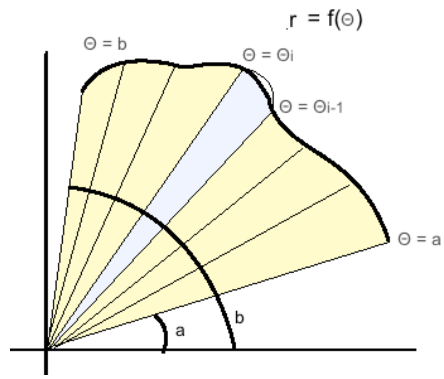
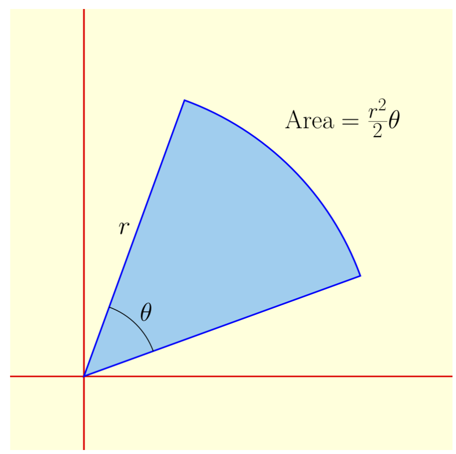
Polar area



The typical polar function is one which specifies $r = f(\theta)$ (θ as independent, r as dependent). The sorts of regions whose areas we might naturally compute this way are ones like those depicted in the top picture at right. As $r > 0$ reflects a distance from a point back to the origin (not the x-axis), slices look like wedges out of a near-circular region. We need to know how to compute areas of true wedges (taken from true circles), such as those depicted in the second picture at right. Its area satisfies a proportion:



$$\frac{\text{Area(wedge)}}{\text{Area(full circle)}} = \frac{\text{measure(central angle)}}{\text{measure(angle for one rotation)}},$$

or

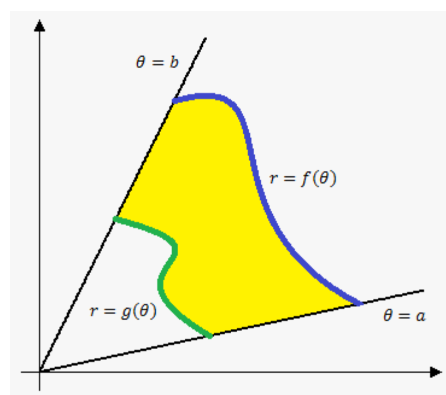
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \qquad \Rightarrow \qquad A = \frac{1}{2} r^2 \theta.$$

Using this, a typical nearly-wedge-shaped slice in the top figure would have area approximately equal to

$$\frac{1}{2}[f(\theta)]^2(\theta_i - \theta_{i-1}) \;=\; \frac{1}{2}[f(\theta)]^2 \Delta\theta,$$

and an approximation to the full area could be obtained via the sum

$$\sum_{i=1}^{n} \frac{1}{2}\,[f(\theta_i)]^2 \Delta\theta.$$



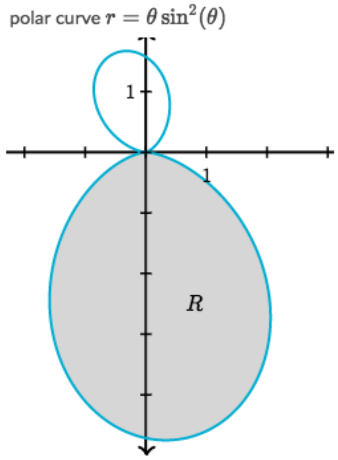
The approximation improves as $\Delta\theta \to 0$, giving the actual area as

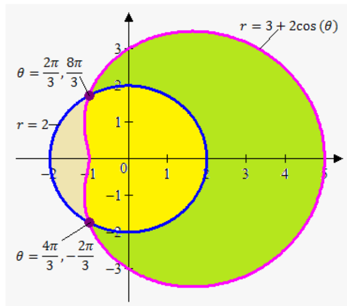
$$\int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta.$$

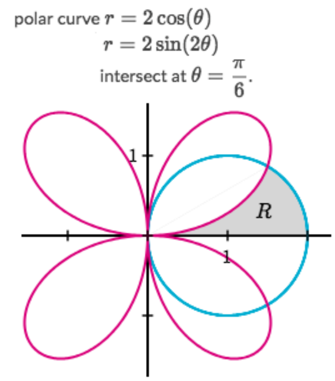
Adapting this to the computation of area for a region between two polar curves (see the bottom figure), we have

$$\text{Area of shaded region} \;=\; \int_a^b \frac{1}{2}\Big([f(\theta)]^2 - [g(\theta)]^2\Big)\, d\theta.$$

Examples of areas of polar regions

1. 

2. 

3. 

Lengths of polar arcs

Key idea: Combine polar-to-rectangular conversion with polar functions to get a parametrization. That is, insert $r = f(\theta)$ into $x = r\cos\theta$, $y = r\sin\theta$. Then find $dy/dx = (dy/d\theta)/(dx/d\theta)$.