

Systems of DEs

In class yesterday I offered several examples of how systems of DEs arise in applications, namely

1. A **predator-prey** system. For $x(t)$, $y(t)$ representing the sizes of a prey and predator population, respectively,

$$\begin{aligned} dx/dt &= ax - bxy \\ dy/dt &= -cy + dxy \end{aligned} \quad (1)$$

We said system (1) is in **normal form**.

2. The **two-body** problem. With the Earth at the origin and a satellite at position $(x(t), y(t))$ relative to the Earth, we used the inverse square law and Newton's 2nd Law to derive the system

$$\begin{aligned} d^2x/dt^2 &= -\frac{GMx}{(x^2 + y^2)^{3/2}} \\ d^2y/dt^2 &= -\frac{GM y}{(x^2 + y^2)^{3/2}} \end{aligned} \quad (2)$$

System (2) is not in normal form, mainly because it is 2nd order. We turned it into a 1st-order system by introducing two more dependent variables, u and v , new names for dx/dt , dy/dt , thereby arriving at the normal form

$$\frac{d}{dt} \begin{bmatrix} x \\ u \\ y \\ v \end{bmatrix} = \begin{bmatrix} u \\ -\frac{GMx}{(x^2 + y^2)^{3/2}} \\ v \\ -\frac{GM y}{(x^2 + y^2)^{3/2}} \end{bmatrix}.$$

Watch <https://drive.google.com/file/d/1-4WJrIPYuyF88kFcEnrS6Dc24wW6QCHf/view?usp=sharing> to see examples of other higher-order DEs converted to 1st-order systems.

Linear first-order systems of DEs

In both the pendulum and RLC circuit examples of the video, we were able to take a 2nd-order DE to a first-order system in normal form. For the RLC circuit DE, the resulting normal form looks like

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{f}(t), \quad (3)$$

where $\mathbf{A}(t)$ is a (constant) 2-by-2 matrix, and $\mathbf{f}(t)$ is a 2-by-1 vector. (3) is a system because it has multiple (at least two) dependent variables, but those have been grouped together into a single entity/vector \mathbf{x} , so that (3) has the form

$$\frac{d}{dt} \mathbf{x} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t). \quad (4)$$

The vector \mathbf{x} containing the dependent variables appears, along with its first derivative, in two locations, the very same locations as y, y' in the 1st-order linear problems we studied in Chapter 2:

$$y' = a(t)y + f(t).$$

Systems with normal forms like (4) are called **1st order linear systems**. It is not possible to put the converted pendulum DE into this form, which means it is **nonlinear**.

The tank-mixing problem of Section 3.1.1. Kapitula derives a 1st-order system for a two-tank mixing problem. At one stage in the derivation (bottom of p. 135) he has

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{100}x_1 + \frac{3}{200}x_2 + 12 \\ \frac{1}{100}x_1 - \frac{8}{200}x_2 + 35 \end{bmatrix}.$$

I wish to point out that, while one would be *technically correct* in writing this as

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{3}{200}x_2 - \frac{7}{100}x_1 + 12 \\ -\frac{8}{200}x_2 + \frac{1}{100}x_1 + 35 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{200} \\ -\frac{8}{200} \end{bmatrix} + x_1 \begin{bmatrix} -\frac{7}{100} \\ \frac{1}{100} \end{bmatrix} + \begin{bmatrix} 12 \\ 35 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{200} & -\frac{7}{100} \\ -\frac{8}{200} & \frac{1}{100} \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 12 \\ 35 \end{bmatrix}, \end{aligned}$$

this does not get us to the form (4), as the vector being differentiated on the left

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{is not the same as the one that appears on the right} \quad \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

It serves you well to pay attention to this detail as you write a linear system in matrix form. When we make sure it is the same \mathbf{x} in both places, as in the middle of p. 136, we wind up with a different matrix (the columns are swapped) which is in form (4), making the system *linear*.

Another multi-tank problem. In Exercise 3.1.3, p. 142, the details are given for a closed system involving 3 tanks. Let us define $x_1(t), x_2(t), x_3(t)$ to be the amounts of salt in Tanks A, B, C at time t , respectively. With these definitions, you should verify that the appropriate mixture model can be written as

$$\mathbf{x}' = \begin{bmatrix} -\frac{8}{350} & \frac{8}{100} & 0 \\ \frac{3}{350} & -\frac{8}{100} & \frac{5}{400} \\ \frac{5}{350} & 0 & -\frac{5}{400} \end{bmatrix} \mathbf{x} + \mathbf{0},$$

making it a **homogeneous** 1st-order linear system. As the vector \mathbf{x} represents a single entity enclosing the three unknown quantities

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \text{it can also represent their three initial values} \quad \mathbf{x}(0) = \begin{bmatrix} 600 \\ 1700 \\ 3200 \end{bmatrix}.$$