MATH 162: Calculus II
Framework for Wed., Feb. 21
Differentiation and Integration of Power Series

While power series are allowed to have nonzero numbers as centers, for today's results we will assume all power series we discuss are centered about $x = 0$; that is, are of the form

$$\sum_{n=0}^{\infty} c_n x^n \;=\; c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots . \tag{1}$$

We will also assume each has a positive radius of convergence $R > 0$, so that the series converges at least for those x satisfying $-R < x < R$.

Differentiation of Power Series about $x = 0$

Theorem (Term-by-Term Differentiation, p. 549**)**: Let $f(x)$ take the form of the power series in (1), with radius of convergence $R > 0$. Then the series

$$\sum_{n=1}^{\infty} n c_n x^{n-1}$$

converges for all x satisfying $-R < x < R$, and

$$f'(x) \;=\; \sum_{n=1}^{\infty} n c_n x^{n-1}, \qquad \text{for all } x \text{ satisfying } -R < x < R.$$

Remarks:

- Since the hypotheses of this theorem now apply to $f'(x)$, we can continue to differentiate the series to find derivatives of f of all orders, convergent at least on the interval $-R < x < R$. For instance,

$$\begin{aligned}
f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \;=\; (2\cdot 1)c_2 + (3\cdot 2)c_3 x + (4\cdot 3)c_4 x^2 + \cdots \\
f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2) c_n x^{n-3} \;=\; (3\cdot 2\cdot 1)c_3 + (4\cdot 3\cdot 2)c_4 x + (5\cdot 4\cdot 3)c_5 x^2 + \cdots \\
&\;\;\vdots \qquad\qquad\qquad \vdots \\
f^{(j)}(x) &= \sum_{n=j}^{\infty} n(n-1)(n-2)\cdots(n-j+1) c_n x^{n-j} \\
&= j!\, c_j + [(j+1)j\cdots 2]\, c_{j+1} x + [(j+2)(j+1)\cdots 3]\, c_{j+2} x^2 + \cdots .
\end{aligned}$$

- The easiest place to evaluate a power series is at its center. In particular, if f has the form (1), we may evaluate f and all of its derivatives at zero to get:

$$\begin{aligned}
f(0) &= c_0, \\
f'(0) &= c_1, \\
f''(0) &= (2\cdot 1)\, c_2 \qquad \Rightarrow \quad c_2 = \frac{1}{2} f''(0), \\
f'''(0) &= (3\cdot 2\cdot 1)\, c_3 \quad \Rightarrow \quad c_3 = \frac{1}{3!} f'''(0),
\end{aligned}$$

and so on. So, we arrive at the following relationship between the coefficients c_j and the derivatives of f at the center:

Corollary: Let $f(x)$ be defined by the power series (1). Then

$$c_n = \frac{f^{(n)}(0)}{n!}, \qquad \text{for all } n = 0, 1, 2, \ldots.$$

Example: We know that $f(x) = (1-x)^{-1}$ has the power series representation $\sum_{n=0}^{\infty} x^n$ for x in the interval $(-1, 1)$. By the term-by-term differentiation theorem, $f'(x) = (1-x)^{-2}$ has the power series representation

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right) \\
&= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \cdots \qquad \text{(the theorem justifies this step)} \\
&= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \\
&= \sum_{n=1}^{\infty} n x^{n-1},
\end{aligned}$$

with this series representation holding at least for $-1 < x < 1$.

Integration of Power Series about $x = 0$

If we can differentiate a series expression for f term-by-term in order to arrive at a series expression for f', it may not be surprising that we may integrate a series term-by-term as well.

Theorem (Term-by-Term Integration, p. 550**)**: Suppose that $f(x)$ is defined by the power series (1) and the radius of convergence $R > 0$. Then

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}, \qquad \text{for all } x \text{ satisfying } -R < x < R.$$

Example: We know that $f(x) = (1+x)^{-1}$ has the power series representation $\sum_{n=0}^{\infty}(-1)^n x^n$ for x in the interval $(-1, 1)$. Moreover,

$$\int_0^x \frac{dt}{1+t} \;=\; \Big[\ln|1+t|\Big]_0^x \;=\; \ln|1+x|.$$

By the term-by-term integration theorem, for $-1 < x < 1$ we also have

$$\begin{aligned}
\int_0^x \frac{dt}{1+t} \;&=\; \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n t^n\right) dt \;=\; \int_0^x (1 - t + t^2 - t^3 + t^4 - t^5 + \cdots)\, dt \\
&=\; \int_0^x dt - \int_0^x t\,dt + \int_0^x t^2\,dt - \int_0^x t^3\,dt + \cdots \qquad \text{(the theorem justifies this step)} \\
&=\; \sum_{n=0}^{\infty}(-1)^n \left(\int_0^x t^n\,dt\right) \\
&=\; \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\Big[t^{n+1}\Big]_0^x \\
&=\; \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \\
&=\; x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\,.
\end{aligned}$$

That is,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \;=\; \ln(1+x),$$

at least for all x in the interval $-1 < x < 1$. In fact, though the theorem does not go so far as to guarantee convergence at the value $x = 1$, since the series on the left converges at $x = 1$ (Why?), one might suspect that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \;=\; \ln 2.$$

This, indeed, is the case.

Example: Use the power series representation for $(1+x^2)^{-1}$ about zero to get a power series representation for $\arctan x$.