Comparing the Growth of Functions as Inputs (x or n) $\to \infty$

Suppose f and g are real-valued functions on a domain that includes nonnegative real numbers. We say that

- f **is of order at most** g, written $f(x)$ is $O(g(x))$, iff there exists $C > 0$ and $k \geqslant 0$ such that

  $$|f(x)| \leqslant C|g(x)|, \qquad \text{for all real numbers } x > k.$$

  We call C, k **witnesses** to this **Big-**O relationship.
- f **is of order at least** g, written $f(x)$ is $\Omega(g(x))$, iff there exists $C > 0$ and $k \geqslant 0$ such that

  $$|f(x)| \geqslant C|g(x)|, \qquad \text{for all real numbers } x > k.$$

- f **is of order** g, written $f(x)$ is $\Theta(g(x))$, iff f is simultaneously of order at most g and of order at least g.

Note: Similar definitions hold for sequences (functions from $\mathbb{N}$ to $\mathbb{R}$).

Examples:

1. Find witnesses that demonstrate $f(x) = 3x^3 + 2x + 7$ is $O(x^3)$.

got witnesses $k=3$, $C=4$

From graph, it appears $|3x^3+2x+7| \leq 4|x^3|$ when $x>3$

Or, without graphing:

$$|3x^3+2x+7| \underset{\text{if } x>0}{=} 3x^3+2x+7 \underset{x\geq 1}{\leq} 3x^3+2x^3+7x^3 = 12x^3 = 12|x^3|$$

witnesses $C=12$, $k=1$

2. Show that $f(x) = \dfrac{15\sqrt{x}(2x+9)}{x+1}$ is $\Theta(x^{1/2})$.

To show $O(\sqrt{x})$, want a series of $\leq$ starting w/ $\dfrac{15\sqrt{x}(2x+9)}{x+1}$

$$\frac{15\sqrt{x}(2x+9)}{x+1} \underset{\text{if } x\geq 9}{\leq} \frac{15\sqrt{x}(3x)}{x+1} \leq \frac{15\sqrt{x}(3\cancel{x})}{\cancel{x}} = 45\sqrt{x}$$

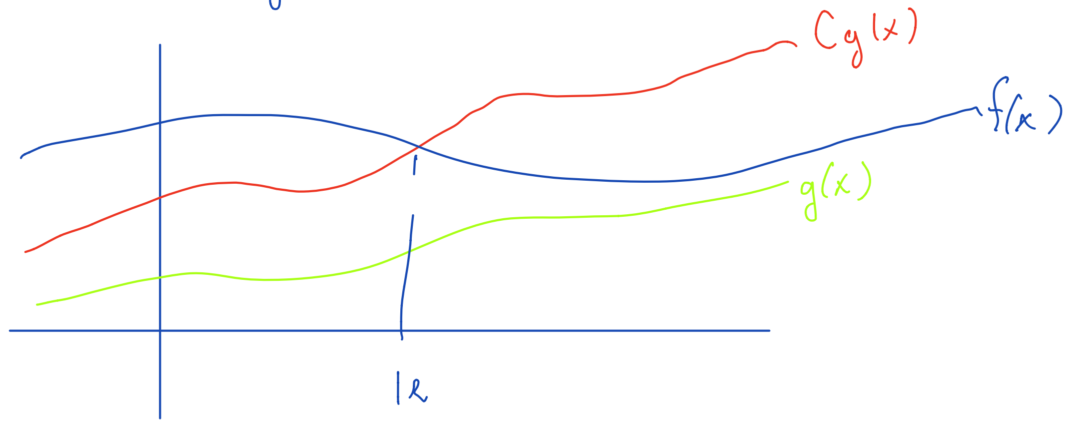
$\Rightarrow$ left fn. is $O(\sqrt{x})$ with witnesses $C=45$, $k=9$

Now show f is $\Omega(\sqrt{x})$. Start

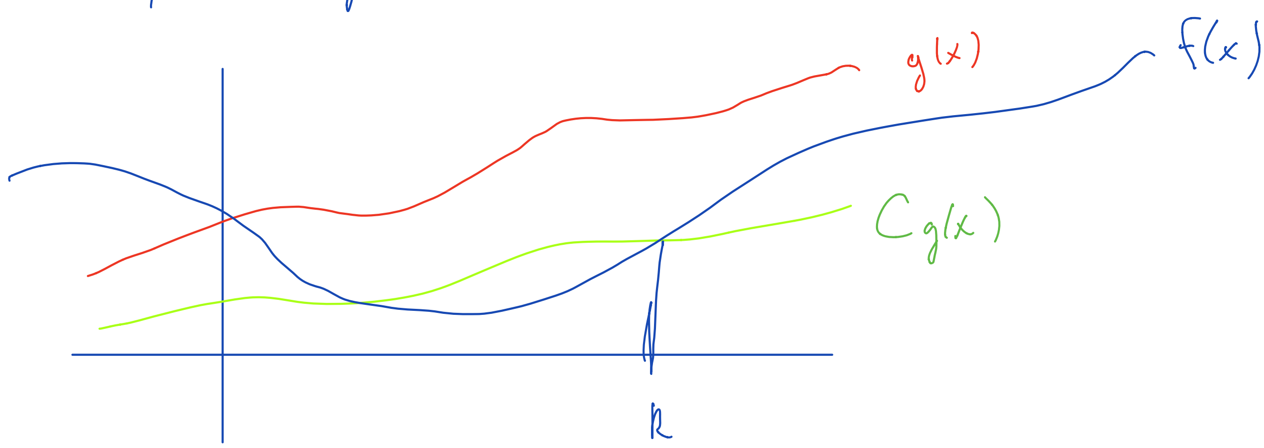
$$\frac{15\sqrt{x}\,(2x+9)}{x+1} \underset{x\geq 0}{\geq} \frac{15\sqrt{x}\,(2x)}{x+1} \underset{x\geq 1}{\geq} \frac{15\sqrt{x}\,(2x)}{2x} = 15\sqrt{x}$$

