1. At interior points x_i, $i = 1, 2, 3, 4$, we have

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) \quad \Rightarrow \quad \begin{aligned} -u_0 + 2u_1 - u_2 &= h^2 f(x_1) \\ -u_1 + 2u_2 - u_3 &= h^2 f(x_2) \\ -u_2 + 2u_3 - u_4 &= h^2 f(x_3) \\ -u_3 + 2u_4 - u_5 &= h^2 f(x_4) \end{aligned}$$

Our BCs are approximated by

$$\alpha = u'(0) = \frac{u_1 - u_0}{h} \quad \Rightarrow \quad u_1 - u_0 = h\alpha$$

$$\beta = u'(1) = \frac{u_5 - u_4}{h} \quad \Rightarrow \quad u_5 - u_4 = h\beta$$

Arranged in matrix-vector form, these equations/constraints become

$$\begin{bmatrix} -1 & 1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \alpha h \\ h^2 f(x_1) \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{n-1}) \\ \beta h \end{bmatrix}$$

2. By Taylor's Theorem, for $h > 0$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \mathcal{O}(h^4)$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \mathcal{O}(h^4)$$

$$f(x+h) + f(x-h) = 2f(x) + f''(x)h^2 + \mathcal{O}(h^4)$$

$$\Rightarrow f''(x)h^2 = f(x-h) - 2f(x) + f(x+h) + \mathcal{O}(h^4)$$

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} + \mathcal{O}(h^2).$$

3. Assuming separation $u(x,t) = X(x)T(t)$, $u_t = k u_{xx}$ becomes

$$X\dot{T} = k X'' T \qquad \text{or} \qquad \frac{X''}{X} = \frac{\dot{T}}{kT} = \lambda,$$

and

$$u(0,t) = 0 \text{ becomes } X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u_x(1,t) = 0 \text{ becomes } X'(1)T(t) = 0 \Rightarrow X'(1) = 0$$

An eigenpair (λ, v) of the spatial operator $A[X] = X''$ would, under the BCs, have

$$\begin{aligned}
\lambda \|v\|^2 &= \langle \lambda v \mid v \rangle = \langle v'' \mid v \rangle = \int_0^1 v''(x)\bar{v}(x)\,dx \\
&= v'(x)\bar{v}(x)\Big|_0^1 - \int_0^1 v'(x)\bar{v}'(x)\,dx \\
&= v'(1)\bar{v}(1) - v'(0)\bar{v}(0) - \|v'\|^2 \\
&= (0)\bar{v}(1) - v'(0)(0) - \|v'\|^2 \\
&= -\|v'\|^2
\end{aligned}$$

So,
$$\lambda = \frac{-\|v'\|^2}{\|v\|^2} \le 0$$

Can $\lambda = 0$ be an eigenvalue? For this value,

$$v'' = 0 \Rightarrow v(x) = ax + b$$

$$\left.\begin{aligned} v(0) = 0 &\Rightarrow b = 0 \\ v'(1) = 0 &\Rightarrow a = 0 \end{aligned}\right\} \Rightarrow \lambda = 0 \text{ is not an eigenvalue.}$$

Having that $\lambda = -\omega^2 < 0$ for real $\omega > 0$,

$$X'' + \omega^2 X = 0 \Rightarrow v(x) = A\cos(\omega x) + B\sin(\omega x), \text{ and}$$
$$v'(x) = -\omega A\sin(\omega x) + \omega B\cos(\omega x).$$

Applying the boundary conditions,

$$0 = v(0) = A, \text{ leading to } v(x) = B\sin(\omega x).$$

$$0 = v'(1) = \omega B\cos(\omega) \Rightarrow \omega = \omega_n = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \frac{(2n-1)\pi}{2}, \dots$$

Thus, spatial eigenfns are $\left\{ \sin\left(\frac{(2n-1)\pi}{2} x\right) \right\}_{n=1}^{\infty}$.

4. The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x) \right]$$

with coefficients computed from integrals over $[-1, 1]$ results in a function that generally agrees with f on that interval and is periodic with period 2: