$ if, for every $\epsilon > 0$, there is a number M such that $|a_n - l| < \epsilon$ for all $n > M$.
- We write $\lim_{n\to\infty} a_n = +\infty$ if for each $N > 0$ we can find $M > 0$ (depends on N) such that $a_n > N$ for all $n > M$.We say a_n diverges to $+\infty$
- We write $\lim_{n\to\infty} a_n = -\infty$ if for each $N > 0$ we can find $M > 0$ (depends on N) such that $a_n < -N$ for all $n > M$.We say a_n diverges to $-\infty$.
- Oscillated sequences such $\{(-1)^n\}$ are divergent.

Theorem 1 (Limit is Unique) *If a sequence $\{a_n\}$ has a limit, then this limit is unique.*

Example 1.1

Using definition of the limit of sequence (ϵ and $M(\epsilon)$ language) prove that:

1. $\lim_{n\to\infty} \frac{2n^2-1}{n^2+2} = 2$
2. $\lim_{n\to\infty} \frac{n+4\sqrt{n}+3}{\sqrt{n}+2} = +\infty$
3. $\lim_{n\to\infty} \frac{-n^2+6\sqrt{n}+3}{n+2} = -\infty$
4. $\lim_{n\to\infty} 3 + cos(n\pi) =$ does not exist.

Solution

1. We should prove, for all $\epsilon > 0$ there is $M \in \mathbf{N}$ such that $n > M \Rightarrow \left|\frac{2n^2-1}{n^2+2} - 2\right| < \epsilon$

$$\left|\frac{-5}{n^2+2}\right| < \epsilon \Rightarrow \frac{5}{n^2+2} < \epsilon \Rightarrow n^2 > \frac{5}{\epsilon} - 2 \Rightarrow n > \sqrt{\frac{5}{\epsilon} - 2}$$

   Always ϵ presents a very small positive number, so, $M > \left[\sqrt{\frac{5}{\epsilon} - 2}\right] + 1$, is enough big to satisfy the definition.

2. We should prove, for all $N > 0$ there is $M \in \mathbf{N}$ such that $n > M \Rightarrow |a_n| > N$

$$\left|\frac{(\sqrt{n}+2)^2-1}{\sqrt{n}+2}\right| > N \Rightarrow \left|(\sqrt{n}+2) - \frac{1}{\sqrt{n+2}}\right| > N \Rightarrow \sqrt{n} > N \Rightarrow n > N^2$$

   Now, select $M > [N^2] + 1$.This means the given sequence increase without bound and is divergent.

3. We should prove, for all $N > 0$ there is $M \in \mathbf{N}$ such that $n > M \Rightarrow a_n < -N$

$$\frac{-n^2+6\sqrt{n}+3}{n+2} < -N \Rightarrow \frac{n^2-6\sqrt{n}-3}{n+2} > N$$

$$\frac{n^2-6\sqrt{n}-3}{n+2} > \frac{n^2-6\sqrt{n}-3}{n+n} > \frac{n^2-6\sqrt{n}-\sqrt{n}}{2n} = \frac{n}{2} - \frac{7}{2\sqrt{n}} > \frac{n}{2} - \frac{7}{2} > N \Rightarrow n > 2N+7$$

   Now, select $M > [2N + 7] + 1$.

4. Suppose $\lim_{n\to\infty} 3 + cos(n\pi) = l$ and select $\epsilon = 0.1$,by definition, there is $M \in \mathbf{N}$ such that $n > M \Rightarrow |a_n - l| < \epsilon$. Since $cos(n\pi) = (-1)^n$, $a_{2n} = 4$ and $a_{2n+1} = 2$. So, $|4 - l| < \epsilon$ and $|2 - l| < \epsilon$
$$2 = |(4-l) - (2-l)| \leq |(4-l)| + |(2-l)| \ < 2\epsilon \Rightarrow 2 < 2(0.1)$$
that is contradiction.This proves that, the sequence is divergent and there is no limit.

Example 1.2

Is the sequence $a_n = \frac{2n^2+3n-1}{n^2+6n+12}$ *convergent or divergent?*
Solution
$$\lim_{n\to\infty} \frac{2n^2+3n-1}{n^2+6n+12} = \lim_{n\to\infty} \frac{n^2(2+\frac{3}{n}-\frac{1}{n^2})}{n^2(1+\frac{6}{n}+\frac{12}{n^2})} = 2$$
So, $\{a_n\}$ is converge to 2.

