400A - Diffusive energy transport

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Materials: Chapters 3 and 5 of Onno Pols' lecture notes, chapter 5 of Kippenhan & Weigert's book.

Energy transport in stars

Summary of where we are

Mass conservation:

$$\frac{dm}{dr} = 4\pi\rho r^2 \quad . \tag{1}$$

Hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho \quad , \tag{2}$$

N.B.: This is the particular form that momentum conservation takes since we can assume accelerations are $\sim$0 because stars do not evolve on timescales comparable to the dynamical timescale

Equation of state:

$$P \equiv P(\rho, \mu, T) = P_{\mathrm{rad}} + P_{\mathrm{gas}} = \frac{1}{3}aT^4 + \frac{\rho}{\mu m_u}k_B T + P_{\mathrm{QM}} \quad . \tag{3}$$

These three equations have as variables m $\equiv$ m(r) (or equivalently r$\equiv$ r(m)), P, ρ, and T. By adding an EOS to relate P and ρ (and the composition of the star which enters in the mean molecular weight μ), in the general case, we have also added a new variable T, so the system is not close again, and we need to look for one more equation to be able to solve it! In today's lecture we will actually derive two, but one of them does not cover all possible physical situations in stars.

Energy flow in stars

To find another equation, we can consider how energy flows in a star. We have already seen that the surface temperature (e.g., of the Sun), and the average temperature (estimated using the virial theorem) are *not* the same, so we know that the star is *not* in global thermal equilibrium. We also know, from similar arguments, that the average temperature is higher than the surface temperature, so there should be an energy flow from the interior outwards.

This energy flow can occur in three forms in a star:

1. **Diffusion**: thermal energy can be moved by the random motion of particles. This is often the dominant energy transport mechanism in a star – unless for some reason it is insufficient. The diffusion of energy can be:
    - **Radiative**, if the particles carrying the energy through their random motion are photons
    - **Conductive**: if it is the thermal motion of gas particles (typically electrons, unless you are in a neutron star) that carries the energy
2. **Convection**: in this case thermal energy is transported by bulk motion of matter. This occurs as an instability if the other means of energy transport are insufficient to carry the energy flux required by the local stellar structure, and it will be the topic of a future lecture.
3. **Neutrino losses**: except for collapsing iron cores and neutron stars, the stellar gas is transparent to neutrinos. As soon as one is produced, it can free stream out of the star carrying away energy effectively instantaneously (or more precisely on a timescale $\sim \mathrm{R}/\mathrm{v}_\nu \simeq \mathrm{R}/\mathrm{c}$ comparable to the light crossing time if neutrinos are non interacting and we neglect their mass, so they are ultra-relativistic particles with $\mathrm{v}_\nu \simeq \mathrm{c}$).

The next *two* stellar structure equations to add to our system will come from combining all of these together and applying conservation of energy.

Diffusion processes

In general, the "diffusion approximation" is useful to describe the net flux of "something" when the average path of the carrier of said "something" is small compared to the lengthscale over which the "something" is transported, that is the mean free path ℓ is much smaller than the size L of the region $\ell \ll$ L.

In this approximation, the net flux of this "something" is related to the density of "something" by Fick's law:

$$\mathbf{J} = -D\nabla n \quad , \tag{4}$$

where J is the flux (which is in general a vector quantity), D is the so called "diffusion coefficient", and ∇ n is the gradient of the volumetric density of the "something" we are considering. The negative sign appears because the flux is opposite to the gradient ("something" flows from where there is a lot to where there is less!). By dimensional analysis, regardless of what "something" is (and what are its dimensions), you can see that:

- [J] = [something]/(L^2t) with L length dimension and t time;
- [∇] $\equiv$ [d/dx] = 1/L;
- [n] = [something]/L^3

Therefore, the diffusion coefficient has the dimensions of [D] = L^2/t $\equiv$ v $\times$ L, so we can expect D to be proportional to the average velocity of the "something" that is diffusing times the length scale for the motion of this something (which by hypothesis is small compared to the length scale over which we want to study the motion of this "something"). In fact this is almost correct!

To get the correct pre-factor, let's say "something" are particles moving in 3 dimensions and focus on one direction, say z. If all these particles have a mean free path ℓ and an average velocity v, then the number that crosses a given coordinate z with a positive δ z as a function of time is:

$$\frac{dN}{dt}(z) = \frac{1}{2}n\frac{1}{3}v \quad , \tag{5}$$

where the first 1/2 factor comes from the fact that their motion is random, so half the particles have negative δ z and half have a positive one, and we just want to count the latter, and the factor of 1/3 in front of their average velocity comes from the fact that the motion is isotropic (see derivation of the pressure).

