$\star$42 (b) $\mathcal{L}[f](s) = 7\dfrac{2}{(s-5)^3} - 12e^{-4s}\dfrac{1}{s}$.

(e) Writing

$$f(t) = -5\delta(t-2) + e^6 e^{3(t-2)} H(t-2) \quad \rightsquigarrow \quad \mathcal{L}[f](s) = -5e^{-2s} + e^6 e^{-2s}\frac{1}{s-3}.$$

(f) Writing

$$f(t) = \left[(t-1)^2 + 2(t-1) - 2\right] H(t-1) + 3t^3 \quad \rightsquigarrow \quad \mathcal{L}[f](s) = e^{-s}\left[\frac{2}{s^3} + 2\frac{1}{s^2} - 2\frac{1}{s}\right] + 3\frac{6}{s^4}.$$

(g) Writing

$$\begin{aligned} f(t) &= 4\sin(t)\,[1 - H(t-2\pi)] + 3(t-2\pi)\,[H(t-2\pi) - H(t-2\pi-4)] + 12H(t-2\pi-4) \\ &= 4\sin(t) + [3(t-2\pi) - 4\sin(t-2\pi)]\,H(t-2\pi) - 3(t-2\pi-4)H(t-2\pi-4). \end{aligned}$$

yields

$$\mathcal{L}[f](s) = 4\frac{1}{s^2+1} + e^{-2\pi s}\left[3\frac{1}{s^2} - 4\frac{1}{s^2+1}\right] - 3e^{-(2\pi+4)s}\frac{1}{s^2}.$$

$\star$43 (a) We have

$$\frac{3s+7}{s^2+s-2} = \frac{10}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2} \quad \rightsquigarrow \quad f(t) = \frac{10}{3}e^{t} - \frac{1}{3}e^{-2t}.$$

(b) Here

$$\frac{6s-5}{s^2+6s+13} = 6\frac{s+3}{(s+3)^2+4} - \frac{23}{2}\frac{2}{(s+3)^2+4} \quad \rightsquigarrow \quad f(t) = 6e^{-3t}\cos(2t) - \frac{23}{2}e^{-3t}\sin(2t).$$

(f) Since

$$\frac{5s^2-7}{(s+4)(s-1)^2} = \frac{73}{25}\frac{1}{s+4} + \frac{52}{25}\frac{1}{s-1} - \frac{2}{5}\frac{1}{(s-1)^2},$$

we have

$$f(t) = \left[\frac{73}{25}e^{-4(t-4)} + \frac{52}{25}e^{t-4} - \frac{2}{5}(t-4)e^{t-4}\right] H(t-4).$$

$\star$44 (a) The characteristic equation,

$$0 \;=\; \lambda^2 + 6\lambda + 5 \;=\; (\lambda+5)(\lambda+1) \qquad \text{has roots} \qquad \lambda = -1,\ -5,$$

so the homogeneous solution is

$$y_h(t) \;=\; c_1 e^{-5t} + c_2 e^{-t}.$$

Employing the method of undetermined coefficients, the form of the nonhomogeneous term $f(t) = 7e^{2t}$ is that of a constant times an exponential. We propose a similar form for a particular solution, noting that no similar term appears in the homogeneous solution:

$$y_p(t) = Ae^{2t} \qquad \Rightarrow \qquad y_p' = 2Ae^{2t} \text{ and } y_p'' = 4Ae^{2t}.$$

Inserting this into the left-hand side of the DE, we have

$$y_p'' + 6y_p' + 5y_p \;=\; 4Ae^{2t} + 6(2Ae^{2t}) + 5Ae^{2t} \;=\; 21Ae^{2t}.$$

This is the same as the target function $7e^{2t}$ on the right-hand side, so long as we take $A = 1/3$. Putting $y_p(t) = (1/3)e^{2t}$ together with y_h gives general solution

$$y(t) \;=\; c_1e^{-5t} + c_2e^{-t} + \frac{1}{3}\,e^{2t},$$

which has derivative

$$y'(t) \;=\; -5c_1e^{-5t} - c_2e^{-t} + \frac{2}{3}\,e^{2t}.$$

Now, we apply the ICs:

$$\left.\begin{array}{ll} y(0) = \frac{4}{3}: & c_1 + c_2 + \frac{1}{3} = \frac{4}{3} \\ y'(0) = -3: & -5c_1 - c_2 + \frac{2}{3} = -3 \end{array}\right\} \quad \Rightarrow \quad c_1 = \frac{2}{3},\; c_2 = \frac{1}{3}.$$

