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Analytical self-consistent solution for the structure of polytropic accretion discs with boundary layers

G. S. Bisnovatyi-Kogan

Space Research Institute, Profsoyuznaya 84/32, Moscow 117810, Russia

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ABSTRACT

Analytical solutions are presented for polytropic accretion discs with boundary layers near the star. The solutions are obtained by the method of matched asymptotic expansions. The total angular momentum flux over the accretion disc and the thickness of the boundary layer are determined self-consistently in these solutions.

Key words: accretion, accretion discs – binaries: close.

1 INTRODUCTION

The boundary layer (BL) is formed near the surface of the stellar equator when there is a disc accretion on to the slowly rotating non-magnetized star (Shakura 1973). The energy release inside the BL can be comparable to the energy release in the disc. This means that the BL luminosity can make a significant signature in the spectra of low-mass X-ray binaries, cataclysmic variables and other sources, where disc accretion on to weakly magnetized stars is expected.

Solution of the BL structure problem is also necessary for completing the solution of the structure of the disc itself, because the integration constant in the latter must be found by fitting a BL solution. A successful approach to a self-consistent solution of the problem was proposed by Regev (1983). He had found, numerically, the structure of the disc with a BL for some set of parameters, finding the integration constant by fitting conditions. He used the method of matched asymptotic expansions (MAE) (Nayfeh 1973), which is well-suited to this problem. Other approaches to the problem of BL structure have been used by Papaloizou & Stanley (1986), who carried out a numerical investigation of the polytropic case, and by Shakura & Sunyaev (1988), Colpi, Nanmurelli & Calvani (1991) and Glatzel (1992), where different analytical solutions were found. These solutions for joint disc–BL structure have been found, for particular choices of the viscosity coefficient and temperature distribution over the disc, by Colpi et al. (1991) and Glatzel (1992). Shakura & Sunyaev (1988) have found an analytical solution for an isothermal BL, but have not considered the fitting problem, i.e. a self-consistent disc structure.

Here, the problem of the self-consistent disc structure together with the boundary layer is solved analytically using an MAE method, following Regev (1983). There are no special restrictions, except that a polytropic equation of state is assumed, avoiding the need to consider the thermal processes, and the viscosity coefficient is taken to be the usual α-approximation without other simplifications.

It is established that the total angular momentum flux over the disc, determined by the integration constant, is slightly less than the advective flux through the inner boundary of the disc, when the star rotates sufficiently slowly. The MAE approximation is unsuitable for the accretion on to a rapidly rotating star, and in this case other methods are used.

2 EQUATIONS FOR THE THIN DISC STRUCTURE

The hydrodynamical equations describing the structure of a stationary thin accretion disc, in usual notation, have the form

$$\frac{\mathrm{d}(\dot{M}u_r)}{\mathrm{d}r} - 2\pi\Sigma r(\Omega^2 r + g_r) + 2\pi r\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}r} = 0, \tag{1}$$

$$\frac{\dot{M}}{2\pi}\frac{\mathrm{d}j}{\mathrm{d}r} = \alpha\frac{\mathrm{d}}{\mathrm{d}r}\left(\Sigma u_{s0} z_0 r^3 \frac{\mathrm{d}\Omega}{\mathrm{d}r}\right), \tag{2}$$

$$\rho g_z = \frac{\partial P}{\partial z}, \tag{3}$$

$$\frac{\mathrm{d}\dot{M}}{\mathrm{d}r} = 0. \tag{4}$$

Following Shakura (1973), we write the viscosity coefficient μ, in the α-approximation,

$$\mu = \alpha\rho u_{s0} z_0, \tag{5}$$

where u_{s0} is the sound velocity in the equatorial plane $z=0$, and z_0 is the semi-thickness of the disc. The average values, surface density Σ, mass flux over the disc $\dot{M}$ and integrated pressure $\mathscr{P}$, and the values of angular velocity Ω and specific angular momentum j are determined as follows:

$$\Sigma=\int_{-z_0}^{z_0}\rho\,\mathrm{d}z,\qquad \dot{M}=2\pi r\int_{-z_0}^{z_0}\rho u_r\,\mathrm{d}z=2\pi r\Sigma u_r,$$
$$\mathscr{P}=\int_{-z_0}^{z_0}P\,\mathrm{d}z,\qquad \Omega=\frac{u_\phi}{r},\qquad j=ru_\phi. \tag{6}$$

