3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the 2^{nd}-order linear recursion relation

$$a_n = 6a_{n-1} - 5a_{n-2}, \quad \text{for } n \geqslant 2, \quad \text{with } a_0 = 0,\ a_1 = 4.$$

Take $P(n)$: $a_n = 5^n - 1$. Then $\forall n \in \mathbb{N}$, $P(n)$ (use strong mathematical induction).

P_n is the claim that the formula $a_n = 5^a - 1$ is closed formula for the n^{th} term of the sequence solving our recurrence relation.

Base case: $n=2$ $\quad P_2$: $a_2 = 5^2 - 1$ satisfies

$a_2 = 6(a_1) - 5(a_0)$ or $24 = a_2 \stackrel{?}{=} 6(4) - 5(0)$ ✓

strong induction: Assume $P_2, P_3, \ldots, P_k$ hold for some $k \geq 2$.

Thus $a_{k+1} = 6(\underline{a_k}) - 5(\underline{a_{k-1}}) = 6(5^k - 1) - 5(5^{k-1} - 1)$

$= 6(5^k) - 6 - 5(5^{k-1}) + 5 = 6(5^k) - 5^k - 1 = 5^k(6-1) - 1$

$= \underline{5^{k+1} - 1}$ destination (hoped-for)

4. Use strong mathematical induction to show the product of n numbers requires $n-1$ multiplications, regardless of grouping.

To multiply $r_1\ r_2\ r_3 \cdots r_n$ requires $n-1$ multiplications

What domain for n? $\quad n \geq 1$ ✓

Base step: list of 1 number requires 0 multiplications

Suppose for inductive step that $\underbrace{P_1, P_2, \ldots, P_k}_{\text{I.H.}}$ holds for some $k \geq 1$.

Now take a list of $k+1$ numbers

$(r_1 \mathbin{)(} r_2\ r_3 \cdots r_k\ r_{k+1})$ last (final outer set of parens) group

This final grouping produces 2 sublists, one of size m, the other of size j where

$\left.\begin{array}{l} 1 \leq m < k+1 \\ 1 \leq j < k+1 \end{array}\right\} m + j = k+1$

By I.H., 1^{st} sublist requires $(m-1)$ multiplies, 2^{nd} requires $(j-1)$ multiplies total # of multiplies:

$(m-1)+(j-1)+1 = k$ multiples

5. A simple polygon with $n \geqslant 3$ sides can be trtiangulated into $n-2$ triangles (use strong mathematical induction, and the fact that every simple polygon with at least four sides has an interior diagonal).

6. Given a strictly decreasing sequence of *positive* integers $r_1, r_2, r_3, \ldots$ (so $r_{i+1} < r_i$ for each i), the sequence terminates (use the well-ordering principle).

For instance

$\underbrace{111, 59, 37, 22, 18, \_\_, \_\_, \_\_}_{\text{subset of } \mathbb{N}}$

last one (end!)

Ex.) $n = 57,\ d = 5$

$\Rightarrow q = 11,\ r = 2$?

Ex.) $n = -33,\ d = 4$

$\Rightarrow q = -9,\ r = 3$?

$\{\ldots, 47, 52, 57, 62, 67, \ldots\}$

Division Algorithm

7. Given any integer n and any positive integer d, there exist integers q and r such that $n = dq + r$ and $0 \leqslant r < d$ (use the well-ordering principle).

quotient

$$d \,\overline{)\,73\,}^{\,q} \quad -(q \cdot d) \quad r$$

remainder

Statement to prove: Given integers n and d, $d > 0$, there exist unique integers q and r satisfying:

- $n = qd + r$
- $0 \leq r < d$

prove using well-ordering principle:

Consider the set $\underline{A} = \{n - qd \mid q \in \mathbb{Z}\} \cap \mathbb{N}$

This $A \subseteq \mathbb{N}$, so by well-ordering principle, A has a smallest element, call it

$$r = n - qd \text{ for some } q \in \mathbb{Z}.$$

Now we have r and q. Note that, by construction,

$$n = r + qd$$

And $r \geq 0$ since r comes from A (containing only nat. nos.)

and either $r < d$ or it isn't.

But, if $r \geq d$ (or $d \leq r$), we'd have

$$d \leq r = n - qd$$

So subtracting d from all these expressions

$$0 \leq r - d = n - (q+1)d$$

which makes $r - d$ another element inside A, even smaller than r, and that's impossible, since r is already smallest. ∎