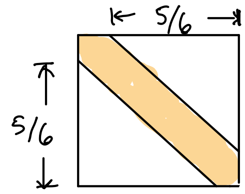
3.55 (a) $f_{X,Y}(x,y) = 1$ for $(x,y) \in [5,6] \times [5,6]$.

(b) $\Pr\big((X,Y) \in [5, 5.5] \times [5, 5.5]\big) = 1/4$.

(c) For them to arrive within 10 minutes of each other, (X,Y) must be a point in the orange shaded region, which has area (= probability)

$$1 - (2)\left(\tfrac{1}{2}\right)\left(5/6\right)^2 = \frac{11}{36}.$$



$[5,6] \times [5,6]$

3.57 (a) Let $A = \{(x,y) \mid x^2+y^2 \le R^2\}$. For the pdf to be a constant k, we require

$$1 = k \cdot \text{Area}(A) = \pi R^2 k \implies k = \frac{1}{\pi R^2}.$$

Thus

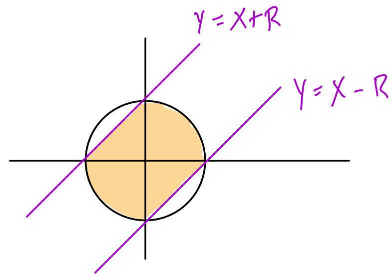
$$f(x,y) = \begin{cases} \dfrac{1}{\pi R^2}, & (x,y) \in A, \\ 0, & \text{otherwise} \end{cases}$$

(b) Let $H = \{(x,y) \mid x^2+y^2 \le \frac{R}{2}\}$. $\Pr\big((x,y) \in H\big) = \dfrac{\text{Area}(H)}{\text{Area}(A)} = \dfrac{\pi (R/2)^2}{\pi R^2} = \dfrac{1}{4}$.

(c) $|X - Y| \le R \iff X - R \le Y \le X + R$

The desired probability corresponds to the area of the orange shaded region:

$$\left(\frac{1}{2}\pi R^2 + R^2\right) \cdot \frac{1}{\pi R^2} = \frac{1}{2} + \frac{1}{\pi}.$$



(d) Let $x \in [-R, R]$. Then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2}$$

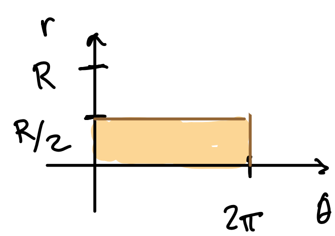
(e) X and Y are not independent. By symmetry, $f_Y(y) = \dfrac{2}{\pi R^2}\sqrt{R^2-y^2}$ for $y \in [-R, R]$.

But $f_X(x) f_Y(y) = \dfrac{4}{\pi^2 R^4}\sqrt{(R^2-x^2)(R^2-y^2)} \ne \dfrac{1}{\pi R^2}$.

3.58 This gives a different distribution. Under this approach,

$$\Pr\big((x,y) \in H\big) = \frac{1}{2\pi R} \cdot \text{Area(shaded region of } r\theta\text{-plane)}$$

$$= \frac{1}{2\pi R} \cdot (2\pi)\left(\tfrac{1}{2}R\right) = \frac{1}{2}$$



a different result than in part (b) of Exercise 3.57.

3.63 $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$, so $M_X(t) = e^{e^t\lambda_1 - \lambda_1}$, and $M_Y(t) = e^{e^t\lambda_2 - \lambda_2}$.

By independence of X, Y,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{e^t\lambda_1} \cdot e^{-\lambda_1} \cdot e^{e^t\lambda_2} \cdot e^{-\lambda_2}$$

$$= e^{e^t\lambda_1 + e^t\lambda_2} \cdot e^{-(\lambda_1+\lambda_2)} = e^{e^t(\lambda_1+\lambda_2) - (\lambda_1+\lambda_2)}.$$

