Can lattice points approximate irrationals?

Exploring number theory through the regular system of points

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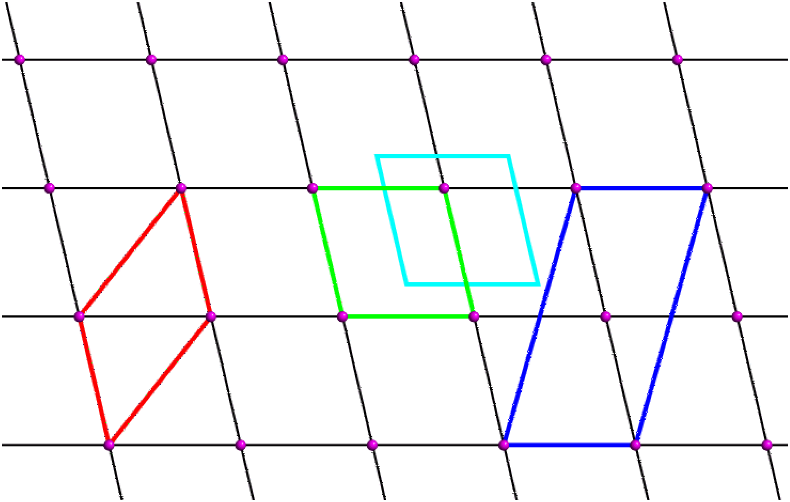
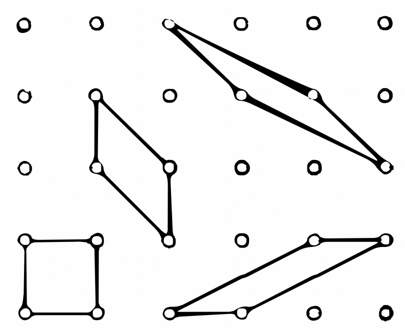
A look into lattice points

a.k.a. regular system of points in a plane



Generating lattice structure

Hero: Unit cells



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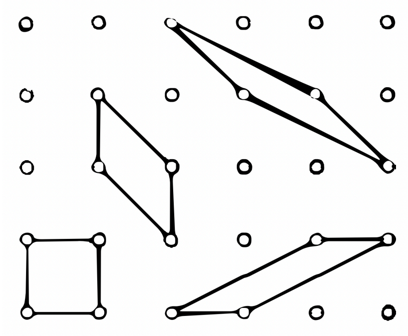

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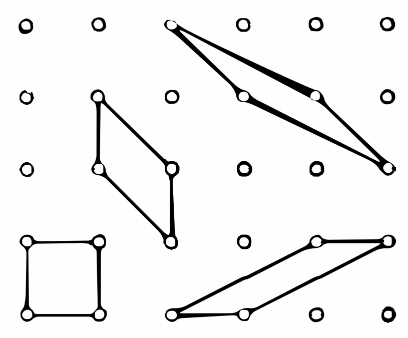

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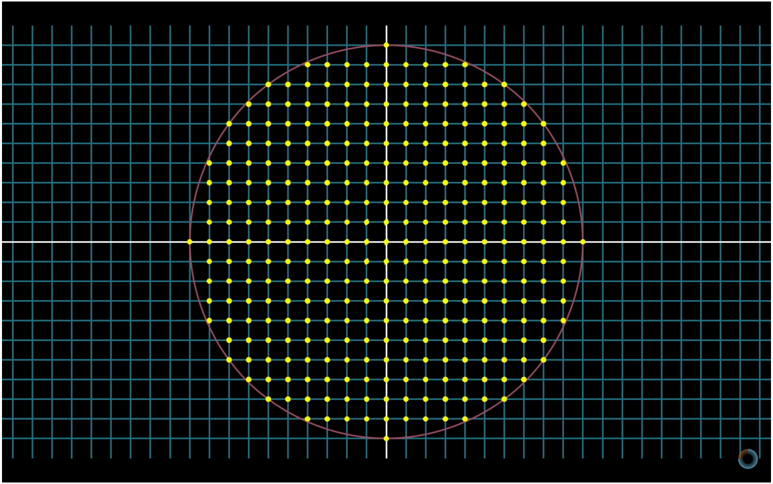
Given any lattice structure, the unit cell corresponding to it is not unique.

