1. (a) Echelon form is not unique, so the answer (to part (a)) that follows is not the only correct one. For all correct answers, however, the 4^{th} column will be free.

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 6 & -4 & 5 \\ 0 & 2 & -1 & 4 \\ 0 & 2 & -1 & 7 \end{bmatrix} \overset{2r_1+r_2 \to r_2}{\sim} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 2 & -1 & 7 \end{bmatrix} \overset{r_4 - r_3 \to r_4}{\sim} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\overset{r_4 - r_2 \to r_4}{\sim} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{r_2 \leftrightarrow r_3}{\sim} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Augmenting A with the zero vector is a scenario where the Gaussian elimination we have already performed is easily adapted:

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 2 & 6 & -4 & 5 & 0 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & 2 & -1 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 + 3x_2 - 2x_3 + x_4 = 0 \\ \Rightarrow 2x_2 - x_3 + 4x_4 = 0 \\ \Rightarrow 3x_4 = 0 \\ \end{array}$$

So, $x_3 = t$ is free, and using backward substitution,

$$3x_4 = 0 \implies x_4 = 0$$

$$x_2 = \tfrac{1}{2}(x_3 - 4x_4) = \tfrac{1}{2}(t - 0) = \tfrac{1}{2}t$$

$$x_1 = -3x_2 + 2x_3 - x_4 = -3(\tfrac{1}{2}t) + 2t - 0 = \tfrac{1}{2}t$$

We have solutions of the homogeneous system $A\vec{x} = \vec{0}$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \\ 0 \end{bmatrix} = \left(\tfrac{1}{2}t\right) \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad t \text{ is any real.}$$

2. The augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 4 \end{array}\right]$ is already in RREF. We see

$x_3 = s$ and $x_4 = t$ are free variables, and solving for basic variables x_1, x_2:

$$x_1 = s - t + 1 \qquad \text{and} \qquad x_2 = -3s + 2t + 4$$

So, solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - t + 1 \\ -3s + 2t + 4 \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix},$$

for s, t any reals.

3. Since

$$\begin{bmatrix} 4 & -1 & 2 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -2 & 1 \\ 2 & -5 \end{bmatrix} = \left[\overbrace{4\begin{bmatrix} 4 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \end{bmatrix}}^{\text{linear comb.} \to 1^{st}\text{ col.}} \quad \overbrace{3\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 2 \\ -2 \end{bmatrix}}^{\text{linear comb.} \to 2^{nd}\text{ col.}}\right]$$

$$= \begin{bmatrix} 22 & 1 \\ -6 & 16 \end{bmatrix},$$

$$B = \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 22 & 1 \\ -6 & 16 \end{bmatrix} = \begin{bmatrix} -19 & -2 \\ 8 & -10 \end{bmatrix}$$

4. (a) $|AB| = \det(A)\det(B) = (2)(-7) = -14$

(b) $|-3A| = (-3)^3 \det(A) = (-27)(2) = -54$

(c) $|A^T| = |A| = 2$

(d) $|B^{-1}| = 1/|B| = -\frac{1}{7}$

(e) $|A^4| = \det(A)^4 = 2^4 = 16$

5. $$\left[\begin{array}{cc|c} 2 & -7 & h \\ -4 & k & -12 \end{array}\right] \overset{2r_1 + r_2 \to r_2}{\sim} \left[\begin{array}{cc|c} 2 & -7 & h \\ 0 & k-14 & 2h-12 \end{array}\right]$$

So that there is the potential of infinitely many solutions, we require

$$k - 14 = 0 \quad \Rightarrow \quad \boxed{k = 14.}$$

Infinitely many solutions arises from having a free column, so long as the system is consistent, which additionally means we require

$$2h - 12 = 0 \quad \Rightarrow \quad \boxed{h = 6}$$

6. $A(3\vec{u} - 3\vec{v}) = 3A(\vec{u} - \vec{v}) = 3(A\vec{u} - A\vec{v})$

$$= 3\left(\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix}\right) = 3\begin{bmatrix} 7 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 21 \\ -6 \\ -18 \end{bmatrix}$$

