Linear 1^{st}-order homogeneous systems with nonreal eigenvalues

There are certain things we build on:

- **Euler's Formula**: Given a real number θ, and $i = \sqrt{-1}$, it says $e^{i\theta} = \cos\theta + i\sin\theta$. A corollary to it is that $e^{-i\theta} = \cos\theta - i\sin\theta$, making $e^{i\theta}$ and $e^{-i\theta}$ complex conjugates. For an explanation of why this amazing formula holds, and secondarily to justify in part your study of Maclaurin series in MATH 172, watch
  https://drive.google.com/file/d/1a7x1QIdNYGis6np3V9rXkq8xYh0wE3yD/view?usp=sharing

  Here are that video's finished notes
  http://scofield.site/courses/m231/coronaDays/complexEvalStuff/eulersFormula.jpg

- When a matrix $\mathbf{A}$ has real entries by a nonreal eigenvalue $\alpha + i\beta$, where α, β are real numbers, there will be at least one corresponding eigenvector $\mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$, $\mathbf{v}$ have real entries. Correspondingly, the complex conjugate $\alpha - i\beta$ is also an eigenvalue of $\mathbf{A}$, and has $\mathbf{u} - i\mathbf{v}$ as an eigenvector. For example, if

$$-3 + 2i \qquad \text{is an eigenvalue with eigenvector} \qquad \begin{bmatrix} 2-3i \\ 1-i \\ 3i \end{bmatrix},$$

  then we can identify

$$\alpha = -3, \quad \beta = 2, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix},$$

  and conclude that

$$\alpha - i\beta = -3 - 2i \qquad \text{is also an eigenvalue with eigenvector} \qquad \mathbf{u} - i\mathbf{v} = \begin{bmatrix} 2+3i \\ 1+i \\ -3i \end{bmatrix}.$$

- We have demonstrated and made use of the fact that, if the matrix $\mathbf{A}$ has eigenpair $(\lambda, \mathbf{v})$, then $e^{\lambda t}\mathbf{v}$ is a solution of the homogeneous linear 1^{st}-order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, But it is not possible to make physical sense of such a solution

$$e^{(\alpha+i\beta)t}(\mathbf{u} + i\mathbf{v}) \qquad \text{and its counterpart} \qquad e^{(\alpha-i\beta)t}(\mathbf{u} - i\mathbf{v}),$$

  when we are talking about nonreal eigenpairs of $\mathbf{A}$. In
  https://drive.google.com/file/d/1elJcAEDA807WxB6JigSH3NYf5YgcY9OC/view?usp=sharing
  I justify why it is reasonable and valid to trade out those nonreal solutions for these *real* substitutes:

$$e^{\alpha t}\left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right] \qquad \text{and} \qquad e^{\alpha t}\left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right].$$

Here are that video's finished notes
http://scofield.site/courses/m231/coronaDays/complexEvalStuff/tradeNonrealSolnsForReal.jpg

Some examples:

1. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} -21 & -30 & -32 \\ -4 & -7 & -7 \\ 24 & 30 & 35 \end{bmatrix}\mathbf{x}$

   The details of solving this system are worked out in four videos:
   https://drive.google.com/file/d/1xyn44wYgDn6_wlVggm1Z7-zYeQ_wwbjo/view?usp=sharing
   https://drive.google.com/file/d/1Jh6WfH3gxzkwmDnYT74a51OSeZZ7idFM/view?usp=sharing
   https://drive.google.com/file/d/1s5bfHJw6M0g0O04CUApkWrQq_BSQJiNY/view?usp=sharing
   https://drive.google.com/file/d/1fOstxjsy8leV-mo3vGAcItdQZCxiXY7t/view?usp=sharing
   These videos will be played during class. Here are the three pages of end notes:

   - page 1: http://scofield.site/courses/m231/coronaDays/complexEvalStuff/A3by3with2complexEvals_p1.jpg
   - page 2: http://scofield.site/courses/m231/coronaDays/complexEvalStuff/A3by3with2complexEvals_p2.jpg
   - page 3: http://scofield.site/courses/m231/coronaDays/complexEvalStuff/A3by3with2complexEvals_p3.jpg

   Mid-way through page 3, I have written the general solution. It has another form which puts the three individual building-block solutions together into a fundamental matrix $\mathbf{\Phi}(t)$:

$$\begin{aligned} \mathbf{x}_h(t) &= c_1 \begin{bmatrix} -11e^{3t} \\ -4e^{3t} \\ 12e^{3t} \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -10\cos(3t) \\ -3\cos(3t) - \sin(3t) \\ 10\cos(3t) \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -10\sin(3t) \\ -3\sin(3t) + \cos(3t) \\ 10\sin(3t) \end{bmatrix} \\ &= \begin{bmatrix} -11e^{3t} & -10e^{2t}\cos(3t) & -10e^{2t}\sin(3t) \\ -4e^{3t} & -3e^{2t}\cos(3t) - e^{2t}\sin(3t) & -3e^{2t}\sin(3t) + e^{2t}\cos(3t) \\ 12e^{3t} & 10e^{2t}\cos(3t) & 10e^{2t}\sin(3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c} \end{aligned}$$

2. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} -5 & -10 \\ 5 & 9 \end{bmatrix}\mathbf{x}$

   A video that finds the solution to this system is found at
   https://drive.google.com/file/d/17bM_j4Zoc8kCpXKnQRxurSaLlm_kdptR/view?usp=sharing
   For a snapshot of the resulting notes, consult page 3 above. Again, the solution found there can be written as

$$\mathbf{x}_h(t) = \begin{bmatrix} -7e^{2t}\cos t - e^{2t}\sin t & -7e^{2t}\sin t + e^{2t}|cost \\ 5e^{2t}\cos t & 5e^{2t}\sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}$$