1. This A has RREF $\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Its rank is 2, and columns 1 and 3 are pivot columns.

(a) basis for col(A):

$$\begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

(b) basis for row(A):

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

(c) basis for null(A) (see below):

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} x_2 &= r \\ x_4 &= s \\ x_5 &= t \end{aligned}\right\} \text{free} \qquad \begin{aligned} x_1 &= -2r - 2s - t \\ x_3 &= s - 2t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

2. (a) $\overrightarrow{PQ} = \begin{bmatrix} 4-2 \\ 1-2 \\ -5+1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$.

(b) $\text{dist}(Q, P) = \|\overrightarrow{PQ}\| = \sqrt{2^2 + (-1)^2 + (-4)^2} = \sqrt{21}$

(c) Our desired unit vector is $\dfrac{1}{\|\overrightarrow{PQ}\|}\overrightarrow{PQ} = \dfrac{1}{\sqrt{21}}\begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$.

(d) The line can be expressed as

$$\overrightarrow{OP} + t \cdot \overrightarrow{PQ} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}, \quad t \in \mathbb{R}, \qquad \text{or} \qquad \left.\begin{aligned} x &= 2+2t \\ y &= 2-t \\ z &= -1-4t \end{aligned}\right\} \ t \in \mathbb{R}.$$

(e) For each point (x, y, z) in the plane,

$$0 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} \circ \begin{bmatrix} x-2 \\ y-2 \\ z+1 \end{bmatrix} = 2x - 4 - y + 2 - 4z - 4$$

$$\Rightarrow \quad 2x - y - 4z = 6$$

(f) The plane in (e) does not contain $(0,0,0)$, so it is not a subspace of $\mathbb{R}^3$. Therefore, this plane cannot be the orthogonal complement of any collection of vectors S. In fact, none of the vectors in standard position pointing to a point in this plane are orthogonal to $\overrightarrow{PQ}$.

3. (a) This is an o.n. basis, since the matrix

$$B = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}$$

is orthogonal — that is, $B^T B = I$.

(b) The matrix A for which $T(\vec{x}) = A\vec{x}$ is found as the product

$$\begin{bmatrix} 2 & -2 & 4 & 0 \\ -2 & -2 & -2 & 4 \\ 4 & -4 & 2 & -2 \\ -2 & 4 & -2 & 0 \\ 6 & 0 & 2 & 6 \end{bmatrix} \cdot B^{-1} = \begin{bmatrix} 2 & -2 & 4 & 0 \\ -2 & -2 & -2 & 4 \\ 4 & -4 & 2 & -2 \\ -2 & 4 & -2 & 0 \\ 6 & 0 & 2 & 6 \end{bmatrix} \cdot B^T \quad \text{(since } B \text{ is orthogonal)}$$

$$= \begin{bmatrix} 2 & -2 & 4 & 0 \\ -2 & -2 & -2 & 4 \\ 4 & -4 & 2 & -2 \\ -2 & 4 & -2 & 0 \\ 6 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} -1/2 & -1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 0 & 2 \\ -1 & 5 & 1 & -1 \\ -4 & -2 & -4 & 2 \\ 4 & 0 & 2 & -2 \\ -7 & -1 & 1 & -5 \end{bmatrix}.$$

(c) Since $\{\vec{u}_1, \vec{u}_2\}$ make an orthogonal basis of W,

$$\begin{aligned} \operatorname{proj}_W \vec{b} &= \operatorname{proj}_{\vec{u}_1} \vec{b} + \operatorname{proj}_{\vec{u}_2} \vec{b} \\ &= (\vec{b} \circ \vec{u}_1)\,\vec{u}_1 + (\vec{b} \circ \vec{u}_2)\,\vec{u}_2 \qquad \left(\text{Simpler formula since } \|\vec{u}_1\| = \|\vec{u}_2\| = 1\right) \\ &= -\vec{u}_1 + 5\vec{u}_2 \\ &= (-1)\begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} + 5 \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \\ -2 \end{bmatrix}. \end{aligned}$$

4. <u>One approach</u>: Find the columns in A by finding what A does to $\hat{\imath}$ and $\hat{\jmath}$.

$$\text{proj}_{\langle 3,4\rangle}\langle 1,0\rangle = \frac{\langle 1,0\rangle \cdot \langle 3,4\rangle}{\|\langle 3,4\rangle\|^2}\langle 3,4\rangle = \frac{3}{25}\langle 3,4\rangle = \left\langle \frac{9}{25}, \frac{12}{25}\right\rangle$$

$$\text{proj}_{\langle 3,4\rangle}\langle 0,1\rangle = \frac{\langle 0,1\rangle \cdot \langle 3,4\rangle}{\|\langle 3,4\rangle\|^2}\langle 3,4\rangle = \frac{4}{25}\langle 3,4\rangle = \left\langle \frac{12}{25}, \frac{16}{25}\right\rangle$$

So,
$$A = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}.$$

<u>Another approach</u>:

(a) Rotate the vector $\langle 3,4\rangle$ clockwise so it faces "east": $\begin{bmatrix} \cos(0.9273) & \sin(0.9273) \\ -\sin(0.9273) & \cos(0.9273) \end{bmatrix}$

(b) Wipe out the 2nd component (i.e., project onto the x-axis): $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) Rotate back: $\begin{bmatrix} \cos(0.9273) & -\sin(0.9273) \\ \sin(0.9273) & \cos(0.9273) \end{bmatrix}$

$$\Rightarrow \quad A = \begin{bmatrix} \cos(0.9273) & -\sin(0.9273) \\ \sin(0.9273) & \cos(0.9273) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos(0.9273) & \sin(0.9273) \\ -\sin(0.9273) & \cos(0.9273) \end{bmatrix}$$

$$= \cdots = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}.$$