$\star$35 (a) As one approach to this problem, it is permissible to take derivatives of this alleged solution $u(t)$, then show that

- the DE becomes an identity when u, u'' are inserted, and
- the ICs are met by this $u(t)$.

I don't carry out that solution, here. Instead, I skip steps (they were done in class) to find $u_h(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$, and use the method of undetermined coefficients to find $u_p(t)$, proposing

$$u_p(t) \;=\; A\cos(\omega t) + B\sin(\omega t)$$

(Note that, without any u' term in the DE, one might anticipate that no $\sin(\omega t)$ term is needed; indeed, below, I will solve for B only to find that it is zero), so that

$$\begin{aligned} u_p'(t) &= -\omega A\sin(\omega t) + \omega B\cos(\omega t) \\ u_p''(t) &= -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) \end{aligned}$$

Inserting these into the DE, we obtain

$$\begin{aligned} F_0\cos(\omega t) &= m\Big[-\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)\Big] + k\Big[A\cos(\omega t) + B\sin(\omega t)\Big] \\ &= A(k - m\omega^2)\cos(\omega t) + B(k - m\omega^2)\sin(\omega t) \\ &= Am(\omega_0^2 - \omega^2)\cos(\omega t) + Bm(\omega_0^2 - \omega^2)\sin(\omega t) \qquad \text{(using that } \omega_0^2 = k/m\text{)}. \end{aligned}$$

Since $m(\omega_0^2 - \omega^2) \neq 0$, we can equate coefficients to obtain

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \qquad \text{and} \qquad B = 0.$$

Prior to applying ICs, we have general solution

$$u(t) \;=\; u_h(t) + u_p(t) \;=\; c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t),$$

which has derivative

$$u'(t) \;=\; -\omega_0 c_1\sin(\omega_0 t) + \omega_0 c_2\cos(\omega_0 t) - \frac{F_0\omega}{m(\omega_0^2 - \omega^2)}\sin(\omega t).$$

Applying ICs leads to the two equations in unknowns c_1, c_2:

$$\left.\begin{aligned} 0 &= c_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)} \\ 0 &= \omega_0 c_2 \end{aligned}\right\} \quad\Rightarrow\quad c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0.$$

Thus,

$$u(t) \;=\; -\frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t),$$

as asserted.

(b) The values of A, B that fit the bill are

$$A = \frac{\omega_0 + \omega}{2}\, t \qquad \text{and} \qquad B = \frac{\omega_0 - \omega}{2}\, t.$$

So,

$$\begin{aligned}
\cos(\omega t) - \cos(\omega_0 t) &= \cos(A - B) - \cos(A + B) \\
&= (\cos A \cos B + \sin A \sin B) - (\cos A \cos B - \sin A \sin B) \\
&= 2 \sin A \sin B \;=\; 2 \sin\left(\frac{\omega_0 + \omega}{2}\, t\right) \sin\left(\frac{\omega_0 - \omega}{2}\, t\right).
\end{aligned}$$

(c) Combining parts (a) and (b), we have solution of the IVP

$$u(t) \;=\; \frac{2F_0}{m(\omega_0 + \omega)(\omega_0 - \omega)} \sin\left(\frac{\omega_0 + \omega}{2}\, t\right) \sin\left(\frac{\omega_0 - \omega}{2}\, t\right) \;=\; \frac{1}{880} \sin(440t) \sin(t)\,.$$

The graphs of $\sin(440t)$ and $\sin t$, which I do not include here, are standard sine curves, with the notable difference that $\sin(440t)$ has by far the short period, so oscillates very quickly. The product of a quickly-oscillating sine curve with a slowly-oscillating one produces the graph with corresponding audible beats (when listened to).

★36 (a) The homogeneous solution $u_h(t)$, or general solution of

$$mu'' + ku = 0 \qquad \text{is} \qquad u_h(t) \;=\; c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$

so while you might usually propose

$$u_p(t) \;=\; A \cos(\omega_0 t) + B \sin(\omega_0 t)$$

when $f(t) = F_0 \cos(\omega_0 t)$, such a proposal could not work, as it matches (identically) u_h. By adding the additional factor t, we obtain a proposal that can work.