This is the mgf for another Poisson r.v. with parameter $\lambda_1 + \lambda_2$. Thus

$$X + Y \sim \text{Pois}(\lambda_1 + \lambda_2).$$

3.67 Each $X_i \sim \text{Gamma}(\alpha, \lambda)$. By independence,

$$M_S(t) = \prod_{i=1} \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} = \left(\frac{\lambda}{\lambda - t}\right)^{n\alpha} \Rightarrow S \sim \text{Gamma}(n\alpha, \lambda).$$

$$M_{\bar{X}}(t) = M_{\frac{1}{n}S}(t) = M_S\left(\tfrac{1}{n}t\right) = \left(\frac{\lambda}{\lambda - t/n}\right)^{n\alpha} = \left(\frac{n\lambda}{n\lambda - t}\right)^{n\alpha} \Rightarrow \bar{X} \sim \text{Gamma}(n\alpha, n\lambda).$$

4.1 Let $\bar{x} = \frac{1}{n}\sum X_i$ be the first sample moment about the origin (a.k.a. the sample mean). The population mean for $\text{Binom}(1, \pi)$ is $1 \cdot \pi = \pi$. Our estimate is $\hat{\pi} = \bar{x}$.

4.4 For $X \sim \text{NBinom}$, $E(X) = \frac{s}{\pi} - s$. So, we set

$$\frac{s}{\hat{\pi}} - s = \bar{x} \implies \hat{\pi} = \frac{s}{s + \bar{x}}.$$

4.7 favstats reveals $\bar{x} \doteq 0.6091$, $s \doteq 0.248$, $n = 134$ $\Rightarrow v = 0.06105$.

From the formulas:

$$\hat{\alpha} = \bar{x}\left(\frac{\bar{x}(1-\bar{x})}{v} - 1\right) \doteq 1.7665$$

$$\hat{\beta} = (1-\bar{x})\left(\frac{\bar{x}(1-\bar{x})}{v} - 1\right) \doteq 1.1337$$

The beta distribution using these shape parameters gives a very poor fit to the data. By filtering out the players with FTPct = 0, the new parameter estimates from remaining players are

$$\hat{\alpha} = 4.824, \quad \hat{\beta} = 2.387,$$

and the fit is vastly improved.

4.9 Take $\bar{x} = \frac{1}{n}\sum x_i$ and $v = \frac{1}{n}\sum (x_i - \bar{x})^2$.

For an exponential distribution, $\hat{\lambda} = \frac{1}{\bar{x}} \doteq 0.00763$

For a gamma distribution,

$$\frac{\hat{\alpha}}{\hat{\lambda}} = \bar{x} \implies \hat{\alpha} = \hat{\lambda}\bar{x}$$

$$\frac{\hat{\alpha}}{\hat{\lambda}^2} = v \implies \frac{\hat{\lambda}\bar{x}}{\hat{\lambda}^2} = \frac{\bar{x}}{\hat{\lambda}} = v$$

$$\implies \hat{\lambda} = \frac{\bar{x}}{v} \doteq 1.858 \quad \text{and} \quad \hat{\alpha} = \hat{\lambda}\bar{x} = \frac{\bar{x}^2}{v} \doteq 0.0141.$$

4.14 We know $SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2.$

Thus, $\Pr(|\bar{X} - \mu| < 3) = \text{pnorm}(3, 0, 2) - \text{pnorm}(-3, 0, 2)$
$\doteq 0.866.$

4.16 (a) The 10 different SRS, along w/ resulting sample means:

Sample	$\bar{X}$	Sample	$\bar{X}$
1, 6	3.5	6, 8	7
1, 6	3.5	6, 9	7.5
1, 8	4.5	6, 8	7
1, 9	5	6, 9	7.5
6, 6	6	8, 9	8.5

$$\implies \mu_{\bar{X}} = (3.5 + 7 + 7.5)(2/10) + (4.5 + 5 + 6 + 8.5)(1/10) = 6.$$

$$Var(\bar{X}) = (3.5^2 + 7^2 + 7.5^2)(2/10) + (4.5^2 + 5^2 + 6^2 + 8.5^2)(1/10) - \mu_{\bar{X}}^2$$
$$= 2.85.$$

