1. This A has RREF

$$\begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Its rank is 2, and columns 1 and 3 are pivot columns.

(a) basis for col(A):

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

(b) basis for row(A):

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

(c) basis for null(A) (see below):

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} x_2 &= r \\ x_4 &= s \\ x_5 &= t \end{aligned}\right\} \text{free} \qquad \begin{aligned} x_1 &= 2r - 2s - 3t \\ x_3 &= s - t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

2. (a) $$\overrightarrow{PQ} = \begin{bmatrix} 4-1 \\ 1-3 \\ -4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}.$$

(b) $$\text{dist}(Q, P) = \|\overrightarrow{PQ}\| = \sqrt{3^2 + (-2)^2 + 3^2} = \sqrt{22}.$$

(c) Our desired unit vector is $$\frac{1}{\|\overrightarrow{PQ}\|}\overrightarrow{PQ} = \frac{1}{\sqrt{22}}\begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}.$$

(d) The line can be expressed as

$$\overrightarrow{OP} + t \cdot \overrightarrow{PQ} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}, \quad t \in \mathbb{R}, \qquad \text{or} \qquad \left.\begin{aligned} x &= 1 + 3t \\ y &= 3 - 2t \\ z &= -1 - 3t \end{aligned}\right\} t \in \mathbb{R}.$$

(e) For each point (x, y, z) in the plane,

$$0 = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} \circ \begin{bmatrix} x-1 \\ y-3 \\ z+1 \end{bmatrix} = 3x - 3 - 2y + 6 - 3z - 3$$

$$\Rightarrow \quad 3x - 2y - 3z = 0.$$

(f) The plane in (e) contains $(0,0,0)$, and is a subspace of $\mathbb{R}^3$. All the vectors in this plane are orthogonal to $\overrightarrow{PQ}$. It is true that this plane is $\{\overrightarrow{PQ}\}^\perp$.

3. (a) This is a basis, since the matrix

$$B = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

has RREF equal to I. And though $\|\vec{u}_1\| = \|\vec{u}_2\| = \|\vec{u}_3\| = \|\vec{u}_4\| = 1$, and the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_4$ are mutually orthogonal, $\vec{u}_3$ is not orthogonal to any of the others. So, it is not an o.n. basis.

(b) The matrix A for which $T(\vec{x}) = A\vec{x}$ is found as the product

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & -2 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 2 & 1 \\ 2 & -1 & 0 & -1 \end{bmatrix} \cdot B^{-1} = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & -2 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 2 & 1 \\ 2 & -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -4 & 1 \\ -3 & 2 & -3 & 4 \\ 4 & -2 & 2 & -6 \\ 3 & -1 & 1 & -1 \\ -1 & -2 & 1 & -2 \end{bmatrix}.$$

(c) Since $\{\vec{u}_1, \vec{u}_2\}$ make an orthogonal basis of W,

$$\begin{aligned} \text{proj}_W \vec{b} &= \text{proj}_{\vec{u}_1} \vec{b} + \text{proj}_{\vec{u}_2} \vec{b} \\ &= (\vec{b} \cdot \vec{u}_1)\, \vec{u}_1 + (\vec{b} \cdot \vec{u}_2)\, \vec{u}_2 \qquad \text{(simpler formula since } \|\vec{u}_1\| = \|\vec{u}_2\| = 1) \\ &= \vec{u}_1 + \vec{u}_2 \\ &= \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}. \end{aligned}$$

4. <u>One approach</u>: Find the columns in A by finding what A does to $\hat{\imath}$ and $\hat{\jmath}$.

$$\text{proj}_{\langle 2,-1\rangle} \langle 1,0\rangle = \frac{\langle 1,0\rangle \cdot \langle 2,-1\rangle}{\|\langle 2,-1\rangle\|^2} \langle 2,-1\rangle = \frac{2}{5}\langle 2,-1\rangle = \left\langle \frac{4}{5}, \frac{-2}{5} \right\rangle$$

$$\text{proj}_{\langle 2,-1\rangle} \langle 0,1\rangle = \frac{\langle 0,1\rangle \cdot \langle 2,-1\rangle}{\|\langle 2,-1\rangle\|^2} \langle 2,-1\rangle = \frac{-1}{5}\langle 2,-1\rangle = \left\langle \frac{-2}{5}, \frac{1}{5} \right\rangle$$

So, $$A = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

<u>Another approach</u>:

(a) Rotate the vector $\langle 2,-1\rangle$ counterclockwise so it faces "east": $\begin{bmatrix} \cos(0.4636) & -\sin(0.4636) \\ \sin(0.4636) & \cos(0.4636) \end{bmatrix}$

(b) Wipe out the 2^{nd} component (i.e., project onto the x-axis): $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) Rotate back: $\begin{bmatrix} \cos(0.4636) & \sin(0.4636) \\ -\sin(0.4636) & \cos(0.4636) \end{bmatrix}$

$$\Longrightarrow A = \begin{bmatrix} \cos(0.4636) & \sin(0.4636) \\ -\sin(0.4636) & \cos(0.4636) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(0.4636) & -\sin(0.4636) \\ \sin(0.4636) & \cos(0.4636) \end{bmatrix}$$

$$= \cdots = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$