Mathematical Induction

- It is a technique for proving a statement $\forall n \in \mathbb{Z}^+ \, P(n)$.
- Can be adapted to prove the correctness of some algorithms.
- As a rule of inference, it is

$$(P(1) \wedge \forall k\,(P(k) \to P(k+1))) \to \forall n\, P(n).$$

  $P(1)$ is called the **basis step**, $P(k) \to P(k+1)$ is called the **inductive step**, and the assumption that the hypothesis $P(k)$ of the inductive step holds is called the **inductive hypothesis**.

Induction is not helpful in discovering in discovering new mathematical statements which are true. Once a pattern or truth has been conjectured, however, induction can often establish that it is true.

Examples:

1. $$\sum_{j=1}^{n}(2j+1) \;=\; 1+3+5+\cdots+(2n+1) \;=\; ?.$$
2. For all positive integers, $23^n - 1$ is divisible by 11.
3. For all positive integers, $n < 2^n$.
4. For all $n \in \mathbb{N} - \{0,1,2,3\}$, $2^n < n!$.
5. If B is a set with $|B| = n$, then $|\mathcal{P}(B)| = 2^n$, for all $n \in \mathbb{N}$.
6. Show that $3n^3 + 2n + 7 \leqslant 4n^3$ for $n = 3,4,5,\ldots$.

   Note: $P(n)$ is the statement $3n^3 + 2n + 7 \leqslant 4n^3$, and we are not asserting that $P(1)$ is true. $P(3)$ is our base step:

$$P(3): \qquad 3(3)^3 + 2(3) + 7 \leqslant 4(3)^3, \qquad \text{since} \qquad 94 \leqslant 108.$$

   Now, from the induction hypothesis $P(k)$ (assumed to hold for an integer $k \geqslant 3$), we must prove $P(k+1)$. That is, we get to assume

$$P(k): \qquad 3k^3 + 2k + 7 \leqslant 4k^3,$$

   and we must show

$$P(k+1): \qquad 3(k+1)^3 + 2(k+1) + 7 \leqslant 4(k+1)^3,$$

   Now

$$\begin{aligned}
3(k+1)^3 + 2(k+1) + 7 &= 3(k+1)(k+1)(k+1) + 2(k+1) + 7 \;=\; 3(k^2+2k+1)(k+1) + 2k + 2 + 7 \\
&= 3(k^3+3k^2+3k+1) + 2k + 2 + 7 \;=\; 3k^3 + 9k^2 + 11k + 12 \\
&= (3k^3 + 2k + 7) + 9k^2 + 9k + 5 \\
&\leqslant 4k^3 + 9k^2 + 9k + 5 \qquad \text{(by the induction hypothesis)} \\
&= 4(k^3+3k^2+3k+1) - 3k^2 - 3k + 1 \;=\; 4(k+1)^3 + 1 - 3k(k+1) \\
&\leqslant 4(k+1)^3,
\end{aligned}$$

since adding $1-3k(k+1)$ to any quantity makes it smaller, if $k>2$.

7. Let $\phi=\frac{\sqrt{5}+1}{2}$. Note that $\phi^{-1}=\frac{\sqrt{5}-1}{2}$. Let F_n stand for n^{th} Fibonacci number: $F_0=0, F_1=1$, $F_2=1, F_3=2, F_4=3$, etc. Prove that $F_n=\frac{1}{\sqrt{5}}\left[\phi^n-(-\phi^{-1})^n\right]$ for all $n \in \mathbb{N}$.
8. One can tile an $2^n \times 2^n$ checkerboard with one space removed using tiles shaped like 
9. **Induction misused**. Let $P(n)$ be the statement "Any collection of $n \geqslant 2$ distinct lines in the plane, no two of which are parallel, shares a common point.

The following is an attempt to prove $\forall n \in \mathbb{Z}^+, P(n)$:
Base case: $P(2)$ says 2 non-parallel lines in the plane have a common point. This seems true enough without requiring proof.
Inductive step: We assume $P(k)$ is true for some integer $k \geqslant 2$. The case $P(k+1)$ has us considering $(k+1)$ non-parallel lines in the plane: $\{\ell_1, \ell_2, \ldots, \ell_k, \ell_{k+1}\}$. Now the collection $\{\ell_1, \ell_2, \ldots, \ell_k\}$ has k non-parallel lines so by the induction hypothesis, this collection has a common point, call it P_1. As well, the induction hypothesis applies to the collection $\{\ell_2, \ell_3, \ldots, \ell_k, \ell_{k+1}\}$, so these lines have a common point, call it P_2. But two points in a plane uniquely determine a line, and since no two lines found in both collections can be the same, it must be that points P_1 and P_2 are really the same point. Thus, our original collection $\{\ell_1, \ell_2, \ldots, \ell_k, \ell_{k+1}\}$ shares a commont point, showing that $P(k+1)$ holds.
Thus, by induction, $P(n)$ holds for all $n=2,3,4,\ldots$.