(b) From Coro. 4.3.3, we have

$$E(\bar{X}) = \mu = (1+6+6+8+9)(1/5) = 6,$$

a match with part (a). Furthermore,

$$Var(\bar{X}) = \frac{\sigma^2}{n} \cdot \frac{N-n}{N-1} = \frac{\sigma^2}{2} \cdot \frac{5-2}{5-1} = \frac{3}{8}\sigma^2,$$

where $\sigma^2 = (1^2+6^2+6^2+8^2+9^2)(1/5) - 6^2 = 7.6.$ So,

$$Var(\bar{X}) = 3/8\ \sigma^2 = 2.85, \quad \text{also matching part (a).}$$

(c) We may treat an iid sample as if we were rolling 5-sided dice, yielding pairings:

(1,1)	(1,6)	(1,6)	(1,8)	(1,9)
(6,1)	(6,6)	(6,6)	(6,8)	(6,9)
(6,1)	(6,6)	(6,6)	(6,8)	(6,9)
(8,1)	(8,6)	(8,6)	(8,8)	(8,9)
(9,1)	(9,6)	(9,6)	(9,8)	(9,9)

The 5-by-5 table of means corresponds directly to these pairings

1	3.5	3.5	4.5	5
3.5	6	6	7	7.5
3.5	6	6	7	7.5
4.5	7	7	8	8.5
5	7.5	7.5	8.5	9

So,

$$\mu = (1+8+9)(1/25) + (4.5+5+8.5)(2/25) + (3.5+6+7+7.5)(4/25) = 6,$$

and

$$\sigma^2 = (1^2+8^2+9^2)(1/25) + (4.5^2+5^2+8.5^2)(2/25) + (3.5^2+6^2+7^2+7.5^2)(4/25) - 6^2$$

$$= 3.8.$$

4.39 (a) $|\vec{u}_1|^2 = \left(\frac{1}{\sqrt{n}}\right)^2 + \cdots + \left(\frac{1}{\sqrt{n}}\right)^2 = n \cdot \left(\frac{1}{n}\right) = 1 \Rightarrow |\vec{u}_1| = 1.$

For $i = 2, 3, \ldots, n,$

$$|\vec{u}_i|^2 = \frac{1}{i(i-1)}\left[(i-1)^2 + \sum_{j=1}^{i-1} 1^2\right] = \frac{1}{i(i-1)}\left[(i-1)^2 + (i-1)\right]$$

$$= \frac{i-1}{i(i-1)}\left[(i-1)+1\right] = \frac{1}{i}(i) = 1 \Rightarrow |\vec{u}_i| = 1.$$

Moreover, for $1 < i < j$,

$$\vec{u}_i \circ \vec{u}_j = 1 - i + \sum_{m=1}^{i-1} 1 = (1-i) + (i-1) = 0.$$

It is more transparent that each $\vec{u}_i \circ \vec{u}_1 = 0$, $i > 1$.

(b) Let $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$. Then,

$$\vec{x} \circ \vec{u}_1 = \frac{1}{\sqrt{n}}(x_1 + x_2 + \cdots + x_n) = \sqrt{n} \cdot \frac{1}{n} \sum x_i = \bar{x}\sqrt{n}.$$

(c) With $\vec{x}$ as in part (b),

$$(\vec{x} \circ \vec{u}_1)\vec{u}_1 = (\bar{x}\sqrt{n}) \cdot \frac{1}{\sqrt{n}} \langle 1, 1, \ldots, 1 \rangle = \langle \bar{x}, \bar{x}, \ldots, \bar{x} \rangle.$$

(d) For $\vec{x} = \langle 3, 4, 4, 7, 7 \rangle$, $\bar{x} = \frac{1}{5}(25) = 5$, and

$$\vec{v} = \langle 3, 4, 4, 7, 7 \rangle - \langle 5, 5, 5, 5, 5 \rangle = \langle -2, -1, -1, 2, 2 \rangle.$$