Thus, the solution of the IVP is

$$y(t) \;=\; \frac{2}{3}\,e^{-5t} + \frac{1}{3}\,e^{-t} + \frac{1}{3}\,e^{2t},$$

(b) Here, our homogeneous solution (found above) *is* the general solution

$$y(t) \;=\; c_1e^{-5t} + c_2e^{-t}, \qquad \text{and has derivative} \qquad y'(t) \;=\; -5c_1e^{-5t} - c_2e^{-t}.$$

Applying the ICs:

$$\left.\begin{array}{ll} y(0) = \frac{4}{3}: & c_1 + c_2 = \frac{4}{3} \\ y'(0) = -3: & -5c_1 - c_2 = -3 \end{array}\right\} \quad \Rightarrow \quad c_1 = \frac{5}{12},\; c_2 = \frac{11}{12}.$$

Thus, the solution of the IVP is

$$y(t) \;=\; \frac{5}{12}\,e^{-5t} + \frac{11}{12}\,e^{-t}.$$

(c) Taking Laplace transforms of both sides, we have

$$s^2\mathcal{L}[y_{\mathrm{p}}](s) - sy(0) - y'(0) + 6\left[s\mathcal{L}[y_{\mathrm{p}}](s) - y(0)\right] + 5\mathcal{L}[y_{\mathrm{p}}](s) = \frac{7}{s-2},$$

or, after inserting the 0 initial data and writing $Y = \mathcal{L}[y_\mathrm{p}](s)$,

$$(s^2{+}6s{+}5)Y \;=\; \frac{7}{s-2}, \qquad \text{which we solve to get} \qquad Y(s) \;=\; \frac{7}{(s-2)(s+5)(s+1)}.$$

In order to take the inverse Laplace transform, we employ partial fractions

$$\frac{7}{(s-2)(s^2+6s+5)} \;=\; \frac{A}{s-2}+\frac{B}{s+5}+\frac{C}{s+1}, \quad \text{which holds if} \quad A=\frac{1}{3},\ B=\frac{1}{4},\ C=-\frac{7}{12}.$$

So, the particular solution of this nonhomogeneous DE which satisfies zero initial conditions is

$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} - \frac{7}{12}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \;=\; \frac{1}{3}e^{2t} + \frac{1}{4}e^{-5t} - \frac{7}{12}e^{-t}.$$

(d) The strategy worked in the combination of parts (b) and (c), since the sum of those solutions

$$\left(\frac{5}{12}e^{-5t} + \frac{11}{12}e^{-t}\right) + \left(\frac{1}{3}e^{2t} + \frac{1}{4}e^{-5t} - \frac{7}{12}e^{-t}\right) \;=\; \frac{1}{3}e^{2t} + \frac{2}{3}e^{-5t} + \frac{1}{3}e^{-t},$$

the same as our answer to (a).

Speaking generally, it will work whenever your method of finding a particular solution results in *the one that satisfies zero initial conditions*, something that the method of undetermined coefficients seldom does.

(e) It is the one term that does not decay exponentially, $(1/3)e^{2t}$.

★45 (a) Upon taking the Laplace transform we have

$$(s^2{+}7s{+}12)\mathcal{L}[y_\mathrm{p}](s) = \frac{3}{s+2}, \qquad \text{or, writing } Y = \mathcal{L}[y_\mathrm{p}](s), \qquad (s^2{+}7s{+}12)Y = \frac{3}{s+2},$$

so that after some algebraic manipulation

$$Y = \frac{3}{2}\frac{1}{s+2} - 3\frac{1}{s+3} + \frac{3}{2}\frac{1}{s+4}.$$

The particular solution is then

$$y_\mathrm{p}(t) = \frac{3}{2}\mathrm{e}^{-2t} - 3\mathrm{e}^{-3t} + \frac{3}{2}\mathrm{e}^{-4t}.$$

Now, to address the IVP in problem 2, the homogeneous problem is

$$y'' + 7y' + 12y = 0.$$

Since the characteristic equation is

$$0 = \lambda^2 + 7\lambda + 12 = (\lambda+3)(\lambda+4),$$

the homogeneous solution is

$$y_{\mathrm{h}}(t) = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-4t}.$$

Having found the particular solution above, the general solution is

$$\begin{aligned} y(t) &= c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-4t} + \frac{3}{2}\mathrm{e}^{-2t} - 3\mathrm{e}^{-3t} + \frac{3}{2}\mathrm{e}^{-4t} \\ &= (c_1 - 3)\,\mathrm{e}^{-3t} + \left(c_2 + \frac{3}{2}\right)\mathrm{e}^{-4t} + \frac{3}{2}\mathrm{e}^{-2t}. \end{aligned}$$