The equations (1)–(4) determine the variables (ρ, Σ, $\dot{M}$, j), where (u_r, Ω) are obtained from (6) with

$$u_r,\ \dot{M}<0. \tag{7}$$

The solution of the equation of vertical balance (3) gives the variables

$$u_{z0}=\left(\frac{\partial P_0}{\partial\rho_0}\right)_S^{1/2},\qquad z_0,\qquad \mathscr{P} \tag{8}$$

as functions of (r, Σ), when the equation of state is given in the form $P(\rho)$ (e.g. a polytrope). Here, ρ_0 and P_0 are the density and the pressure in the equatorial plane, respectively.

When the derivatives of 'pressure' $\mathscr{P}$ and velocity u_r are neglected in (1), the boundary conditions are determined by the values of the mass flux and Keplerian angular velocity at infinity, by placing a condition on the angular velocity and its derivative near the accreting body (black hole, neutron star or other), determining the total angular momentum flux through the disc, and assuming zero density and pressure on the outer boundary of the disc.

$$\dot{M}=\dot{M}_\infty,\qquad \Omega(r\to\infty)=\left(\frac{GM}{r^3}\right)^{1/2},\qquad \rho(\pm z_0)=0,$$

and

$$\Omega(r_{\rm in})=\Omega_{\rm in},\qquad \left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)(r_{\rm in})=\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)_{\rm in}, \tag{9}$$

where M is the mass of the star. If all terms remain in equation (1), then an additional boundary condition is necessary. It follows from this requirement that the solution must go smoothly through the sonic (singular) point in the radial motion of the accretion disc to the black hole, or be adjusted continuously to the conditions at the outer boundary on the stellar equator. The stellar angular velocity may be much less than the Keplerian velocity of the accretion disc. In that case, a boundary layer is established between the disc and the star, which needs to be treated separately.

In most cases, the external force is represented by the gravity of the star, which may be treated as a point mass with a potential

$$\phi_g=-\frac{GM}{(r^2+z^2)^{1/2}}. \tag{10}$$

For a thin disc, the components of the gravitational acceleration, g_i, are written as

$$g_i=-\frac{\partial\phi_g}{\partial x_i}=\left[-\frac{GM}{r^2}\left(1-\frac{3}{2}\frac{z^2}{r^2}\right),\quad 0,\quad -\frac{GM_z}{r^3}\left(1-\frac{3}{2}\frac{z^2}{r^2}\right)\right]. \tag{11}$$

3 POLYTROPIC APPROXIMATION: ZEROTH-ORDER SOLUTION FOR ACCRETION DISC

For a polytropic equation of state

$$P=K\rho^{(1+1/n)}, \tag{12}$$

taking the main term in the gravitational force g_z from (11), we get the solution of equation (3) (Hōshi 1977):

$$\rho=\rho_0\left(1-\frac{z^2}{z_0^2}\right)^n, \tag{13}$$

where the density in the equatorial plane, ρ_0, is related to r and z_0 by the relation

$$\rho_0=\left[\frac{GM}{2K(n+1)}\right]^n\frac{z_0^{2n}}{r^{3n}}. \tag{14}$$

Some physical quantities from equations (6)–(8) can be expressed in terms of r and ρ_0:

$$z_0=\left[\frac{2K(n+1)}{GM}\right]^{1/2}\rho_0^{1/2n}r^{3/2}=\beta_{z0}\rho_0^{1/2n}r^{3/2},$$

$$\Sigma=\sqrt{\pi}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\rho_0z_0=\beta_\Sigma\rho_0^{(2n+1)/2n}r^{3/2},$$

