MATH 162: Calculus II

Framework for Mon., Apr. 9

Constrained Optimization of Functions of 2 Variables

Today's Goal: To be able to find absolute extrema for functions of two variables on closed and bounded domains.

> **Definition**: A function f of two variables is said to have
>
> 1. a *global maximum* (or *absolute maximum*) at the point $(a, b) \in \text{dom}(f)$ if $f(a, b) \geq f(x, y)$ for all points (x, y) in $\text{dom}(f)$.
> 2. a *global minimum* (or *absolute minimum*) at the point $(a, b) \in \text{dom}(f)$ if $f(a, b) \geq f(x, y)$ for all points (x, y) in $\text{dom}(f)$.
>
> The value $f(a, b)$ is correspondingly called the *global* (or *absolute*) *maximum or minimum* value of f.

Before we embark on a process of looking for absolute extrema, it would be nice to know that what we are looking for is out there to be found. The following theorem supplies such an assurance in the special case that the domain of f is closed and bounded.

> **Theorem**: (*Extreme Value Theorem*) Suppose $f(x, y)$ is a continuous function on a closed and bounded region R of the xy-plane. Then there exist points (a, b) and (c, d) in R for which
>
> $$f(a, b) \geq f(x, y) \qquad\qquad f(c, d) \leq f(x, y)$$
>
> for all (x, y) in R.

Remarks:

- In some simple cases, it is even possible that (a, b) and (c, d) from the theorem are the same point.
- In the theorem, we would call "continuity of f over a closed, bounded region R" as sufficient conditions to guarantee that f reaches maximum and minimum values in R. It is possible for f to attain such extrema even if one or both of these conditions (the "continuity" or the "closed and boundedness of R") is not in place.

Examples:

1. Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ along the line $x + 3y = 10$.

2. Find the absolute extrema for $f(x,y) = x^2 - y^2 - 2x + 4y$ on the region of the xy-plane bounded below by the x-axis, above by the line $y = x + 2$, and on the right by the line $x = 2$.

3. Find the absolute extrema for $f(x,y) = x^2 + 2y^2$ on the closed disk $x^2 + y^2 \leq 1$.

4. Find the maximum of $f(x,y,z) = 36 - x^2 - y^2 - z^2$ subject to the constraint $x + 4y - z = 21$.

5. Find the point on the graph of $z = x^2 + y^2 + 10$ closest to the plane $x + 2y - z = 0$.