Solution to E1

TLS

September 30, 2022

E1: Paul the Octopus

It would seem natural to take as our null hypothesis that Paul's "choice" of a team is a 50-50 venture, equivalent to flipping a fair coin. It is, generally speaking, not right to use the same data **both** to

- decide that the alternative hypothesis should go in one direction, and
- as the data that leads to the test statistic that determines the P-value.

In that spirit, I use a 2-sided alternative hypothesis:

$$\mathbf{H}_0 : \pi = \frac{1}{2} \qquad \text{vs.} \qquad \mathbf{H}_a : \pi \neq \frac{1}{2}.$$

The count X of successes should have a binomial distribution and, in particular, the **null distribution** (the distribution of X when the null hypothesis holds) is $\text{Binom}(14, \frac{1}{2})$. We employ `dbinom()` and include in our sum all probabilities of the pmf that are at least as small as $\Pr(X = 12)$. A quick look at the values of the pmf

```
dbinom(0:14, 14, 1/2)
```

```
##  [1] 6.103516e-05 8.544922e-04 5.554199e-03 2.221680e-02 6.109619e-02
##  [6] 1.221924e-01 1.832886e-01 2.094727e-01 1.832886e-01 1.221924e-01
## [11] 6.109619e-02 2.221680e-02 5.554199e-03 8.544922e-04 6.103516e-05
```

shows that $\Pr(X = k) \leq \Pr(X = 12)$ whenever $k = 0, 1, 2, 12, 13, 14$. Thus, our P-value is

```
sum( dbinom( c(0:2, 12:14), 14, 1/2 ) )
```

```
## [1] 0.01293945
```

At the 5% level, we would reject the null hypothesis.

Of course, setting $\alpha = 0.05$ means we commit Type I error about 5% of the time. Paul's results are extreme enough we would only see results this extreme, given the predictions were mere guess-work, a little more often than 1% of the time. Still, there are other possibilities besides Paul being a soccer savant, able to predict game outcomes when it would seem he is unaware such a game is even being played. So, despite rejecting the null, we can acknowledge

- that the null hypothesis may, in fact be true. It may be that we have simply witnessed an event which, though rare, does occur about 1% of the time.
- that there are human handlers involved. While we don't suspect Paul of doing things to inform his guesses, it is possible the handlers are. Maybe they use their knowledge of the teams from the match to predict who will win, and they add a scent to one flag to influence Paul in that direction.
- or, maybe in every World Cup event running for the last N years there has been a Paul the Octopus, or a Harry the Rabbit, or a Sarah the Gazelle, who has been made the object of a similar "predict-the-winner" quiz. Even a rare event, given enough opportunity, will occur eventually. That it would be in 2010 instead of 1974 is not remarkable. This alternative is fairly similar to the Baltimore Stockbroker parable.

2.20 Defn. Events A, B are independent precisely when $P(B \mid A) = P(B)$.

Assume that A, B are independent, and show

(a) A and B^c are independent:

$$P(B^c \mid A) = 1 - P(B \mid A) = 1 - P(B) = P(B^c).$$

(b) A^c and B^c are independent

$$P(B^c \mid A^c) = 1 - P(B \mid A^c) = 1 - \frac{P(A^c \mid B) P(B)}{P(A^c)}$$

$$= 1 - \frac{P(A^c)\, P(B)}{P(A^c)} = 1 - P(B) = P(B^c).$$

2.24 Use S to be the event "is smoker"

W to be the event "is a woman"

M to be the event "is a man"

C to be the event "has cancer"

We have been told

$\Pr[C \mid M \cap S] = 23 \cdot \Pr[C \mid M \cap S^c]$,

$\Pr[C \mid M \cap S] = 23 \cdot \Pr[C \mid M \cap S^c]$,

$\Pr[S \mid M] = 0.231$, $\Pr[S \mid W] = 0.183$ $\Rightarrow$ $\Pr[S^c \mid M] = 0.769$, $\Pr[S^c \mid W] = 0.817$.

(a) $$\Pr[W \mid S] = \frac{\#\text{ of women who smoke}}{\#\text{ of smokers}} = \frac{21.1}{21.1 + 24.8} \doteq 0.46$$

