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An analytical model for a Keplerian disc and a boundary layer around a rapidly rotating star

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SUMMARY

We present an analytical model for a disc accreting on to a rapidly rotating star through a boundary layer. The disc is isothermal along the radial direction, and a polytropic equation of state is used to model the transport of energy along the vertical direction; the kinematic viscosity of the fluid is constant.

If the star rotates near breakup, the disc angular velocity close to the stellar surface has no maximum, but it settles smoothly to the stellar value, the flow always remaining subsonic. The process is accompanied by a negligible transfer of angular momentum from the disc to the star: both features are general and do not depend on the model assumptions.

We are able to determine the analytical relation between the specific angular momentum accreted by the star j_* and the stellar angular velocity Ω_*, for our isothermal model. The comparison with the recent calculations of Paczyński and Popham & Narayan has then shown that the magnitude of $j_*(\Omega_*)$ and its functional form depend rather sensitively on the way energy is transported in the disc along the radial direction, and on the viscosity law.

The result might be relevant for models of neutron star formation from accretion induced collapse of massive rapidly rotating white dwarfs.

1 INTRODUCTION

The study of the structure of accretion discs around compact objects is central to our understanding of many astrophysical sources such as X-ray binaries and cataclysmic variables (Pringle 1981).

It is generally believed that if the accreting object is stellar and it is slowly rotating, a narrow boundary layer forms between the disc and the star. In this region, the matter that grazes the stellar surface decelerates rather rapidly, the angular velocity decreasing from the near Keplerian to the stellar value. The falling of material through the boundary layer may result in a large deposition of energy that if liberated, can be as large as half of the accretion energy (Sunyaev & Shakura 1986). In the case of white dwarf accretors (non-magnetic cataclysmic variables) this energy can emerge in UV or X-rays depending on the accretion rate (Pringle 1977; Pringle & Savonije 1979; Tylenda 1981; Regev & Hougerat 1988; Van der Woerd 1988; Regev & Shara 1989). A further consequence of accretion is the transport of angular momentum from the disc to the star that can be spun up to large angular velocities. If the star rotates just near breakup, the weak unbalance between the disc rotation and the stellar rotation might instead result in a small dissipation of energy and in a negligible or even negative deposition of angular momentum (Lynden-Bell & Pringle 1974; Paczyński 1991; Popham & Narayan 1991).

Most calculations of the structure of thin boundary layers examined the disc equations in the neighbourhood of the star's radius either using the method of asymptotic matching expansion (Regev 1983; Regev & Hougerat 1988) or solving the equations with the inclusion of an outer boundary condition that accounts for the presence of a Keplerian disc farther out from the star (Shakura & Sunyaev 1988). This procedure, although quite general, may not fully exploit the whole information contained in the original equations. As a result, some consistency relation among the physical parameters of the model is missing in the analysis. Of importance is the relation between the stellar angular velocity Ω_* and the disc angular momentum $\dot{J}_*$ accreted by the star, and it is a direct consequence of the condition of angular momentum conservation in the disc. This correspondence provides information on the magnitude and sign of the accreted angular momentum $\dot{J}_*$.

Recently, the problem of relating $\dot{J}_*$ and Ω_* for a star accreting through a geometrically thin disc has been tackled (Paczyński 1991; Popham & Narayan 1991, hereafter PN). These authors explored a polytropic α model treating the central object, the boundary layer and the disc as a unique structure. Examining this relation they found that accretion of mass is possible even when the star is rotating slightly above breakup, since angular momentum is transported outward from the star to the disc through viscous stresses (equivalently, $\dot{J}_* < 0$). This result is of particular relevance in the context of models for the formation of neutron stars from accretion induced collapse of massive white dwarfs [we defer to PN and Narayan & Popham (1989) for a discussion of the problem].

In this paper, we present an analytical model for the structure of a steady geometrically thin disc accreting on to a very rapidly rotating star through a boundary layer, in a self-consistent way: the purpose is to determine the analytical relation between $\dot{J}_*$ and Ω_*. For this reason, we introduce a number of approximations: we first simplify the thermal structure of the disc–boundary layer introducing a polytropic relation between the pressure and the density along the vertical direction, and we assume that efficient mechanisms of energy transfer along horizontal direction are able to keep the accreting matter isothermal. We also neglect viscosity gradients and the effects of a radial motion, the flow maintaining always subsonic (we refer to Shakura & Sunyaev 1988). Under these simplifying assumptions, the accretion process is described by a one-dimensional model in which all physical quantities are vertically integrated and the laws of mass, momentum and angular momentum are expressed by a set of ordinary non-linear differential equations. The star is uniformly rotating; its angular velocity Ω_* is a free parameter and the radius R_* and mass M_* enter as normalization factors.

The study of this basic model has offered the possibility of determining the properties of the accretion flow in a consistent way and in the particular limit of rapid rotation, via analytical techniques. We then concentrate on two questions. (i) How sensitive is the disc–boundary layer structure on the processes of energy transfer? (ii) How much disc angular momentum the star can accrete and how does it depend on the assumptions of the model? In considering these two questions, we will rely on the comparison models developed by Paczyński (1991) and PN.

