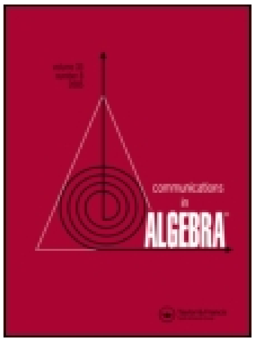

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FAITHFULLY FLAT HOPF BI-GALOIS EXTENSIONS

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This article is devoted to faithfully flat Hopf bi-Galois extensions defined by Fischman, Montgomery, and Schneider. Let H be a Hopf algebra with bijective antipode. Given a faithfully flat right H-Galois extension A/R and a right H-comodule subalgebra $C \subset A$ such that A is faithfully flat over C, we provide necessary and sufficient conditions for the existence of a Hopf algebra W so that A/C is a left W-Galois extension and A a (W, H)-bicomodule algebra. As a consequence, we prove that if $R = k$, there is a Hopf algebra W such that A/C is a left W-Galois extension and A a (W, H)-bicomodule algebra if and only if C is an H-submodule of A with respect to the Miyashita–Ulbrich action.

Key Words: Hopf algebra; Hopf bi-Galois extension; Hopf–Galois object.

2000 Mathematics Subject Classification: 16W30.

1. INTRODUCTION

Let H be a Hopf algebra over a field k, A a right H-comodule algebra, and A/k a right H-Galois extension. Van Oystaeyen and Zhang [16] constructed a Hopf algebra L along with a coaction $A \to A \otimes L$ such that A/k is an L-Galois extension under the additional assumption that A and H are commutative. In [3], Greither proved that the construction in [16] is an involution, that is, applying it to the L-Galois extension A/k we get the original Hopf algebra H with the correct comodule structure back. Later, Schauenburg introduced bi-Galois objects in [8, 11]: Let W, H be Hopf algebras. A W-H-bi-Galois object is a (W, H)-bicomodule algebra A which is simultaneously a left W-Galois object and a right H-Galois object. He showed that, for every right H-Galois object A, there exists a unique Hopf algebra W such that A is a left W-Galois object and a (W, H)-bicomodule algebra. In [2], Fischman, Montgomery, and Schneider introduced faithfully flat bi-Galois extension so that Schauenburg's bi-Galois object is a special case of it. As one of the main results, they proved that the coinvariants in a faithfully flat bi-Galois extension (U, W, H, A) are β-Frobenius extensions (see [6, 7]) provided the Hopf algebras extension $U \subset W$ being of right integral type.

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In this article, we generalize Schauenburg's Hopf bi-Galois extensions theory to Hopf Galois extensions of arbitrary subalgebras.

In Section 2, some results on Hopf Galois extensions are reviewed. In particular, we describe the structure theorem for Hopf modules in the form of categories equivalence that will be needed in the sequel.

In Section 3, we use the notion of Galois extension in the sense of Schneider [12]. Let H be a Hopf algebra, $\overline{H} = H/I$ a quotient coalgebra, and a quotient right H-module of H. Assume that A is a right H-comodule algebra, then it has a right $\overline{H}$-comodule structure in a natural way. If the canonical map $A \otimes_{A^{co\overline{H}}} A \to A \otimes \overline{H}$, $x \otimes y \mapsto xy_{\langle 0\rangle} \otimes \overline{y_{\langle 1\rangle}}$ is an isomorphism, then $A/A^{co\overline{H}}$ is called an $\overline{H}$-Galois extension. It is showed that under certain conditions, the property of being a Galois extension is inherited by comodule subalgebras. That is, if A/R is a Galois extension, then B/S is also a Galois extension where B is a comodule subalgebra of A and $S = R \cap B$.

Section 4 is devoted to the conditions under which a left Hopf Galois extension can be constructed from a faithfully flat right Hopf Galois extension. Our main result, Theorem 4.5, shows that: Let H be a Hopf algebra with bijective antipode and A/R a faithfully flat right H-Galois extension. For a right H-comodule subalgebra C of A such that A is faithfully flat over C, there exists a Hopf algebra W such that A/C is a left W-Galois extension and A a (W, H)-bicomodule algebra if and only if $(A^{\bullet} \otimes_C A^{\bullet}_{\bullet})^{coH} \overset{\varphi}{\cong} W \otimes R_{\bullet}$ as right R-module for some algebra W which satisfies two conditions. Assume further that $R = k$. The necessary conditions in Theorem 4.5 can be simplified to C being an H-submodule of A with respect to the Miyashita–Ulbrich action.

2. PRELIMINARIES

Throughout this article, k means a field. Unless specified otherwise, all modules, algebras, coalgebras, and Hopf algebras are over k, and unadorned $\otimes$ is $\otimes_k$. For a coalgebra (C, Δ, ε) and a left (right) C-comodule (M, ρ_M), we use Sweedler's notation

$$\Delta(c) = c_{(1)} \otimes c_{(2)}, \qquad \rho_M(m) = m_{\langle -1\rangle} \otimes m_{\langle 0\rangle} (\rho_M(m) = m_{\langle 0\rangle} \otimes m_{\langle 1\rangle}),$$

where $c \in C, m \in M$.

Let W, H be Hopf algebras. Recall that the right H-comodule algebra (A, ρ) is called a right H-Galois extension of $R = A^{coH}$, if the Galois map

$$\beta : A \otimes_R A \longrightarrow A \otimes H, \qquad a \otimes b \longmapsto ab_{\langle 0\rangle} \otimes b_{\langle 1\rangle}$$

is a bijection. Similarly, if A is a left W-comodule algebra via $\rho_l : A \to W \otimes A$ and $C = {}^{coW}A$ such that the Galois map

$$\beta_l : A \otimes_C A \longrightarrow W \otimes A, \qquad a \otimes b \longmapsto a_{\langle -1\rangle} \otimes a_{\langle 0\rangle} b$$

is bijective, then A/C is called a left W-Galois extension. For convenience, we call briefly a right (left) Hopf Galois extension A of k a "right (left) Galois object."

Suppose that A is a (W, H)-bicomodule algebra. It is easy to see that $R = A^{coH}$ is a left W-comodule subalgebra and $C = {}^{coW}A$ is a right H-comodule subalgebra of

A. Fischman, Montgomery, and Schneider [2, Definition 2.6] defined the notion of faithfully flat bi-Galois extension below.

Definition 2.1. Let W and H be Hopf algebras with bijective antipodes, $U \subset W$ a Hopf subalgebra such that $U \subset W$ is a faithfully flat extension of algebras, and A a (W, H)-bicomodule algebra. Set $\overline{W} = W/WU^+$, $B = {}^{co\overline{W}}A$, $C = {}^{coW}A$, $R = A^{coH}$, and $S = R \cap B$. If

(1) A/R and B/S are faithfully flat right H-Galois extensions and
(2) A/C is a left W-Galois extension,

then (U, W, H, A) is called a faithfully flat bi-Galois extension.