(b) Proposing this $u_p(t)$ leads to derivatives

$$\begin{aligned}
u_p'(t) &= A \cos(\omega_0 t) - \omega_0 A t \sin(\omega_0 t) + B \sin(\omega_0 t) + \omega_0 B t \cos(\omega_0 t) \\
u_p''(t) &= -2\omega_0 A \sin(\omega_0 t) - \omega_0^2 A t \cos(\omega_0 t) + 2\omega_0 B \cos(\omega_0 t) - \omega_0^2 B t \sin(\omega_0 t)
\end{aligned}$$

Inserting u_p, u_p' and u_p'' into the original (nonhomogeneous) DE, we have

$$\begin{aligned}
F_0 \cos(\omega_0 t) &= m\Big[-2\omega_0 A \sin(\omega_0 t) - \omega_0^2 A t \cos(\omega_0 t) + 2\omega_0 B \cos(\omega_0 t) - \omega_0^2 B t \sin(\omega_0 t)\Big] \\
&\qquad + k\Big[A t \cos(\omega_0 t) + B t \sin(\omega_0 t)\Big] \\
&= A(k - m\omega_0^2)\, t \cos(\omega_0 t) + 2m\omega_0 B \cos(\omega_0 t) \\
&\qquad + B(k - m\omega_0^2)\, t \sin(\omega_0 t) - 2m\omega_0 A \sin(\omega_0 t) \\
&= 2m\omega_0 B \cos(\omega_0 t) - 2m\omega_0 A \sin(\omega_0 t),
\end{aligned}$$

using that $k = m\omega_0^2$. Equating coefficients of like terms, we get that $A = 0$ and $B = F_0/(2m\omega_0)$. Thus,

$$u(t) \;=\; u_h(t) + u_p(t) \;=\; c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}\,t\sin(\omega_0 t).$$

(c) Let's focus on the steady state solution $u_p(t)$. In the case $\omega \neq \omega_0$,

$$u_p(t) \;=\; \frac{F_0}{m(\omega_0^2-\omega^2)}\cos(\omega t)$$

is just a periodic function with amplitude $F_0/(m(\omega_0^2-\omega^2))$. So long as that amplitude is not so huge that it damages the spring to be stretched so far, repeated stretchings to such an amplitude might go on indefinitely. On the other hand, when $\omega = \omega_0$, the amplitude $F_0 t/(2m\omega_0)$ of

$$u_p(t) \;=\; \frac{F_0}{2m\omega_0}t\sin(\omega t)$$

grows larger with t, with limit $(+\infty)$ as $t \to \infty$. Thus, the spring will assuredly break at some point, being stretched to ever larger lengths.

$\star$37 [Part (f) below is optional.]

(a) Using the method of undetermined coefficients, we propose the form of $y_p(t)$ to be

$$\begin{aligned} y_p &= A\cos(\omega t) + B\sin(\omega t), &&\text{so that}\\ y_p' &= -A\omega\sin(\omega t) + B\omega\cos(\omega t), &&\text{and}\\ y_p'' &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t). \end{aligned}$$

We require y_p satisfy the nonhomogeneous DE, so

$$\begin{aligned} f_0\cos(\omega t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) + 2b[-A\omega\sin(\omega t) + B\omega\cos(\omega t)]\\ &\quad +\omega_0^2[A\cos(\omega t) + B\sin(\omega t)]\\ &= (A\omega_0^2 + 2bB\omega - A\omega^2)\cos(\omega t) + (B\omega_0^2 - 2bA\omega - B\omega^2)\sin(\omega t). \end{aligned}$$

Equating coefficients yields

$$\left.\begin{aligned} (\omega_0^2-\omega^2)A + 2b\omega B &= f_0\\ -2b\omega A + (\omega_0^2-\omega^2)B &= 0 \end{aligned}\right\} \;\Rightarrow\; \begin{bmatrix} \omega_0^2-\omega^2 & 2b\omega\\ -2b\omega & \omega_0^2-\omega^2 \end{bmatrix}\begin{bmatrix} A\\ B\end{bmatrix} = \begin{bmatrix} f_0\\ 0\end{bmatrix}.$$