The paper is organized as follows: in Section 2 we describe the model. We derive the equations that are found to depend on two parameters: the disc angular momentum $\dot{J}_*$ accreted by the star and the ratio λ of the Keplerian velocity to the sound velocity in R_*. We show that the Keplerian disc is the asymptotic solution to the model equations; this introduces a unique relation between $\dot{J}_*$, λ and the star's angular velocity Ω_*. In Section 3 we provide the analytical solution to the equations valid in the limit of very rapid rotation and we determine the profile of the averaged density and angular velocity. The expression for $\dot{J}_*$ as a function of Ω_* and λ is derived and we show that disc models with negative j_* rotate supercritically. We then solve numerically the equations in the slow-rotation regime for various values of λ and j_*. In Section 4 we compare our results with those of Paczyński and PN, and we finally present the conclusion.

2 THE MODEL

2.1 Model equations

In this section we describe a model of an accretion disc and a boundary layer around a rotating star. We introduce a system cylindrical coordinates (R, z) with origin at the star's centre; R is the distance from the rotation axis (along z), and z the distance from the equatorial plane where the disc lies. The star has radius R_*, mass M_*, and rotates uniformly with angular velocity Ω_*. The accretion is stationary and the mass flow $\dot{M}$ is a constant. We derive the model equations treating the disc–boundary layer as a unique structure.

The equations for vertical hydrostatic equilibrium and mass conservation are

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GM_* z}{R^3} \tag{1}$$

$$\dot{M} = -2\pi \int_{-z_0}^{z_0} dz\, \rho R u_{\rm R}, \tag{2}$$

where $u_{\rm R}$ is the radial component of the velocity vector, and $z_0 = z_0(R)$ the vertical height of the disc, i.e. the distance above the disc plane at which the density ρ and pressure P vanish. The motion of matter that slowly spirals inwards, is described by the radial momentum equation and by the equation for angular momentum conservation that read

$$\Omega^2 R = \frac{GM_*}{R^2} + \frac{1}{\rho}\frac{\partial P}{\partial R} \tag{3}$$

$$\rho R u_{\rm R} \frac{\partial}{\partial R}(\Omega R^2) = \frac{\partial}{\partial R}\left(\rho \nu_{\rm t} R^3 \frac{\partial \Omega}{\partial R}\right), \tag{4}$$

with Ω the angular velocity of the accreting material and $\nu_{\rm t}$ the kinematic viscosity. In equation (3) radial hydrostatic equilibrium is maintained by the balance of the gravitational acceleration with the centrifugal acceleration and the pressure gradient. This last term, $\rho^{-1}\partial P/\partial R$, introduces the deviation from a purely Keplerian motion and it becomes important just close to the star's surface where the centrifugal support of the disc is weakened as Ω decreases to the stellar value Ω_*. Accretion is made possible by viscosity that transports angular momentum outwards. Here, we neglect viscosity gradients: thus, $\nu_{\rm t}$ is taken to be *constant* throughout the disc–boundary layer structure [we defer to Shakura & Sunyaev (1988), and Papaloizou & Stanley (1986) for an extended discussion on the viscosity]. The above equations (1–4) are derived under the assumption that the disc–boundary layer is geometrically thin ($z_0/R \ll 1$) and that the radial flow is subsonic (see Regev 1983; PN).

To close the system of equations we need to specify the thermal structure of the disc–boundary layer. In this paper we assume that a polytropic relation between the pressure and density holds along the vertical direction;

$$P = K\rho^{(1+1/n)}, \tag{5}$$

where n is the polytropic index and K is a function of R. We further require that the disc is *isothermal* along radial direction: this implies the presence of very efficient mechanisms of energy transport that dissipate the accretion energy to keep matter nearly isothermal. Under these simplifying assumptions an analytical and self-consistent description of the disc–boundary layer structure is possible.

Vertical integration of equation (1) with the polytropic relation (5) gives automatically the density as a function of z;

$$\rho(R,z)=\left[\frac{GM_*}{K(R)2(n+1)}\frac{z_0^2(R)}{R^3}\right]^n\left[1-\left(\frac{z}{z_0(R)}\right)^2\right]^m. \tag{6}$$

For a perfect fluid, isothermal along R, we also have

$$\frac{P(R,0)}{\rho(R,0)}=\frac{GM_*}{2(n+1)}\frac{z_0^2}{R^3};\qquad \frac{P(R,0)}{\rho(R,0)}=\frac{k_B T}{m}=\text{const}, \tag{7}$$

where T is the temperature and m the atomic mass. Equations (6) and (7) then determine uniquely z_0 as a function of R:

$$z_0(R)=R^{3/2}\left[\frac{k_B T}{m}\frac{2(n+1)}{GM_*}\right]^{1/2}. \tag{8}$$

The remaining equations of structure (2–4) take the final form, after integrating along z;

$$\frac{d\ln\Sigma}{dR}=\frac{\Omega^2 R}{\sigma}-\frac{GM_*}{\sigma R^2} \tag{9}$$