Example 1.3

Is the sequence $a_n = \frac{n}{\sqrt{n+2}}$ *convergent or divergent?*
Solution
$$\lim_{n\to\infty} \frac{n}{\sqrt{n+2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n(1+\frac{2}{n})}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+\frac{2}{n}}} = \frac{\infty}{\sqrt{1+0}} = \infty$$
So, $\{a_n\}$ is divergent.

Theorem 2 (Bounded Sequence) *Every convergent sequence is a bounded sequence. There are two numbers* m*(lower bound) and* M*(upper bound) such that* $m \leq \{a_n\} \leq M$.

Remark *Example 1.1.4 shows that, there exist bounded, but divergence sequence.*

Rate of Grow
A sequence $\{b_n\}$ **grows faster than** $\{a_n\}$ if
$$\lim_{n\to\infty} \frac{b_n}{a_n} = \infty \ or \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$
In this case we write $\{a_n\} \ll \{b_n\}$. We can write
$$\frac{slow}{fast} \to 0 \ and \ \frac{fast}{slow} \to \infty.$$
Sequence $\{a_n\}$ and $\{b_n\}$ **grow at the same rate** if for some $L, 0 < L < \infty$,
$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$
We call $\{a_n\}$ and $\{b_n\}$ **Equivalent Sequences** if $L = 1$.

Theorem 3 (Orders of Growth)

$$lnlnn \ll lnn \ll \cdots \ll \sqrt[3]{n} \ll \sqrt{n} \ll n \ll n^2 \ll \cdots \ll 2^n \ll e^n \ll n! \ll n^n \ll e^{n^2}$$

Example 1.4

*By using **Rate of Grow**, evaluate limit of sequences.*

1. $\lim_{n\to\infty} \frac{3+4lnn}{3n+5}$
2. $\lim_{n\to\infty} \frac{\sqrt{n+2}+1}{lnn+5}$
3. $\lim_{n\to\infty} \frac{2^n+3^n}{3^{n+1}+2^{n+5}}$
4. $\lim_{n\to\infty} \frac{3\cdot 4^n}{(n+3)!}$

Solution

1. $\lim_{n\to\infty} \frac{3+4lnn}{3n+5} = \lim_{n\to\infty} \frac{4lnn}{3n} = \frac{4}{3}\lim_{n\to\infty} \frac{lnn}{n} = 0$
2. $\lim_{n\to\infty} \frac{\sqrt{n+2}+1}{lnn+5} = \lim_{n\to\infty} \frac{\sqrt{n}}{lnn} = +\infty$
3. $\lim_{n\to\infty} \frac{2^n+3^n}{3^{n+1}+2^{n+5}} = \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3}$
4. $\lim_{n\to\infty} \frac{3\cdot 4^n}{(n+3)!} = 3\lim_{n\to\infty} \frac{4^n}{(n+3)!} = 0$

Theorem 4 (Squeeze Theorem) *If $a_n \leq b_n \leq c_n$ for $n \geq m$ and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l,$$

then $\lim_{n\to\infty} b_n = l$.

Example 1.5

*By using **Squeeze Theorem**, evaluate limit of sequences.*

1. $\lim_{n\to\infty} \frac{n!}{n^n}$
2. $\lim_{n\to\infty} \sqrt[n]{2^n+3^n+4^n}$
3. $\lim_{n\to\infty} \frac{3}{n}cos^3(n!)$
4. $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$

Solution

1. $a_n = \frac{n!}{n^n} = \frac{1}{n}(\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdots\cdot n})$

$$0 < a_n \leq \frac{1}{n}$$

We know that $\frac{1}{n} \to 0$. Therefore $a_n \to 0$ by the Squeeze Theorem.

2. $a_n = \sqrt[n]{2^n+3^n+4^n}$

$$4^n \leq 2^n + 3^n + 4^n \leq 4^n + 4^n + 4^n = 3\cdot 4^n$$

$$\lim_{n\to\infty} \sqrt[n]{4^n} = 4, \lim_{n\to\infty} \sqrt[n]{3\cdot 4^n} = 4\lim_{n\to\infty} \sqrt[n]{3} = 4\cdot 1 = 4$$

Therefore $a_n \to 4$ by the Squeeze Theorem.

