1. (a)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \qquad 0 = \begin{vmatrix} 2-\lambda & 3 \\ 1 & -5-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 13$$

The zeros of the characteristic polynomial, the eigenvalues, are

$$\lambda = \frac{-3}{2} \pm \frac{1}{2}\sqrt{9+52} = \frac{1}{2}\left(-3 \pm \sqrt{61}\right),$$

both real, with one positive and the other negative, since $\sqrt{61} > 3$.

(b) Because the two eigenvalues are real of opposite sign, the origin is a "saddle point", inherently unstable.

2. Solve first for a basis on null$(A + 2I)$ (basis eigenvector(s)):

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 3 & -3 & 0 \end{array}\right] \hookrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad v_1 = v_2 \text{ in eigenvectors } \vec{v}$$

one free column, so GM = 1. $\lambda = -1$ is degenerate.

$\vec{v} = \langle 3, 3 \rangle$ is a basis e-vector (i.e., all others are scalar multiples of it).

So, we need a generalized e-vector $\vec{w}$ solving $(A + I)\vec{w} = \vec{v}$.

$$\left[\begin{array}{cc|c} 3 & -3 & 3 \\ 3 & -3 & 3 \end{array}\right] \hookrightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad w_1 - w_2 = 1 \text{ for the components of any valid } \vec{w}.$$

I will take $\vec{w} = \langle 1, 0 \rangle$, as it satisfies $w_1 - w_2 = 1$.

The eigenvector soln:

$$e^{-t}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

The generalized eigenvector soln:

$$e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} (3t+1)e^{-t} \\ 3te^{-t} \end{bmatrix}$$

So, the general soln is

$$\vec{x}(t) = \underbrace{\tilde{c}_1 \cdot 3}_{\text{OK to absorb into one arbitrary } c_1}\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} + c_2\begin{bmatrix} (3t+1)e^{-t} \\ 3te^{-t} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-t} & (3t+1)e^{-t} \\ e^{-t} & 3te^{-t} \end{bmatrix}}_{\text{This is my } \Phi(t), \text{ though it is not the only correct one.}}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

3. Here, for nonreal eigenpairs, it is natural to identify $\alpha = 1.5$, $\beta = 2$, $\vec{u} = \langle -1, 2, 3 \rangle$ and $\vec{w} = \langle -2, 2, -1 \rangle$. This leads to two of the required three solns.,

$$e^{\alpha t}\left[\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\right] \quad \text{and} \quad e^{\alpha t}\left[\sin(\beta t)\vec{u} + \cos(\beta t)\vec{w}\right].$$

Combining with the third solution, arising from the real eigenpair, we get general solution

$$\vec{x}(t) = c_1 e^{-2.5t}\begin{bmatrix}4\\1\\0\end{bmatrix} + c_2 e^{1.5t}\left(\cos(2t)\begin{bmatrix}-1\\2\\3\end{bmatrix} - \sin(2t)\begin{bmatrix}-2\\2\\-1\end{bmatrix}\right) + c_3 e^{1.5t}\left(\sin(2t)\begin{bmatrix}-1\\2\\3\end{bmatrix} + \cos(2t)\begin{bmatrix}-2\\2\\-1\end{bmatrix}\right)$$

4(a) In standard form $y' - \left(\frac{1}{x}\right)y = -2\ln x$, we recognize this as a linear, nonhomog. 1st order DE, with $p(x) = -1/x$ and $f(x) = -2\ln x$.

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}.$$

So, the homogeneous solution is $y_h(x) = C\cdot\frac{1}{\mu(x)} = Cx$.

And by the variation of parameters formula,

$$y_p(x) = \frac{1}{\mu(x)}\int f(x)\mu(x)\,dx = x\int \frac{-2\ln x}{x}\,dx$$

$u = \ln x$
$\Rightarrow du = \frac{1}{x}dx$

$$= -2x\int u\,du = -xu^2 = -x(\ln x)^2.$$

The general solution, then, is

$$y(x) = y_h(x) + y_p(x) = \boxed{Cx - x(\ln x)^2.}$$

4(b) This is a separable DE. $y^{-2}dy = 6x\,dx \Rightarrow \int y^{-2}dy = \int 6x\,dx$

$\Rightarrow -\frac{1}{y} = 3x^2 + C$. We can apply the IC now or later. Doing it now,

$-\frac{1}{1/25} = 3 + C \Rightarrow C = -28$. So, $\boxed{y(x) = \frac{1}{28 - 3x^2}}$

5. Let
$$\left.\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \end{aligned}\right\} \Rightarrow \quad \begin{aligned} dx_1/dt &= x_2 \\ dx_2/dt &= x_3 \\ dx_3/dt &= y''' = 2x_1\cos t - \frac{3}{t}x_3 + \ln(t^2+1) \end{aligned}$$

In matrix vector form, with $\vec{x} = \langle x_1, x_2, x_3 \rangle$ as the vector of unknowns,

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2\cos t & 0 & -\frac{3}{t} \end{bmatrix}}_{A(t)} \vec{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \ln(t^2+1) \end{bmatrix}}_{\vec{f}(t)}$$

with initial condition

$$\vec{x}(1) = \begin{bmatrix} y(1) \\ y'(1) \\ y''(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$