$$\mathscr{P}=\sqrt{\pi}\,\frac{\Gamma(n+2)}{\Gamma(n+\frac{5}{2})}K\rho_0^{1+1/n}z_0=\beta_{\mathscr{P}}\rho_0^{(2n+3)/2n}r^{3/2}, \tag{15}$$

$$u_{s0}=\left[\frac{(n+1)}{n}K\right]^{1/2}\rho_0^{1/2n}=\beta_{s0}\rho_0^{1/2n},$$

where

$$\beta_\Sigma=\sqrt{\pi}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\left[\frac{2K(n+1)}{GM}\right]^{1/2},$$

and

$$\beta_{\mathscr{P}}=\sqrt{\pi}\,\frac{\Gamma(n+2)}{\Gamma(n+\frac{5}{2})}K\left[\frac{2K(n+1)}{GM}\right]^{1/2}. \tag{16}$$

For the isothermal case, corresponding to $n=\infty$,

$$P=K\rho, \tag{17}$$

and we have, instead of (13),

$$\rho=\rho_0\exp\left(-\frac{GMz^2}{2Kr^3}\right)=\rho_0\exp\left(-\frac{z^2}{z_i^2}\right),\qquad z_i=\left(\frac{2Kr^3}{GM}\right)^{1/2}. \tag{18}$$

Formally, an isothermal disc has infinite thickness z_0, but the density falls exponentially with the characteristic height z_i, from (18). Instead of (15), we have

$$\Sigma = \left(\frac{2\pi K}{GM}\right)^{1/2} \rho_0 r^{3/2},$$

$$\mathscr{P} = K\left(\frac{2\pi K}{GM}\right)^{1/2} \rho_0 r^{3/2}, \qquad u_{s0} = \sqrt{K}. \tag{19}$$

Equation (2) can be integrated, giving (Shakura 1973)

$$\frac{\dot{M}}{2\pi}(j - j_0) = \alpha \Sigma u_{s0} z_0 r^3 \frac{d\Omega}{dr}. \tag{20}$$

The integration constant j_0, after multiplication by $\dot{M}$, gives the total (advective plus viscous) flux of angular momentum within the accretion disc. A positive value of j_0 corresponds to a negative total flux (due to a negative $\dot{M}$), which means that the central body accretes angular momentum. When j_0 is negative, the central body decreases its total angular momentum, while increasing its mass.

For a thin disc, neglecting u_r and $\mathscr{P}$ and taking only the main term for g_r from (11), we get, from (1),

$$\Omega^{(0)} = \left(\frac{GM}{r^3}\right)^{1/2} = \Omega_K, \qquad j^{(0)} = j_K = (GMr)^{1/2}. \tag{21}$$

The solution (21) is used up to the inner boundary of the accretion disc at $r = r_{in}$. The integration constant j_0 can be scaled by the Keplerian angular momentum $j_K(r_{in})$, so that

$$j_0 = \xi j_K(r_{in}) = \xi\sqrt{GMr_{in}}. \tag{22}$$

Substituting (22) into (20) we get, taking into account (16) and (21), the relation for the density in the equatorial plane:

$$\rho_0^{(0)} = b_{\rho 0}\left[-\frac{\dot{M}}{r^3}\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right)\right]^{2n/(2n+3)},$$

$$b_{\rho 0} = \left[\frac{1}{6\pi^{3/2}\alpha}\frac{\sqrt{n}}{(n+1)^{3/2}}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{GM}{K^{3/2}}\right]^{2n/(2n+3)}. \tag{23}$$

