JOURNAL OF ALGEBRA 17, 352–368 (1971)

Characters of Hopf Algebras*

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Communicated by D. Buchsbaum

Received December 8, 1969

In the development of the character theory of finite groups, one of the key facts used in proving the orthogonality relations is the invariance property of $\sum g \in kG$. (See, e.g., the proofs of Lemma 5.1.3 in [2] and of Eq. (31.8) in [1].) Since finite-dimensional Hopf algebras [5] and the dual algebras of certain infinite-dimensional Hopf algebras [7] contain elements with properties analogous to those of $\sum g$, it is reasonable to expect that an orthogonality relation will hold for characters of such Hopf algebras. In Section 2 of this paper we prove such an orthogonality relation.

In order to include a character theory for infinite dimensional Hopf algebras, we must consider characters as elements of the Hopf algebra which are associated with comodules over the Hopf algebra, rather than as functionals on the Hopf algebra which are associated with modules over the Hopf algebra. This point of view allows a simultaneous treatment of the characters of compact Lie groups and completely reducible affine algebraic groups (taking as the Hopf algebra the algebra of representative functions), of the characters of semisimple Lie algebras over an algebraically closed field of characteristic 0 (taking $U(L)^\circ$ as the Hopf algebra; the characters studied here differ from those of Exposé 18 of [6] by a scalar multiple), and of the characters of finite groups (taking as the Hopf algebra the dual Hopf algebra to the group algebra).

We then use our results to prove that the dimension of a simple comodule of an involutory cosemisimple Hopf algebra over an algebraically closed field is not divisible by the characteristic of the field. In Section 3 we prove that the antipode γ of a cosemisimple Hopf algebra is bijective, and that γ^2 maps each simple subcoalgebra onto itself. In Section 4 we give a generalization of Maschke's Theorem: If the characteristic of the field does not divide the dimension of a finite dimensional involutory Hopf algebra, then the Hopf algebra and its dual are semisimple; if the characteristic does divide the

* This research was supported by National Science Foundation Grant GP-9584.

dimension, then the Hopf algebra and its dual cannot both be semisimple. In Sections 5 and 6 we use our results in a detailed study of finite-dimensional semisimple Hopf algebras, paying particular attention to the antipode of such a Hopf algebra. It is known that the antipode of a commutative or cocommutative Hopf algebra is of order 2 [8, p. 74]; it is also known that there exist (infinite-dimensional) Hopf algebras with antipodes of arbitrarily high order (see [4] and [8, p. 89]. Sweedler also has an unpublished example of a Hopf algebra of dimension 4 with antipode of order 4). We give a bound for the order of the antipode of a finite-dimensional semisimple Hopf algebra, and give various conditions for the antipode to be order 2 or of order 4.

Throughout this paper we will use freely the techniques and results of [5] and [7].

1. Coalgebras and Hopf Algebras

A *coalgebra* over the field k is a vector space C together with maps $\delta : C \to C \otimes C$ and $\epsilon : C \to k$ satisfying

$$(I \otimes \delta)\delta = (\delta \otimes I)\delta$$

and

$$I = (I \otimes \epsilon)\delta = (\epsilon \otimes I)\delta.$$

The maps δ and ϵ are sometimes called the *structure maps* of the coalgebra C. If $c \in C$, we will usually denote $\delta(c)$ by $\sum c_{(1)} \otimes c_{(2)}$, $(I \otimes \delta)\,\delta(c)$ by $\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc.

If C is a coalgebra, then $C^* = \hom(C, k)$ is an algebra, with multiplication defined by $(c^*d^*)(c) = (c^* \otimes d^*)\,\delta(c)$, where $c^*, d^* \in C^*$, $c \in C$. The map ϵ is an identity for this algebra. The coalgebra C is called *simple* if there are no proper subcoalgebras contained in C. C is simple if and only if the algebra C^* is simple. The coalgebra C is called *(co)semisimple* if it is spanned by its simple subcoalgebras. If C is semisimple, then it is the direct sum of its simple subcoalgebras, and the algebra C^* is the direct product of finite-dimensional simple ideals.

Denote the map from $V \otimes \cdots \otimes V$ (n times) to itself which sends $v_1 \otimes \cdots \otimes v_n$ to $v_i \otimes v_j \otimes \cdots \otimes v_k$ by $(i, j, \ldots, k)$. The coalgebra C is called *(co)commutative* if $\delta = (2, 1)\delta$, that is, if $\sum c_{(1)} \otimes c_{(2)} = \sum c_{(2)} \otimes c_{(1)}$. The coalgebra C is commutative if and only if the algebra C^* is commutative.

A *left C-comodule* is a vector space V together with a map $\psi : V \to C \otimes V$ satisfying

$$(\delta \otimes I)\psi = (I \otimes \psi)\psi$$

and

$$(\epsilon \otimes I)\psi = I.$$

If $\mathbf{v} \in V$ we will usually denote $\psi(\mathbf{v})$ by $\sum v_{(1)} \otimes \mathbf{v}_{(2)}$, $(I \otimes \psi)\,\psi(\mathbf{v})$ by $\sum v_{(1)} \otimes v_{(2)} \otimes \mathbf{v}_{(3)}$, etc. If V and W are left C-comodules, a linear map $\mathbf{f} : V \to W$ is a *C-comodule morphism* if $(I \otimes \mathbf{f})\,\psi_V = \psi_W \mathbf{f}$. Right C-comodules and morphisms of right C-comodules are defined in an analogous fashion. If V is a left C-comodule, then defining $\mathbf{v} \cdot c^* = \sum c^*(v_{(1)})\,\mathbf{v}_{(2)}$, where $c^* \in C^*$, $\mathbf{v} \in V$, makes V into a right C^*-module. In a similar fashion, right C-comodules have a left C^*-module structure. Sometimes the name *rational* C^*-module is used for a C^*-module whose module structure arises from a C-module structure. The following Lemma is immediate.

