1. $$\begin{vmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ -2 & k & 0 & 2 \\ -4 & 0 & -1 & 6 \end{vmatrix} = (3)(-1)^4 \begin{vmatrix} 2 & 0 & -1 \\ -2 & 0 & 2 \\ -4 & -1 & 6 \end{vmatrix} + (-1)(-1)^5 \begin{vmatrix} 2 & 1 & -1 \\ -2 & k & 2 \\ -4 & 0 & 6 \end{vmatrix}$$

$$= 3(-1)(-1)^5 \begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} + (-1)^3 \begin{vmatrix} -2 & 2 \\ -4 & 6 \end{vmatrix} + k(-1)^4 \begin{vmatrix} 2 & -1 \\ -4 & 6 \end{vmatrix} = 6 + 4 + 8k = 10 + 8k$$

Solve $10 + 8k = 0 \implies k = -5/4$.

2. (a) Since

$$\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a free column (two, in fact), $\text{null}(A) = \text{null}(A - 0 \cdot I)$ is nontrivial. Thus, 0 is an eigenvalue of A.

(b) And because rref(A) has pivots in columns 1 and 3, it follows that $\{\langle 1, 2, 0, 0 \rangle, \langle 0, 0, 1, 2 \rangle\}$ (columns 1 and 3 from A itself) form a basis for col(A).

3. (a) True. Any free column is in the span of its preceding columns.
   (b) True.
   (c) False. There are plenty of singular square matrices.
   (d) True. A trivial null space coincides with no free columns.
   (e) False. The column space of an $l \times k$ matrix is a subspace of $\mathbb{R}^l$.
   (f) True. The condition ensures RREF of A doesn't have a row of zeros.

4. Call the given matrix A. Eigenvectors corresponding to $\lambda = -$ are in $\text{null}(A + I)$. So, we solve $(A + I)\vec{v} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -4 & 0 & -2 & 0 \\ -10 & 0 & -5 & 0 \\ 8 & 0 & 4 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \underbrace{0 \quad\quad 0}_{v_2,\, v_3 \text{ free}} & & 0 \end{array}\right] \rightarrow v_1 + \tfrac{1}{2} v_3 = 0$$

Eigenvectors look like

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{2} v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -\tfrac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Thus, basis vectors for the eigenspace E_{-2} are $\langle 0,1,0 \rangle$ and $\langle -1,0,2 \rangle$

5. $$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 5 \\ -5 & -3-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda) + 25$$

$$= \lambda^2 - 2\lambda + 10 \implies \lambda = \tfrac{1}{2}\left(2 \pm \sqrt{4-40}\right)$$

$$= \tfrac{1}{2}(2 \pm 6i) = 1 \pm 3i$$

eigenvalues are $1+3i$ and $1-3i$.

6. The system in matrix form is

$$\underbrace{\begin{bmatrix} 3 & -2 & 2 & 0 \\ -1 & 2 & 6 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}_{\vec{b}}$$

To solve, we build the augmented matrix

$$\begin{array}{c} \left[\begin{array}{cccc|c} 3 & -2 & 2 & 0 & -1 \\ -1 & 2 & 6 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 \end{array}\right] \\ \begin{array}{cccc} x & y & z & w \end{array} \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \tfrac{1}{2} & 1 \\ 0 & 1 & 0 & \tfrac{5}{8} & \tfrac{13}{8} \\ 0 & 0 & 1 & -\tfrac{1}{8} & -\tfrac{3}{8} \end{array}\right]$$

w is free

Solutions satisfy

$$x = 1 - \tfrac{1}{2} w$$
$$y = \tfrac{13}{8} - \tfrac{5}{8} w$$
$$z = -\tfrac{3}{8} + \tfrac{1}{8} w$$

with w free (anything in $\mathbb{R}$)

$$\implies \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ \tfrac{13}{8} \\ -\tfrac{3}{8} \\ 0 \end{bmatrix} + w \begin{bmatrix} -\tfrac{1}{2} \\ -\tfrac{5}{8} \\ \tfrac{1}{8} \\ 1 \end{bmatrix}$$

with $w \in \mathbb{R}$.