If there is a gradient ∂ n $/\partial$ z of particles along the z direction, the particles moving with positive δ z come from n(z-ℓ) while those moving with the negative δ z come from n(z+ℓ), so the net flux across z is

$$J = \frac{dN}{dt}(z-\ell) - \frac{dN}{dt}(z+\ell) = \frac{1}{6}v\left(n(z-\ell) - n(z+\ell)\right) \simeq \frac{1}{6}v\left(-2\ell\frac{\partial n}{\partial z}\right) = -\frac{1}{3}v\ell\frac{\partial n}{\partial z} \ , \tag{6}$$

where the only approximation we use is a first order Taylor expansion of the density n(z) assuming that the mean free path ℓ is small compared to the scale of interest! Eq. 4 is the generalization in 3 dimensions of the above, where $\nabla = (\partial_{\mathrm{x}}, \partial_{\mathrm{y}}, \partial_{\mathrm{z}})$ for Cartesian coordinates, and D=vℓ/3.

Energy transport by radiative diffusion

We have already calculated that mean free path for photons and estimated that it is very small compared to the typical size of stars (and the typical size of a resolution element in a numerical simulation of a star!). Therefore, we can treat the energy transport by photons in the diffusion approximation.

N.B.: if the star were a perfect black body, there would not be any transport of energy by photons, because by definition the radiation field would be isotropic, and the gradient of photon energy density would be zero! In reality, we have already seen that stars are *not* black bodies at the surface (in the atmospheric layers where ℓ_γ is not small) and neither they are in the interior because there is a small deviation from LTE of the order of ℓ_γ dT/dr~10^{-11}. While this is a small enough deviation that we can assume LTE to write down an EOS, it is also big enough to introduce a non-negligible flux of energy in the stars!

If the "something" that we are considering in our diffusion equation is energy, then in Eq. 4 J$\rightarrow$ F_{rad} is a energy flux of radiative energy, and n$\rightarrow$ u is the energy density. Moreover, in the diffusion coefficient D the mean velocity of photons is v$\rightarrow$ c, and we have already written $\ell_\gamma = 1/\kappa_{\mathrm{rad}}\rho$ as a function of ρ.

N.B.: today we will introduce different kinds of opacity κ, κ_{rad} is the one impeding the diffusion of photons.

Thus, the radiative diffusion equation is

$$F_{\mathrm{rad}} = -\frac{1}{3}\frac{c}{\kappa_{\mathrm{rad}}\rho}\frac{du}{dr} \ , \tag{7}$$

where we use the spherical symmetry of the problem to explicit the gradient and turn it into a total derivative. The radiation energy density is u=aT^4. We can then explicit these into our equation obtaining:

$$F_{\mathrm{rad}} = -\frac{4ac}{3c\rho T^3}\frac{1}{\kappa_{\mathrm{rad}}}\frac{dT}{dr} \quad , \tag{8}$$

which can be turned into an equation for the temperature gradient. This is a *local* quantity and it is valid in a region of the star where the dominant energy transport is radiative diffusion only:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\rho}{T^3}\kappa_{\mathrm{rad}}F_{\mathrm{rad}} \propto \kappa_{\mathrm{rad}}F_{\mathrm{rad}}. \quad . \tag{9}$$

In a radiative region the temperature is proportional to the opacity κ_{rad} times the radiative energy flux!

We can further rewrite the flux F_{rad} = $L_{\mathrm{rad}}/(4\pi\ r^2)$. This introduces the *local* luminosity $L_{\mathrm{rad}} \equiv L_{\mathrm{rad}}(r)$ which is the rate (that is per unit time) at which radiation transports energy through a surface of radius r within the star (or in other words, the "power" that is in the photon field at the location r):

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\rho\kappa_{\mathrm{rad}}}{r^2}\frac{L_{\mathrm{rad}}}{T^3} \quad . \tag{10}$$

This is, for the case of radiative energy transport only, the extra differential equation relating T and ρ, but unfortunately it also brings in a new variable, the local radiative luminosity L_{rad}.

N.B.: If radiative energy transport is the **only** energy transport mechanism at radius r, then $L_{\mathrm{rad}}(r) \equiv L(r)$ where L(r) is the *total* luminosity. This is in general not true because of the contribution by neutrinos and convection: $L(r) = L_{\mathrm{rad}} + L_{\nu} + L_{\mathrm{conv}}$.

