1. $(\mathbf{A}+\mathbf{B})^2$ and $(\mathbf{B}+\mathbf{A})^2$ are equal because matrix addition is commutative (making $\mathbf{A}+\mathbf{B}$ the same as $\mathbf{B}+\mathbf{A}$). Since we are squaring equal things, the results are the same.

   $(\mathbf{A}+\mathbf{B})^2$ and $(\mathbf{A}+\mathbf{B})(\mathbf{B}+\mathbf{A})$ are equal because the two factors in the right-hand expression are equivalent ways to write the same thing (commutativity) of addition).

   $(\mathbf{A}+\mathbf{B})^2$, $\mathbf{A}(\mathbf{A}+\mathbf{B})+\mathbf{B}(\mathbf{A}+\mathbf{B})$, and $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$ are equal since

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^2 &= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) \\ &= \mathbf{A}(\mathbf{A}+\mathbf{B})+\mathbf{B}(\mathbf{A}+\mathbf{B}) && \text{(distributive law)} \\ &= \mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2 && \text{(distributive law again).}\end{aligned}$$

   But $\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$ is different from the others. Comparing it with $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$, we see that they would be equal only if $\mathbf{AB}=\mathbf{BA}$, which is rarely true.

2. (a) Here

$$\mathbf{AB}_1 = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \qquad \text{while} \qquad \mathbf{B}_1\mathbf{A} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

   So, if $\mathbf{AB}_1=\mathbf{B}_1\mathbf{A}$, then $b=0=c$. Similarly,

$$\mathbf{AB}_2 = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \qquad \text{while} \qquad \mathbf{B}_2\mathbf{A} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}.$$

   So, if $\mathbf{AB}_2=\mathbf{B}_2\mathbf{A}$, then $a=d$. Put together, this means

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = a\mathbf{I}.$$

   (b) The matrix $\mathbf{A}=a\mathbf{I}$ from part (a) will, indeed, commute with all other 2-by-2 matrices, since

$$(a\mathbf{I})\mathbf{B} = a(\mathbf{IB}) = a\mathbf{B} = a(\mathbf{BI}) = \mathbf{B}(a\mathbf{I}).$$

   (c) One can argue, as in part (b), that an n-by-n matrix $\mathbf{A}=a\mathbf{I}_n$ (i.e., a scalar multiple of the n-by-n identity matrix) will commute with any other n-by-n matrix $\mathbf{B}$. Part (a) might make one further suspect that if there are any nonzero entries in $\mathbf{A}$ *off* the main diagonal, then it will *not* commute with all $\mathbf{B}$.

   Note to grader: If a student demonstrates that $A=a\mathbf{I}_3$ commutes with another 3-by-3 matrix (providing only one or two instances to show it), take that as adequate response for the problem. Nicer: If a similar conjecture to mine about when $\mathbf{A}$ will not commute is made, and then demonstrated. Give a "happy face" (or some sort of praise) if you see that sort of thing.

3. (a) We have

$$\begin{aligned}\mathbf{AB} = \mathbf{AC} \quad &\Rightarrow \quad \mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}(\mathbf{AC}) \\ &\Rightarrow \quad \mathbf{B} = \mathbf{C}.\end{aligned}$$

It is not the case, however, that one can go from

$$\mathbf{AB} = \mathbf{AC} \qquad \text{to} \qquad \mathbf{A}^{-1}\mathbf{AB} = \mathbf{ACA}^{-1}.$$

(b) There are a *lot* of possible matrices **B**, **C** that satisfy the constraints of this problem. There are some requirements, however. If we write

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix},$$

then

$$\mathbf{AB} = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{AC} = \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}.$$

Thus, any correct pair **B**, **C** will have the same first row for both, but will have a different 2^{nd} row (the one from the other).

4. (a) $\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(b) Here, **P** should be the transpose of the **P** from part (a), namely $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(c) $\mathbf{B} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

5. For $t \in \mathbb{R}$ the line is

$$\boldsymbol{x} = t\begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \rightsquigarrow \quad x_1 = 3t,\ x_2 = -2t.$$

Solving for t yields

$$x_1 = -\frac{3}{2}x_2 \quad \rightsquigarrow \quad 2x_1 + 3x_2 = 0.$$

6. (a) This is **false**. For example, if **A** has a column filled with zeros, then its columns are linearly dependent. It is easy to construct such a matrix so that it has more rows than columns.

(b) This statement is **false**. As written, the matrix might have a row of zeros at the bottom, and then it would *not* span $\mathbb{R}^m$.

(c) This statement is **true**, since the statement limits us to looking at free columns in $\mathbf{A}$, not in the augmented matrix.

(d) This statement is **false**. As written, it is possible our two vectors are parallel, in which case their span is a *line*, not a plane, through the origin in $\mathbb{R}^3$.

(e) This statement is **true**. All vectors in its span are simple rescalings, and hence parallel to the original vector. So they all lie along the same line. The origin is obtained when you rescale the original by 0.

(f) This statement is **true**. Consistency *means* weights can be found so that $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$.

(g) This statement is **false**. In order for the columns of $\mathbf{A}$ to span $\mathbb{R}^m$, there must be a pivot in every row. For that to happen, we need $m < n$ (that is, you can't afford to have columns run out before rows do). But this would mean there is a free column in $\mathbf{A}$, making its columns linearly dependent.