Stat 343, Thu 17-Sep-2020 -- Thu 17-Sep-2020
Probability and Statistics
Fall 2020

Thursday, September 17th 2020

Wk 3, Th
Topic:: mean, variance of random variable

Crazy example X has pmf

x	-1	0	1
$f_X(x)$	$1/6$	$1/6$	$2/3$

When $X = -1$
$Y = 1/2$

$X = 0$
$Y = 2$

$X = 1$
$Y = 3/2$

A 2^{nd} r.v. $Y = \dfrac{2+X}{1+X^2}$

Its pmf

y	$1/2$	$3/2$	2
$f_Y(y)$	$1/6$	$2/3$	$1/6$

$$\mu_Y = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) + \left(\frac{3}{2}\right)\left(\frac{2}{3}\right) + (2)\left(\frac{1}{6}\right)$$

$$= \sum_x \left(\underline{\frac{2+x}{1+x^2}}\right) \cdot \underline{f_X(x)}$$

$$F = \frac{9}{5}C + 32$$

The corollary: $Y = 2X - 5$

implies $E(Y) = 2E(X) - 5$

Mean and Variance of a random variable X

Definition 1 (2.5.1): Given a discrete random variable X with pmf $f_X(x)$, the **expected value** of X, also known as its **mean**, and variously denoted $\mathrm{E}(X)$ or μ_X, is given by

$$\mathrm{E}(X) = \sum_x x f_X(x).$$

- weighted averages: calculating your semester gpa

Credits	gpa
3	3.5
4	4.0
4	2.5

$$\frac{3(3.5)+4(4)+4(2.5)}{11}$$

- Problem B.19: Given the pmf

x	0	1	2	3	4
$f(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$

X^2	0	1	4	9	16
	1/6	1/3	1/4	1/6	1/12

(a) Find $\sum_x f(x)$.

$$= \frac{1}{6} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = 1$$

(b) Find $\sum_x x f(x)$.

$$= (0)\left(\frac{1}{6}\right) + (1)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{4}\right) + (3)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{12}\right)$$

$$= \frac{5}{3}\ ?$$

Example: Given our theoretical pmf for X, the sum of pips from the roll of two fair dice, calculate $\mathrm{E}(X)$.

```
x = 2:12
probability = c(1:6,5,4,3,2,1) / 36
sum( x * probability )
```

Question: What is $\mathrm{E}(X^2)$ for X as in Exercise B.12? Is it the same/different from $(\mathrm{E}(X))^2$?

$$0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{6} + 4^2 \frac{1}{12} = \frac{25}{6}$$

$$\mathrm{E}(X^2) = \frac{25}{6} \underset{\text{different}}{\longleftrightarrow} (\mathrm{E}(X))^2 = \left(\frac{5}{3}\right)^2 = \frac{25}{9}.$$

Lemma 1 (2.5.3): Suppose X is a discrete random variable, and $Y = t(X)$. Then

$$\mathrm{E}(Y) = \sum_x t(x) \cdot f_X(x).$$

Corollary 1 (2.5.4): If $Y = aX + b$ (that is, Y is a linear transformation of the random variable X), then $\mathrm{E}(Y) = a \cdot \mathrm{E}(X) + b$.

Question: What is $\mathrm{E}(X)$ for $\boxed{X \sim \mathrm{Binom}(n, \pi)}$? Do some test cases, then make a conjecture.

```
enn = 5
x = 0:enn
testProb = 0.2
sum( x * dbinom(x, size=enn, prob = testProb )
```

Conjecture $\mathrm{E}(X) = n\pi$

General result: the mean of a binomial random variable $X \sim \mathrm{Binom}(n, \pi)$

$$\mathrm{E}(X) = \sum_{x=0}^{n} x \binom{n}{x} \pi^x (1-\pi)^{n-x} = \sum_{x=0}^{\infty} x \frac{n!}{x!(n-x)!} \pi^x (1-\pi)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} \pi^x (1-\pi)^{n-x} = \sum_{x=1}^{n} \frac{n \cdot (n-1)!}{(x-1)!(n-x)!} \pi^x (1-\pi)^{n-x}$$

Note: $n - x = (n-1) - (x-1)$

$$= \sum_{x=1}^{n} n\pi \frac{(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!} \pi^{x-1} (1-\pi)^{[(n-1)-(x-1)]}$$

$\tilde{n} = n - 1$

$\tilde{x} = x - 1$

$$= \sum_{x=1}^{n} n\pi \binom{n-1}{x-1} \pi^{x-1} (1-\pi)^{[(n-1)-(x-1)]}$$

$$= n\pi \boxed{\sum_{\tilde{x}=0}^{\tilde{n}} \binom{\tilde{n}}{\tilde{x}} \pi^{\tilde{x}} (1-\pi)^{\tilde{n}-\tilde{x}}} = n\pi$$

Definition 2 (2.5.7): Let X be a discrete r.v. The **variance** of X, denoted by $\mathrm{Var}(X)$ or by σ_X^2, is that variables mean squared deviation from the mean. More explicitly, that is

$$\mathrm{Var}(X) \;=\; \mathrm{E}((X-\mu_X)^2).$$

Example. Compute by hand the variance for X when

(a) $X \sim \mathrm{Binom}(2, 0.3)$

(b) $X \sim \mathrm{Binom}(3, 0.5)$

Theorem 1 (2.5.8): Let X be a discrete r.v. Then $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$.

Q: For $X \sim \mathrm{NBinom}(3, \pi = 0.2)$, can we estimate

$$E(X) = \sum_x x f_X(x)$$