Copy A

1. $y' = f(t, y)$ with $f(t,y) = t^2 + \sqrt{y}$. From the IC, $t_0 = 1$, $y_0 = 3$.

$$y_1 = y_0 + hf(t_0, y_0) = 3 + (0.25)(1^2 + \sqrt{3}) \doteq 3.6830$$

$$t_1 = t_0 + h = 1.25$$

$$y_2 = y_1 + hf(t_1, y_1) = 3.6830 + (0.25)(1.25^2 + \sqrt{3.6830}) \doteq 4.5534$$

$$t_2 = t_1 + h = 1.5$$

$$y_3 = y_2 + hf(t_2, y_2) = 4.5534 + (0.25)(1.5^2 + \sqrt{4.5534}) \doteq 5.6494$$

$$t_3 = t_2 + h = 1.75$$

$$y_4 = y_3 + hf(t_3, y_3) = 5.6494 + (0.25)(1.75^2 + \sqrt{5.6494}) \doteq 7.0092$$

$$t_4 = t_3 + h = 2.0$$

$y(2) \approx 7.0092$

2. (a) $\alpha = 2$, $\beta = 3$, $\vec{u} = \langle 7, -1 \rangle$, $\vec{w} = \langle 0, -2 \rangle$. So the general soln. is

$$\vec{x}(t) = c_1 e^{2t}\left(\cos(3t)\begin{bmatrix} 7 \\ -1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin(3t)\begin{bmatrix} 7 \\ -1 \end{bmatrix} + \cos(3t)\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 7e^{2t}\cos(3t) & 7e^{2t}\sin(3t) \\ e^{2t}[2\sin(3t) - \cos(3t)] & -e^{2t}[2\cos(3t) + \sin(3t)] \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(b) It's easiest to learn the e-values through the relation $A\vec{v} = \lambda\vec{v}$:

$$\begin{bmatrix} 4 & 4 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} = -2\begin{bmatrix} 2 \\ -3 \end{bmatrix} \Rightarrow \lambda = -2 \text{ for e-vector } \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \lambda = 8 \text{ for e-vector } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, $$\vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 e^{8t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-2t} & e^{8t} \\ -3e^{-2t} & e^{8t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

3. (a) Since $\alpha = 2 > 0$, solutions are origin-fleeing (the origin is unstable). Since the e-values are nonreal (w/ $\alpha \neq 0$), the origin is a spiral point. Since $a_{21} = \frac{15}{14} > 0$, trajectories spiral counterclockwise. These lead to Figure A.

(b) Since the eigenvalues are real but of opposite sign, the origin is an (unstable) saddle. The straight-line trajectories are in the directions of the eigenvectors. See Figure B.

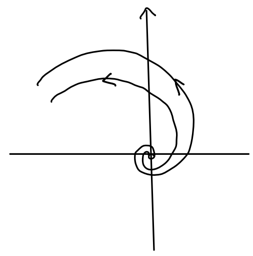


Figure A

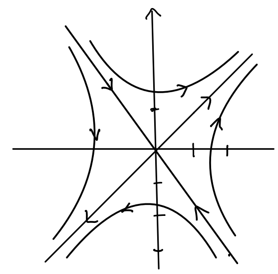


Figure B

4. (a) This DE is first-order linear. It's normal form is

$$y' = -\frac{2}{t}y + \frac{\sin t}{t^2}, \quad \text{making } a(t) = -\frac{2}{t}, \quad f(t) = \frac{\sin t}{t^2}$$

So, the homogeneous soln. is $x_h(t) = C\varphi(t)$, where

$$\varphi(t) = e^{\int -2t^{-1}dt} = e^{-2\ln|t|} = e^{\ln t^{-2}} = t^{-2}.$$

Using variation of parameters,

$$x_p(t) = \varphi(t)\int \frac{f(t)}{\varphi(t)}\,dt = t^{-2}\int \sin t\,dt = -t^{-2}\cos t$$

The soln.:

$$x(t) = x_h(t) + x_p(t) = Ct^{-2} - \frac{\cos t}{t^2}.$$

(b) This DE is nonlinear, but separable.

$$\frac{t^3+t}{4}y^{-3}$$

$$\frac{dy}{dt} = \frac{t^3+t}{4y^3} \quad \Rightarrow \quad \int 4y^3\,dy = \int (t^3+t)\,dt$$

$$\Rightarrow \quad y^4 = \frac{1}{4}t^4 + \frac{1}{2}t^2 + C$$

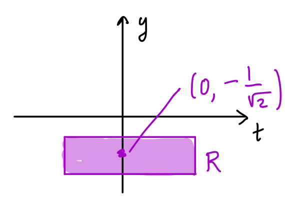
Explicit expressions for y might be either $y(t) = \pm\sqrt[4]{\frac{1}{4}t^4 + \frac{1}{2}t^2 + C}$.

But, for the IC to be satisfied, we require the negative 4th root, and $C = \frac{1}{4}$:

$$x(t) = -\sqrt[4]{\frac{1}{4}t^4 + \frac{1}{2}t^2 + \frac{1}{4}}.$$

5. Here $y' = g(t,y)$, with $g(t,y) = \dfrac{t^3+t}{4y^3}$.

The partial derivative $\dfrac{\partial g}{\partial y} = \dfrac{-3(t^3+t)}{4y^4}$



Both g and $\partial g/\partial y$ are continuous except at $y=0$, so we can draw a box/rectangle R around the point $(t_0, y_0) = (0, -1/\sqrt{2})$ throughout which both g, $\partial g/\partial y$ are continuous. By the Fundamental Theorem on Existence and Uniqueness, the IVP in 4(b) has exactly one solution.

6. Letting $x_1 = y$, $x_2 = y'$, $x_3 = y''$, we have

$x_1' = x_2$ and $x_2' = x_3$ naturally from our definitions, and

$y''' = -2ty'' + 3y' + 4y + \cos(2t)$ becomes $x_3' = -2t x_3 + 3x_2 + 4x_1 + \cos(2t)$.

So,

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 4x_1 + 3x_2 - 2t x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 3 & -2t \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \cos(2t) \end{bmatrix}.$$

The IC becomes

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$