1st-Order Linear Homogeneous systems of DEs: Degenerate Case

In all examples of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ ($\mathbf{A}$ necessarily square—let's say n-by-n) we have investigated thus far,

- eigenvalues λ_i have had AM = GM, and as a result,
- there has been a fundamental matrix solution $\mathbf{\Phi}(t)$ whose columns were $e^{\lambda_j t}\mathbf{v}_j$ for some eigenpair $(\lambda_j, \mathbf{v}_j)$.

So, what happens when some eigenvalue has GM < AM? For instance, in the problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} -5 & -8 & -9 \\ 9 & 16 & 18 \\ -6 & -10 & -11 \end{bmatrix},$$

the characteristic polynomial of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 + 3\lambda - 2 = -(\lambda+2)(\lambda-1)^2,$$

so $\lambda = -2$ is an eigenvalue with AM $= 1$ and $\lambda = 1$ is an eigenvalue with AM $= 2$. When you find the null space of $(\mathbf{A} + 2\mathbf{I})$, it must be 1-dimensional since the number of free columns in $\mathbf{A} - \lambda\mathbf{I}$ is stuck between 1 and the algebraic multiplicity of the eigenvalue λ. We find that a basis eigenvector of that null space is $\langle 2, -3, 2\rangle$. As for the other eigenvalue $\lambda = 1$, all we know going in is that $(\mathbf{A} - \mathbf{I})$ will have either 1 or 2 free columns (i.e., either a 1- or 2-dimensional null space). Indeed,

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} -6 & -8 & -9 \\ 9 & 15 & 18 \\ -6 & -10 & -12 \end{bmatrix} \qquad \text{has RREF} \qquad \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix},$$

so it has a 1-dimensional null space with basis eigenvector $\langle 1, -3, 2\rangle$. There are no more (linearly independent) eigenpairs to be had. The ones we've found give us solutions

$$e^{-2t}\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \quad \text{and} \quad e^{t}\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix},$$

but we need to find one more linearly independent solution to fill out a fundamental matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} 2e^{-2t} & e^{t} & \text{a 3rd soln} \\ -3e^{-2t} & -3e^{t} & \downarrow \\ 2e^{-2t} & 2e^{t} & \end{bmatrix}.$$

We have come as far as we have on the success of guess-and-check: we guessed a function of the form $e^{rt}\mathbf{v}$ might just solve a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, plugged it in, and found it does if $(\lambda, \mathbf{v})$ form an eigenpair. Let us guess, again, that maybe a solution could take the form

$$\mathbf{x}(t) = e^{rt}(\mathbf{u} + t\mathbf{v}), \qquad \text{so that the derivative is} \qquad \mathbf{x}'(t) = re^{rt}(\mathbf{u} + t\mathbf{v}) + e^{rt}\mathbf{v} = e^{rt}[(r\mathbf{u} + \mathbf{v}) + t\mathbf{v}].$$

Then inserting our guess into $\mathbf{x}' = \mathbf{A}\mathbf{x}$ gives us

$$e^{rt}[(r\mathbf{u} + \mathbf{v}) + t\mathbf{v}] = \mathbf{A}[e^{rt}(\mathbf{u} + t\mathbf{v}] = e^{rt}(\mathbf{A}\mathbf{u} + t\mathbf{A}\mathbf{v}).$$

After dividing through by e^{rt} we see both sides have a t-term and a constant term. Setting the coefficients of those two terms equal results in two equations:

$$r\mathbf{u} + \mathbf{v} = \mathbf{A}\mathbf{u} \qquad \text{and} \qquad r\mathbf{v} = \mathbf{A}\mathbf{v}.$$

Note the second of these equations is the eigenvalue-eigenvector equation all over again, while the first can be rearranged to say $\mathbf{v} = \mathbf{A}\mathbf{u} - r\mathbf{u} = (\mathbf{A} - r\mathbf{I})\mathbf{u}$. What this says is that our guess, $\mathbf{x}(t) = e^{rt}(\mathbf{u} + t\mathbf{v})$, *can work* if

- $(r, \mathbf{v})$ form an eigenpair of $\mathbf{A}$, and
- using this eigenpair $(r, \mathbf{v})$, we find a vector $\mathbf{u}$ which solves $(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{v}$.

Now, the equation $(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{v}$ can be *inconsistent* when $(r, \mathbf{v})$ forms an eigenpair; in fact, under such conditions it is *usually* inconsistent. However, in the case when the eigenvalue r has GM $= 1$ and AM > 1,[1] there solutions $\mathbf{u}$, infinitely many of them, which satisfy the equation; they are called **generalized eigenvectors** corresponding to eigenvalue r. Moreover, the proposed solution $\mathbf{x}(t) = e^{rt}(\mathbf{u} + t\mathbf{v})$, when used to fill out the columns of $\mathbf{\Phi}(t)$, contributes a linearly independent column from the others, helping us get a nonzero determinant (Wronskian).