Thus, by Cramer's rule,

$$A \;=\; \frac{\begin{vmatrix} f_0 & 2b\omega\\ 0 & \omega_0^2-\omega^2\end{vmatrix}}{\Delta} = \frac{(\omega_0^2-\omega^2)f_0}{\Delta}, \qquad\text{and}$$

$$B \;=\; \frac{\begin{vmatrix} \omega_0^2-\omega^2 & f_0\\ -2b\omega & 0\end{vmatrix}}{\Delta} = \frac{2b\omega f_0}{\Delta},$$

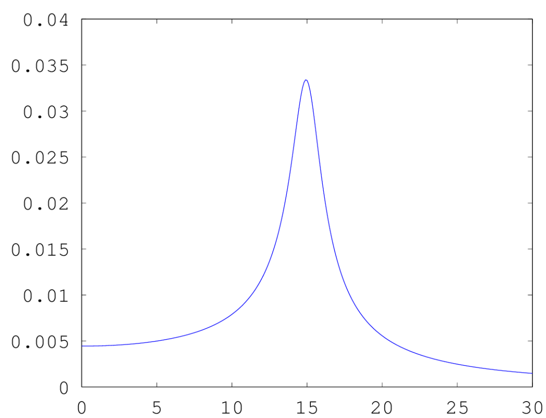
with $\Delta = (\omega_0^2 - \omega^2)^2 + 4b^2\omega^2$. Thus,

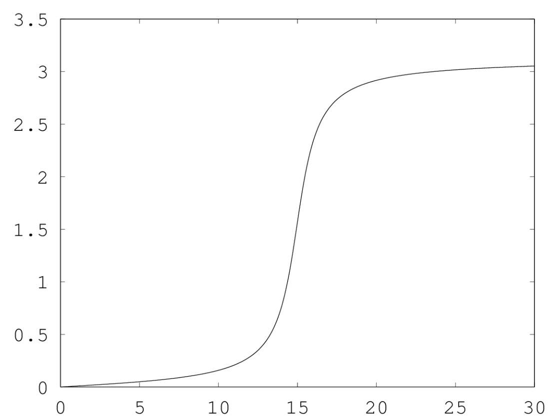
$$y_p(t) \;=\; \frac{(\omega_0^2 - \omega^2) f_0}{\Delta} \cos(\omega t) + \frac{2b\omega f_0}{\Delta} \sin(\omega t).$$

(b) We have $y_p(t) = R\cos(\omega t - \phi)$, with

$$\begin{aligned}
R &= \sqrt{\left(\frac{(\omega_0^2 - \omega^2) f_0}{\Delta}\right)^2 + \left(\frac{2b\omega f_0}{\Delta}\right)^2} \;=\; \frac{f_0}{\Delta}\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} \;=\; \frac{f_0}{\sqrt{\Delta}}, \\
\cos\phi &= \frac{(\omega_0^2 - \omega^2) f_0/\Delta}{R} \;=\; \frac{\omega_0^2 - \omega^2}{\sqrt{\Delta}}, \\
\sin\phi &= \frac{2b\omega f_0/\Delta}{R} \;=\; \frac{2b\omega}{\sqrt{\Delta}}.
\end{aligned}$$

(c) The amplitude curve $R = R(\omega)$ is on the left, while the phase curve $\phi = \phi(\omega)$ is on the right.





The phase curve pictured is one I obtained using the arccosine function, specifically

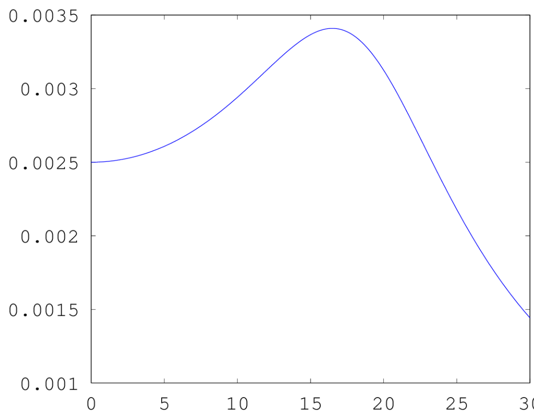
$$\phi(\omega) \;=\; \arccos\left((15^2 - \omega^2)/\sqrt{((15^2 - \omega^2)^2 + 4\omega^2}\right).$$