This definition can be explained as follows:

$$
\begin{array}{ccccc}
S & \subseteq & B \overset{\rho_r}{\underset{i_1}{\rightrightarrows}} B \otimes H & & C \\
\cap & & \cap \qquad\quad \cap & & \cap \\
R & \subseteq & A \overset{\rho_r}{\underset{i_1}{\rightrightarrows}} A \otimes H & & A \\
 & & \downdownarrows \qquad\qquad & & \downdownarrows \\
 & & \overline{W} \otimes A \qquad & & W \otimes A
\end{array}
$$

Note that the bi-Galois object in the sense of Schauenburg is a special case of faithfully flat bi-Galois extension above, where $U = W$ (so $\overline{W} = k$) and $B = A$, $C = R = k$, that is, the coinvariants in A for W and H are trivial.

Example 2.2 [2, Example 2.7]**.** Assume that $B \subset A$ and H are Hopf algebras and $\pi : A \to H$ is a surjective Hopf algebra map such that π restricted to B is also surjective. Then A is a right H-comodule via $\rho = (A \otimes \pi) \circ \Delta_A$. Let $R = A^{coH}$ and $S = B^{coH}$. Then (B, A, H, A) is a faithfully flat bi-Galois extension provided A/R and B/S are faithfully flat right H-Galois extensions.

Example 2.3. Let H be a finite-dimensional Hopf algebra and A a right H-Galois object. By [8], there exists a Hopf algebra L such that A is an L-H-bi-Galois object and the antipode of L is bijective. If L has a nontrivial Hopf ideal I, $U = W = L/I$ and $B = {}^{coW}A$, then the Galois map $A \otimes_B A \longrightarrow W \otimes A$ is an isomorphism ([5, Theorem 1.7]). Thus (U, W, H, A) and (L, L, H, A) are faithfully flat bi-Galois extensions.

The following proposition will be used in Section 4.

Proposition 2.4 [13, Remark 3.4]**.** *Let A/R be a right H-Galois extension. For $h \in H$, we write $\beta^{-1}(1_A \otimes h) =: h^{[1]} \otimes_R h^{[2]}$. For $g, h \in H$, $r \in R$, and $a \in A$, we have*

$$h^{[1]}h^{[2]}{}_{\langle 0\rangle} \otimes h^{[2]}{}_{\langle 1\rangle} = 1_A \otimes h, \tag{2.1}$$

$$h^{[1]} \otimes_R h^{[2]}{}_{\langle 0\rangle} \otimes h^{[2]}{}_{\langle 1\rangle} = h_{(1)}{}^{[1]} \otimes_R h_{(1)}{}^{[2]} \otimes h_{(2)}, \tag{2.2}$$

$$h^{[1]}{}_{\langle 0\rangle} \otimes_R h^{[2]} \otimes h^{[1]}{}_{\langle 1\rangle} = h_{(2)}{}^{[1]} \otimes_R h_{(2)}{}^{[2]} \otimes S_H(h_{(1)}), \tag{2.3}$$

$$h^{[1]}h^{[2]} = \varepsilon(h)1_A, \tag{2.4}$$

$$(gh)^{[1]} \otimes_R (gh)^{[2]} = h^{[1]}g^{[1]} \otimes_R g^{[2]}h^{[2]}, \tag{2.5}$$

$$rh^{[1]} \otimes_R h^{[2]} = h^{[1]} \otimes_R h^{[2]}r, \tag{2.6}$$

$$a_{\langle 0\rangle}a_{\langle 1\rangle}{}^{[1]} \otimes_R a_{\langle 1\rangle}{}^{[2]} = 1_A \otimes_R a. \tag{2.7}$$

For a left Hopf–Galois extension, we get the similar conclusions.

Proposition 2.5. *If A/C is a left W-Galois extension, we write $\beta_l{}^{-1}(w \otimes 1_A) =: w^{(1)} \otimes_C w^{(2)}$. For $w, v \in W$, $c \in C$, and $a \in A$, the following identities hold*:

$$w^{(1)}{}_{\langle -1\rangle} \otimes w^{(1)}{}_{\langle 0\rangle}w^{(2)} = w \otimes 1_A,$$

$$w^{(1)}{}_{\langle -1\rangle} \otimes w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(2)} = w_{(1)} \otimes w_{(2)}{}^{(1)} \otimes_C w_{(2)}{}^{(2)}, \tag{2.8}$$

$$w^{(2)}{}_{\langle -1\rangle} \otimes w^{(1)} \otimes_C w^{(2)}{}_{\langle 0\rangle} = S_W(w_{(2)}) \otimes w_{(1)}{}^{(1)} \otimes_C w_{(1)}{}^{(2)},$$

$$w^{(1)}w^{(2)} = \varepsilon(w)1_A, \tag{2.9}$$

$$(wv)^{(1)} \otimes_C (wv)^{(2)} = w^{(1)}v^{(1)} \otimes_C v^{(2)}w^{(2)}, \tag{2.10}$$

$$cw^{(1)} \otimes_C w^{(2)} = w^{(1)} \otimes_C w^{(2)}c, \tag{2.11}$$

$$a_{\langle -1\rangle}{}^{(1)} \otimes_C a_{\langle -1\rangle}{}^{(2)}a_{\langle 0\rangle} = a \otimes_C 1_A.$$

Proof. The proof is similar to that of Proposition 2.4. □

Denote by $\mathcal{M}_R$ the category of right R-modules and $\mathcal{M}_A^H$ the category of (A, H)-Hopf modules. Then the functor

$$\mathcal{M}_R \longrightarrow \mathcal{M}_A^H, \qquad N \longmapsto N \otimes_R A$$

is left adjoint to the functor

$$\mathcal{M}_A^H \longrightarrow \mathcal{M}_R, \qquad M \longmapsto M^{coH}.$$

Similarly, a functor

$${}_R\mathcal{M} \longrightarrow {}_A\mathcal{M}^H, \qquad N \longmapsto A \otimes_R N$$

is defined.

We require some known facts about Hopf modules.

Proposition 2.6 [12, Theorem 1]**.** *Assume the antipode of H is bijective. Then the following statements are equivalent*:

(1) *A is injective as right H-comodule and $\beta : A \otimes_R A \longmapsto A \otimes H$ is surjective*;
(2) *$\mathcal{M}_R \longrightarrow \mathcal{M}_A^H$, $N \longmapsto N \otimes_R A$ is an equivalence*;

(3) ${}_R\mathcal{M} \longrightarrow {}_A\mathcal{M}^H$, $N \longmapsto A \otimes_R N$ *is an equivalence*;
(4) *A is faithfully flat as left R-module and β is an isomorphism*;
(5) *A is faithfully flat as right R-module and β is an isomorphism.*

Specially, the module action induces an isomorphism $M^{coH} \otimes_R A \longrightarrow M$ in $\mathcal{M}_A^H$, whose inverse is given by

$$\delta_M : M \longrightarrow M^{coH} \otimes_R A, \qquad m \longmapsto m_{\langle 0\rangle} {m_{\langle 1\rangle}}^{[1]} \otimes_R {m_{\langle 1\rangle}}^{[2]}.$$

3. FAITHFULLY FLAT BI-GALOIS EXTENSIONS

In this section, we study the algebra extensions in a faithfully flat bi-Galois extension.

