1. We have

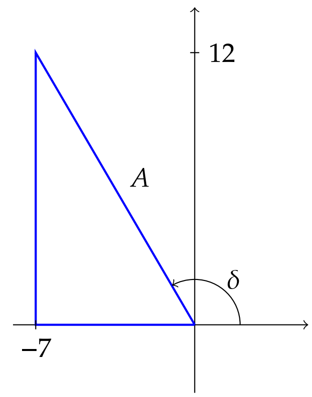
$$A = \sqrt{(-7)^2 + 12^2} = \sqrt{193}$$

and

$$\cos\delta = \frac{-7}{\sqrt{193}}, \quad \sin\delta = \frac{12}{\sqrt{193}} \quad \Rightarrow \quad \delta \doteq 2.10.$$

Thus,

$$-7\cos(3t) + 12\sin(3t) \;\approx\; \sqrt{193}\cos(3t - 2.10).$$



2. (a) The function $2t^2 - 5t + 1$ can be shifted left one unit:

$$2t^2 - 5t + 1\Big|_{t\mapsto t+1} \;=\; 2(t+1)^2 - 5(t+1) + 1 \;=\; 2t^2 - t - 2.$$

This altered function, when shifted *right* one unit, returns us to the original polynomial. And so

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\Big\{(2t^2 - 5t + 1)u(t-1)\Big\} = \mathcal{L}\Big\{\Big(2t^2 - t - 2\Big|_{t\mapsto t-1}\Big)u(t-1)\Big\} \\ &= \Big(2\mathcal{L}\big\{t^2\big\} - \mathcal{L}\{t\} - 2\mathcal{L}\{1\}\Big)e^{-s} \;=\; \left(\frac{4}{s^3} - \frac{1}{s^2} - \frac{2}{s}\right)e^{-s}.\end{aligned}$$

(b) Here, $f(t) \;=\; \Big(4t^2e^{-8t}\Big) * \Big(\cos(2t)\Big)$, and so

$$\mathcal{L}\{f(t)\} \;=\; 4\mathcal{L}\Big\{t^2e^{-8t}\Big\}\cdot\mathcal{L}\{\cos(2t)\} \;=\; \frac{8}{(s+8)^3}\cdot\frac{s}{s^2+4} \;=\; \frac{8s}{(s+8)^3(s^2+4)}.$$

(c) First, we ignore the exponential e^{-s}. By partial fractions,

$$\frac{2}{(s^2+6s+10)(s+2)} \;=\; \frac{As+B}{s^2+6s+10} + \frac{C}{s+2}.$$

Multiplying through by the common denominator gives

$$2 \;=\; (As+B)(s+2) + C(s^2+6s+10) \;=\; (A+C)s^2 + (2A+B+6C)s + (2B+10C).$$

Equating coefficients of s-terms, we have a matrix problem:

$$\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 6\\ 0 & 2 & 10\end{bmatrix}\begin{bmatrix}A\\ B\\ C\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix} \quad\Rightarrow\quad \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 2 & 1 & 6 & 0\\ 0 & 2 & 10 & 2\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & -1\\ 0 & 1 & 0 & -4\\ 0 & 0 & 1 & 1\end{array}\right] \quad\Rightarrow\quad A=-1, B=-4, C=1.$$

So,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{(s^2+6s+10)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{-\frac{s+4}{s^2+6s+10} + \frac{1}{s+2}\right\} \\ &= -\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= -e^{-3t}\cos(t) - e^{-3t}\sin(t) + e^{-2t}.\end{aligned}$$

As to the exponential factor,

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{2}{(s^2+6s+10)(s+2)}\right\} \;=\; u(t-1)\Big[-e^{-3(t-1)}\cos(t-1) - e^{-3(t-1)}\sin(t-1) + e^{-2(t-1)}\Big].$$

3. (a) In finding the homogeneous part y_h of the solution, our characteristic equation has a double root:

$$(r+2)^2 = 0 \qquad \Rightarrow \qquad r = --2,\ -2 \qquad \Rightarrow \qquad y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

We propose a particular solution that, like the right-hand side, is a 2nd-degree polynomial:

$$y_p(t) = At^2 + Bt + C \qquad \Rightarrow \qquad y_p' = 2At + B, \qquad y_p'' = 2A.$$

Then

$$y_p'' + 4y_p' + 4y_p \;=\; 2A + 4(2At + B) + 4(At^2 + Bt + C) \;=\; 4At^2 + (8A + 4B)t + (2A + 4B + 4C).$$

