

## Linear 1<sup>st</sup>-order homogeneous systems with nonreal eigenvalues

There are certain things we build on:

- **Euler's Formula:** Given a real number  $\theta$ , and  $i = \sqrt{-1}$ , it says  $e^{i\theta} = \cos \theta + i \sin \theta$ .  
A corollary to it is that  $e^{-i\theta} = \cos \theta - i \sin \theta$ , making  $e^{i\theta}$  and  $e^{-i\theta}$  complex conjugates.  
For an explanation of why this amazing formula holds, and secondarily to justify in part your study of Maclaurin series in MATH 172, watch  
<https://drive.google.com/file/d/1a7x1QIdNYGis6np3V9rXkq8xYh0wE3yD/view?usp=sharing>  
Here are the **finished notes** from the video.
- When a matrix  $\mathbf{A}$  has real entries by a nonreal eigenvalue  $\alpha + i\beta$ , where  $\alpha, \beta$  are real numbers, there will be at least one corresponding eigenvector  $\mathbf{u} + i\mathbf{v}$ , where  $\mathbf{u}, \mathbf{v}$  have real entries. Correspondingly, the complex conjugate  $\alpha - i\beta$  is also an eigenvalue of  $\mathbf{A}$ , and has  $\mathbf{u} - i\mathbf{v}$  as an eigenvector. For example, if

$$-3 + 2i \quad \text{is an eigenvalue with eigenvector} \quad \begin{bmatrix} 2 - 3i \\ 1 - i \\ 3i \end{bmatrix},$$

then we can identify

$$\alpha = -3, \quad \beta = 2, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix},$$

and conclude that

$$\alpha - i\beta = -3 - 2i \quad \text{is also an eigenvalue with eigenvector} \quad \mathbf{u} - i\mathbf{v} = \begin{bmatrix} 2 + 3i \\ 1 + i \\ -3i \end{bmatrix}.$$

- We have demonstrated and made of the fact that, if the matrix  $\mathbf{A}$  has eigenpair  $(\lambda, \mathbf{v})$ , then  $e^{\lambda t}\mathbf{v}$  is a solution of the homogeneous linear 1<sup>st</sup>-order system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ . But it is not possible to make physical sense of such a solution

$$e^{\alpha+i\beta t}(\mathbf{u} + i\mathbf{v}) \quad \text{and its counterpart} \quad e^{\alpha-i\beta t}(\mathbf{u} - i\mathbf{v}),$$

when we are talking about nonreal eigenpairs of  $\mathbf{A}$ . In

<https://drive.google.com/file/d/1e1JcAEDA807WxB6JigSH3NYf5YgcY90C/view?usp=sharing>

I justify why it is reasonable and valid to trade out those nonreal solutions for these *real* substitutes:

$$e^{\alpha t} [\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}] \quad \text{and} \quad e^{\alpha t} [\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}].$$

Here are the **finished notes** from that video.

Some examples:

$$1. \frac{d}{dt}\mathbf{x} = \begin{bmatrix} -21 & -30 & -32 \\ -4 & -7 & -7 \\ 24 & 30 & 35 \end{bmatrix}$$

Videos will be played during class, but here are the three pages of end notes:

- [page 1](#)
- [page 2](#)
- [page 3](#)

$$2. \frac{d}{dt}\mathbf{x} = \begin{bmatrix} -5 & -10 \\ 5 & 9 \end{bmatrix}$$

For end notes, consult page 3 above.