Insight into Problem 1

Say you have a piecewise-defined function to transform such as

$$f(t) \;=\; \begin{cases} 0, & t < 5, \\ t^2 - 10t + 40, & t \geqslant 5, \end{cases}$$

and the goal is to find the Laplace transform $F(s)$. I displayed this method in class: to find $F(s)$ via the definition. Assuming $s > 0$,

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}\,dt \;=\; \int_0^5 0\,dt + \int_5^\infty (t^2 - 10t + 40)e^{-st}\,dt \\
&= \int_5^\infty t^2 e^{-st}\,dt + \int_5^\infty (40 - 10t)e^{-st}\,dt \\
&= \left[-\frac{1}{s}t^2 e^{-st}\right]_5^\infty + \frac{2}{s}\int_5^\infty te^{-st}\,dt + \int_5^\infty (40-10t)e^{-st}\,dt \\
&= \frac{25}{s}e^{-5s} + \frac{2}{s}\int_5^\infty te^{-st}\,dt + \int_5^\infty (40-10t)e^{-st}\,dt \\
&= \frac{25}{s}e^{-5s} + \int_5^\infty \left[40 + \left(\frac{2}{s} - 10\right)t\right]e^{-st}\,dt \\
&= \frac{25}{s}e^{-5s} + 40\int_5^\infty e^{-st}\,dt + \int_5^\infty \left(\frac{2}{s} - 10\right)te^{-st}\,dt \\
&= \frac{25}{s}e^{-5s} + 40\int_5^\infty e^{-st}\,dt + \int_5^\infty \left(\frac{2}{s} - 10\right)te^{-st}\,dt \\
&= \frac{25}{s}e^{-5s} + 40\int_5^\infty e^{-st}\,dt + \left[-\frac{1}{s}\left(\frac{2}{s} - 10\right)te^{-st}\right]_5^\infty + \frac{1}{s}\int_5^\infty \left(\frac{2}{s} - 10\right)e^{-st}\,dt \\
&= \frac{25}{s}e^{-5s} + \frac{5}{s}\left(\frac{2}{s} - 10\right)e^{-5s} + \left(\frac{2}{s^2} - \frac{10}{s} + 40\right)\int_5^\infty e^{-st}\,dt \\
&= \frac{25}{s}e^{-5s} + \frac{5}{s}\left(\frac{2}{s} - 10\right)e^{-5s} + \left(\frac{2}{s^2} - \frac{10}{s} + 40\right)\left[-\frac{1}{s}e^{-st}\right]_5^\infty \\
&= \frac{25}{s}e^{-5s} + \frac{5}{s}\left(\frac{2}{s} - 10\right)e^{-5s} + \frac{1}{s}\left(\frac{2}{s^2} - \frac{10}{s} + 40\right)e^{-5s} \\
&= \left(\frac{25}{s} + \frac{10}{s^2} - \frac{50}{s} + \frac{2}{s^3} - \frac{10}{s^2} + \frac{40}{s}\right)e^{-5s} \\
&= \left(\frac{2}{s^3} + \frac{15}{s}\right)e^{-5s}
\end{aligned}$$

There is, however, an easier way to do this particular problem, if you can make a fundamental insight. The function f sort of gets "switched on" at time $t = 5$. Moreover, at the moment it is comes on, it is a shifted version of another quadratic function. That is,

$$f(t) \;=\; u(t-5)(t^2 - 10t + 40) \;=\; u(t-5)[(t-5)^2 + 15] \;=\; u(t-5)\Big[(t^2+15)\big|_{t\mapsto t-5}\Big].$$

By the 2nd shifting theorem, since we know

$$\mathcal{L}\left\{t^2 + 15\right\} \;=\; \frac{2}{s^3} + \frac{15}{s},$$

we have that

$$\mathcal{L}\left\{u(t-5)\left[(t^2+15)\big|_{t\mapsto t-5}\right]\right\} \;=\; e^{-5s}\left(\frac{2}{s^3}+\frac{15}{s}\right).$$

Insight into Problem 3

This problem refers to functions such as $u_4(t)$ and $u_7(t)$. There are a lot of introductory DE textbooks, and nearly all of them present the unit step function along with shifts of that function. Some of these texts, such as ours, refer to a shift to the right c units of $u(t)$ by $u(t-c)$. Others, choose to employ a subscript indicating how much of a right shift there is. The writer of this problem was using one of these latter sorts of textbooks. The upshot is that

$$u_4(t) \qquad \text{and} \qquad u(t-4)$$

are the same thing. You can find the Laplace transform of such functions via the definition:

$$\mathcal{L}\{u_7(t)\} \;=\; \mathcal{L}\{u(t-7)\} \;=\; \int_0^\infty u(t-7)e^{-st}\,dt \;=\; \int_7^\infty e^{-st}\,dt \;=\; -\frac{1}{s}e^{-st}\Big|_7^\infty \;=\; \frac{1}{s}e^{-7s}.$$

Insight into Problem 12

If this problem said solve the IVP

$$f'-f=8t, \quad \text{subject to IC} \quad f(0)=-5,$$

you could use the Laplace transform method for solving IVPs. Starting by taking the Laplace transform of both sides, you would have