$$\frac{d\Omega}{dR}=\frac{\dot{J}_*-\dot{M}\Omega R^2}{2\pi\nu_t\Sigma R^3} \tag{10}$$

$$\dot{M}=2\pi R|u_R|\Sigma, \tag{11}$$

where Σ is defined as

$$\Sigma(R)\equiv\int_{-z_0}^{z_0}\rho\,dz=2\rho(R,0)\,z_0 I_n \tag{12}$$

with $I_n=(2^n n!)^2/(2n+1)!$ (Hoshi 1984); $\dot{J}_*$ is a constant obtained from the radial integration of equation (4), and is equal to the difference between the rate of angular momentum advected inward by the accreting fluid and the rate of angular momentum transported outward by shear stresses. We note that there is no constraint on the magnitude and sign of $\dot{J}_*$ whose value is determined by imposing the appropriate boundary conditions.

In equation (9), σ is the averaged pressure to density ratio $\sigma\equiv W/\Sigma$ where

$$W(R)\equiv\int_{-z_0}^{z_0}P\,dz=2P(R,0)z_0 I_{n+1}; \tag{13}$$

As the disc–boundary layer is isothermal, σ is a constant and gives a measure of the sound speed. We note that the polytropic exponent enters only the equations (12) and (13) that just give the vertically averaged values of the density and pressure.

The model equations (9) and (10) for the unknown $\Omega(R)$, $\Sigma(R)$ are first order non-linear differential equations, in the parameters $\dot{J}_*$ and σ. Their solutions can be determined once the value of Ω and Σ are given at some radius R_0; $u_R(R)$ is then determined from the solution to equations (9–10) using equation (11).

2.2 Normalization

Equations (9) and (10) can take a simple dimensionless form, once the following normalizations are introduced:

$$R=R_* x;\qquad \Sigma=\frac{\dot{M}}{2\pi\nu_t}\tilde{\Sigma};\qquad \Omega=\Omega_K^*\omega\equiv\left(\frac{GM_*}{R_*^3}\right)^{1/2}\omega \tag{14}$$

and

$$\lambda=\frac{GM_*}{R_*}\frac{1}{\sigma};\qquad \dot{J}_*=\dot{M}(GM_*R_*)^{1/2}j_*. \tag{15}$$

The constant λ is proportional to the square of the Keplerian to sound velocity ratio as evaluated at the star's surface; since $z_0/R\sim(x/\lambda)^{1/2}$, the thin disc approximation requires $\lambda\gg 1$. The constant j_* is the dimensionless angular momentum deposited by the disc on to the star. Equations (9) and (10) then read:

$$\frac{d\ln\tilde{\Sigma}}{dx}=\lambda\left(\omega^2 x-\frac{1}{x^2}\right) \tag{16}$$

$$\frac{d\omega}{dx}=\frac{j_*-x^2\omega}{\tilde{\Sigma}x^3} \tag{17}$$

For a particular choice of j_*, i.e. for $j_*=0$, the solution to equations (16) and (17) is the *Keplerian* solution for which

$$\omega=x^{-3/2};\qquad \tilde{\Sigma}=\tilde{\Sigma}_0=2/3. \tag{18}$$

Thus, the model equations retains two key relations of the steady thin discs (see Frank, King & Raine 1985). For $j_*\neq 0$, one can easily verify that this solution is an *asymptotic* solution to equations (16) and (17): one has indeed

$$\omega=\frac{1}{x^{3/2}}-\frac{2j_*}{\tilde{\Sigma}_0 x^2}\qquad \text{for}\qquad x\gg 1 \tag{19}$$

and

$$\tilde{\Sigma}=\tilde{\Sigma}_0+\frac{8}{3}\frac{j_*\lambda}{x^{3/2}}\qquad \text{for}\qquad x\gg 1. \tag{20}$$

This property allows us to determine in a unified way the structure of the boundary layer (at $x\sim 1$) and of the disc, Keplerian at large x: the solution to the model equations is uniquely determined when the values of ω and $\tilde{\Sigma}$ are given at some outer radius x_0 according to equation (19) and (20). The distinction between disc and boundary layer can then be made only if desirable.

It is apparent that of the four degrees of freedom [j_*, λ; $\omega(x_0)$ and $\tilde{\Sigma}(x_0)$] only j_* and λ specify uniquely the solution *if* we require the solution to be Keplerian at large radii ($x\gg 1$). Once j_* and λ are given, the functions $\omega[x;(j_*,\lambda)]$ and $\tilde{\Sigma}[x;(j_*,\lambda)]$ are then determined in the full range of integration $(1,\infty)$, and in particular at the star's surface ($x=1$). This yields a unique relation between Ω_* and (j_*,λ): $\Omega_*=\Omega_*(j_*,\lambda)$.

In the next section we carry out a detailed analysis of equations (16) and (17): we show that the equations verify appropriate scaling relations and on this basis we provide the analytical solution to the disc–boundary layer structure in the limit of very rapid rotation.

