### Today's outline

We considered first the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$, determining that it is divergent because of these observations:

- This series has the same fate as $\sum_{n=3}^{\infty} \frac{\ln n}{n}$.
- $\ln n > 1$ for each $n \geqslant 3$, so the terms (all positive) of $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ are never smaller than the terms of $\sum_{n=3}^{\infty} \frac{1}{n}$.
- The series $\sum_{n=3}^{\infty} \frac{1}{n}$ has the same fate as $\sum_{n=1}^{\infty} \frac{1}{n}$, and the latter is the harmonic series (divergent $p$-series with $p = 1$). The direct comparison test allows us to conclude $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges as well.

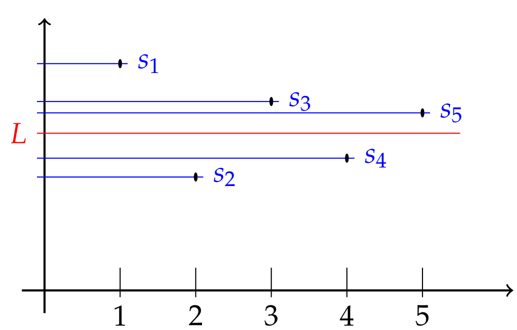
**Side Note**: I hinted that there may be other conclusions possible besides the ones stated in the book's version of the **Limit Comparison Test**, but all of the versions I looked for online say even *less* than the version given in the text. While looking, I found this useful(?) page containing practice problems for comparison tests with solutions.

Next, we turned to series $\sum_n a_n$ where terms are both positive and negative (infinitely often). An example of such a series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

In this particular series every other term is positive/negative, meriting be called an **alternating series**. The terms in the sequence of partial sums

$$\begin{aligned}
s_1 &= 1, \\
s_2 &= 1 - \tfrac{1}{2} = 0.5, \\
s_3 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} \doteq 0.833, \\
s_4 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} \doteq 0.583, \\
s_5 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} \doteq 0.783, \\
s_6 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} \doteq 0.617,
\end{aligned}$$




successively bounce above/below a value

(represented by the horizontal line) whose position/height is difficult to ascertain, but represents $L = \lim_n s_n$.

Side Note: Each new term $a_n = (-1)^{n+1}/n$ is of sufficient magnitude to make $s_{n+1}$ jump to the opposite side of $L$ as the one $s_n$ is on, which means the difference $|s_n - L| < \frac{1}{n+1}$; that is, no partial sum is ever farther from $L$ than the first term that partial sum omits.

Not all series with infinitely many positive and negative terms are truly *alternating*, but for those that are, it is worth checking whether their partial sums behave in the same way as the series we just considered. That is what the alternating series test is about.

**Theorem 1 (Alternating Series Test):** If $b_1, b_2, b_3, \ldots$ is a sequence

- with all positive terms,
- that is strictly decreasing, and
- has limit 0,

then the corresponding alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \;=\; b_1 - b_2 + b_3 - b_4 + b_5 + \cdots$$

converges to a number $s$. Moreover, for each $n$, the difference between the $n^{\text{th}}$ partial sum and the full series sum $s$ has upper bound $|s_n - s| < b_{n+1}$.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but the series $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. That makes it an example of a **conditionally convergent** series, one where $\sum_n a_n$ converges but $\sum_n |a_n|$ diverges. Another conditionally convergent series is $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$, a fact we can demonstrate, in part, using the Alternating Series Test.

In general, when considering a series $\sum_n a_n$ whose terms are both positive and negative infinitely often, it is natural to look at the fate of $\sum_n |a_n|$ because of the absolute convergence test.

**Theorem 2 (Absolute Convergence Test):** If the series $\sum_n |a_n|$ converges, then so does $\sum_n a_n$.

We applied this test to the series $\displaystyle\sum_{n=1} \frac{\sin n}{n^2}$, and since $\displaystyle\sum_{n=1} \frac{|\sin n|}{n^2}$ converges, so does the original. (It is an example of an **absolutely convergent** series.)