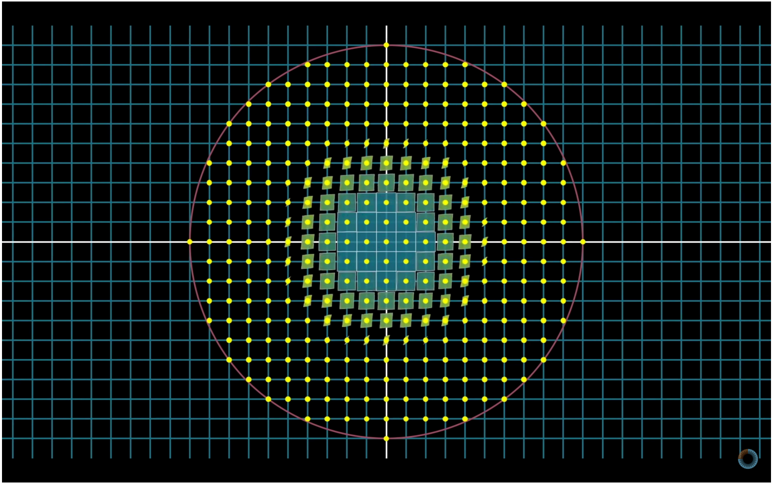
Circle enters the room

Hunting lattice points inside a circle



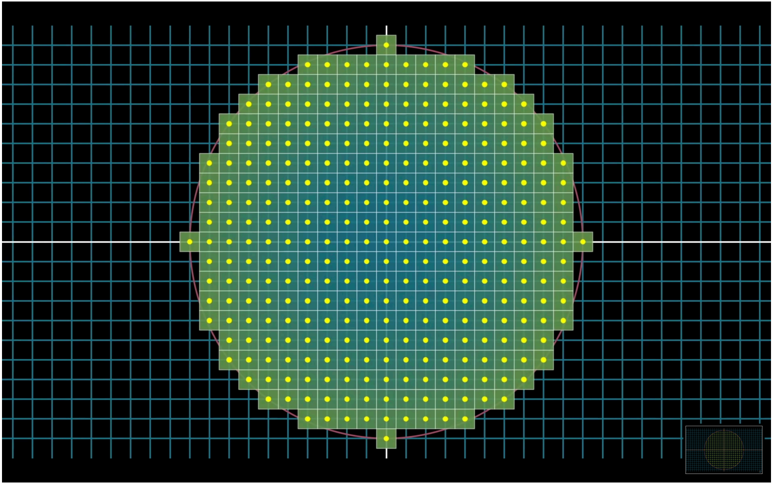
Circle entered the room

Approximating the area of the circle through lattice points



Approximating the area

π: I am coming!



Is it a good approximation?

Suppose we assign the lattice points integer coordinates. Let the number of lattice points inside and on the circle is $N(r)$ for a given integer radius r.

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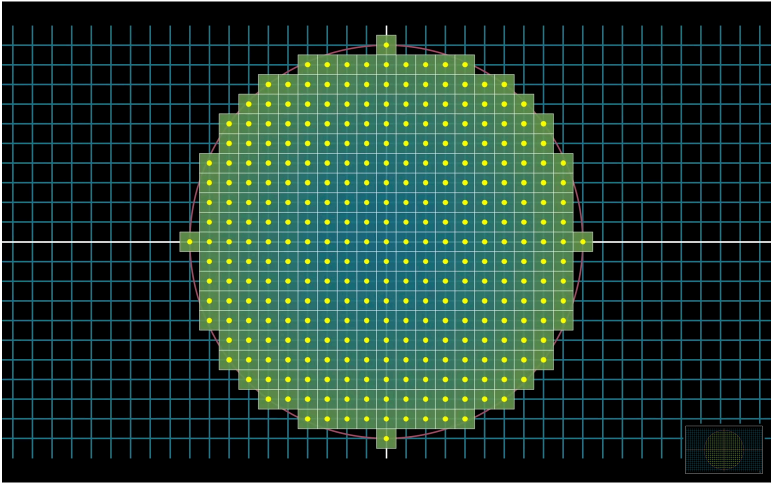
This is the time we have to look back the previous diagram to estimate some bound on $\varepsilon(r)$.

The bad squares

Can we estimate the area of the bad squares which is yielding the error?



Estimating the bad area

The bad squares are entirely contained inside the annulus of smaller radius $r - \sqrt{2}$ and bigger radius $r + \sqrt{2}$. Hence the bad area

$$\varepsilon(r) < E(r) = \pi[(r + \sqrt{2})^2 - (r - \sqrt{2})^2] = 4\sqrt{2}\pi r$$

Now it's moment to plug back!

