1. One **unsatisfactory** answer is

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 4i & -4i \\ -1 & -3-i & -3+i \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3/2 - i/2 \\ -3/2 + i/2 \end{bmatrix},$$

due to the references to $i = \sqrt{-1}$.

Better if you identify $\alpha + \beta i = -2 + 3i$ (so $\alpha = -2$, $\beta = 3$) and $\mathbf{u} + i\mathbf{w} = \langle -4i, -3+i, 1\rangle$ (so that $\mathbf{u} = \langle 0, -3, 1\rangle$ and $\mathbf{w} = \langle -4, 1, 0\rangle$), and then replace the last two columns of the matrix above with $e^{\alpha t}[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}]$ and $e^{\alpha t}[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}]$. You get

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-t} & 4e^{-2t}\sin(3t) & -4e^{-2t}\cos(3t) \\ -e^{-t} & (-3\cos(3t) - \sin(3t))e^{-2t} & (\cos(3t) - 3\sin(3t))e^{-2t} \\ 2e^{-t} & e^{-2t}\cos(3t) & e^{-2t}\sin(3t) \end{bmatrix}$$

The vector $\mathbf{c}$ that makes $\mathbf{\Phi}(0)\mathbf{c} = \langle 5, 7, -1\rangle$ is $\mathbf{c} = \langle 1, -3, -1\rangle$, so the solution is

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & 4e^{-2t}\sin(3t) & -4e^{-2t}\cos(3t) \\ -e^{-t} & (-3\cos(3t) - \sin(3t))e^{-2t} & (\cos(3t) - 3\sin(3t))e^{-2t} \\ 2e^{-t} & e^{-2t}\cos(3t) & e^{-2t}\sin(3t) \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

2. There is no inflow of salt, just outflow. The rate at which salt leaves at any moment is the product of the concentration and the volume flow rate.

$$\frac{dy}{dt} = -(20)\frac{y}{250} = -\frac{2}{25}y, \qquad y(0) = 8000.$$

3. (a) The eigenvalues are found by solving

$$0 = \begin{vmatrix} 7-\lambda & 16 \\ -1 & -1-\lambda \end{vmatrix} = (7-\lambda)(-1-\lambda) + 16 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

showing $\lambda = 3$ to have algebraic multiplicity 2. Solving for null $(\mathbf{A} - 3\mathbf{I})$

$$\left[\begin{array}{cc|c} 4 & 16 & 0 \\ -1 & -4 & 0 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since there is just one free column, the geometric multiplicity is 1, and $\lambda = 3$ is degenerate; a basis vector of its eigenspace is $\mathbf{v} = \langle 4, -1\rangle$. So, along with $e^{3t}\mathbf{v}$, we seek a second solution of the form $e^{3t}(\mathbf{w} + t\mathbf{v})$, where $\mathbf{w}$ solves $(\mathbf{A} - 3\mathbf{I})\mathbf{w} = \mathbf{v}$:

$$\left[\begin{array}{cc|c} 4 & 16 & 4 \\ -1 & -4 & -1 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

We can use any vector $\mathbf{w} = \langle w_1, w_2\rangle$ for which $w_1 + 4w_2 = 1$; $\mathbf{w} = \langle 1, 0\rangle$ is such a vector. Thus, a fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{bmatrix} 4e^{3t} & (1+4t)e^{3t} \\ -e^{3t} & -te^{3t} \end{bmatrix}.$$

(b) Since the eigenvalues are real and both positive, the equilibrium at the origin is an **unstable node**.

(c) With the original $\mathbf{\Phi}(t)$, we have

$$\begin{aligned}
\mathbf{\Phi}^{-1}(t)\mathbf{f}(t) &= \frac{1}{-4te^{6t} + (1+4t)e^{6t}} \begin{bmatrix} -te^{3t} & -(1+4t)e^{3t} \\ e^{3t} & 4e^{3t} \end{bmatrix} \begin{bmatrix} 2t \\ e^{-t} \end{bmatrix} \\
&= \frac{1}{e^{6t}} \left(2t \begin{bmatrix} -te^{3t} \\ e^{3t} \end{bmatrix} + e^{-t} \begin{bmatrix} -(1+4t)e^{3t} \\ 4e^{3t} \end{bmatrix} \right) \\
&= \begin{bmatrix} -2t^2e^{-3t} - (1+4t)e^{-4t} \\ 2te^{-3t} + 4e^{-4t} \end{bmatrix}.
\end{aligned}$$

Thus,

$$\mathbf{x}_p(t) = \begin{bmatrix} 4e^{3t} & (1+4t)e^{3t} \\ -e^{3t} & -te^{3t} \end{bmatrix} \int \begin{bmatrix} -2t^2e^{-3t} - (1+4t)e^{-4t} \\ 2te^{-3t} + 4e^{-4t} \end{bmatrix} dt.$$

4. (a) Here $a(t) = 2t$, so $\phi(t) = e^{t^2}$. The general solution, then, is $y(t) = ce^{t^2}$.

(b) The problem is linear and nonhomogeneous, with complementary homogeneous problem given in part (a). When we solved part (a), we found $y_h(t)$. Given $f(t) = 12t^2e^{3t}$, the variation of parameters formula gives

$$y_p(t) = e^{t^2} \int 12t^2 e^{3t-t^2}\, dt.$$

The general solution is

$$y_h(t) + y_p(t) = ce^{t^2} + e^{t^2} \int 12t^2 e^{3t-t^2}\, dt.$$

5. (a) We have a linear problem $y' = a(t)y + f(t)$ where the discontinuities of $a(t)$ are at $t = -1, 4$, and the discontinuities of $f(t)$ are at $t = -2, 2$. Thus, the largest open interval containing the initial time $t = 0$ before we run into a discontinuity of a or f is $-1 < t < 2$, or $(-1, 2)$.

(b) The differential equation is first-order and already in normal form $y' = g(t, y)$, where

$$g(t, y) = \frac{t^3}{t^2 - 3t - 4} y + \frac{1}{t^2 - 4}.$$

Euler's Method with this initial time $t_0 = 0$ and stepsize $h = 0.5$ produces approximations to $y(t)$ at $t_1 = 0.5$, $t_2 = 1.0$ and $t_3 = 1.5$. These approximate values are

$$\begin{aligned}
y_1 &= 2 + (0.5)g(0, 2) = 2 + (0.5)(-0.25) = 1.875 \\
y_2 &= 1.875 + (0.5)g(0.5, 1.875) \doteq 1.875 + (0.5)(-0.311310) \doteq 1.719345 \\
y_2 &\doteq 1.719345 + (0.5)g(1.0, 1.719345) \doteq 1.719345 + (0.5)(-0.619891) \doteq 1.409400.
\end{aligned}$$