LEMMA 1.1. *Let V and W be left (right) C-modules. The linear map $\mathbf{f} : V \to W$ is a C-comodule morphism if and only if it is a C^*-module morphism.*

Note that $\delta : C \to C \otimes C$ makes C into both a left C-comodule and a right C-comodule. Therefore we have a right C^*-module and a left C^*-module structure on C. We will denote these actions of C^* on C by $c < c^*$ and $c^* > c$. That is,

$$c < c^* = \sum c^*(c_{(1)})\, c_{(2)}$$

and

$$c^* > c = \sum c^*(c_{(2)})\, c_{(1)} .$$

Since $(C \otimes V) \oplus (C \otimes W) \cong C \otimes (V \oplus W)$, if V and W are left C-modules, then $V \oplus W$ has a natural left C-comodule structure. A nonzero C-comodule is called *simple* if the only subcomodules of V are 0 and V. V is called *completely reducible* if V is the direct sum of simple subcomodules. By [7, p. 326], the coalgebra C is semisimple if and only if every C-comodule is completely reducible.

Let V be a simple left C-comodule. It follows from the discussion on p. 326 of [7] that there exists a unique minimal subcoalgebra $D \subset C$ satisfying $\psi(V) \subset D \otimes V$. The comodule V and the subcoalgebra D are finite dimensional. Consider V as a right D^*-module. The fact that D is minimal with respect to the property $\psi(V) \subset D \otimes V$ implies that D^* acts faithfully on V. The fact that V is a simple comodule implies that V is an irreducible D^*-module. These facts imply that D^* is a simple algebra, which implies that D is a simple coalgebra. Call D the *simple subcoalgebra associated with V*, and call V *a simple left comodule associated with D*. It is easily seen that two simple left comodules are isomorphic if and only if they are associated with the same simple subcoalgebra.

Now suppose that k is algebraically closed, and let $\{\mathbf{v}_1, ..., \mathbf{v}_d\}$ be a basis for V. Let $e_{ij} \in D^*$ satisfy $\mathbf{v}_k \cdot e_{ij} = \delta_{ki}\mathbf{v}_j$. Then $\{e_{ij}\}$ is a basis for D^*. Let $\{a_{ij}\}$ be the basis of D dual to the basis $\{e_{ij}\}$. We have proved:

LEMMA 1.2. *Let C be a coalgebra over the algebraically closed field k, let V be a simple left C-comodule, and let D be the simple subcoalgebra associated with V. Then for each basis $\{\mathbf{v}_1, ..., \mathbf{v}_d\}$ of V, there is a basis $\{a_{ij}\}$ of D such that*

$$\psi(\mathbf{v}_i) = \sum_{j=1}^{d} a_{ij} \otimes \mathbf{v}_j,$$

$$\delta(a_{ij}) = \sum_{k=1}^{d} a_{ik} \otimes a_{kj},$$

and

$$\epsilon(a_{ij}) = \delta_{ij}.$$

The basis $\{a_{ij}\}$ is called a *matric basis* of the coalgebra D.

Let V be a left C-comodule. Pick a basis $\{\mathbf{v}_1, ..., \mathbf{v}_d\}$ for V and let $\{\mathbf{v}_1^*, ..., \mathbf{v}_d^*\}$ be the basis of V^* dual to this basis. Define $\chi(V) \in C$ by

$$\chi(V) = \sum_i (I \otimes \mathbf{v}_i^*)\, \psi(\mathbf{v}_i).$$

It easily checked that $\chi(V)$ does not depend on the particular choice of basis $\{\mathbf{v}_1, ..., \mathbf{v}_d\}$. Call $\chi(V)$ the *character of the comodule* V. Note that $\chi(V \oplus W) = \chi(V) + \chi(W)$. If V is a simple left C-comodule, then $\chi(V)$ is called an *irreducible character*, and $\chi(V) = \sum_{i=1}^{d} a_{ii}$, where $\{a_{ij}\}$ is a matric basis of the simple coalgebra associated with V. Call $d = \dim V$ the *degree* of $\chi(V)$.

A *bialgebra* over k is a vector space H together with an algebra structure (with unit) on H and a coalgebra structure on H such that the coalgebra structure maps are (unit-preserving) algebra homomorphisms. A bialgebra will be called *semisimple* if it is semisimple as an algebra, *cosemisimple* if it is semisimple as a coalgebra, *commutative* if it is commutative as an algebra, and *cocommutative* if it is commutative as a coalgebra.

If H is a bialgebra, the set of maps $\hom(H, H)$ form a semigroup with multiplication defined by $(f^* g)(h) = \sum f(h_{(1)})g(h_{(2)})$. The map $h \mapsto \epsilon(h)1$ is an identity for this semigroup. If $I \in \hom(H, H)$ has a two-sided inverse γ, then H is called a *Hopf algebra*, and γ is called the *antipode* of H. It can be shown [8] that $\gamma : H \to H$ is an algebra and coalgebra antiendomorphism, and that if H is commutative or cocommutative, $\gamma^2 = I$. A Hopf algebra for which $\gamma^2 = I$ is called *involutory*.

LEMMA 1.3. *Let H be a Hopf algebra over the field k, and let K/k be a field extension. Then H is cosemisimple if and only if $K \otimes H$ is cosemisimple.*

Proof. This follows immediately from Eq. (3.2) of [7].

2. CHARACTERS

In this section we prove the basic orthogonality relations for matric bases and characters of a cosemisimple Hopf algebra over an algebraically closed field. These are used to prove that the dimension of a simple comodule of an involutory cosemisimple Hopf algebra over an algebraically closed field is not divisible by the characteristic of the field.

Let A be a Hopf algebra, and let V be a finite dimensional left A-comodule. The map $V^* \to \hom(V, A)$ given by $\mathbf{v}^* \mapsto \sum \mathbf{v}^*(\mathbf{v}_{(2)})\, v_{(1)}$, where $\mathbf{v} \in V$, $\mathbf{v}^* \in V^*$, can be identified using the canonical isomorphism $\hom(V, A) \simeq V^* \otimes A$ with a map $V^* \to V^* \otimes A$. It is easily checked that this map gives a right A-comodule structure on V^*. (This comodule structure is the comodule structure giving the left A^*-module structure on V^* which is adjoint to the right A^*-module structure on V.) If V is a simple left A-comodule with basis $\{\mathbf{v}_i\}$, let $a_{ij} \in A$ satisfy $\psi(\mathbf{v}_i) = \sum a_{ij} \otimes \mathbf{v}_j$, as described in Lemma 1.2. Let $\{\mathbf{v}_i^*\}$ be the basis of V^* dual to $\{\mathbf{v}_i\}$. Then $\psi(\mathbf{v}_i^*) = \sum \mathbf{v}_j^* \otimes a_{ji}$.