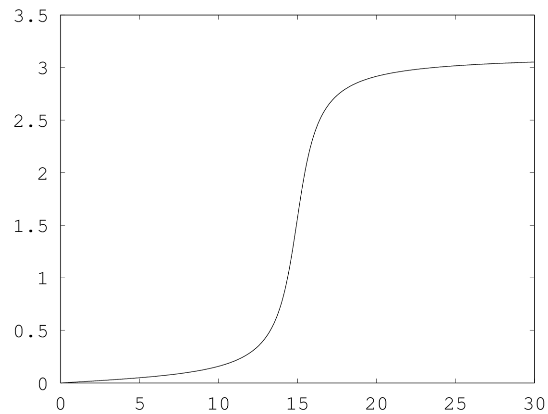
I think it presents the truest picture, but students have not been told (by me) that this expression is to be preferred over the use of arcsine or arctangent

$$\phi(\omega) \;=\; \arcsin\left(2\omega/\sqrt{((15^2 - \omega^2)^2 + 4\omega^2}\right) \qquad \text{or} \qquad \phi(\omega) \;=\; \arctan\left(2\omega/(15^2 - \omega^2)\right).$$

Note to Grader: You should plot both of these to see just how different their pictures are. I'll not mention this again, but it will affect answers in part (d), and perhaps in (f).

(d) The amplitude curve $R = R(\omega)$ is on the left, while the phase curve $\phi = \phi(\omega)$ is on the right.





(e) The derivative,

$$\frac{dR}{d\omega} = \frac{d}{d\omega}\frac{f_0}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}} = \frac{[2\omega(\omega_0^2-\omega^2)-4b^2\omega]f_0}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{3/2}},$$

has a denominator which cannot be zero. Setting the numerator equal to zero, we get

$$\omega\left[\omega^2+2b^2-\omega_0^2\right] = 0 \qquad \Rightarrow \qquad \omega = \pm\sqrt{\omega_0^2-2b^2},$$

where we have tossed out the possibility that $\omega = 0$. We may also toss out the negative ω value, leaving only

$$\omega = \sqrt{\omega_0^2-2b^2}.$$

When this formula is applied to the amplitude curves in parts (c) and (d) above, we get that the peak amplitudes occur at $\omega \doteq 14.933$ (part (c), when $\omega_0 = 15$ and $b = 1$) and $\omega \doteq 16.492$ (part (d), when $\omega_0 = 20$ and $b = 8$).

(f) For fixed values of the constants (including ω), one can plot the graph of the forcing function $f_0\cos(\omega t)$ and superimpose a plot of the solution $y(t) = R\cos(\omega t-\phi)$. The two will be periodic curves having the same period. They will likely have different amplitudes, and reach their peaks at different times. The lag between peaks is ϕ. The plots of $\phi(\omega)$ indicate how this lag varies as one tweaks the frequency ω of the forcing function.

$\star$38 Let $y_1(t) = u_1(t)$, $y_2(t) = u_1'(t)$, $y_3(t) = u_2(t)$, and $y_4(t) = u_2'(t)$. (Other definitions are possible, such as setting $y_1 = u_1$, $y_2 = u_2$, $y_3 = u_1'$ and $y_4 = u_2'$; such definitions do not alter the answer substantively, but do affect the layout of the matrix **A** and vector

$\mathbf{b}(t)$.) Then

$$\begin{aligned}
\frac{d}{dt}\mathbf{y} &= \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} u_1' \\ u_1'' \\ u_2' \\ u_2'' \end{bmatrix} = \begin{bmatrix} y_2 \\ \frac{k_2}{m_1}(u_2 - u_1) + \frac{c_2}{m_1}(u_2' - u_1') - \frac{k_1}{m_1}u_1 - \frac{c_1}{m_1}u_1' \\ y_4 \\ \frac{f(t)}{m_2} - \frac{k_2}{m_2}(u_2 - u_1) - \frac{c_2}{m_2}(u_2' - u_1') \end{bmatrix} \\
&= \begin{bmatrix} y_2 \\ -\frac{k_1+k_2}{m_1}u_1 - \frac{c_1+c_2}{m_1}u_1' + \frac{k_2}{m_1}u_2 + \frac{c_2}{m_1}u_2' \\ y_4 \\ \frac{k_2}{m_2}u_1 + \frac{c_2}{m_2}u_1' - \frac{k_2}{m_2}u_2 - \frac{c_2}{m_2}u_2' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix} \\
&= \begin{bmatrix} y_2 \\ -\frac{k_1+k_2}{m_1}y_1 - \frac{c_1+c_2}{m_1}y_2 + \frac{k_2}{m_1}y_3 + \frac{c_2}{m_1}y_4 \\ y_4 \\ \frac{k_2}{m_2}y_1 + \frac{c_2}{m_2}y_2 - \frac{k_2}{m_2}y_3 - \frac{c_2}{m_2}y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix} = \mathbf{A}\mathbf{y} + \mathbf{b}(t),
\end{aligned}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix}.$$

