1. (a) $h(t) = 5e^{-2t}\int_0^t \cos(w)\, e^{2w}\, dw = \int_0^t 5e^{-2t}\cos(w)\, e^{2w}\, dw$

$$= \int_0^t 5e^{-2t+2w}\cos(w)\, dw = \int_0^t 5e^{-2(t-w)}\cos(w)\, dw$$

$$\stackrel{?}{=} (f * g)(t) = \int_0^t f(w)\, g(t-w)\, dw$$

Match if we take

$$\left.\begin{array}{ll} f(t) = \cos t, & g(t) = 5e^{-2t} \\ \text{or} & \\ f(t) = 5\cos t, & g(t) = e^{-2t} \end{array}\right\} \text{both sets are correct}$$

(b) $\left.\begin{array}{l} f(t) = e^t \sin t \\ g(t) = e^{2t} \end{array}\right\}$ Find $\mathcal{L}\{(f*g)(t)\}(s)$

Nasty approach:

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\left\{\int_0^t f(w)\, g(t-w)\, dw\right\}$$

$$\overset{\text{use defn.}}{=} \int_0^\infty e^{-st}\left(\int_0^t f(w)\, g(t-w)\, dw\right) dt$$

Yikes!!!

Much Nicer: Convolution Theorem

$$\underbrace{\mathcal{L}\{(f*g)(t)\}}_{\text{convolution on time side}} = \underbrace{\mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}}_{\text{multiplication on freq. side}}$$

So, we need

1·t here means right shift by 1 here

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{t}\sin t\} = \mathcal{L}\{\sin t\}(s)\Big|_{s \mapsto s-1}$$

$$= \frac{1}{s^2+1}\Big|_{s\mapsto s-1} = \frac{1}{(s-1)^2+1} = \frac{1}{s^2-2s+2}$$

and $$\mathcal{L}\{g(t)\} = \mathcal{L}\{e^{2t}\} = \mathcal{L}\{e^{2t}\cdot 1\}$$

$$= \mathcal{L}\{1\}(s)\Big|_{s\mapsto s-2} = \frac{1}{s}\Big|_{s\mapsto s-2} = \frac{1}{s-2}$$

Conv. Thm.

$$\mathcal{L}\{(f*g)(t)\} = \frac{1}{s^2-2s+2}\cdot\frac{1}{s-2}.$$

2. $\mathcal{L}^{-1}\left\{\frac{3}{s^2+2s+5}\right\} = \cdots$ See my posted soln.

Harder version

$$\mathcal{L}^{-1}\left\{\frac{3s-1}{s^2+2s+5}\right\}$$

Look at denom.: irreducible (nonreal roots)

Complete the square

$$\frac{3s-1}{s^2+2s+5} = \frac{3s-1}{(s^2+2s+1)+4} = \frac{3s-1}{\underbrace{(s+1)}_{\text{shift in } s}{}^2+4}$$

$$= \frac{3(s-1)-1}{s^2+4}\Big|_{s\mapsto s+1} = \frac{3s-4}{s^2+4}\Big|_{s\mapsto s-(-1)}$$

$$= \frac{3s}{s^2+4} - \frac{4}{s^2+4} \Bigg|_{s \mapsto s-(-1)}$$

Now, taken to the t-side

$$\mathcal{L}^{-1}\{\text{original}\} = \mathcal{L}^{-1}\Bigg\{ \underbrace{\frac{3s}{s^2+4}}_{3\cos(2t)} - \underbrace{\frac{4}{s^2+4}}_{2\sin(2t)} \Bigg|_{s \mapsto s-\underline{(-1)}} \Bigg\}$$

$$= \boxed{e^{-t}\left[3\cos(2t) - 2\sin(2t)\right]} \quad \text{Ans.}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-\underline{a})$$

3. Have $f(t) = 3t^2 - 2$

Want $\mathcal{L}\{U(t-3)\,f(t)\}$

which easier?

Have discussed $\underbrace{\mathcal{L}\{U(t-3)\,f(t-3)\}}$

Shift Thm. applies directly

1^{st} task: Find $g(t)$ so $g(t-3) = f(t)$

$$\Rightarrow \quad g(t) = f(t+3)$$

$$= 3(t+3)^2 - 2 = 3(t^2+6t+9) - 2$$

$$= 3t^2 + 18t + 25.$$

Why helps?

$$\mathcal{L}\{U(t-3)\,f(t)\} = \mathcal{L}\{U(t-3)\,g(t-3)\}$$

$$= e^{-3s} \cdot \mathcal{L}\{g(t)\}$$

And

$$\mathcal{L}\{3t^2 + 18t + 25\} = 3\,\mathcal{L}\{t^2\} + 18\,\mathcal{L}\{t\} + 25\,\mathcal{L}\{1\}$$

$$= 3 \cdot \frac{2!}{s^3} + 18 \cdot \frac{1!}{s^2} + 25 \cdot \frac{0!}{s^1}$$

$$= \frac{6}{s^3} + \frac{18}{s^2} + \frac{25}{s}$$

Ans.

$$\mathcal{L}\{\mathcal{U}(t-3)\,f(t)\} = e^{-3s}\left(\frac{6}{s^3} + \frac{18}{s^2} + \frac{25}{s}\right)$$