

# 400A - EOS (polytropic & ideal gas)

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**Materials:** Chapter 3 of Onno Pols' lecture notes, Sec. 2.2 and Chapter 5 and 13 of Kippenhahn's book, Sec. 2-4 of Clayton's book.

## Equation of state 1/2

### Summary of where we are

We have introduced now two equations of the stellar structure (**N.B.:** at the end of the course we'll need 5 to close the system and get a solvable model), and we know we can write them using the Lagrangian mass coordinate as independent variable. But to reason conceptually about the problem, we can stick to writing them as a function of  $r$ : the Lagrangian formalism is important when doing numerical work, but less so for analytic reasoning, when we can easily switch back and forth between  $r$ ,  $m$ , or any other variable as independent.

#### Mass continuity:

$$\frac{dm}{dr} = 4\pi r^2 \rho . \quad (1)$$

#### Hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm}{r^2} \rho , \quad (2)$$

These two equations express the conservation of mass and momentum, respectively (**N.B.:** it is often useful to write all equations in the form of conservation laws to solve them numerically), and have as variables the function  $m \equiv m(r)$  (or equivalently  $r \equiv r(m)$ ),  $P$  and  $\rho$ . So clearly we cannot yet solve them, we need some relation between the pressure and the density - a so-called *equation of state* (EOS).

## General consideration on the EOS

In general, this is a relation between the *local* value of pressure  $P$ , density  $\rho$ , and temperature  $T$ , which may also depend on the composition, that is the ensemble of fraction by mass of elements ( $X_i = \text{mass of } i\text{-th element}/\text{total mass}$ ):

$$P \equiv P(\rho, T, \{X_i\}) \quad (3)$$

For a given composition  $\{X_i\}$ , given two out of three between  $P$ ,  $\rho$ ,  $T$ , one can obtain the third using the EOS.

The aim of this lecture is to introduce first simple EOS which do not introduce any other variables (so independent of  $T$ ), and then go over a more appropriate EOS which is a good approximation for most stars, the ideal gas EOS. We will see other EOS (for degenerate or radiation-dominated gas) in a [future lecture](#).

## Global and local thermal equilibrium

Before we get into the EOS though, we have to ask ourselves whether we can use thermodynamics/statistical mechanics to describe the *microscopic* properties of the stellar gas and find some relationship between  $P$ ,  $\rho$  (and other thermodynamic quantities such as  $T$ , entropy, etc). Are these even well defined?

Thermal equilibrium means by definition that all the particles in a system (including photons, if there is radiation) have the same temperature  $T$ , and their velocity distribution are described by a Maxwell-Boltzmann distribution (for particles) and the black-body spectrum (for radiation). This situation can only be achieved if there are sufficient interactions so each particle can exchange energy with the others (collisions among particles, and scattering/absorption for photon-particles interactions).

- **Q:** are stars in global thermal equilibrium? If they were, they would have a single temperature throughout the star, does this sound realistic?

**Stars are not in global thermal equilibrium.** They are in "*gravothermal equilibrium*": the hydrostatic requirement and the self-gravity determine the thermal state of each layer so that the pressure gradient can balance gravity.

However, *locally*, we can still assume that there are stellar layers with a thickness  $dr \ll R$  that are in thermal equilibrium (with  $R$  outer radius of the star, which we have seen to be an idealization, since in reality  $R \equiv R(\lambda)$  depends on the wavelength  $\lambda$  at which one observes). For this to happen, this layer needs still to be much larger than the mean-free-path for particle-particle and particle-photon interactions. This allows for many interactions between the particles to occur and establish *local* thermal equilibrium within the layer of thickness  $dr$ . Let's consider the mean-free-path for particle-photon interactions first:

$$\ell_\gamma = \frac{1}{n\sigma} \equiv \frac{1}{\kappa\rho} , \quad (4)$$

where  $n$  is the number density of particles ( $[n] = \text{cm}^{-3}$ ) and  $\sigma$  is the interaction cross section ( $[\sigma] = \text{cm}^2$ ), or equivalently,  $\rho$  is the mass density (appearing in the two equations we already have), and  $\kappa$  is the "opacity", which is a way to measure the probability of a photon to interact with a certain amount of mass.

