

Stat 343, Tue 6-Oct-2020 -- Tue 6-Oct-2020
Probability and Statistics
Fall 2020

Tuesday, October 06th 2020

Wk 6, Tu
Topic:: Moments, power series
Read:: FASt 3.3

Mean/Variance of Continuous r.v.s

If X is a continuous random variable with pdf $f_X(x)$, we have defined its expected value and variance to be

$$\begin{aligned}\mu_X &= E(X) := \int_{-\infty}^{\infty} x f_X(x) dx \\ \sigma_X^2 &= \text{Var}(X) := E((X - \mu_X)^2).\end{aligned}$$

The definition for $E(X)$ is analogous to that for a discrete r.v.s, with an integral replacing a sum, and a pdf taking over the former role of the pmf.

Some other facts that we demonstrated for mean/variance of discrete r.v.s naturally hold, as well, for their continuous counterparts.

Facts about expected values and variances: If X is a continuous r.v. with pdf $f_X(x)$, then

- if $Y = t(X)$ (a transformation/function of X), then $E(Y) = \int_{-\infty}^{\infty} t(x) f_X(x) dx$
- $E(aX + b) = a E(X) + b$ and $\text{Var}(aX + b) = a^2 \text{Var}(X)$ (effect of linear operations on X).
- $\text{Var}(X) = E(X^2) - (E(X))^2$

Moments

$$\mu_1 = E(X^1) = \text{1st moment about origin}$$

$$\mu_2 = E(X^2)$$

Quantities like $E(X^2)$ and $E((X - \mu_X)^2)$ can be generalized.

Definition 1: Let X be a random variable with expected value μ . We define

- (i) the k^{th} **moment about the origin** to be $\mu_k := E(X^k)$, when this number is defined.
- (ii) the k^{th} **moment about the mean** to be $\mu'_k := E((X - \mu)^k)$, when this number is defined.

Note that

$$E(X^2) - [E(X)]^2$$

- $\mu = \mu_1$ is the first moment about the origin.
- $\text{Var}(X) = \mu_2 - (\mu_1)^2$, or the 2nd moment about the mean μ'_2 is the difference of the 2nd moment about the origin μ_2 and the square of the first moment about the origin.
- The definition of moments closely matches definitions given in calculus textbooks when studying centers of mass.
- For $k \geq 1$, if the k^{th} moment about the origin μ_k exists, then all lower moments, μ_i with $i \leq k$ exist as well. This is half of Lemma 3.3.2.

$$\mu'_2 = \mu_2 - (\mu_1)^2$$

The other half of the lemma asserts that, not only do the moments μ'_i about the mean exist for $i \leq k$, but there is a formula for μ'_k based on the various moments about the origin:

$$\mu'_k = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \mu_i \mu^{k-i}.$$

$$\mu_0 = E(X^0) = 1$$

The above formula applied to μ'_3, μ'_4 :

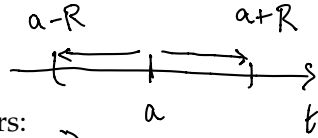
$$\begin{aligned} \mu'_3 &= \sum_{i=0}^3 \binom{3}{i} (-1)^{3-i} \mu_i \mu^{3-i} = \binom{3}{0} (-1)^3 \mu_0 \mu^3 + \binom{3}{1} (-1)^2 \mu_1 \mu^2 \\ &\quad + \binom{3}{2} (-1)^1 \mu_2 \mu + \binom{3}{3} (-1)^0 \mu_3 \mu^0 \\ \mu'_3 &= -\mu^3 + 3\mu^2 - 3\mu_2 \mu + \mu_3 \end{aligned}$$

Uses of higher moments about the mean (see Definition 3.3.3)

- **coefficient of skewness** γ_1 : symmetric distributions have $\gamma_1 = 0$

- coefficient of kurtosis γ_2 : normal distributions have $\gamma_2 = 0$

Power Series



Notes and reminders:

- The appearance of any power series centered at $t = a$ is ...
- Radius and interval of convergence
- Any function f that is differentiable to arbitrary order at a point $t = a$ has a formal power series at $t = a$, called its Taylor series:

Take f : apply differentiation at 'a'
 $f(a), f'(a), f''(a), \dots$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Taylor series at a

There are things we would like to be true, but are not generally:

- It is *not* generally true that the domain (interval of convergence) of the power series is the same as that for f .
- When t is in both domains, the value to which the power series converges need not be the same as $f(t)$.

Despite the uncertainties of such facts, there are some pairings of functions with their Taylor series about which we have a good understanding of when they are equal:

$$f(t) = \frac{1}{1-t} \Rightarrow f(0) = 1$$

$$f'(t) = \frac{1}{(1-t)^2} \Rightarrow f'(0) = 1$$

$$f''(t) = \frac{2}{(1-t)^3} \Rightarrow f''(0) = 2$$

Term-by-term differentiation

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k, \quad -1 < t < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad -\infty < t < \infty$$

$$\text{If } f(t) = \sum_{n=0}^{\infty} c_n t^n, \text{ then } f'(t) = \frac{d}{dt} (c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots)$$

$$= c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \dots$$

$$= \sum_{n=1}^{\infty} n c_n t^{n-1}$$

When $a = 0$
Maclaurin

$$\sum_{n=0}^{\infty} c_n (t-a)^n = c_0 + c_1 (t-a) + c_2 (t-a)^2 + \dots + c_n (t-a)^n + \dots$$

"infinite polynomials"