# MATH 162: Calculus II
## Framework for Tues., Apr. 17
## Triple Integrals, Rectangular Coordinates

**Today's Goal**: To be able to set up and evaluate triple integrals.

**Important Note**: In conjunction with this framework, you should look over Section 13.5 of your text.

## Defining Triple Integrals

Suppose

- $D$ is a "nice" bounded region in 3-dimensional space.
- We subdivide $D$, creating a partition $P$ of $D$, where $P$ consists of $n$ "boxes" wholly contained in $D$.
- In the $k$th box ($1 \le k \le n$), we choose a point $(x_k, y_k, z_k)$.

We then look at sums of the form

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

where $\Delta V_k$ denotes the volume of the $k$th box.

As with Riemann sums over partitions of regions of the plane, there are many functions and regions $D$ for which the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k$$

exists, in which case we say that $f$ is *integrable* over $D$. This limit is denoted by

$$\iiint_D f(x, y, z)\, dV,$$

read as the *triple integral* of $f$ over $D$.

# Comparisons to Double Integrals

1. **Evaluation**.

**Double integrals:**

Here our principle tool is Fubini's Theorem. We have two cases.

**Case:** $\displaystyle\iint_R g(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} g(x,y)\,dy\,dx.$

Here, $a$ and $b$ reflect the lowest and highest of a continuum of $x$-values encountered in the region $R$, while the lower and upper boundaries of $R$ may be identified as functions of $x$.

**Case:** $\displaystyle\iint_R g(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} g(x,y)\,dx\,dy.$

Here, $c$ and $d$ reflect the lowest and highest of a continuum of $y$-values encountered in the region $R$, while the left and right boundaries of $R$ may be identified as functions of $y$.

For most regions $R$, either case is applicable (sometimes one of the options requires a sum of integrals instead of a single one), meaning that the double integral may be written in either of *two orders* (i.e., either with $y$ as the inner integral, as in $dy\,dx$, or in the order $dx\,dy$).

**Triple integrals:**

Given a bounded region $D$ of 3-dimensional space, the triple integral $\iiint_D f(x,y,z)\,dV$ may be written as an iterated integral in *six different orders*. Here are *two of the possibilities*:

- $\displaystyle\iiint_D f(x,y,z)\,dV = \int_c^d \int_{g_1(y)}^{g_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$

  Here, $c$ and $d$ are the lowest and highest in a continuum of $y$-values encountered as one passes through the region $D$. For any fixed $y \in [c,d]$, we imagine a 2-dimensional (planar) region that results from slicing through $D$ with a plane parallel to the $xz$-plane. This planar region has a starting and ending $z$-value, given by $g_1(y)$ and and $g_2(y)$ respectively. At the inner-most level (the inner-most integral, which is in $x$), $y$ and $z$ are held fixed while $x$ is allowed to vary. The interval of possible $x$-values starts at $h_1(y,z)$ and ends at $h_2(y,z)$.

- $\displaystyle\iiint_D f(x,y,z)\,dV = \int_r^s \int_{g_1(z)}^{g_2(z)} \int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy\,dx\,dz.$

  The explanation of our region is similar to the above, but this time $r$ and $s$ represent lowest and highest $z$-values encountered in $D$; for a fixed $z$, $g_1(z)$ and $g_2(z)$ give lowest and highest $x$-values; for both $x$ and $z$ fixed, $h_1(x,z)$ and $h_2(x,z)$ give lowest and highest $y$-values.

2. **Interpretations**.

   (a) **Areas, volumes and higher**.

   **Double integrals:**

   When $g(x, y)$ is nonnegative, the double integral $\iint_R f(x, y)\, dA$ gives the volume under the surface $z = g(x, y)$ over the region $R$ of the $xy$-plane. If $g$ changes sign in the region $R$, then $\iint_R g(x, y)\, dA$ represents a *difference of volumes.*

   A special case is when $g(x, y) \equiv 1$. As we have seen $\iint_R g(x, y)\, dA = \iint dA$ gives the *area* of $R$ (numerically equal to the volume under a surface over $R$ whose height is uniformly 1).

   **Triple integrals:**

   When $f(x, y, z)$ is nonnegative, we can be sure that $\iiint_D f(x, y, z)\, dV$ is nonnegative as well. But since the graph of $w = f(x, y, z)$ is 4-dimensional, we would have to think of this value as a type of 4-dimensional volume (or difference of volumes, if $f$ changes sign in $D$).

   When $f(x, y) \equiv 1$, then $\iiint_R f(x, y, z)\, dV = \iiint dV$ gives the *volume* of the 3-dimensional region $D$.

   (b) **Average values**. The average value of $g(x, y)$ over a region $R$ of the $xy$-plane was defined to be $\iint_R g(x, y)\, dA / \iint_R dA$. Similarly, we define the average value of $f(x, y, z)$ over a region $D$ of 3-dimensional space to be $\iiint_D f(x, y, z)\, dV / \iiint_D dV$.

   (c) **Density integrals**. When $g(x, y)$ gives the amount of a substance per unit area, then $\iint_R g(x, y)\, dA$ tallies the amount of that substance found in a region $R$ of the $xy$-plane. Similarly, when $f(x, y, z)$ gives the amount of a substance per unit volume, then $\iiint_D f(x, y, z)\, dV$ tallies the amount of that substance found in a region $D$ of 3D space.

**Examples**:

1. Evaluate $\iiint_D z\, dV$ over the region enclosed by the three coordinate planes and the plane $x + y + z = 1$.
2. Find the average $z$-value in the region from problem 1.
3. Find limits of integration for $\iiint_D \sqrt{x^2 + z^2}\, dy\, dz\, dx$ where $D$ is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.
4. Write a triple integral for $f(x, y, z)$ over the region bounded by the ellipsoid $9x^2 + 4y^2 + z^2 = 1$.
5. What solid is it for which the iterated triple integral $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dz\, dx\, dy$ gives its volume. What do other iterated triple integrals for the same expression look like?