Proposition 3.1. *If (U, W, H, A) is a faithfully flat bi-Galois extension and A is flat as right C-module, then*:

(1) *The canonical map $A \otimes_C B \longrightarrow W \square_{\overline{W}} A$, $x \otimes y \mapsto x_{\langle -1\rangle} \otimes x_{\langle 0\rangle} y$ is bijective*;
(2) *A/B is a left $\overline{W}$-Galois extension and A_B is faithfully flat provided that A_C is faithfully flat*;
(3) *If furthermore ${}_BA$ is flat, R/S is a left $\overline{W}$-Galois extension.*

Proof. (1) and (2) follow by applying [14, Theorem 1.4] to W^{opcop}, U^{opcop}, A^{op}, and C^{op}. Note that we are using [14, Remark 1.9(3)] stating that if $U \subset W$ is faithfully flat, then W is faithfully flat as a $\overline{W}$-comodule. We next prove (3). We claim that the following diagram commutes:

$$\begin{array}{ccccccc}
(R \otimes_S R) \otimes_R A & \xrightarrow{f_1} & R \otimes_S A & \xrightarrow{\quad f_2 \quad} & (R \otimes_S B) \otimes_B A & \xrightarrow{\delta_A^{-1} \otimes_B A} & A \otimes_B A \\
\Big\downarrow{\scriptstyle \beta \otimes_R A} & & & & & & \Big\downarrow{\scriptstyle \beta'} \\
(\overline{W} \otimes R) \otimes_R A & \xrightarrow{\quad g_1 \quad} & \overline{W} \otimes (R \otimes_R A) & & \xrightarrow{\qquad g_2 \qquad} & & \overline{W} \otimes A
\end{array}$$

where β' and β are the Galois maps, f_i $(i = 1, 2)$, g_j $(j = 1, 2)$ are the natural isomorphisms, and δ_A is the map in category $\mathcal{M}_B^H$ defined as in Proposition 2.6. β' is an isomorphism since A/B is a left $\overline{W}$-Galois extension by (2). $\delta_A^{-1} \otimes_B A$ is bijective, for B/S is a faithfully flat right H-Galois extension and ${}_BA$ is flat. For any $(r \otimes_S 1_A) \otimes_R a \in (R \otimes_S R) \otimes_R A$,

$$\begin{aligned}
\beta' \circ (\delta_A^{-1} \otimes_B A) \circ f_2 \circ f_1((r \otimes_S 1_A) \otimes_R a) &= \beta' \circ (\delta_A^{-1} \otimes_B A)((r \otimes_S 1_A) \otimes_B a) \\
&= \beta'(r \otimes_B a) = \overline{r_{\langle -1\rangle}} \otimes r_{\langle 0\rangle} a. \\
g_2 \circ g_1 \circ (\beta \otimes_R A)((r \otimes_S 1_A) \otimes_R a) &= g_2 \circ g_1((\overline{r_{\langle -1\rangle}} \otimes r_{\langle 0\rangle}) \otimes_R a) \\
&= \overline{r_{\langle -1\rangle}} \otimes r_{\langle 0\rangle} a.
\end{aligned}$$

Hence, the the diagram is commutative and $\beta \otimes_R A$ is an isomorphism. Since ${}_RA$ is faithfully flat, $\beta : R \otimes_S R \longrightarrow \overline{W} \otimes R$ is an isomorphism and R/S is a left $\overline{W}$-Galois extension. □

Proposition 3.2. *Let H be a Hopf algebra with bijective antipode and A/R a faithfully flat right H-Galois extension. For a right H-comodule subalgebra B of A such that ${}_BA$ and A_B are faithfully flat, set $S = B \cap R$, then the following are equivalent*:

(1) *$S \subseteq B$ is a faithfully flat right H-Galois extension*;
(2) *$R \otimes_S B \cong A$ via multiplication*;
(3) *$B \otimes_S R \cong A$ via multiplication.*

If any of the above holds, R is faithfully flat as left and right S-module.

Proof. (1) $\Rightarrow$ (2). By Proposition 2.6, the categories $\mathcal{M}_S$ and $\mathcal{M}_B^H$ are equivalent. Now $A \in \mathcal{M}_B^H$ and $A^{coH} = R$, (2) follows.

(2) $\Rightarrow$ (1). For any right S-module X,

$$X \otimes_R A \cong X \otimes_R (R \otimes_S B) \cong X \otimes_S B.$$

Since ${}_RA$ is faithfully flat, ${}_SB$ is also faithfully flat. For any left B-module Y, we have

$$A \otimes_B Y \cong (R \otimes_S B) \otimes_B Y \cong R \otimes_S Y.$$

Thus R_S is faithfully flat since A_B is faithfully flat. Note

$$\begin{aligned} R \otimes_S (B \otimes_S B) &\cong A \otimes_R A \cong A \otimes H \cong R \otimes_S (B \otimes H), \\ r \otimes_S (x \otimes_S y) &\mapsto rx \otimes_R y \mapsto rxy_{\langle 0\rangle} \otimes y_{\langle 1\rangle} \mapsto r \otimes_S (xy_{\langle 0\rangle} \otimes y_{\langle 1\rangle}), \end{aligned}$$

so $B \otimes_S B \cong B \otimes H$, $x \otimes_S y \mapsto xy_{\langle 0\rangle} \otimes y_{\langle 1\rangle}$, and (1) is true. Similarly, (1) and (3) are equivalent. If any of (1)–(3) holds, then R is faithfully flat as right S-module by the proof of (2)$\Rightarrow$(1). Similarly, R is faithfully flat as left S-module. □

We introduce Hopf-separable extension below.

Definition 3.3. Let H be a Hopf algebra and A a right H-comodule algebra. A separable extension $C \subset A$ is called a (right) Hopf-separable extension if:

(1) C is a comodule subalgebra of A;
(2) The multiplication map $m_A : A^\bullet \otimes_C A^\bullet \longrightarrow A^\bullet$ is a split epimorphism in ${}_A\mathcal{M}_A^H$.

Note that when $C \subset A$ is a Hopf-separable extension, the separability idempotent $e = \sum a_i \otimes_C b_i \in (A^\bullet \otimes_C A^\bullet)^{coH}$. Similarly, we may define left Hopf-separable extension.

Lemma 3.4. *Let H be a Hopf algebra, $C \subset A$ a (right) Hopf-separable extension, and $R = A^{coH}$, $D = C^{coH}$. If C/D is a right H-Galois extension and ${}_DC$ is faithfully flat, then $D \subset R$ is a separable extension.*

Proof. Since C/D is a right H-Galois extension and ${}_DC$ is faithfully flat, we have the categories equivalence $\mathcal{M}_D \leftrightarrows \mathcal{M}_C^H$. $A \in \mathcal{M}_C^H$ and $A^{coH} = R$, so $A \cong R \otimes_D C$ via multiplication and the isomorphism is left D-linear. Thus, there is an isomorphism in $\mathcal{M}_C^H$

$$\begin{aligned} A \otimes_C A &\cong (R \otimes_D C) \otimes_C A \cong R \otimes_D A \\ &\cong R \otimes_D (R \otimes_D C) \cong (R \otimes_D R) \otimes_D C. \end{aligned}$$

So $R \otimes_D R \overset{f}{\cong} (A \otimes_C A)^{coH}$ as right D-modules and $f(r \otimes_D s) = r \otimes_C s$. Since the separability idempotent e of the Hopf-separable extension $C \subset A$ is in $(A \otimes_C A)^{coH}$, assume $f(\sum a_i \otimes_D b_i) = e = \sum a_i \otimes_C b_i$. One may check that $\sum a_i \otimes_D b_i$ is the separable idempotent and $D \subset R$ is a separable extension. $\square$

