Form A Solutions

1. (a) We must subtract multiples of 24 from (-73) until our result (remainder) satisfies $0 \le r < 24$: This entails subtracting $q = -4$ multiples of 24: $-73 - (-4)(24) = 23 = r$.

   (b) We note that $3^3 = 27 \equiv -1 \pmod{14}$, and so
$$3^{302} = (3^3)^{100} \cdot 3^2 \equiv (-1)^{100} \cdot 9 \equiv 9 \pmod{14}.$$
   Thus, $3^{302} \bmod 14 = 9$.

2. (a) The arrival of the extra person offers $n-1$ new pairings/handshakes, the new person with the other $n-1$ people.

   (b) $h_n = h_{n-1} + (n-1)$.

   (c) The recurrence relation of part (b) is linear, but not homogeneous.

3. Since $\sum_{i=1}^{10} ix_i = (1)(0)+(2)(1)+(3)(2)+(4)(4)+(5)(2)+(6)(1)+(7)(1)+(8)(7)+(9)(1)+10x_{10} = 112+10x_{10} \equiv 2 - x_{10} \pmod{11}$, we need $2 - x_{10} \equiv 0 \pmod{11}$. Thus, $x_{10} = 2$.

4. (a) We have
$$\begin{aligned} 2964 &= 1(1776) + 1188 && (1)\\ 1776 &= 1(1188) + 588 && (2)\\ 1188 &= 2(588) + 12 && (3)\\ 588 &= 49(12) + 0 \end{aligned}$$
   So, $\gcd(2964, 1776) = 12$.

   (b) We rearrange equations (1)–(3) above to say
$$\begin{aligned} 1188 &= 2964 - 1776 && (4)\\ 588 &= 1776 - 1188 && (5)\\ 12 &= 1188 - 2(588). && (6) \end{aligned}$$
   Then, we insert (5) into (6) to obtain
$$12 \;=\; 1188 - 2[1776 - 1188] \;=\; 3(1188) - 2(1776),$$
   and finally insert (4) into that expression to get
$$12 \;=\; 3[2964 - 1776] - 2(1776) \;=\; 3(2964) - 5(1776).$$
   Thus, we make take $s = 3$ and $t = -3$.

5. (a) Since $91 = (7)(13)$, with prime factors, we have
$$\varphi(91) \;=\; \varphi(7)\varphi(13) \;=\; (6)(12) \;=\; 72.$$

   (b) Euler's Theorem states that
$$a^{\varphi(n)} \equiv 1 \pmod{n},$$
   when a and n are relatively prime (i.e., when $\gcd(a, n) = 1$). Here $a = 77$ and $n = 4669$ share the common factor 7, so they are *not* relatively prime. Euler's Theorem does not apply in our setting.

6. The additive inverse of 6 $\pmod{18}$ is 12, and the multiplicative inverse of 5 $\pmod{18}$ is 11, prompting us to add 12 to both sides and then multiply by 11. The effect on our two equivalent quantities:

$$\begin{aligned} 11[(5x+6)+12] \equiv 11(4+12) \pmod{18} \quad &\Rightarrow \quad 11(5x+18) \equiv 11(16) \pmod{18} \\ &\Rightarrow \quad 11(5x+0) \equiv 176 \pmod{18} \\ &\Rightarrow \quad 55x \equiv 14 \pmod{18} \\ &\Rightarrow \quad 1x \equiv 14 \pmod{18}. \end{aligned}$$

The solution is $x = 14$.

7.

for 5 pts: We may apply the Master Theorem, taking $a = 3$, $b = 2$, $c = 7$ and $d = 0$. Since $a > b^d$ (i.e, $3 > 1$) we have that $f(n)$ is $O(n^{\log_2 3})$.