3. $a_n = \frac{3}{n}cos^3(n!)$

$$-\frac{3}{n} \leq \frac{3}{n}cos^3(n!) \leq \frac{3}{n}$$

Since $-\frac{3}{n} \to 0$ and $\frac{3}{n} \to 0$, $a_n \to 0$, by the Squeeze Theorem.

4. $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$

$$\frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}}$$

$$n\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq n\frac{1}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1, \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$$

Therefore $a_n \to 1$ by the Squeeze Theorem.

Theorem 5 (Sub-Sequences) *If a sequence converges then all sub-sequences converge and all convergent sub-sequences converge to the same limit.*
If $\{a_n\}$is a sequence that either has a sub-sequence that diverges or two convergent sub-sequences with different limits then $\{a_n\}$ is divergent.

Example 1.6

Use Theorem 4,to proof divergence of sequences.

1. $a_n = tan(4n+1)\frac{\pi}{3}$
2. $a_n = \frac{ncos(n\pi)+2}{2n+3}$
3. $a_n = \frac{n}{2n+3} + cos\frac{2n\pi}{3}$
4. $a_n = \left[2 + \frac{sinn}{n}\right]$

Solution

1. Select two sub-sequences with different limits. Select a_{3n} and a_{3n+1},

$$a_{3n} = tan(4(3n)+1)\frac{\pi}{3} = tan(4n\pi + \frac{\pi}{3}) = tan(\frac{\pi}{3}) = \sqrt{3} \to \sqrt{3}$$

$$a_{3n+1} = tan(4(3n+1)+1)\frac{\pi}{3} = tan(4n\pi + \frac{5\pi}{3}) = -\sqrt{3} \to -\sqrt{3}$$

Therefore,$\{a_n\}$is a sequence that has two convergent sub-sequences with different limits, so $\{a_n\}$ is divergent.

2. Select two sub-sequences with different limits. Select a_{2n} and a_{2n+1}. We know that $cos(n\pi) = (-1)^n$.

$$a_{2n} = \frac{2ncos(2n\pi)+2}{4n+3} = \frac{2n+2}{4n+3} = \frac{1}{2} \to \frac{1}{2}$$

$$a_{2n+1} = \frac{(2n+1)cos((2n+1)\pi)+2}{4n+5} = \frac{-2n+1}{4n+5} = -\frac{1}{2} \to -\frac{1}{2}$$

Therefore,$\{a_n\}$is a sequence that has two convergent sub-sequences with different limits, so $\{a_n\}$ is divergent.

3. Select two sub-sequences with different limits. Select a_{3n} and a_{3n+1}.

$$a_{3n} = \frac{3n}{6n+3} + cos\frac{2(3n)\pi}{3} = \frac{3n}{6n+3} + cos2n\pi \to \frac{1}{2} + 1 = \frac{3}{2}$$

$$a_{3n+1} = \frac{3n+1}{6n+5} + cos\frac{2(3n+1)\pi}{3} = \frac{3n+1}{6n+5} + cos(2n\pi + \frac{2\pi}{3}) \to \frac{1}{2} - \frac{1}{2} = 0$$

Therefore,$\{a_n\}$is a sequence that has two convergent sub-sequences with different limits, so $\{a_n\}$ is divergent.

4. By Squeeze Theorem we know:

$$-1 \leq sinn \leq 1 \Rightarrow \frac{sinn}{n} \to 0 \Rightarrow 2 + \frac{sinn}{n} \to 2$$

Select two types of a_n terms, where $sinn > 0$ and where $sinn < 0$

If $sinn > 0$ we have $2 + \frac{sinn}{n} > 2$, so $a_n \to 2$ and if $sinn < 0$ we have $2 + \frac{sinn}{n} < 2$, so $a_n \to 1$.

Therefore,$\{a_n\}$is divergent.

Example 1.7

Prove the divergence of the sequence. $1, \frac{1}{2}, 3, 1, \frac{1}{4}, 6, 1, \frac{1}{8}, 9, 1, \frac{1}{16}, 27, \cdots$.

Solution Here we can see 3 sub-sequences $a_{3n+1} \to 1, a_{3n+2} \to 0$ and $a_{3n} \to +\infty$, so the sequence is unbounded and therefore divergence.

Asymptotic Sequences

We say $\{a_n\}$ is asymptotic to $\{b_n\}$ and write $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$.