Theorem 3.5. *Let (U, W, H, A) be a faithfully flat bi-Galois extension, and $B \subset A$ a (right) Hopf-separable extension. Then $S \subset R$ is a separable extension. Moreover, if U is a normal Hopf subalgebra of W and A is flat as right C-module, then $B \subset A$ and $S \subset R$ are β-Frobenius extensions.*

Proof. By Lemma 3.4, $S \subset R$ is a separable extension since $S = B^{coH}$, $R = A^{coH}$, and B/S is a faithfully flat Galois extension. If U is a normal Hopf subalgebra of W, then $\overline{W} = W/WU^+$ is a Hopf algebra and $B \subset A$ a separable $\overline{W}$-Galois extension. Hence $\overline{W}$ is finite-dimensional by [1], $U \subset W$ is of right integral type. So $B \subset A$ and $S \subset R$ are β-Frobenius extensions by [2, Theorem 3.1]. $\square$

4. THE MAIN CONSTRUCTIONS

In this section, H means a Hopf algebra with bijective antipode, and A/R is a faithfully flat right H-Galois extension.

Lemma 4.1. *Let A/R be a faithfully flat right H-Galois extension, A/C a left W-Galois extension, and A a (W, H)-bicomodule algebra. Then for any $n \in \mathbb{N}$ and k-module V, we have the bijection*

$$\Phi : Hom_k(W^{\otimes n}, V \otimes R) \cong Hom_{-C}^{-H}(\underbrace{A \otimes_C A \otimes_C \cdots \otimes_C A}_{n}, V \otimes A),$$

given by

$$\Phi(f)(x_1 \otimes_C \cdots \otimes_C x_n) = f(x_{1\langle -1\rangle} \otimes \cdots \otimes x_{n\langle -1\rangle})x_{1\langle 0\rangle} \cdots x_{n\langle 0\rangle}.$$

Moreover, the following universal property holds. Given a k-module V and a map $\phi \in Hom_{-C}^{-H}(A, V \otimes A)$, there is a unique map $f : W \longrightarrow V \otimes R$ such that $\phi = g\rho_l$ as *in the diagram*

$$\begin{tikzcd} A \arrow[r, "\rho_l"] \arrow[dr, "\phi"] & W \otimes A \arrow[d, "g"] & W \arrow[d, "f"] \\ & V \otimes A & V \otimes R\,, \end{tikzcd}$$

where $g(w \otimes a) = f(w)a$.

Proof. Note that the left Galois map $\beta_l : A^{\bullet} \otimes_C A^{\bullet}_{\bullet} \longrightarrow W \otimes A^{\bullet}_{\bullet}$ is evidently a Hopf module map in $\mathcal{M}_A^H$ and β_l is a left C-module map. In $\mathcal{M}_A^H$, we have

$$\begin{aligned} W^{\otimes n} \otimes A^{\bullet}_{\bullet} &\cong A^{\bullet} \otimes_C (W^{\otimes(n-1)} \otimes A^{\bullet}_{\bullet}) \cong A^{\bullet} \otimes_C A^{\bullet} \otimes_C (W^{\otimes(n-2)} \otimes A^{\bullet}_{\bullet}) \\ &\cong \cdots \cong \underbrace{A^{\bullet} \otimes_C \cdots \otimes_C A^{\bullet}}_{n} \otimes_C A^{\bullet}_{\bullet}. \end{aligned}$$

By Proposition 2.6,

$$\begin{aligned} Hom(W^{\otimes n}, V \otimes R) &\cong Hom_{-R}(W^{\otimes n} \otimes R, V \otimes R) \cong Hom_{-A}^{-H}(W^{\otimes n} \otimes A, V \otimes A) \\ &\cong Hom_{-A}^{-H}(\underbrace{A \otimes_C \cdots \otimes_C A}_{n} \otimes_C A, V \otimes A) \\ &\cong Hom_{-C}^{-H}(\underbrace{A \otimes_C \cdots \otimes_C A}_{n}, V \otimes A). \end{aligned}$$

The bijection has the claimed form obviously. □

Theorem 4.2. *Let A/R be a faithfully flat right H-Galois extension, A/C a left W-Galois extension, and A a (W, H)-bicomodule algebra. Then for any k-module V, k-linear map $f : W \longrightarrow V$, $\overline{f} : W \longrightarrow V \otimes R$, $w \longmapsto f(w) \otimes 1$, and $\lambda = \Phi(\overline{f}) : A \longrightarrow V \otimes A$, we have the following conclusions*:

1. *Assume V is a coalgebra. Then f is a coalgebra map if and only if λ is a comodule structure on A*;
2. *Assume V is an algebra. Then f is an algebra map if and only if λ is*;
3. *When V is a bialgebra, f is a bialgebra map if and only if λ is a comodule algebra structure on A. In particular, if $R = k$, the bialgebra structure of W is uniquely determined by A and C.*

Proof. (1) By Lemma 4.1, $\lambda(x) = f(x_{-1}) \otimes x_0$ for any $x \in A$. f is a coalgebra map, i.e., $\Delta \circ f = (f \otimes f) \circ \Delta : W \to V \otimes V$ if and only if $\Phi(\overline{\Delta \circ f}) = \Phi(\overline{(f \otimes f) \circ \Delta}) : A \to V \otimes V \otimes A$. For any $x \in A$,

$$\begin{aligned} \Phi(\overline{\Delta \circ f})(x) &= \Delta \circ f(x_{-1}) \otimes x_0 = (\Delta \otimes A)\lambda(x), \\ \Phi(\overline{(f \otimes f) \circ \Delta})(x) &= (f \otimes f) \circ \Delta(x_{-1}) \otimes x_0 = (V \otimes \lambda)\lambda(x). \end{aligned}$$

Hence f is a coalgebra map if and only if λ is a comodule structure on A. Similarly, we can prove (2). (3) follows from (1) and (2). $\square$

For any right H-comodule subalgebra C of A, the centralizer $Z(C)$ of C in $(A \otimes_C A)^{coH}$ is an algebra via the multiplication $(\sum x_i \otimes_C y_i)(\sum z_j \otimes_C l_j) = \sum x_i z_j \otimes_C l_j y_i$ for any $\sum x_i \otimes_C y_i, \sum z_j \otimes_C l_j \in Z(C)$. Let E be the subalgebra of $Z(C)$ such that $\sum x_i y_i \in k1_A$ for any $\sum x_i \otimes_C y_i \in E$.

Lemma 4.3. *Suppose that W is a subalgebra of E and A is flat over C. If there exists a right R-module morphism $\varphi : (A^{\bullet} \otimes_C A^{\bullet}_{\bullet})^{coH} \to W \otimes R_{\bullet}$, define a map $\rho_l : A \longrightarrow W \otimes A$ by*

$$\rho_l(x) = \varphi(x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]}){x_{\langle 1\rangle}}^{[2]}.$$

Then ρ_l is a right C-module morphism.

