$\star$11 The answer requires construction of a matrix, which here I call $\Phi(t)$. Prof. Kapitula calls it the Wronskian, but I have defined the Wronskian to be the determinant of $\Phi(t)$, not simply the matrix $\Phi(t)$.

(a) We have

$$\Phi(t) = \begin{pmatrix} \mathrm{e}^t & \mathrm{e}^{2t} & \mathrm{e}^{3t} \\ \mathrm{e}^t & 2\mathrm{e}^{2t} & 3\mathrm{e}^{3t} \\ \mathrm{e}^t & 4\mathrm{e}^{2t} & 9\mathrm{e}^{3t} \end{pmatrix} \quad \rightsquigarrow \quad \Phi(0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}.$$

Since $\Phi(0) \xrightarrow{\text{RREF}} \mathbf{I}_3$, the functions are linearly independent. This is not the only way to arrive at this conclusion. One might evaluate $\Phi(t)$ at some other value than $t = 0$, and look at RREF; finding it to be $\mathbf{I}_3$ once again, the conclusion is the same. Instructions in Kapitula's book for determining linear independence of functions take this route.

Instead of RREF, one might compute the Wronskian, either at a specific value of t (again, easiest at $t = 0$), or for general t as I do below. (This is the method I have shown to the class for ascertaining linear independence of functions.)

$$\begin{aligned} W(t) &= \begin{vmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{vmatrix} = e^t \begin{vmatrix} 2e^{2t} & 3e^{3t} \\ 4e^{2t} & 9e^{3t} \end{vmatrix} - e^t \begin{vmatrix} e^{2t} & e^{3t} \\ 4e^{2t} & 9e^{3t} \end{vmatrix} + e^t \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} \\ &= e^t(18e^{5t} - 12e^{5t}) - e^t(9e^{5t} - 4e^{5t}) + e^t(3e^{5t} - 2e^{5t}) = 6e^{6t} - 5e^{6t}e^{6t} = 2e^{6t}. \end{aligned}$$

Since this Wronskian is nonzero (here at all choices of t, but it is sufficient to demonstrate that it is nonzero at some particular t-value), the collection of functions is linearly independent.

(b) We have

$$\Phi(t) = \begin{pmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{pmatrix} \quad \rightsquigarrow \quad \Phi(0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since $\Phi(0) \xrightarrow{\text{RREF}} I_3$, the functions are linearly independent.

Using the Wronskian at $t = 0$ yields the same result:

$$W(0) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 1.$$

The fact this determinant is nonzero implies linear independence.

$\underline{\star 12}$ We wish to see if there is a choice of vector $\mathbf{c} = (c_1, c_2, c_3)$ different from $\mathbf{c} = \mathbf{0}$ such that $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$. But

$$\begin{aligned} c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 &= c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) \\ &= (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3 \\ &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3, \end{aligned}$$

where $d_1 = c_1 + c_2$, $d_2 = c_1 + c_3$, and $d_3 = c_2 + c_3$. We know $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, so the only way we get

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 = \mathbf{0}$$

is if $d_1 = d_2 = d_3 = 0$. Thus, the only way we get

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$

is if

$$\begin{aligned} c_1 + c_2 &= 0, \\ c_1 + c_3 &= 0, \\ c_2 + c_3 &= 0, \end{aligned} \qquad \text{or, in matrix form,} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2,$$

we know this matrix problem has just one solution—namely, $c_1 = c_2 = c_3 = 0$. Thus, it is true that $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ are linearly independent.

$\underline{\star 13}$ (a) FALSE. The vectors described do form a plane in $\mathbb{R}^3$, but this plane does not include the origin, and is not a *subspace* of $\mathbb{R}^3$. The column space of any 3-by-5 matrix *must* be a subspace of $\mathbb{R}^3$.

(b) FALSE. The null space always has at least one vector, the zero vector.

(c) FALSE. Since there is a nonzero vector in null $(\mathbf{A})$, the columns of $\mathbf{A}$ are linearly dependent; in particular, $-4\mathbf{A}_1 + 5\mathbf{A}_2 + 9\mathbf{A}_3 = \mathbf{0}$.

$\underline{\star 14}$ We use the notation $\mathrm{m_a}(\lambda)$ to denote the algebraic multiplicity of the eigenvalue λ.

(b) $n = 9$ with $\mathrm{m_a}(0) = 2$, $\mathrm{m_a}(1) = 4$, $\mathrm{m_a}(-3) = 1$, $\mathrm{m_a}(2 + \mathrm{i}3) = 1$, $\mathrm{m_a}(2 - \mathrm{i}3) = 1$. The matrix is not invertible.

(c) $n = 8$ with $\mathrm{m_a}(-2) = 3$, $\mathrm{m_a}(-5) = 1$, $\mathrm{m_a}(-3 + \mathrm{i}4) = 2$, $\mathrm{m_a}(-3 - \mathrm{i}4) = 2$. The matrix is invertible.

$\underline{\star 15}$ (a) $5\mathbf{A}\mathbf{v}_1 - 3\mathbf{A}\mathbf{v}_2 = 5(-3)\mathbf{v}_1 - 3(7)\mathbf{v}_2 = \begin{pmatrix} -33 \\ -243 \end{pmatrix}$.

(b) Since

$$\begin{pmatrix} 6 \\ 4 \end{pmatrix} = -\frac{36}{31}\mathbf{v}_1 + \frac{38}{31}\mathbf{v}_2,$$

we have

$$\mathbf{A}\mathbf{x} = -\frac{36}{31}(-3)\mathbf{v}_1 + \frac{38}{31}(7)\mathbf{v}_2 = \begin{pmatrix} 582/31 \\ 2668/31 \end{pmatrix}.$$