1. The characteristic equation: $r^2 - 2r + 10 = 0$

$$\Rightarrow \quad r = \frac{2}{2} \pm \frac{\sqrt{4-(4)(10)}}{2} = 1 \pm 3i$$

$\Rightarrow$ the homogeneous DE has independent solns. $y_1 = e^t\cos(3t)$, $y_2 = e^t\sin(3t)$

$\Rightarrow$ $y_h(t) = c_1 e^t \cos(3t) + c_2 e^t \sin(3t)$ (No overlap with $26\cos(2t) + 26\sin(2t)$)

For particular soln. pose the form

$$y_p(t) = A\cos(2t) + B\sin(2t) \quad \Rightarrow \begin{cases} y_p' = -2A\sin(2t) + 2B\cos(2t) \\ y_p'' = -4A\cos(2t) - 4B\sin(2t) \end{cases}$$

So

$$\begin{aligned} y_p'' - 2y_p' + 10y_p &= -4A\cos(2t) - 4B\sin(2t) \\ &\quad -2\left[-2A\sin(2t) + 2B\cos(2t)\right] \\ &\quad +10\left[A\cos(2t) + B\sin(2t)\right] \\ &= \underbrace{(6A - 4B)}_{\text{must } = 26}\cos(2t) + \underbrace{(4A - 6B)}_{\text{must } = 26}\sin(2t) \end{aligned}$$

$$\begin{bmatrix} 6 & -4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 26 \\ 26 \end{bmatrix} \Rightarrow A = \frac{\begin{vmatrix} 26 & -4 \\ 26 & 6 \end{vmatrix}}{\begin{vmatrix} 6 & -4 \\ 4 & 6 \end{vmatrix}} = 5, \quad B = \frac{\begin{vmatrix} 6 & 26 \\ 4 & 26 \end{vmatrix}}{52} = 1$$

Together, $y(t) = y_h(t) + y_p(t) = c_1 e^t\cos(3t) + c_2 e^t \sin(3t) + 5\cos(2t) + \sin(2t)$

$$y'(t) = c_1 e^t\cos(3t) - 3c_1 e^t\sin(3t) + c_2 e^t \sin(3t) + 3c_2 e^t\cos(3t) - 10\sin(2t) + 2\cos(2t)$$

The ICs: $9 = y(0) = c_1 + 5 \Rightarrow c_1 = 4$

$15 = y'(0) = c_1 + 3c_2 + 2 \Rightarrow c_2 = 3$

$$\Rightarrow \boxed{y(t) = 4e^t\cos(3t) + 3e^t\sin(3t) + 5\cos(2t) + \sin(2t)}$$

2. (a) Since $\mathcal{L}\left[3\cos(2t)\right] = 3\,\mathcal{L}\left[\cos(2t)\right] = \dfrac{3s}{s^2+4}$

and $\mathcal{L}\left[5t^2 + 7\right] = 5\,\mathcal{L}\left[t^2\right] + 7\,\mathcal{L}\left[1\right] = \dfrac{10}{s^3} + \dfrac{7}{s}$

we have $\mathcal{L}\left[(f * g)(t)\right] = \boxed{\dfrac{3s}{s^2+4}\left(\dfrac{10}{s^3} + \dfrac{7}{s}\right)}$

(b) From the definition,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}(4t+3)\,dt \qquad \begin{array}{l|l} dv = e^{-st}\,dt & u = 4t+3 \\ v = -\frac{1}{s}e^{-st} & du = 4\,dt \end{array}$$

$$\overset{\text{by parts}}{=} -\frac{1}{s}e^{-st}(4t+3)\Big|_0^2 + \frac{4}{s}\int_0^2 e^{-st}\,dt$$

$$= -\frac{11}{s}e^{-2s} + \frac{3}{s} - \frac{4}{s^2}\left[e^{-st}\right]_0^2 = \boxed{\frac{3}{s} - \frac{11}{s}e^{-2s} + \frac{4}{s^2}\left(1-e^{-2s}\right)}$$

