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The Order of the Antipode of Finite-dimensional Hopf Algebra

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ABSTRACT **Examples of finite-dimensional Hopf algebras over a field, whose antipodes have arbitrary even orders ≥ 4 as mappings, are furnished. The dimension of the Hopf algebra is q^{n+1}, where the antipode has order $2q$, $q \geq 2$, and n is an arbitrary positive integer. The algebras are not semisimple, and neither they nor their dual algebras are unimodular.**

I consider Hopf algebras H over a field K, following the terminology of [1]. H has an associative multiplication $m:H \otimes H \rightarrow H$ and a unit $\mu:K \rightarrow H$ [I identify $\mu(1) = 1$, the unit of H]. H is also a coalgebra, i.e., it has a comultiplication $\Delta:H \rightarrow H \otimes H$ which is coassociative [i.e., $(\Delta \otimes I_H)\Delta = (I_H \otimes \Delta)\Delta$, I_H the identity mapping of H], and a counit $\epsilon:H \rightarrow K$, satisfying $\sum_h \epsilon(h_1)h_2 = h = \sum_h \epsilon(h_2)h_1$ for all h in H, where $\Delta h = \sum_h h_1 \otimes h_2$. Δ and ϵ are algebra homomorphisms, i.e., H is a bialgebra. Finally H has an antipode S, i.e., a linear mapping from H to H satisfying $\sum_h S(h_1)h_2 = \epsilon(h)1 = \sum_h h_1S(h_2)$ for all h in H. All tensor products are taken over K.

In most familiar examples, S has order ≤ 2, i.e., $S^2 = I_H$, where $S^2 = S \cdot S$ is them apping composite. For example, if H is commutative (as an algebra) or cocommutative (as a coalgebra), then $S^2 = I$. This is the case, for example, with group algebras, polynomial functions on an algebraic group, the universal enveloping algebra of a Lie algebra, representative functions on a topological group, etc. In general, S is an antihomomorphism of the algebra H. In [1, pp. 89–90] and [2], examples are given of infinite-dimensional H for which S has any desired even order, or has infinite order (i.e., no finite power of S is I_H). For finite-dimensional H, there is an example due to M. Sweedler (and described in [3]) of a 4-dimensional H with S of order 4. In this example, K is of characteristic not-2. In [4], D. Radford also exhibits a 4-dimensional H with S of order 4 for K of any characteristic. For characteristic $K \neq 2$, Radford's example is isomorphic to Sweedler's (I shall exhibit later a specific isomorphism). In this note, I shall produce a finite-dimensional H of dimension q^{n+1} over K, n an arbitrary positive integer, whose antipode S has order $2q$, where q is any prescribed integer ≥ 2. The only restriction on K is that it should contain a primitive q-th root of unity, so that, in particular, the characteristic K is zero or is relatively prime to q.

THE CONSTRUCTION

Let q be an integer ≥ 2, and let K be a field that contains a primitive q-th root ω of unity. Let n be any positive integer. We form the free algebra $R = K[X_1, \ldots, X_n, Y]$ in $n + 1$ noncommuting indeterminates $X_1, \ldots, X_n, Y$. Since R is free, we define an algebra homomorphism Δ from R into $R \otimes R$ by specifying its action on the free generators. Let $\Delta X_i = X_i \otimes X_i$ for $1 \leq i \leq n$, and $\Delta Y = Y \otimes X_1 + 1 \otimes Y$. One easily checks that Δ is coassociative, as it suffices to show that the algebra homomorphisms $(\Delta \otimes I_R)\Delta$ and $(I_R \otimes \Delta)\Delta$ agree on the free generators. This is clear on the X_i, and on Y both yield $Y \otimes X_1 \otimes X_1 + 1 \otimes Y \otimes X_1 + 1 \otimes 1 \otimes Y$. We define an algebra homomorphism ϵ from R to K by specifying $\epsilon(X_i) = 1$ for $1 \leq i \leq n$, and $\epsilon(Y) = 0$. One easily checks that $(\epsilon \otimes I_R)\Delta = I_R = (I_R \otimes \epsilon)\Delta$ (identifying $K \otimes R$ and $R \otimes K$ with R), as these are algebra homomorphisms agreeing on the X_i and Y, so that ϵ is a counit. Hence, R is a bialgebra.

