1. (a) We have

$$h(t) = 5e^{-2t}\int_0^t \cos(w)e^{2w}\,dw = \int_0^t 5\cos(w)e^{-2(t-w)}\,dw = (f \star g)(t),$$

where $f(t) = 5\cos(t)$, and $g(t) = e^{-2t}$.

(b) Since

$$\mathcal{L}\left\{e^t \sin t\right\}(s) = \mathcal{L}\{\sin t\}(s)\Big|_{s\mapsto s-1} = \frac{1}{s^2+1}\Big|_{s\mapsto s-1} = \frac{1}{(s-1)^2+1} = \frac{1}{s^2-2s+2},$$

and

$$\mathcal{L}\left\{e^{2t}\right\}(s) = \frac{1}{s-2},$$

the Convolution Theorem says

$$\mathcal{L}\left\{(e^t \sin t) \star e^{2t}\right\}(s) = \mathcal{L}\left\{e^t \sin t\right\} \cdot \mathcal{L}\left\{e^{2t}\right\} = \frac{1}{(s-2)(s^2-2s+2)}.$$

2. We a denominator $s^2 + 2s + 5$ that is an irreducible quadratic, and so, we complete the square:

$$\frac{3}{s^2+2s+5} = \frac{3}{(s^2+2s+1)+4} = \frac{3}{(s+1)^2+4} = \frac{3}{s^2+4}\Big|_{s\mapsto s-(-1)}.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+2s+5}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\Big|_{s\mapsto s-(-1)}\right\} = \frac{3}{2}e^{-t}\sin(2t).$$

3. In preparation for using a shift theorem, we look for the function $g(t)$ which, when shifted right 3 units, becomes $f(t)$. This means

$$g(t) = f(t+3) = 3(t+3)^2 - 2 = 3(t^2+6t+9) - 2 = 3t^2 + 18t + 25.$$

Now

$$\mathcal{L}\{U(t-3)f(t)\} = \mathcal{L}\{U(t-3)g(t-3)\} = e^{-3s}\cdot\mathcal{L}\{g(t)\} = e^{-3s}\left(\frac{6}{s^3} + \frac{18}{s^2} + \frac{25}{s}\right).$$

4. (a) This $\delta(t-2)$ represents a shock, or blow to the spring assembly, occurring at time $t = 2$. It is a force delivering finite energy.

(b) We have $H(s) = \dfrac{1}{3s^2+4s+1}$.

(c) This term, like all terms in the DE, is a force. It is, particularly, the damping force on spring motion.

(d) We may take the mass $m = 3$ and the spring constant $k = 1$, giving the natural frequency as $\omega_0 = \sqrt{k/m} = 1/\sqrt{3}$.

(e) We may take Laplace transforms of both sides,

$$\mathcal{L}\{3y'' + 4y' + y\} = \mathcal{L}\{\delta(t-2)\}$$

and, because the ICs in Sub-Problem (2) are zeroed, this becomes

$$3s^2Y + 4sY + Y = e^{-2s}, \qquad \text{or} \qquad Y(s) = e^{-2s}\frac{1}{3s^2+4s+1}.$$

One can use partial fractions on the transfer function to obtain

$$H(s) \;=\; \frac{1}{(3s+1)(s+1)} \;=\; \frac{3/2}{3s+1} - \frac{1/2}{s+1},$$

so that

$$\mathcal{L}^{-1}\{H(s)\} = \frac{1}{2}e^{-t/3} - \frac{1}{2}e^{-t}.$$

Since $Y(s)$ is not just $H(S)$, but has an exponential factor, too, we use a shift theorem to get

$$y(t) \;=\; \mathcal{L}^{-1}\left\{e^{-2s}H(s)\right\} \;=\; \frac{1}{2}U(t-2)\left[e^{-(t-2)/3} - e^{-(t-2)}\right].$$

