How means and variances behave for simple combinations of r.v.s

Theorem 1: Some important results we can verify through simulations. Assume X and Y are random variables (r.v.s), and take μ_X, μ_Y to be their expected values, σ_X, σ_Y to be their standard deviations. Also, assume a, b are arbitrary constants.

(i) $\boldsymbol{E}(aX + b) = a\boldsymbol{E}(X) + b$.
(This can also be written, $\mu_{aX+b} = a\mu_X + b$.)

(ii) $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.

(iii) $\boldsymbol{E}(X + Y) = \boldsymbol{E}(X) + \boldsymbol{E}(Y)$.
Several corollaries include

   (a) $\boldsymbol{E}(aX \pm bY) = a\boldsymbol{E}(X) \pm b\boldsymbol{E}(Y)$.

   (b) $\boldsymbol{E}\left(\sum_i X_i\right) = \sum_i \boldsymbol{E}(X_i)$

   (c) $\boldsymbol{E}\Big(\frac{1}{2}X + \frac{1}{2}Y\Big) = \frac{1}{2}\Big(\boldsymbol{E}(X) + \boldsymbol{E}(Y)\Big)$.

   (d) $\boldsymbol{E}\Big(\frac{1}{n}\sum_i X_i\Big) = \frac{1}{n}\sum_i \boldsymbol{E}(X_i)$

If, in addition, the variables X, Y are independent, then

(iv) $\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.
This can also be written as $\sigma_{X\pm Y} = \sqrt{\sigma_X^2 + \sigma_Y^2}$. Several corollaries include

   (a) If $X_1, \ldots, X_n$ are independent r.v.s, then

$$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots \mathrm{Var}(X_n).$$

   (b) If $X_1, \ldots, X_n$ is an i.i.d. random sample from a population with standard deviation σ, then

$$\mathrm{Var}(\sum_i X_i) = n\sigma^2, \qquad \text{and} \qquad \mathrm{Var}(\frac{1}{n}\sum_i X_i) = \frac{1}{n}\sigma^2.$$

Theorem 2 (Central Limit Theorem): Let $X_1, X_2, \ldots, X_n$ represent an i.i.d. random sample, taken from a population with mean μ and standard deviation σ. Then their sum $S = \sum_i X_i$

(i) has mean $n\mu$ and standard deviation $\sigma\sqrt{n}$.

(ii) has an approximate normal distribution as n grows large.

(iii) has (exactly) a normal distribution, regardless of the size of n, if the original population is normal.

Moreover, if we consider the sample mean

$$\overline{X} = \frac{1}{n}\sum_i X_i = \frac{S}{n},$$

these corresponding statements hold: $\overline{X}$

(iv) has mean μ and standard deviation $\sigma/\sqrt{n}$.

(v) has an approximate normal distribution as n grows large.

(vi) has (exactly) a normal distribution, regardless of the size of n, if the original population is normal.