Some Uses of Determinants

Cramer's Rule

Previously, we have learned how to calculate determinants for (square) matrices. But it has seemed, to now, that the main question is whether a determinant is zero or not, as $\det(\mathbf{A}) \neq 0$ tells us a unique solution exists to the vector equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{1}$$

Cramer's Rule actually uses the *values* of determinants to solve the linear vector equation (1). We will describe its use, and then explain why it works.

Let us assume the coefficient matrix $\mathbf{A}$ in (1) has nonzero determinant, so that this system has precisely one solution $\mathbf{x}$. Cramer's Rule requires the use of matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n$, where each $\mathbf{A}_j$, $1 \leqslant j \leqslant n$ is built from the original $\mathbf{A}$ and the vector $\mathbf{b}$. These are constructed as follows: the j^{th} column of $\mathbf{A}$ is replaced by $\mathbf{b}$ to form $\mathbf{A}_j$. In the case when $\mathbf{A}$ is 3-by-3, these $\mathbf{A}_j$ are

$$\mathbf{A}_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}.$$

Armed with these $\mathbf{A}_j$, $1 \leqslant j \leqslant n$, the solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ has its j^{th} coordinate given by

$$x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}, \qquad j = 1, 2, \ldots, n. \tag{2}$$

It should be clear from this formula why it is necessary that $\mathbf{A}$ have a nonzero determinant.

Example 1: A system of 4 linear equations in 4 unknowns

Use Cramer's Rule to solve the system of equations

$$\begin{aligned} x + 3y + z - w &= -9 \\ 2x + y - 3z + 2w &= 51 \\ x + 4y + 2w &= 31 \\ -x + y + z - 3w &= -43 \end{aligned}$$

Here, $\mathbf{A}$ and $\mathbf{b}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -9 \\ 51 \\ 31 \\ -43 \end{bmatrix}, \qquad \text{so} \qquad |\mathbf{A}| = \begin{vmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{vmatrix} = -46.$$

Thus,

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} -9 & 3 & 1 & -1 \\ 51 & 1 & -3 & 2 \\ 31 & 4 & 0 & 2 \\ -43 & 1 & 1 & -3 \end{vmatrix} = \frac{-230}{-46} = 5,$$

$$y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & -9 & 1 & -1 \\ 2 & 51 & -3 & 2 \\ 1 & 31 & 0 & 2 \\ -1 & -43 & 1 & -3 \end{vmatrix} = \frac{-46}{-46} = 1,$$

$$z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & -9 & -1 \\ 2 & 1 & 51 & 2 \\ 1 & 4 & 31 & 2 \\ -1 & 1 & -43 & -3 \end{vmatrix} = \frac{276}{-46} = -6,$$

$$w = \frac{|\mathbf{A}_4|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & 1 & -9 \\ 2 & 1 & -3 & 51 \\ 1 & 4 & 0 & 31 \\ -1 & 1 & 1 & -43 \end{vmatrix} = \frac{-506}{-46} = 11,$$

yielding the solution $\mathbf{x} = (x, y, z, w) = (5, 1, -6, 11)$.

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Cramer's Rule requires only a square system with a nonsingular coefficient matrix. It works for various numbers n of equations and unknowns.

Example 2: Other systems of n linear equations in n unknowns

Try these yourself. Using Cramer's Rule, solve these two systems of linear equations

(a)
$$\begin{aligned} 2x - 3y &= 25 \\ 5x + 4y &= 28 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= -4 \\ 2x_1 - x_2 + 4x_3 &= -1 \\ 3x_1 + 2x_2 - x_3 &= -12 \end{aligned}$$

The answers are $\mathbf{x} = (8, -3)$, and $\mathbf{x} = (-3, -1, 1)$, respectively.

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One might reasonably ask whether Cramer's Rule fills an important niche. After all, we have encountered other methods for solving Equation (1). Gaussian elimination, in its refined form understood as **LU**-factorization, doesn't require $|\mathbf{A}| \neq 0$, nor even that $\mathbf{A}$ be square, making it far more broadly applicable. Yet, there are many who use Cramer's Rule.

Why does it work? A key observation is this one. If you replace column j of the identity matrix with a vector $\mathbf{x}$, the resulting matrix has determinant equal to the new diagonal element:

$$\underbrace{\begin{vmatrix} 1 & \cdots & x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & & x_j & & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & x_n & \cdots & 1 \end{vmatrix}}_{n\times n} = \underbrace{\begin{vmatrix} 1 & \cdots & x_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & & x_j & & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & x_n & \cdots & 1 \end{vmatrix}}_{(n-1)\times(n-1)} = \cdots = \underbrace{\begin{vmatrix} x_j & 0 & 0 & \cdots & 0 \\ x_{j+1} & 1 & 0 & \cdots & 0 \\ x_{j+2} & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ x_n & 0 & 0 & \cdots & 1 \end{vmatrix}}_{(n-j+1)\times(n-j+1)} = x_j.$$

Let us write $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$, and assume that $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{b}$. Then we have

$$\mathbf{A} \begin{bmatrix} 1 & \cdots & x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & & x_j & & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & x_n & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{j-1} & (\mathbf{Ax}) & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} = \mathbf{A}_j.$$

