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Stat 343, Mon 19-Oct-2020 -- Mon 19-Oct-2020
Probability and Statistics
Fall 2020


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Monday, October 19th 2020
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Wk 8, Mo
Topic:: Joint continuous distributions
Read::  FASt 3.8
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Joint Distributions for Continuous r.v.s

Definition 1: Let X, Y be r.v.s. The **bivariate distribution function**, or **joint cdf**, is the function F given by

$$F(x, y) \;:=\; P(X \leq x, Y \leq y),$$

where it is understood that these conditions are happening simultaneously (i.e., $X \leq x$ and $Y \leq y$). More generally, if $X_1, \ldots, X_n$ are r.v.s and $\mathbf{X} = (X_1, \ldots, X_n)$, the joint cdf is the function

$$F(\mathbf{x}) \;=\; P(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

We call $\mathbf{X}$ a **random vector**.

Definition 2: Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector. If there exists a nonnegative function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ such that

$$F(\mathbf{x}) \;=\; \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} f(t_1, t_2, \ldots, t_n)\, dt_n \cdots dt_2\, dt_1,$$

then $X_1, \ldots, X_n$ are said to be **jointly continuous r.v.s** with **joint probability density function** $f(\mathbf{x})$; $\mathbf{X}$ is said to be a **continuous random vector**.

We will generally consider bivariate distributions for random vectors $\mathbf{X} = (X_1, X_2)$, but most results carry over naturally to multivariate distributions.

pdf (1) $f_{X,Y}(x,y) \geq 0$ (2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

Theorem 1: Suppose $X_1, \ldots, X_n$ are jointly continuous r.v.s with joint cdf $F(x_1, \ldots, x_n)$ and joint density function $f(x_1, \ldots, x_n)$. Then

(i) given a subset $A \subset \mathbb{R}^n$,

$$P(\mathbf{X} \in A) \;=\; \int_A f(\mathbf{x})\, d\mathbf{x}.$$

(ii) $F(-\infty, x_2, \ldots, x_n) = F(x_1, -\infty, \ldots, x_n) = \cdots = F(x_1, x_2, \ldots, -\infty) = 0.$

(iii) $F(\infty, \infty, \cdots, \infty) \;=\; \int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} \;=\; 1.$

Definition 3: Let f be the joint pdf for continuous r.v.s X, Y. The **marginal distribution** of X has pdf

$$f_X(x) \;=\; \int_{-\infty}^{\infty} f(x,y)\, dy.$$

Similar definitions for f_Y, or when there are more r.v.s.

Definition 4: Let f be the joint pdf for continuous r.v.s X, Y. The **conditional distribution** of X given $Y = y$ has pdf

$$f_{X|Y=y}(x) \;=\; f_{X|Y}(x|y) \;=\; \frac{f(x,y)}{f_Y(y)}.$$

Similar definitions for f_Y, or when there are more r.v.s.

Definition 5: Say two continuous r.v.s X and Y with joint pdf $f(x,y)$ are **independent** if $f(x,y) = f_X(x) f_Y(y)$, for all x, y.

Lemma 1: If X, Y are independent continuous r.v.s, then for each x, y,

(i) $f_X(x) = f_{X|Y}(x|y)$, and
(ii) $f_Y(y) = f_{Y|X}(y|x)$.

Lemma 2: Let X, Y be independent r.v.s, t and s transformations. Then $t(X)$, $s(Y)$ are independent.

Theorem 2: Let X, Y be r.v.s. Then

(i) $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$.
(ii) $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$, if X, Y are independent.
(iii) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$, with $\mathrm{Cov}(X,Y) = 0$ when X, Y are independent.

Theorem 3: Let M_X, M_Y be moment generating functions, defined on an interval containing 0, for independent r.v.s X, Y. Then $M_{X+Y}(t) = M_X(t)M_Y(t)$, with M_{X+Y} defined on the intersection of intervals of definition for M_X, M_Y.

Definition 6: Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random vector with joint pdf $f_{\mathbf{X}}(\mathbf{x})$. Suppose there is a single density function $f(x)$ that serves as the pdf for the marginal distribution for each X_j, so that

$$f_{\mathbf{X}}(\mathbf{x}) \;=\; f_{\mathbf{X}}(x_1, x_2, \ldots, x_n) \;=\; \prod_{j=1}^{n} f(x_i).$$

Then the r.v.s $X_1, \ldots, X_n$ are said to be **independent and identically distributed**, or i.i.d..

In particular, if the X_j are independent with each $X_j \sim \mathsf{Exp}(\lambda)$, we will denote this by $\mathbf{X} \overset{\text{i.i.d.}}{\sim} \mathsf{Exp}(\lambda)$.

Lemma 3: Suppose $X_1, \ldots, X_n$ are i.i.d. and that each $\mathrm{E}\left(X_i\right) = \mu$, each $\mathrm{Var}(X_i) = \sigma^2$. Let $S = X_1 + X_2 + \cdots + X_n$, and $\overline{X} = \frac{1}{n}S$. Then

(i) $\mathrm{E}\left(S\right) = n\mu$ and $\mathrm{Var}(S) = n\sigma^2$.