Thus,

i	$\vec{u}_i$	$\vec{x} \circ \vec{u}_i$	$\vec{v} \circ \vec{u}_i$	$\lvert proj(\vec{x} \to \vec{u}_i)\rvert$	$\lvert proj(\vec{v} \to \vec{u}_i)\rvert$
1	$1/\sqrt{5}\langle 1,1,1,1,1 \rangle$	$5^{3/2}$	0	$5^{3/2}$	0
2	$1/\sqrt{2}\langle 1,-1,0,0,0 \rangle$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
3	$1/\sqrt{6}\langle 1,1,-2,0,0 \rangle$	$-1/\sqrt{6}$	$-1/\sqrt{6}$	$1/\sqrt{6}$	$1/\sqrt{6}$
4	$1/\sqrt{12}\langle 1,1,1,-3,0 \rangle$	$-10/\sqrt{12}$	$-10/\sqrt{12}$	$10/\sqrt{12}$	$10/\sqrt{12}$
5	$1/\sqrt{20}\langle 1,1,1,1,-4 \rangle$	$-10/\sqrt{20}$	$-10/\sqrt{20}$	$10/\sqrt{20}$	$10/\sqrt{20}$

For $i \geq 2$, $\vec{x} \circ \vec{u}_i = \vec{v} \circ \vec{u}_i$. This is true for other $\vec{x} \in \mathbb{R}^5$, too, since

$$\vec{v} \circ \vec{u}_i = (\vec{x} - \vec{\bar{x}}) \circ \vec{u}_i = \vec{x} \circ \vec{u}_i - \vec{\bar{x}} \circ \vec{u}_i = \vec{x} \circ \vec{u}_i - 0$$

since $\vec{\bar{x}}$, being parallel to $\vec{u}_1$, is orthogonal to each $\vec{u}_i$ with $i \geq 2$.

4.40 (a) $\bar{x} = (3+4+5+8)/4 = 5$

$$s^2 = \frac{1}{3}\left[(3-5)^2 + (4-5)^2 + (5-5)^2 + (8-5)^2\right] = \frac{14}{3}.$$

(b) $\vec{p}_1 = \langle 5, 5, 5, 5 \rangle$, as determined in Exercise 4.39.

$$\vec{p}_2 = \frac{1}{\sqrt{2}}(3-4)\vec{u}_2 = \langle -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0 \rangle$$

$$\vec{p}_3 = \frac{1}{\sqrt{6}}(3+4-10)\vec{u}_3 = \langle -\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0 \rangle$$

$$\vec{p}_4 = \frac{1}{\sqrt{12}}(3+4+5-24)\vec{u}_4 = \langle -1, -1, -1, 3 \rangle$$

and $\sum \vec{p}_i = \langle 3, 4, 5, 8 \rangle$ as predicted.

(c) $\ell_1 = |\vec{p}_1| = \sqrt{4(5^2)} = 10$

$\ell_2 = |\vec{p}_2| = \sqrt{2(1/2)^2} = 1/\sqrt{2}$

$\ell_3 = |\vec{p}_3| = \sqrt{2(\frac{1}{2})^2 + 1^2} = \sqrt{3/2}$

$\ell_4 = |\vec{p}_4| = \sqrt{3(-1)^2 + 3^2} = 2\sqrt{3}$

(d) $\sum_{i=2}^{4} \ell_i^2 = \frac{1}{2} + \frac{3}{2} + 12 = 14 = 3\left(\frac{14}{3}\right) = 3s^2.$

C.10 $\text{proj}(\langle 1,2,3\rangle \to \langle 1,1,1\rangle) = \frac{\langle 1,2,3\rangle \circ \langle 1,1,1\rangle}{|\langle 1,1,1\rangle|^2}\langle 1,1,1\rangle = \langle 2,2,2\rangle$

This is the closest vector to $\langle 1,2,3\rangle$ in $W = \text{span}(\langle 1,1,1\rangle)$, so

$$\langle 1,2,3\rangle - \text{proj}(\langle 1,2,3\rangle \to \langle 1,1,1\rangle) = \langle -1,0,1\rangle$$

is perpendicular/orthogonal to $\langle 1,1,1\rangle$, and

$$\text{span}\{\langle 1,1,1\rangle, \langle 1,2,3\rangle\} = \text{span}\{\langle 1,1,1\rangle, \langle -1,0,1\rangle\}.$$

To get these orthogonal vectors to be of length 1, we divide by their lengths:

$$\vec{u}_1 = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle, \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\langle -1,0,1\rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle.$$