MATH 231, Worksheet
Date: May 8, 2020

Find the inverse Laplace transform for each function.

1. $F(s) = \dfrac{e^{-\pi s/2}}{s^2+9}$
2. $F(s) = \dfrac{1}{s^2(s^2+4)}$
3. $F(s) = \dfrac{s}{s^2+6s+11}$
4. $F(s) = e^{-s}\dfrac{s}{s^2+6s+11}$
5. $F(s) = \dfrac{1}{(s-1)^3} + \dfrac{1}{s^2+2s-8}$
6. $F(s) = e^{-2s}\dfrac{1}{(s-1)^3} + e^{-s}\dfrac{1}{s^2+2s-8}$

Overriding theme in course, made possible by a focus on linear problems

① Take a homogeneous version of the problem at hand

- solve it
- soln. often contains "freedoms"
- variously called: Null space, homogeneous soln., span of basis solutions

② Find one soln. to the given problem: particular soln.

Full soln. is sum of results from ① and ②.

Even Ch. 5 is in this vein.

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \ y'(0) = y_1$$

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{f(t)\}$$

$$a(s^2Y - sy_0 - y_1) + b(sY - y_0) + cY = F$$

$$(as^2 + bs + c)Y = F(s) + asy_0 + ay_1 + by_0$$

$$Y(s) = \frac{F(s)}{as^2 + bs + c} + \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\{H(s)F(s)\}}_{\substack{\text{soln. to} \\ ay''+by'+cy=0, \\ \text{zero ICs}}} + \underbrace{\mathcal{L}^{-1}\{H(s)(asy_0 + ay_1 + by_0)\}}_{\substack{\text{soln. to} \\ ay''+by'+cy=0,\ y(0)=y_0, \\ y'(0)=y_1}}$$

particular

homog. soln.
freedoms disappeared
because of ICs

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1. $\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s} \cdot \underline{\frac{1}{s^2+9}}\right\} = \underline{u\left(t - \frac{\pi}{2}\right) \cdot \frac{1}{3}\sin\left(3\left(t - \pi/2\right)\right)}.$

exponential
on s-side

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{1}{3}\sin(3t)$$

$$\boxed{\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as} \cdot \mathcal{L}\{f(t)\}}$$

s-side product

$$\mathcal{L}^{-1}\left\{ \mathcal{L}\{\delta(t-\pi/2)\} \cdot \mathcal{L}\{\tfrac{1}{3}\sin(3t)\} \right\}$$

$$= \delta(t-\pi/2) * \tfrac{1}{3}\sin(3t) \quad \text{(by convolution theorem)}$$

$$= \int_0^t \underbrace{\delta(w-\pi/2)}_{\text{activated when } w=\pi/2} \tfrac{1}{3}\sin(3(t-w))\,dw$$

$$= \begin{cases} 0, & t < \pi/2 \\ \tfrac{1}{3}\sin(3(t-\pi/2)), & t \geq \pi/2 \end{cases}$$

2. $\mathcal{L}^{-1}\left\{ \dfrac{1}{s^2(s^2+4)} \right\}$

comes from $A \cdot 1$ (t-side)

$B \cdot t$

One start $\dfrac{1}{s^2(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4}$

$$= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$= \frac{A\cancel{s}}{s^{\cancel{2}}} + \frac{B}{s^2} + \frac{Cs}{s^2+4} + \frac{D}{s^2+4}$$

t side: cosine

t side: sine

Another: See individually

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$$

Use convolution

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \cdot \frac{1}{s^2+4} \right\} = t * \tfrac{1}{2}\sin(2t)$$

$$= \int_0^t (t-w)\,\tfrac{1}{2}\sin(2w)\,dw$$

$$= \int_0^t w \cdot \tfrac{1}{2}\sin\big(2(t-w)\big)\,dw$$

3. $$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+6s+11} \right\}$$

roots of $s^2+6s+11=0$ $\qquad s = \frac{-6}{2} \pm \frac{1}{2}\sqrt{36-44}$

nonreal

$\Longrightarrow$ $s^2+6s+11$ irreducible

(no partial fractions)

$$\frac{s}{s^2+6s+11} = \frac{s}{s^2+6s+9+2} = \frac{s+3-3}{(s+3)^2+2}$$

$$= \frac{s+3}{(s+3)^2+2} - \underbrace{\frac{\sqrt{2}}{(s+3)^2+2}} \cdot \frac{3}{\sqrt{2}}$$

comes from

$e^{at}\cos(bt)$

w/ $a=-3$

$b=\sqrt{2}$

comes from

$e^{-3t}\sin(\sqrt{2}\,t)$

Answer: $\mathcal{L}^{-1}\{\ \} = e^{-3t}\cos(\sqrt{2}\,t) - \frac{3}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}\,t)$.

4. $\mathcal{L}^{-1}\left\{ e^{-s} \cdot \dfrac{s}{s^2+6s+11} \right\}$

Start: $\mathcal{L}^{-1}\left\{ \dfrac{s}{s^2+6s+11} \right\}$ = from # 3

Presence of exponential e^{-1s} on s-side

Ans. $u(t-1) \cdot e^{-3(t-1)}\left[\cos\left(\sqrt{2}(t-1)\right) - \frac{3}{\sqrt{2}} \sin\left(\sqrt{2}(t-1)\right)\right]$