Span, linear combination, and column space

We have seen that any system of linear equations can be restated in other, equivalent ways. That is, the problem

Find all solutions to the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

can be formulated either as the matrix equation

Solve

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{2}$$

or as the vector equation

Solve

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \tag{3}$$

If a solution exists, in any of the formulations (1)–(3), we say the problem is **consistent**.

Under formulation (3), we have employed the terms **linear combination** and **span**. Specifically, if weights $x_1, x_2, \ldots, x_n$ exist that make the two sides of (3) equal, then

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{is a linear combination of} \quad \mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \ldots \quad \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

We are saying the same thing when we use the words, "**b** is **in the span of** the columns of **A**," or more concisely, "**b** is in the **column space** of **A**," a phrase which can be taken to mean $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent.

Linear independence

The one vector which, regardless of the specific contents of $\mathbf{A}$, is guaranteed to be in its column space, is the zero vector $\mathbf{0} \in \mathbb{R}^m$. Indeed, one can take all weights to be zero to produce it.

$$\mathbf{0} \;=\; 0 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \cdots + 0 \cdot \mathbf{a}_n \;=\; \mathbf{A0}.$$

But one wonders if some other vector $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ of weights, *not all zero*, can result in the zero vector:

$$\mathbf{0} \;=\; c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n \;=\; \mathbf{Ac}.$$

If there is such a nonzero vector of weights—that is, if the null space of $\mathbf{A}$ contains a nontrivial $\mathbf{c}$—we say the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are **linearly dependent**. If the only way to write $\mathbf{0}$ as a linear combination of the columns of $\mathbf{A}$ is through taking each weight $c_1 = c_2 = \cdots = c_n = 0$, then the columns of $\mathbf{A}$ are said to be **linearly independent**.

Basis and dimension

Section 1.7.1 describes **vector spaces** as collections of objects endowed with an algebraic structure which plants them in their own self-contained universe. Each of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, etc. fits the definition. The vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

span $\mathbb{R}^3$ (that is, every vector is a linear combination of these), and are linearly independent; we call such a collection a **basis** of $\mathbb{R}^3$. There are other bases one can use in $\mathbb{R}^3$. For instance, it is also true that the collection

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

spans $\mathbb{R}^3$ and is linearly independent. In fact, there are (infinitely) many collections of three vectors from $\mathbb{R}^3$ which both span and are linearly independent. But you could not span all of $\mathbb{R}^3$ with just two vectors (too few), and four vectors from $\mathbb{R}^3$ are always linearly dependent. Any basis for $\mathbb{R}^3$ will always involve three vectors, making the **dimension** of $\mathbb{R}^3$ equal to 3.

Similar considerations lead us to conclude that any linearly independent spanning set of vectors from $\mathbb{R}^2$ contains 2 vectors, so the dimension of $\mathbb{R}^2$ is 2. A linearly independent spanning set of vectors in $\mathbb{R}^4$ contains 4 vectors, so the dimension of $\mathbb{R}^4$ is 4. In general, the dimension of $\mathbb{R}^n$ is n.

Subspaces, rank and nullity

Within $\mathbb{R}^3$, the collection S of vectors consisting of those with *third coordinate zero* form their own self-contained universe in the sense that, if you take a linear combination of vectors from S, you'll get another vector in S. Since the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

lie in S, are linearly independent, and *span* S (every vector in S is a linear combination of these two with weights chosen appropriately), these vectors form a basis of S, making S an object of dimension 2. Since S lies inside $\mathbb{R}^3$, we call it a 2-dimensional **subspace** of $\mathbb{R}^3$.

Two dimensional subspaces of $\mathbb{R}^3$ are quite abundant, when you look for them. In fact, if you take any two linearly independent vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$, their *span*, the collection of destinations you can reach through linear combinations

$$c\mathbf{u} + d\mathbf{v},$$

will form a plane that passes through the origin lying in $\mathbb{R}^3$ and, once again, serves as a self-contained universe of objects. Conversely, if you start with any plane in $\mathbb{R}^3$ containing the origin, any basis (linearly independent spanning set) of that plane will contain exactly two vectors, no more and no less.

The one-dimensional subspaces consist of those vectors spanned by a single nonzero vector, the *lines* which pass through the origin. As with two-dimensional subspaces of $\mathbb{R}^3$, the 1-dimensional subspaces abound. Together, these amount to almost all the subspaces of $\mathbb{R}^3$, though there are two more lurking in plain sight. One is $\mathbb{R}^3$, itself, the only 3-dimensional subspace that the 3-dimensional vector space $\mathbb{R}^3$ can contain. And the other is the **trivial subspace**, the self-contained universe consisting of only the zero-vector itself $\{\mathbf{0}\}$, said to be 0-dimensional.