3 THE DISC–BOUNDARY LAYER STRUCTURE

3.1 Scaling relations

The model equations satisfy a particular scaling that allows λ to enter the model only as a normalization factor of the coordinate x. Setting

$$x=\lambda\chi \qquad \omega=\lambda^{-3/2}\omega' \qquad \tilde{\Sigma}=\tilde{\Sigma} \qquad j_*=\lambda^{1/2}j'_* \tag{21}$$

the equations (16) and (17) then read

$$\frac{d\ln\tilde{\Sigma}}{d\chi}=\chi\omega'^2-\frac{1}{\chi^2} \tag{22}$$

$$\frac{d\omega'}{d\chi}=\frac{j'_*-\chi^2\omega'}{\tilde{\Sigma}\chi^3}. \tag{23}$$

The equations (22) and (23) are now written in a form suitable for an analytical study.

3.2 The regime of rapid rotation

3.2.1 *The linear analysis*

In an accretion disc around a rotating star, the deviation from the Keplerian behaviour is expected to be significant only in the very vicinity of the star's surface where large pressure gradients establish, as Ω adjusts to the stellar value Ω_*. It is therefore useful to introduce two new variables that give the measure of this deviation:

$$U(\chi)\equiv 1-\omega'\chi^{3/2}; \qquad Z(\chi)\equiv\frac{1}{\tilde{\Sigma}_0}-\frac{1}{\tilde{\Sigma}}=\frac{3}{2}-\frac{1}{\tilde{\Sigma}}. \tag{24}$$

The model equations take the form

$$\frac{dZ}{d\chi}=\frac{U}{\chi^2}\left(Z-\frac{3}{2}\right)(2-U), \tag{25}$$

and

$$\frac{dU}{d\chi}=\frac{j'_*}{\chi^{3/2}}\left(Z-\frac{3}{2}\right)+\frac{Z}{\chi}(U-1). \tag{26}$$

If $Z(\chi)$ and $U(\chi)$ do not deviate significantly from Keplerian behaviour ($U\ll 1$ and $Z\ll 1$) they satisfy the *linearized* equations

$$\frac{dZ}{d\chi}=-3\frac{U}{\chi^2} \tag{27}$$

$$\frac{dU}{d\chi}=-\frac{3}{2}\frac{j'_*}{\chi^{3/2}}-\frac{Z}{\chi} \tag{28}$$

that can be cast into a unique equation for U:

$$\frac{d^2U}{d\chi^2}+\frac{1}{\chi}\frac{dU}{d\chi}-\frac{3U}{\chi^3}=\frac{3}{4}\frac{j'_*}{\chi^{5/2}}. \tag{29}$$

The most general solution to equation (29) is found to be the sum of the particular integral given by the modified Struve function $\boldsymbol{L}_0$, and of the two general integrals given by the modified Bessel functions $\boldsymbol{I}_0$ and $\boldsymbol{K}_0$:

$$U(\chi)=\sqrt{3}j'_*\left[\Gamma\left(\frac{3}{2}\right)\right]^2\boldsymbol{L}_0\left(2\sqrt{\frac{3}{\chi}}\right)+\mathscr{C}_1\boldsymbol{I}_0\left(2\sqrt{\frac{3}{\chi}}\right)+\mathscr{C}_2\boldsymbol{K}_0\left(2\sqrt{\frac{3}{\chi}}\right) \tag{30}$$

(the details of the derivation are contained in the Appendix).

Asymptotically, i.e. for $\chi\rightarrow\infty$, the behaviour of the function $\boldsymbol{I}_0$ is $\boldsymbol{I}_0(2\sqrt{3/\chi})\rightarrow 1$, indicating that a motion proportional to the Keplerian one is solution to the model equations; the function $\boldsymbol{K}_0$ diverges logarithmically, $\boldsymbol{K}_0(2\sqrt{3/\chi})\rightarrow -\ln(2\sqrt{3/\chi})$. Thus, if the solution $U(\chi)$ merges into the Keplerian solution ($U\rightarrow 0$ for $\chi\rightarrow\infty$), this forces the choice of $\mathscr{C}_1=0=\mathscr{C}_2$.

Restoring physical variables (equation 21 with 24), the solution to equation (29) with the correct asymptotic behaviour is

$$U(x)=\sqrt{3}\frac{j_*}{\lambda^{1/2}}\left[\Gamma\left(\frac{3}{2}\right)\right]^2\boldsymbol{L}_0\left(2\sqrt{\frac{3\lambda}{x}}\right)$$
$$=\sqrt{3}\frac{j_*}{\lambda^{1/2}}\frac{\pi}{4}\left(\frac{3\lambda}{x}\right)^{1/2}\sum_{i=0}^{\infty}\left(\frac{3\lambda}{x}\right)^i\frac{1}{[\Gamma(i+3/2)]^2} \tag{31}$$