Consider k, m, r are constants and $r > 0$. We can write:

$$a_n = \sqrt{n+k} - \sqrt{n+m} \sim \frac{k-m}{2\sqrt{n}} \to 0 \tag{1}$$

$$a_n = \sqrt[r]{n+k} - \sqrt[r]{n+m} \sim \frac{k-m}{r\sqrt[r]{n^{r-1}}} \to 0 \tag{2}$$

Example 1.8

Find limit of the sequence.

1. $\lim_{n\to\infty} n(\sqrt{n^2+1} - n)$
2. $\lim_{n\to\infty} n^2(\sqrt[3]{n^3+1} - n)$
3. $\lim_{n\to\infty} \sqrt{n}(\sqrt[3]{n+12} - \sqrt[3]{n+4})$
4. $\lim_{n\to\infty} 2\sqrt{n+1} - \sqrt{n+3} - \sqrt{n}$

Solution

1. Use (1.1) equivalency, substitute n^2 to n

$$\lim_{n\to\infty} n(\sqrt{n^2+1} - n) = \lim_{n\to\infty} n(\sqrt{n^2+1} - \sqrt{n^2}) = \lim_{n\to\infty} n(\frac{1}{2\sqrt{n^2}}) = \frac{1}{2}$$

2. Use (1.2) equivalency, substitute n^3 to n

$$\lim_{n\to\infty} n^2(\sqrt[3]{n^3+1} - n) = \lim_{n\to\infty} n^2(\sqrt[3]{n^3+1} - \sqrt[3]{n^3}) = \lim_{n\to\infty} n^2(\frac{1}{3\sqrt[3]{n^6}}) = \frac{1}{3}$$

3. Use (1.2) equivalency,

$$\lim_{n\to\infty} \sqrt{n}(\sqrt[3]{n+12} - \sqrt[3]{n+4}) = \lim_{n\to\infty} \sqrt{n}(\frac{12-4}{3\sqrt[3]{n^2}}) = 0$$

4. By dividing into subtraction, we will use (1.1) equivalency, two times.

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n+3} + \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{1-3}{2\sqrt{n}} + \frac{1}{2\sqrt{n}} = 0$$

Example 1.9

Find limit of the sequence

$$\lim_{n\to\infty} \sqrt{n^2+6n+1} - n$$

Solution

$$\lim_{n\to\infty} \sqrt{n^2+6n+1} - n = \lim_{n\to\infty} \sqrt{n^2+6n+9-8} - n =$$

$$\lim_{n\to\infty} \sqrt{(n+3)^2-8} - n = \lim_{n\to\infty} (n+3) - n = 3$$

Asymptotic Sequences

Consider a, b, c, k are constants and $a > 0, k > 0$. We can write:

$$\sqrt{an^2+bn+c} \sim \sqrt{a}(n + \frac{b}{2a}) \tag{3}$$

$$\sqrt[k]{an^k + bn^{k-1} + c + \cdots} \sim \sqrt[k]{a}(n + \frac{b}{ka}) \tag{4}$$

Example 1.10

Find the limit of the sequence or show that it diverges.

1. $\lim_{n\to\infty} \sqrt{4n^2+6n+1} - \sqrt{4n^2+2}$
2. $\lim_{n\to\infty} \sqrt[3]{2n^3+18n^2+n+1} - \sqrt[3]{2n^3+24n^2-5}$
3. $\lim_{n\to\infty} \sqrt{2n^2+12n-1} - \sqrt{2n^2-8n+11}$

Solution Use (1.3) and (1.4) equations.

1. $\lim_{n\to\infty} \sqrt{4n^2+6n+1} - \sqrt{4n^2+2} = \lim_{n\to\infty} 2(n+\frac{6}{8}) - 2n = \frac{3}{2}$
2. $\lim_{n\to\infty} \sqrt[3]{2n^3+18n^2+n+1} - \sqrt[3]{2n^3+24n^2-5} = \lim_{n\to\infty} \sqrt[3]{2}((n+\frac{18}{6}) - (n+\frac{24}{6})) = -\sqrt[3]{2}$
3. $\lim_{n\to\infty} \sqrt{2n^2+12n-1} - \sqrt{2n^2-8n+11} = \lim_{n\to\infty} \sqrt{2}((n+\frac{12}{4}) - (n-\frac{8}{4})) = 5\sqrt{2}$

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2 Monotonic Sequences

We say that $\{x_n\}$ is **increasing** if $x_n \leq x_{n+1}$ for all n and strictly increasing if $x_n < x_{n+1}$ for all n. Similarly, We say that $\{x_n\}$ is **decreasing** if $x_{n+1} \leq x_n$ for all n and strictly decreasing if $x_{n+1} < x_n$ for all n.Sequences which are either increasing or decreasing are called **monotone**.

