★1 (a) It is

$$\begin{aligned} 2x_2 + x_3 + 3x_4 &= 3, \\ 2x_1 + x_2 + 2x_3 - x_4 &= 4, \\ x_1 - 3x_2 + x_3 + x_4 &= 7, \\ 2x_1 + x_3 - 2x_4 &= 2. \end{aligned}$$

(b) We have

$$\left[\begin{array}{cccc|c} 0 & 2 & 1 & 3 & 3 \\ 2 & 1 & 2 & -1 & 4 \\ 1 & -3 & 1 & 1 & 7 \\ 2 & 0 & 1 & -2 & 2 \end{array}\right] \overset{\mathbf{r}_1 \leftrightarrow \mathbf{r}_3}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 2 & 1 & 2 & -1 & 4 \\ 0 & 2 & 1 & 3 & 3 \\ 2 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\overset{(-2)r_1 + \mathbf{r}_2 \to \mathbf{r}_2}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 7 & 0 & -3 & -10 \\ 0 & 2 & 1 & 3 & 3 \\ 2 & 0 & 1 & -2 & 2 \end{array}\right] \overset{(-2)r_1 + \mathbf{r}_4 \to \mathbf{r}_4}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 7 & 0 & -3 & -10 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 6 & -1 & -4 & -12 \end{array}\right]$$

$$\overset{\mathbf{r}_2 \leftrightarrow \mathbf{r}_3}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 7 & 0 & -3 & -10 \\ 0 & 6 & -1 & -4 & -12 \end{array}\right] \overset{(-7/2)\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 6 & -1 & -4 & -12 \end{array}\right]$$

$$\overset{(-3)\mathbf{r}_2 + \mathbf{r}_4 \to \mathbf{r}_4}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 0 & -4 & -13 & -21 \end{array}\right] \overset{(-8/7)\mathbf{r}_3 + \mathbf{r}_4 \to \mathbf{r}_4}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 0 & 0 & 17/7 & 17/7 \end{array}\right]$$

(c) There is, unfortunately, not just one sequence of EROs leading to RREF, though the end result must always be the same. Here is one sequence the produces the desired result.

viii. ERO2: rescale row 2 by a factor of (1/2); i.e., $(1/2)\mathbf{r}_2 \to \mathbf{r}_2$

ix. ERO2: rescale row 3 by a factor of (−2/7); that is, $(-2/7)\mathbf{r}_3 \to \mathbf{r}_3$

x. ERO2: rescale row 4 by a factor of (7/17); $(7/17)\mathbf{r}_4 \to \mathbf{r}_4$

xi. ERO3: $\mathbf{r}_1 - \mathbf{r}_4 \to \mathbf{r}_1$

xii. ERO3: $\mathbf{r}_2 - (3/2)\mathbf{r}_4 \to \mathbf{r}_2$

xiii. ERO3: $\mathbf{r}_3 - (27/7)\mathbf{r}_4 \to \mathbf{r}_3$

xiv. ERO3: $\mathbf{r}_2 - (1/2)\mathbf{r}_3 \to \mathbf{r}_2$

xv. ERO3: $\mathbf{r}_1 - \mathbf{r}_3 \to \mathbf{r}_1$

xvi. ERO3: $\mathbf{r}_1 + 3\mathbf{r}_2 \to \mathbf{r}_1$

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 0 & 0 & 17/7 & 17/7 \end{array}\right] \begin{array}{c} (1/2)\mathbf{r}_2 \to \mathbf{r}_2 \\ (-2/7)\mathbf{r}_3 \to \mathbf{r}_3 \\ \sim \\ (7/17)\mathbf{r}_4 \to \mathbf{r}_4 \end{array} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 1 & 0.5 & 1.5 & 1.5 \\ 0 & 0 & 1 & 27/7 & 41/7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} -\mathbf{r}_4 + \mathbf{r}_1 \to \mathbf{r}_1 \\ (-3/2)\mathbf{r}_4 + \mathbf{r}_2 \to \mathbf{r}_2 \\ \sim \\ (-27/7)\mathbf{r}_4 + \mathbf{r}_3 \to \mathbf{r}_3 \end{array} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & 6 \\ 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \mathbf{r}_2 - (1/2)\mathbf{r}_3 \to \mathbf{r}_2 \\ \sim \\ \mathbf{r}_1 - \mathbf{r}_3 \to \mathbf{r}_1 \end{array} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \mathbf{r}_1 + 3\mathbf{r}_2 \to \mathbf{r}_1 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

(d) The (only) solution is $\mathbf{x} = (1, -1, 2, 1)$.

