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Stat 343, Mon 26-Oct-2020 -- Mon 26-Oct-2020
Probability and Statistics
Fall 2020


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Monday, October 26th 2020
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Wk 9, Mo
Topic:: Method of moments
Read::  FASt 4.2
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4.2 **Method of Moments**

The following chart gives the parameters for some distributions we have studied, and how the means (μ_1, the first moment) and variances (μ_2', the second moment about the mean) relate to those parameters.

Distribution	Params	Mean	Variance
Binom	π	$n\pi$	$n\pi(1-\pi)$
Pois	λ	λ	λ
Unif	a, b	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$
Exp	λ	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Norm	μ, σ	μ	σ^2
Gamma	α, λ	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Weib	α, β	$\beta\,\Gamma(1+\frac{1}{\alpha})$	$\beta^2\left[\Gamma(1+\frac{2}{\alpha}) - \Gamma(1+\frac{1}{\alpha})^2\right]$
Beta	α, λ	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Sampled data x_i, $i = 1, \ldots, n$, can also be used to calculate a sample mean and sample variance.

$$\text{sample mean } \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\text{sample variance } \overset{s^2}{\cancel{\overline{s}^2}} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

The method of moments is a way of using data to choose model parameters within a specific model family in hopes the model with these parameters represents a good (perhaps the best) model fit within that family to the data. The approach comes down to matching the theoretical mean and variance (as well as other moments, if necessary) with other k^{th} sample moments:

$$\hat{\mu}_k := \frac{1}{n}\sum_{i=1}^{n} x_i^k \qquad (k^{\text{th}} \text{ sample moment about 0})$$

$$\hat{\mu}_k' := \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^k \qquad (k^{\text{th}} \text{ sample moment about sample mean})$$

Note that $v := \mu_2'$ is not the same as s^2:

Most of our distributions require two parameters. It is natural to choose our estimated mean/-variance as

$$\hat{\mu} = \overline{x} \qquad \text{and} \qquad \hat{\sigma}^2 = v,$$

then solve for the appropriate parameters (the difficult part).

Have data

$x_1, x_2, x_3, x_4, \ldots, x_n$ (n data values)

Data has a mean

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \leftarrow 1^{st} \text{ data moment}$$

If fitting data w/ exponential distribution, estimate mean $\hat{\mu} = \frac{1}{\lambda}$ to be $\bar{x}$.

Choose rate param. $\lambda = \frac{1}{\hat{\mu}} = \frac{1}{\bar{x}}$.

But, most families require 2 parameters.

When needing to estimate 2 params, natural

estimate mean $\hat{\mu} = \bar{x}$

variance $\hat{\mu}_2' = s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

n

Define k^{th} data moments

about 0: $\frac{1}{n} \sum_{i=1}^{n} x_i^k$

about mean: $\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^k$

so this is the 2^{nd} data moment about the mean.

Call it $v = \frac{n-1}{n} s^2$

Task:

Given data points

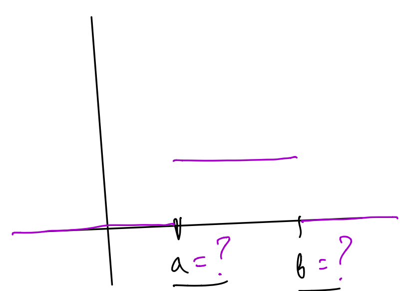
31, 22, 47, 85, 11

What does the method of moments suggest is the right unif dist. to consider as a model for this data?

From data

$\bar{x} = 39.2 = \hat{\mu}$

$v = 663.36 = \hat{\sigma}^2$



Unlike normal dist, μ, σ^2 are not the params of unif. dist.

Have relation for Unif(a, b)

$$\hat{\mu} = \frac{a+b}{2} \qquad v = \hat{\sigma}^2 = \frac{(b-a)^2}{12}$$

Alg. problem: Solve simult. eqns. for a, b.

$$\boxed{\frac{a+b}{2} = 39.2 \quad , \quad \frac{(b-a)^2}{12} = 663.36}$$

Still a task here

How about a beta dist?

Estimate $\hat{\mu}, \hat{\sigma}^2 \quad (\bar{X}, v)$

Problem: Find α, β (params. of a beta dist.) from

$$\mu = \frac{\alpha}{\alpha+\beta}, \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\hat{\mu} = \frac{\alpha}{\alpha+\beta} \Rightarrow \qquad \bar{X} = \frac{\alpha}{\alpha+\beta}$$

$$\frac{1}{\bar{X}} = \frac{\alpha+\beta}{\alpha} = \frac{\alpha}{\alpha} + \frac{\beta}{\alpha}$$

$$= 1 + \frac{\beta}{\alpha}$$

So $$\frac{\beta}{\alpha} = \frac{1}{\bar{X}} - 1 =: R$$

$$R = \frac{\beta}{\alpha} \quad \text{or} \quad \beta = \alpha R$$

Substitute into

$$v = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha^2 R}{(\underline{\alpha+\alpha R})^2(\alpha+\alpha R+1)}$$

$$= \frac{\cancel{\alpha^2} R}{[\cancel{\alpha}(1+R)]^2(\alpha+\alpha R+1)}$$

$$v = \frac{R}{\left(\frac{1}{\bar{x}}\right)^2[\alpha(1+R)+1]} = \frac{\bar{x}^2 R}{1+\alpha/\bar{x}}$$

$$v\left(1+\frac{\alpha}{\bar{x}}\right) = \bar{x}^2 R$$

$$\frac{\alpha}{\bar{x}} = \frac{\bar{x}^2 R}{v} - 1$$

$$\underline{\alpha} = \bar{x}\left(\frac{\bar{x}^2 R}{v} - 1\right), \qquad \underline{\beta} = \alpha R$$

$$\bar{x}, v \downarrow \qquad R = \frac{1}{\bar{x}} - 1$$