Theorem 6 (Monotonic Sequence Theorem) *Every bounded, monotonic sequence is convergent.*

1. *Suppose $\{x_n\}$ is a bounded and increasing sequence. Then the $sup(x_n)$ is the limit of $\{x_n\}$.*
2. *Suppose $\{x_n\}$ is a bounded and decreasing sequence. Then the $inf(x_n)$ is the limit of $\{x_n\}$.*

Example 2.11

Prove that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded. Find $inf(a_n)$ and $sup(a_n)$.
Solution

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}).\frac{(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} =$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

So, $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$,

$$a_{n+1} < a_n$$

$$inf = 0 < a_n \ \leq sup = \sqrt{2} - 1$$

Example 2.12

Prove that $a_n = \frac{2n+1}{n+5} cos(\frac{\pi}{2n+2})$ is monotone and find limit.
Solution

$$lim_{n\to\infty} \frac{2n+1}{n+5} cos(\frac{\pi}{2n+2}) = 2 \cdot cos(0) = 2$$

We know that $(\frac{ax+b}{cx+d})' = \frac{ad-bc}{(cx+d)^2}$, therefore if $f(x) = \frac{2x+1}{x+5}$ then $f'(x) > 0$. It shows that $x_n = \frac{2n+1}{n+5}$ is increasing. Also by growing n, the $arc = \frac{\pi}{2n+2}$ is variate from $\frac{\pi}{4}$ to zero, therefore $y_n = cos(\frac{\pi}{2n+2}) > 0$ and increasing between $\frac{\sqrt{2}}{2}$ to 1. So, $a_n = x_n \cdot y_n$ is multiply of two positive and increasing sequences, therefore a_n is increasing and converges to 2 and bounded $a_1 = \frac{\sqrt{2}}{4} \leq a_n < 2$.

Example 2.13

Prove that $a_n = (1 + \frac{1}{n})^n$ is increasing and bounded.
SolutionWe can use binomial theorem to expand the expression

$$a_n = (1 + \frac{1}{n})^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2!}(\frac{1}{n})^2 + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}(\frac{1}{n})^n$$

$$a_n = 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n \cdot n} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n \cdot n \cdot n} + \cdots + \frac{1}{n!}\frac{n(n-1)(n-2)\cdots 1}{n \cdot n \cdot n \cdots n}$$

$$a_n = 2 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{n-1}{n})$$

Similarly for a_{n+1} we have

$$a_{n+1} = 2+\frac{1}{2!}(1-\frac{1}{n+1})+\frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})+\cdots+\frac{1}{(n+1)!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n}{n+1})$$

Because $(1-\frac{k}{n+1})-(1-\frac{k}{n}) = \frac{k}{n(n+1)} \geq 0$ we have $a_{n+1}-a_n \geq 0$, therefore it is increasing sequence.
Next, we need to show it is bounded.

$$a_n = 2+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$

$$a_n \leq 2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}} < 3$$

So, a_n is increasing and bounded.

Theorem 7 (Leonhard Euler "e" number) *The sequence $a_n = (1+\frac{1}{n})^n$ is increasing and bounded, and so has a limit which denote by e. The value of $e = 2.718281828459045\cdots$.*

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

Remark Suppose $a_n \to 0$ and $b_n \to +\infty$, then $\lim_{n\to\infty}(1+a_n)^{b_n} = exp(\lim_{n\to\infty} a_n \cdot b_n)$

Example 2.14

Evaluate following limits.

