1. Here
$$\mathbf{B} = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix} - \begin{bmatrix}3 & 1\\-1 & -4\\1 & -3\end{bmatrix}\begin{bmatrix}3 & -2\\-2 & -1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix} - \begin{bmatrix}7 & -7\\5 & 6\\9 & 1\end{bmatrix} = \begin{bmatrix}-6 & 9\\-2 & -2\\-4 & 5\end{bmatrix}.$$

2. There is more than one correct answer. Here is one such sequence:
$$\begin{bmatrix}1&2&1&2\\0&1&2&2\\0&0&0&3\\0&0&0&0\\0&0&0&0\end{bmatrix} \begin{array}{c}(1/3)\mathbf{r}_3 \to \mathbf{r}_3\\ \sim\end{array} \begin{bmatrix}1&2&1&2\\0&1&2&2\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \begin{array}{c}-2\mathbf{r}_3+\mathbf{r}_2 \to \mathbf{r}_2\\ \sim \\ -2\mathbf{r}_3+\mathbf{r}_1 \to \mathbf{r}_1\end{array} \begin{bmatrix}1&2&1&0\\0&1&2&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \begin{array}{c}-2\mathbf{r}_2+\mathbf{r}_1 \to \mathbf{r}_1\\ \sim\end{array} \begin{bmatrix}1&0&1&0\\0&1&2&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

3. The precondition to a system of n equations in n unknowns being *inconsistent* is that the matrix be singular. So, we calculate the determinant of the matrix (I'm expanding in cofactors along the first row)
$$\begin{aligned}\begin{vmatrix}5&-6&-7\\4&-3&k\\1&6&13\end{vmatrix} &= 5(-1)^2\begin{vmatrix}-3&k\\6&13\end{vmatrix} - 6(-1)^3\begin{vmatrix}4&k\\1&13\end{vmatrix} - 7(-1)^4\begin{vmatrix}4&-3\\1&6\end{vmatrix}\\ &= 5(-39-6k)+6(52-k)-7(24+3)\\ &= -195-30k+312-6k-168-21 \quad = \quad -72-36k.\end{aligned}$$

Solving to make this determinant zero, we have $36k = -72$, or $k = -2$. To ensure the system is consistent, we must have $k \neq -2$.

4. (a) We form a matrix whose columns are the given vectors and take it to RREF:
$$\begin{bmatrix}2&-6&-8&-22\\2&-6&-8&-22\\-1&2&3&7\end{bmatrix} \sim \begin{bmatrix}1&0&-1&1\\0&1&1&4\\0&0&0&0\end{bmatrix}$$

   This shows that only two of the four vectors are linearly independent, as there are just 2 pivot columns. Thus, $\mathcal{W}$ is a 2-dimensional subspace of $\mathbb{R}^3$, a **plane**.

   (b) We keep linearly independent columns of the matrix above as a basis. One option is the first two columns: $\mathbf{v}_1$ and $\mathbf{v}_2$.

5. (a) The system has augmented matrix
$$\begin{bmatrix}4&-5&3&4&3\\-1&1&2&2&4\\3&-4&5&6&7\\-3&3&6&6&12\end{bmatrix} \sim \begin{bmatrix}1&0&-13&-14&-23\\0&1&-11&-12&-19\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}.$$

   Columns 3 and 4—or variables x_3, x_4—are free. Rows 1 and 2 of RREF say
$$x_1 = 13x_3 + 14x_4 - 23 \qquad \text{and} \qquad x_2 = 11x_3 + 12x_4 - 19.$$

So, solutions of the system take the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -23 \\ -19 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 11 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 14 \\ 12 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{with } x_3, x_4 \in \mathbb{R}.$$

(b) The matrix is the same as in part (a), so its null space is revealed in the answer to part (a) as the part with freedoms. A basis for the null space is

$$\begin{bmatrix} 13 \\ 11 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 14 \\ 12 \\ 0 \\ 1 \end{bmatrix}.$$

6. We have

$$0 = \begin{vmatrix} -7-\lambda & -3 \\ 9 & -5-\lambda \end{vmatrix} = (-7-\lambda)(-5-\lambda) + 27 = \lambda^2 - 12\lambda + 62,$$

$$\Rightarrow \lambda = \frac{1}{2}\left(12 \pm \sqrt{144 - (4)(62)}\right) = 6 \pm i\frac{\sqrt{104}}{2} = 6 \pm i\sqrt{26}.$$

7. If we call the given matrix $\mathbf{A}$, then using the given eigenvalue λ, the problem "solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$" has augmented matrix

$$\left[\mathbf{A} + 2\mathbf{I} \,\middle|\, \mathbf{0}\right] = \begin{bmatrix} 4 & 0 & -8 & 0 \\ 4 & 0 & -8 & 0 \\ 2 & 0 & -4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

RREF has two free columns (so that is the dimension of the eigenspace), and we take x_2 and x_3, components of an eigenvector, as free, leading to eigenvectors of the form

$$\begin{bmatrix} 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

span the eigenspace. They are also linearly independent, making this collection a *basis*.