As for the IVP, the initial data leads to the linear system

$$1 = y(0) = c_1 + c_2, \;\; 3 = y'(0) = -3c_1 - 4c_2 \quad \rightsquigarrow \quad c_1 = 7, \; c_2 = -6.$$

In conclusion, the solution to the IVP is

$$\begin{aligned} y(t) &= 7\mathrm{e}^{-3t} - 6\mathrm{e}^{-4t} + \frac{3}{2}\mathrm{e}^{-2t} - 3\mathrm{e}^{-3t} + \frac{3}{2}\mathrm{e}^{-4t} \\ &= 4\mathrm{e}^{-3t} - \frac{9}{2}\mathrm{e}^{-4t} + \frac{3}{2}\mathrm{e}^{-2t}. \end{aligned}$$

(b) For the particular solution, we apply the Laplace transform to the differential equation, assuming zero initial data. This leads to

$$(s^2 + 8s + 20)\mathcal{L}[y_{\mathrm{p}}](s) = 4\mathrm{e}^{-3s} - 12\mathrm{e}^{-5s},$$

so that after some algebraic manipulation

$$Y(s) := \mathcal{L}[y_{\mathrm{p}}](s) = 2\mathrm{e}^{-3s}\frac{2}{(s+4)^2 + 4} - 6\mathrm{e}^{-5s}\frac{2}{(s+4)^2 + 4}.$$

The particular solution is then

$$y_{\mathrm{p}}(t) = 2\mathrm{e}^{-4(t-3)}\sin(2(t-3))H(t-3) - 6\mathrm{e}^{-4(t-5)}\sin(2(t-5))H(t-5),$$

Now consider the homogeneous problem

$$y'' + 8y' + 20y = 0.$$

Since the characteristic equation is

$$0 = \lambda^2 + 8\lambda + 20 = (\lambda + 4)^2 + 4,$$

the homogeneous solution is

$$y_{\mathrm{h}}(t) = c_1 \mathrm{e}^{-4t}\cos(2t) + c_2 \mathrm{e}^{-4t}\sin(2t).$$

Combining with the particular solution above, the general solution is

$$y(t) = c_1 e^{-4t}\cos(2t) + c_2 e^{-4t}\sin(2t) + 2e^{-4(t-3)}\sin(2(t-3))H(t-3) - 6e^{-4(t-5)}\sin(2(t-5))H(t-5).$$

As for the IVP, the initial data leads to the linear system

$$-2 = y(0) = c_1,\ 6 = y'(0) = -4c_1 + 2c_2 \quad \rightsquigarrow \quad c_1 = -2,\ c_2 = -1.$$

In conclusion, the solution to the IVP is

$$y(t) = -2e^{-4t}\cos(2t) - e^{-4t}\sin(2t) + 2e^{-4(t-3)}\sin(2(t-3))H(t-3) - 6e^{-4(t-5)}\sin(2(t-5))H(t-5).$$

★46 (a) The characteristic polynomial is

$$p(\lambda) = \lambda^2 + 10\lambda + 25 = (\lambda + 5)^2,$$

which has negative (real) roots. So the solution to the homogeneous problem is transient. The transfer function is

$$H(s) = \frac{1}{s^2 + 10s + 25} = \frac{1}{(s+5)^2}.$$

The corresponding **impulse response** function

$$h(t) = te^{-5t},$$

and the particular solution in problem 2 is

$$y_p(t) = (h * f)(t) = \int_0^t (t-u)e^{-5(t-u)} f(u)\,du.$$

(b) The characteristic polynomial is

$$p(\lambda) = \lambda^2 + 8\lambda + 17 = (\lambda + 4)^2 + 1,$$

with roots $(-4) \pm i$. Corresponding to these roots, homogeneous solutions $e^{-4t}\cos t$ and $e^{-4t}\sin t$ are transient, so the solution to the homogeneous problem is transient. The transfer function is

$$H(s) = \frac{1}{s^2 + 8s + 17} = \frac{1}{(s+4)^2 + 1}.$$

This gives rise to impulse response

$$h(t) = e^{-4t}\sin(t),$$

and particular solution

$$y_p(t) = (h * f)(t) = \int_0^t e^{-4(t-u)}\sin(t-u)f(u)\,du.$$