$\underline{\star 40}$ (a) We know $\mathcal{L}\{t\} = 1/s^2$, so by one of the shift theorems, $\mathcal{L}\{H(t-1)(t-1)\} = \mathcal{L}\left\{H(t-1)\left(t\big|_{t\mapsto t-1}\right)\right\} = e^{-s}/s^2$. Thus,

$$\mathcal{L}\{t - H(t-1)(t-1)\} = \mathcal{L}\{t\} - \mathcal{L}\{H(t-1)(t-1)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} = \frac{1}{s^2}(1 - e^{-s}).$$

(b) Since $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, the same shift theorem (as for part (a)) gives us that

$$\mathcal{L}\left\{H\left(t - \frac{\pi}{4}\right)\cos\left(t - \frac{\pi}{4}\right)\right\} = \frac{se^{-\pi s/4}}{s^2+1}.$$

(c) Let us define $g(t)$ to be the function one obtains by shifting the polynomial $t^2 + 3t - 8$ three units to the left; that is,

$$g(t) = t^2+3t-8\Big|_{t\mapsto t+3} = (t+3)^2+3(t+3)-8 = t^2+6t+9+3t+9-8 = t^2+9t+10.$$

We have

$$\mathcal{L}\{g(t)\} = \mathcal{L}\left\{t^2+9t+10\right\} = \mathcal{L}\left\{t^2\right\} + 9\mathcal{L}\{t\} + 10\mathcal{L}\{1\} = \frac{2}{s^3} + \frac{9}{s^2} + \frac{10}{s},$$

and $t^2 + 3t - 8 = g(t-3)$ (i.e., you get back to the original polynomial by shifting $g(t)$ three units to the right), so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{H(t-3)\,g(t-3)\} = e^{-3s}\left(\frac{2}{s^3} + \frac{9}{s^2} + \frac{10}{s}\right).$$

(d) One may use this as another exercise in applying the shift theorem used in previous parts. In that vein, it is advantageous to rewrite

$$f(t) = [H(t-\pi) - H(t-2\pi)](t-\pi) = H(t-\pi)\left(t\Big|_{t\mapsto t-\pi}\right) - H(t-2\pi)\left(t\Big|_{t\mapsto t-2\pi}\right) - \pi H(t-2\pi).$$

I will take another approach, calculating the Laplace transform directly from the definition:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,dt \;=\; \int_\pi^{2\pi} e^{-st}(t-\pi)\,dt \\
&= \int_\pi^{2\pi} te^{-st}\,dt - \pi\int_\pi^{2\pi} e^{-st}\,dt \\
&= -\frac{1}{s}te^{-st}\Big|_\pi^{2\pi} + \frac{1}{s}\int_\pi^{2\pi} e^{-st}\,dt - \pi\int_\pi^{2\pi} e^{-st}\,dt \\
&= \frac{\pi}{s}\left(e^{-\pi s} - 2e^{-2\pi s}\right) + \left(\frac{1}{s} - \pi\right)\int_\pi^{2\pi} e^{-st}\,dt \\
&= \frac{\pi}{s}\left(e^{-\pi s} - 2e^{-2\pi s}\right) - \frac{1}{s}\left(\frac{1}{s} - \pi\right)\left[e^{-st}\right]_\pi^{2\pi} \\
&= \frac{\pi}{s}\left(e^{-\pi s} - 2e^{-2\pi s}\right) - \frac{1}{s}\left(\frac{1}{s} - \pi\right)\left(e^{-2\pi s} - e^{-\pi s}\right) \\
&= \frac{\pi}{s}e^{-\pi s} - 2\frac{\pi}{s}e^{-2\pi s} - \frac{1}{s^2}e^{-2\pi s} + \frac{1}{s^2}e^{-\pi s} + \frac{\pi}{s}e^{-2\pi s} - \frac{\pi}{s}e^{-\pi s} \\
&= -\frac{\pi}{s}e^{-2\pi s} - \frac{1}{s^2}e^{-2\pi s} + \frac{1}{s^2}e^{-\pi s}.
\end{aligned}$$

Either way this is done, the results should be equivalent.