(b) $$\Pr[S \mid W \cap C] = \frac{\Pr[S \cap W \cap C]}{\Pr[W \cap C]} = \frac{\Pr[C \mid S \cap W] \cdot \Pr[S \cap W]}{\Pr[C \cap W \cap S] + \Pr[C \cap W \cap S^c]}$$

$$= \frac{\Pr[C \mid S \cap W] \Pr[S \cap W]}{\Pr[C \mid S \cap W] \cdot \Pr[S \cap W] + \Pr[C \mid S^c \cap W] \cdot \Pr[S^c \cap W]}$$

$$= \frac{13 \cdot \Pr[C \mid S^c \cap W] \Pr[S \cap W]}{13 \cdot \Pr[C \mid S^c \cap W] \cdot \Pr[S \cap W] + \Pr[C \mid S^c \cap W] (1 - \Pr[S \cap W])}$$

$$= \frac{13 \Pr[S \cap W]}{13 \Pr[S \cap W] + \Pr[S^c \cap W]} = \frac{13 \Pr[S \mid W] \Pr[W]}{13 \Pr[S \mid W] \Pr[W] + \Pr[S^c \mid W] \Pr[W]}$$

$$= \frac{13 \Pr[S \mid W]}{13 \Pr[S \mid W] + \Pr[S^c \mid W]} = \frac{13(0.183)}{13(0.183) + 0.817} \approx 0.744$$

(c) Similarly to part (b),

$$\Pr[S \mid M \cap C] = \frac{23(0.231)}{23(0.231) + 0.769} = \boxed{0.874}$$

2.25 Let A = "produced on Monday or Thursday"
D = "part is defective".

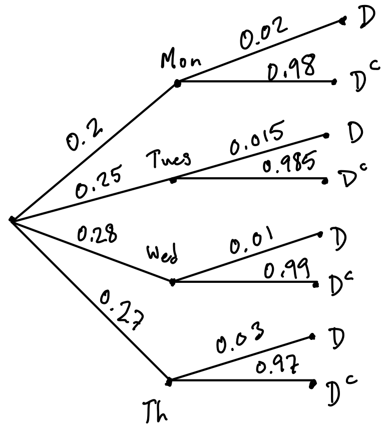
(a) $\Pr[A] = 0.2 + 0.27 = 0.47.$

(b)
$$\Pr[A \mid D] = \frac{\Pr[A \cap D]}{\Pr[D]}$$

$$= \frac{\Pr[D \text{ and Mon}] + \Pr[D \text{ and Th}]}{\Pr[D \text{ and Mon}] + \Pr[D \text{ and Tu}] + \Pr[D \text{ and Wed}] + \Pr[D \text{ and Th}]}$$

$$= \frac{(0.2)(0.02) + (0.27)(0.03)}{(0.2)(0.02) + (0.25)(0.015) + (0.28)(0.01) + (0.27)(0.03)}$$

$$\doteq 0.6488$$



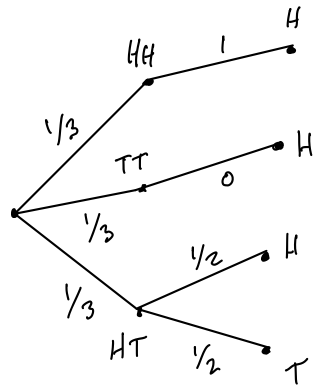
(c)
$$\Pr[A \mid D^c] = \frac{\Pr[A \cap D^c]}{\Pr[D^c]} = \frac{\Pr[A] - \Pr[A \cap D]}{1 - \Pr[D]}$$

$$= \frac{0.47 - [(0.2)(0.02) + (0.27)(0.03)]}{1 - [(0.2)(0.02) + (0.25)(0.015) + (0.28)(0.01) + (0.27)(0.03)]}$$

$$\doteq 0.4666$$

2.38 (a)
$$\Pr(HH \mid H) = \frac{\Pr(HH \text{ and } H)}{\Pr(H)}$$

$$= \frac{(1/3)(1)}{(1)(1/3) + (1/2)(1/3)} = \frac{2}{3}$$



(b)
$$\Pr(2^{nd} \text{ flip is H} \mid 1^{st} \text{ flip is H}) = \frac{\Pr(1^{st} \text{ and } 2^{nd} \text{ flips H})}{\Pr(1^{st} \text{ flip is H})}$$

$$= \frac{(1/3)(1)(1) + (1/3)(1/2)^2}{(1/3)(1) + (1/3)(1/2)} = 0.833$$

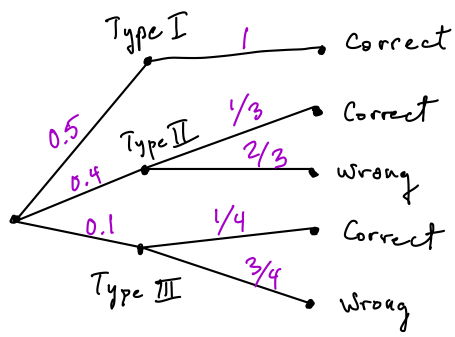
(c) $\Pr(HH \mid 1^{st} \text{ and } 2^{nd} \text{ flips are } H) = \dfrac{\Pr(HH \text{ and } 1^{st} \text{ and } 2^{nd} \text{ flips are } H)}{\Pr(1^{st} \text{ and } 2^{nd} \text{ flips are } H)}$

$$= \frac{1/3}{(1/3)(1)(1) + (1/3)(1/2)(1/2)} = \frac{1}{1 + 0.25} = 0.8$$

2.49 (a) $1 - \text{pbinom}(11, 20, 0.25) = 0.000935$

(b) $1 - \text{pbinom}(11, 20, 1/3) = 0.013$

(c) Say the questions are Type I if you know the answer, Type II if you can eliminate one choice, and Type III if you must guess blindly. Then The probability tree for the 1^{st} question is given at right.