By using (23) in (15), we get

$$z_0^{(0)} = b_{z0}\left[-\dot{M}\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right)\right]^{1/(2n+3)} r^{(3/2)(2n+1)/(2n+3)},$$

$$\Sigma^{(0)} = b_\Sigma\left[-\dot{M}\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right)\right]^{(2n+1)/(2n+3)} r^{-(3/2)(2n-1)/(2n+3)},$$

$$\mathscr{P}^{(0)} = b_{\mathscr{P}}\left(-\frac{\dot{M}}{r^{3/2}}\right)\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right), \tag{24}$$

$$u_{s0}^{(0)} = b_{s0}\left[-\frac{\dot{M}}{r^3}\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right)\right]^{1/(2n+3)},$$

where

$$b_{z0} = [2(n+1)]^{1/2}\left[\frac{1}{6\pi^{3/2}\alpha}\frac{\sqrt{n}}{(n+1)^{3/2}}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{1/(2n+3)} \times K^{n/(2n+3)}(GM)^{-(2n+1)/[2(2n+3)]},$$

$$b_\Sigma = [2\pi(n+1)]^{1/2}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\left[\frac{1}{6\pi^{3/2}\alpha}\frac{\sqrt{n}}{(n+1)^{3/2}} \times \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{(2n+1)/(2n+3)} K^{-2n/(2n+3)}(GM)^{(2n-1)/[2(2n+3)]},$$

$$b_{\mathscr{P}} = \frac{\sqrt{nGM}}{\alpha 3\pi\sqrt{2}(n+\frac{3}{2})}, \qquad b_{s0} = b_{\rho 0}^{1/2n}\left(K\frac{n+1}{n}\right)^{1/2}. \tag{25}$$

In the formula for viscosity coefficient (5), z_0 must be replaced by z_i from (18) in the isothermal case,

$$\mu = \alpha\rho u_{s0} z_i. \tag{26}$$

Then, instead of equations (23)–(25), we get a solution

$$\rho_0^{(0)} = \frac{GM}{6\pi^{3/2}\alpha K^{3/2}}\left(-\frac{\dot{M}}{r^3}\right)\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right),$$

$$\Sigma^{(0)} = \frac{\sqrt{GM}}{\alpha 3\pi\sqrt{2K}}\left(-\frac{\dot{M}}{r^{3/2}}\right)\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right), \tag{27}$$

$$\mathscr{P}^{(0)} = \frac{\sqrt{GM}}{\alpha 3\pi\sqrt{2}}\left(-\frac{\dot{M}}{r^{3/2}}\right)\left(1 - \xi\sqrt{\frac{r_{in}}{r}}\right).$$

The formulae for the values of z_i in (18) and u_{s0} in (19) remain the same in all approximations for the isothermal case.

For accretion on to a black hole in the presence of a free inner boundary with $\xi = 1$, the zeroth-order approximation is not valid in the vicinity of r_{in}, where other terms in (1) must be taken into account. For accretion on to a slowly rotating star with angular velocity smaller than the Keplerian velocity on the equator, the drop in the angular velocity, from Keplerian in the disc to the stellar velocity in the equatorial regions, takes place in a thin boundary layer, which must be considered separately, with proper account taken of the pressure term in (1).

4 STRUCTURE OF THE BOUNDARY LAYER: FITTING WITH THE DISC SOLUTION

Inside the BL, variables change considerably over the small thickness of the layer $H_b \ll r_{in}$. Matter has no room to accelerate in the radial direction, so the radial velocity term in (1) is negligible, but the pressure term is comparable to the gravitational and centrifugal forces (Regev 1983; Papaloizou & Stanley 1986; Regev & Hougerat 1988). The thickness H_b of the BL is smaller than its vertical size z_0, which also remains small. The adopted inequalities for the boundary layer parameters

$$H_b \ll z_0 \ll r_* \tag{28}$$

will be confirmed by the results. The radius of the star, r_*, differs from the radius at which $d\Omega/dr = 0$, by the very small value H_b. In the asymptotic consideration of the BL, we use r_* as an inner boundary for the disc solution

$$r_{in} = r_*. \tag{29}$$

The variable x in

$$r = r_* + \delta x, \qquad \delta = \frac{H_b}{r_*} \ll 1 \tag{30}$$

is used inside the BL, instead of r. The inner solution within the BL is sought in the region $0 < x < \infty$, while the outer solution, from (23) and (24), is valid in $r_* < r < \infty$. According to the method of matched asymptotic expansion (see Nayfeh 1973), the inner and outer solutions are fitted so that values for the outer solution at $r = r_*$ are equal to the corresponding values of the inner solution at $x = \infty$. This condition is valid asymptotically at $\delta \to 0$.

