MATH 162: Calculus II

Framework for Tues., Mar. 13

Cross Products

Today's Goal: To define the cross product and learn of some of its properties and uses

The Cross Product

Definition: For nonzero, non-parallel 3D vectors $\mathbf{u}$ and $\mathbf{v}$, we define the *cross product* of $\mathbf{u}$ and $\mathbf{v}$ to be

$$\mathbf{u} \times \mathbf{v} \ := \ (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n},$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, and in the direction determined by the "right-hand rule."

If either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, we define $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. Similarly, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, we take $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Notes:

- There is no corresponding concept for 2D vectors.
- The dot product between two vectors produces a scalar. The cross product of two vectors yields another vector.
- Properties
  1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
  2. $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$
  3. $(r\mathbf{u}) \times (s\mathbf{v}) \ = \ (rs)(\mathbf{u} \times \mathbf{v})$
  4. The cross product is not associative! This means that, in general, it is *not* the case that
  $$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \qquad \text{and} \qquad \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$
  are equal.
- The cross product $\mathbf{u} \times \mathbf{v}$ may be computed from the following symbolic determinant:
$$\mathbf{u} \times \mathbf{v} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \ := \ \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k},$$
  where $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

Applications

- $\mathbf{r} \times \mathbf{F}$ is the *torque* vector resulting from a force $\mathbf{F}$ applied at the end of a lever arm $\mathbf{r}$.
- $|\mathbf{u} \times \mathbf{v}|$ (the length of the cross product $\mathbf{u} \times \mathbf{v}$) is the area of a parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.
- $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ (the absolute value of the scalar $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$) is the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
- Finding normal vectors to planes.

  Example: The vectors $\mathbf{u} = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle -2, 3, 1 \rangle$

  - are not parallel,
  - so they determine a family of parallel planes.

  Find a vector that is normal to these planes. Then determine an equation for the particular one of these planes passing through the point $(1, 1, 1)$.