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Look at the equation: $N(r), r \in \mathbb{N}$. Hence, $\frac{N(r)}{r^2}$ is a rational approximation of the irrational number π with the fact that we can approximate as close as we want, with increasing r, subsequently decreasing the error term.

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Specifically, we will focus on **unit lattices**. Unit lattices are the lattice structures which are generated by the parallelograms or squares whose area is 1 unit^2.

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The distance of a lattice point to its first (nearest) neighbour (denote that distance as c).

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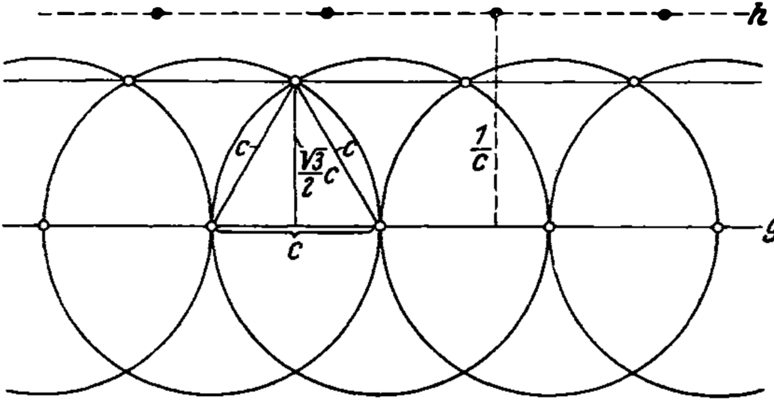
Question

Can we find some upper bound to it?

Finding max of the min distance



Finding max of the min distance

Thus we have,

$$\frac{1}{c} \geq \frac{c}{2}\sqrt{3}$$

which yields an upper bound:

$$\boxed{c \leq \sqrt{\frac{2}{\sqrt{3}}}}$$

An easy exercise is to check when does the equality occur!

Approximation back again!

Theorem (Dirichlet's Approximation)

For a given irrational α, the inequality,

$$\left|\alpha - \frac{x}{y}\right| < \frac{1}{y^2}$$

*is satisfied by **infinitely many** integers x, y.*

Consider the quadratic form of two variables

$$Q(x, y) = ax^2 + 2hxy + by^2$$

where $a, h, b \in \mathbb{R}$. Inspecting $Q(x, y)$, we get a representation of it in the following way:

Approximation back again!

Focusing object: Quadratic forms

Consider the quadratic form of two variables

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where $a, h, b \in \mathbb{R}$. Inspecting $Q(x, y)$, we get a representation of it in the following way:

$$Q(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The determinant of the matrix in between is called *Discriminant* (denoted by D) of $Q(x, y)$. For our problem, set $D = 1$. Then $a \neq 0$. WLOG, assume $a > 0$.

Setting up a linear transformation

Notice that, now $Q(x, y)$ can be written as

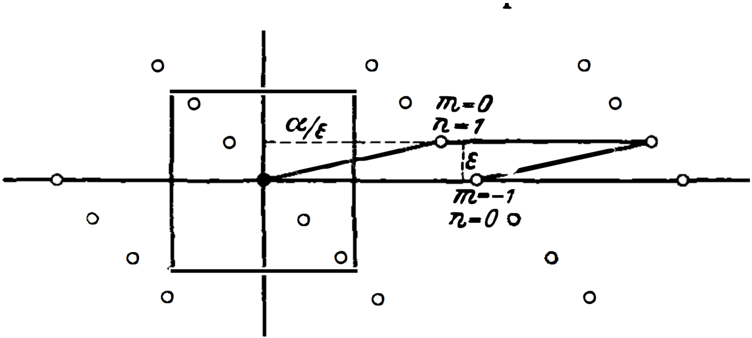
$$Q(x, y) = \left(\sqrt{a}x + \frac{h}{\sqrt{a}}y\right)^2 + \left(\frac{1}{\sqrt{a}}y\right)^2$$

Now set

$$\tilde{x} = \sqrt{a}x + \frac{h}{\sqrt{a}}y$$
$$\tilde{y} = \frac{1}{\sqrt{a}}y$$

Note that, this is a linear transformation with determinant 1 (thus, area scaled is 1, i.e., the unit cell square is sheared into a parallelogram), pictorially shown in the next slide.

