1. Here

$$\mathbf{B} = \begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix} - \begin{bmatrix}-4 & 4\\ -4 & 3\\ -1 & -3\end{bmatrix}\begin{bmatrix}3 & -2\\ 2 & 2\end{bmatrix} = \begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix} - \begin{bmatrix}-4 & 16\\ -6 & 14\\ -9 & -4\end{bmatrix} = \begin{bmatrix}5 & -12\\ 8 & -9\\ 12 & 10\end{bmatrix}.$$

2. There is more than one correct answer. Here is one such sequence:

$$\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & 1 & 2 & 5\\ 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \begin{array}{c}(1/2)\mathbf{r}_3 \to \mathbf{r}_3\\ \sim\end{array} \begin{bmatrix}1 & 2 & 1 & 3\\ 0 & 1 & 2 & 5\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \begin{array}{c}-5\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2\\ \sim\\ -3\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1\end{array} \begin{bmatrix}1 & 2 & 1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \begin{array}{c}-2\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1\\ \sim\end{array} \begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

3. The precondition to a system of n equations in n unknowns being *inconsistent* is that the matrix be singular. So, we calculate the determinant of the matrix (I'm expanding in cofactors along the first row)

$$\begin{aligned}\begin{vmatrix}6 & 6 & 5\\ -7 & 5 & k\\ -8 & 16 & -5\end{vmatrix} &= 6(-1)^2\begin{vmatrix}5 & k\\ 16 & -5\end{vmatrix} + 6(-1)^3\begin{vmatrix}-7 & k\\ -8 & -5\end{vmatrix} + 5(-1)^4\begin{vmatrix}-7 & 5\\ -8 & 16\end{vmatrix}\\ &= 6(-25-16k) - 6(35+8k) + 5(-112+40)\\ &= -150 - 96k - 210 - 48k - 560 + 200 \quad = \quad -720 - 144k.\end{aligned}$$

Solving to make this determinant zero, we have $144k = -720$, or $k = -5$. To ensure the system is consistent, we must have $k \neq -5$.

4. (a) We form a matrix whose columns are the given vectors and take it to RREF:

$$\begin{bmatrix}2 & -4 & 0 & 8\\ 1 & -2 & 0 & 4\\ -3 & 7 & 1 & -15\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -2\\ 0 & 1 & 1 & -3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

This shows that only two of the four vectors are linearly independent, as there are just 2 pivot columns. Thus, $\mathcal{W}$ is a 2-dimensional subspace of $\mathbb{R}^3$, a **plane**.

(b) We keep linearly independent columns of the matrix above as a basis. One option is the first two columns: $\mathbf{v}_1$ and $\mathbf{v}_2$.

5. (a) The system has augmented matrix

$$\begin{bmatrix}4 & -5 & 4 & 3 & 2\\ -1 & 1 & 2 & 3 & 2\\ 3 & -4 & 6 & 6 & 4\\ -2 & 2 & 4 & 6 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -14 & -18 & -12\\ 0 & 1 & -12 & -15 & -10\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Columns 3 and 4—or variables x_3, x_4—are free. Rows 1 and 2 of RREF say

$$x_1 = 14x_3 + 18x_4 - 12 \qquad \text{and} \qquad x_2 = 12x_3 + 15x_4 - 10.$$

So, solutions of the system take the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -12 \\ -10 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 14 \\ 12 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 18 \\ 15 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{with } x_3, x_4 \in \mathbb{R}.$$

(b) The matrix is the same as in part (a), so its null space is revealed in the answer to part (a) as the part with freedoms. A basis for the null space is

$$\begin{bmatrix} 14 \\ 12 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 18 \\ 15 \\ 0 \\ 1 \end{bmatrix},$$

6. We have

$$0 = \begin{vmatrix} 3-\lambda & -6 \\ 3 & 2-\lambda \end{vmatrix} = (7-\lambda)(2-\lambda) + 18 = \lambda^2 - 5\lambda + 24,$$

$$\Rightarrow \lambda = \frac{1}{2}\left(5 \pm \sqrt{25 - (4)(24)}\right) = \frac{5}{2} \pm i\frac{\sqrt{71}}{2}.$$

7. If we call the given matrix $\mathbf{A}$, then using the given eigenvalue λ, the problem "solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$" has augmented matrix

$$\left[\mathbf{A} - 2\mathbf{I} \,\middle|\, \mathbf{0}\right] = \begin{bmatrix} 2 & 0 & -2 & 0 \\ -4 & 0 & 4 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

RREF has two free columns (so that is the dimension of the eigenspace), and we take x_2 and x_3, components of an eigenvector, as free, leading to eigenvectors of the form

$$\begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

span the eigenspace. They are also linearly independent, making this collection a *basis*.