Or, using the table entry $\mathcal{L}[f(t-a)\,\mathcal{U}(t-a)] = e^{-sa}\,\mathcal{L}[f(t)]$:

$$f(t) = [1-\mathcal{U}(t-2)](4t+3) = 4t+3 - \mathcal{U}(t-2)[4(t-2)+11]$$

$$= 4t+3 - \mathcal{U}(t-2)\cdot\left(4t+11\,\big|_{t\mapsto t-2}\right)$$

$$\Rightarrow F(s) = 4\mathcal{L}[t] + 3\mathcal{L}[1] - e^{-2s}\cdot\mathcal{L}[4t+11] = \boxed{\frac{4}{s^2} + \frac{3}{s} - e^{-2s}\left(\frac{4}{s^2}+\frac{11}{s}\right)}$$

(c) Using partial fractions

$$\frac{2s+1}{s(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5} \quad\Rightarrow\quad 2s+1 = A(s^2+4s+5) + Bs^2 + Cs$$

$$= \underbrace{(A+B)}_{\text{must}=0}s^2 + \underbrace{(4A+C)}_{\text{must}=2}s + \underbrace{5A}_{\text{must}=}$$

$5A = 1 \quad\Rightarrow\quad A = 1/5$

$A+B = 0 \quad\Rightarrow\quad B = -1/5$

$4A+C = 2 \quad\Rightarrow\quad C = 6/5$

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s(s^2+4s+5)}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{-1/5\,s + 6/5}{(s+2)^2+1}\right] = \frac{1}{5} + \mathcal{L}^{-1}\left[\frac{-1/5\,(s+2) + 8/5}{(s+2)^2+1}\right]$$

$$= \frac{1}{5} - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] + \frac{8}{5}\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1}\right] = \boxed{\frac{1}{5} - \frac{1}{5}e^{-2t}\cos t + \frac{8}{5}e^{-2t}\sin t}$$

3. (a) $$H(s) = \frac{1}{s^2+3s+2}$$

(b) $$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+3s+2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\cdot\frac{1}{s+1}\right] \qquad \left(\text{can also write as } \frac{A}{s+2}+\frac{B}{s+1} \text{ using partial fractions}\right)$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] * \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-2t} * e^{-t}$$

$$= \int_0^t e^{-2w}e^{-(t-w)}\,dw = e^{-t}\int_0^t e^{-w}\,dw = -e^{-t}\left[e^{-w}\right]_0^t = -e^{-t}\left(e^{-t}-1\right)$$

$$= \boxed{e^{-t} - e^{-2t}}$$

(c) The characteristic equation, $r^2+3r+2=0$, has 2 distinct real (and negative) roots. So, it is <u>overdamped</u>.

(d) $$y(t) = (f * h)(t) = \int_0^t f(t-w)\, h(w)\, dw$$
$$= \int_0^t f(t-w)\left(e^{-w} - e^{-2w}\right) dw$$

4. Here, $y'' - \frac{12}{t^2} y = 5t^4 + 2t^{-2} = f(t)$,

and
$$W = \begin{vmatrix} t^4 & t^{-3} \\ 4t^3 & -3t^{-4} \end{vmatrix} = -3-4 = -7$$

$$y_p(t) = t^4 \int \frac{1}{7}\left(5t^4+2t^{-2}\right)t^{-3}\,dt - t^{-3}\int \frac{1}{7}\left(5t^4+2t^{-2}\right)t^4\,dt$$
$$= t^4 \int\left(\frac{5}{7}t + \frac{2}{7}t^{-5}\right)dt - t^{-3}\int\left(\frac{5}{7}t^8 + \frac{2}{7}t^2\right)dt$$
$$= t^4\left(\frac{5}{14}t^2 - \frac{1}{14}t^{-4}\right) - t^{-3}\left(\frac{5}{63}t^9 + \frac{2}{21}t^3\right)$$
$$= \frac{5}{14}t^6 - \frac{1}{14} - \frac{5}{63}t^6 - \frac{2}{21} = \boxed{\frac{5}{18}t^6 - \frac{1}{6}}$$