N.B.: Yes, we are introducing yet two other variables, L_{rad} and κ_{rad} here. We will have to write an equation for the former, but fortunately for us κ_{rad} is determined by atomic physics, as we will see in the next lecture. While this is an active topic of research (including classified research for military purposes...), for stellar physics application we have lookup tables for $\kappa_{\mathrm{rad}} \equiv \kappa_{\mathrm{rad}}(T,\rho)$, and thus we will not count it as a new variable after discussing the physics it represents.

Because of the assumption underpinning the diffusion approximation, this is *not* the right equation whenever $\ell_{\mathrm{nil}\gamma}$ is not negligible compared to the scale over which one wants to consider the gradient: in the stellar atmosphere we need a more detailed approach requiring to treat the radiative transfer.

Now, before looking at the equation for L_{rad}, it is useful to consider next the case where energy is carried not by photons, but by the local motion of particles, that is **conduction**.

Energy transport by conduction

Energy transport by diffusion, and especially conduction that is diffusion of energy through particle motion, is not limited to stars. For example, in a piece of metal left half in the Sun and half in the shade, the thermal motion of particles (atoms, electrons, ions) carries energy from the hotter parts to the colder ones, and the transfer occurs through collisions between the loose electrons in the metallic energy bands.

Conduction, although always present, is important only in certain kind of stars. To demonstrate this, we can consider the diffusion coefficient $D \simeq v\ell/3$ and compare it to the radiative diffusion coefficient $D_{rad} = c\ell_\gamma/3 = c/(3\kappa\rho)$. In the diffusion coefficient D, the velocity that appears is the thermal velocity of the particles ($v^2 \simeq 2k_BT/m$ for a non-relativistic gas): at a given temperature T, the least massive particles are faster, and will contribute more to the conduction of thermal energy. In a star, this means the electrons are going to dominate conduction whenever there is some.

The other thing to consider is the mean free path ℓ, but since the collisional (Coulomb-scattering) $\sigma \leq 10^{-18}$ cm^2, the mean free path $\ell = 1/(n_e\sigma)$ $\ll \ell_\gamma$. Thus, since $v \leq c$ and $\ell \ll \ell_\gamma$ for most stars energy conduction by particle (electron) collisions is sub-dominant compared to radiative energy.

Things are different though for degenerate electron gas (so inside white dwarfs and neutron stars, but also evolved stellar cores that are dense enough for degeneracy to occur). In the case of degeneracy, the thermal velocities increase (up to $v \simeq c$ for an degenerate gas of ultra-relativistic electrons!), and the mean-free path for electron-electron scattering also increases, because for such a scattering to be possible the final state must be available for an electron to populate it, but in the case of (partial) degeneracy (most) states in the "Fermi sea", with $\varepsilon \leq \varepsilon_{Fermi}$ are **not** available.

In general though, in (partially) degenerate layers of the star we cannot neglect conduction, and it can dominate over radiative diffusion even! To consider it, we can follow the same reasoning as above and write an equation for the conductive flux

$$F_{\rm cond} = -\frac{1}{3}\frac{c}{\kappa_{\rm cond}\rho}\frac{dT}{dr} \quad , \tag{11}$$

where we are implicitly defining a "conductive opacity" $\kappa_{\rm cond}$ and assuming that the energy density of the gas is proportional to the temperature T (not a big assumption, since we know we are very close to LTE, so we can define a local T). With this implicit definition of $\kappa_{\rm cond}$ then we can just sum the contribution to the energy flux from radiative diffusion and conduction: F = $F_{\rm rad}$ + $F_{\rm cond}$ and

$$F = -\frac{1}{3}\frac{c}{\kappa\rho}\frac{dT}{dr} \quad , \tag{12}$$

where now

$$\frac{1}{\kappa} = \frac{1}{\kappa_{\rm rad}} + \frac{1}{\kappa_{\rm cond}} \quad . \tag{13}$$

In the absence of convection (which we will treat later) and neutrinos (which leave the star instantaneously without further interaction, unless it's a neutron star), this $F_{\rm rad}$ + $F_{\rm conv}$ is the total energy flux.

From Eq. 13 we can infer an interpretation of these radiative and conductive opacities, which is corroborating the definition of κ: the equation corresponds to the combination of two resistances in parallel! $\kappa_{\rm i}$ is the "resistance" to the flow of energy carried by radiation (i=rad) or particle collisions (i=cond). The lowest resistance allows for the largest energy flux, and the star will use that mechanism as the dominant energy transport.