Taking the determinant of the two ends of this equation, we get the desired formula for the j^{th} component of the solution to (1)

$$\det(\mathbf{A})x_j = \det(\mathbf{A}_j) \qquad \Rightarrow \qquad x_j = \frac{\det(\mathbf{A}_j)}{\det(\mathbf{A})}.$$

Taking Cramer's Rule to the extreme, it provides a route to finding the entries of $\mathbf{A}^{-1}$. The j^{th} column $\mathbf{x}_j$ of $\mathbf{A}^{-1}$ is, after all, the solution of the vector equation

$$\mathbf{Ax} = \mathbf{e}_j,$$

where the right-hand vector $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the j^{th} column of $\mathbf{I}$. The n entries in $\mathbf{x}_j$ may be found using Cramer's Rule, after which one moves on to find the entries of the next column of $\mathbf{A}^{-1}$ in the same manner. It sounds rather inefficient, but the resulting formula (see the top of p. 275) has theoretical value. It can be used, for instance, to establish this fact:

Fact 1: When $\mathbf{A}$ is a square, lower (resp. upper) triangular matrix which is nonsingular, then $\mathbf{A}^{-1}$ is also lower (resp. upper) triangular.

Cross products

This topic, the cross product of vectors in $\mathbb{R}^3$, is taught in most calculus books. For $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, we define the cross product to be

$$\mathbf{u} \times \mathbf{v} \;=\; \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}, \tag{3}$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. This definition has some nice properties, two of which, likely demonstrated in some calculus course you took, are that

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\text{span}(\{\mathbf{u}, \mathbf{v}\})$. Indeed, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$, in that order, form a right-hand system.
- The parallelogram (see figure below) that has, as two of its sides, $\mathbf{u}$ and $\mathbf{v}$, has an area $A = \|\mathbf{u} \times \mathbf{v}\|$.

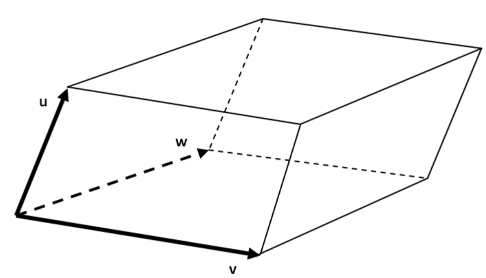
The definition (3) is, in form, just like a cofactor expansion. In fact, we write

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

because the definition (3) comes out of carrying out the expansion of this determinant in cofactors along the first row, despite the fact that entries along that row are vectors, not scalars. A strange feature of cross products, namely that to change the order of factors means a change in sign of the result, or $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$, is now explained by our knowledge that a row swap in a determinant has this effect.

Areas, volumes, and the like

Given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, one can place each of these vectors in standard position to obtain



- parallelograms, one that has $\mathbf{u}$, $\mathbf{v}$ as two of its sides, another with $\mathbf{u}$, $\mathbf{w}$ as two sides, and a third with two of its sides $\mathbf{v}$, $\mathbf{w}$.
- a parallelepiped, a 6-faced figure with three of its faces being the parallelograms mentioned above.

It was likely demonstrated in calculus that the volume of the parallelepiped is given by

$$V \;=\; |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \;=\; \left| \det \left(\begin{bmatrix} \mathbf{u}^T \rightarrow \\ \mathbf{v}^T \rightarrow \\ \mathbf{w}^T \rightarrow \end{bmatrix} \right) \right| \;=\; \left| \det \left(\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right) \right|.$$

Note that the determinant in this expression is a number; the absolute value of that number makes it nonnegative.

The idea of a parallelepiped in $\mathbb{R}^n$, one that arises from n vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in the same way that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ gave rise to a parallelepiped in space in the figure above, seems an unnecessary sidelight at this point. But such ideas can be explored, and determinants again are used to calculate their n-dimensional volumes.

The Jacobian

You have sometime been exposed to polar coordinates, where location is in the plane is described by (r, θ) which represent "distance" from the origin and bearing, as opposed to (x, y) which relate to projections onto x- and y-axes. The relationships from polar to rectangular coordinates are

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

If you have done a double integral in polar coordinates, you may recall that a differential area element $dA = r\,dr\boldsymbol{d}\theta$. The extra factor r comes from a determinant of partial derivatives known as the Jacobian. Specifically,

$$dA \;=\; dx\,dy \;=\; \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta \;=\; \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta \;=\; r\,dr\,d\theta.$$

When applying a change of variable to an integral in just $n = 1$ variable, we encounter the Jacobian, there, too. For example, to assist in calculating the integral

$$\int 9x\sqrt{x^2+3}\,dx,$$

we might make the substitution $u = x^2 + 3$. To carry this out, we note that $du/dx = 2x$, and we separate out that factor from the integrand:

$$9x\sqrt{x^2+3} \;=\; \frac{9}{2}\sqrt{x^2+3}\cdot(2x) \;=\; \frac{9}{2}\sqrt{u}\,\frac{du}{dx}.$$

This du/dx is the Jacobian of the mapping from x to u.

The Jacobian appears as a factor whenever a change of variables is made in an integral. When the integral involves n variables, the Jacobian is an n-by-n determinant. It accounts for the expansion/contraction seen in a small "volume" element in one set of variables in comparison with the corresponding volume in the other variables.