$$\Pr(1^{st} \text{ question is correct}) = \left(\tfrac{1}{2}\right)(1) + \left(\tfrac{2}{5}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{10}\right)\left(\tfrac{1}{4}\right)$$
$$\doteq 0.6583.$$

So,
$$\Pr(\text{pass test}) = 1 - \text{pbinom}(11, 20, 0.6583) \doteq 0.786.$$

2.51 $X \sim \text{Geom}(\pi) \Rightarrow \Pr(X = x) = (1-\pi)^x \pi$

(a) $\Pr(X \geq k) = \left[(1-\pi)^k + (1-\pi)^{k+1} + \ldots\right]\pi$

$$= (1-\pi)^k \pi \left[1 + (1-\pi) + (1-\pi)^2 + \cdots\right] = \frac{(1-\pi)^k \pi}{1-(1-\pi)} = (1-\pi)^k$$

(b)
$$\Pr(X = x \mid X \geq k) = \frac{\Pr(X \geq k \text{ and } X = x)}{\Pr(X \geq k)}$$

$$= \left\{ \begin{matrix} 0, & x < k \\ \dfrac{\pi(1-\pi)^x}{(1-\pi)^k}, & x \geq k \end{matrix} \right\} = \left\{ \begin{matrix} 0, & x < k \\ \pi(1-\pi)^{x-k}, & x \geq k \end{matrix} \right\} = \Pr(X = x-k).$$

(c) Saying $X \geq k$ is like starting over.

2.57 (a) Note that $\text{qbinom}(0.025, 200, 0.5) = 86$

and $\text{pbinom}(85, 200, 0.5) = 0.02$ while

$\text{pbinom}(86, 200, 0.5) = 0.028$

Thus, the rejection region corresponding to $\alpha = 0.05$ is

$$0 \le x \le 85 \quad \text{or} \quad 115 \le x \le 200.$$

Code:

```
rejectRegion = c(0:85, 115:200)
sum(dbinom(rejectRegion, 200, 0.55))
```

Ans. 0.262

(b) `sum(dbinom(c(0:179, 221:400), 400, 0.55))` Ans. 0.481

(c) If we execute the command

```
1 - sum(dbinom(qbinom(0.025, n, 0.5) : qbinom(0.975, n, 0.5), n, 0.55))
```

with $n = 1062$, the result is 0.898. With $n = 1063$, the answer is 0.904. So, 1063 flips are needed.

2.72 X has the pmf

x	-1	1
$\Pr(X=x)$	$20/38$	$18/38$

So, $$E(X) = (-1)\left(\frac{20}{38}\right) + \frac{18}{38} = \frac{-1}{19} \doteq -0.05263$$

(that is, expect to lose about 5 cents, on average, per play), and

$$E(X^2) = (-1)^2\left(\frac{20}{38}\right) + \frac{18}{38} = 1$$

$$\Rightarrow \quad Var(X) = E(X^2) - E(X)^2 = 1 - \left(-\frac{1}{19}\right)^2 \doteq \boxed{0.99723}$$

2.75 Let $Z \sim Geom(1/2)$. $P(X = 2^k) = P(Z = k)$

$$f_Z(z) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^z = \frac{1}{2^{z+1}}$$

So, $$E(X) = 2^0 \cdot \frac{1}{2} + 2^1 \cdot \frac{1}{2^2} + 2^2 \cdot \frac{1}{2^3} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

2.81 (a) $E(X + Y) = E(X) + E(Y) = 20$

$Var(X + Y) = Var(X) + Var(Y) = 4$, because X, Y are independent.

(b) $E(X+X) = E(2X) = 2E(X) = 20$

$Var(X+X) = Var(2X) = 4\,Var(X) = 8$

(c) When Y is independent of X, one may well have a small value of X paired with a large value of Y, making $X+Y$ moderately-sized. The same cannot be said of $X+X$, which is always twice as large as X. These observations lead to $X+X$ having greater variability than $X+Y$. Independence, however, is not a factor in the expected value of a sum,