Consider a stationary accretion disc BL where equations (1), (4) and (20) are valid, with the integration constant ξ_b for ξ in (22). The thickness of the disc remains small in the boundary layer, so relations (12)–(16) for the polytropic case, and (17)–(19) for the isothermal case, are valid. Taking account of only the main terms in the asymptotic expansion inside the BL, we get the equations, from (1) and (20) (Regev 1983),

$$\frac{d\mathscr{P}}{dx} = -\Omega_{K*}^2 H_b \Sigma (1 - \omega^2), \tag{31}$$

$$\frac{d\omega}{dx} = -\frac{\dot{M} H_b}{2\pi \Sigma \nu_b r_*^2}(\xi_b - \omega). \tag{32}$$

Here

$$\omega = \frac{\Omega}{\Omega_{K*}}, \qquad \Omega_{K*}^2 = \frac{GM}{r_*^3}. \tag{33}$$

The viscosity coefficient μ in the BL is expressed through the coefficient of kinematical viscosity ν_b:

$$\mu = \rho \nu_b. \tag{34}$$

While the radial extent of the BL, H_b, is much less than its vertical size z_0, the formula (5) cannot be used for a viscosity coefficient approximation. Inside the BL, we thus use the α-approximation in the form

$$\nu_b = \alpha_b u_{s0} H_b. \tag{35}$$

4.1 Polytropic case

It is convenient to use the variables (Σ, ω) in the equations (31) and (32). We get, from (15),

$$\mathscr{P} = d_{\mathscr{P}} \Sigma^{(2n+3)/(2n+1)},$$

$$u_{s0} = d_{s0} \Sigma^{1/(2n+1)}, \tag{36}$$

$$d_{s0} = \left[\frac{(n+1)}{n} K\right]^{1/2} \left[\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{1/(2n+1)} \times \left[\frac{GM}{r_*^3 K(n+1)}\right]^{1/2(2n+1)},$$

$$d_{\mathscr{P}} = \frac{(n+1)}{(n+\frac{3}{2})} K \left[\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{2/(2n+1)} \times \left[\frac{GM}{r_*^3 K(n+1)}\right]^{1/(2n+1)} = \frac{n}{(n+\frac{3}{2})} d_{s0}^2. \tag{37}$$

From the matching conditions in MAE, we need to get the Keplerian angular velocity $\Omega = \Omega_{K*}$, $\omega = 1$ from the inner solution at $x \to \infty$ in order to fit the outer solution (21) at the inner boundary $r = r_*$. This implies $\xi_b = 1$ for the constant in (32). We get, after transition to the variable Σ in (31) and (32),

$$\frac{d\Sigma}{dx} = -\frac{1}{d_{\mathscr{P}}}\left(\frac{2n+1}{2n+3}\right) \Omega_{K*}^2 H_b (1 - \omega^2) \Sigma^{(2n-1)/(2n+1)}, \tag{38}$$

$$\frac{d\omega}{dx} = -\frac{\dot{M}}{2\pi \alpha_b r_*^2 d_{s0}} (1 - \omega) \Sigma^{-2(n+1)/(2n+1)}. \tag{39}$$

Dividing (38) by (39), we get

$$D_b \frac{d\Sigma}{d\omega} = -(1 + \omega) \Sigma^{(4n+1)/(2n+1)}, \tag{40}$$

$$D_b = -\left(\frac{2n+3}{2n+1}\right) \frac{d_{\mathscr{P}}}{d_{s0}} \frac{\dot{M}}{2\pi \alpha_b H_b} \frac{1}{r_*^2 \Omega_{K*}^2}.$$