(in equation (31) we used the identity $[\Gamma(3/2)]^2=\pi/4$). If only the first term of the series is retained, we recover the asymptotic expression for ω, in accordance with equation (19). The constant j_* is uniquely related to the value of U at the star's surface and it can be determined examining the behaviour of the Struve function $\boldsymbol{L}_0$ about $x\sim 1$ (i.e. for $\chi\equiv x/\lambda\rightarrow 0$, being $\lambda\gg 1$): denoting with U_* the value of U at $x=1$, we find the explicit expression of j_* that depends linearly on U_* and exponentially on λ:

$$j_*=\left\{\frac{2(\pi)^{1/2}}{3[\Gamma(3/2)]^2}\right\}U_*[3\lambda]^{3/4}\exp[-2(3\lambda)^{1/2}]. \tag{32}$$

About $x\sim 1$ the solution (30) takes the form

$$U(x)\sim U_*x^{1/4}\exp[2(3\lambda)^{1/2}(x^{-1/2}-1)]. \tag{33}$$

Expanding x about unity, i.e. setting $x=1+l$, we have $U(l)=U_*\exp[-(3\lambda)^{1/2}l]$: this indicates that the scale over which the velocity deviates from Keplerian behaviour is

$$l_\omega\sim(3\lambda)^{-1/2}\ll 1. \tag{34}$$

The formal solution for $Z(x)$ can be found from equation (28) and it is given in Appendix. About $x\sim 1$ the function $Z(x)\sim(3\lambda)^{1/2}U(x)^{-1/2}$, yielding

$$\tilde{\Sigma}\sim\tilde{\Sigma}_0\left[1-(3\lambda)^{1/2}\frac{2}{3}\left(1-\frac{\Omega_*}{\Omega^*_\mathrm{K}}\right)\right]^{-1}. \tag{35}$$

As $\tilde{\Sigma}>0$, the validity of equation (31) is restricted to the range

$$\sqrt{\lambda}U_*\equiv\sqrt{\lambda}\left(1-\frac{\Omega_*}{\Omega^*_\mathrm{K}}\right)<1. \tag{36}$$

The above analysis applies only when the star is rotating close to breakup, for $\Omega_* \sim \Omega_K^*[1-\lambda^{-1/2}]$. In this limit, the analytical solution to the linearized equation describes the disc–boundary layer structure in the full range $(1, \infty)$, and the relation $\Omega_*(j_*, \lambda)$ is given in analytical form by equation (32).

3.2.2 *The analytical results*

Equations (33–35) show that $U(x)$ and $Z(x)$ rise exponentially close to the star's surface, on a scale $\sim l_\omega$, and the density $\tilde{\Sigma}$ at $x=1$ remains finite, although very large. The angular velocity instead increases monotonically as $[1-U(x)]/x^{3/2}$, and deviates from Keplerian behaviour close to $x \sim 1$ where it remains constant and equal to ω_*. This feature comes naturally from our calculation, and the indication for such a behaviour was given by Lynden-Bell & Pringle (1974). We note that this smooth settling of ω to the stellar value is due to the large enhancement of the surface density $\tilde{\Sigma}$ about $x \sim 1$ that yields $d\omega/dx \sim 0$ in the boundary layer.

In the case of rapid rotation ($\Omega_* \sim \Omega_K^*[1-\lambda^{-1/2}]$) the result of our analysis shows that the accreted angular momentum j_* depends linearly on U_*; thus, $j_* \propto (1-\Omega_*/\Omega_K^*)$. As the Keplerian solution is the exact solution to the disc equations for $j_*=0$, j_* takes negative values only when the star is rotating above Ω_K^*. As a consequence of equation (31) the disc rotates everywhere supercritically yielding an unphysical result; negative values of j_* are therefore discarded. It is, however, apparent that star–disc models that are in *near* Keplerian rotation even for $j_*=0$ (as those of Paczyński and PN for which Keplerian rotation is reached only asymptotically) can produce self-consistent solutions with $j_*<0$, for $\Omega_*<\Omega_K^*$. Within our formalism, we note that near Keplerian discs can be obtained in correspondence of a positive value of the constant $\mathscr{C}_1$. At large distances, this choice gives an angular velocity proportional to the Keplerian value ($\Omega \rightarrow [1-\mathscr{C}_1]\Omega_K$). In this case, negative values of j_* correspond to sub-Keplerian solutions, provided $j_* > -(\lambda/3)^{1/2} 4\mathscr{C}_1/\pi$.

For our choice of $\mathscr{C}_1=0$, equation (32) shows further that j_* takes exceedingly small values being $j_* \propto \exp[-2(3\lambda)^{1/2}]$, and $\lambda \gg 1$. This extreme dependence of the magnitude of j_* on λ is an indirect consequence of the assumption that the disc is everywhere isothermal. The effect of j_* on the solution is to introduce a finite deviation from Keplerian behaviour near the stellar radius; this deviation is large even though j_* is exponentially small.