(e) This problem, having an exponential factor in the time domain, involves the other shift theorem. Since $\mathcal{L}\{\sin(4t)\} = 4/(s^2+16)$, we have

$$\mathcal{L}\left\{e^{3t}\sin(4t)\right\} = \frac{4}{s^2+16}\Big|_{s\mapsto s-3} = \frac{4}{(s-3)^2+16} = \frac{4}{s^2-6s+25}.$$

(f) What we want here is

$$\mathcal{L}\left\{H(t-5)\left(4t^2e^{-2t}\Big|_{t\mapsto t-5}\right)\right\}.$$

In stages, we have

$$\mathcal{L}\left\{4t^2\right\} = 4\mathcal{L}\left\{t^2\right\} = 4\cdot\frac{2!}{s^3} = \frac{8}{s^3},$$

and so

$$\mathcal{L}\left\{4t^2e^{-2t}\right\} = \frac{8}{s^3}\Big|_{s\mapsto s-(-2)} = \frac{8}{(s+2)^3}.$$

Finally, then,

$$\mathcal{L}\left\{H(t-5)\left(4t^2e^{-2t}\Big|_{t\mapsto t-5}\right)\right\} = \frac{8e^{-5s}}{(s+2)^3}.$$

$\star$41 (a) Completing the square, we have

$$F(s) = \frac{2(s-1)}{s^2-2s+1+1} = \frac{2(s-1)}{(s-1)^2+1} = \frac{2s}{s^2+1}\Big|_{s\mapsto s-1}.$$

And, since

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} \;=\; 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \;=\; 2\cos(t),$$

the appropriate shift theorem gives us that

$$\mathcal{L}^{-1}\left\{\left.\frac{2s}{s^2+1}\right|_{s\mapsto s-1}\right\} \;=\; 2e^t\cos(t).$$

(b) The frequency domain function in part (b) is identical, but for the exponential factor e^{-2s}, to the one in part (a). Using the answer to part (a), we then have

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{2(s-1)}{s^2-2s+2}\right\} \;=\; 2H(t-2)\,e^{t-2}\cos(t-2).$$

(c) Employing partial fraction expansion, we have

$$\frac{4}{s^2-4} \;=\; \frac{4}{(s+2)(s-2)} \;=\; \frac{A}{s+2}+\frac{B}{s-2} \;=\; \frac{A(s-2)+B(s+2)}{(s+2)(s-2)} \;=\; \frac{(A+B)s+(-2A+2B)}{s^2-4}.$$

Equating coefficients in the numerators for the various powers of s, we obtain equations $A+B=0$ and $2B-2A=4$, which means $A=-1$ and $B=1$. Thus,

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2-4}\right\} \;=\; \mathcal{L}^{-1}\left\{\frac{1}{s-2}-\frac{1}{s+2}\right\} \;=\; \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} \;=\; e^{2t}-e^{-2t}.$$

(d) First, I find

$$\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^4}\right\} \;=\; \mathcal{L}^{-1}\left\{\left.\frac{4}{6}\cdot\frac{3!}{s^4}\right|_{s\mapsto s-2}\right\} \;=\; \frac{2}{3}\mathcal{L}^{-1}\left\{\left.\frac{3!}{s^4}\right|_{s\mapsto s-2}\right\} \;=\; \frac{2}{3}t^3e^{2t}.$$

Next, I consider the other term absent its exponential factor:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} \;&=\; \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} \;=\; \mathcal{L}^{-1}\left\{\frac{1/3}{s-1}-\frac{1/3}{s+2}\right\} \quad \text{(by partial fractions)}\\ &=\; \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} \;=\; \frac{1}{3}\left(e^t-e^{-2t}\right).\end{aligned}$$

Thus, using the appropriate shift theorem, we have

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+s-2}\right\} \;=\; \frac{1}{3}H(t-2)\left(e^{t-2}-e^{-2(t-2)}\right).$$

Putting these results together, we have

$$\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^4}+e^{-2s}\frac{1}{s^2+s-2}\right\} \;=\; \frac{2}{3}t^3e^{2t}+\frac{1}{3}H(t-2)\left(e^{t-2}-e^{-2(t-2)}\right).$$

(e) Since $\mathcal{L}^{-1}\left\{e^{-as}/s\right\} = H(t-a)$, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}\right\} &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\} \\ &= H(t-1)+H(t-2)-H(t-3)-H(t-4).\end{aligned}$$

(f) The denominator is reducible, so we employ partial fraction expansion to obtain

$$\frac{s-2}{s^2-4s+3} = \frac{1/2}{s-1}+\frac{1/2}{s-3}.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = \frac{1}{2}\left(e^t+e^{3t}\right).$$