MATH 162: Calculus II
Framework for Tues., Feb. 27
Functions of Multiple Variables

Definition: A *function* (or *function of n variables*) f is a rule that assigns to each ordered n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ in a certain set D a real number $f(x_1, x_2, \ldots, x_n)$. The set D is called the *domain* of the function.

Example: Most real-life functions are, in fact, functions of multiple variables. Here are some:

1. $v(r, h) = \pi r^2 h$
2. $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
3. $g(m_1, m_2, R) = G m_1 m_2 / R^2$ (G is a constant)
4. $P(n, T, V) = nRT/V$ (R is a constant)

Of course, one can hold fixed the values of all but one of the input variables and thereby create a function of a single variable. For instance, the way the volume of a right-circular cylinder whose height is 3 varies with its radius is given by the formula

$$V(r) = 3\pi r^2, \quad r \geq 0.$$

Graphing

Functions of a single variable

To graph a function of a single variable requires two coordinate axes. When we write $y = f(x)$, it is implied that x is a possible input and the y-value is the corresponding output. We think of the domain (the set of all possible inputs) of f as consisting of some part of the real line, the graph of f (often called a "curve") as having a point at location $(x, f(x))$ for each x in the domain of f. Keep in mind that, given an arbitrary equation involving x and y, it is not always the case that

(i) we *want* to make y be the dependent variable, and

(ii) if we *do* solve for y, the result is a *function*.

Example: $x^2 + y^2 = 4$

Functions of multiple variables

We have the following analogies for functions of multiple variables:

- When nothing explicit is said about the inputs to a function of multiple variables, we take the domain to be as inclusive as possible.

  Examples:

  $$f(x, y) = \sqrt{xy}$$

  $$f(x, y) = xy(x^2 + y)^{-1}$$

- The graphs of functions of n variables are n-dimensional objects drawn in a coordinate frame involving $(n + 1)$ mutually-perpendicular coordinate axes. (Think of a curve which is the graph of $y = f(x)$ as a 1-dimensional object weaving through 2-dimensional space.)

  As a corollary: *It is not possible to produce the graph of a function of 3 or more variables.* A possible work-around: level sets.

  > **Definition**: Let f be a function of n variables, and c be a real number. The set of all n-tuples $(x_1, \ldots, x_n)$ for which $f(x_1, \ldots, x_n) = c$ is called the c *level set of f*.

  Examples:

  $$f(x, y) = y^2 - x^2$$

  $$f(x, y, z) = z - x^2 - 2y^2$$

- Not every equation involving x, y and z yields z as a single function of x and y.

  Examples:

  $$x - y^2 - z^2 = 0$$

  $$x^2 + y^2 + z^2 = 4$$

- One may assume a missing variable is implied and takes on all real values.

  Example: The meaning of $x = 1$ in 1, 2 and 3 dimensions.

  One may need more than one equation/inequality to describe certain regions of space.

  Example: $x^2 + (y - 1)^2 \leq 1, \quad z = -1$