So, our **modified algorithm** for solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is this:

- Solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ for the eigenvalues λ of $\mathbf{A}$.
- For each eigenvalue λ, find a *basis* of null $(\mathbf{A} - \lambda\mathbf{I})$. Any vector $\mathbf{v}$ in this basis is an eigenvector of $\mathbf{A}$ corresponding to λ, and the vector function $e^{\lambda t}\mathbf{v}$ contributes favorably to the general solution in the sense that it can be a column of $\mathbf{\Phi}(t)$ linearly independent with other similarly-formed columns.
- When λ has GM $= 1$ (so null $(\mathbf{A} - \lambda\mathbf{I})$ has dimension 1, and only one basis vector $\mathbf{v}$), and AM $\geqslant 2$, find one (any) solution of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$ and obtain another solution (another column of $\mathbf{\Phi}(t)$, linearly independent from the others) as $e^{\lambda t}(\mathbf{u} + t\mathbf{v})$.

For most situations, the algorithm above will generate enough columns for the matrix $\mathbf{\Phi}(t)$ to make it square. Most, but not all. We have not described a sufficiently-complete algorithm to produce a square matrix $\mathbf{\Phi}(t)$ when some eigenvalue has AM $\geqslant 3$. Such situations are rare enough, we will leave them as ”for further investigation on your own sometime.”

[1] If $2 \leqslant$ GM $<$ AM, then the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$ is consistent when you start with the *right sort* of eigenvector $\mathbf{v}$ corresponding to λ; not just any eigenvector $\mathbf{v}$ will do. This situation is complicated enough that we will not explore it as a class this semester.

So, let us return to the system of 1st-order DEs (the example above)

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} -5 & -8 & -9 \\ 9 & 16 & 18 \\ -6 & -10 & -11 \end{bmatrix}.$$

We learned that $\mathbf{A}$ has

$$\text{eigenvalue } (-2) \ (\text{AM} = 1) \qquad \text{with basis eigenvector} \qquad \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix},$$

and

$$\text{eigenvalue } 1 \ (\text{AM} = 2, \text{GM} = 1) \qquad \text{with basis eigenvector} \qquad \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix},$$

and these eigenpairs generated two columns for $\mathbf{\Phi}(t)$. Our algorithm indicates we should find $\mathbf{u}$ satisfying

$$(\mathbf{A} - \mathbf{I})\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

We do this using Gaussian elimination: the augmented matrix

$$\begin{bmatrix} -6 & -8 & -9 & 1 \\ 9 & 15 & 18 & -3 \\ -6 & -10 & -12 & 2 \end{bmatrix} \qquad \text{has RREF} \qquad \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see the 3rd column, corresponding to u_3 in the vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, is free. This implies there are infinitely many solutions $\mathbf{u}$. But, as we *need just one more column* for $\mathbf{\Phi}(t)$, we exercise this freedom in *choosing a value* for u_3. When we choose $u_3 = 0$, the solution becomes $\mathbf{u} = \langle 1/2, -1/2, 0 \rangle$. If we choose $u_3 = 1$, then $\mathbf{u} = \langle 1, -2, 1 \rangle$. Either is fine to use, as both do what is required. Using the second one of these, our corresponding solution, which we will use as the third column in $\mathbf{\Phi}(t)$, is

$$e^t \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right) = e^t \begin{bmatrix} 1 + t \\ -2 - 3t \\ 1 + 2t \end{bmatrix}.$$

From this, we have general solution

$$\mathbf{x}_h(t) = \begin{bmatrix} 2e^{-2t} & e^t & e^t(1+t) \\ -3e^{-2t} & -3e^t & -e^t(2+3t) \\ 2e^{-2t} & 2e^t & e^t(1+2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + c_3 e^t \begin{bmatrix} 1 + t \\ -2 - 3t \\ 1 + 2t \end{bmatrix}.$$

Further investigations?

1. Once we had, in hand, sufficiently many columns/solutions to make a square fundamental matrix $\mathbf{\Phi}(t)$ above, there were still $3! = 6$ different orderings for those three columns. Why are all equally useful?

2. Suppose my example problem had come with the initial condition $\mathbf{x}(0) = \langle 5, -8, 6 \rangle$. Starting from the general solution above, determine constants $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ which produce the solution of the IVP.

   Then, explore what would have happened if we had used a different choice of $\mathbf{u}$, say, $\mathbf{u} = \langle 1/2, -1/2, 0 \rangle$. Find the corresponding fundamental matrix $\mathbf{\Phi}(t)$ for *this* choice of $\mathbf{u}$, then use it to find the solution of the same IVP. Convince yourself that, while $\mathbf{\Phi}(t)$ and $\mathbf{c}$ have changed, the resulting answer is the same, as the Existence/Uniqueness says it should be. Can you see how the change in $\mathbf{c}$ offsets the change in $\mathbf{\Phi}(t)$?

3. We have explored the conditions on r, $\mathbf{u}$, and $\mathbf{v}$ that, when met, make
$$e^{rt}\left(\mathbf{u} + t\mathbf{v}\right)$$
   a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Consider expressions of the form
$$e^{rt}\left(\mathbf{w} + t\mathbf{u} + \frac{t^2}{2!}\mathbf{v}\right).$$
   What conditions on r, $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ would make it a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$? What conditions on r, $\mathbf{z}$, $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ make
$$e^{rt}\left(\mathbf{z} + t\mathbf{w} + \frac{t^2}{2!}\mathbf{u} + \frac{t^3}{3!}\mathbf{v}\right)$$
   into a solution?