Proof. It is easy to see that $A \otimes_C A \otimes_R A$ is a right H-comodule with structure map $a \otimes_C b \otimes_R x \mapsto a_{\langle 0\rangle} \otimes_C b_{\langle 0\rangle} \otimes_R x \otimes a_{\langle 1\rangle} b_{\langle 1\rangle}$. For any $x \in A$, consider the element $x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {x_{\langle 1\rangle}}^{[2]} \in A \otimes_C A \otimes_R A$. Then

$$\begin{aligned}
& x_{\langle 00\rangle} \otimes_C {{x_{\langle 1\rangle}}^{[1]}}_{\langle 0\rangle} \otimes_R {x_{\langle 1\rangle}}^{[2]} \otimes x_{\langle 01\rangle} {{x_{\langle 1\rangle}}^{[1]}}_{\langle 1\rangle} \\
&= x_{\langle 0\rangle} \otimes_C {{x_{\langle 2\rangle}}^{[1]}}_{\langle 0\rangle} \otimes_R {x_{\langle 2\rangle}}^{[2]} \otimes x_{\langle 1\rangle} {{x_{\langle 2\rangle}}^{[1]}}_{\langle 1\rangle} \\
&\overset{(2.3)}{=} x_{\langle 0\rangle} \otimes_C {x_{\langle 3\rangle}}^{[1]} \otimes_R {x_{\langle 3\rangle}}^{[2]} \otimes x_{\langle 1\rangle} S_H(x_{\langle 2\rangle}) \\
&= x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {x_{\langle 1\rangle}}^{[2]} \otimes 1_H \in A \otimes_C A \otimes_R A \otimes H.
\end{aligned}$$

Since ${}_R A$ is faithfully flat,

$$x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {x_{\langle 1\rangle}}^{[2]} \in (A \otimes_C A)^{coH} \otimes_R A. \tag{4.1}$$

Thus the definition of ρ_l is reasonable.

For any $c \in C$,

$$\begin{aligned}
\rho_l(xc) &= \varphi(x_{\langle 0\rangle} c_{\langle 0\rangle} \otimes_C {c_{\langle 1\rangle}}^{[1]} {x_{\langle 1\rangle}}^{[1]}) {x_{\langle 1\rangle}}^{[2]} {c_{\langle 1\rangle}}^{[2]} \\
&= \varphi(x_{\langle 0\rangle} \otimes_C c_{\langle 0\rangle} {c_{\langle 1\rangle}}^{[1]} {x_{\langle 1\rangle}}^{[1]}) {x_{\langle 1\rangle}}^{[2]} {c_{\langle 1\rangle}}^{[2]} \\
&\overset{(2.7)}{=} \varphi(x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]}) {x_{\langle 1\rangle}}^{[2]} c = \rho_l(x) c.
\end{aligned}$$

So ρ_l is a right C-module morphism. $\square$

Lemma 4.4. *In Lemma 4.3, assume furthermore that $\varphi|_W = id_W \otimes 1_A$. Denote any element $w \in W$ by $w = w^{(1)} \otimes_C w^{(2)} \in A \otimes_C A$. Define a map $\Psi : A \to A \otimes_C A \otimes_R A$, $x \mapsto x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {x_{\langle 1\rangle}}^{[2]}$. Then:*

(1) *Ψ is a morphism of (R, C)-bimodules and $\Psi(w^{(1)})w^{(2)} = w^{(1)} \otimes_C w^{(2)} \otimes_R 1_A$;*
(2) *$\beta_l : A \otimes_C A \to W \otimes A$, $x \otimes_C y \mapsto \rho_l(x)y$ is an isomorphism in $\mathcal{M}_A^H$, and the inverse is given by $w^{(1)} \otimes_C w^{(2)} \otimes a \mapsto w^{(1)} \otimes_C w^{(2)} a$;*
(3) *ρ_l is an algebra map and $\rho_l(w^{(1)})w^{(2)} = w \otimes 1_A$.*

Proof. (1) Ψ is obviously left R-linear. Ψ being right C-linear is proved similarly to Lemma 4.3. Now, we compute:

$$\begin{aligned}
\Psi(w^{(1)})w^{(2)} &= w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 1\rangle}{}^{[2]} w^{(2)} \\
&\overset{(2.7)}{=} w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(2)}{}_{\langle 0\rangle} w^{(2)}{}_{\langle 1\rangle}{}^{[1]} w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 1\rangle}{}^{[2]} w^{(2)}{}_{\langle 1\rangle}{}^{[2]} \\
&= w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(2)}{}_{\langle 0\rangle} (w^{(1)}{}_{\langle 1\rangle} w^{(2)}{}_{\langle 1\rangle})^{[1]} \otimes_R (w^{(1)}{}_{\langle 1\rangle} w^{(2)}{}_{\langle 1\rangle})^{[2]} \\
&= w^{(1)} \otimes_C w^{(2)} \otimes_R 1_A. \qquad (4.2)
\end{aligned}$$

(2) By Proposition 2.6, in $\mathscr{M}_A^H$,

$$\beta_l : A \otimes_C A \overset{\delta_{A\otimes_C A}}{\cong} (A \otimes_C A)^{coH} \otimes_R A \overset{\varphi\otimes_R A}{\cong} (W \otimes R) \otimes_R A \cong W \otimes A.$$

Therefore, (2) holds.

(3) For any $x, y \in A$, assume

$$\varphi(x_{\langle 0\rangle} \otimes_C x_{\langle 1\rangle}{}^{[1]}) \otimes_R x_{\langle 1\rangle}{}^{[2]} = \sum u_i \otimes 1_A \otimes_R a_i \in W \otimes R \otimes_R A;$$
$$\varphi(y_{\langle 0\rangle} \otimes_C y_{\langle 1\rangle}{}^{[1]}) \otimes_R y_{\langle 1\rangle}{}^{[2]} = \sum v_j \otimes 1_A \otimes_R b_j \in W \otimes R \otimes_R A.$$

Applying $\varphi^{-1} \otimes_R A$ to the two equations above,

$$\begin{aligned}
x_{\langle 0\rangle} \otimes_C x_{\langle 1\rangle}{}^{[1]} \otimes_R x_{\langle 1\rangle}{}^{[2]} &= \sum u_i{}^{(1)} \otimes_C u_i{}^{(2)} \otimes_R a_i \in A \otimes_C A \otimes_R A; \\
y_{\langle 0\rangle} \otimes_C y_{\langle 1\rangle}{}^{[1]} \otimes_R y_{\langle 1\rangle}{}^{[2]} &= \sum v_j{}^{(1)} \otimes_C v_j{}^{(2)} \otimes_R b_j \in A \otimes_C A \otimes_R A.
\end{aligned} \qquad (4.3)$$

We have

$$\begin{aligned}
\rho_l(xy) &= \varphi(x_{\langle 0\rangle} y_{\langle 0\rangle} \otimes_C y_{\langle 1\rangle}{}^{[1]} x_{\langle 1\rangle}{}^{[1]}) x_{\langle 1\rangle}{}^{[2]} y_{\langle 1\rangle}{}^{[2]} \\
&= \varphi(\sum u_i{}^{(1)} v_j{}^{(1)} \otimes_C v_j{}^{(2)} u_i{}^{(2)}) a_i b_j \\
&= \varphi(\sum (u_i v_j)^{(1)} \otimes_C (u_i v_j)^{(2)}) a_i b_j \\
&= \sum u_i v_j \otimes a_i b_j \\
&= \rho_l(x)\rho_l(y).
\end{aligned}$$