★2 (b) The linear system is

$$x_1 = -4, \quad x_2 + 2x_3 = 7,$$

which has the solution

$$x_1 = -4,\ x_2 = 7 - 2t,\ x_3 = t \quad \rightsquigarrow \quad \boldsymbol{x} = \begin{pmatrix} -4 \\ 7-2t \\ t \end{pmatrix}, \quad t \in \mathbb{R}.$$

(c) The system is not consistent.

(d) The linear system is

$$x_1 + 3x_4 = -1, \quad x_2 + x_3 + 4x_4 = 3,$$

which has the solution

$$x_1 = -1 + 3s,\ x_2 = 3 - s - 4t,\ x_3 = s,\ x_4 = t \quad \rightsquigarrow \quad \boldsymbol{x} = \begin{pmatrix} -1+3s \\ 3-s-4t \\ s \\ t \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

★3 (a) Gaussian elimination on the augmented matrix gives

$$\left(\begin{array}{cc|c} 1 & 4 & -3 \\ -2 & -8 & r \end{array}\right) \xrightarrow{2\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & 0 & r-6 \end{array}\right).$$

The equivalent linear system is now

$$x_1 + 4x_2 = -3, \quad 0 = r - 6,$$

and is consistent if and only if $r = 6$. In this case the solution is given by

$$x_1 = -3 - 4t,\ x_2 = t \quad \rightsquigarrow \quad \boldsymbol{x} = \begin{pmatrix} -3-4t \\ t \end{pmatrix}, \quad t \in \mathbb{R}.$$

(c) Gaussian elimination on the augmented matrix gives

$$\left(\begin{array}{cc|c} 1 & 4 & -3 \\ -3 & r & -9 \end{array}\right) \xrightarrow{3\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & 12+r & -18 \end{array}\right).$$

The equivalent linear system is now

$$x_1 + 4x_2 = -3, \quad (12+r)x_2 = -18.$$

The system is consistent if and only if $r \neq -12$. If $r \neq -12$ the solution is

$$x_1 = \frac{72}{12+r} - 3,\ x_2 = -\frac{18}{12+r} \quad \rightsquigarrow \quad \boldsymbol{x} = \begin{pmatrix} 72/(12+r) - 3 \\ -18/(12+r) \end{pmatrix}.$$

$\underline{\star 4}$ Upon performing Gaussian elimination we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right) \xrightarrow{-\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & a & b \end{array}\right) \xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right) \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right).$$

(a) The system will be consistent, and there will be a free variable, if and only if $a = 5$ and $b = 4$.

(b) The system will be inconsistent if $a = 5$ and $b \neq 4$.

$\underline{\star 5}$ (a) FALSE. The system will have no solutions, one solution, or an infinite number of solutions.

(b) TRUE. The free variable implies that the value of (at least) one of the variables is arbitrary.

(c) FALSE. The solution may be unique, which means that there is no free variable.

(d) FALSE. The presence of a zero row does not imply the existence of a free variable.

(e) FALSE. It may be the case that the RREF of the coefficient matrix has a zero row, but that the RREF of the augmented matrix does not have a zero row, which would imply that the linear system is inconsistent.

$\underline{\star 6}$ The question is answered by determining whether or not the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent, where $\boldsymbol{A} = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3)$. We have

$$(\boldsymbol{A}|\boldsymbol{b}) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & -3 & -1 & 3 \\ 0 & 1 & 1 & -5/7 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

which means that the linear system is inconsistent. Consequently, the vector $\boldsymbol{b}$ is not a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

$\underline{\star 7}$ (b) Since

$$(\boldsymbol{A}|\boldsymbol{I}_2) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|cc} 1 & -3/2 & 0 & 1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right),$$

$\boldsymbol{A}^{-1}$ does not exist.

(c) We have

$$(\boldsymbol{A}|\boldsymbol{I}_3) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & -2 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right) \quad \rightsquigarrow \quad \boldsymbol{A}^{-1} = \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & -1/2 & -2 \\ 0 & 1 & 3 \end{pmatrix}.$$

$\underline{\star 8}$ (a) FALSE. The matrix must also be square.

(b) FALSE. The matrix must also be square.

(c) FALSE. See Problem **1.4.4**.

(d) FALSE. If $\boldsymbol{A}$ is invertible, the RREF is $\boldsymbol{I}_n$.

(e) FALSE. The matrix must also be square.