1. We have

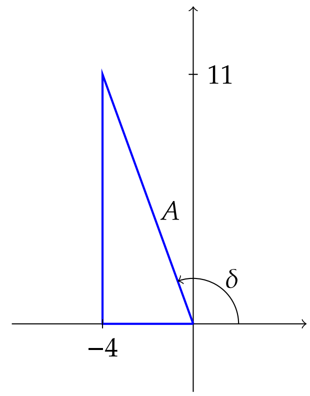
$$A = \sqrt{(-4)^2 + 11^2} = \sqrt{137}$$

and

$$\cos\delta = \frac{-4}{\sqrt{137}}, \quad \sin\delta = \frac{11}{\sqrt{137}} \quad \Rightarrow \quad \delta \doteq 1.92.$$

Thus,

$$-4\cos(2t) + 11\sin(2t) \approx \sqrt{137}\cos(2t - 1.92).$$




2. (a) The function $3t^2 + 2t - 2$ can be shifted left two units:

$$3t^2 + 2t - 2\Big|_{t\mapsto t+2} = 3(t+2)^2 + 2(t+2) - 2 = 3t^2 + 14t + 14.$$

This altered function, when shifted *right* two units, returns us to the original polynomial. And so

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\left\{(3t^2 + 2t - 2)u(t-2)\right\} = \mathcal{L}\left\{\left(3t^2 + 14t + 14\Big|_{t\mapsto t-2}\right)u(t-2)\right\}\\ &= \left(3\mathcal{L}\left\{t^2\right\} + 14\mathcal{L}\{t\} + 14\mathcal{L}\{1\}\right)e^{-2s} = \left(\frac{6}{s^3} + \frac{14}{s^2} + \frac{14}{s}\right)e^{-2s}.\end{aligned}$$

(b) Here, $f(t) = \left(5e^{3t}\sin(2t)\right) * \left(t^4\right)$, and so

$$\mathcal{L}\{f(t)\} = 5\mathcal{L}\left\{e^{3t}\sin(2t)\right\}\cdot\mathcal{L}\left\{t^4\right\} = \frac{10}{(s-3)^2+4}\cdot\frac{4!}{s^5} = \frac{240}{s^5(s^2-6s+13)}\cdot\frac{4!}{s^5}.$$

(c) First, we ignore the exponential $e^{-3s}$. By partial fractions,

$$\frac{5}{(s^2+4s+8)(s+1)} = \frac{As+B}{s^2+4s+8} + \frac{C}{s+1}.$$

Multiplying through by the common denominator gives

$$5 = (As+B)(s+1) + C(s^2+4s+8) = (A+C)s^2 + (A+B+4C)s + (B+8C).$$

Equating coefficients of $s$-terms, we have a matrix problem:

$$\begin{bmatrix}1&0&1\\1&1&4\\0&1&8\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix} = \begin{bmatrix}0\\0\\5\end{bmatrix} \quad\Rightarrow\quad \left[\begin{array}{ccc|c}1&0&1&0\\1&1&4&0\\0&1&8&5\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&-3\\0&0&1&1\end{array}\right] \quad\Rightarrow\quad A=-1, B=-3, C=1.$$

So,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5}{(s^2+4s+8)(s+1)}\right\} &= \mathcal{L}^{-1}\left\{-\frac{s+3}{s^2+4s+8} + \frac{1}{s+1}\right\}\\ &= -\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= -e^{-2t}\cos(2t) - \frac{1}{2}e^{-2t}\sin(2t) + e^{-t}.\end{aligned}$$

As to the exponential factor,

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{5}{(s^2+4s+8)(s+1)}\right\} = u(t-3)\left[-e^{-2(t-3)}\cos(2(t-3)) - \frac{1}{2}e^{-2(t-3)}\sin(2(t-3)) + e^{-(t-3)}\right].$$

3. (a) In finding the homogeneous part $y_h$ of the solution, our characteristic equation has a double root:

$$(r+3)^2 = 0 \qquad \Rightarrow \qquad r = --3,\ -3 \qquad \Rightarrow \qquad y_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}.$$

We propose a particular solution that, like the right-hand side, is a $2^{\text{nd}}$-degree polynomial:

$$y_p(t) = At^2 + Bt + C \qquad \Rightarrow \qquad y_p' = 2At + B, \qquad y_p'' = 2A.$$

Then

$$y_p'' + 6y_p' + 9y_p \;=\; 2A + 6(2At + B) + 9(At^2 + Bt + C) \;=\; 9At^2 + (12A + 9B)t + (2A + 6B + 9C).$$