for 10 pts: Here,

$$\begin{aligned} f(2^k) &= 3f(2^{k-1}) + 7 \;=\; 3[3f(2^{k-2}) + 7] + 7 \;=\; 3^2 f(2^{k-2}) + (3)(7) + 7 \\ &= 3^2[3f(2^{k-3}) + 7] + (3)(7) + 7 \;=\; 3^3 f(2^{k-3}) + (3^2)(7) + (3)(7) + 3 \\ &= 3^3 f(2^{k-3}) + 7[3^2 + 3 + 1] \;=\; \cdots \;=\; 3^k f(1) + 7[3^{k-1} + 3^{k-2} + \cdots + 3^2 + 3 + 1] \\ &= 3^k f(1) + 7\frac{3^k - 1}{3 - 1} \;=\; 4 \cdot 3^k + \frac{7}{2}(3^k - 1) \;=\; \frac{15}{2} \cdot 3^k - \frac{7}{2}. \end{aligned}$$

for 8 pts: We have

$$\begin{aligned} a_n &= 2a_{n-1} - 3 \;=\; 2[2a_{n-2} - 3] - 3 \;=\; 2^2 a_{n-2} - (2)(3) - 3 \\ &= 2^2[2a_{n-3} - 3] - (2)(3) - 3 \;=\; 2^3 a_{n-3} - (2^2)(3) - (2)(3) - 3 \\ &= 2^3 a_{n-3} - 3[2^2 + 2 + 1] \;=\; \cdots \;=\; 2^n a_0 - 3[2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1] \\ &= 2^n a_0 - 3 \cdot \frac{2^n - 1}{2 - 1} \;=\; 5 \cdot 2^n - 3(2^n - 1) \;=\; 2^{n+1} + 3. \end{aligned}$$

8.
- In the first option, we assume $a \mid b$ and $b \mid c$. By definition, this means $\exists k_1 \in \mathbb{Z}$ and $\exists k_2 \in \mathbb{Z}$ such that $ak_1 = b$ and $bk_2 = c$. Thus,

$$c \;=\; bk_2 \;=\; (ak_1)k_2 \;=\; a(k_1k_2).$$

  Since the product k_1k_2 of integers k_1, k_2 is an integer, this says that $a \mid c$.

- The given congruences, $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ mean, by definition, that $m \mid a - b$ and $m \mid b - c$—that is, $\exists k_1 \in \mathbb{Z}$ and $\exists k_2 \in \mathbb{Z}$ such that $mk_1 = a - b$ and $mk_2 = b - c$. We must show that $m \mid a - c$. But,

$$a - c \;=\; (a - b) + (b - c) \;=\; mk_1 + mk_2 \;=\; m(k_1 + k_2).$$

  Since the sum $k_1 + k_2$ of integers k_1, k_2 is an integer, this shows that $m \mid a - c$.

9. Our recurrence relation is linear, homogeneous, with constant coefficients. For solving these, we assume solutions exist of the from $a_n = r^n$. Substituting this into the recurrence relation turns

$$a_n \;=\; 6a_{n-1} - 9a_{n-2} \qquad \text{into} \qquad r^n \;=\; 6r^{n-1} - 9r^{n-2}, \;\text{ or }\; r^{n-2}(r^2 - 6r + 9) \;=\; 0.$$

We are looking for nontrivial solutions, thereby ruling out $r = 0$, and solve the quadratic equation

$$r^2 - 6r + 9 \;=\; (r - 3)^2 \;=\; 0,$$

arriving at the repeated root $r = 3$. It is true, the sequence

$$3^n : \ 1,\ 3,\ 3^2,\ 3^3,\ \ldots$$

satisfies the recurrence relation, but it does not satisfy the initial conditions. As in the past, we know a repeated root also generates a related sequence, in this case

$$n3^n : \ 0 \cdot 0,\ 1 \cdot 3,\ 2 \cdot 3^2,\ 3 \cdot 3^3,\ \ldots$$

which also satisfies the recurrence relation, but not the initial values. We now seek a linear combination,

$$a_n \ = \ \alpha 3^n + \beta n 3^n,$$

with constants α and β to be determined by applying the known initial values:

$$\left.\begin{array}{rcccr} 2 & = & a_0 & = & \alpha \cdot 3^0 + \beta \cdot 0 \\ 3 & = & a_1 & = & \alpha \cdot 3 + \beta \cdot 3 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rcr} \alpha & = & 2 \\ \beta & = & -1 \end{array}\right.$$

Thus, $a_n = 2 \cdot 3^n - n3^n = (2 - n)3^n$.