- **Q:** what are the units of  $\kappa$ ? How can we interpret it?

Let's for simplicity assume that a star is all made of hydrogen (a result first derived  $\sim 100$  years ago by Cecilia Payne in her PhD thesis in 1925). Let's also assume that it is fully ionized (i.e., protons and electrons are separated from each other). The very minimum opacity in this case is given by scattering of photons onto the electrons and is:  $\kappa = 0.2(1+X) \text{ cm}^2 \text{ g}^{-1}$  (see also [lecture on opacities](#)), where  $X$  is the mass fraction hydrogen, so for our rough approximation,  $\kappa \simeq 0.4$ . The average density of the Sun is  $\rho_{\odot, \text{avg}} \simeq 1.4 \text{ g cm}^{-3}$ , therefore  $\ell_\gamma \simeq 2 \text{ cm}$ . The mean-free-path for particle-particle interactions is much smaller (because the cross section  $\sigma$  is larger): the interaction is electromagnetic and has infinite range, in principle all charged particles feel all other (we will return on this at the end of this lecture). Therefore, we can conservatively ask what is the temperature variation across  $\ell_\gamma$ :

$$\Delta T \simeq \ell_\gamma \frac{dT}{dr} \leq \ell_\gamma \frac{T_{\text{center}} - T_{\text{surface}}}{R} \simeq \ell_\gamma \frac{T_{\text{center}}}{R} \quad (5)$$

(where we neglect  $T_{\text{surface}}$  because we expect it much lower than  $T_{\text{center}}$ , since we already know  $P_{\text{center}} \gg P_{\text{surface}}$ ). For the Sun we will see that we can estimate  $T_{\text{center}} \simeq 10^7 \text{ K}$ , and we know  $R_{\odot} \simeq 10^{11} \text{ cm}$ , so  $\Delta T \simeq 10^{-4} \text{ K} \ll T_{\text{surf}} \ll T_{\text{center}}$ .

Thus, we can define layers of the star with  $\ell_\gamma \ll dr \ll R$  where locally the temperature variation across the layer is negligible, and the assumption of *local thermal equilibrium* (LTE) holds well. (**N.B.:** you can revise this a posteriori once we have an EOS and a way to estimate of  $T_{\text{center}}$ !)

This may not be true close to the surface, where  $\ell_\gamma$  progressively increases (because  $\rho$  decreases). There we may need to care about the wavelength (or frequency) dependence of the cross section  $\sigma \equiv \sigma(\lambda)$  and thus the mean free path: a star has a different "surface" for different photon frequencies! In those layers, the assumption of LTE does not work.

As long as we discuss regions of the star with a thickness large w.r.t.  $\ell_\gamma$  but small enough that the T variation across them are negligible, we can use thermodynamics!

Also, as we have seen before, the stars are usually in equilibrium and change very slowly. Any *local* thermal imbalance will be restored on the *local* thermal timescale, which can be estimate in multiple ways, but it is usually very short compared to the star evolutionary timescale (we will see this in more detail later). Thus the assumption of LTE holds locally (as the "L" indicates) at any time! Therefore, while we cannot really define a physically meaningful T for the entire star (beyond some rough averaged quantity), we can meaningfully define a *local*  $T(r)$  or  $T(m)$  temperature that characterizes the gas *and* the radiation at that radius/mass coordinate. The local validity of the LTE assumption allows us to define thermodynamic quantities from statistical mechanics and look for a relation between them, that is an EOS.

## Polytropic EOS

Polytropic EOS as a special case of *barotropic* EOS, which are all the EOSs for which *the density depends only on the pressure and not on the temperature or composition*:  $\rho \equiv \rho(P)$ . These have applications beyond stellar physics (e.g., for atmospheric physics). A polytropic EOS is thus a barotropic EOS where the  $\rho \equiv \rho(P)$  is in the form of a powerlaw, although usually written (following the general form of Eq. 3):

$$P = K\rho^\Gamma \equiv K\rho^{1+1/n} , \quad (6)$$

where K is a constant, and by definition  $\Gamma = \partial \ln P / \partial \ln \rho = 1 + 1/n$ . The form of the EOS in Eq. 6 may seem particularly artificial, but it is helpful because it does not introduce any new variable, and thus "closes" (together with Eq. 1 and Eq. 2) the system of equations describing a star! This is the system of equations used to create, for example, analytic stellar

models (e.g., the Lane-Emden equation), which is one of the topics for the projects. These can be useful to initialize roughly correct stellar structures in multi-D hydrodynamic codes for particular applications (e.g., a common envelope evolution simulation!), but they are usually insufficient for modern stellar physics application.

