

# Experiments in Finding 42 as Sum of 3 Cubes

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## 1 Introduction

The sum of 3 cubes of signed integers has some interesting properties as is summarised in [Wik19]. In particular for numbers  $k$  up to 100 with  $k \not\equiv \pm 4 \pmod 9$ , after finding the solution for 33 given in [Boo19], there is only 42 as the outstanding number that is not known to be the sum of three cubes. This article gives some thoughts on the constraints for the  $k = 42$  case and how to encode into a general positive integer. Let

$$x_0^3 + x_1^3 + x_2^3 = 42 \quad (1.1)$$

in the equation we are trying to solve for integers  $x_0, x_1, x_2$ .

## 2 Interesting modular arithmetic results

Taking the mod of a cube provides some interesting results:

Table 2.1: Modular Arithmetic Results

|               | $x$ |    |    |   |    |    |   |    |   |    |
|---------------|-----|----|----|---|----|----|---|----|---|----|
|               | 1   | 2  | 3  | 4 | 5  | 6  | 7 | 8  | 9 | 42 |
| $x^3 \pmod 7$ | 1   | 1  | -1 | 1 | -1 | -1 | 0 |    |   | 0  |
| $x^3 \pmod 6$ | 1   | 2  | 3  | 4 | 5  | 0  |   |    |   | 0  |
| $x^3 \pmod 9$ | 1   | -1 | 0  | 1 | -1 | 0  | 1 | -1 | 0 | -3 |
| $x^3 \pmod 3$ | 1   | -1 | 0  | 1 | -1 | 0  | 1 | -1 | 0 | 0  |

from the  $\pmod 7$  table one can see that one of the integer powers say  $x_3$  is divisible by 7 so put  $x_0 = 7a$ . Also note from the  $\pmod 9$  table that we have, since  $42 \pmod 9 \equiv -3$

$$X_i^3 \pmod 9 \equiv -1 \quad \text{for } i = 0, 1, 2 \quad (2.1)$$

which in turn gives

$$X_i^3 \pmod 3 \equiv -1 \quad \text{for } i = 0, 1, 2 \quad (2.2)$$

so we get for  $i = 0, 1, 2$

$$x_i \mod 6 \equiv x_i^3 \quad (2.3)$$

$$\equiv 2 \text{ or } 5 \quad (2.4)$$

$$\equiv 2 \text{ or } -1 \quad (2.5)$$

since  $42 \mod 6 \equiv 0$  we get

$$x_0 + x_1 + x_2 \mod 6 \equiv x_0^3 + x_1^3 + x_2^3 \quad (2.6)$$

$$\equiv 0 \quad (2.7)$$

also note that

$$x_0 \mod 6 \equiv 7a \quad (2.8)$$

$$\equiv (7 \mod 6) a \quad (2.9)$$

$$\equiv a \quad (2.10)$$

We also get from the  $\mod 7$  table since  $x_0 \equiv 0 \mod 7$  that

$$x_1^3 \mod 7 \equiv -x_2^3 \quad (2.11)$$

$$\equiv \pm 1 \quad (2.12)$$

so without loss of generality we put

$$x_1^3 \mod 7 \equiv 1 \quad (2.13)$$

$$x_2^3 \mod 7 \equiv -1 \quad (2.14)$$

### 3 Forming Candidates from a big integer

When working with a Set Partical Swarm Optimiser (SPSO) the cost function is presented with a random big integer that represents in coded form the candidate to cost after it has been modified to meet given constraints. I believe The adoption of the modular constraints avoids attracting the SPSO towards low scoring solutions away from the solution we are here interested in. This section describes how this is done for Equation 1.1.

#### 3.1 partitioning the big integer

For SPSO the parameters big integer is a positive one and can be regarded as an array of bits with least significant bits to the left of the array in our representation; using this the big integer is partitioned as<sup>1</sup>

$$b = j_0 | j_1 | j_2 | f \quad (3.1)$$

where  $j_i i = 0, 1, 2$  are positive integers occupying the same number of bits say  $N$  with possible padding with zeros on the right as represented.  $f$  is regarded as an array of flags taking on the value 1 or 0.