The solution of (40) must fit the boundary condition on the stellar surface $\Sigma = \Sigma_*$ at $\omega = \omega_*$. Taking into account that the surface density rapidly grows into the star, we may put, with sufficient accuracy, $\Sigma_* = \infty$ and obtain the solution, using (36) and (40), in the form

$$\Sigma = d_\Sigma \frac{\sqrt{K \Omega_{K*}^{1/n}}}{(\Omega_{K*} r_*)^{(2n+1)/n}} \left(-\frac{\dot{M}}{\alpha_b H_b}\right)^{(2n+1)/2n} \times \left[(\omega - \omega_*)\left(1 + \frac{\omega + \omega_*}{2}\right)\right]^{-(2n+1)/2n},$$

$$d_\Sigma = \sqrt{n+1}\, n^{-(2n+1)/4n} (2\pi)^{-(4n+3)/4n} \left[\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{1/2n}. \tag{41}$$

For fitting of the inner BL solution and the outer solution for the accretion disc, we must make the surface densities (41) at $\omega = 1$, and (24) at $r = r_{in}$ equal. This fitting uniquely determines the outer integration constant ξ. After some algebraic calculations, we get

$$1 - \xi = d_n \alpha \alpha_b^{-(2n+3)/2n} \left(\frac{r_*}{H_b}\right)^{(2n+3)/2n} \left[-\frac{\dot{M} K^n}{r_*^2 (\Omega_{K*} r_*)^{2n+1}}\right]^{3/2n} \times [(1 - \omega_*)(3 + \omega_*)]^{-(2n+3)/2n},$$

$$d_n = 2^{(2n+3)/2n} 3\sqrt{\pi} (2\pi)^{-(2n+9)/4n} (n+1)^{3/2} \left[\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{3/2n}. \tag{42}$$

The value of H_b is still not determined. It must be found from equation (39), where Σ is substituted from the solution (41). Before doing this, we estimate by order of magnitude the values of u_{s0}/u_K from (15), and u_r/u_K from (20):

$$u_K=\Omega_K r, \qquad \frac{u_{s0}}{u_K}\sim\frac{z_0}{r},$$

$$\frac{u_r}{u_K}\sim\alpha\left(\frac{z_0}{r}\right)^2\left(1-\xi\sqrt{\frac{r_{\rm in}}{r}}\right)^{-1}. \tag{43}$$

The equation (39) contains a non-physical logarithmic divergence, a result of using the MAE method. For an approximate estimation of the value of H_b, we use a characteristic thickness over which the ω-variation happens, and for this, using the definition (30), the following relation may be written:

$$-\frac{2\pi\alpha_b r_* d_{s0}}{\dot M}\Sigma^{2(n+1)/(2n+1)}|_{\omega=1}=1. \tag{44}$$

With the help of (36) and (41), we get, from (44),

$$\frac{H_b}{r_*}\approx d_H\alpha_b^{-1/(n+1)}\left[-\frac{\dot M K^n}{r_*^2(\Omega_{K*}r_*)^{2n+1}}\right]^{1/(n+1)}$$

$$\times[(1-\omega_*)(3+\omega_*)]^{-1},$$

$$d_H=2(2\pi)^{-3/[2(n+1)]}n^{-(2n+1)/[2(n+1)]}(n+1)^{n/(n+1)}$$

$$\times\left[\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{1/(n+1)}. \tag{45}$$

Using (45) in (42), we finally get the expression for $(1-\xi)$:

$$1-\xi\approx d_\xi\alpha\alpha_b^{-(2n+3)/[2(n+1)]}\left[-\frac{\dot M K^n}{r_*^2(\Omega_{K*}r_*)^{2n+1}}\right]^{1/[2(n+1)]}, \tag{46}$$

$$d_\xi=d_n d_H^{-(2n+3)/2n}=3\sqrt{\pi}(2\pi)^{-(2n+5)/[4(n+1)]}n^{1/[4(n+1)]}$$

$$\times(n+1)^{n/[2(n+1)]}\left[\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right]^{1/[2(n+1)]}.$$

Estimating the values by order of magnitude, taking account of (43), we get, combining (45) and (46),

$$(1-\xi)\sim\frac{\alpha}{\alpha_b}\frac{z_0}{r_*}, \qquad \frac{H_b}{r_*}\sim\frac{1}{1-\omega_*}\left(\frac{z_0}{r_*}\right)^2. \tag{47}$$

Thus we get a complete analytical solution for the accretion disc structure with the boundary layer near the star, for a viscosity inside the layer from (35), and a polytropic equation of state everywhere.

