Comparing the Growth of Functions as Inputs (x or n) $\to \infty$

Suppose f and g are real-valued functions on a domain that includes nonnegative real numbers. We say that

- f **is of order at most** g, written $f(x)$ is $O(g(x))$, iff there exists $C > 0$ and $k \geqslant 0$ such that

$$|f(x)| \leqslant C|g(x)|, \qquad \text{for all real numbers } x > k.$$

  We call C, k **witnesses** to this **Big-**O relationship.
- f **is of order at least** g, written $f(x)$ is $\Omega(g(x))$, iff there exists $C > 0$ and $k \geqslant 0$ such that

$$|f(x)| \geqslant C|g(x)|, \qquad \text{for all real numbers } x > k.$$

- f **is of order** g, written $f(x)$ is $\Theta(g(x))$, iff f is simultaneously of order at most g and of order at least g.

Note: If f and g are sequences (i.e., functions from $\mathbb{N}$ into $\mathbb{R}$), we apply these same notions, writing things like $f(n)$ is Big-O $g(n)$ if there exist positive numbers C, k such that $|f(n)| \leqslant C|g(n)|$ for all $n > k$, and so on.

Some Facts:

1. If $m \geqslant n$ and f is a polynomial of degree n, then $f(x)$ is $O(x^m)$.
2. $n!$ is $O(n^n)$ and, as a consequence, $\log_b n!$ is $O(n \log_b n)$, for any $b > 1$.
3. It can be shown that $n < 2^n$ for $n \geqslant 1$ and, as a consequence, $\log_b n$ is $O(n)$ for all $b > 1$.
4. If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.
5. If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 f_2)(x)$ is $O(g_1(x)g_2(x)))$.
6. As a result of Facts 3 and 5, we have

$$n \log_b n \text{ is } O(n^2), \qquad x^p (\log_b x)^q \text{ is } O(x^{p+q}), \qquad \text{etc.}$$

7. If $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$, the $f(x)$ is $O(h(x))$.
8. Let $c > b > 1$, and $d > 0$. For comparing of a power function x^d with an exponential growth runction b^x, we have

$$x^d \text{ is } O(b^x), \qquad \text{but not vice versa.}$$

   For comparing the two exponential growth functions c^x, b^x we have

$$b^x \text{ is } O(c^x), \qquad \text{but not vice versa.}$$

9. It requires calculus, but it can be shown that for any $b > 0$, $c > 0$, $(\log_b x)^c$ is $O(x)$.

There is, therefore, this increasing sequence of orders: $1, \quad \log_b n, \quad (\log_b n)^2, \quad (\log_b n)^3, \quad \ldots, n,$ $n \log_b n, \quad n(\log_b n)^2, \quad \ldots, n^2, \quad n^2 \log_b n, \quad n^3, \quad \ldots, 2^n, \quad 3^n, \quad \ldots, n!.$

Examples:

1. It is a fact that, for all real numbers $x > 2$,

$$10|x^6| \leqslant |17x^6 - 45x^3 + 2x + 8| \leqslant 30|x^6|.$$

   Given this, what sort of Big-O, Big-Ω and/or Big-Θ statements are possible here?

2. Show that $f(x) = \dfrac{15\sqrt{x}(2x+9)}{x+1}$ is $\Theta(x^{1/2})$.

3. Find witnesses that demonstrate $f(x) = 3x^3 + 2x + 7$ is $O(x^3)$.

Theorem 1: Let $f(x)$ be a polynomial of degree n—that is,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

with $a_n \neq 0$. Then

- $f(x)$ is $O(x^s)$ for all integers $s \geqslant n$.
- $f(x)$ is not $O(x^r)$ for all integers $r < n$.
- $f(x)$ is $\Omega(x^r)$ for all integers $r \leqslant n$.
- $f(x)$ is not $\Omega(x^s)$ for all integers $s > n$.
- $f(x)$ is $\Theta(x^n)$.