Moreover, since we have *defined* $\kappa_{\rm cond}$ so that the conductive flux has the same form as the radiative flux, we can (using Eq. 13 and L(r) = $L_{\rm rad}$ + $L_{\rm cond}$) continue the analogy and write down:

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\rho\kappa}{r^2}\frac{L}{T^3} \quad , \tag{14}$$

which is the radiative+conductive energy transport equation that related T, ρ, and the new variable L we introduced and depends on the opacity (radiative and conductive combined in parallel) κ, which we treat as a parameter dependent on atomic and condensed matter physics ($\kappa \equiv \kappa$(T,ρ)).

N.B.: For conduction, we have considered the motion of electrons as the ions are "frozen in place" since $v_{\rm e} \gg v_{\rm ions}$. However, this will quickly lead to a *local* charge imbalance! In stars where conduction is important (typically at least partially degenerate) there will be a small but non-zero electric field created by this charge imbalance that slows down the electrons, until their motion is such that there is a net transfer of their thermal energy without any net motion of electrons!

Local energy conservation

Let's finally write an equation for the *local* luminosity in a star L that we have introduced above. Since the luminosity is just the local "power", it makes sense to look into the *local* energy conservation to derive such equation. For a unit mass, the "first law of thermodynamics" states that the change d u in internal energy (the specific internal energy) is equal to the heat added/extracted d q plus the work done on the unit mass Pd v with v=$1/\rho$ the specific volume:

$$du = dq + Pdv \equiv dq - \frac{P}{\rho^2}d\rho \quad , \tag{15}$$

where we express things as a function of the density ρ which already appears in the other equations.

- **Q**: if we compress the gas (dρ > 0 because ρ increases), without adding/extracting heat (d q = 0) what happens to the internal energy?

The heat term in a star can only be due to:

- energy generation by an internal source (nuclear fusion!), which can release per unit mass and time energy equal to $\varepsilon_{\rm nuc}$ ($[\varepsilon_{\rm nuc}]$ =[E]/([t][M])).
- energy loss by some particle escaping, this can be for example neutrinos ν. Neutrinos in a star can come from nuclear reactions and they effectively just reduce $\varepsilon_{\rm nuc} \rightarrow \varepsilon_{\rm nuc}$ - $\varepsilon_{\nu,\,\rm nuc}$, or they can come from so-called **cooling processes**, for example e$^-$ +$\gamma \rightarrow$ e$^-$ + ν + anti-ν, which really decrease the energy by extracting internal energy, since as soon as they are produced neutrinos will leave the star with no further interaction (with the exception of neutron stars). The neutrino energy cooling rate per unit mass is indicated by ε_ν and it has always a **negative** contribution to the heat (it's a loss term for the star)
- energy can flow in and out from the boundary of a thin shell of matter. Above, we have defined: L = 4π r^2 F (where now both L and F include the contribution from conduction and radiation). Therefore, the energy per unit time coming from below is L$\equiv$ L(m) and the energy per unit time leaking from above is L(m+dm).

Putting all these together we have, at a given mass location m

$$dq(m) = \varepsilon_{\rm nuc}(m)dt - \varepsilon_\nu(m)dt + (L(m) - L(m+dm))dt \simeq \varepsilon_{\rm nuc}(m)dt - \varepsilon_\nu(m)dt - \frac{dL}{dm}dt \quad . \tag{16}$$

Thus, substituting in the local energy conservation we obtain:

$$\frac{dL}{dm} = \varepsilon_{\rm nuc}(m) - \varepsilon_{\nu}(m) - \frac{du}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt} \quad . \tag{17}$$

Often the last two terms are combined together to define:

$$\varepsilon_{\rm grav} = -\frac{du}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt} = -T\frac{ds}{dt} \quad . \tag{18}$$

which being a term dependent on dt it is usually small for a star in a static ($\partial_{\rm t} \equiv 0$) configuration. However, a star may occasionally be out of thermal equilibrium (du/dt $\neq 0$) and/or expanding or contracting (dρ/dt$\neq$0). This will change the internal state of the gas, and that is why it is often convenient to write things in terms of the (specific) entropy s. Moreover, since most often this occurs because of contraction/expansion of a star, historically this has been called ε "grav", although it really has more to do with the internal energy of the gas. With this definition, the next equation of stellar structure becomes

$$\frac{dL}{dm} = \varepsilon_{\rm nuc} - \varepsilon_{\nu} + \varepsilon_{\rm grav} \quad . \tag{19}$$

N.B.: In regions where no energy is produced ($\varepsilon_{\rm nuc} = 0$), there are no neutrino losses (ε_{ν}=0) and in thermal equilibrium ($\varepsilon_{\rm grav} = {\rm T}\partial\,{\rm s}/\partial\,{\rm t} = 0$), the luminosity is constant as one moves inward or outward in mass coordinate: dL/dm = 0 $\Rightarrow$ L = constant.