Because our target function—what we want this result to equal—is $18t^2 + 15t - 11$, we can make this work by choosing $A, B, C$ so that

$$\left.\begin{array}{l} 9A = 18 \\ 12A + 9B = 15 \\ 2A + 6B + 9C = -11 \end{array}\right\} \quad \Rightarrow \quad A = 2,\ B = -1,\ C = -1.$$

Thus, $y_p(t) = 2t^2 - t - 1$, and $y(t) \;=\; y_h(t) + y_p(t) \;=\; c_1 e^{-3t} + c_2 t e^{-3t} + 2t^2 - t - 1.$

(b) The homogeneous problem has characteristic equation

$$r^2 + 6r + 13 = 0 \qquad \Rightarrow \qquad r_{1,2} = \frac{-6}{2} \pm \frac{1}{2}\sqrt{36 - (4)(13)} = -3 \pm 2i.$$

So, our

$$y_1(t) = e^{-3t}\cos(2t), \quad y_2(t) = e^{-3t}\sin(2t) \quad \Rightarrow \quad y_h(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t),$$

and

$$\begin{aligned} |\mathbf{\Phi}(t)| &= \begin{vmatrix} e^{-3t}\cos(2t) & e^{-3t}\sin(2t) \\ e^{-3t}[-3\cos(2t) - 2\sin(2t)] & e^{-3t}[-3\sin(2t) + 2\cos(2t)] \end{vmatrix} \\ &= e^{-6t}\left[-3\cos(2t)\sin(2t) + 2\cos^2(2t) + 3\cos(2t)\sin(2t) + 2\sin^2(2t)\right] \;=\; 2e^{-6t}[\cos^2(2t) + \sin^2(2t)] \\ &= 2e^{-6t}. \end{aligned}$$

Thus,

$$u_1(t) \;=\; \int \frac{[-e^{-3t}\sin(2t)][4e^{-3t}\sec(2t)]}{2e^{-6t}}\,dt \;=\; \int \frac{-2\sin(2t)}{\cos(2t)}\,dt \;=\; \ln|\cos(2t)|,$$

$$u_2(t) \;=\; \int \frac{[e^{-3t}\cos(2t)][4e^{-3t}\sec(2t)]}{2e^{-6t}}\,dt \;=\; 2\int dt \;=\; 2t,$$

and

$$y_p(t) \;=\; u_1 y_1 + u_2 y_2 \;=\; e^{-3t}\cos(2t)\ln|\cos(2t)| + 2te^{-3t}\sin(2t).$$

So, our general solution is

$$y(t) \;=\; c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t) + e^{-3t}\cos(2t)\ln|\cos(2t)| + 2te^{-3t}\sin(2t).$$

[redacted]

4. (a) Resonance occurs when $\omega = \omega_0$, the natural frequency. That frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{49}{25}} = \frac{7}{5}.$$

(b) Critical damping for $mu'' + \gamma u' + ku = 0$ occurs when the discriminant (from the quadratic formula) is zero. That is, when

$$\gamma^2 - 4mk = 0 \implies \gamma = 2\sqrt{mk} = 2\sqrt{(25)(49)} = 70.$$

5. $y'' + 5y' + 4y = g(t)$ has Laplace transforms (right and left sides) which, after accounting for the zero ICs, is

$$(s^2 + 5s + 4)Y(s) = G(s) \implies Y(s) = H(s)G(s),$$

where $H(s) = \dfrac{1}{s^2 + 5s + 4}$ is the transfer function, and $G(s) = \mathcal{L}\{g(t)\}$.

Using that multiplication on the frequency side corresponds to convolution on the time side, we have

$$Y(s) = H(s)G(s) \implies y(t) = (h * g)(t),$$

where the impulse response $h(t) = \mathcal{L}^{-1}\{H(s)\}$. By partial fractions,

$$\frac{1}{s^2+5s+4} = \frac{A}{s+4} + \frac{B}{s+1}, \quad \text{where (after some work)}, \ A = -\frac{1}{3},\ B = \frac{1}{3}.$$

So,

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = \underbrace{\boxed{\frac{1}{3}\left(e^{-t} - e^{-4t}\right)}}_{\text{This answers part (b).}}.$$

Finally, as answer to (a),

$$y(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right) * g(t) = \boxed{\int_0^t \frac{1}{3}\left(e^{-w} - e^{-4w}\right) g(t-w)\,dw}.$$