(ii) $\mathrm{E}\left(\overline{X}\right) = \mu$ and $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

Lemma 4: Suppose $\mathbf{X} \overset{\text{i.i.d.}}{\sim} \mathsf{Norm}(\mu, \sigma)$, and define S, $\overline{X}$ as in the previous lemma. Then

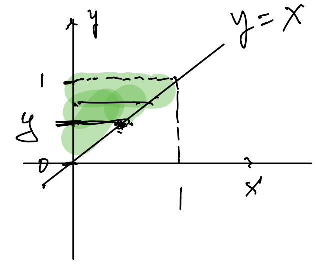
(i) $S \sim \mathsf{Norm}(n\mu, \sigma\sqrt{n})$, and
(ii) $\overline{X} \sim \mathsf{Norm}(\mu, \sigma/\sqrt{n})$.

Proof: By induction on Theorem 3, we have that

$$M_S(t) \;=\; \prod_{j=1}^{n} M_{X_j}(t) \;=\; \prod_{j=1}^{n} e^{\mu t + \sigma^2 t^2/2} \;=\; e^{n\mu t + n\sigma^2 t^2/2},$$

which is the mgv for a normal r.v. with distribution $\mathsf{Norm}(n\mu, n\sigma^2)$. This proves (i).
For (ii), Theorem 3.3.6 (p. 133) gives that $M_{\overline{X}}(t) = M_{X/n}(t) = M_X(t/n) = e^{\mu t + (\sigma^2/n)t^2/2}$, which is the mgf for an r.v. distributed as $\mathsf{Norm}(\mu, \sigma^2/n)$. $\square$

Say $f(x,y) = \begin{cases} x y^2 \cdot k & \text{for } 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$



What k makes this a PDF?

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^1 \int_0^y f(x,y)\, dx\, dy = \int_0^1 \int_0^y (xy^2 \cdot k)\, dx\, dy$$

inside integral $= \displaystyle\int_0^y xy^2 \cdot k\, dx = ky^2 \cdot \frac{1}{2} x^2 \Big|_0^y$

$$= ky^2 \cdot \frac{1}{2} y^2$$

Full double integral

$$\int_0^1 \int_0^y kxy^2\, dx\, dy = \int_0^1 ky^4 \cdot \frac{1}{2}\, dy$$

$$= \frac{1}{2} k \int_0^1 y^4\, dy = \frac{1}{10} k\, y^5 \Big|_0^1 = \frac{1}{10} k$$

$$= \frac{k}{10}$$

To be a pdf, we want this to be 1: $\Longrightarrow$ $k = 10$.

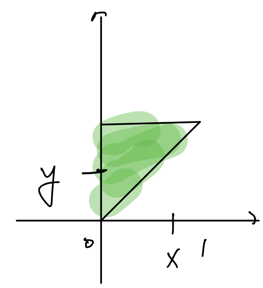
Find $f_X(x)$ the marginal distribution of X

With <u>discrete</u> $f_{X,Y}(x,y)$

$$f_X(x) = \sum_y f_{X,Y}(x,y)$$

So, for continuous joint r.v.s X, Y w/ pdf $f_{X,Y}(x,y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$



$$= \int_x^1 10xy^2\,dy$$

$$= \frac{10}{3} x y^3 \Big|_x^1$$

$$= \frac{10}{3}(x - x^4)$$

Try $$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^y 10xy^2\,dx$$

$$= \frac{10}{2} x^2 y^2 \Big|_0^y = 5y^4.$$

Q: Are X, Y independent? In ~~discrete~~ case $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$10x^2y \stackrel{?}{=} \frac{10}{3}(x - x^4) \cdot 5y^4$$ Not a chance!

Define conditional distribution

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

When X, Y are independent (so $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$)

get

$$f_{X|Y=y}(x) = \frac{f_X(x) \cdot \cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x).$$

PDF $\to$ CDF

cdf $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(t_1,t_2)\, dt_1\, dt_2$$

Given CDF $\to$ PDF

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X,Y}(x,y)$$

Q: Can $F(x,y) = \begin{cases} 1 - e^{-\lambda(x+y)}, & 0 \leq x,y < \infty \\ 0, & \text{otherwise} \end{cases}$

serve as the CDF for jointly distributed continuous vars. X, Y?

If so, the corresp. PDF (for $x, y > 0$) would be

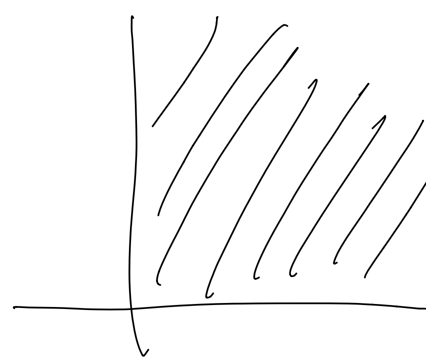
$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[1 - e^{-\lambda(x+y)}\right] = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(1 - e^{-\lambda(x+y)}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\lambda e^{-\lambda(x+y)}\right)$$

$$= \underline{\lambda e^{-\lambda(x+y)} \cdot (-\lambda)}$$

$$= \underline{-\lambda^2 e^{-\lambda(x+y)}} \quad (\text{for } x, y \geq 0)$$

$$< 0$$



A: No!

Generalizable to n variables $\underline{X_1, X_2, X_3, \ldots, X_n}$

pdf $f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) \longrightarrow$ Corresp. CDF

Say $\vec{X} = \langle X_1, X_2, \ldots, X_n \rangle$ is random vector.