Naturally, analogous statements can be made within $\mathbb{R}^4$, which has 0-, 1-, 2-, 3- and 4-dimensional subspaces.

Example 1:

Let's consider one subspace of $\mathbb{R}^4$, the column space of the 4-by-5 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 & 1 & 8 \\ 3 & 6 & 2 & 4 & 7 \\ -1 & -2 & 2 & -4 & 3 \\ 2 & 4 & -1 & 5 & 0 \end{bmatrix}.$$

Naturally, the columns of $\mathbf{A}$ span the column space, and they reside in $\mathbb{R}^4$. But they may not all be linearly independent. Indeed, by looking at RREF, we see they are not:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 & 1 & 8 \\ 3 & 6 & 2 & 4 & 7 \\ -1 & -2 & 2 & -4 & 3 \\ 2 & 4 & -1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The 2^{nd}, 4^{th} and 5^{th} columns of $\mathbf{A}$ are each in the spans of preceding columns, which means all the destinations in $\mathbb{R}^4$ made possible through taking linear combinations of the columns of $\mathbf{A}$ are uniquely writeable as linear combinations of just its first and third columns. Thus, the column space of $\mathbf{A}$ is a 2-dimensional subspace of $\mathbb{R}^4$ with basis

$$\begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix}.$$

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Example 2:

Consider now the null space of the matrix $\mathbf{A}$ appearing in the previous example. To find it, we solve $\mathbf{Ax} = \mathbf{0}$, and so deal with the augmented matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 & 1 & 8 & 0 \\ 3 & 6 & 2 & 4 & 7 & 0 \\ -1 & -2 & 2 & -4 & 3 & 0 \\ 2 & 4 & -1 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The final column is for the right-hand side zero vector, so there are five columns associated with variables $x_1, x_2, \ldots, x_5$. Of these, x_2, x_4 and x_5 are *free*. If we give to these the new names

$$x_2 = r, \qquad x_4 = s, \qquad \text{and} \qquad x_5 = t,$$

the first two rows of RREF yield

$$\left.\begin{array}{l} x_1 + 2x_2 + 2x_4 + x_5 = 0 \\ x_3 - x_4 + 2x_5 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = -2r - 2s - t \\ x_3 = s - 2t \end{array}\right.$$

Thus, the null space of $\mathbf{A}$ consists of vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r - 2s - t \\ r \\ s - 2t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \qquad r, s, t \ \text{real nos.},$$

making the collection of three vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

a linearly independent spanning set (i.e., a basis) generating the null space of $\mathbf{A}$.

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One observation that holds not just in these two examples, but holds for all matrices, is that the count of free columns in $\mathbf{A}$, a number known as the **rank** of $\mathbf{A}$, always reveals the dimension of the column space of $\mathbf{A}$. On the other hand, the number of free columns in $\mathbf{A}$, what we call the **nullity** of $\mathbf{A}$, always reveals the dimension of the null space of $\mathbf{A}$.

Some additional questions to ponder

1. Can a single vector make up a linearly independent collection? If so, how?
2. How can you check if two vectors are linearly independent?
3. How do you check whether an arbitrary collection of vectors is linear independent?
4. How do you check whether a vector $\mathbf{b}$ is in the column space of a given matrix $\mathbf{A}$?
5. Say you have a collection of vectors which is linearly dependent. Can you, by adding one or more vectors to this collection, obtain a larger collection which is linearly independent?
6. Say you have a collection of vectors which is linearly independent. Can you, by removing one or more vectors from the collection, obtain a smaller collection that is linearly dependent?
7. What does the span of a collection of two linearly independent vectors in $\mathbb{R}^3$ look like geometrically? How does your answer change if the two vectors are parallel?
8. If a collection of two or more vectors is linearly dependent, must one of the vectors be in the *span* of the others?
9. If $\mathbf{A}$ is m-by-n, which of $\mathbb{R}^m$ or $\mathbb{R}^n$ is the Euclidean space that contains the column space of $\mathbf{A}$? Which one contains the null space of $\mathbf{A}$?
10. An argument was made above that lines and planes which include the zero vector are self-contained universes. Is this not true of all lines and planes? In particular, if you consider a line such as $x + 2y = 1$ which lies in the plane but does not contain the origin, and you take a

vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ whose head lies on this line (when placed in standard position), would scalar multiples of this vector also lie on the line?