Lie algebra

Proposition. $H = U(L)$, L is a Lie algebra. Suppose $char\ k = 0$. We claim that all the Hopf subalgebras of H are $U(P)$, P is a Lie subalgebra of L.

Proof: Suppose H' is Hopf subalgebra of H and $H' \bigcap L = P$. Then obviously P is a Lie subalgebra of L and $U(P) \subset H'$. Note that

$$\Delta(x_{\alpha_1}...x_{\alpha_n}) = \sum_{\substack{0 \le i \le n, \\ \sigma(1)<\sigma(2)<...<\sigma(i), \\ \sigma(i+1)<\sigma(i+2)<...<\sigma(n)}} x_{\alpha_{\sigma(1)}}...x_{\alpha_{\sigma(i)}} \otimes x_{\alpha_{\sigma(i+1)}}...x_{\alpha_{\sigma(n)}}$$

$\forall h = \sum_{i(),n_i} x_{i(1)}x_{i(2)}...x_{i(n_i)} \in H', i(1) \le i(2) \le ...i(n_i)$, we prove that it is in $U(P)$ by induction on the degree (the highest degree of the monomials) and the biggest x_α that appears in the highest-degree terms. Consider $deg = n > 1$, and assume a highest-degree term with the biggest x_α in h is $x_{j(1)}...x_{j(n_j)}$.

Let $x^*_{j(n_j)}$ be the linear function on H such that $< x^*_{j(n_j)}, x_{j(n_j)} >= 1$ and 0 on the other elements in the standard basis. Then $x^*_{j(n_j)} \rightharpoonup h \in H'$ has degree<n. So $f = x^*_{j(n_j)} \rightharpoonup h \in U(P)$ by induction. Choose a linear function $f^*, < f^*, f >= 1$. Then $h \leftharpoonup f^* \in H'$ has the monomial $x_{j(n_j)}$ and $deg(h \leftharpoonup f^*) < n$. Thus, $h \leftharpoonup f^* \in U(P)$. But we can find $d \in k, h - df(h \leftharpoonup f^*) \in H'$ has smaller biggest x_α in some highest-degree terms, and is in $U(P)$ by induction. Therefore, $h \in U(P)$.

One example of my idea : suppose $h = x_{\alpha_7} + (x_{\alpha_1}x_{\alpha_2} + x_{\alpha_3}x_{\alpha_4})(x_{\alpha_5} + x_{\alpha_6}) \in H'$. Then $f = x_{\alpha_1}x_{\alpha_2} + x_{\alpha_3}x_{\alpha_4}$. $h \leftharpoonup f^* = x_{\alpha_5} + x_{\alpha_6}$. $h - f(h \leftharpoonup f^*) = x_{\alpha_7}$.

Therefore, $H' = U(P)$. $\square$