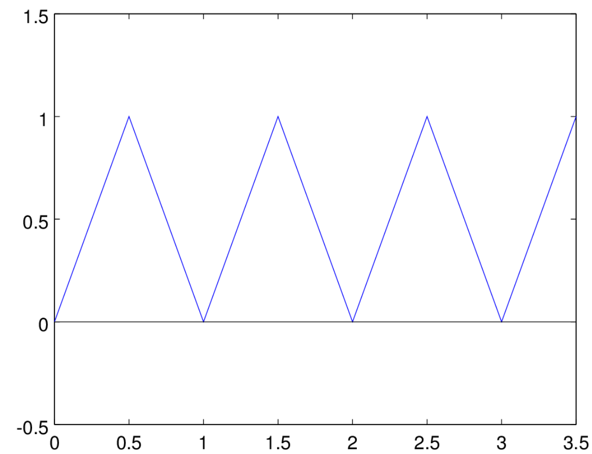
$$\begin{aligned}
\mathcal{L}\{f'-f\} \;=\; \mathcal{L}\{8t\} \qquad &\Rightarrow \qquad \mathcal{L}\{f'\}-\mathcal{L}\{f\} \;=\; 8\mathcal{L}\{t\} \\
&\Rightarrow \qquad sF(s)-f(0)-F(s) \;=\; 8\cdot\frac{1}{s^2} \\
&\Rightarrow \qquad sF(s)+5-F(s) \;=\; \frac{8}{s^2} \\
&\Rightarrow \qquad (s-1)F(s) \;=\; \frac{8}{s^2}-5 \\
&\Rightarrow \qquad F(s) \;=\; \frac{8}{s^2(s-1)}-\frac{5}{s-1}.
\end{aligned}$$

At this point, you would find the solution $f(t)$ by taking the inverse Laplace transform of $F(s)$.

But, your problem is different from this one. Yours gives an initial condition at some time $t>0$ instead of $t=0$. Thus, you should do forego using the Laplace transform on this problem, instead employing methods from Chapter 2 to solve it.

Insight into Problem 15

When your forcing function is periodic, you can use the *definition* along with *substitution* to find the Laplace transform. For Problem 15, the forcing function $S(t)$ alternates back-and-forth between the values 1 and 0, and has period 2. Suppose, instead, we had the *saw tooth* function (periodic) which looks like



$$r(t) = \begin{cases} 2t, & 0 \leqslant t < 1/2, \\ 2(1-t), & 1/2 \leqslant t < 1, \end{cases} \qquad r(t+1) = r(t),$$

making r periodic with period 1.

Using the definition of Laplace transform, we have

$$\begin{aligned}
\mathcal{L}\{r(t)\} &= \int_0^\infty r(t)e^{st}\,dt \\
&= \int_0^1 r(t)e^{-st}\,dt + \int_1^\infty r(t)e^{-st}\,dt \quad \text{(split off integral over single period)} \\
&= \int_0^1 r(t)e^{-st}\,dt + \int_0^\infty r(u+1)e^{-s(u+1)}\,du \quad \text{(making substitution } u = t-1\text{)} \\
&= \int_0^1 r(t)e^{-st}\,dt + e^{-s}\int_0^\infty r(u+1)e^{-su}\,du \\
&= \int_0^1 r(t)e^{-st}\,dt + e^{-s}\int_0^\infty r(u)e^{-su}\,du \quad \text{(using the fact } r(t+1) = r(t)\text{)} \\
&= \int_0^1 r(t)e^{-st}\,dt + e^{-s}\int_0^\infty r(t)e^{-st}\,dt \quad \text{(since } u, t \text{ are just dummy variables)} \\
&= \int_0^1 r(t)e^{-st}\,dt + e^{-s}\mathcal{L}\{r(t)\} \quad \text{(by definition of Laplace transform).}
\end{aligned}$$

Subtracting the final term from both sides, we have

$$\mathcal{L}\{r(t)\} - e^{-s}\mathcal{L}\{r(t)\} = \int_0^1 r(t)e^{-st}\,dt.$$

Recognizing the left-hand side is $R(s) - e^{-s}R(s) = (1-e^{-s})R(s)$, this really says

$$\begin{aligned}
(1-e^{-s})R(s) &= \int_0^1 r(t)e^{-st}\,dt \\
&= \int_0^{1/2} 2te^{-st}\,dt + \int_{1/2}^1 2(1-t)e^{-st}\,dt \\
&= \frac{2}{s^2} - \frac{1}{s}e^{-s/2} - \frac{2}{s^2}e^{-s/2} + \frac{2}{s^2}e^{-s} + \frac{1}{s}e^{-s/2} - \frac{2}{s^2}e^{-s/2} \quad \text{(integrating by parts)} \\
&= \frac{2}{s^2} - \frac{4}{s^2}e^{-s/2} + \frac{2}{s^2}e^{-s}.
\end{aligned}$$

Thus, the Laplace transform of $r(t)$ is

$$R(s) = \frac{1}{1-e^{-s}}\left(\frac{2}{s^2} - \frac{4}{s^2}e^{-s/2} + \frac{2}{s^2}e^{-s}\right).$$