In our model, the radial flow is assumed to be subsonic. As $\tilde{\Sigma}$ increases exponentially in the boundary layer and the sound speed c_s is constant, the condition $|u_R|/c_s = \nu_t[R\tilde{\Sigma}\sqrt{\sigma}]^{-1} \ll 1$ is verified when

$$\nu_t \ll 7.7\times 10^{15}\,\frac{x}{\sqrt{\lambda}}\left(\frac{R_*^2\Omega_K^*}{1.2\times10^{16}\ \mathrm{cm^2\,s^{-1}}}\right) \times\left[1-(3\lambda)^{1/2}\frac{2}{3}\left(1-\frac{\Omega_*}{\Omega_K^*}\right)\right]^{-1}\ \mathrm{cm^2\,s^{-1}}. \tag{37}$$

In the case of *slow* rotation ($\Omega_*<\Omega_K^*$ or equivalently $U_* \lesssim 1$), the structure of the disc–boundary layer can only be determined numerically from equations (25) and (26). Nonetheless, the solution to the linearized equations should provide the correct behaviour of the solution to the complete equations at $x \gg 1$. Indeed, equation (31) is the asymptotic solution to equations (25) and (26) for the same choice of $\mathscr{C}_1$ and $\mathscr{C}_2$ (set equal to zero). In the next Section we present the results of the numerical integration to the full equations (25) and (26).

3.3 The regime of slow rotation: numerical analysis

We solve numerically the model equations (25) and (26) employing a Runge–Kutta integrator with adaptive stepsize control, and the analytical solution was used as a check on the numerical code. Introducing a suitable choice of variable we integrated the equations in the full range $(1, \infty)$.

We verified that the non-linear terms in equations (25) and (26) become important just when $U(x)$ grows exponentially, in the linear regime (equation 31). This enables us to estimate the range of validity of the linearized solution and to determine approximately the radial extension of the boundary layer, for a given value of j_*. We find that the solution to the full non-linear equations is well reproduced by equation (31) until

$$U(x) \sim j_*\,\frac{3(\pi)^{1/2}}{8}\exp[2(3\lambda/x)^{1/2}](3\lambda)^{-3/4}x^{1/4} \sim 1. \tag{38}$$

Therefore, the radius x_{bl} at which $U(x)$ deviates from Keplerian behaviour is

$$x_{bl} \sim \frac{12\lambda}{\ln^2(1/j_*)} \tag{39}$$

to logarithmic accuracy. Since the disc is geometrically thin ($\lambda \gg 1$), the constant j_* has to be small, i.e. $j_* \ll 1$ for x_{bl} to be close to unity.

From the numerical solution, we find that $Z(x)$ saturates quickly (about x_{bl}) to its maximum value of 3/2 for which $\tilde{\Sigma}=\infty$, then it remains constant until the stellar surface is approached. This step rise is consistent with equation (16) that gives a radial scale of variation

$$l_\Sigma \equiv \left(\frac{1}{\Sigma}\frac{d\Sigma}{dr}\right)^{-1} = \frac{R_*}{\lambda}\,[1-\Omega_*^2/\Omega_K^2(R_*)]^{-1} \ll 1 \tag{40}$$

(Papaloizou & Stanley 1986). The angular velocity $\omega(x)$ increases always monotonically deviating from Keplerian behaviour about x_{bl}; thereafter it remains constant and equal to ω_*. The effect of increasing j_* (λ is kept constant and large) acts only as to increase to very large values the radius at which the deviations from Keplerian behaviour become important, in accordance to equation (39): the density $\tilde{\Sigma}$ becomes infinitely large about x_{bl} while the angular velocity ω approaches monotonically the stellar value ω_*. In all cases examined (with $\lambda \gg 1$ and $j_* \lesssim 1$), the solution always describes a disc that accretes on a star of radius $x_{bl}R_* \gg R_*$, and it thus gives a physically unacceptable result.

We were therefore led to explore the solutions to equations (25–26) about $\lambda \sim 10$. In this case, the disc is isothermal and thin *locally*. Fig. 1 shows the result of the integration obtained for $\lambda=10$ and $j_*=10^{-3}$; we report ω and Z as a function of x. We find that the surface density is large but finite, and $\omega(x)$ joins monotonically the stellar value

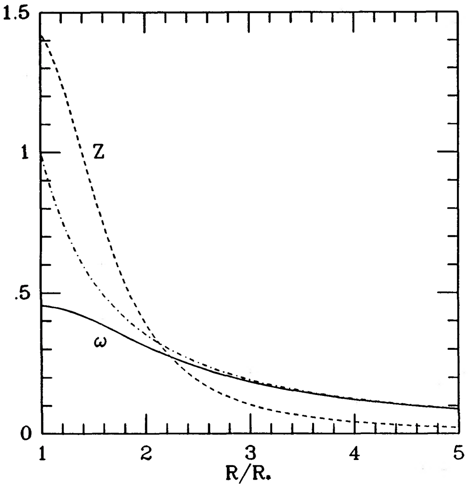

Figure 1. The solid line denotes ω as a function of R/R_*, and the dashed line denotes Z as a function of R/R_*; the model parameters are $\lambda=10$ and $j_*=10^{-3}$. The dashed-dotted line shows the Keplerian disc angular velocity $\omega_K \equiv (R/R_*)^{-3/2}$ as a function of R/R_*.

