Stat 343, Tue 13-Oct-2020 -- Tue 13-Oct-2020

Probability and Statistics

Fall 2020

Tuesday, October 13th 2020

Wk 7, Tu

Topic:: Gamma, Weibull, beta distributions

Read:: FASt 3.4

Say $X \sim \text{Gamma}(\alpha = 3, \lambda = 0.5)$

Find $P(X \leq 10) = \text{pgamma}(10, \text{shape} = 3, \text{rate} = 0.5)$

$$= \int_0^{10} \underbrace{x^2 e^{-x/2} \, dx}_{} \cdot \text{rescaling factor}$$

Some Other Continuous Distributions

Continuous distributions thus far

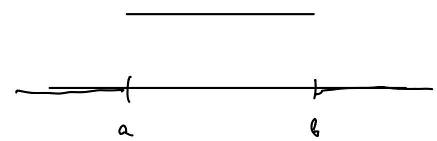
We have followed a process for each of the continuous distributions studied to date:

1. Select a type of kernel function, with set parameters.
2. Find a factor that rescales the area under the kernel function to be 1. Usually this factor has depended on the values of the parameters.

Specific instances:

- For parameters $a, b \in \mathbb{R}$, $a < b$, we considered kernel functions

$$k(x) \;=\; \begin{cases} 1, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$



  - appropriate scaling factor: $\frac{1}{b-a}$
  - result: a pdf $f(x; a, b) = \frac{1}{b-a} k(x)$, generating the **uniform distributions**

- For parameters $\lambda > 0$ we considered kernel functions

$$k(x) \;=\; \begin{cases} e^{-\lambda x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

  - appropriate scaling factor: λ
  - result: a pdf $f(x; \lambda) = \lambda k(x)$, generating the **exponential distributions**

- For parameters $\mu \in \mathbb{R}$, $\sigma > 0$, we consider kernel functions $k(x) \;=\; e^{-(x-\mu)^2/(2\sigma^2)}$.
  - appropriate scaling factor: $\frac{1}{\sigma\sqrt{2\pi}}$
  - result: a pdf $f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} k(x)$, generating the **normal distributions**

Gamma distributions

Consider, now, this kernel function, with parameters λ, $\alpha > 0$:

$$k(x) \;=\; \begin{cases} 0, & \text{if } x < 0, \\ x^{\alpha-1} e^{-\lambda x}, & \text{if } x \ge 0, \end{cases} \tag{1}$$

We have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_0^\infty x^{\alpha-1} e^{-\lambda x}\,dx \;=\; \int_0^\infty \left(\frac{u}{\lambda}\right)^{\alpha-1} e^{-u} \left(\frac{1}{\lambda}\right) du \quad \text{substituting } u = \lambda x, \\ &= \lambda^{-\alpha} \underbrace{\boxed{\int_0^\infty u^{\alpha-1} e^{-u}\,du}}_{\Gamma(\alpha)} \;=\; \lambda^{-\alpha}\Gamma(\alpha), \end{aligned}$$

For integers $\alpha \ge 1$

$\Gamma(\alpha) = (\alpha-1)!$

where $\Gamma(\alpha)$ is a special function arising in many applications given by

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

Some properties of the Gamma function are listed in Lemma 3.4.9, p. 162.

The Γ function is *not* as foreign as you might first think. Try out these commands:

```
gamma(2)
gamma(3)
gamma(4)
gamma(5)
gamma(6)
gf_point( c(0,factorial(0:5)) ~ 0:6) %>%
  gf_segment(0+factorial(0:5) ~ (1:6)+(1:6)) %>%
  gf_fun(gamma(x) ~ x, xlim=c(0,6), color="blue")
```

So, the kernel function k defined in Equation (1), when rescaled by the factor $\lambda^\alpha/\Gamma(\alpha)$, becomes a density function.

Definition 1: Let X be the (continuous) r.v. whose pdf is

params for this family
α (shape param)
λ (rate param)

$$f(x;\alpha,\lambda) := \boxed{\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.}$$

Such an X is said to be a **gamma random variable** with **shape** parameter α and **rate** parameter λ; this is denoted by $X \sim \mathsf{Gamma}(\alpha,\lambda)$.

By taking $\beta = 1/\lambda$, we may also write

$$f(x;\alpha,\beta) := \boxed{\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}.}$$

We have exchanged the parameter λ for another, β, called the **scale** parameter. R accepts either choice possibility, if the name is specified.

```
gf_dist("gamma", params=list(shape=2, rate=3))
gf_dist("gamma", params=list(shape=2, scale=1/3))
```

Note that the exponential family of distributions arises as a special case of gamma distributions when $\alpha = 1$.