More importantly, there are various *physical* situations (as we will return on during [the second lecture on EOSs](#)) in which EOS of this form occur, and are useful to describe real observed stars.

- fully convective stars (see [relative lecture](#))
- stars supported by quantum mechanical effects such as white dwarfs (WDs)

Often, different polytropes in the form of Eq. 6 can be used for different layers of the stars (piece-wise polytropes) as useful approximation. In this cases, it is important to ensure the *continuity* of P at the points where different polytropes are stitched together.

- **Q:** why do we want P to be a continuous function in stars?

## Ideal gas

In general, it is not possible to have an EOS independent of T and/or the composition  $\{X_i\}$ . Let's consider a *perfect gas*, that is a gas of particles that do interact (otherwise they could never exchange energy and be in thermal equilibrium!), but such as that the energy exchanged in the particle-particle interactions is much small than their kinetic energy (due to their thermal motion).

To obtain the pressure of such gas, we may need to consider the distribution in velocity space of these particles. Let's for a moment consider particles that all have the same mass m, we can then equivalently consider the distribution in momentum  $p=mv$  of the particles – this is convenient to generalize later to relativistic particles, and we will see how to deal with mixtures of gases (each with particles  $m_i$ ) further down.

Since the particles of the ideal gas move isotropically within their volume, the momentum distribution of particles is a Maxwell-Boltzmann distribution. The number of particles with momentum between  $p$  and  $p+dp$  is

$$n(p)dp = \frac{n}{(2\pi mk_B T)^{3/2}} \exp\left(-\frac{p^2}{2mk_B T}\right) 4\pi p^2 dp , \quad (7)$$

where on the r.h.s.,  $n$  is the total number density, the prefactor of the exponential is the normalization constant, and the exponential comes from assuming a Gaussian distribution of kinetic energies for each momentum component (you can demonstrate that a Maxwellian distribution is equivalent to a Gaussian in each direction  $p_x$ ,  $p_y$ ,  $p_z$ , by going to spherical-polar coordinates in momentum space, thus introducing a Jacobian for the change of variables, and integrating over the  $p_\theta$ ,  $p_\varphi$  components).

From this, we can calculate the pressure of the gas. By definition this will be isotropic, and so we can imagine to put a "box" with unit linear size through our gas (the orientation of the walls does not matter). To determine the gas pressure we want the force exerted by the gas on the walls. By Newton's third law this is equal to the change in momentum of each gas particles as they hit the walls. We will first consider the momentum change for a generic single particle, and then integrate over the distribution in Eq. 7 to get the whole pressure.

Let's call the imaginary wall the  $xy$  plane and assume the collisions to be elastic (because we are considering an ideal gas, by definition any exchange of energy is negligible). For a generic particle of the gas, it will collide with the wall at an angle  $\theta \in [0, \pi/2]$  and the collision will change its momentum from  $(p_x, p_y, p_z)$  to  $(p_x, p_y, -p_z)$ : only the component perpendicular to the wall changes sign. Thus

$$\Delta p = 2p \cos(\theta) \text{ with } p = \sqrt{p_x^2 + p_y^2 + p_z^2} \quad (8)$$

The time between two collisions of a particle on the same wall is

$$\Delta t = \frac{2L}{v \cos(\theta)} = \frac{2}{v \cos(\theta)} , \quad (9)$$

where we used the  $L=1$  assumption. Thus the force exerted on this imaginary wall of a unit box is  $F = \Delta p / \Delta t = vp \cos^2(\theta)$ , dividing by  $L^2 = 1$  we obtain the contribution to the pressure from one particle coming from one specific direction  $\theta$  (w.r.t. the wall of the box), and introducing the distribution of particles in angle and momentum we have

$$dP = vp \cos^2(\theta) n(\theta, p) d\theta dp , \quad (10)$$

But we can assume that the motion of the particles is isotropic, so  $n(\theta, p)d\theta = n(p)\sin(\theta)d\theta$ , and thus

$$dP = vp \cos^2(\theta)n(p) \sin(\theta)d\theta dp = \frac{1}{3} \int_0^{+\infty} n(p)p v dp , \quad (11)$$

which combined with Eq. 7 gives the pressure.