<sup>1</sup>In coding this up the positive big integer is represented as an array of 64 bit words; for the sake of computational speed the partition is applied at word boundaries.

From this we get three signed integers

$$k_i = -1^{f[i]} j_i \quad \text{for } i = 0, 1, 2 \quad (3.2)$$

that provide a starting point in representing the candidate integers for 1.1 by modifying the  $k_i$

$$a = 6k_0 + c_0 \quad (3.3)$$

$$x_1 = 6k_1 + c_1 \quad (3.4)$$

$$x_2 = 6k_2 + c_2 \quad (3.5)$$

where the  $c_i$  are yet to be chosen based on  $f$  to meet the mod 6 constraints.

Once the  $k_i$  are found the  $j_i$  are replaced by  $|k_i|$  and the  $f[i]$  replaced by  $\text{sign}(k_i)$  if required to give the modified big integer that gives the constraint satisfying parameters to be used in the next iteration in the SPSO.

### 3.2 Choosing the $c_i$ to satisfy the mod 6 Constraints

From 2.10 and 2.5 we have  $c_0$  is either 2 or -1 so use  $f[3]$  to choose the option and put

$$c_0 = \begin{cases} 2 & \text{if } f[3] = 1 \\ -1 & \text{otherwise} \end{cases} \quad (3.6)$$

the  $x_1, x_2$  must satisfy 2.5 as well as 2.7. It transpires by inspection that  $x_1$  can take on either mod 6 option which in turn determines the  $x_2$  options; to this extent put

$$c_1 = \begin{cases} 2 & \text{if } f[4] = 1 \\ -1 & \text{otherwise} \end{cases} \quad (3.7)$$

and then we have that  $c_2$  satisfies the constraint 2.7 if we use the table 3.1.

Table 3.1:  $c_2$  Values

| $f[3]$ | $f[4]$ | $c_2$ |
|--------|--------|-------|
| 1      | 1      | 2     |
| 1      | 0      | -1    |
| 0      | 1      | -1    |
| 0      | 0      | 2     |

### 3.3 Changing the $k_i$ to Satisfy the mod 7 Constraints

Finally to satisfy the mod 7 constraints in 2.13 and 2.14 we shift the mod 7 values by adding or subtracting to  $k_1$  and  $k_2$ ; this does not change the constants  $c_i$  and it moves the mod 7 results in reverse order. Since

$$6(x + d) + m \pmod{7} \equiv (6x + m) + 6d \quad (3.8)$$

$$\equiv (6x + m) - 1d \quad (3.9)$$

$$\equiv (6x + m) - d \quad (3.10)$$

Let the substitutions be

$$j_1 \leftarrow j_1 + d_1 \quad (3.11)$$

$$j_2 \leftarrow j_2 + d_2 \quad (3.12)$$

where  $d_1$  and  $d_2$  are chosen to be positive or zero to avoid generating negative  $j_i$ . then by inspection of the  $\text{mod } 7$  table we can satisfy the constraints 2.13 and 2.14 using tables 3.2 and 3.3

Table 3.2:  $d_1$  Values to satisfy 2.13

| $x_1 \text{ mod } 7$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------------|---|---|---|---|---|---|---|
| $d_1, f_1 = 0$       | 3 | 0 | 0 | 1 | 0 | 1 | 2 |
| $d_1, f_1 = 1$       | 1 | 0 | 0 | 1 | 0 | 3 | 2 |

Table 3.3:  $d_2$  Values to satisfy 2.14

| $x_2 \text{ mod } 7$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------------|---|---|---|---|---|---|---|
| $d_2, f_2 = 0$       | 1 | 2 | 3 | 0 | 1 | 0 | 0 |
| $d_2, f_2 = 1$       | 3 | 2 | 1 | 0 | 1 | 0 | 0 |

## References

- [Boo19] Andrew R. Booker. Cracking the problem with 33. <https://arxiv.org/abs/1903.04284>, March 2019.
- [Wik19] Wikipedia. Sum of three cubes. [https://en.wikipedia.org/wiki/Sums\\_of\\_three\\_cubes](https://en.wikipedia.org/wiki/Sums_of_three_cubes), May 2019.