Continuous Distributions

Example: A first pdf. Let

$$f(x) \;=\; \begin{cases} ax(1-x), & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

(a) Draw a graph of f in RStudio, using $a = 2$ for convenience.

(b) Determine the value of a so that

$$\int_{-\infty}^{\infty} f(x)\,dx \;=\; 1.$$

When $a=2$, got $\int_{-\infty}^{\infty} f(x) = \frac{1}{3}$

Take $a=6$

> **Definition 1:** A function $f\colon \mathbb{R} \to \mathbb{R}$ is a **probability density function**, or **pdf**, if it has the properties
>
> - $f(x) \geq 0$ for all $x \in \mathbb{R}$, and
> - $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Example: Is the following function a pdf for some choice of a?

A: Yes, w/ $a=2$

$$f(x) \;=\; \begin{cases} 0, & x < 0 \\ ae^{-2x}, & x \geq 0 \end{cases}$$

ae^{bx}

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} ae^{-2x}\,dx = \lim_{B\to\infty} \int_0^B ae^{-2x}\,dx$$

$$\lim_{B\to\infty} \frac{ae^{-2x}}{-2}\Bigg|_0^B = \frac{a}{2} = 1$$

What if we replace (-2) with $b \leq 0$?

A continuous random variable X

- can take on values throughout an interval
- satisfies $P(X = x) = 0$
- has a **cumulative distribution function**, or **cdf**, defined to be $F_X(x) \;=\; P(X \leq x)$. Note that F is
  - monotone increasing (in x), making F (almost everywhere) differentiable (a deep insight from 20th Century analysis).
  - the derivative $f = F'$ is (almost everywhere) nonnegative, and

$$F(x) \;=\; P(X \leq x) \;=\; \int_{-\infty}^{x} f(t)\,dt.$$

$x=5$

$x=6$

  - as a consequence of the above, $\int_{-\infty}^{\infty} f(x)\,dx \;=\; \lim_{x\to\infty} F(x) = 1.$

So, the derivative of F is a pdf.

- satisfies $P(a \le X \le b) = F(b) - F(a) = \int_a^b f(x)\,dx$.

As a result, one can *define* a random variable X

- starting with a pdf and using integration to get its cdf, or
- starting with a cdf (any $F\colon \mathbb{R} \to [0,1]$ with $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$), and using differentiation to get its pdf.

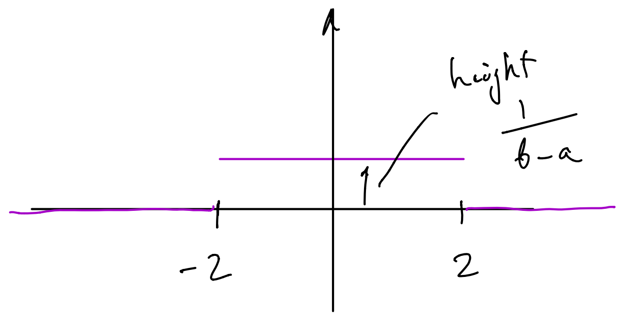
Uniform distributions

This family of distributions arises from having a pdf that is an appropriately-scaled indicator function on a finite interval $[a,b]$. That is, $X \sim \text{Unif}(a,b)$ if

$$f(x) = \left\{ \begin{array}{ll} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{array} \right\} = \frac{1}{b-a}\chi_{[a,b]}(x) = \frac{1}{b-a}[\![a \le x \le b]\!]. \qquad \text{pdf}$$

Example $X \sim \text{Unif}(-2,2)$.

(a) Plot the pdf $f_X(x)$.



$= F_X(-3)$

(b) Find $P(X < -3)$, $P(X \le 0)$, and $P(X < 1)$.

$= 0 \qquad = \frac{1}{2} \qquad = 3/4$

$P(X > 5)$
$= 1 - P(X \le 5)$

(c) Use commands in R to redo part (b).

punif(-3, -2, 2)
punif(0, -2, 2)
punif(1, -2, 2)

1 - punif(5, -2, 2)

For $-2 \le x \le 2$

$F_X(x) = \int_{-2}^{x} \frac{1}{4}\,dt$

(d) Give a formula for the cdf $F_X(x)$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{4}(x+2), & \text{if } -2 \le x \le 2 \\ 1, & \text{if } x > 2 \end{cases}$$