(the radial flow remaining always subsonic). The major effect of decreasing λ is to decrease x_{bl} while allowing for progressively higher values of j_* (equation 39); the solutions are physically acceptable. If we further decrease $\lambda \lesssim 5$ while simultaneously increasing $j_* \sim 1$, the angular velocity displays naturally a maximum about x_{bl}; it then drops rapidly to the stellar value $\omega_* \sim 0$. However, for these values of λ we are close to the limit of applicability of the thin disc approximation.

Since $\omega_* \sim x_{bl}^{-3/2}$, we can determine the $\omega_*(j_*, \lambda)$ relation from equation (39): we have

$$j_* - \exp[-2(\omega_*)^{1/3}\sqrt{3\lambda}]. \tag{41}$$

Thus, for $\lambda \sim 1$ and $\omega_* \ll 1$

$$j_* \sim 1 - 2\sqrt{\lambda/3}\,\omega_*. \tag{42}$$

In the case of very slow rotation ($\omega_* \ll 1$) the parameter j_* is close to unity, and matter accreting rather abruptly on the stellar surface deposits its near Keplerian angular momentum. With increasing ω_* in the interval (0, 1) j_* decreases rather continuously to vanish at $\omega_* = 1$.

4 DISCUSSION

In the above sections we explored our idealized isothermal model and determined in particular the disc angular velocity Ω as a function of radius R, and the relation between the stellar angular velocity Ω_* and the specific angular momentum accreted by the star j_*.

We have shown that near breakup j_* is proportional to $[1-\Omega_*/\Omega_K^*]$, and displays an exponential dependence on the parameter λ, the sound to Keplerian velocity ratio measuring the thickness of the disc (see equation 32). The same dependence of j_* on λ is retained in the slow rotation models. It is also found that j_* decreases monotonically with increasing Ω_*: the functional form of j_* on Ω_* is a new result of our study.

The comparison of our model with that of Paczyński and PN offers the possibility to a further analysis of the above relations, and we proceed on summarizing first the various approximations introduced in the three models.

In our analysis, we introduced three major assumptions: (1) that the disc–boundary layer is geometrically thin, (2) that efficient mechanisms of energy transfer keep the disc *isothermal* along radial direction (this condition may hold at least locally), and (3) that the kinematic viscosity is constant. In Paczyński's analysis of a thin disc the inclusion (in the radial motion equation) of terms subleading in z_0/R allows to treat consistently the matching between the disc and the star. Furthermore, the thermal energy transport along the radial direction is approximated by an *adiabatic* law between pressure and density, and the viscosity is modelled following the α-prescription. PN's model is similar to that of Paczyński but it includes the radial velocity gradient in the centrifugal force balance equation (equation 3) thus allowing for supersonic inflows. We then ask how the different approximations affect the $\Omega(R)$ and the $j_*(\Omega_*, \lambda)$ relations.

Here, we distinguish two regimes: (a) the regime of rapid rotation for which $\Omega_* \sim \Omega_K^*[1-\lambda^{-1/2}]$, and (b) the one of slow rotation for which $\Omega_* < \Omega_K^*$.

Rapid rotation. In this limit, the disc angular velocity $\Omega(R)$ close to the stellar surface has no maximum, but increases monotonically until $R \sim R_*$; thereafter it remains constant and equal to the stellar value. The flow is always subsonic and a negligible amount of disc angular momentum is transferred to the star. This behaviour is also reproduced by the adiabatic models of Paczyński and PN for which j_* is negligibly small. This seems to be a general feature of a disc–boundary layer structure around a rapidly rotating star.

Our model gives rise to a disc in 'strict' Keplerian rotation when $j_*=0$ due to conditions (2) and (3): as a consequence j_* has a positive definite sign (negative values of j_* would correspond to unstable discs in supercritical rotation, i.e. with $\Omega(R) > \Omega_K(R)$ for any R). In our model, the 'equilibrium' angular velocity Ω_{equil}^* defined as the value at which $j_*=0$ thus coincides with the Keplerian angular velocity in R_*, Ω_K^*. Paczyński and PN by contrast found physical solutions with negative j_* in correspondence of $\Omega_* < \Omega_K^*$. In the interval $\Omega_{equil}^* \le \Omega_* \le \Omega_K^*$ the solutions describe a 'nearly' Keplerian disc accreting the central star and close to R_* angular momentum is transported outward by viscous stresses. Within our formalism we have shown that self-consistent solutions with negative values of j_* are indeed possible, for which the disc at large distances is in sub-Keplerian rotation; the value of Ω_{equil}^*, however, depends on the detail of the model. In Paczyński and PN, Ω_{equil}^* is found to be slightly above $\Omega_{breakup}$, the critical value (smaller than Ω_K^*) at which the star loses its centrifugal equilibrium if it were isolated. It would be of some importance to know whether Ω_{equil} is always above $\Omega_{breakup}$ in a more general context.