Let V, W be left A-comodules with V finite dimensional. We identify $\hom(V, W)$ with $V^* \otimes W$ by the canonical isomorphism. Define the map $\psi_{V,W} : \hom(V, W) \to \hom(V, W) \otimes A$ by

$$\psi_{V,W}(\mathbf{v}^* \otimes \mathbf{w}) = \sum (\mathbf{v}_{(1)}{}^* \otimes \mathbf{w}_{(2)}) \otimes v_{(2)}{}^* \gamma(w_{(1)}). \tag{2.1}$$

It is easily checked that $\psi_{V,W}$ makes $\hom(V, W)$ into a right A-comodule. If $\mathbf{f} \in \hom(V, W)$ then (identifying $\hom(V, W) \otimes A$ with $\hom(V, W \otimes A)$) we have (using $\sum \mathbf{v}^*(\mathbf{v}_{(2)})\, v_{(1)} = \sum \mathbf{v}_{(1)}{}^*(\mathbf{v})\, v_{(2)}{}^*$) that

$$(\psi_{V,W}\mathbf{f})(\mathbf{v}) = \sum (\mathbf{f}(\mathbf{v}_{(2)}))_{(2)} \otimes v_{(1)}\, \gamma((f(\mathbf{v}_{(2)}))_{(1)}). \tag{2.2}$$

PROPOSITION 2.3. *Let V, W be left A-comodules with V finite dimensional, and let $\mathbf{f} \in \hom(V, W)$. Then the following statements are equivalent:*

(a) $\mathbf{f}$: $V \to W$ *is a left A-comodule morphism;*

(b) $\mathbf{f}$: $V \to W$ *is a right A^*-module morphism;*

(c) $\psi_{V,W}\mathbf{f} = \mathbf{f} \otimes 1$;

(d) $a^* \cdot \mathbf{f} = a^*(1)\mathbf{f}$ *for all $a^* \in A^*$.*

Proof. Clearly (a) is equivalent to (b) and (c) is equivalent to (d). We will show that (a) and (c) are equivalent. Suppose $\mathbf{f}$ is a comodule morphism. This means that $\sum f(\mathbf{v})_{(1)} \otimes \mathbf{f}(\mathbf{v})_{(2)} = \sum v_{(1)} \otimes \mathbf{f}(\mathbf{v}_{(2)})$. From (2.2) we get that

$$\begin{aligned}(\psi_{V,W}\mathbf{f})(\mathbf{v}) &= \sum (\mathbf{f}(\mathbf{v}_{(2)}))_{(2)} \otimes v_{(1)}\, \gamma((f(\mathbf{v}_{(2)}))_{(1)}) \\ &= \sum \mathbf{f}(\mathbf{v}_{(3)}) \otimes v_{(1)}\, \gamma(v_{(2)}) \\ &= \mathbf{f}(\mathbf{v}) \otimes 1.\end{aligned}$$

Conversely, suppose that $\psi_{V,W}\mathbf{f} = \mathbf{f} \otimes 1$. This says that

$$\sum (\mathbf{f}(\mathbf{v}_{(2)}))_{(2)} \otimes v_{(1)}\, \gamma((f(\mathbf{v}_{(2)}))_{(1)}) = \mathbf{f}(\mathbf{v}) \otimes 1.$$

Applying $(1 \otimes \mu_A)(2, 3, 1)(\psi_W \otimes 1)$ to both sides of this equation we get

$$\sum \mathbf{f}(\mathbf{v}_{(2)})_{(3)} \otimes v_{(1)}\, \gamma(f(\mathbf{v}_{(2)}))_{(1)})(f(\mathbf{v}_{(2)}))_{(2)} = \sum \mathbf{f}(\mathbf{v})_{(2)} \otimes f(\mathbf{v})_{(1)}$$

or

$$\sum \mathbf{f}(\mathbf{v}_{(2)}) \otimes v_{(1)} = \sum \mathbf{f}(\mathbf{v})_{(2)} \otimes f(\mathbf{v})_{(1)},$$

which implies that $\mathbf{f}$ is an A-comodule morphism. This completes the proof of the Proposition.

Assume now that k is algebraically closed, and that the Hopf algebra A is cosemisimple. The latter assumption implies by the results of [7] that there exists an element $\lambda \in A^*$ such that $a^*\lambda = a^*(1)\lambda$ for all $a^* \in A^*$, and $\lambda(1) = 1$.

Let V, W be nonisomorphic simple left A-comodules. If $\mathbf{f} \in \hom(V, W)$, then $\lambda \cdot \mathbf{f}$ satisfies $a^* \cdot (\lambda \cdot \mathbf{f}) = a^*(1)(\lambda \cdot \mathbf{f})$ for all $a^* \in A^*$, so by Proposition 2.3 $\lambda \cdot \mathbf{f}: V \to W$ is an A^*-module morphism. Similarly, if $\mathbf{g} \in \hom(V, V)$ then $\lambda \cdot \mathbf{g}$: $V \to V$ is an A^*-module morphism. Since V and W are irreducible A^*-modules and K is algebraically closed, Schur's Lemma implies that $\lambda \cdot \mathbf{f} = 0$ and that $\lambda \cdot \mathbf{g}$ is a scalar which we will denote $z(\mathbf{g})$.

Fix bases $\{\mathbf{v}_i\}$ and $\{\mathbf{w}_k\}$ of V and W, let $\{\mathbf{v}_i^*\}$ and $\{\mathbf{w}_k^*\}$ be the bases of V^* and W^* dual to these bases, and let $\{a_{ij}\}$, $\{b_{kl}\}$ be the sets of elements of A described in Lemma 1.2 satisfying $\psi(\mathbf{v}_i) = \sum_j a_{ij} \otimes \mathbf{v}_j$ and $\psi(\mathbf{w}_k) = \sum_l b_{kl} \otimes \mathbf{w}_l$. Consider the linear map from V to W identified with $\mathbf{v}_m^* \otimes \mathbf{w}_n$. We have shown that $\lambda \cdot (\mathbf{v}_m^* \otimes \mathbf{w}_n) = 0$. Computing directly, we get

$$\lambda \cdot (\mathbf{v}_m^* \otimes \mathbf{w}_n) = \sum_{i,j} \lambda(a_{im}\gamma(b_{nj}))\, \mathbf{v}_i^* \otimes \mathbf{w}_j .$$

