EMMY NOETHER'S FIRST THEOREM FOR FIRST ORDER LAGRANGIANS

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1. Introduction. The purpose of this note is to formulate and prove one direction of Emmy Noether's first theorem [3] on the relation between invariances and conservation laws in the simplest case of a first order Lagrangian and a single independent variable. We follow the presentation in Arnol'd [1] (for a historical assessment of Arnol'd's approach see Kosmann-Schwarzbach [2]). For a modern approach to Emmy Noether's theorems in [3] we refer to Olver [4].

2. Notation and basic definitions. In this note M will be an n-dimensional smooth (C^∞) manifold and $L : TM \times \mathbb{R} \to \mathbb{R}$ be a C^2-function. We will call (M, L) a **mechanical system** with **configuration manifold** M and **Lagrangian** (function) L. If L does not depend on the last component (i.e. $L(\cdot, t) = \tilde{L}$ for some $\tilde{L} : TM \to \mathbb{R}$ and all t), then (M, L) is said to be **autonomous** (and we will identify L with $\tilde{L}$), otherwise **non-autonomous**. In the sequel, we will work in local coordinates $((x, \xi), x \in M, \xi \in T_xM))$.

For a Riemannian[1] manifold $(M, \langle\cdot,\cdot\rangle)$ we call $T : TM \to \mathbb{R}$, $T(\xi) = \frac{1}{2}\langle\xi, \xi\rangle$ the **kinetic energy** of M. If L is of the form

$$L(x, \xi) = T(\xi) - V(x),$$

for a C^2-function $V : M \to \mathbb{R}$ (**potential energy**), then we call (M, L) (more accurately, $(M, \langle\cdot,\cdot\rangle, L)$) a **natural system**. In this case, the function $E : TM \to \mathbb{R}$, $E(x, \xi) = T(\xi) + V(x)$, is called the **total energy** of (M, L).

Now let

$$\mathcal{M} = \{u \in C^1([a, b], M) \mid u(a) = u_0, u(b) = u_1\},$$

where $u_0, u_1 \in M$. Let us consider the functional $\mathcal{J} : \mathcal{M} \to \mathbb{R}$,

$$\mathcal{J}[u] = \int_a^b L(u(t), \dot{u}(t), t)dt.$$

THEOREM 2.1. *A function u is a critical point of the functional $\mathcal{J}$ (i.e. a zero of the Gâteaux derivative $\delta\mathcal{J}$ of $\mathcal{J}$) if and only if u satisfies the so-called Euler-Lagrange*

[1] All following notions and arguments carry over to pseudo-Riemannian manifolds.

equation

$$\frac{d}{dt}\frac{\partial L}{\partial \xi}(u(t),u'(t),t) - \frac{\partial L}{\partial x}(u(t),u'(t),t) = 0 \tag{2.1}$$

for all $t \in (a,b)$.

Definition 2.2. *(i) A C^1-function $u : [a,b] \to M$ is called a* **motion** *of (M,L) if it satisfies the Euler-Lagrange equation (2.1).*
(ii) A function $I : TM \times \mathbb{R} \to \mathbb{R}$ is called a **first integral** *for (M,L) if for each motion u the function $t \mapsto I(u(t),u'(t),t) \in \mathbb{R}$ is constant.*

Definition 2.3. *(i) Let $J \subseteq \mathbb{R}$ an open interval with $0 \in J$. A family $\mathcal{G} = (g^s)_{s\in J}$ of C^k-diffeomorphisms $M \to M$ is called a* **local one-parameter group** *of C^k-diffeomorphisms on M if*

1. *$J \times M \ni (s,x) \mapsto g^s x \in M$ is C^k,*
2. *$g^0 = id_M$, and*
3. *$g^{s+t} = g^s g^t$ for all $s,t \in J$ such that $s+t \in J$.*

(ii) The C^{k-1} vector field $v : M \to TM$, $v(x) = \frac{d}{ds}|_{s=0} g^s(x) \in T_xM$ is called the **symmetry vector field** *corresponding to $\mathcal{G}$.*

Definition 2.4. *(i) $L : TM \to \mathbb{R}$ is said to be invariant w.r.t. a C^1-map $g : M \to M$ if $L \circ Dg = L$.*
(ii) L is invariant w.r.t. a local one-parameter group of C^1-diffeomorphisms $(g^s)_{s\in J}$ if it is invariant w.r.t. g^s for each $s \in J$, i.e.

