Planar Systems

The homogeneous linear 1^{st}-order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

- has an equilibrium point at $\mathbf{x}_0$ if $\left.\frac{d\mathbf{x}}{dt}\right|_{\mathbf{x}=\mathbf{x}_0} = \mathbf{0}$. That is, if $\mathbf{x}_0$ is in the null space of $\mathbf{A}$. Note that the origin $\mathbf{0}$ is always an equilibrium point, and if $\mathbf{A}$ is nonsingular, it is the only one.
- is **planar** if it has exactly two dependent variables—i.e., $\mathbf{x} = \langle x_1, x_2 \rangle$. Necessarily, $\mathbf{A}$ is a 2-by-2 matrix. In a planar system, the equilibrium at the origin is described as
  - a **globally asymptotically stable node** if the eigenvalues of $\mathbf{A}$ are *real* and $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - an **unstable node** if the eigenvalues of $\mathbf{A}$ are *real* and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - an **unstable saddle point** if the eigenvalues of $\mathbf{A}$ are *real* and their product $\lambda_1\lambda_2 < 0$.
  - a **globally asymptotically stable spiral point** if the eigenvalues of $\mathbf{A}$ are *nonreal* and $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - an **unstable spiral point** if the eigenvalues of $\mathbf{A}$ are *nonreal* and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - a **stable center** if the eigenvalues of $\mathbf{A}$ are *nonreal* and the origin is neither an unstable spiral point nor a globally asymptotically stable spiral point.

A coordinate frame that gives an axis to each of the dependent variables but and no others (no axis for the independent variable t) is called **phase space** (the **phase plane**, for planar systems). A **phase portrait** is a sketch of trajectories/solutions in phase space.

Today's Work We will be considering the homogeneous planar systems numbered below. Each one of you should log into `https://b.socrative.com/login/student/`, identifying yourself by name using the convention "lastFirst" (Pat Walsh would log in as "walshPat"), and going to room "SCOFIELD3894". Give answers to the questions you find there, discussing them first with your team. You may also use the two apps
`http://scofield.site/teaching/demos/eigenstuff.html` and
`http://scofield.site/teaching/demos/PhasePortrait2D.html`
for finding eigenpairs and plotting phase portraits, respectively.

1. $\mathbf{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \mathbf{x}$
2. $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$
3. $\mathbf{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{x}$
4. $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}$
5. $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$
6. $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$