MATH 162: Calculus II

Framework for Wed., Apr. 11

Double and Iterated Integrals, Rectangular Regions

Today's Goal: To understand the meaning of double integrals over bounded rectangular regions R of the plane, and to be able to evaluate such integrals.

Important Note: In conjunction with this framework, you should look over Section 13.1 of your text.

Riemann Sums

We assume that $f(x, y)$ is a function of 2 variables, and that $R = \{(x, y) \in \mathbb{R}^2 \,|\, a \leq x \leq b,\ c \leq y \leq d\}$ (i.e., R is some bounded rectangular region of the plane whose sides are parallel to the coordinate axes). Supppose we

- divide R up into n smaller rectangles, labeling them R_1, R_2, $\ldots$, R_n,
- choose, from each rectangle R_k, some point (x_k, y_k), and
- use the symbol ΔA_k to denote the area of rectangle R_k.

Then the sum

$$\sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

is called a *Riemann sum* of f over the region R.

Definition: The collection of smaller rectangles R_1, $\ldots$, R_n is called a *partition* P of R. The maximum, taken over all lengths and widths of these rectangles, is called the *norm* of the partition P, and is denoted by $\|P\|$.

The function f is said to be *integrable over* R if the limit

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k,$$

taken over all partitions P of R, exists. The value of this limit, called the *double integral of* f over R, is denoted by

$$\iint_R f(x, y)\, dA.$$

The following theorem tells us that, for certain functions, integrability is assured

Theorem: Suppose $f(x, y)$ is a continuous function over the closed and bounded rectangle $R = \{(x, y \in \mathbb{R}^2 \,|\, a \leq x \leq b,\ c \leq y \leq d\}$ in the plane. Then f is integrable over R.

Remarks

- The double integral of f over R is a definite integral, and has a numeric value.
- When $f(x,y)$ is a nonnegative function, $\iint_R f(x,y)\,dA$ may be interpreted as the volume under the surface $z = f(x,y)$ over the region R in the xy-plane.
- When $f(x,y)$ is a constant function (i.e., f has the same value, say c, for each input point (x,y)), then
$$\iint\limits_R f(x,y)\,dA \;=\; c\cdot Area(R).$$

Iterated Integrals and Fubini's Theorem

Except in the very special case of constant functions f, the definition of $\iint_R f(x,y)\,dA$ does not, by itself, provide us with much help for evaluating a double integral. (If you are viewing this document on the web, click here for an alternate point of view, expressed by Peter A. Lindstrom of Genesee Community College.) However, the following theorem indicates that the double integral may be evaluated as an *iterated integral*.

Theorem: (**Fubini**) Suppose $f(x,y)$ is continuous throughout the rectangle $R = \{(x,y) \in \mathbb{R}^2 \,|\, a \le x \le b,\ c \le y \le d\}$. Then

$$\iint\limits_R f(x,y)\,dA \;=\; \int_a^b \int_c^d f(x,y)\,dy\,dx \;=\; \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Examples: Evaluate the double integral, with R as specified.

1. $\displaystyle\iint\limits_R (2x + x^2 y)\,dA,$ where $R: -2 \le x \le 2,\ -1 \le y \le 1.$

2. $\displaystyle\iint\limits_R y\sin(xy)\,dA,$ where $R: 1 \le x \le 2,\ 0 \le y \le \pi.$