Slow rotation. In our model, conditions (1) and (2) are satisfied for quite large values of the parameter λ. In this case, the boundary layer is found to have an unphysical radial extension, unless $j_* \sim 0$. If λ is decreased to values $\lesssim 10$ the conditions (1–3) apply locally, close to the stellar surface. In this case we find that the angular velocity increases monotonically with decreasing radius, and the amount of angular momentum accreted by the star is limited to values $j_* \lesssim 10^{-3}$. Only very close to the limit of validity of the approximations introduced (for $\lambda \sim 3$) $\Omega(R)$ displays a maximum followed by a rapid drop and $j_* \sim 1$. In the comparison models the presence of this maximum appears instead naturally for a wide interval of stellar angular velocities. Furthermore in these models, the functional form of the $j_*(\Omega_*)$ relation is very diverse if compared with equation (41), since j_* is found to take a rather stable value about unity (independent of Ω_*) until it drops suddenly below zero near breakup.

The above analysis indicates that the magnitude and functional dependence of j_* on Ω_* are determined not only by the form of the viscosity law but also by the way energy is transported (adiabatically or isothermally) in the disc, along radial direction. This study expresses the need for a careful examination of the disc–boundary layer structure in one-dimensional models that include a realistic treatment of the transport processes. In particular, a more accurate modelling of the disc–star boundary will enable us to determine the value of the 'equilibrium' rotational period above which the star angular momentum is transferred to the disc, as this problem is of particular relevance for neutron star formation models that advocate accretion induced collapse of a massive white dwarf.

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APPENDIX

In this section we sketch the derivation of the general solution to equation (29), written in terms of the rescaled variables (equation 21). Equation (29) can be recast into the following form

$$\frac{d^2F}{d\chi^2} + \frac{F}{4\chi^2} - \frac{3F}{\chi^3} = \frac{3}{4}\frac{j'_*}{4\chi^2} \qquad \text{(A1)}$$

where $F \equiv U\chi^{1/2}$. To determine the particular integral of equation (A1) we expand F is series of powers:

$$F(\chi) = \sum_{n=0}^{\infty} \frac{f_n}{\chi^n} \qquad \text{(A2)}$$

Combining equation (A1) with (A2), we find the expression for the coefficients f_n

$$f_n = 3^n f_0 \left(\frac{\Gamma(3/2)}{[\Gamma(n+3/2)]}\right)^2; \qquad f_0 = 3j'_* \qquad \text{(A3)}$$

yielding

$$F(\chi) = 3j'_*\left[\Gamma\left(\frac{3}{2}\right)\right]^2 \sum_{n=0}^{\infty} \left(\frac{3}{\chi}\right)^n \frac{1}{[\Gamma(n+3/2)]^2}. \qquad \text{(A4)}$$

The solution for $U(\chi) = F(\chi)/\chi^{1/2}$ is identified with the modified Struve function $\boldsymbol{L}_0$ (Abramowitz & Stegun 1964) and takes the form

$$U(\chi) = \sqrt{3}j'_*\left[\Gamma\left(\frac{3}{2}\right)\right]^2 \boldsymbol{L}_0\left(2\sqrt{\frac{3}{\chi}}\right)$$

$$= \sqrt{3}j'_*\left[\Gamma\left(\frac{3}{2}\right)\right]^2 \left(\frac{3}{\chi}\right)^{1/2} \sum_{n=0}^{\infty} \left(\frac{3}{\chi}\right)^n \frac{1}{[\Gamma(n+3/2)]^2}. \qquad \text{(A5)}$$

The general integral of equation (A1) (i.e. for $j'_* = 0$) can be found as follows. Let us introduce the new variables

$$\zeta \equiv 2\sqrt{3/\chi} \qquad \text{and} \qquad G \equiv F\zeta/2\sqrt{3}. \qquad \text{(A6)}$$

The function G satisfies the equation

$$\zeta\frac{d^2G}{d\zeta^2} + \frac{dG}{d\zeta} - \xi G = 0 \qquad \text{(A7)}$$

having for solution the modified Bessel functions of zero order $\boldsymbol{I}_0$ and $\boldsymbol{K}_0$, and of argument $2\sqrt{3/\chi}$ (we refer to Abramowitz & Stegun 1964). Restoring variables, the complete solution to equation (29) is given by equation (30).

The function $Z(\chi)$ can be determined from the following equation

$$Z(\chi) = -\frac{3}{2}\frac{j'_*}{\chi^{1/2}} - \chi\frac{dU}{d\chi}. \qquad \text{(A8)}$$

Using the recurrence relations of the Struve functions, we find

$$Z(\chi) = 9j'_*\left[\Gamma\left(\frac{3}{2}\right)\right]^2 \frac{1}{\chi^{3/2}} \sum_{n=0}^{\infty} \left(\frac{3}{\chi}\right)^n$$

$$\times \frac{1}{\Gamma(n+3/2)\Gamma(n+5/2)}. \qquad \text{(A9)}$$

In the limit $\chi \to 0$, the function $Z(\chi)$ has the following behaviour

$$Z(\chi) \to \sqrt{3}\,U(\chi)/\chi^{1/2}. \qquad \text{(A10)}$$

Asymptotically ($\chi \to \infty$), and restoring the physical dimensionless variables we recover the result of equation (20), just retaining the first term of the series in equation (A9).

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