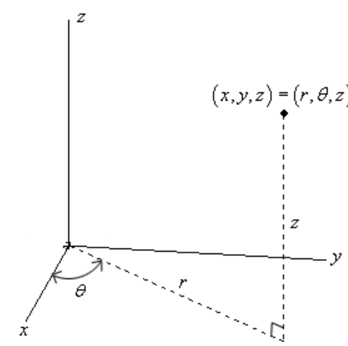
MATH 162: Calculus II
Framework for Mon., Apr. 30
Integration in Cylindrical Coordinates

Today's Goal: To develop an understanding of cylindrical and spherical coordinates, and to learn to set up and evaluate triple integrals in cylindrical coordinates.

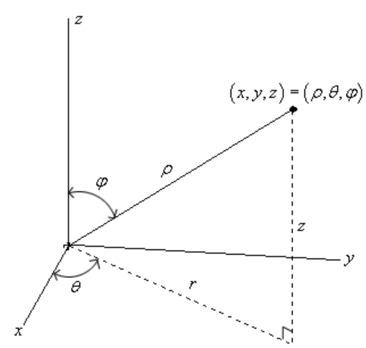
Important Note: In conjunction with this framework, you should look over Section 13.7 of your text.

Coordinate Systems for 3D Space



- **Rectangular Coordinates**: Generally uses letters (x, y, z). It tells how far to travel in directions parallel to three orthogonal coordinate axes in order to arrive at the specified point.

- **Cylindrical Coordinates** Generally uses letters (r, θ, z). Here z should be understood in exactly the same way as it is for rectangular coordinates, while r and θ are polar coordinates for the shadow point in the xy-plane. (See the top figure.) Each of the three coordinates may take any value in $\mathbb{R}$.



- **Spherical Coordinates** Generally uses letters (ρ, ϕ, θ). The meaning of θ is precisely the same as with cylindrical coordinates. If a ray is drawn from the origin to the point in question, then ρ is the distance along that ray to the point, while ϕ is the angle that ray makes with the z-axis. It is possible to identify all points of 3D-space using values for the three coordinates which satisfy

$$\rho \geq 0, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \pi.$$

Conversions between Coordinates

The Pythagorean Theorem and trigonometry give

$$r \;=\; \sqrt{x^2+y^2} \;=\; \rho\sin\phi, \quad \text{and} \quad \rho \;=\; \sqrt{x^2+y^2+z^2}.$$

Thus,

$$\begin{aligned} x &= r\cos\theta \;=\; \rho\sin\phi\cos\theta, \\ y &= r\sin\theta \;=\; \rho\sin\phi\sin\theta, \\ z &= \rho\cos\phi \quad \text{(by trigonometry)}. \end{aligned}$$

Triple Integrals in Cylindrical Coordinates

When computing the triple integral $\iiint_D f(x,y,z)\,dV$, one can choose any of the three coordinate systems discussed above. Cylindrical coordinates are attractive when

- the boundary of the shadow region in the plane may be expressed nicely as a combination of polar functions, and/or
- the integrand $f(r\cos\theta, r\sin\theta, z)$ is simple.

For iterated integrals in cylindrical coordinates, the volume element is $dV = r\,dz\,dr\,d\theta$, or some permutation of this. Thus,

$$\iiint\limits_D f(x,y,z)\,dV \;=\; \iiint\limits_D f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta.$$

Examples:

1. Show that the volume of a sphere with radius a is what we think it should be.
2. Find the volume of a "cored apple"—a sphere of radius a from which a cylindrical region of radius b ($b < a$) has been removed.
3. Find the volume of a typical cone whose radius at the base is a and whose height is h.