Let I be the ideal of R generated by the union of the following four sets of elements of R:

(*1*) $\{X_i^q - 1 | 1 \leq i \leq n\}$

(*2*) $\{X_iX_j - X_jX_i | 1 \leq i, j \leq n\}$

(*3*) $\{YX_i - \omega X_iY | 1 \leq i \leq n\}$

(*4*) $\{Y^q\}$

We assert that I is a bi-ideal of R, i.e., $\Delta I \subseteq I \otimes R + R \otimes I$ and $\epsilon(I) = 0$. As Δ and ϵ are algebra homomorphisms, and since $J = I \otimes R + R \otimes I$ is an ideal of $R \otimes R$, it suffices to check these conditions on the generators of I. Since $\epsilon(X_i) = 1$, we have $\epsilon(X_i^q - 1) = \epsilon(X_iX_j - X_jX_i) = 0$, and since $\epsilon(Y) = 0$, we have $\epsilon(YX_i - \omega X_iY) = \epsilon(Y^q) = 0$. Working modulo J, $\Delta(X_i^q - 1) = X_i^q \otimes X_i^q - 1 \otimes 1 \equiv 1 \otimes 1 - 1 \otimes 1 = 0$. For (*2*), $\Delta(X_iX_j - X_jX_i) = X_iX_j \otimes X_iX_j - X_jX_i \otimes X_jX_i \equiv X_iX_j \otimes X_iX_j - X_iX_j \otimes X_iX_j = 0$. Next, for (*3*), $\Delta(YX_i - \omega X_iY) = YX_i \otimes X_1X_i + X_i \otimes YX_i - \omega X_iY \otimes X_iX_1 - \omega X_i \otimes X_iY \equiv \omega X_iY \otimes X_1X_i + \omega X_i \otimes X_iY - \omega X_iY \otimes X_iX_1 - \omega X_i \otimes X_iY = 0$.

Finally, for (4), $\Delta(Y^q) = (Y \otimes X_1 + 1 \otimes Y)^q \equiv Y^q \otimes X_1^q + \sum_{r=1}^{q-1} c_r(Y^{q-r} \otimes X_1^{q-r}Y^r) + 1 \otimes Y^q$. For each $1 \leq r \leq q-1$, in each of the $\binom{q}{r}$ products of $q-r$ factors of $Y \otimes X_1$ and r factors of $1 \otimes Y$, we shift the Y's to the right past the X_1's, starting with the last (i.e., furthest to the right) Y appearing. Let m_1 be the number of shifts of the last Y past the X_1's, m_2 the number of shifts of the next to the last Y, etc. We are doing this in the second factors in the tensor product, of which there are $q - r$ X_1's and r Y's. Hence, straightening each of the $\binom{q}{r}$-products to "standard" form we get

$$c_r = \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq q-r} \omega^{m_1+m_2+\ldots+m_r}$$

We assert that $c_r = 0$. We use the following change of summation device. Let $n_1 = m_1$, $n_2 = m_2 + 1$, $n_3 = m_3 + 2, \ldots, n_r = m_r + r - 1$. Then $0 \leq n_1 < n_2 < \ldots < n_r \leq q - 1$, and

$$c_r = \sum_{0 \leq n_1 < n_2 < \ldots < n_r \leq q-1} \omega^{n_1+n_2+\ldots+n_r-1-2-\ldots-(r-1)}$$

$$= \omega^{-(r-1)r/2} \sum_{0 \leq n_1 < n_2 < \ldots < n_r \leq q-1} \omega^{n_1}\omega^{n_2}\ldots\omega^{n_r}$$

But this sum is the coefficient of λ^{q-r} in the characteristic polynomial

$$\lambda^q - 1 = (\lambda - 1)(\lambda - \omega)(\lambda - \omega^2)\ldots(\lambda - \omega^{q-1})$$

of ω. Hence, $c_r = 0$ for $1 \leq r \leq q-1$, and $\Delta(Y^q) \equiv Y^q \otimes X_1^q + 1 \otimes Y^q \equiv 0 \pmod{J}$. (I thank J. Milnor for pointing out the change of summation device used above.)