★39 (a) The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1),$$

so the homogeneous solution is

$$y_{\mathrm{h}}(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t}.$$

The particular solution will be found using variation of parameters. We have

$$\mathbf{\Phi}(t) = \begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{pmatrix} \quad \rightsquigarrow \quad \det \mathbf{\Phi}(t) = 2.$$

The particular solution is given by

$$
\begin{aligned}
y_{\mathrm{p}}(t) &= -\left(\int^t \frac{(6/(1+\mathrm{e}^s))\mathrm{e}^s}{2}\,\mathrm{d}s\right)\mathrm{e}^{-t} + \left(\int^t \frac{(6/(1+\mathrm{e}^s))\mathrm{e}^{-s}}{2}\,\mathrm{d}s\right)\mathrm{e}^t \\
&= -3 + 3\mathrm{e}^t \ln(1+\mathrm{e}^{-t}) - 3\mathrm{e}^{-t}\ln(1+\mathrm{e}^t).
\end{aligned}
$$

The general solution is the sum of the homogeneous and particular solutions,

$$y(t) = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^t - 3 + 3\mathrm{e}^t\ln(1+\mathrm{e}^{-t}) - 3\mathrm{e}^{-t}\ln(1+\mathrm{e}^t).$$

(b) The homogeneous solution is given by

$$y_{\mathrm{h}}(t) = c_1 t + c_2 t\ln(t).$$

The particular solution will be found using variation of parameters. We have

$$\mathbf{\Phi}(t) = \begin{pmatrix} t & t\ln(t) \\ 1 & 1+\ln(t) \end{pmatrix} \quad \rightsquigarrow \quad \det\mathbf{\Phi}(t) = t.$$

Upon rewriting the ODE as

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 4\frac{\ln(t)}{t},$$

the particular solution is given by

$$
\begin{aligned}
y_{\mathrm{p}}(t) &= -\left(\int^t \frac{(4\ln(s)/s)s\ln(s)}{s}\,\mathrm{d}s\right)t + \left(\int^t \frac{(4\ln(s)/s)s}{s}\,\mathrm{d}s\right)t\ln(t) \\
&= -\frac{4}{3}[\ln(t)]^3 t + 2[\ln(t)]^2 t\ln(t) = \frac{2}{3}t[\ln(t)]^3.
\end{aligned}
$$

The general solution is the sum of the homogeneous and particular solutions,

$$y(t) = c_1 t + c_2 t\ln(t) + \frac{2}{3}t[\ln(t)]^3.$$

(c) The homogeneous solution is given by

$$y_{\mathrm{h}}(t) = c_1 t + c_2 t^2.$$

The particular solution will be found using variation of parameters. We have

$$\mathbf{\Phi}(t) = \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} \quad \rightsquigarrow \quad \det\mathbf{\Phi}(t) = t^2.$$

Upon rewriting the ODE as

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{t}{1+t^2},$$

the particular solution is given by

$$
\begin{aligned}
y_{\mathrm{p}}(t) &= -\left(\int^t \frac{[s/(1+s^2)]s^2}{s^2}\,\mathrm{d}s\right)t + \left(\int^t \frac{[s/(1+s^2)]s}{s^2}\,\mathrm{d}s\right)t^2 \\
&= -\frac{1}{2}t\ln(1+t^2) + t^2\tan^{-1}(t).
\end{aligned}
$$

The general solution is the sum of the homogeneous and particular solutions,

$$
y(t) = c_1 t + c_2 t^2 - \frac{1}{2}t\ln(1+t^2) + t^2\tan^{-1}(t).
$$