Analytical solutions for an accretion disc with boundary layer, with a simplifying assumption about the viscosity coefficient and temperature distribution, were obtained directly, without expansions, by Colpi et al. (1991) and Glatzel (1992). The MAE method, used by Regev (1983) for the solution of the accretion disc BL problem, enables one to obtain a self-consistent analytical solution for a disc with boundary layer, for a viscosity coefficient in the usual α-representation.

4.2 Isothermal case

For an isothermal disc with the same viscosity as in (35), we get, using (19), (31) and (32),

$$\mathscr{P}=K\Sigma,$$

$$\frac{d\Sigma}{dx}=-\frac{\Omega_{K*}^2}{K}H_b\Sigma(1-\omega^2), \tag{48}$$

$$\frac{d\omega}{dx}=-\frac{\dot M(1-\omega)}{2\pi\alpha_b\sqrt{K}\Sigma r_*^2}.$$

Dividing the last two equations, and using the approximate boundary condition $\Sigma_*=\infty$, we get the solution (Shakura & Sunyaev 1988)

$$\Sigma=\left(-\frac{\dot M\sqrt{K}}{2\pi\alpha_b H_b}\right)\frac{1}{r_*^2\Omega_{K*}^2}\left[(\omega-\omega_*)\left(1+\frac{\omega+\omega_*}{2}\right)\right]^{-1}. \tag{49}$$

From (48) and (49), we obtain the characteristic scale, which we identify with H_b:

$$\frac{H_b}{r_*}\approx\frac{2K}{r_*^2\Omega_{K*}^2}[(1-\omega_*)(3+\omega_*)]^{-1}. \tag{50}$$

Matching (27) and (49), taking account of (50) as for a polytrope, we get the expression for the integration constant

$$1-\xi=\frac{3}{\sqrt{2}}\frac{\sqrt{K}}{r_*\Omega_{K*}}\frac{\alpha}{\alpha_b}. \tag{51}$$

Comparing (50) and (51) with (47), we see that the order of magnitude estimates for the polytropic case become exact for the isothermal case, apart from a numerical coefficient close to unity.

In the isothermal case, there is a simple solution of equation (48), when (49) is substituted for Σ:

$$(1-\omega)^{-2}(\omega-\omega_*)^{(3+\omega_*)/(1+\omega_*)}(2+\omega+\omega_*)^{-(1-\omega_*)/(1+\omega_*)}$$

$$=\exp\left[2\frac{H_b}{K}\Omega_{K*}^2x(1-\omega_*)(3+\omega_*)\right]. \tag{52}$$

We can see from (52) that the scale (50) naturally appears in the exponent. It is clear that the solution has a physical sense only over several characteristic scales H_b. It is proved (see, for example, Nayfeh 1973, ch. 4, pp. 110–158) that the regions of applicability of the inner and outer solutions, obtained by the MAE method, do overlap and the formula, constructed from the inner (i) and outer (e) solutions, gives good interpolation also for the intermediate region. The formula, describing the function f in the whole region, has a structure

$$f=f_i+f_e-(f_i)_e. \tag{53}$$

Here, $(f_i)_e$ is the value of the inner solution on the outer edge [equal to the value of the outer solution on the inner edge $(f_e)_i$]. The solutions for $\Sigma(r)$ are constructed from $\Sigma^{(0)}(r)$ in (24) (e), and the solution of the equation (38), taking account of (30) and (41), for $\Sigma_i(r)$. The choice of ξ from (46) gives the equality $(\Sigma_i)_e=(\Sigma_e)_i$. The solution for $\Omega(r)$ is constructed using $\Omega_K(r)$ from (21) as (e), and the solution of (39), taking account of (30), (33) and (41), as $\Omega_i(r)$. It is clear from the above consideration that $(\Omega_i)_e = (\Omega_e)_i = \Omega_{K*}$.

