5.2.2 (e)

Find $\mathcal{L}\{-5\delta(t-2) + e^{3t}\underbrace{u(t-2)}\}$ Heaviside shifted 2 right

$$= \underbrace{\mathcal{L}\{-5\delta(t-2)\}}_{\text{easier?}} + \underbrace{\mathcal{L}\{e^{3t}u(t-2)\}}_{\text{more (?) difficult of two parts}}$$

Last Fri or Mon.

$$\mathcal{L}\{u(t-a)\,f(t)\}$$

Have an entry in table for $\mathcal{L}\{u(t-a)\,f(t-a)\}$

$$= e^{-as}\,\mathcal{L}\{f(t)\}$$

Problems like

shift built into both

$$\mathcal{L}\{u(\overbrace{t-1})\,\sin(2(\overbrace{t-1}))\} = e^{-s}\,\mathcal{L}\{\sin(2t)\}$$

already in form of table entry.

$$= e^{-s}\frac{2}{s^2+4} \checkmark$$

But

$$\mathcal{L}\{u(t-3)\,\underbrace{(t^2-4t)}\}$$

want to consider this the post-shift-3-units-to-right fn.

Q: what is the pre-shifted fn.?

A: Find it by shifting the "post-shifted" fn. 3 to the left — entails replacing t's by $t+3$

$$t^2 - 4t \Big|_{t \mapsto t+3} = (t+3)^2 - 4(t+3)$$

$$= t^2 + 6t + 9 - 4t - 12$$

$$= \underbrace{t^2 + 2t - 3}_{\text{preshifted fn.}}$$

Table entry really says

$$\mathcal{L}\{u(t-a)\cdot(\underset{\text{fn.}}{\text{post-shifted}})\} = e^{-as}\cdot\mathcal{L}\{\underset{\text{fn.}}{\text{pre-shifted}}\}$$

or

$$\mathcal{L}\{u(t-3)(t^2-4t)\} = e^{-3s}\cdot\mathcal{L}\{t^2+2t-3\}$$

$$= \underline{e^{-3s}\left(\frac{2}{s^3} + \frac{2}{s^2} - \frac{3}{s}\right)}$$

5.2.2.(e)

$$\mathcal{L}\{u(t-2)\underbrace{e^{3t}}_{\substack{\text{post shift -2}\\ \text{-to-right fn.}}}\}$$

Q: pre-shift fn? A: Take $e^{3t}\Big|_{t \mapsto t+2}$

$$\mathcal{L}\{u(t-5)\,\underbrace{e^{7t+1}}_{\text{post}}\}$$

pre-shift $\quad e^{7t+1}\Big|_{\underbrace{t\mapsto t+5}_{\text{shift to left 5}}} = e^{7(t+5)+1} = e^{7t+36}$

$$= e^{36}\cdot e^{7t}$$

$$\mathcal{L}\{u(t-5)\,e^{7t+1}\} = e^{-5s}\cdot\mathcal{L}\{\underbrace{\overset{\text{constant}}{e^{36}}\cdot e^{7t}}_{\text{pre-shift}}\}$$

$$= e^{-5s}\cdot e^{36}\cdot\mathcal{L}\{e^{7t}\}$$

$$= e^{-5s}\cdot e^{36}\cdot\frac{1}{s-7}$$

$$= e^{36-5s}\cdot\frac{1}{s-7}$$

Ex.] $\quad y''-y=-20\,\delta(t-3),\quad y(0)=1,\ y'(0)=0$

Have advocated breaking into 2 problems

zeroes the nonhomog.-term ① $\quad y''-y=0,\quad y(0)=1,\ y'(0)=0$

zeroes the ICs ② $\quad y''-y=-20\,\delta(t-3),\ y(0)=0,\ y'(0)=0 \quad\leftarrow$ Exercise 5.5.2

Attacking ① using Ch.4 approach

$y'' - y = 0 \longrightarrow$ char. eq. $r^2 - 1 = 0$

$(r+1)(r-1) = 0$

$\Longrightarrow$ roots $r = -1, 1$

Both $\underbrace{e^{-t}, e^{t}}_{}$ satisfy the DE

neither satisfies the ICs in ① however

Take linear comb.

$$y_1(t) = c_1 e^{-t} + c_2 e^{t} \longleftarrow$$ insert $t=0$ for ICs

has deriv.

$$y_1' = -c_1 e^{-t} + c_2 e^{t} \longleftarrow$$

Apply ICs

$$1 = y(0) = c_1 \cdot e^0 + c_2 e^0 = c_1 + c_2$$

$$0 = y'(0) = -c_1 \cdot e^0 + c_2 e^0 = -c_1 + c_2$$

$$\Longrightarrow c_1 = c_2 = 1/2$$

Soln. to ①

$$\boxed{y_1(t) = \frac{1}{2} e^{-t} + \frac{1}{2} e^{t}}$$

Attack ②

$$y'' - y = -20\delta(t-3)$$

$$\mathcal{L}\{\qquad\} = \mathcal{L}\{-20\delta(t-3)\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -20\ \mathcal{L}\{\delta(t-3)\}$$

$$s^2 Y - \underbrace{\cancel{s y(0)} - \cancel{y'(0)}}_{\text{(3) has these} \,=\, 0} - Y = -20 e^{-3s}$$

$$(s^2 - 1) Y = -20 e^{-3s}$$

$$Y = e^{-3s} \cdot \frac{-20}{s^2 - 1} \qquad \text{Need to take } \mathcal{L}^{-1} \text{ of this}$$

exponential on s-side — think table entry $\mathcal{L}\{u(t-a) f(t-a)\}$

First, figure out what $\dfrac{-20}{s^2-1}$ comes from (i.e., its $\mathcal{L}^{-1}$)

$$\left(\frac{-20}{s^2 - 1} = \frac{A}{s-1} + \frac{B}{s+1} \right) \times (s^2 - 1)$$

$$-20 = A(s+1) + B(s-1)$$

Insert $s = -1$:

$$-20 = A(-1+1) + B(-1-1) \implies B = 10$$

Insert $s = 1$:

$$-20 = A(1+1) + B(1-1) \implies A = -10$$

$$\mathcal{L}^{-1}\left\{\frac{-20}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{-10}{s-1} + \frac{10}{s+1}\right\} = -10\, \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 10\, \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -10 e^{t} + 10 e^{-t}$$

Want: $\mathcal{L}^{-1}\left\{ e^{-3s} \cdot \underbrace{\frac{-20}{s^2-1}} \right\} = \boxed{u(t-3)\cdot\left(10e^{-(t-3)} - 10e^{t-3}\right)}$

comes from

$-10e^{t} + 10e^{-t}$

$y_2(t) =$ this

Soln. to original problem

$$y_1(t) + y_2(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} + 10\,u(t-3)\left(e^{-(t-3)} - e^{t-3}\right)$$