MATH 162: Calculus II

Framework for Mon., Apr. 23

Double Integrals in Polar Coordinates

Today's Goal: To learn the mechanics of setting up double integrals in polar coordinates, and learn to recognize situations in double integrals that may be easier in polar form.

Important Note: In conjunction with this framework, you should look over Section 13.4 of your text.

Polar Rectangles

While the specific integrand $f(x, y)$ plays a large role in how difficult it is to *evaluate* a double integral $\iint_R f(x, y)\, dA$, it is the region R *alone* that determines what limits of integration one uses in formulating an iterated integral. When R is the rectangular region $R:\ a \leq x \leq b,\ c \leq y \leq d$, setting up an iterated integral is quite easy (the limits are simply the bounding x and y-values for the rectangle).

Correspondingly, if our region R is a *polar rectangle*

$$a \leq r \leq b,\ \alpha \leq \theta \leq \beta,$$

then it will be easy to find limits of integration for an iterated integral in polar coordinates.

Some examples of polar rectangles:

- $0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$
- $1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi$
- $1 \leq r \leq 2,\ \dfrac{\pi}{4} \leq \theta \leq \pi$

Double Integrals in Polar Coordinates over a Polar Rectangle $R:$ $r_1 \leq r \leq r_2,\ \alpha \leq \theta \leq \beta$

- If the integrand is $f(x, y)$ (i.e., if it is given in terms of rectangular coordinates), one must find the appropriate expression in polar coordinates by substituting $r\cos\theta$ for x, $r\sin\theta$ for y.
- The dA in $\iint_R f(x, y)\, dA$ becomes $dx\, dy$ or $dy\, dx$ when written as an iterated integral in rectangular form.

  In polar form, $dA = r\, dr\, d\theta$ because of the need for the *area expansion factor* r.

Example: Compute the volume under the hemisphere $z = \sqrt{1 - x^2 - y^2}$ above the polar rectangle $R:\ 0 \leq r \leq 1/2,\ 0 \leq \theta \leq \pi$ in the plane.

Bounded Regions

Our regions of integration for double integrals are not always rectangles (neither in the usual sense, nor in the polar sense). Often the region R of integration for a double integral $\iint_R f(x,y)\,dA$ is described as "the region bounded by the curves ... ." In such instances, one step in setting up an iterated integral involves finding points where the curves intersect.

Example: Find the area of the region outside the circle $r = 2$ and inside the circle $r = 4\sin\theta$.

More Examples

1. Find the volume of the region bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$.
2. Evaluate the iterated integral $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2\,dx\,dy$.