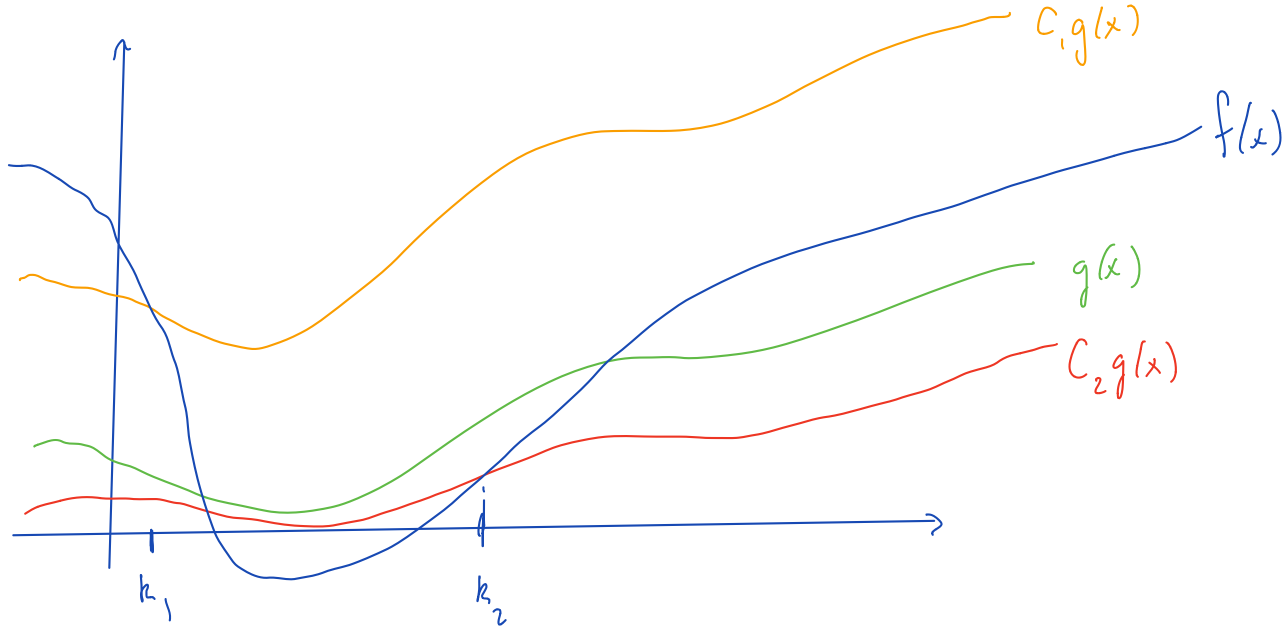
So f is $\Omega(\sqrt{x})$ with witnesses: $C = 15$, $k = 1$

Big O: f is $O(g(x))$



f is $\Omega(g(x))$





f is $O(g(x))$ since witnesses C_1, k_1 say so

f is $\Omega(g(x))$ since witnesses C_2, k_2 say so

C_1, C_2, $\max(k_1, k_2)$ are witnesses to f being $\Theta(g(x))$

There is, therefore, this increasing sequence of orders: $1, \quad \log_b n, \quad (\log_b n)^2, \quad (\log_b n)^3, \quad \ldots, n, \quad n \log_b n, \quad n(\log_b n)^2, \quad \ldots, n^2, \quad n^2 \log_b n, \quad n^3, \quad \ldots, 2^n, \quad 3^n, \quad \ldots, n!$.

Theorem 1: Let $f(x)$ be a polynomial of degree n—that is,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

with $a_n \neq 0$. Then

- $f(x)$ is $O(x^s)$ for all integers $s \geqslant n$.
- $f(x)$ is not $O(x^r)$ for all integers $r < n$.
- $f(x)$ is $\Omega(x^r)$ for all integers $r \leqslant n$.
- $f(x)$ is not $\Omega(x^s)$ for all integers $s > n$.
- $f(x)$ is $\Theta(x^n)$.