A Bit of Set Theory

Notation

set =

- specifying what a collection contains: enumeration, set builder notation

$A = \{0, 1, 3, 5\}$, $A = \{1, 2, 3, 4, \ldots\}$ $A = \{x \mid \underbrace{x^2 > 16}_{\text{criterion}}\}$

- some common sets of numbers: $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$, $[a, b)$

$\mathbb{N}$ integers $[2, 5)$

- "Let $x \in A$."

Let $x \in \mathbb{Z}$ (know x is an integer)

- size (or **cardinality**) of A: $|A|$

$|A|$ = # of elements in set A; $|\mathbb{Z}| = \infty$

$|\{0, 1, 2\}| = 3$

- containment: $A \subset B$ and $A \supset B$

means all elements of A are found in B $A \subseteq B$

$A \subsetneq B$

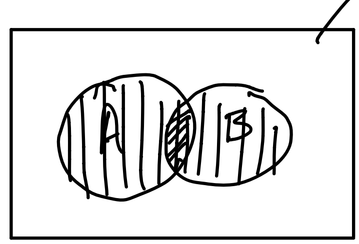
- disjoint sets

A, B disjoint if $\boxed{A \cap B = \emptyset}$

- set operations: $A \cup B$, $A \cap B$, $A \setminus B$, A^c, $A \times B$

Visualizing these using Venn diagrams

S = universe = sample space



$\emptyset$ = empty set = $\{\}$

$A \cap B$ elements found in both A, B

$A \cup B$ elements in A, in B or in both

$A \setminus B$ elements in A but not in B $(A - B)$

$A^c = S \setminus A$

Theorem 1 (DeMorgan's laws): Let A, B be sets living inside the same universal set.

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

$$(A \cup B)^c = A^c \cap B^c$$

Inclusion-Exclusion Principle

Lemma 1: If A, B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

Functions

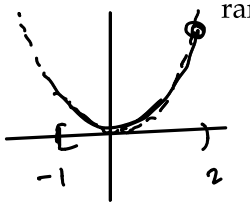
- some terms: domain, codomain, range

- notation: $\boxed{f\colon A \to B}$ Says f is a function with domain A and range a subset of B

  for $x \in A$, $f(x)$ is the image of x under f

  for $S \subset A$, $f(S) = \{f(x) \mid x \in S\}$, called the **image of S under f**

  $\operatorname{ran} f = \{f(x) \mid x \in A\}$



$f(x) = x^2$ Then $f([-1, 2)) = [0, 4)$

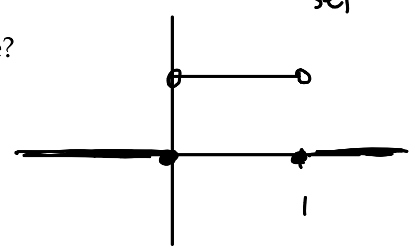
- Given any set A (and implied universal set S) define the **indicator function** or **characteristic function** on A given by

$$\chi_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } S \setminus A = A^c. \end{cases}$$

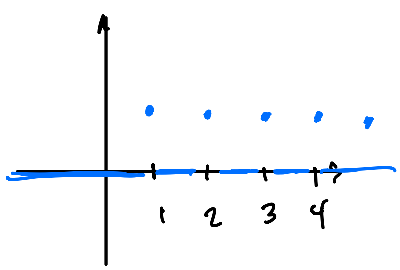
$$\chi_{(0,1)}(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{off the set} \end{cases}$$

Question: What are the domain and codomain of χ_A?

Question: Could what we call the codomain be different than the range?



Graph $\chi_{\mathbb{N}}(x)$



$\chi_{\mathbb{N}}(x) = [\![x \in \mathbb{N}]\!]$

- A **random variable** X **Example 1:**

X : sample space $\longrightarrow \mathbb{R}$

Roll two dice. The sample space

$$S = \{(1,1), (1,2), \ldots, (1,6), (2,1), \ldots, (6,6)\}.$$

Take $X\colon S \to \mathbb{R}$ to be the sum of pips.

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Sums and products

$= a_1 + a_2 + a_3 + \cdots + a_n$

Recall the meanings of

$$\sum_{j=1}^{n} a_j, \quad \text{and} \quad \sum_{j=1}^{\infty} a_j. = a_1 + a_2 + a_3 + \cdots$$

Some useful relationships:

- $\sum_{j=1}^{n} a = na$

$$\sum_{j=1}^{10} 3 = \underbrace{3 + 3 + 3 + \cdots + 3}_{10 \text{ terms}} = 30$$

- $\sum_{j=1}^{n} ba_j = b \sum_{j=1}^{n} a_j$

like $\int c f(x)\, dx = c \int f(x)\, dx$

- $\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$ H.W.
- $\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{j=1}^{n} j^3 = \left[\frac{1}{2}n(n+1)\right]^2$

Notes:

- Recall geometric series
- Product notation: $\prod_{j=1}^{n} a_j$ $= a_1 \cdot a_2 \cdot a_3 \cdots a_n$

$$\prod_{j=1}^{n} e^{a_j} = e^{a_1} \cdot e^{a_2} \cdots e^{a_n} = e^{a_1 + a_2 + \cdots + a_n}$$

$$= e^{\sum_{j=1}^{n} a_j}$$