It is clear that $\rho_l(1_A) = 1_W \otimes 1_A$. So ρ_l is an algebra map. For any $w \in W$,

$$\begin{aligned}
\rho_l(w^{(1)})w^{(2)} &= [(W \otimes m_A)(\varphi \otimes_R A)\Psi(w^{(1)})]w^{(2)} \\
&= (W \otimes m_A)(\varphi \otimes_R A)(\Psi(w^{(1)})w^{(2)}) \overset{(4.2)}{=} w \otimes 1_A. \qquad (4.4)
\end{aligned}$$

□

Theorem 4.5. *Let C be a right H-comodule subalgebra of A such that A is faithfully flat over C. Then there exists a Hopf algebra W such that $C \subset A$ is a left W-Galois extension and A is a (W, H)-bicomodule algebra if and only if $(A^\bullet \otimes_C A^\bullet_\bullet)^{coH} \overset{\varphi}{\cong} W \otimes R_\bullet$*

as right R-modules for some subalgebra W of E such that:

(1) $\varphi|_W = id_W \otimes 1$;
(2) *Denote any element* $w \in W$ *by* $w = w^{(1)} \otimes_C w^{(2)} \in A \otimes_C A$. *Let* ρ_l *be defined as in Lemma* 4.3. *Then* $\rho_l(w^{(1)}) \otimes_C w^{(2)} \in W \otimes W$.

Proof. Assume $C \subset A$ is a left W-Galois extension and A a (W, H)-bicomodule algebra, then the Galois map $A^{\bullet} \otimes_C A^{\bullet}_{\bullet} \to W \otimes A^{\bullet}_{\bullet}$ is an isomorphism in $\mathcal{M}^H_A$. Thus $(A \otimes_C A)^{coH} \overset{\varphi}{\cong} W \otimes R$ as right R-modules, where φ is the restriction of the Galois map to $(A \otimes_C A)^{coH}$. By (2.9), (2.10), and (2.11) in Proposition 2.5, we can regard W as a subalgebra of E and $\varphi|_W = id_W \otimes 1$. For any $w \in W$,

$$\begin{aligned}
\rho_l(w^{(1)}) \otimes_C w^{(2)} &= \varphi({w^{(1)}}_{\langle 0\rangle} \otimes_C {{w^{(1)}}_{\langle 1\rangle}}^{[1]}){{w^{(1)}}_{\langle 1\rangle}}^{[2]} \otimes_C w^{(2)} \\
&= {w^{(1)}}_{\langle -1\rangle} \otimes {w^{(1)}}_{\langle 0\rangle}{{w^{(1)}}_{\langle 1\rangle}}^{[1]}{{w^{(1)}}_{\langle 1\rangle}}^{[2]} \otimes_C w^{(2)} \\
&\overset{(2.4)}{=} {w^{(1)}}_{\langle -1\rangle} \otimes {w^{(1)}}_{\langle 0\rangle} \otimes_C w^{(2)} \\
&\overset{(2.8)}{=} w_{(1)} \otimes {w_{(2)}}^{(1)} \otimes_C {w_{(2)}}^{(2)} \in W \otimes W.
\end{aligned}$$

Conversely, suppose that $(A \otimes_C A)^{coH} \overset{\varphi}{\cong} W \otimes R$ as right R-modules for some subalgebra W of E.

(i) $(W, \Delta_l, \varepsilon_l)$ is a coalgebra, where

$$\begin{aligned}
\Delta_l(w) \otimes 1_A &= [(W \otimes \rho_l)\rho_l(w^{(1)})]w^{(2)}, \\
\varepsilon_l(w)1_A &= w^{(1)}w^{(2)}.
\end{aligned}$$

Now we prove that the definition of comultiplication is reasonable. Let Ψ be defined as in Lemma 4.4. For $x \in A$,

$$\begin{aligned}
&x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {{x_{\langle 1\rangle}}^{[2]}}_{\langle 0\rangle} \otimes {{x_{\langle 1\rangle}}^{[2]}}_{\langle 1\rangle} \\
&\overset{(2.2)}{=} x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle(1)}}^{[1]} \otimes_R {x_{\langle 1\rangle(1)}}^{[2]} \otimes x_{\langle 1\rangle(2)} \\
&= x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {x_{\langle 1\rangle}}^{[2]} \otimes x_{\langle 2\rangle}.
\end{aligned}$$

From here, we have

$$\begin{aligned}
&x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {{x_{\langle 1\rangle}}^{[2]}}_{\langle 0\rangle} \otimes_C {{{x_{\langle 1\rangle}}^{[2]}}_{\langle 1\rangle}}^{[1]} \otimes_R {{{x_{\langle 1\rangle}}^{[2]}}_{\langle 1\rangle}}^{[2]} \\
&= x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]} \otimes_R {x_{\langle 1\rangle}}^{[2]} \otimes_C {x_{\langle 2\rangle}}^{[1]} \otimes_R {x_{\langle 2\rangle}}^{[2]}.
\end{aligned} \tag{4.5}$$

Now, consider $w = w^{(1)} \otimes_C w^{(2)}$. Apply Ψ on the first tensorand to get

$${w^{(1)}}_{\langle 0\rangle} \otimes_C {{w^{(1)}}_{\langle 1\rangle}}^{[1]} \otimes_R {{w^{(1)}}_{\langle 1\rangle}}^{[2]} \otimes_C w^{(2)}.$$

Apply Ψ on the third tensorand, we obtain

$$\begin{aligned} & w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 1\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(2)} \\ & \overset{(4.5)}{=} w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(1)}{}_{\langle 2\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 2\rangle}{}^{[2]} \otimes_C w^{(2)}. \end{aligned} \tag{4.6}$$

Apply Ψ on the first tensorand of (4.2) to get

$$\begin{aligned} & w^{(1)}{}_{\langle 00\rangle} \otimes_C w^{(1)}{}_{\langle 01\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 01\rangle}{}^{[2]} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 1\rangle}{}^{[2]} w^{(2)} \\ & = w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes_R w^{(1)}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(2)} \otimes_R 1_A. \end{aligned} \tag{4.7}$$

From here,

$$\begin{aligned} [(W \otimes \rho_l)\rho_l(w^{(1)})]w^{(2)} &= (W \otimes \rho_l)(\varphi(w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]})w^{(1)}{}_{\langle 1\rangle}{}^{[2]})w^{(2)} \\ &= (W \otimes \varphi)\Big(\varphi(w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]})w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 1\rangle}{}^{[1]}\Big) \\ &\quad \times w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 1\rangle}{}^{[2]} w^{(2)} \\ &\overset{(4.6)}{=} (W \otimes \varphi)\Big(\varphi(w^{(1)}{}_{\langle 00\rangle} \otimes_C w^{(1)}{}_{\langle 01\rangle}{}^{[1]})w^{(1)}{}_{\langle 01\rangle}{}^{[2]} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]}\Big) \\ &\quad \times w^{(1)}{}_{\langle 1\rangle}{}^{[2]} w^{(2)} \\ &\overset{(4.7)}{=} (W \otimes \varphi)(\varphi(w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]})w^{(1)}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(2)}) \\ &= (W \otimes \varphi)(\rho_l(w^{(1)}) \otimes_C w^{(2)}). \end{aligned}$$