$$\frac{d}{ds}L(g^s(x), Dg^s(\xi)) = 0.$$

Remark 2.5. *The following example shows that if $g \circ u$ is a motion for each motion u, then L is not necessarily invariant w.r.t. g (cf. [1] Footnote 38, p. 88). Let $M = \mathbb{R}$, $L : TM = \mathbb{R} \times \mathbb{R} \ni (x,\xi) \mapsto \xi^2 \in \mathbb{R}$ and $g : \mathbb{R} \ni x \mapsto 2x \in \mathbb{R}$. Then obivously $L \circ Dg \neq L$. On the other hand, $u : [a,b] \to \mathbb{R}$ is a motion if and only if it is of the form $u = \alpha(\cdot) + \beta$ for some $\alpha, \beta \in \mathbb{R}$, thus $g \circ u = 2u$ is a motion for each motion u of (M,L).*

3. The autonomous case. We now show

THEOREM 3.1. *(Emmy Noether's first theorem) Let $L : TM \to \mathbb{R}$ be invariant w.r.t. a local one-parameter group of C^2-diffeomorphisms $(g^s)_{s\in J}$ and $v : M \to TM$ its symmetry vector field. Then $I : TM \to \mathbb{R}$, defined by*

$$I(x,\xi) = \frac{d}{dt}|_{t=0} L(x, \xi + tv(x)) = D_\xi L(x,\xi)v(x).$$

is a first integral of (M, L).

Proof. Let $u : (a,b) \to M$ be a solution of the Euler-Lagrange equation (2.1), $w = v \circ u$ and

$$\Phi : J \times (a,b) \ni (s,t) \mapsto g^s(u(t)) \in M.$$

Then

$$\Phi(0,t) = u(t), \frac{\partial \Phi}{\partial t}(0,t) = u'(t). \tag{3.1}$$

Since (g^s) is C^2, the mixed derivatives $\frac{\partial^2\Phi}{\partial s \partial t}$ and $\frac{\partial^2\Phi}{\partial t \partial s}$ exist and coincide. Moreover,

$$\frac{\partial^2 \Phi}{\partial s \partial t}(0,t) = v(u(t))' = w'(t). \tag{3.2}$$

Since

$$\frac{d}{ds} L(g^s(x), Dg^s(\xi)) = 0,$$

we obtain for $x = u(t)$, $\xi = u'(t)$

$$\frac{d}{ds} L(\Phi(s,t), \frac{\partial \Phi}{\partial t}(s,t)) = 0,$$

thus

$$D_x L(\Phi(s,t), \frac{\partial \Phi}{\partial t}(s,t)) \cdot \frac{\partial \Phi}{\partial s}(s,t) + D_\xi(\Phi(s,t), \frac{\partial \Phi}{\partial t}(s,t)) \cdot \frac{\partial^2 \Phi}{\partial s \partial t}(s,t) = 0.$$

By (3.1) and (3.2) we obtain for $s = 0$:

$$D_x L(u(t), u'(t)) \cdot w(t) + D_\xi L(u(t), u'(t)) \cdot w'(t) = 0.$$

Since u satisfies the Euler-Lagrange equation (2.1), we finally obtain

$$\begin{aligned}
\frac{d}{dt} I(u(t), u'(t)) &= \frac{d}{dt}(D_\xi L(u(t), u'(t)) w(t)) \\
&= (\frac{d}{dt}(D_\xi L(u(t), u'(t))) \cdot w(t) + D_\xi L(u(t), u'(t)) \cdot w'(t) \\
&= D_x L(u(t), u'(t)) \cdot w(t) + D_\xi L(u(t), u'(t)) \cdot w'(t) \\
&= 0. \ \square
\end{aligned}$$

Remark 3.2. *Suppose that in some (local) system of coordinates $x_1, ..., x_n$ on M, L does not depend on x_1. Then L is invariant w.r.t. to the local one-parameter group $g^s : x_1 \to x_1 + s, x_k \to x_k, k \geq 2$. By Theorem 2.1, $I = \frac{\partial L}{\partial \xi_1}$ is a first integral. In mechanics, x_1 is called a cyclic coordinate and I is a cyclic integral corresponding to x_1.*

Let $(M, \langle \cdot, \cdot \rangle, L)$ be a natural system. Then the first integral I from Theorem 3.1 is equal to $\langle v(x), \xi \rangle$ and thus linear in the (velocity) component ξ. The following examples are to illustrate this property. ($|\cdot|$ will denote the Euclidean distance).

Example 3.3. *Let $M = \mathbb{R}^3$, $L = T - V$ with*

$$T(\xi) = \langle \xi, \xi \rangle := \frac{1}{2} m |\xi|^2$$

and

$$V(x_1, x_2, x_3) = U(x_1, x_2)$$

with a smooth $U : \mathbb{R}^2 \to \mathbb{R}$. Then L is invariant w.r.t. (g^s), $g^s(x) = x + se_3$, so that $v = e_3$. The corresponding first integral I is $I(x, \xi) = \langle \xi, e_3 \rangle = m\xi \cdot e_3$, i.e. if u is a motion, then $p_3 = mu'(t) \cdot e_3$ is conserved.