We now form the bialgebra $H = R/I$. Let $x_1, \ldots, x_n, y$ be the cosets $X_1 + I, \ldots, X_n + I, Y + I$, respectively, so that H is generated as an algebra by $x_1, \ldots, x_n, y$. We now define an algebra antihomomorphism S of R to R by specifying $S(X_i) = X_i^{q-1}$ and $S(Y) = -\omega^{-1} X_1^{q-1}Y$. To see that S leaves I invariant, it suffices to check the generators of I. We work modulo I. $S(X_i^q - 1) = (X_i^q)^{q-1} - 1 \equiv 1 - 1 = 0$. For (2), $S(X_iX_j - X_jX_i) = X_j^{q-1}X_i^{q-1} - X_i^{q-1}X_j^{q-1} \equiv 0$. Next, for (3), $S(YX_i - \omega X_iY) = X_i^{q-1}(-\omega^{-1}X_1^{q-1}Y) - \omega(-\omega^{-1} X_1^{q-1}Y)(X_i^{q-1}) = -\omega^{-1}X_i^{q-1}X_1^{q-1}Y + X_1^{q-1}YX_i^{q-1} \equiv -\omega^{-1}X_1^{q-1}X_i^{q-1}Y + \omega^{q-1}X_1^{q-1}X_i^{q-1}Y = 0$. Finally, for (4), $S(Y^q) = (-1)^q(X_1^{q-1}Y)^q \equiv (-1)^qX_1^tY^q \equiv 0$ for some t. Hence, S induces an algebra antiendomorphism of H, which we also denote by S. We also let Δ and ϵ be the induced comultiplication and counit of H.

We assert that S is an antipode of H. As the set of elements h of H satisfying $\sum_h S(h_1)h_2 = \epsilon(h)1 = \sum_h h_1S(h_2)$ is a subalgebra, it suffices to check this on the generators $x_1, \ldots, x_n, y$ of H. For each x_i, $x_iS(x_i) = S(x_i)x_i = x_i^q = 1 = \epsilon(x_i)1$. For y,

$$\sum_y S(y_1)y_2 = S(y)x_1 + y = -\omega^{-1}x_1^{q-1}yx_1 + y = -x_1^qy + y$$
$$= -y + y = 0 = \epsilon(y)1, \text{ and}$$
$$\sum_y y_1S(y_2) = yx_1^{q-1} + S(y) = \omega^{q-1}x_1^{q-1}y - \omega^{-1}x_1^{q-1}y = 0$$
$$= \epsilon(y)1.$$

Hence, S is an antipode for H, and H is a Hopf algebra.

As H is not commutative, S does not have odd order. We assert S has order $2q$. First note $S^2(x_i) = S(x_i^{q-1}) = (x_i^{q-1})^{q-1} = x_i^{q^2-2q+1} = x_i$. Now

$$S^2(y) = S(-\omega^{-1}x_1^{q-1}y) = -\omega^{-1}(-\omega^{-1}x_1^{q-1}y)x_1$$
$$= \omega^{-2}x_1^{q-1}(\omega x_1y) = \omega^{-1}y$$
$$S^4(y) = S^2(\omega^{-1}y) = \omega^{-1}(\omega^{-1}y) = \omega^{-2}y, \ldots, S^{2q-2}(y) = \omega^{-(q-1)}y = \omega y, \text{ and}$$
$$S^{2q}(y) = \omega^{-q}y = y.$$

Hence, S has order $2q$.

Finally, note that H has dimension q^{n+1} over K, as it has a basis of "standard" monomials $x_1^{e_1}x_2^{e_2}\ldots x_n^{e_n}y^f$ for $0 \leq e_1, e_2, \ldots, e_n, f \leq q - 1$.

FURTHER REMARKS

I first remark that for the case $q = 2$ and $n = 1$, our Hopf algebra H is 4-dimensional with basis 1, $x_1 = x$, y and $w = xy$ with multiplication table

TABLE 1. *Multiplication table of H (with $q = 2$ and $n = 1$)*

	x	y	w
x	1	w	y
y	$-w$	0	0
w	$-y$	0	0

The characteristic of K is not-2, the costructure is given by $\Delta x = x \otimes x$, $\Delta y = y \otimes x + 1 \otimes y$, $\Delta w = w \otimes 1 + x \otimes w$, $\epsilon(x) = 1$, $\epsilon(y) = \epsilon(w) = 0$, and the antipode S is described by $S(x) = x$, $S(y) = w$, $S(w) = -y$. This is M. Sweedler's example, as described in [3], and so is a special case of our construction.

