Lines and Planes

We have derived the following representations.

- **Lines**. The line through point $P = (x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

  component form: $$\begin{aligned} x &= x_0 + v_1 t, \\ y &= y_0 + v_2 t, \qquad -\infty < t < \infty, \\ z &= z_0 + v_3 t, \end{aligned}$$

  vector form: $$\mathbf{r}(t) = (x_0 + v_1 t)\mathbf{i} + (y_0 + v_2 t)\mathbf{j}(z_0 + v_3 t)\mathbf{k}, \qquad -\infty < t < \infty.$$

- **Planes**. The plane through point $P = (x_0, y_0, z_0)$ perpendicular to $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

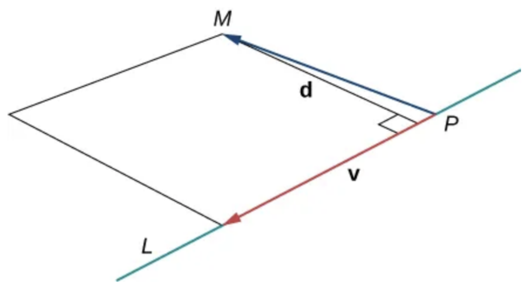
  $$\mathbf{n} \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0, \qquad \text{or} \qquad ax + by + cz = d,$$

  where $d = ax_0 + by_0 + cz_0$.

Formulas and Algorithms for Lines and Planes

- **Distance from a point M to a line L.**

  $$\frac{\|\overrightarrow{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}$$



- **Distance from a point M to a plane** containing the point P with normal vector $\mathbf{n}$.

  $$\frac{|\overrightarrow{PM} \cdot \mathbf{n}|}{\|\mathbf{n}\|}.$$

- **Angle between two planes**.

  Definition: The *angle between planes* is taken to be the angle $\theta \in [0, \pi/2]$ between normal vectors to the planes.

  By this definition, if $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors to the two planes, then the angle between the planes is

  $$\theta = \begin{cases} \arccos\left(\dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}\right), & \text{if } \mathbf{n}_1 \cdot \mathbf{n}_2 \geqslant 0, \\[2ex] \pi - \arccos\left(\dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}\right), & \text{if } \mathbf{n}_1 \cdot \mathbf{n}_2 < 0. \end{cases}$$

- **Line of intersection between two non-parallel planes**.

  It should not be difficult to find a point on the desired line. If the two planes have equations $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$, then it is quite likely the line of intersection will eventually pass through a point P where the x-coordinate is zero. Assuming this is so, we may do the usually steps of solving the simultaneous equations in 2 unknowns

  $$\begin{aligned} b_1y + c_1z &= d_1 \\ b_2y + c_2z &= d_2 \end{aligned}$$

  for the corresponding y and z coordinates of this point. (If the solution process fails to yield corresponding y and z coordinates, one can instead look for the point P for which the y or, alternatively, the z-coordinate is zero.)

  Once a point P on our line of intersection is found, we next need a vector that is parallel to our line. Such a vector would be perpendicular to normal vectors to both planes, and so could be any multiple of

  $$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$