1. $\lim_{n\to\infty}(1+\frac{5n+3}{n^3+n+2})^{n^2}$
2. $\lim_{n\to\infty}(\frac{4n+3}{4n+1})^{3n+2}$
3. $\lim_{n\to\infty}(cos\frac{1}{3n})^{n^2}$

Solution

1. $\lim_{n\to\infty}(1+\frac{n+3}{5n^3+n+2})^{n^2} = exp(\lim_{n\to\infty}(\frac{5n+3}{n^3+n+2})\cdot n^2) == exp(\lim_{n\to\infty}\frac{5n^3+3n^2}{n^3+n+2}) = e^5$
2. $\lim_{n\to\infty}(\frac{4n+3}{4n+1})^{3n+2} = \lim_{n\to\infty}(1+(\frac{4n+3}{4n+1}-1))^{3n+2} = \lim_{n\to\infty}(1+\frac{2}{4n+1})^{3n+2} =$

$$exp(\lim_{n\to\infty}(\frac{2}{4n+1})\cdot 3n+2) = exp(\lim_{n\to\infty}\frac{6n+4}{4n+1}) = e^{\frac{6}{4}}$$

3. $\lim_{n\to\infty}(cos\frac{1}{3n})^{n^2} = \lim_{n\to\infty}(1+(cos\frac{1}{3n}-1))^{n^2} =$

$$exp(\lim_{n\to\infty}(cos\frac{1}{3n}-1)\cdot n^2) = exp(\lim_{n\to\infty}(-\frac{1}{2}\cdot\frac{1}{9n^2})\cdot n^2) = e^{-\frac{1}{18}}$$

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3 Recurrent Sequences

Example 3.15

Consider $\{x_n\}$ *defined by recurrent formula* :

$$x_1 = 10, x_{n+1} = \sqrt{21 + 4x_n}$$

Prove $\{x_n\}$ *is convergent and find its limit.*

Solution By induction we will show that x_n is bounded and $7 \leq x_n \leq 10$ and is a decreasing sequence, then by Monotonic Sequence Theorem x_n is convergent.This is true for initial steps $x_1 = 10 \geq 7, x_2 = \sqrt{61} \geq 7$, assume $x_n \geq 7$, then

$$x_n \geq 7 \Rightarrow 21 + 4x_n \geq 49 \Rightarrow x_{n+1} \geq 7.$$

Now, we will prove that x_n is decreasing

$$\frac{x_{n+1}}{x_n} = \sqrt{\frac{21 + 4x_n}{x_n^2}} = \sqrt{\frac{21}{x_n^2} + \frac{4}{x_n}} \leq \sqrt{\frac{21}{49} + \frac{4}{7}} = 1 \Rightarrow \frac{x_{n+1}}{x_n} \leq 1 \Rightarrow x_{n+1} \leq x_n$$

So, x_n is decreasing and has lower bound, assume the limit is equal to l

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \sqrt{21 + 4x_n} = \lim_{n\to\infty} x_n$$

$$\sqrt{21 + 4l} = l \Rightarrow l = 7$$

Example 3.16

Consider $\{x_n\}$ *defined by recurrent formula* :

$$x_1 = 1, x_{n+1} = \sqrt{10 + 3x_n}$$

Prove $\{x_n\}$ *is convergent and find its limit.*

Solution By induction we will show that x_n is bounded and $1 \leq x_n \leq 5$ and is a increasing sequence, then by Monotonic Sequence Theorem x_n is convergent.This is true for initial steps $x_1 = 1 \leq 5, x_2 = \sqrt{13} \leq 5$, assume $x_n \leq 5$, then

$$x_n \leq 5 \Rightarrow 10 + 3x_n \leq 25 \Rightarrow x_{n+1} \leq 5.$$

Now, we will prove that x_n is increasing

$$\frac{x_{n+1}}{x_n} = \sqrt{\frac{10 + 3x_n}{x_n^2}} = \sqrt{\frac{10}{x_n^2} + \frac{3}{x_n}} \geq \sqrt{\frac{10}{25} + \frac{3}{5}} = 1 \Rightarrow \frac{x_{n+1}}{x_n} \geq 1 \Rightarrow x_{n+1} \geq x_n$$

So, x_n is increasing and has upper bound, assume the limit is equal to l

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \sqrt{10 + 3x_n} = \lim_{n\to\infty} x_n$$

$$\sqrt{10 + 3l} = l \Rightarrow l = 5$$

Example 3.17

Newton Sequence to Approximate $\sqrt{a}$
Let $x_1 = 2$ *and* $x_{n+1} = \frac{1}{2}(x_n + \frac{5}{x_n})$ *for* $n \geq 2$.