N.B.: Once again, we found another equation but it comes with new unknowns. $\varepsilon_{\rm grav}$ is fortunately only dependent on the thermodynamics of the gas, so with the EOS we can calculate that (the specific entropy is yet a function of ρ and T). The other two terms instead are input physics for the star. We will later unpack more $\varepsilon_{\rm nuc}$ by discussing nuclear energy generation – but ultimately it will depend on cross sections for nuclear interactions which in stellar physics are taken as known input physics (again coming often from military research). Similarly, ε_{ν} depends on neutrino physics and contains many neutrino loss terms. We will discuss also these a bit more later on, but effectively in stellar physics ε_{ν} is also a quantity that we assume to know as a function of T and ρ, borrowing the work of neutrino physicists.

So, in total at this point, we have κ, ε_{ν}, $\varepsilon_{\rm nuc}$ assumed to be known input physics, and we have an equation for the local conservation of energy, and the energy transport in the case of diffusion (mediated by photons or particles, i.e. conduction).

We still need an equation for the convective energy transport, and while unpacking $\varepsilon_{\mathrm{nuc}}$ we will write a set of equations for the chemical evolution due to nuclear burning, but we are getting close!

Homework

Eddington Luminosity

Consider an optically thick, hot, and stratified gas: this could be (some layers of) a star, or a sufficiently dense accretion or decretion flow to/from a compact object. Because of the assumption of optical thickness, we can assume that the layer is in LTE and the radiation field is well approximated by a black body, thus we know that the radiation pressure is $\mathrm{P_{rad}} = \mathrm{aT^4}/3$. If the gas is sufficiently hot, this is the only pressure term we need to consider ($\mathrm{P_{rad}} \gg \mathrm{P_{gas}}$).

1. Write $\mathrm{dP_{rad}/dr}$ as a function of L, κ, and ρ and r expressing dT/dr assuming energy is transported throughout our layer of gas by radiative diffusion.

2. Impose hydrostatic equilibrium for this gas, and derive the functional form for the luminosity (call it $\mathrm{L_{Edd}}$) required for radiation pressure in an optically thick gas to balance out gravity.

The expression that you found was first derived by Arthur Eddington, assuming that $\kappa \equiv \kappa_{\mathrm{es}} = 0.2(1{+}\mathrm{X})\ \mathrm{g\ cm^{-2}}$. In this derivation you did not need to assume anything for κ: the expression you derived is sometimes referred to as "modified Eddington Luminosity". Because of its dependence on κ, which we will see can vary throughout the star, it can occasionally occur that $\mathrm{L_{Edd}} < \mathrm{L}$: in this case radiative energy transport and hydrostatic equilibrium cannot be simultaneously satisfied - and this occurs in the envelope of massive stars for example. For an (optically thick) accretion flow, this luminosity corresponds to the limit when the in-falling material liberates gravitational potential in the form of heat to the point that the photons produced balance out the gravitational pull that brings in the in-falling material in the first place.

N.B.: the only central hypothesis necessary to derive the Eddington luminosity here is that the photons are a black body, that is an optically thick environment is necessary.

Exercise 5.3 in Onno Pols' lecture notes

Without solving the stellar structure equations, we can already derive useful scaling relations. In this question you will use the equation for radiative energy transport with the equation for hydrostatic equilibrium to derive a scaling relation between the mass and the luminosity of a star.

1. Derive how the central temperature, T_{center}, scales with the mass, M, radius, R, and luminosity, L, for a star in which the energy transport is by radiation. To do this, use the stellar structure equation for the temperature gradient in radiative equilibrium (**hint:** use the dT/dr form).

2. Assume that $r \sim R$ and that the temperature is proportional to T_{center}, $L(m) \sim L$ and estimating $dT/dr \;\; T_{center} / R$.

3. Derive how T_{center} scales with M and R, using the hydrostatic equilibrium equation, and assuming that the ideal gas EOS holds.

4. Combine the results obtained in 1. and 2., to derive how L scales with M and R for a star whose energy transport is radiative.

You have arrived at a mass-luminosity relation *without assuming anything about how the energy is produced*, only about how it is transported (by radiation). This shows that the luminosity of a star is not determined by the rate of energy production in the centre, but by how fast it can be transported to the surface!