### Non-relativistic, classical gas

Let's consider a non-relativistic gas of classical particles (no quantum effects). Then  $p = mv \Leftrightarrow v = p/m$  (**N.B.:** we are considering an ideal gas, so each particle is freely moving, no potential of interaction!). Carrying out the integral above using the Maxwell-Boltzmann distribution for  $n(p)$  and  $v = p^{nil}/m$  gives

$$P = nk_B T . \quad (12)$$

### Mixture of (non-relativistic, classical) gases

Let's now say that we have multiple gas mixed, for example, a gas of ions of various species and electrons. Each gas will contribute to the pressure:  $P_{tot} = P_{ion, tot} + P_e = \sum_i P_{ion, i} + P_e = (\sum_i n_i + n_e) k_B T$ , where  $n_i$  is the number density of the ions  $i$ , which have mass  $m_u A_i$  with  $m_u$  the *atomic mass unit*:

$$m_u = 1/N_A g \simeq 1.66 \times 10^{-24} g . \quad (13)$$

Thus, we can relate the number density of the ions of species  $i$  with the mass density that already appears in the equations we already have with  $n_i = X_i \rho / (A_i m_u)$ . Note that we are implicitly using the fact that everything has the same  $T$  because of the assumption of LTE!

We can re-write the total contribution of the ions defining the ion *mean molecular weight* such that  $\mu_{ion} \times m_u$  = "average mass of ions", that is  $\mu_{ion} n_{ion} = \rho / m_u$  or  $n_{ion} = \sum_i n_i = \sum_i X_i \rho / (A_i m_u) \equiv \rho / (\mu_{ion} m_u)$  and:

$$\frac{1}{\mu_{ion}} = \sum_i \frac{X_i \rho}{A_i} . \quad (14)$$

Similarly, we can define an electron mean molecular weight noticing that to maintain a total charge of zero per unit volume, since each ion carries charge  $+Z_i e$  and each electron as charge  $-e$ :  $(\sum_i Z_i n_i - n_e)e = 0$ . Thus

$$\frac{1}{\mu_e} = \sum_i \frac{Z_i X_i \rho}{A_i} . \quad (15)$$

and we can define a combined mean molecular weight:

$$\frac{1}{\mu} = \frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_e} , \quad (16)$$

So that the total pressure of a mixture of ideal, classical and non-relativistic gas is

$$P = \frac{\rho}{\mu m_u} k_B T \quad (17)$$

The introduction of the *mean molecular weight* allows us to treat a mixture of gases (assumed to be in LTE) as a single gas!

**N.B.:** this holds as long as every species satisfies our assumption of ideal, non-relativistic, classical gas.

### Physical interpretation of $\mu$

The *mean molecular weight* we have introduced above may seem a bit arbitrary, but it has a clear physical interpretation: it is the average number of particles per unit atomic mass  $m_u$ .

For a fully ionized gas (i.e., where every ion is stripped of *all* its electrons):

$$\mu = \frac{1}{\sum_i X_i \left( \frac{Z_i + 1}{A_i} \right)} , \quad (18)$$

In fact, for every  $i$ -th species that corresponds to a fraction  $X_i$  of the total mass, that is  $X_i/A_i$  in number density, we have  $Z_i$  electrons plus one nucleus that contribute. This approximation is usually good in the stellar interior, but as one moves outwards in the star,  $P$  decreases, therefore by Eq. 17,  $T$  decreases, and elements recombine, meaning the term  $Z_i + 1$  is reduced. This can have important consequences in certain layers of the stars ("partial ionization layers").