By (1) and (2), $[(W \otimes \rho_l)\rho_l(w^{(1)})]w^{(2)} \in W \otimes W \otimes 1_A$, hence the definition of comultiplication is reasonable and

$$\Delta_l(w) = \rho_l(w^{(1)}) \otimes_C w^{(2)} = (\rho_l \otimes_C A)(w). \tag{4.8}$$

Δ_l is coassociative. For $x \in A$, suppose $\varphi(x_{\langle 0\rangle} \otimes_C x_{\langle 1\rangle}{}^{[1]}) \otimes_R x_{\langle 1\rangle}{}^{[2]} = \sum u_i \otimes 1_A \otimes_R a_i \in W \otimes R \otimes_R A$. Then $\rho_l(x) = \sum u_i \otimes a_i$. By the definition of Δ_l,

$$\begin{aligned} (\Delta_l \otimes A)\rho_l(x) &= (\Delta_l \otimes A)(\textstyle\sum u_i \otimes a_i) \\ &= \textstyle\sum [(W \otimes \rho_l)\rho_l(u_i{}^{(1)})]u_i{}^{(2)} a_i \overset{(4.3)}{=} (W \otimes \rho_l)\rho_l(x). \end{aligned}$$

Hence,

$$(\Delta_l \otimes A)\rho_l = (W \otimes \rho_l)\rho_l. \tag{4.9}$$

$$(\varepsilon_l \otimes A)\rho_l(x) = \textstyle\sum u_i{}^{(1)} u_i{}^{(2)} a_i \overset{(4.3)}{=} x. \tag{4.10}$$

For any $w \in W$,

$$
\begin{aligned}
(\Delta_l \otimes W)\Delta_l(w) &\overset{(4.8)}{=} (\Delta_l \otimes W)(\rho_l \otimes_C A)(w) \\
&= (\Delta_l \otimes A \otimes_C A)(\rho_l \otimes_C A)(w) \\
&= ((\Delta_l \otimes A)\rho_l \otimes_C A)(w) \\
&\overset{(4.9)}{=} (W \otimes \rho_l)\rho_l(w^{(1)}) \otimes_C w^{(2)} \\
&= (W \otimes \rho_l \otimes_C A)(\rho_l(w^{(1)}) \otimes_C w^{(2)}) \\
&= (W \otimes \Delta_l)\Delta_l(w).
\end{aligned}
$$

So Δ_l is coassociative.

$$
\begin{aligned}
(W \otimes \varepsilon_l)\Delta_l(w) &\overset{(4.8)}{=} (W \otimes \varepsilon_l)(\rho_l(w^{(1)}) \otimes_C w^{(2)}) \\
&= (A \otimes_C m_A)(\rho_l(w^{(1)})w^{(2)}) \overset{Lem.4.4(3)}{=} w.
\end{aligned}
$$

$$
\begin{aligned}
(\varepsilon_l \otimes W)\Delta_l(w) &= (\varepsilon_l \otimes A \otimes_C A)(\rho_l(w^{(1)}) \otimes_C w^{(2)}) \\
&= (\varepsilon_l \otimes A)\rho_l(w^{(1)}) \otimes_C w^{(2)} \overset{(4.10)}{=} w.
\end{aligned}
$$

Thus, $(W, \Delta_l, \varepsilon_l)$ is a coalgebra.

(ii) (A, ρ_l) being a left W-comodule algebra follows from Lemma 4.4(3) and (4.9), (4.10).

It is easy to see that Δ_l and ε_l are algebra maps. Therefore, W is a bialgebra. For any $c \in C$,

$$
\begin{aligned}
\rho_l(c) &= \varphi(c_{\langle 0\rangle} \otimes_C {c_{\langle 1\rangle}}^{[1]}){c_{\langle 1\rangle}}^{[2]} \\
&= \varphi(1_A \otimes_C c_{\langle 0\rangle}{c_{\langle 1\rangle}}^{[1]}){c_{\langle 1\rangle}}^{[2]} = \varphi(1_A \otimes_C 1_A)c = 1_A \otimes c.
\end{aligned}
$$

So $C \subseteq {}^{coW}A$. By Lemma 4.4(2), $A \otimes_C A \cong W \otimes A$, $x \otimes y \mapsto \rho_l(x)y$ is bijective. And A is faithfully flat over C, so $C = {}^{coW}A$ by [14, Remark 1.2]. Thus A/C is a left W-Galois extension and W is a Hopf algebra by [9]. The antipode of W is defined by $S_W(w) \otimes 1_A = {w^{(2)}}_{\langle -1\rangle} \otimes w^{(1)}{w^{(2)}}_{\langle 0\rangle}$. For any $x \in A$,

$$
\begin{aligned}
(W \otimes \rho)\rho_l(x) &= \varphi(x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]}){{x_{\langle 1\rangle}}^{[2]}}_{\langle 0\rangle} \otimes {{x_{\langle 1\rangle}}^{[2]}}_{\langle 1\rangle} \\
&= \varphi(x_{\langle 0\rangle} \otimes_C {x_{\langle 1\rangle}}^{[1]}){x_{\langle 1\rangle}}^{[2]} \otimes x_{\langle 2\rangle} = (\rho_l \otimes H)\rho(x).
\end{aligned}
$$

Therefore A is a (W, H)-bicomodule algebra. □

Recall that the Miyashita–Ulbrich action of H on the centralizer $Z_A(R)$ of R in A, makes $Z_A(R)$ into a commutative algebra in YD_H^H, where the Miyashita–Ulbrich action is defined by $x \leftharpoonup h = h^{[1]}xh^{[2]}$ for any $x \in Z_A(R)$, $h \in H$.

The following lemma describes $Z(C)$ when $C \subseteq Z_A(R)$.

Lemma 4.6. *Assume the right H-comodule subalgebra $C \subset Z_A(R)$ and A is faithfully flat over C. Then $Z(C) = (A \otimes_C A)^{coH}$ if and only if C is an H-submodule of $Z_A(R)$ with respect to the Miyashita–Ulbrich action.*

Proof. Suppose $Z(C) = (A \otimes_C A)^{coH}$. For any $c \in C$ and $h \in H$,

$$\begin{aligned} 1_A \otimes_C c \leftharpoondown h &\overset{(2.1)}{=} h^{[1]}h^{[2]}{}_{\langle 0\rangle} \otimes_C c \leftharpoondown h^{[2]}{}_{\langle 1\rangle} \\ &= h^{[1]}h^{[2]}{}_{\langle 0\rangle} \otimes_C h^{[2]}{}_{\langle 1\rangle}{}^{[1]} c h^{[2]}{}_{\langle 1\rangle}{}^{[2]} \\ &\overset{(4.1)}{=} h^{[1]}ch^{[2]}{}_{\langle 0\rangle} \otimes_C h^{[2]}{}_{\langle 1\rangle}{}^{[1]} h^{[2]}{}_{\langle 1\rangle}{}^{[2]} = c \leftharpoondown h \otimes_C 1_A. \end{aligned}$$

By the faithfully flatness of A over C, $c \leftharpoondown h \in C$ and C is an H-submodule of $Z_A(R)$.