In [4], D. Radford gives the following example. Starting with the free algebra $K[Z, A]$ in noncommuting free generators Z and A, let $\Delta Z = Z \otimes Z$, $\Delta A = A \otimes Z + 1 \otimes A$, $\epsilon(Z) = 1$, $\epsilon(A) = 0$. Factor out the ideal generated by the three elements $Z^2 - 1$, $A^2 - A$, and $ZA - Z + AZ + 1$. This is a bi-ideal, and denoting the cosets of Z, A, and $B = ZA$ by z, a, and $b = za$, respectively, the factor algebra H_1 has a basis $1, z, a, b$, with multiplication table

TABLE 2. *Multiplication table of H_1*

	z	a	b
z	1	b	a
a	$-1 + z - b$	a	$-a$
b	$1 - z - a$	b	$-b$

costructure $\Delta z = z \otimes z$, $\Delta a = a \otimes z + 1 \otimes a$, $\Delta b = b \otimes 1 + z \otimes b$, $\epsilon(z) = 1$, $\epsilon(a) = \epsilon(b) = 0$, and antipode S given by $S(z) = z$, $S(a) = 1 - z + b$, $S(b) = -a$.

Define a linear map F from H to H_1 by $F(1) = 1$, $F(x) = z$, $F(y) = -1 + z + 2a$, $F(w) = 1 - z + 2b$. Then one may directly check that F is a Hopf-algebra isomorphism of H onto H_1 (this is for characteristic K not-2).

Returning to our general construction H, it is clear that H is not semisimple. In fact, the ideal N of dimension $q^n(q-1)$ spanned by monomials $x_1^{e_1}\ldots x_n^{e_n}y^f$ for $0 \leq e_1, \ldots, e_n \leq q-1$, $1 \leq f \leq q-1$ is an ideal with a basis of nilpotent elements, so is a nilpotent ideal. N has a complementary subalgebra T in H spanned by the monomials $x_1^{e_1}\ldots x_n^{e_n}$, $0 \leq e_1, \ldots, e_n \leq q-1$. T is isomorphic to the group algebra over K of a direct sum of n cyclic groups of order q, so that since the characteristic of K is zero or is relatively prime to q, T is a semisimple (in fact, separable) commutative algebra. Hence, N is the radical of H, and H/N is separable.

In [5, Corollaries 5.6 and 5.7], some bounds are given for the order of the antipode when H satisfies some unimodular condition. A finite-dimensional H is called unimodular if its space of left integrals coincides with its space of right integrals (in H), see [6]. A left integral u in H satisfies $hu = \epsilon(h)u$ for all h in H, and a right integral v in H satisfies $vh = \epsilon(h)v$ for all h in H, see [1, 7]. The left integrals and the right integrals are always one-dimensional subspaces of H (see [1, p. 101]). We note that our example H is not unimodular, and neither is the Hopf algebra H^*, which has operations dual to those of H (see [1]). In fact, for H, $\left(\sum_{0 \leq e_1,\ldots,e_n \leq q-1} x_1^{e_1}\ldots x_n^{e_n}\right)y$ is a left integral, and $\left(\sum_{0 \leq e_1,\ldots,e_n \leq q-1} \omega^{e_1+\cdots+e_n}x_1^{e_1}\ldots x_n^{e_n}\right)y$ is a right integral. If these were linearly dependent, then since the various monomials in the sums are linearly independent, comparison of the y terms ($e_1 = e_2 = \ldots = e_n = 0$) would show they were equal, but then $\omega x_1 y \neq x_1 y$ (the $e_1 = 1$, $e_2 = \ldots = e_n = 0$ terms) is a contradiction. Hence, H is not unimodular. However, the bound of [5, Corollary 5.6] for unimodular H, that order $S < 4$ (dim H), is still satisfied for our H, reading $2q < 4q^{n+1}$, which is true even for $n = 1$.

Using our basis of standard monomials for H, let $\{(x_1^{e_1}\ldots x_n^{e_n}y^f)^* | 0 \leq e_1,\ldots,e_n, f \leq q-1\}$ be the dual basis in H^*. Then one may check that y^* is a left integral for H^*, and that $(x_1^{q-1}y)^*$ is a right integral for H^*. Hence, H^* is also not unimodular (and, hence, not semisimple, as an algebra). [5, Corollary 5.7] says that if H and H^* are unimodular, then the order of the antipode is 1, 2, or 4. My example (where neither H nor H^* is unimodular) for $q > 2$ shows the necessity of some further assumptions, such as unimodularity conditions, to bound the order of the antipode this strongly.

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