Because our target function—what we want this result to equal—is $12t^2 + 20t + 10$, we can make this work by choosing A, B, C so that

$$\left.\begin{array}{l} 4A = 12 \\ 8A + 4B = 20 \\ 2A + 4B + 4C = 10 \end{array}\right\} \quad \Rightarrow \quad A = 3,\ B = -1,\ C = 2.$$

Thus, $y_p(t) = 3t^2 - t + 2$, and $y(t) \;=\; y_h(t) + y_p(t) \;=\; c_1 e^{-2t} + c_2 t e^{-2t} + 3t^2 - t + 2.$

(b) The homogeneous problem has characteristic equation

$$r^2 + 4r + 13 = 0 \qquad \Rightarrow \qquad r_{1,2} = \frac{-4}{2} \pm \frac{1}{2}\sqrt{16 - (4)(13)} = -2 \pm 3i.$$

So, our

$$y_1(t) = e^{-2t}\cos(3t), \quad y_2(t) = e^{-2t}\sin(3t) \quad \Rightarrow \quad y_h(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t),$$

and

$$\begin{aligned} |\mathbf{\Phi}(t)| &= \begin{vmatrix} e^{-2t}\cos(3t) & e^{-2t}\sin(3t) \\ e^{-2t}[-2\cos(3t) - 3\sin(3t)] & e^{-2t}[-2\sin(3t) + 3\cos(3t)] \end{vmatrix} \\ &= e^{-4t}\left[-2\cos(3t)\sin(3t) + 3\cos^2(3t) + 2\cos(3t)\sin(3t) + 3\sin^2(3t)\right] \;=\; 3e^{-4t}[\cos^2(3t) + \sin^2(3t)] \\ &= 3e^{-4t}. \end{aligned}$$

Thus,

$$u_1(t) \;=\; \int \frac{[-e^{-2t}\sin(3t)][9e^{-2t}\sec(3t)]}{3e^{-4t}}\,dt \;=\; \int \frac{-3\sin(3t)}{\cos(3t)}\,dt \;=\; \ln|\cos(3t)|,$$

$$u_2(t) \;=\; \int \frac{[e^{-2t}\cos(3t)][9e^{-2t}\sec(3t)]}{3e^{-4t}}\,dt \;=\; 3\int dt \;=\; 3t,$$

and

$$y_p(t) \;=\; u_1 y_1 + u_2 y_2 \;=\; e^{-2t}\cos(3t)\ln|\cos(3t)| + 3te^{-2t}\sin(3t).$$

So, our general solution is

$$y(t) \;=\; c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + e^{-2t}\cos(3t)\ln|\cos(3t)| + 3te^{-2t}\sin(3t).$$

4. (a) Resonance occurs when $\omega = \omega_0$, the natural frequency. That frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$$

(b) Critical damping for $mu'' + \gamma u' + ku = 0$ occurs when the discriminant (from the quadratic formula) is zero. That is, when

$$\gamma^2 - 4mk = 0 \implies \gamma = 2\sqrt{mk} = 2\sqrt{(9)(4)} = 12.$$

5. $y'' + 5y' + 4y = g(t)$ has Laplace transforms (right and left sides) which, after accounting for the zero ICs, is

$$(s^2 + 5s + 4)Y(s) = G(s) \implies Y(s) = H(s)G(s),$$

where $H(s) = \dfrac{1}{s^2 + 5s + 4}$ is the transfer function, and $G(s) = \mathcal{L}\{g(t)\}$.

Using that multiplication on the frequency side corresponds to convolution on the time side, we have

$$Y(s) = H(s)G(s) \implies y(t) = (h * g)(t),$$

where the impulse response $h(t) = \mathcal{L}^{-1}\{H(s)\}$. By partial fractions,

$$\frac{1}{s^2+5s+4} = \frac{A}{s+4} + \frac{B}{s+1},$$ where (after some work), $A = -\frac{1}{3}$, $B = \frac{1}{3}$.

So,

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = \underbrace{\boxed{\frac{1}{3}\left(e^{-t} - e^{-4t}\right)}}_{\text{This answers part (b).}}.$$

Finally, as answer to (a),

$$y(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right) * g(t) = \boxed{\int_0^t \frac{1}{3}\left(e^{-w} - e^{-4w}\right) g(t-w)\,dw}.$$