The shear



Ignore the parameters in the diagrams.

The shear

If we allow x, y to run over integers (i.e., the pre-image lattice is ordinary unit square lattice), then the equation of the transformations represent set of points lying on the line

$$\tilde{x} = h\tilde{y} + \sqrt{a}m$$

and

$$\tilde{y} = \frac{1}{\sqrt{a}}n$$

where $m, n \in \mathbb{Z}$. Which roughly looks like the dots in the above diagram.

Key lemma

Consider,

$$Q(x, y) = \tilde{x}^2 + \tilde{y}^2$$

Thus $\sqrt{Q(x, y)}$ represents the distance from the origin to the point $(\tilde{x}, \tilde{y})$. The above "first neighbour distance" bound says that there will exist a point with the minimum distance $c \leq \sqrt{\frac{2}{\sqrt{3}}}$.

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Lemma

Take integer lattice $\mathbb{Z}^2$. After we transform it through a linear transformation as discussed above whose determinant is 1. Then there exists $x, y \in \mathbb{Z}$ such that

$$Q(x, y) \leq \frac{2}{\sqrt{3}}$$

where $(x, y) \mapsto (\tilde{x}, \tilde{y})$ and $Q(x, y) = \tilde{x}^2 + \tilde{y}^2$.

The final bash

Let $\epsilon > 0$. Consider the quadratic form

$$Q(x, y) = \left(\frac{\alpha y - x}{\epsilon}\right)^2 + \epsilon^2 y^2$$

whose determinant is

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From the previous lemma, it follows that, $\exists x, y \in \mathbb{Z}$ such that

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Consequently, as a *fortiori* we have two simultaneous inequalities (projection of radius is lesser than or equal to the radius!):

$$\left|\frac{\alpha y - x}{\epsilon}\right| \leq \sqrt{\frac{2}{\sqrt{3}}}, \quad |\epsilon y| \leq \sqrt{\frac{2}{\sqrt{3}}}$$

The final bash

Assume $y \neq 0$. Rearranging the inequalities a bit and using the fact that $|y| \geq 1$ and $y \in \mathbb{Z} \backslash \{0\}$, we have

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{\epsilon}{|y|}\sqrt{\frac{2}{\sqrt{3}}} \leq \epsilon\sqrt{\frac{2}{\sqrt{3}}}, \quad |y| \leq \frac{1}{\epsilon}\sqrt{\frac{2}{\sqrt{3}}}$$

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- Fix some $\epsilon > 0$.
- By the argument we are guranteed to get integer solutions x, y which satisfies both the inequalities.

Then rational number $\frac{x}{y}$ approximates the irrational α.

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- Take $\epsilon < \frac{1}{2}\left|\alpha - \frac{x_0}{y_0}\right|$
- Again, the previous argument ensures the existence of new integer solutions x_1, y_1, which indeed is better than x_0, y_0.

This process can be continued indefinitely to get as much good accuracy as we want.

Arriving to the end of the story!

Or, it is the start?

Theorem (Weaker form of Dirichlet's Approximation)

Suppose α is an irrational number. $\forall \epsilon > 0 \quad \exists x, y \in \mathbb{Z}$ such that the two simultaneous inequalities hold:

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{\epsilon}{|y|}\sqrt{\frac{2}{\sqrt{3}}}, \quad |y| \leq \frac{1}{\epsilon}\sqrt{\frac{2}{\sqrt{3}}}$$

By the above refinement process, we are guaranteed to generate an ***infinite sequence*** *of $x, y \in \mathbb{Z}$, such that*

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{2}{\sqrt{3}}\frac{1}{y^2}$$

Arriving to the end of the story!

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Example

We want to approximate $\sqrt{43}$. Take $\epsilon = 0.5$, then $\frac{x}{y} = \frac{6}{1}$ is the only solution satisfying both the inequalities. Reducing ϵ a bit, say $\epsilon = 0.2 < \frac{1}{2}\left|\sqrt{43} - \frac{6}{1}\right|$, we get two solutions $\frac{x}{y} = \frac{13}{2}$ and $\frac{33}{5}$, which are guaranteed to have higher accuracy of approximation than before.

Remark

Not only the above, as we go along the sequence, **eventually** the approximation gets better as well as **quickly** since the error bound is proportional to the inverse of the square of the denominator, than randomly setting a denominator and then finding a multiple of that which “fits” with the irrational number.

Thank You! Identity.

References and Acknowledgements:

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