$\underline{\star 10}$ Upon row reduction we have

$$A = \begin{pmatrix} 1 & -2 & 8 \\ -2 & 1 & -7 \\ -5 & 3 & r \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 8 \\ 0 & 1 & -3 \\ 0 & 0 & r+19 \end{pmatrix}$$

(a) If $r \neq -19$, then $A \xrightarrow{\text{RREF}} I_3$, and the vectors are linearly independent.

(b) If $r = -19$, then the homogenous solution to $Ax = 0$ is

$$x_{\text{h}} = t \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

This yields

$$-2v_1 + 3v_2 + v_3 = 0 \quad \rightsquigarrow \quad v_3 = 2v_1 - 3v_2 \in \text{Span}\{v_1, v_2\}.$$

(c) The answer to each is found after first row reducing A.

$\underline{\star 11}$ We have

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ -1 & -1 & 3 & 1 \\ 4 & 6 & 2 & 0 \\ 6 & 8 & -5 & 7 \end{pmatrix} \xrightarrow{\text{RREF}} I_4,$$

so the linear system $Ax = b$ is consistent for any $b \in \mathbb{R}^4$. The set of vectors is therefore a spanning set.

$\underline{\star 12}$ Since

$$\begin{pmatrix} 2s - 3t \\ -s + 4t \\ 7t \end{pmatrix} = s \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix} \quad \rightsquigarrow \quad S = \text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix} \right\},$$

S is a subspace.

$\underline{\star 13}$ If $0 \in S$, then

$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \\ 1 & 9 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

Since

$$\left(\begin{array}{cc|c} 4 & 2 & 0 \\ -3 & -1 & -1 \\ 1 & 9 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right),$$

the linear system is not consistent. Since $0 \notin S$, the set is not a subspace.

$\star$14 (a) $n > m$.

(b) $m > n$.

$\star$15 (b) Since

$$A \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

bases are given by

$$\text{Col}(A) : \left\{ \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\}; \quad \text{Null}(A) : \left\{ \begin{pmatrix} 10 \\ 2 \\ 0 \\ 7 \end{pmatrix} \right\}.$$

We have $\text{rank}(A) = 3$ and $\dim[\text{Null}(A)] = 1$.

(c) Since

$$A \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & -13/20 \\ 0 & 1 & 0 & 21/20 \\ 0 & 0 & 1 & 15/20 \end{pmatrix},$$

bases are given by

$$\text{Col}(A) : \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \right\}; \quad \text{Null}(A) : \left\{ \begin{pmatrix} 13 \\ -21 \\ -15 \\ 20 \end{pmatrix} \right\}.$$

We have $\text{rank}(A) = 3$ and $\dim[\text{Null}(A)] = 1$.

$\star$16 (a) $\text{rank}(A) = 5$.

(b) $\dim[\text{Null}(A)] = 3$.

$\star$17 Since

$$\begin{pmatrix} 1 & -2 & 3 \\ 5 & 3 & 1 \\ -2 & 0 & -5 \end{pmatrix} \xrightarrow{\text{RREF}} I_3,$$

the vectors are linearly independent, and $\dim[\text{Span}(S)] = 3 = \dim[\mathbb{R}^3]$. Since S is also a collection of 3-vectors, S is a basis for $\mathbb{R}^3$.

$\star$18 Letting each vector in S be a column for a matrix A, we find

$$A \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since all of the columns of $\boldsymbol{A}$ are not pivot columns, the vectors are not linearly independent. A basis for S is the pivot columns of $\boldsymbol{A}$,

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}\right\} \quad \rightsquigarrow \quad \dim[\mathrm{Span}(S)] = 2.$$

<u>★19</u> (a) FALSE. $\mathrm{rank}(\boldsymbol{A}) = 5$.

(b) FALSE. The vectors also need to be 7-vectors.

(c) FALSE. $\dim[\mathrm{Null}(\boldsymbol{A})] = 3$.

(d) TRUE. There are three columns associated with free variables.

<u>★20</u> (a) Going down the first column,

$$\begin{aligned} \det(\boldsymbol{A}) &= \det\begin{pmatrix} b & b^2 \\ c & c^2 \end{pmatrix} - \det\begin{pmatrix} a & a^2 \\ c & c^2 \end{pmatrix} + \det\begin{pmatrix} a & a^2 \\ b & b^2 \end{pmatrix} \\ &= bc(c-b) - ac(c-a) + ab(b-a) \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

(b) We need $\det(\boldsymbol{V}) \neq 0$, which means $a \neq b$, $a \neq c$, $b \neq c$.

<u>★21</u> A matrix $\boldsymbol{B}$ has a nontrivial null-space if and only if $\det(\boldsymbol{B}) = 0$. We have

$$0 = \det(\boldsymbol{A}(\lambda)) = (3-\lambda)^2 - 4 \quad \rightsquigarrow \quad \lambda = 1, 5.$$

As for the corresponding nontrivial solution,

$$\boldsymbol{A}(1) \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \rightsquigarrow \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \boldsymbol{A}(5) \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \rightsquigarrow \quad \boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$