We conclude that

$$\lambda(a_{im}\gamma(b_{nj})) = 0. \tag{2.4}$$

Consider now the map from V to itself identified with $\mathbf{v}_m^* \otimes \mathbf{v}_n$. Denote $z(\mathbf{v}_m^* \otimes \mathbf{v}_n)$ by z_{mn}. A similar argument yields that

$$\lambda(a_{im}\gamma(a_{nj})) = z_{mn}\delta_{ij} . \tag{2.5}$$

From Eq. (2.5) we get

$$1 = \lambda(\epsilon(a_{11})1) = \lambda\left(\sum_m a_{1m}\gamma(a_{m1})\right) = \sum_m z_{mm}\,. \tag{2.6}$$

From Eq. (2.4) we get

$$\lambda(\chi(V)\,\gamma(\chi(W))) = \lambda\left(\sum_{m,n} a_{mm}\gamma(b_{nn})\right) = 0.$$

From Eqs. (2.5) and (2.6) we get

$$\lambda(\chi(V)\,\gamma(\chi(V))) = \lambda\left(\sum_{m,n} a_{mm}\gamma(a_{nn})\right) = \sum_m z_{mm} = 1.$$

We have proved:

THEOREM 2.7. *Let A be a cosemisimple Hopf algebra over the algebraically closed field k, and let $\lambda \in A^*$ satisfy $a^*\lambda = a^*(1)\lambda$ and $\lambda(1) = 1$. If V, W are simple nonisomorphic left A-comodules then*

$$\lambda(\chi(V)\,\gamma(\chi(W))) = 0$$

and

$$\lambda(\chi(V)\,\gamma(\chi(V))) = 1.$$

Suppose now that A is involutory. Since $\gamma^2 = I$,

$$\epsilon(a)1 = \gamma(\epsilon(a)1) = \gamma(\sum a_{(1)}\,\gamma(a_{(2)})) = \sum a_{(2)}\,\gamma(a_{(1)}).$$

In particular,

$$\delta_{nm}1 = \epsilon(a_{nm})1 = \sum_i a_{im}\gamma(a_{ni}).$$

This fact together with Eq. (2.5), gives

$$\delta_{nm} = \lambda(\delta_{nm}1) = \lambda\left(\sum_i a_{im}\gamma(a_{ni})\right) = \sum_i z_{mn} = z_{mn}\dim V.$$

In particular we have $1 = z_{11}\dim V$, which proves:

THEOREM 2.8. *Let A be an involutory cosemisimple Hopf algebra over the algebraically closed field k of characteristic p. If V is a simple left A-comodule, then p does not divide* $\dim V$.

3. The Square of the Antipode

In this section we prove that if A is a cosemisimple Hopf algebra, then γ^2 maps each simple subcoalgebra of A onto itself. This allows us to derive a description of the automorphism γ^2 in terms of the matrix (z_{mn}) which appears in the orthogonality relation Eq. (2.5).

Let A be a Hopf algebra over k. Call $a \in A$ *invariant* if $\sum a_{(1)} \otimes a_{(2)} = \sum a_{(2)} \otimes a_{(1)}$. Denote the set of all invariant elements of A by $i(A)$. Since $\sum (ab)_{(1)} \otimes (ab)_{(2)} = \sum a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$, $i(A)$ is a subalgebra of A. It is easily checked that the character of a comodule is invariant.

Lemma 3.1. *Let A be a cosemisimple Hopf algebra over the algebraically closed field k. The set of characters of the simple comodules of A is a basis for $i(A)$. If C is a simple subcoalgebra of A, then $C \cap i(A)$ is one-dimensional, and consists of all scalar multiples of the character of a simple comodule associated with C.*

Proof. We first show that the set of characters of simple comodules of A is linearly independent. Suppose $t_1\chi_1 + \cdots + t_n\chi_n = 0$, where the χ_i *are* characters of non-isomorphic simple comodules. Then

$$0 = \lambda((t_1\chi_1 + \cdots + t_n\chi_n)\,\gamma(\chi_i)) = t_i\,,$$

by Theorem 2.7.

Suppose that $a \in i(A)$. Write $a = a_1 + \cdots + a_n$, where the a_i lie in distinct simple subcoalgebras. Since the linear span of a set of distinct simple subcoalgebras is the direct sum of these subcoalgebras, each $a_i \in i(A)$. We will complete the proof by showing that a_i is a scalar multiple of the character of the simple comodule associated with the simple subcoalgebra C_i containing a_i. C_i^* is a finite-dimensional simple algebra over k, and since a_i is invariant the functional it induces on C_i^* satisfies $a_i(c^*d^*) = a_i(d^*c^*)$ for all $c^*, d^* \in C_i^*$. This implies that a_i is a multiple of the trace map on C_i^*, or equivalently, that a_i is a multiple of the character of a simple comodule associated with C_i. This completes the proof of the Lemma.

Corollary 3.2. *Let A be a cosemisimple Hopf algebra over the algebraically closed field k. If $a \in i(A)$, then $a = \sum \lambda(a\gamma(\chi))\chi$.*

Theorem 3.3. *Let A be a cosemisimple Hopf algebra over k. Then the antipode γ of A is bijective, and for each simple subcoalgebra $C \subset A$, $\gamma^2(C) = C$.*

Proof. We first show that γ is injective. By [7] pp. 330-331, the map $j\colon A \to A^*$ defined by $j(a)(b) = \lambda(\gamma(a)b)$ is injective. If $\gamma(a) = 0$ then

$j(a)(b) = \lambda(0b) = 0$, and the injectivity of j implies that $a = 0$. Now consider $\gamma^*: A^* \to A^*$, the algebra antiendomorphism which is the transpose of $\gamma: A \to A$. Since γ is injective, γ^* is surjective. Therefore, for any $a^* \in A^*$, $a^* = \gamma^*(b^*)$ for some $b^* \in A^*$, so $\gamma^*(\lambda)\, a^* = \gamma^*(\lambda)\, \gamma^*(b^*) = \gamma^*(b^*\lambda) = b^*(1)\, \gamma^*(\lambda) = a^*(1)\, \gamma^*(\lambda)$. Also $\gamma^*(\lambda)(1) = \lambda(\gamma(1)) = \lambda(1) = 1$. Therefore $\lambda = \gamma^*(\lambda)\lambda = \gamma^*(\lambda)$.

