Covariance

context: Two jointly distributed r.v.s X, Y

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y).$$

From the proof of Theorem 2.6.8 (iii):

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\left[E(XY) - E(X)E(Y)\right]$$

$$\overset{\text{under independence}}{=} \mathrm{Var}(X) + \mathrm{Var}(Y)$$

W/out independence, still have this

$$\boxed{\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)}$$

$$\begin{aligned}
\mathrm{Var}(X-Y) &= E\left((X-Y)^2\right) - \left[E(X-Y)\right]^2 \\
&= E\left(X^2 - 2XY + Y^2\right) - \left[E(X) - E(Y)\right]^2 \\
&= E\left(X^2 - 2XY + Y^2\right) - \left[E(X)^2 - 2E(X)E(Y) + E(Y)^2\right] \\
&= E(X)^2 - 2E(XY) + E(Y^2) - E(X)^2 + 2E(X)E(Y) - E(Y)^2
\end{aligned}$$

$$\boxed{\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y)}$$

True generally

If, in addition, X, Y independent, get 0 Covariance so

$$\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

Thm.:
$$\begin{aligned}
\mathrm{Cov}(X,Y) &= E\left((X-\mu_X)(Y-\mu_Y)\right) \\
&= E\left((X-E(X))(Y-E(Y))\right)
\end{aligned}$$

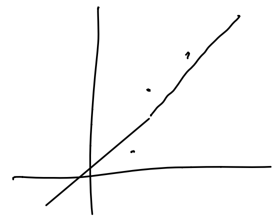
When is Covariance nearly 0?

large positive?

large negative?

Using covariance, we can define a "normalized version": correlation

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{\sigma_{XY}}{\sigma_X \, \sigma_Y}$$



Facts about ρ (correlation between X and Y)

1) $-1 \le \rho \le 1$

2) If $\rho = -1$, then $P(Y = aX + b) = 1$ for some choice of $a < 0$ and b.

Some covariance results

1. $Cov(aX, bY) = ab\, Cov(X, Y)$

2. $$\underbrace{Var(X_1 + X_2 + \cdots + X_n)}_{\text{Useful}} = \boxed{\sum_{j=1}^{n} \sum_{k=1}^{n} Cov(X_j, X_k)}$$

Covariance matrix

$$\begin{bmatrix} Cov(X_1,X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,X_n) \\ \vdots & & & \\ Cov(X_n,X_1) & \cdots & & Cov(X_n,X_n) \end{bmatrix}$$

$Cov(X_j, X_j) = Var(X_j)$