Assume that C is an H-submodule of $Z_A(R)$. Recall that, for any $c \in C$, $x \in A$, the Miyashita–Ulbrich action satisfies $cx = x_{\langle 0\rangle}(c \leftharpoondown x_{\langle 1\rangle}) = x_{\langle 0\rangle}x_{\langle 1\rangle}{}^{[1]}cx_{\langle 1\rangle}{}^{[2]}$. Then for $\Sigma x_i \otimes y_i \in (A \otimes_C A)^{coH}$,

$$\begin{aligned} \Sigma cx_i \otimes_C y_i &= \Sigma x_{i\langle 0\rangle}x_{i\langle 1\rangle}{}^{[1]}cx_{i\langle 1\rangle}{}^{[2]} \otimes_C y_i \\ &= \Sigma x_{i\langle 0\rangle} \otimes_C x_{i\langle 1\rangle}{}^{[1]}cx_{i\langle 1\rangle}{}^{[2]}y_i \\ &= \Sigma x_{i\langle 0\rangle} \otimes_C y_{i\langle 0\rangle}y_{i\langle 1\rangle}{}^{[1]}x_{i\langle 1\rangle}{}^{[1]}cx_{i\langle 1\rangle}{}^{[2]}y_{i\langle 1\rangle}{}^{[2]} \\ &= \Sigma x_i \otimes_C y_i c \in A \otimes_C A, \end{aligned}$$

so $Z(C) = (A \otimes_C A)^{coH}$. □

Corollary 4.7. *Assume $R = k$. For a right H-comodule subalgebra C of A such that A is faithfully flat over C, there exists a Hopf algebra W such that A/C is a left W-Galois extension and A is a (W, H)-bicomodule algebra if and only if C is an H-submodule of A with respect to the Miyashita–Ulbrich action. Moreover, W is unique up to isomorphism.*

Proof. Note that when $R = k$, $Z_A(R) = A$ and $C \subset Z_A(R)$. If C is an H-submodule of A, then $Z(C) = (A \otimes_C A)^{coH}$ by Lemma 4.6. For any $\sum x_i \otimes_C y_i \in (A \otimes_C A)^{coH}$, $\sum x_i y_i \in R = k1_A$. We may suppose that $W = (A \otimes_C A)^{coH}$ and $\varphi = id_W \otimes 1_A$. Then for any $w = w^{(1)} \otimes_C w^{(2)} \in W$,

$$\begin{aligned} \rho_l(w^{(1)}) \otimes_C w^{(2)} &= \varphi(w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]})w^{(1)}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(2)} \\ &= w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes w^{(1)}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(2)}. \end{aligned}$$

By Theorem 4.5 and (4.1), we only need to prove $\rho_l(w^{(1)}) \otimes_C w^{(2)} \in A \otimes_C A \otimes (A \otimes_C A)^{coH}$.

$$\begin{aligned} & w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 0\rangle} \otimes_C w^{(2)}{}_{\langle 0\rangle} \otimes w^{(1)}{}_{\langle 1\rangle}{}^{[2]}{}_{\langle 1\rangle} w^{(2)}{}_{\langle 1\rangle} \\ &\overset{(2.2)}{=} w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes w^{(1)}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(2)}{}_{\langle 0\rangle} \otimes w^{(1)}{}_{\langle 2\rangle} w^{(2)}{}_{\langle 1\rangle} \end{aligned}$$

$$= w^{(1)}{}_{\langle 00\rangle} \otimes_C w^{(1)}{}_{\langle 01\rangle}{}^{[1]} \otimes w^{(1)}{}_{\langle 01\rangle}{}^{[2]} \otimes_C w^{(2)}{}_{\langle 0\rangle} \otimes w^{(1)}{}_{\langle 1\rangle} w^{(2)}{}_{\langle 1\rangle}$$
$$= w^{(1)}{}_{\langle 0\rangle} \otimes_C w^{(1)}{}_{\langle 1\rangle}{}^{[1]} \otimes w^{(1)}{}_{\langle 1\rangle}{}^{[2]} \otimes_C w^{(2)} \otimes 1.$$

Conversely, if A/C is a left W-Galois extension, then $(A \otimes_C A)^{coH} \cong W \otimes R \cong W$. Thus $Z(C) = (A \otimes_C A)^{coH}$ by (2.11). By Lemma 4.6, C is an H-submodule of A with respect to the Miyashita–Ulbrich action. The uniqueness of W follows from Theorem 4.2. □

Let A be a W-H-bi-Galois object, $Quot_{ff}(W)$ the set of coideal left ideals $I \subset W$ such that W is left faithfully coflat over W/I, and $Sub^H_{ff}(A)$ the set of H-comodule subalgebras $C \subset A$ such that A is left faithfully flat over C. According to Schauenburg [10, Proposition 3.2], there are mutually inverse bijections $Quot_{ff}(W) \underset{G}{\overset{F}{\rightleftarrows}} Sub^H_{ff}(A)$: $F(I) = {}^{coW/I}A$ for any $I \in Quot_{ff}(W)$, and $W/G(C) = (A \otimes_C A)^{coH}$ for any $C \in Sub^H_{ff}(A)$. Note that faithfully coflatness assumptions in $Quot_{ff}(W)$ are guaranteed in the case that W is finite-dimensional by Skryabin [15, Theorem 6.1].

Schauenburg [10, Theorem 3.8] proved that $I \in Quot_{ff}(W)$ is a Hopf ideal if and only if $F(I) \in Sub^H_{ff}(A)$ is an H-submodule of A. Here we give the following result which is equivalent to [10, Theorem 3.8] by Lemma 4.6.

Corollary 4.8. *Let A be a W-H-bi-Galois object, $Quot^H_{ff}(W)$ the set of Hopf ideals in $Quot_{ff}(W)$, and $Subcen^H_{ff}(A)$ the subset of $Sub^H_{ff}(A)$ such that $(A \otimes_C A)^{coH} = Z(C)$ for any $C \in Subcen^H_{ff}(A)$. Then there are mutually inverse bijections*

$$Quot^H_{ff}(W) \underset{\widetilde{G}}{\overset{\widetilde{F}}{\rightleftarrows}} Subcen^H_{ff}(A),$$

where $\widetilde{F}$ and $\widetilde{G}$ are the restrictions of F and G.

Proof. If $I \in Quot_{ff}(W)$ is a Hopf ideal, $W/I = W/\widetilde{G}\widetilde{F}(I) = (A \otimes_{\widetilde{F}(I)} A)^{coH}$ is a Hopf algebra and $A \otimes_{\widetilde{F}(I)} A \cong W/I \otimes A$. Thus $A/\widetilde{F}(I)$ is a left W/I-Galois extension. By (2.11), $(A \otimes_{\widetilde{F}(I)} A)^{coH} = Z_{A\otimes_{\widetilde{F}(I)}A)^{coH}}(\widetilde{F}(I))$. Conversely, if $C \in Subcen^H_{ff}(A)$, $W/\widetilde{G}(C) = (A \otimes_C A)^{coH}$ is a quotient Hopf algebra by Lemma 4.6 and Corollary 4.7; hence $\widetilde{G}(C)$ is a Hopf ideal in $Quot^H_{ff}(W)$. Since F and G are mutually inverse bijections, $\widetilde{F}$ and $\widetilde{G}$ are also mutually inverse bijections. □

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