We now show that for each simple subcoalgebra $C \subset A$, $\gamma^2(C) = C$. Since A is spanned by its simple subcoalgebras, this will imply that γ is surjective. Suppose first that k is algebraically closed. Since γ is injective and is a coalgebra antiendormorphism, $\gamma(C)$ is a simple subcoalgebra. Similarly $\gamma^2(C)$ is a simple subcoalgebra. Let χ_C be the character of a simple comodule associated with C. Then $\gamma(\chi_C)$ is the character of a simple comodule associated with $\gamma(C)$, and $\gamma^2(\chi_C)$ is the character of a simple comodule associated with $\gamma^2(C)$. In particular $\gamma^2(\chi_C)$ is invariant, so by Corollary 3.2

$$\begin{aligned}\gamma^2(\chi_C) &= \sum \lambda(\gamma^2(\chi_C)\, \gamma(\chi))\chi \\ &= \sum \gamma^*(\lambda)(\chi\gamma(\chi_C))\chi \\ &= \sum \lambda(\chi\gamma(\chi_C))\chi = \chi_C\, .\end{aligned}$$

Therefore by Lemma 3.1, $C \cap i(A) = \gamma^2(C) \cap i(A)$. Since C and $\gamma^2(C)$ are simple subcoalgebras, this implies that $\gamma^2(C) = C$. If k is not algebraically closed, let K be its algebraic closure. If C is a simple subcoalgebra of A, then $K \otimes_k C = C_1' \oplus \cdots \oplus C_n'$, where the C_i' are simple subcoalgebras of the Hopf algebra $K \otimes_k A$. The above argument shows that $\gamma^2(C_i') = C_i'$, which implies that $\gamma^2(K \otimes_k C) = K \otimes_k C$. Since A is the direct sum of its simple subcoalgebras, it follows that $\gamma^2(C) = C$ for each simple subcoalgebra C.

The following fact from the proof will be used in the sequel:

COROLLARY 3.4. *Let A be a cosemismple Hopf algebra over k. For each $a \in A$, there exists $b \in A$ such that $\lambda(ab) \neq 0$.*

PROPOSITION 3.5. *Let A be a cosemisimple Hopf algebra over the algebraically closed field k. Let C be a simple subcoalgebra of A, and let $\{a_{ij}\}$ be a matric basis of C with $\lambda(a_{im}\gamma(a_{nj})) = z_{mn}\delta_{ij}$. Then the matrix (z_{ij}) is invertible with inverse $(\bar{z}_{ij})$, and*

$$\gamma^2(a_{ij}) = \sum_{k,l} z_{ki}\bar{z}_{jl}a_{kl}\, .$$

Proof. The coalgebra automorphism $\gamma^2 \mid C$ induces an algebra automorphism of C^*, which by the Noether–Skolem Theorem is inner. That is, there exists a matrix (x_{ij}) with inverse $(\bar{x}_{ij})$ such that $(\gamma^2 \mid C)^* (a_{ij}^*) = \sum_{k,l} x_{ki}\bar{x}_{jl}a_{kl}^*$. (Recall that $\{a_{ij}^*\}$ is the basis of C^* dual to the basis $\{a_{ij}\}$.) This says $\gamma^2(a_{ij}) = \sum_{k,l} x_{ik}\bar{x}_{lj}a_{kl}$. Now, since $\gamma^*(\lambda) = \lambda$, it follows that $\lambda(\gamma^2(a_{ij})\,\gamma(a_{mn})) = \lambda(a_{mn}\gamma(a_{ij}))$. This implies that

$$\sum_{k,l} x_{ik}\bar{x}_{lj}\lambda(a_{kl}\gamma(a_{mn})) = \lambda(a_{mn}\gamma(a_{ij})).$$

This in turn implies $\sum_l x_{in}\bar{x}_{lj}z_{lm} = z_{ni}\delta_{mj}$. Multiplying both sides of this last equation by x_{jk} and summing on j, we conclude that $x_{in}z_{km} = z_{ni}x_{mk}$.

If $z_{ij} = 0$ for all i, j we would have $\lambda(c\gamma(d)) = 0$ for all $c, d \in C$ which would imply that $\lambda(ca) = 0$ for all $c \in C$, $a \in A$, which would contradict Corollary 3.4. Therefore some $z_{rs} \neq 0$. Now $x_{in} = (z_{rs}^{-1}x_{sr})\,z_{ni}$. Since the matrix (x_{ij}) is invertible, $x_{sr} \neq 0$, and $z_{ni} = (x_{rs}^{-1}z_{rs})\,x_{in}$. The inverse to the matrix (z_{ij}) is given by $\bar{z}_{ij} = (z_{rs}^{-1}x_{sr})\,\bar{x}_{ji}$. It now follows that

$$\gamma^2(a_{ij}) = \sum_{k,l} z_{ki}\bar{z}_{jl}a_{kl}. \qquad \text{Q.E.D.}$$

COROLLARY 3.6. *Let A be a cosemisimple Hopf algebra over the algebraically closed field k. Then the following conditions are equivalent:*

(a) *A is involutory*

(b) *For every simple subcoalgebra C, $\lambda(a_{im}\gamma(a_{nj})) = (1/d)\,\delta_{mn}\delta_{ij}$, where $\{a_{ij}\}$ is a matric basis of C and d is the dimension of a simple comodule associated with C.*

(c) *$\lambda(\sum a_{(2)}\gamma(a_{(1)})) = \epsilon(a)$, for all $a \in A$.*

Proof. We will prove (a) implies (c), (c) implies (b), and (b) implies (a). Suppose A is involutory. Then

$$\begin{aligned}\lambda(\textstyle\sum a_{(2)}\gamma(a_{(1)})) &= \gamma^*(\lambda)(\textstyle\sum a_{(2)}\gamma(a_{(1)}))\\ &= \lambda(\textstyle\sum \gamma^2(a_{(1)})\,\gamma(a_{(2)}))\\ &= \lambda(\textstyle\sum a_{(1)}\gamma(a_{(2)})) = \epsilon(a),\end{aligned}$$

so (a) implies (c).

Now suppose condition (c) holds. Then

$$\begin{aligned}dz_{mn} &= \sum_i \lambda(a_{im}\gamma(a_{ni}))\\ &= \lambda\left(\textstyle\sum (a_{nm})_{(2)}\gamma((a_{nm})_{(1)})\right)\\ &= \epsilon(a_{nm}) = \delta_{nm},\end{aligned}$$

so $z_{mn} = (1/d)\,\delta_{mn}$. Therefore condition (b) holds.

If $z_{ij} = (1/d)\,\delta_{ij}$, then $\bar{z}_{ij} = d\delta_{ij}$, so by the Proposition, $\gamma^2(a_{ij}) = a_{ij}$, so condition (b) implies condition (a).

