1. (a) The transmitted $\mathbf{v}$ is given by

$$\mathbf{v} \;=\; 1\begin{bmatrix}1\\0\\0\\0\\0\\1\\1\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\\0\\1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\\0\\1\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\0\\1\\1\\1\\1\end{bmatrix} \;=\; \begin{bmatrix}1\\1\\0\\1\\0\\0\\1\end{bmatrix}.$$

(b) We have

$$\mathbf{H}\tilde{\mathbf{v}} \;=\; \begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}\begin{bmatrix}1\\0\\1\\1\\1\\0\\0\end{bmatrix} \;=\; \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Since $\mathbf{H}\tilde{\mathbf{v}}$ is not the zero vector, $\tilde{\mathbf{v}}$ is corrupted. If only corrupted in a single entry, it must be the the 3^{rd} entry (as $\mathbf{H}\tilde{\mathbf{v}}$ equals the 3^{rd} column of $\mathbf{H}$). Thus, the originally-intended 7-bit word is $\mathbf{v} = (1,0,0,1,1,0,0)$, from which we extract the 4-bit word $(1,0,0,1)$, (or 1001).

(c) Sadly, the use of the Hamming (7,4) scheme for detecting and correcting errors breaks down if two (or more) bits from a 7-bit transmitted word are corrupt. To see this, notice that if the 7-bit word $\mathbf{v} = (1,0,0,1,1,0,0)$ is corrupted to $\tilde{\mathbf{v}} = (1,1,0,1,0,0,0)$ (*two* altered bits), then we will, indeed, *detect* an error (it is still the case, with this $\tilde{\mathbf{v}}$, that $\tilde{\mathbf{v}} \notin \text{null}\,(\mathbf{H})$), but that our process for correction would make us think that the 7^{th} bit alone was faulty (not the pair of 2^{nd} and 5^{th} bits). With *three* altered bits we might not even detect the error!

2. [Conversion to 1^{st}-order system practice.]

(a) Setting $\boldsymbol{x} = \langle x_1, x_2\rangle$, with

$$x_1 = y, \quad x_2 = y',$$

gives the system

$$\boldsymbol{x}' = \begin{pmatrix}0&1\\-4&0\end{pmatrix}\boldsymbol{x} + \begin{pmatrix}0\\t^2 - e^{3t}\end{pmatrix}, \qquad \text{with IC} \qquad \boldsymbol{x}(0) = \begin{pmatrix}3\\1\end{pmatrix}.$$

(b) Setting $\boldsymbol{x} = \langle x_1, x_2, x_3\rangle$, with

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'',$$

gives the system

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tan(t) & -5 & e^{-t} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 12\cos(4t) \end{pmatrix}, \qquad \text{with IC} \qquad x(1) = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

(c) Setting $x = \langle x_1, x_2, x_3, x_4 \rangle$, with

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad x_4 = y''',$$

gives the system

$$x' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -13 & 7 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^3 + 3t - 9\sin(2t) \end{pmatrix}.$$

3. (a) The (approximate) homogeneous solution is

$$\begin{aligned} \mathbf{x}_h(t) &= c_1 e^{-0.0447t} \begin{bmatrix} 1 \\ -0.69 \\ -0.09 \end{bmatrix} + c_2 e^{-0.02t} \begin{bmatrix} 1 \\ 1.3 \\ -0.19 \end{bmatrix} + c_3 e^{-0.0000306t} \begin{bmatrix} 1 \\ 0.39 \\ 892.56 \end{bmatrix} \\ &= \begin{bmatrix} e^{-0.0447t} & e^{-0.02t} & e^{-0.0000306t} \\ -0.69e^{-0.0447t} & 1.3e^{-0.02t} & 0.39e^{-0.0000306t} \\ -0.09e^{-0.0447t} & -0.19e^{-0.02t} & 892.56e^{-0.0000306t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \end{aligned}$$

(b) All of the eigenvalues of $\mathbf{A}$ are negative, which means that, as $t \to \infty$, the three fundamental solutions all go to $\mathbf{0}$. Thus, each component of $\mathbf{x}_h(t)$ representing, respectively, the amount of lead in the bloodstream, body tissue, and bone, goes to 0.

(c) Writing, as we usually do, the matrix of part (a) as $\Phi(t)$, we must solve

$$\begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix} = \Phi(0) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -0.69 & 1.3 & 0.39 \\ -0.09 & -0.19 & 892.56 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Using Gaussian elimination, we get approximate values $c_1 = 32.66$, $c_2 = 17.333$, $c_3 = 0.006983$. The 3rd (bone) component of the solution $\mathbf{x}_h(t)$, then, is

$$\begin{aligned} x_3(t) &\doteq (32.66)(-0.09)e^{-0.0447t} + (17.333)(-0.19)e^{-0.02t} + (0.006983)(892.56)e^{-0.0000306t} \\ &\doteq -2.9394e^{-0.0447t} - 3.2933e^{-0.02t} + 6.2327e^{-0.0000306t} \end{aligned}$$

The peak value of the function $x_3(t)$, approximately 6.168, occurs around $t = 292.75$, i.e., after 292 days. The approximate time when the value of $x_3(t)$ returns to 0.5 is $t = 82449.6$ days, or about 225 years.