1.Find x_2, x_3.
2.Prove that x_n is convergent.
3.Find the limit of sequence.
Solution $x_1 = 2, x_2 = \frac{161}{72} = 2.2361\cdots, x_3 = \frac{51841}{23184} = 2.236067977$
By induction we will show that x_n is bounded and $1 \leq x_n \leq \sqrt{5}$ and is a increasing sequence, then by Monotonic Sequence Theorem x_n is convergent.This is true for initial steps $x_1 = 1 \leq \sqrt{5}, x_2 \leq \sqrt{5}$. Now, we will prove that x_n is increasing

$$x_{n+1} - x_n = \frac{5 - x_n^2}{2x_n} \geq 0 \Rightarrow x_{n+1} \geq x_n$$

So, x_n is increasing and has upper bound, assume the limit is equal to l

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \frac{1}{2}(l + \frac{5}{l}) = l \Rightarrow \frac{1}{2}(l + \frac{5}{l}) = l \Rightarrow l = \sqrt{5}$$

4 Sequential Characterization of Limits of Functions

Theorem 8 *$lim_{x\to a} f(x) = l$ if and only if $lim_{n\to\infty} f(x_n) = l$ for every sequence $x_n \neq a$, and $x_n \to a$.*
Conclusion *If there are two sequences $\{a_n\}$ and $\{b_n\}$, such that $a_n \neq a$ and $b_n \neq a$ and $lim_{n\to\infty} f(a_n) \neq lim_{n\to\infty} f(b_n)$ then $lim_{x\to a} f(x) =$does not exist.*

Example 4.18

By using sequences, show that $lim_{x\to 0} sin(\frac{1}{x}) =$does not exist.
Solution Let $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$

$$f(a_n) = sin(\frac{1}{a_n}) = sin(2n\pi) = 0 \to 0$$

$$f(b_n) = sin(\frac{1}{b_n}) = sin(2n\pi + \frac{\pi}{2}) = 1 \to 1$$

So, $lim_{x\to 0} sin(\frac{1}{x}) =$does not exist.

Example 4.19

Show that $lim_{x\to 0} \left[x \left[\frac{1}{x}\right]\right] =$does not exist.
Solution Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n+0.5}$, $a_n \to 0$ and $b_n \to 0$.

$$f(a_n) = \left[\frac{1}{n}[n]\right] = \left[\frac{n}{n}\right] = 1 \to 1$$

$$f(b_n) = \left[\frac{1}{n+0.5}[n+0.5]\right] = \left[\frac{n}{n+0.5}\right] = 0 \to 0$$

So, $lim_{x\to 0} \left[x \left[\frac{1}{x}\right]\right] =$does not exist.

Example 4.20

For given function f and $\{a_n\}$ sequence.Find limit of $\{f(a_n)\}$ sequence.

1. $f(x) = \frac{3}{x^2-1} \left| cos(\frac{\pi}{2}x) \right|$ and $a_n = \sqrt{n^2+6n+10} - n - 2$

2. $f(x) = \frac{3x^2-4\left[x^2\right]}{x-2}$ and $a_n = \frac{2n+1}{n+3}$

Solution

1. Use asymptotic sequences to evaluate limit of a_n ,

$$a_n = \sqrt{n^2+6n+10} - n - 2 \sim (n+3) - n - 2 \to 1 \text{ and}$$
$$a_n = \sqrt{(n+3)^2+1} - n - 2 > (n+3) - n - 2 = 1$$

$$lim_{n\to\infty} f(a_n) = lim_{x\to 1^+} f(x) = -lim_{x\to 1^+} \frac{3}{x^2-1} cos(\frac{\pi}{2}x) = -\frac{3}{2} lim_{x\to 1^+} \frac{cos(\frac{\pi}{2}x)}{x-1} =$$
$$\frac{3}{2} lim_{x\to 1^+} \frac{sin(\frac{\pi}{2}x - \frac{\pi}{2})}{x-1} = \frac{3}{2} lim_{x\to 1^+} \frac{sin(\frac{\pi}{2}(x-1))}{x-1} = \frac{3}{2} lim_{x\to 1^+} \frac{\frac{\pi}{2}(x-1)}{x-1} = \frac{3\pi}{4}$$

2. $a_n \to 2$ and $a_n < 2$.

$$lim_{n\to\infty} f(a_n) = lim_{x\to 2^-} f(x) = lim_{x\to 2^-} \frac{3x^2 - 4\left[x^2\right]}{x-2} =$$

$$lim_{x\to 2^-} \frac{3x^2-12}{x-2} = lim_{x\to 2^-} \frac{3(x-2)(x+2)}{x-2} = 12$$