4. Maschke's Theorem

Maschke's Theorem states that the group algebra of a finite group is semisimple if and only if the characteristic of the field does not divide the order of the group. A more or less direct Hopf algebraic translation of this (Theorem 3.2 of [3]) would be: Let A be a cocommutative (therefore involutory) cosemisimple Hopf algebra over k. Then A is semisimple if and only if the characteristic of k does not divide the dimension of A. In this section we prove: Let A be an involutory cosemisimple Hopf algebra over k. Then A is semisimple if and only if the characteristic of k does not divide the dimension of A.

Let A be a finite-dimensional cosemisimple involutory Hopf algebra over the algebraically closed field k. Then the Hopf algebra A^* is semisimple, so by Proposition 3 of [5] there exists a nonsingular integral Λ^* in A^* with $\Lambda^*(1) = 1$. Let Λ be a left integral in A. Since A is involutory and A^* is unimodular (Proposition 4 of [5]), by the second Corollary to Proposition 8 of [5] $\sum \Lambda_{(2)} \otimes \Lambda_{(1)} = \sum \Lambda_{(1)} \otimes \Lambda_{(2)}$. Therefore by Lemma 3.1 Λ is a linear combination of characters. This implies that $\gamma(\Lambda)$ is a linear combination of irreducible characters, say $\gamma(\Lambda) = \sum_i c_i\chi_i$. Now consider

$$\begin{aligned}\Lambda^*(\chi_j\Lambda) &= \Lambda^*(\chi_j\gamma^2(\Lambda)) \\ &= \Lambda^*\left(\chi_j\gamma\left(\sum_i c_i\chi_i\right)\right) \\ &= c_j\,.\end{aligned}$$

But $\Lambda^*(\chi_j\Lambda) = \epsilon(\chi_j)\,\Lambda^*(\Lambda) = d_j\Lambda^*(\Lambda)$, where d_j is the degree of the irreducible character χ_j. Therefore $c_j = \Lambda^*(\Lambda)\,d_j$. Now

$$\begin{aligned}\Lambda = \gamma^2(\Lambda) &= \sum_i c_i\gamma(\chi_i) \\ &= \Lambda^*(\Lambda)\sum_i d_i\gamma(\chi_i) = \Lambda^*(\Lambda)\sum_i d_i\chi_i\,,\end{aligned}$$

since the irreducible characters χ_i and $\gamma(\chi_i)$ have the same degree. By Proposition 7 of [5] we may pick Λ with $\Lambda^*(\Lambda) = 1$. We have proved

Proposition 4.1. *Let A be a finite-dimensional cosemisimple involutory Hopf algebra over the algebraically closed field k. Then $\sum d_\chi\chi$, the sum ranging over all irreducible characters of A, is a nonsingular left integral in A, where d_χ is the the degree of the irreducible character χ.*

COROLLARY 4.2. *Let A be a finite-dimensional cosemisimple involutory Hopf algebra. Then A is unimodular.*

THEOREM 4.3. *Let H be a finite-dimensional involutory Hopf algebra over the field k.*

(1) *If the characteristic of k does not divide the dimension of H, then H is semisimple.*

(2) *If H and H^* are semisimple, then the characteristic of k does not divide the dimension of H.*

Proof. (1) By Proposition 9 of [5] $T \in H$ defined by $p(T) = \mathrm{Tr}(L(p))$, where $L(p)\colon H^* \to H^*$ is defined by $L(p)(q) = pq$, is an integral in H, so $T = t\Lambda$ for some $t \in k$. Since $\epsilon(T) = \mathrm{Tr}(L(\epsilon)) = (\dim H)1$, if char k does not divide dim H, $0 \neq \epsilon(T) = t\epsilon(\Lambda)$ so then $\epsilon(\Lambda) \neq 0$. Therefore by Proposition 3 of [5] H is semisimple.

(2) Suppose first that k is algebraically closed. If H^* is semisimple, then $\sum d_x\chi$ is a nonsingular integral in H. If H is semisimple, then $0 \neq \epsilon(\sum d_x\chi) = \sum d_x\epsilon(\chi) = \sum d_x^2 = (\dim H)1$, so char k does not divide dim H.

Now suppose k arbitrary, and let K be the algebraic closure of k. The semisimplicity of H and H^* imply that $K \otimes_k H$ and $(K \otimes_k H)^* = K \otimes_k H^*$ are semisimple. Therefore char k = char K does not divide $\dim_K K \otimes_k H = \dim_k H$.

5. FINITE-DIMENSIONAL HOPF ALGEBRAS

Using the techniques of Section 7 of [5], we prove that if γ is the antipode of a finite-dimensional unimodular Hopf algebra, then γ^4 is the inner automorphism induced by a grouplike element. In particular, this implies that the order of the antipode is $< 4(\dim H)$. With suitable restrictions on the characteristic of k, this result can be used to prove that the matrix (z_{mn}) which appears in the orthogonality relation (2.5) can be made diagonal by a suitable choice of a basis of the simple comodule.

Let H be a finite-dimensional unimodular Hopf algebra over k. Let Λ be a two-sided integral in H, and let Λ^* be a left integral in H^*. By Proposition 7 of [5] we can assume that $\Lambda^*(\Lambda) = 1$. If $p, q \in H^*$, then $p(\Lambda^*q) = \epsilon(p)(\Lambda^*q)$, so by the uniqueness of the left integral, Λ^*q is a scalar multiple of Λ^*. Therefore there exists $a \in H$ such that $\Lambda^*q = q(a)\,\Lambda^*$. It is easily checked that the map $H^* \mapsto k$ defined by $q \to q(a)$ is an algebra homomorphism.

Therefore a satisfies $\delta(a) = a \otimes a$. The set of elements which are powers of a is a linearly independent set by Proposition 3.2.1 of [8]. Since H is finite-dimensional this implies that the order t of a is $\leqslant \dim H$. If $t = \dim H$ then H is a group algebra, and so H^* is unimodular. That is, $a = 1$ and so $1 = t = \dim H$. We have shown that the order of a is $< \dim H$ except when $\dim H = 1$.

