MATH 162: Calculus II

Framework for Fri., Feb. 23

Taylor Series

In the section on power series, we seemed to be

- interested in finding power series expressions for various functions f,
- but able to find such series only when the function f, or some order derivative/antiderivative of f, looked enough like $(1-x)^{-1}$ to make this feasible.

Our goal today is to find series expressions for important functions that are not so closely linked to $(1-x)^{-1}$. First, a definition:

Definition: Suppose f is a function which has derivatives of all orders at $x = a$. The *Taylor series for f at $x = a$* is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

What this definition says is that, for appropriate f, we may construct a power series about $x = a$ employing the values of derivatives $f^{(n)}(a)$ in the coefficients. As of yet, no assertion that this power series actually equals f has been made. (See note 2 below.)

Some important notes:

1. The Taylor series, like any power series, has a radius of convergence R, which may be zero.
2. Even if the radius of convergence $R > 0$, the function defined by the Taylor series of f might not equal f except at the single location $x = a$.
3. But, if $R > 0$, then for many "nice" functions f, the Taylor series for f equals f on its entire interval of convergence.
4. If we stop the sum at the term containing $(x-a)^n$ (i.e., consider the partial sum of the series that includes as its last term the one with $(x-a)$ to the n^{th} power), we get a polynomial of n^{th} degree. This polynomial is called the *Taylor polynomial of order n for f at $x = a$.*
5. If $a = 0$, then the Taylor series is called the *MacLaurin series of f.*
6. If f equals any power series about $x = a$ at all, then that series must be the Taylor series.

Some favorite Taylor series (all of these are MacLaurin series)

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
&= 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \qquad -\infty < x < \infty \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots, \qquad -\infty < x < \infty \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots, \qquad -\infty < x < \infty \\
\arctan x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \qquad -1 \le x \le 1 \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots, \qquad -1 < x \le 1
\end{aligned}$$

As with expressions that were similar to $(1-x)^{-1}$, we may substitute into these power series to get power series expressions for other, related functions.

Example: The MacLaurin series converging to $exp(-x^2)$ is

$$exp(-x^2) \;=\; 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots.$$