Example 3.4. *System of two (non-colliding) particles of masses m_1 and m_2. Let $M = (\mathbb{R}^3 \times \mathbb{R}^3) \setminus \{(x, x) \mid x \in \mathbb{R}^3\}$ and*

$$L(x_1, x_2, \xi_1, \xi_2) = \frac{1}{2}(m_1 |\xi_1|^2 + m_2 |\xi_2|^2) - V(x_1, x_2)$$

with $V(x_1, x_2) = U(x_1 - x_2)$ for a smooth $U : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$. For each $w \in \mathbb{R}^3$, L is invariant w.r.t.

$$g^s(x) = x + s \begin{bmatrix} w \\ w \end{bmatrix}.$$

The corresponding symmetry field is the constant vector field $v = \begin{bmatrix} w \\ w \end{bmatrix}$. Thus

$$I = \langle \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \begin{bmatrix} w \\ w \end{bmatrix} \rangle = \begin{bmatrix} m_1 \xi_1 \\ m_2 \xi_2 \end{bmatrix} \cdot \begin{bmatrix} w \\ w \end{bmatrix} = (m_1 \xi_1 + m_2 \xi_2) \cdot w$$

is a first integral each w, i.e. $p_{tot} = m_1 u_1'(t) + m_2 u_2'(t)$ is constant for each motion u of (M, L).

Example 3.5. *Let $M = \mathbb{R}^3$ and*

$$L = \frac{1}{2} m |\xi|^2 - V(x)$$

where V is of the form $V = U \circ |\cdot|$ and $U : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ (this covers e.g. the cases of the gravitational, electrical and Yukawa potential, respectively). Then L is invariant w.r.t. to all $h \in O(3)$. In particular, it is invariant w.r.t. to the one-parameter group

$$g^s = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now

$$v(x) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}.$$

Thus,

$$\mathcal{L}_3 = m \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} = m(\xi_2 x_1 - \xi_1 x_2),$$

the x_3-component of the angular momentum $\mathcal{L}$, is conserved. Analogously, $\mathcal{L}_1 = m(\xi_3 x_2 - \xi_2 x_3)$ and $\mathcal{L}_2 = m(\xi_1 x_3 - \xi_3 x_1)$ are first integrals, i.e. the angular momentum

$$\mathcal{L}(u, u') = m(u \times u')$$

is conserved.

4. The non-autonomous case. For given (in general, non-autonomous) system (M, L) consider the extended (autonomous) Lagrangian system $(\bar{M}, \bar{L})$ with $\bar{M} = M \times \mathbb{R}$, $T\bar{M} \cong TM \times T\mathbb{R}$ and $\bar{L} : TM \times (T\mathbb{R} \setminus 0) \to \mathbb{R}$ defined by

$$\bar{L}(x, \xi, t, \tau) = L(x, \frac{\xi}{\tau}, t)\tau.$$

The proofs of the following results are straightforward.

Lemma 4.1. *If $x : [a, b] \to M$ is a motion in (M, L), then $\bar{x} : [a, b] \ni t \mapsto (x(t), t) \in \bar{M}$ is a motion in $(\bar{M}, \bar{L})$.* □

Proposition 4.2. *If $\bar{I} : T\bar{M} \to \mathbb{R}$ is a first integral of $(\bar{M}, \bar{L})$, then $I : TM \times \mathbb{R} \to \mathbb{R}$, defined in local coordinates by*

$$I(x, \xi, t) = \bar{I}(x, \xi, t, 1),$$

is a first integral of (M, L). □

Example 4.3. *If (M, L) is autonomous, i.e. L does not depend on t, then, by Remark 3.2, t is a cyclic coordinate of $(\bar{M}, \bar{L})$ and the corresponding cyclic integral $\frac{\partial \bar{L}}{\partial \tau}$ is a first integral of $(\bar{M}, \bar{L})$. By Proposition 4.2,*

$$\frac{\partial \bar{L}}{\partial \tau}|_{\tau=1} = L - D_\xi L \cdot \xi$$

is a first integral[2] *of (M, L). If (M, L) is natural, then $D_\xi L \cdot \xi - L = T + V = E$.*

References.

[1] Vladimir I. Arnol'd. *Mathematical Methods of Classical Mechanics.* Springer, 1st edition, 1978.

[2] Yvette Kosmann-Schwarzbach. *The Noether Theorems.* Springer, 2nd edition, 2018.

[3] Emmy Noether. Invariante Variationsprobleme. *Nachrichten von der Kgl. Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse. 1918*, pages 235–257.

[4] Peter Olver. *Applications of Lie Groups to Differential Equations.* Springer, 2nd edition, 1993.

[2]For a motion $u \in C^2$ this can be directly verified.