Let b be the bilinear form on H^* associated with Λ, and let b^* be the bilinear form on H associated with Λ^*. Note that $b^*(\Lambda, 1) = b(\Lambda^*, 1) = b(1, \Lambda^*) = \Lambda^*(\Lambda) = 1$. Therefore

$$\begin{aligned} p(\gamma(h)) &= b(p < \gamma(h), \Lambda^*) \\ &= b(p, \Lambda^* < h) = p(b_r b_l^*(h)), \end{aligned}$$

so

$$\gamma(h) = b_r b_l^*(h) \qquad \text{for all } h \in H. \tag{5.1}$$

Also

$$\begin{aligned} p(\gamma^{-1}(h)a) &= \sum p_{(1)}(\gamma^{-1}(h))\, p_{(2)}(a) \\ &= b(\Lambda^*, p < \gamma^{-1}(h)) \\ &= b(\Lambda^* < h, p) = p(b_l b_l^*(h)), \end{aligned}$$

so

$$\gamma^{-1}(h)a = b_l b_l^*(h) \qquad \text{for all } h \in H. \tag{5.2}$$

Since H is unimodular, $\Lambda h = \epsilon(h)\Lambda$. It follows that

$$\begin{aligned} \gamma^{-1}(p)(h) &= b^*(\Lambda, h < \gamma^{-1}(p)) \\ &= b^*(\Lambda < p, h) = b_l^* b_l(p)(h), \end{aligned}$$

so

$$\gamma^{-1}(p) = b_l^* b_l(p) \qquad \text{for all } p \in H^*. \tag{5.3}$$

Again using the fact that H is unimodular, we conclude from the first Corollary to Proposition 8 of [5] that $\gamma(\Lambda) = \Lambda$. Then

$$\begin{aligned} q(b_l(p)) &= b(p, q) = pq(\Lambda) = pq(\gamma(\Lambda)) \\ &= (\gamma(q)\,\gamma(p))(\Lambda) = b(\gamma(q), \gamma(p)) \\ &= q(\gamma b_r \gamma(p)), \end{aligned}$$

so $b_l(p) = \gamma b_r \gamma(p)$. This implies

$$b_l \gamma^{-1}(p) = \gamma b_r(p) \qquad \text{for all } p \in H^*. \tag{5.4}$$

Since $\delta(a) = a \otimes a$, $\gamma(a) = a^{-1}$. Now consider

$$\begin{aligned} a^{-1}ha &= \gamma^{-1}(\gamma(h)a)a \\ &= b_l b_l{}^*(\gamma(h)a) \\ &= b_l b_l{}^*(\gamma^{-1}(\gamma^2(h))a) \\ &= b_l b_l{}^* b_l b_l{}^* \gamma^2(h) \\ &= b_l \gamma^{-1} b_l{}^* \gamma^2(h) \\ &= \gamma b_r b_l{}^* \gamma^2(h) \\ &= \gamma^4(h), \end{aligned}$$

the second and fourth equalities following from Eq. (5.2), the fifth equality from Eq. (5.3), the sixth equality from Eq. (5.4), and the seventh equality from Eq. (5.1). We have proved

THEOREM 5.5. *Let H be a finite-dimensional unimodular Hopf algebra, let Λ^* be a nonsingular left integral in H^*, and let $a \in H$ satisfy $\Lambda^* p = p(a)\,\Lambda^*$ for all $p \in H^*$. Then $\gamma^4(h) = a^{-1}ha$ for all $h \in H$.*

COROLLARY 5.6. *Let H be a finite-dimensional unimodular Hopf algebra. Then the order of the antipode of H is $< 4(\dim H)$.*

COROLLARY 5.7. *Let H be a finite-dimensional Hopf algebra. If H and H^* are unimodular, then the order of the antipode of H is 1, 2, or 4.*

Suppose now that k is algebraically closed, and let A be a finite-dimensional cosemisimple Hopf algebra over k. A has a basis $\{a_{ij}^{(t)}\}$, $t = 1,\ldots, r$; $i, j = 1,\ldots, d_t$ with

$$\delta(a_{ij}^{(t)}) = \sum_k a_{ik}^{(t)} \otimes a_{kj}^{(t)}$$

and

$$\epsilon(a_{ij}^{(t)}) = \delta_{ij}\,.$$

If Λ^* is an integral in A^* with $\Lambda^*(1) = 1$, then

$$\Lambda^*(a_{ij}^{(t)}\gamma(a_{kl}^{(u)})) = z_{jk}^{(t)}\delta_{il}\delta_{tu}\,.$$

For $t = 1,\ldots, r$, let $(c_{ij}^{(t)})$ be an invertible $d_t \times d_t$ matrix with inverse $(\bar{c}_{ij}^{(t)})$, and let $a_{kl}'^{(t)} = \sum_{i,j} \bar{c}_{ki}^{(t)} c_{jl}^{(t)} a_{ij}^{(t)}$. Then $\{a_{ij}'^{(t)}\}$ is a basis of A satisfying

$$\delta(a_{ij}'^{(t)}) = \sum_k a_{ik}'^{(t)} \otimes a_{kj}'^{(t)},$$

$$\epsilon(a_{ij}'^{(t)}) = \delta_{ij}\,,$$

and

$$\Lambda^*(a_{ij}'^{(t)}\gamma(a_{kl}'^{(u)})) = z_{jk}'^{(t)}\delta_{il}\delta_{tu}$$

$$= \sum_{m,n} c_{mj}^{(t)} z_{mn}^{(t)} \bar{c}_{kn}^{(t)} \delta_{il}\delta_{tu} .$$

Thus, replacing the matric basis $\{a_{ij}^{(t)}\}$ by the matric basis $\{a_{ij}'^{(t)}\}$ has the effect of conjugating the matrices $(z_{ij}^{(t)})$ by the transpose of the matrices $(c_{ij}^{(t)})$. Therefore we may assume that the matrices $(z_{ij}^{(t)})$ are in the Jordan Canonical Form. Considering the basis of A^* dual to the basis $\{a_{ij}^{(t)}\}$, we see that $A^* = \sum_{t=1}^{r} \oplus M(d_t)$, where $M(d)$ denotes the ring of $d \times d$ matrices over k. Proposition 3.5 implies that the automorphism $\gamma^2:A^* \to A^*$ is conjugation by the element $(..., {}^tZ^{(t)},...)$ in A^*, where ${}^tZ$ denotes the transpose of the matrix Z. Let f be the order of the automorphism γ^2. By Corollary 5.6, $f < 2(\dim A)$. Note that $({}^tZ^{(t)})^f$ is a scalar matrix. Since $Z^{(t)}$ is assumed to be in the Jordan Canonical Form, if char $k = 0$ or is $> f$ we can conclude that $Z^{(t)}$ is a diagonal matrix with $z_{ij}^{(t)} = \delta_{ij}\theta_i^{(t)}u^{(t)}$, where the $\theta_i^{(t)}$ are f-th roots of 1. From Eq. (2.6) we get that

$$1 = \sum_i z_{ii}^{(t)} = \sum_i \theta_i^{(t)}u^{(t)}, \quad \text{so} \quad u^{(t)} = \left(\sum_i \theta_i^{(t)}\right)^{-1}.$$