The existence of the overlapping regions, where both solutions (external and internal) are valid, was first proved by Prandtl (1905; cited by Nayfeh 1973) for the problem of the boundary layer near the body in the flux of the rapidly moving viscid fluid, for which he invented the MAE method. The interpolation formula of the type (53) has been introduced by Vasil'eva (1959; cited by Nayfeh 1973), and is considered as a uniformly valid approximation in the whole region, due to existence of the overlapping regions. The direct proof of validity of (53), for the cases where general analytical solutions exist, can be found in the books of Nayfeh (1981, ch. 12) and Zwillinger (1992, section 135, pp. 510–517). These books, as well as Nayfeh (1973), contain examples of the application of MAE method, including quasi-linear equations of the type (1) and (2), and also references to many other applications of the MAE method and its detailed mathematical investigations.

Using the relation $\dot{M} = 2\pi r_* \Sigma v_r$ for the estimation of the radial velocity v_r in the boundary layer, we get, from (36), (37), (41) and (45) for polytropic [or from (49) and (50) for isothermal] cases, $v_r/v_{s0} \sim \alpha_b$. This means that the solution obtained, where radial velocity was neglected, is valid only for sufficiently small viscosity, with $\alpha \ll 1$.

5 DISCUSSION

A self-consistent analytical solution is obtained for the structure of a polytropic accretion disc, with the boundary layer near a slowly rotating star, for the usual α-viscosity law. It follows from the solutions obtained in (46) and (51) that the integration constant ξ is less than 1 by a small value, of the order of the parameter z_0/r_*, when the star rotates slowly. This corresponds to the fact that the total angular momentum flux over the accretion disc is less than the advective angular momentum flux at the inner boundary of the disc. Large deviations of the ξ-value from unity, $\xi = 0.61$, obtained by Regev (1983) for $\omega_* = 0.3$, could indicate the position in the region where the asymptotic expansion begins to be violated.

When the stellar angular velocity approaches a critical value, the boundary layer becomes thick [see (45)] and the MAE method fails. The solution for the accretion disc structure around a star rotating rapidly with Keplerian angular velocity at the equator was considered by other methods (Popham & Naryan 1991; Paczyński 1991). The self-consistent solution for the polytropic star–accretion disc system was obtained for this case by Bisnovatyi-Kogan (1993). The value of ξ can then differ considerably from unity and becomes negative for polytropic indices $n > 2.5$.

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REFERENCES

Bisnovatyi-Kogan G. S., 1993, A&A, 274, 796
Chapman S., Cowling T. G., 1939, The mathematical theory of non-uniform gases. Cambridge Univ. Press, Cambridge
Colpi M., Nanmurelli M., Calvani M., 1991, MNRAS, 253, 55
Glatzel W., 1992, MNRAS, 257, 572
Hōshi R., 1977, Prog. Theor. Phys., 58, 1191
Nayfeh A., 1973, Perturbation methods. John Wiley & Sons, New York
Nayfeh A., 1981, Introduction to Perturbation Techniques. John Wiley & Sons, New York
Paczyński B., 1991, ApJ, 370, 597
Papaloizou J. C. B., Stanley G. Q. G., 1986, MNRAS, 220, 593
Popham R., Narayan R., 1991, ApJ, 370, 604
Prandtl L., 1905, Proc. Third Int. Math. Kongr., Heidelberg, pp. 484–491
Regev O., 1983, A&A, 126, 146
Regev O., Hougerat A. A., 1988, MNRAS, 232, 81
Schade H., 1970, Kontinuumstheorie strömender Medien. Springer-Verlag, Berlin
Shakura N. I., 1973, SvA, 16, 756
Shakura N. I., Sunyaev R. A., 1988, Adv. Space Res., 8 (2), 135
Vasil'eva A. B., 1959, Math. Sbornik, 48, 311 (in Russian)
Zwillinger D., 1992, Handbook of Differential Equations. Academic Press, Boston