We can further simplify the expression for  $\mu$  by noting that for hydrogen  $(Z_i + 1)/A_i = 2$  (i.e., hydrogen contributes 2 particles per unit atomic mass,

one proton and one electron), for helium  $(Z_i + 1)/A_i = 3/4$  (we are actually considering only the stable isotope of helium that contributes 3 particles every 4 atomic mass units, one nucleus and 2 electrons), and for the vast majority of stable metals  $Z_i \gg 1$  and  $Z_i \simeq A_i/2$  (i.e., most metals contribute per each  $A_i$  atomic mass units roughly  $A_i/2$  particles which are the electrons that are typically half as many as the nucleons). Therefore, for fully ionized gas, we can simplify the mean molecular weight to:

$$\mu \simeq \frac{1}{2X + \frac{3Y}{4} + \frac{Z}{2}} , \quad (19)$$

where X, Y, Z are the mass fraction of hydrogen, helium, and the metallicity, respectively.

### Relation between pressure and internal density

Eq. 11 can be used to relate P to the internal energy density of a gas. This can be defined as:

$$du_{\text{int}} = \varepsilon(p)n(p)dp , \quad (20)$$

with  $\varepsilon(p)$  the energy per particle.

If the particles in the (ideal) gas are non-relativistic, then  $\varepsilon = p^2/2m$ , thus in the term  $pv$  in Eq. 11 is  $pv = 2\varepsilon$ , and thus:

$$P = \frac{2}{3}u_{\text{int}} . \quad (21)$$

If instead the gas is ultra-relativistic, then  $\varepsilon = pc$  (neglecting the mass term in the energy since  $pc \gg mc^2$ ), and thus:

$$P = \frac{u_{\text{int}}}{3} . \quad (22)$$

### Can we really use an ideal gas EOS in a plasma?

This is legitimate as long as the interaction energy between the particles are small compared to their kinetic energy. The dominant interaction between the particles (ions and electrons) is going to be through the Coulomb force, leading to energy exchanges of the order of:

$$\varepsilon_{\text{Coulomb}} \simeq \frac{Z^2 e^2}{d} , \quad (23)$$

for particles of charge  $Ze$  (the ions) and average distance  $d \simeq (4\pi n/3)^{-1/3}$  with  $n \simeq \rho/Am_u$  number density. We want to compare this with the kinetic energy, which for point-like particles is  $\varepsilon_{\text{thermal}} = 3k_b T/2$ . The ratio of these two is often called the Coulomb parameter (neglecting constants of order unity):

$$\Gamma_C = \frac{\varepsilon_{\text{Coulomb}}}{\varepsilon_{\text{thermal}}} \simeq \frac{Z^2 e^2}{dk_B T} \simeq \frac{(Ze)^2}{k_B T} \left( \frac{\rho}{Am_u} \right)^{1/3} \simeq 2.275 \times 10^5 Z^2 A^{-1/3} \left( \frac{\rho}{\text{g cm}^{-3}} \right)^{1/3} \left( \frac{T}{\text{K}} \right)^{-1}. \quad (24)$$

We can assume the ideal gas situation if  $\Gamma_C \ll 1$ , which is the case for the average  $T$  and  $\rho$  of the Sun (we will estimate the average temperature of the Sun in the [next lecture](#)). We also see that at progressively lower temperature the Coulomb interaction start to matter (this is important for the crystallization of WDs for instance), or at increasingly high densities.

## Homework

- We have discussed that the internal layers of the star are in LTE using an argument based on the photons mean free path  $\ell_\gamma$ . Assuming a star of constant density (use the mean density for this!), pure hydrogen composition, and that electron scattering is the dominant interaction of the photons in the stellar interior so  $\kappa \simeq \kappa_{\text{es}} = 0.2(1+X)$ , using one-dimensional random-walk arguments, estimate:
  1. how many scatterings a photon created in the center of the Sun will experience before coming out at the surface?
  2. Knowing that photons travel at the speed of light  $c$  and assuming scatterings to be instantaneous, what is the photon diffusion timescale throughout the star? How does it compare to the dynamical timescale?
- Run with **MESA-web** a  $0.3M_\odot$  star up to  $10^8$  yrs, and plot the  $P(\rho)$  profile of the star at this age (**Hint**: it may be useful to plot it on a log-log plot). Do you think it is possible to find a good approximation of this profile with a polytropic relation? Note that **MESA-web** does **not** assume a polytropic EOS! As usual, the deliverables are the plot, the code used to make it, and a small paragraph of text with your answer. **Extra**: you may even try to fit a polytrope throughout the star and provide the polytropic index  $\Gamma$ .