#1 $\Pr[(X,Y)\in A] = \int_0^{1/2}\int_1^2 \frac{3}{5}x(y+x)\,dy\,dx$

$$= \frac{3}{5}\int_0^{1/2} x \int_1^2 (y+x)\,dy\,dx = \frac{3}{5}\int_0^{1/2} x\left[\frac{1}{2}y^2+xy\right]_1^2 dx$$

$$= \frac{3}{5}\int_0^{1/2} x\left[(2+2x)-\left(\frac{1}{2}+x\right)\right] = \frac{3}{5}\int_0^{1/2}\left(x^2+\frac{3}{2}x\right)dx$$

$$= \frac{3}{5}\left[\frac{1}{3}x^3+\frac{3}{4}x^2\right]_0^{1/2} = \frac{3}{5}\left(\frac{1}{24}+\frac{3}{16}\right) = \frac{3}{5}\left(\frac{2+9}{48}\right)$$

$$= \frac{11}{80}$$

#2 $\Pr[(X_1,X_2,X_3)\in A] = \int_{-\infty}^1\int_{-\infty}^1\int_{-\infty}^{1/2} f(x_1,x_2,x_3)\,dx_1\,dx_2\,dx_3 = \int_0^1\int_{1/2}^1\int_0^{1/2}(x_1+x_2)e^{-x_3}\,dx_1\,dx_2\,dx_3$

$$= \int_0^1 e^{-x_3}\int_{1/2}^1\left[\frac{1}{2}x_1^2+x_1x_2\right]_0^{1/2} dx_2\,dx_3 = \int_0^1 e^{-x_3}\int_{1/2}^1\left(\frac{1}{8}+\frac{1}{2}x_2\right)dx_2\,dx_3$$

$$= \int_0^1 e^{-x_3}\left[\frac{1}{8}x_2+\frac{1}{4}x_2^2\right]_{1/2}^1 dx_3 = \int_0^1 e^{-x_3}\left[\left(\frac{1}{8}+\frac{1}{4}\right)-\left(\frac{1}{16}+\frac{1}{16}\right)\right]dx_3$$

$$= \frac{1}{4}\int_0^1 e^{-x_3}\,dx_3 = -\frac{1}{4}\left[e^{-x_3}\right]_0^1 = -\frac{1}{4}\left(e^{-1}-1\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{e}\right).$$

#3 For $x \geq 0$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} 4e^{-2x}\cdot e^{-2y}\,dy = 4e^{-2x}\left(\lim_{A\to\infty}\int_0^A e^{-2y}\,dy\right)$$

$$= 4e^{-2x}\lim_{A\to\infty}\left[-\frac{1}{2}e^{-2y}\right]_0^A = 2e^{-2x}\lim_{A\to\infty}\left(-e^{-2A}+1\right)$$

$$= 2e^{-2x}.$$

The integral producing $f_X(x)$ is zero if $x<0$. Similarly, for $y \geq 0$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} 4e^{-2x}\cdot e^{-2y}\,dx = 4e^{-2y}\left(\lim_{A\to\infty}\int_0^A e^{-2x}\,dx\right)$$

$$= 4e^{-2y} \lim_{A\to\infty} \left[-\frac{1}{2} e^{-2x} \right]_0^A = 2e^{-2y} \lim_{A\to\infty} \left(-e^{-2A} + 1 \right)$$

$$= 2e^{-2y}.$$

Note that, if either $x<0$ or $y<0$, $f_X(x)\, f_Y(y) = 0$. Otherwise,

$$f_X(x)\, f_Y(y) = \left(2e^{-2x}\right)\left(2e^{-2y}\right) = 4e^{-2(x+y)}.$$

Thus, $f(x,y) = f_X(x) f_Y(y)$, and X, Y are independent.

#4 (a) For $y \in [0,2]$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \left(\frac{1}{4}x^2 + \frac{1}{4}y^2 + \frac{1}{6}xy\right)dx = \frac{1}{12}x^3 + \frac{1}{4}xy^2 + \frac{1}{12}x^2y \Big|_0^1$$

$$= \frac{1}{12} + \frac{1}{4}y^2 + \frac{1}{12}y$$

So, for $0 \le x \le 1$, $0 \le y \le 2$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{4}x^2 + \frac{1}{4}y^2 + \frac{1}{6}xy}{\frac{1}{12} + \frac{1}{4}y^2 + \frac{1}{12}y} = \frac{3x^2 + 2xy + 3y^2}{3y^2 + y + 1}$$

and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{3x^2+2xy+3y^2}{3y^2+y+1}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $$\Pr\left(X < \tfrac{1}{2} \,\middle|\, Y = y\right) = \int_{-\infty}^{1/2} f_{X|Y}(x|y)\,dx = \int_0^{1/2} \frac{3x^2+2xy+3y^2}{3y^2+y+1}\,dx$$

$$= \frac{1}{3y^2+y+1} \int_0^{1/2} \left(3x^2 + 2xy + 3y^2\right) dx = \frac{1}{3y^2+y+1} \left[x^3 + x^2y + 3xy^2 \right]_0^{1/2}$$

$$= \frac{1}{3y^2+y+1} \left(\frac{1}{8} + \frac{1}{4}y + \frac{3}{2}y^2 \right)$$

$$= \frac{12y^2 + 2y + 1}{8(3y^2+y+1)}, \quad 0 \le y \le 2.$$

#5 For $0 < x < 1$, $0 < y < 1$,

$$f(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} F(x,y)\right) = \frac{\partial}{\partial y}\left(xy + \frac{1}{2}y^3\right) = x + \frac{3}{2}y^2.$$

So, the joint pdf is

$$f(x,y) = \begin{cases} x + \frac{3}{2}y^2, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

#6 (i) $E(X+Y) = \iint (x+y) f_{X,Y}(x,y)\,dy\,dx = \iint x f_{X,Y}(x,y)\,dy\,dx + \iint y f_{X,Y}(x,y)\,dx\,dy$

$$= \int x\left(\int f_{X,Y}(x,y)\,dy\right)dx + \int y\left(\int f_{X,Y}(x,y)\,dx\right)dy = \int x f_X(x)\,dx + \int y f_Y(y)\,dy$$

$$= E(X) + E(Y).$$

(ii) $E(XY) = \iint xy\, f_{X,Y}(x,y)\,dy\,dx = \iint xy\, f_X(x) f_Y(y)\,dy\,dx = \int x f_X(x)\left(\int y f_Y(y)\,dy\right)dx$

$$= \left(\int y f_Y(y)\,dy\right)\left(\int x f_X(x)\,dx\right) = E(Y)E(X).$$

(iii) $Var(X+Y) = E((X+Y)^2) - [E(X+Y)]^2 = E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2$

$$= E(X^2 + 2XY + Y^2) - [E(X)^2 + 2E(X)E(Y) + E(Y)^2]$$

$$= E(X^2) + 2E(XY) + E(Y^2) - [E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2$$

$$= E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2 + 2[E(XY) - E(X)E(Y)]$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y).$$

By part (ii), $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$ if X, Y are independent.