This proves

THEOREM 5.8. *Let A be a finite-dimensional cosemisimple Hopf algebra over k with antipode γ and integral $\Lambda^* \in A^*$ satisfying $\Lambda^*(1) = 1$. Let f be the order of the automorphism γ^2. Assume that the field k is algebraically closed, and that the characteristic of $k = 0$ or is $> f$. Then A has a basis $\{a_{ij}^{(t)}\}$, $t = 1,..., r$; $i, j = 1,..., d_t$ such that*

$$\delta(a_{ij}^{(t)}) = \sum_k a_{ik}^{(t)} \otimes a_{kj}^{(t)},$$

$$\epsilon(a_{ij}^{(t)}) = \delta_{ij} ,$$

and

$$\Lambda^*(a_{ij}^{(t)}\gamma(a_{kl}^{(u)})) = \delta_{tu}\delta_{il}\delta_{jk}\theta_j^{(t)} \Big/ \left(\sum_m \theta_m^{(t)}\right),$$

where the $\theta_j^{(t)}$ are f-th roots of 1. The automorphism γ^2: $A \to A$ is given by

$$\gamma^2(a_{ij}^{(t)}) = (\theta_i^{(t)}/\theta_j^{(t)})\, a_{ij}^{(t)} .$$

6. Duality

If H is a finite-dimensional Hopf algebra, then the bilinear form b on H^* gives rise to isomorphisms b_l and b_r between H^* and H. In this section we show that if H is semisimple, then $b_l(i(H^*))$ and $b_r(i(H^*))$ are translates of the center of H, and equal the center if and only if H is involutory. Also $b_l(i(H^*)) = b_r(i(H^*))$ if and only if $\gamma^4 = I$.

Let H be a finite-dimensional semisimple Hopf algebra over the algebraically closed field k. By Proposition 3.5 there exists an invertible element $c \in H$ such that $\gamma^2(h) = c^{-1}hc$ for all $h \in H$. If $x \in i(H^*)$, then $\sum x_{(2)} \otimes x_{(1)} = \sum x_{(1)} \otimes x_{(2)}$, so $h > x = x < h$ for all $h \in H$. Let Λ be a nonsingular two-sided integral in H, and let b be the bilinear form on H^* associated with Λ. The fact that Λ is a left integral in H implies that $b(p < \gamma(h), q) = b(p, q < h)$, which implies that $b_l(p < \gamma(h)) = hb_l(p)$ and $\gamma(h)\, b_r(q) = b_r(q < h)$. The fact that Λ is a right integral implies that $b(h > p, q) = b(p, \gamma(h) > q)$, which implies that $b_l(h > p) = b_l(p)\, \gamma(h)$ and $b_r(q)h = b_r(\gamma(h) > q)$.

Let $x \in i(H^*)$, and consider

$$b_l(x < \gamma(h)) = hb_l(x)$$

and

$$b_l(\gamma(h) > x) = b_l(x)\, \gamma^2(h).$$

Since $x < \gamma(h) = \gamma(h) > x$, we conclude that $hb_l(x) = b_l(x)\, \gamma^2(h)$, or $hb_l(x) = b_l(x)\, c^{-1}hc$. This implies that $hb_l(x)\, c^{-1} = b_l(x)\, c^{-1}h$, which implies that $b_l(x)\, c^{-1} \in Z(H)$, the center of the algebra H. Since $\dim i(H^*) = \dim Z(H)$ is the number of simple components of the algebra H, we conclude that $b_l(i(H^*)) = Z(H)c$. Similarly, $b_r(i(H^*)) = c^{-1}Z(H) = Z(H)\, c^{-1}$. We have proved

Proposition 6.1. *Let H be a finite-dimensional semisimple Hopf algebra over the algebraically closed field k, and suppose $\gamma^2(h) = c^{-1}hc$ for all $h \in H$. Then*

$$b_l(i(H^*)) = Z(H)c$$

and

$$b_r(i(H^*)) = Z(H)\, c^{-1}$$

Corollary 6.2. *Let H be a finite-dimensional Hopf algebra over the algebraically closed field k. Then the following statements are equivalent*:

(a) $b_l(i(H^*)) = Z(H)$;

(b) $b_l(i(H^*)) \cap Z(H)$ *contains an invertible element*;

(c) $b_r(i(H^*)) = Z(H)$;

(d) $b_r(i(H^*)) \cap Z(H)$ *contains an invertible element*;

(e) *H is involutory.*

Proof. It is clear that (a) implies (b). If $u \in b_l(i(H^*)) \cap Z(H)$ is invertible, then $u = zc$ with $z \in Z(H)$ invertible. Therefore $c = z^{-1}u \in Z(H)$, so $\gamma^2(h) = c^{-1}hc = h$. Therefore (b) implies (e). If $\gamma^2 = I$, then $c \in Z(H)$, which implies $b_l(i(H^*)) = Z(H)c = Z(H)$. Therefore (e) imples (a). In a similar fashion, (c), (d), and (e) are equivalent.

COROLLARY 6.3. *Let H be a finite-dimensional semisimple Hopf algebra over the algebraically closed field k. Then the following statements are equivalent*:

(a) $b_l(i(H^*)) = b_r(i(H^*))$

(b) $b_l(i(H^*)) \cap b_r(i(H^*))$ *contains an invertible element.*

(c) $\gamma^4 = I$.

Proof. It is clear that (a) implies (b) and that (c) implies (a). If $u = z_1c = z_2c^{-1}$ is invertible, $z_i \in Z(H)$, then z_1 is invertible, and $c^2 = z_1^{-1}z_2 \in Z(H)$, which implies that $\gamma^4 = I$. Thus (b) implies (c).

ACKNOWLEDGMENT

Moss Sweedler made many helpful suggestions and thought-provoking comments during the preparation of this paper; I wish to express my gratitude to him.

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