Form A Solutions

1. (a) Both $f(x) = 3x^2 + 8x + 7$ and $g(x) = x^2$ are nonnegative and have graphs which are parabolas. The graph of f rises more steeply than the graph of g, owing to the leading coefficient of 3 in the x^2-term. But, by taking $C = 5$, the rescaled $5g(x)$ will rise more steeply. In fact, if we take $k = 5$, then $|f(5)| < 5|g(5)|$, and they will not cross again any further along. So, $C = 5$ and $k = 5$ are witnesses to this Big-Oh relationship.

   (b) Since we require positive numbers C and k so that, for $x > k$,
   $$C \cdot 1 \leq |f(x)|,$$
   this means $|f(x)|$ (eventually) rises beyond some fixed positive value and stays above it forever afterward. Constant functions, polynomials, polylogs, exponential growth, and factorial functions (among others) all do this.

   (c) (i) and (iv) are true.

2. One can conclude (ii) and (iii).

3. In order of ascending complexity,
$$n - \sqrt{n}, \quad n\log n, \quad n^2 + \log n, \quad n^2\log n, \quad (n-1)(n^2+\log n), \quad n^8 - 2^n, \quad 2^n(\log n - 1), \quad 3^n$$

4. (a) $P(1)$ says $\quad 1 \cdot 2 \cdot 3 = \dfrac{1 \cdot 2 \cdot 3 \cdot 4}{4}$.

   (b) $P(4)$ says $\quad 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 = \dfrac{4 \cdot 5 \cdot 6 \cdot 7}{4}$.

   (c) $P(k+1)$ says $\quad 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \dfrac{(k+1)(k+2)(k+3)(k+4)}{4}$.

   (d) The basis step is stated in the answer to part (a), and that equation holds.
   For the induction step, we assume $P(k)$ holds for some $k \in \mathbb{Z}^+$—that is,
   $$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}.$$
   Now, we throw in the extra term on the left-hand side required in $P(k+1)$:
   $$\begin{aligned}
   &1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\
   &= [1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2)] + (k+1)(k+2)(k+3) \\
   &= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \qquad \text{(induction hypothesis)} \\
   &= \left(\frac{k}{4} + 1\right)(k+1)(k+2)(k+3) \quad = \quad \left(\frac{k}{4} + \frac{4}{4}\right)(k+1)(k+2)(k+3) \\
   &= \frac{k+4}{4}(k+1)(k+2)(k+3).
   \end{aligned}$$

5. Using Form 1, a term like $2x^5$ requires 5 multiplications (2)(1.5)(1.5)(1.5)(1.5)(1.5), and a careful count shows the whole process requiring 20 flops. Horner's method (a la Form 2) evaluates $f(1.5)$ using 10 flops, half as many.

6. A linear search through an unsorted list of length n is, in the worst case, of complexity $O(n)$. To do the two-pronged approach described above requires complexity $O(\max(n\log n, \log n)) = O(n\log n)$, making it less efficient for a single search (that is, the linear search is more efficient).

7. The strings 100, 101000, and 101010000 are admitted into A on the 1^{st}, 2^{nd} and 3^{rd} recursive steps, respectively.

8. (a) S has a smallest element, by the **Well-Ordering Principle**.

   (b) The basis step is $P(5)$.

   (c) In this instance, we would have $P(n)$ holds for $n = 5, 9, 13, 17, 21, \ldots$.

9. The problem in the proof is that it does not satisfy $P(0) \to P(1)$. That is, a crucial detail of the inductive step is to write $k+1 = i+j$ with both i, j being numbers in $\{0, \ldots, k\}$. That simply isn't possible when $k = 0$, as this would enforce $i = j = 0$, so that $i + j = 0$, not 1.