To evaluate the pdf in `RStudio`, we may employ either the pair of parameters (α,λ), or the pair (α,β):

```
dgamma(1, shape=1.5, rate=2)

[1] 0.4319277
```

```
dgamma(1, shape=1.5, scale=.5)

[1] 0.4319277
```

Theorem 1 (Theorem 3.4.11, p. 163 in FASt): Let $X \sim \text{Gamma}(\alpha, \lambda)$. Then

(i) $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$. fun to do yourself

(ii) $\mathrm{E}(X) = \alpha/\lambda$.

(iii) $\mathrm{Var}(X) = \alpha/\lambda^2$.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \, \text{pdf}(x) \, dx$$

$$E(X) = M_X'(0)$$

$$\text{Var}(X) = M_X''(0) - [M_X'(0)]^2 = \mu_2 - \mu_1^2 = E(X^2) - [E(X)]^2.$$

Lemma 1 (Lemma 3.4.12, p. 164 in FASt): Let $X \sim \text{Gamma}(\alpha, \lambda)$, and $Y = kX$. Then $Y \sim \text{Gamma}(\alpha, \lambda/k)$.

$Y = kX$ is also a gamma r.v. when X is.

Since $X \sim \text{Gamma}(\alpha, \lambda)$, its mgf $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$

So $Y = kX$, its mgf

$$M_Y(t) = E(e^{tY}) = E(e^{t(kX)}) = E(e^{(kt)X})$$

$$= M_X(kt) = \left(\frac{\lambda}{\lambda - kt}\right)^\alpha = \left(\frac{\lambda/k}{\lambda/k - t}\right)^\alpha$$

in form, same as M_X but with rate param. λ replaced by λ/k.

$\Rightarrow$ $Y \sim \text{Gamma}(\alpha, \lambda/k)$.

2 more cont. r.v. families in 3.4.

Weibull distributions

For a different generalization of the exponential distributions, we have the following.

Definition 2: A random variable X is said to have a **Weibull distribution** with **shape** parameter $\alpha > 0$ and **scale** parameter $\beta > 0$ if the pdf for X is

$$f(x;\alpha,\beta) := \begin{cases} 0, & \text{if } x < 0, \\ \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}, & \text{if } x \geq 0, \end{cases}$$

We write $X \sim \mathsf{Weib}(\alpha,\beta)$.

Note that $\mathsf{Weib}(1,\beta) = \mathsf{Exp}(1/\beta)$.

The mean and variance for a Weibull r.v. are given in the following lemma.

Lemma 2 (Lemma 3.4.14, p. 165 in FASt): Suppose $X \sim \mathsf{Weib}(\alpha,\beta)$. Then

(i) $\mathrm{E}(X) = \beta\Gamma(1 + \frac{1}{\alpha})$.
(ii) $\mathrm{Var}(X) = \beta^2 \left[\Gamma(1 + \frac{2}{\alpha}) - \Gamma^2(1 + \frac{1}{\alpha})\right]$.

Beta distributions

Now, for parameters α, $\beta > 0$, consider the kernel function

$$k(x) \;=\; \begin{cases} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

As this function is nonzero only on a finite interval (inside which it is nonnegative), its integral can be rescaled to produce a density function. It can be shown that the appropriate rescaling constant is $\Gamma(\alpha+\beta)/(\Gamma(\alpha)\Gamma(\beta))$.

Definition 3: A r.v. X is said to have a **beta distribution** with **shape** parameters α, $\beta > 0$ if the pdf for X is

$$f(x;\alpha,\beta) \;=\; \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, we write $X \sim \mathsf{Beta}(\alpha,\beta)$.

Some examples of the variety of shapes one can obtain with different choices of parameters α, β are displayed in Figure 3.11, p. 166. One, perhaps surprising, possibility is that, when $\alpha = \beta = 1$, we obtain the $\mathsf{Unif}(0,1)$. We have the following facts.

Lemma 3 (Lemma 3.4.16, p. 167 in FASt): Let $X \sim \mathsf{Beta}(\alpha,\beta)$. Then

(i) $\mathrm{E}(X) = \frac{\alpha}{\alpha+\beta}$.

(ii) $\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

(iii) $M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\Pi_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$.