8. $|A| = (2)(-2) - (-2)(3) = -4 + 6 = 2$

Since $|A| \neq 0$, A^{-1} exists. To find it,

$$\left[\underbrace{\begin{matrix} 2 & -2 \\ -2 & 3 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}\right] \underset{\sim}{r_1 + r_2 \to r_2} \left[\begin{matrix} 2 & -2 \\ 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}\right]$$

$$\underset{\sim}{r_1 + 2r_2 \to r_1} \left[\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 3 & 2 \\ 1 & 1 \end{matrix}\right] \underset{\sim}{\tfrac{1}{2} r_1 \to r_1} \left[\overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I} \;\middle|\; \overbrace{\begin{matrix} 3/2 & 1 \\ 1 & 1 \end{matrix}}^{A^{-1}}\right]$$

We get $$A^{-1} = \begin{bmatrix} 3/2 & 1 \\ 1 & 1 \end{bmatrix}$$

9. Finding $|A|$ via Laplace expansion along the 4^{th} row:

$$|A| = (1)(-1)^5 \begin{vmatrix} 3 & 1 & 2 \\ 0 & 2 & -1 \\ 4 & -1 & 1 \end{vmatrix} + 0 + (3)(-1)^7 \begin{vmatrix} 2 & 3 & 2 \\ -1 & 0 & -1 \\ 3 & 4 & 1 \end{vmatrix} + 0 = (-1)(-17) + (-3)(-6) = \boxed{35}$$

$$\begin{vmatrix} 3 & 1 & 2 \\ 0 & 2 & -1 \\ 4 & -1 & 1 \end{vmatrix} = (3)(-1)^2 \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + 0 + (4)(-1)^4 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = (3)(1) + (4)(-5) = -17$$

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 0 & -1 \\ 3 & 4 & 1 \end{vmatrix} = (3)(-1)^3 \begin{vmatrix} -1 & -1 \\ 3 & 1 \end{vmatrix} + 0 + (4)(-1)^5 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} = (-3)(2) - (4)(0) = -6$$

Or, finding $|A|$ using GE:

$$\begin{vmatrix} 2 & 3 & 1 & 2 \\ -1 & 0 & 2 & -1 \\ 3 & 4 & -1 & 1 \\ 1 & 0 & 3 & 0 \end{vmatrix} \underset{\text{single row swap}}{= (-1)} \begin{vmatrix} 1 & 0 & 3 & 0 \\ -1 & 0 & 2 & -1 \\ 3 & 4 & -1 & 1 \\ 2 & 3 & 1 & 2 \end{vmatrix} \underset{r_1 + r_2 \to r_2}{= (-1)} \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 5 & -1 \\ 3 & 4 & -1 & 1 \\ 2 & 3 & 1 & 2 \end{vmatrix} \underset{\substack{-3r_1 + r_3 \to r_3 \\ -2r_1 + r_4 \to r_4}}{= (-1)} \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 5 & -1 \\ 0 & 4 & -10 & 1 \\ 0 & 3 & -5 & 2 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 5 & -1 \\ 0 & 1 & -5 & -1 \\ 0 & 3 & -5 & 2 \end{vmatrix} = (-1)^2\begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 5 & -1 \\ 0 & 3 & -5 & 2 \end{vmatrix} = (-1)^2\begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 10 & 5 \end{vmatrix}$$

$r_3 - r_4 \rightarrow r_3$ another row swap $-3r_2 + r_4 \rightarrow r_4$

$$= (-1)^2(5)\begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 7 \end{vmatrix} = (-1)^2 \cdot (1)(1)(5)(7) = \underline{35}$$

(same as above)

$r_4 - 2r_3 \rightarrow r_4$