(f) The solution to the full (original) problem is the sum of the solutions to Sub-Problems (1) and (2), and while we have (2) solved above, it seems easier to solve (1) using Chapter 4 methods. The characteristic equation:

$$3\lambda^2 + 4\lambda + 1 = 0 \qquad \text{has roots} \qquad \lambda = -1,\; -\frac{1}{3}.$$

Thus, the general solution is

$$y_h(t) \;=\; c_1 e^{-t} + c_2 e^{-t/3},$$

which, in preparation for initial conditions, has derivative

$$y_h'(t) \;=\; -c_1 e^{-t} - \frac{1}{3}c_2 e^{-t/3}.$$

Applying the ICs leads to two equations

$$\left.\begin{array}{r} -1 = y_h(0) = c_1 + c_2 \\ 1 = y_h'(0) = -c_1 - (1/3)c_2 \end{array}\right\} \quad\Rightarrow\quad \begin{bmatrix} 1 & 1 \\ -1 & -1/3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad\Rightarrow\quad c_1 = -1,\; c_2 = 0.$$

So, $y_h(t) = -e^{-t}$, and

$$y(t) \;=\; -e^{-t} + \frac{1}{2}U(t-2)\left[e^{-(t-2)/3} - e^{-(t-2)}\right].$$

5. (a) The homogeneous DE has characteristic polynomial

$$\lambda^2 + 4\lambda + 4 \;=\; (\lambda+2)^2,$$

so a fundamental set of solutions (repeated root case) is e^{-2t} and te^{-2t}, and the homogeneous solution is

$$y_h(t) \;=\; c_1 e^{-2t} + c_2 t e^{-2t}.$$

The target/nonhomogeneous term $f(t)$ is a first degree polynomial, and there is no overlap with y_h in proposing

$$y_p(t) = At + B \qquad\Rightarrow\qquad y_p'(t) = A,\;\; y_p''(t) = 0.$$

Inserting these into the left-hand side of the DE, we have

$$y_p'' + 4y_p' + 4y_p \;=\; 0 + 4A + 4(At+B) \;=\; 4At + 4(A+B).$$

Since this result should match the target $f(t) = 4t - 3$, we get

$$\left.\begin{array}{l} 4A = 4 \\ 4A + 4B = -3 \end{array}\right\} \quad\Rightarrow\quad A = 1,\; B = -\frac{7}{4}.$$

The general solution, then, is

$$y(t) \;=\; y_h(t) + y_p(t) \;=\; c_1 e^{-t} + c_2 t e^{-t} + t - \frac{7}{4}.$$

(b) Here, the characteristic polynomial

$$\lambda^2 + 9 \quad \text{has roots} \quad \lambda = 0 \pm 3i, \qquad \text{and} \qquad y_h(t) \;=\; c_1\cos(3t) + c_2\sin(3t).$$

This gives us a fundamental matrix

$$\mathbf{\Phi} \;=\; \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}, \quad \text{and Wronskian} \quad \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} \;=\; 3[\cos^2(3t)+\sin^2(3t)] \;=\; 3.$$

Thus, variation of parameters gives us particular solution

$$\begin{aligned} y_p(t) &= \cos(3t)\int \frac{1}{3}\begin{vmatrix} 0 & \sin(3t) \\ 2\sec(3t) & 3\cos(3t) \end{vmatrix}\,dt + \sin(3t)\int \frac{1}{3}\begin{vmatrix} \cos(3t) & 0 \\ -3\sin(3t) & 2\sec(3t) \end{vmatrix}\,dt \\ &= -\frac{2}{3}\cos(3t)\int \frac{\sin(3t)}{\cos(3t)}\,dt + 2\sin(3t)\int dt \\ &= \frac{2}{9}\cos(3t)\ln|\cos(3t)| + 2t\sin(3t). \end{aligned}$$

Thus, the general solution is

$$y(t) \;=\; c_1\cos(3t) + c_2\sin(3t) + \frac{2}{9}\cos(3t)\ln|\cos(3t)| + 2t\sin(3t).$$