The two most instructive examples involving a repeated eigenvalue from class today (Mar. 30, 2017) are likely these.

Example 1: Uses matrix from #7 on today's handout

Consider the 1st order system of linear DEs

$$\mathbf{x}' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} \;=\; \mathbf{A}\mathbf{x}.$$

The fact that our matrix is already in echelon form reveals the eigenvalues are all equal to (-2). One can double check this by viewing the characteristic polynomial

$$p_{\mathbf{A}}(\lambda) \;=\; \det(\mathbf{A} - \lambda\mathbf{I}) \;=\; (-2-\lambda)^3,$$

which has $\lambda = -2$ has a repeated (AM=3) root. We set out to find corresponding eigenvectors by solving $[\mathbf{A} - (-2)\mathbf{I}]\mathbf{v} = \mathbf{0}$. The augmented matrix for this problem is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which is already in RREF and reveals there to be three free columns—that is, the *geometric* multiplicity of the eigenvalue (-2) is 3, just as was its algebraic multiplicity. This is *not* a degenerate matrix.

Since all three values of an eigenvector $\mathbf{v}$ are free we do not specify any of them in terms of others, though it is still useful to separate out the various parts associated with each free variable:

$$\mathbf{v} \;=\; \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \;=\; v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The geometric multiplicity tells us the dimension of the eigenspace corresponding to $\lambda = -2$ and, indeed, we have shown above that all eigenvectors are generated as linear combinations of

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which serves as a basis for this eigenspace. In terms of the system of DEs we wish to solve, we have three separate solutions that arise from eigenpairs, namely

$$\mathbf{x}_1(t) \;=\; e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2(t) \;=\; e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3(t) \;=\; e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The corresponding matrix

$$\Phi(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix},$$

has a nonzero determinant for all t, which means $\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)$ form a fundamental set of solutions. Thus, we have general solution

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

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Example 2: Uses matrix from #3 on today's handout

We now consider the 1^{st} order system of linear DEs

$$\mathbf{x}' = \begin{bmatrix} 14 & 16 & 25 \\ -11 & -13 & -25 \\ 2 & 3 & 8 \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}.$$

We skip the details here, but the characteristic polynomial is

$$p_{\mathbf{A}}(\lambda) = -\lambda^3 + 9\lambda^2 - 27\lambda + 27 = -(\lambda - 3)^3,$$

revealing that $\lambda = 3$ is an eigenvalue of $\mathbf{A}$ with AM=3. We find corresponding eigenvectors solving $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$. The augmented matrix

$$\left[\begin{array}{ccc|c} 11 & 16 & 25 & 0 \\ -11 & -16 & -25 & 0 \\ 2 & 3 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has 1 free column, which means the dimension of the eigenspace corresponding to $\lambda = 3$ is 1 (i.e., GM = 1). Our eigenvectors have the form

$$\mathbf{v} = v_3 \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix},$$

with basis vector $\langle 5, -5, 1 \rangle$. The eigenpair we have contributes just one solution to a fundamental set

$$\mathbf{x}_1(t) = e^{3t} \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix},$$

despite the fact that we need three. There are no other eigenpairs (none that have eigenvectors which are linearly independent with eigenvector already found). But, since GM=1, we can fill in extra solutions using the method described in class. In particular, our second solution will be

$$\mathbf{x}_2(t) \;=\; \left(t \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix} + \mathbf{u} \right) e^{3t},$$

and the 3rd is

$$\mathbf{x}_3(t) \;=\; \left(\frac{1}{2!} t^2 \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix} + t\mathbf{u} + \mathbf{w} \right) e^{3t},$$

which is the right number for a fundamental set of solutions. But we must still determine vectors $\mathbf{u}$ and $\mathbf{w}$. The prescription from class said that $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ each solve similar-looking equations:

$$\begin{aligned} &\mathbf{v} \text{ satisfies } (\mathbf{A} - 3\mathbf{I})\mathbf{v} = \mathbf{0}, \\ &\mathbf{u} \text{ satisfies } (\mathbf{A} - 3\mathbf{I})\mathbf{u} = \mathbf{v}, \\ &\mathbf{w} \text{ satisfies } (\mathbf{A} - 3\mathbf{I})\mathbf{w} = \mathbf{u}. \end{aligned}$$

We already know $\mathbf{v}$, but will need to know $\mathbf{u}$ next before finding $\mathbf{w}$. The augmented matrix for the equation $(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \mathbf{v}$, is

$$\left[\begin{array}{ccc|c} 11 & 16 & 25 & 5 \\ -11 & -16 & -25 & -5 \\ 2 & 3 & 5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -1 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which yields infinitely many vectors $\mathbf{u}$ that would be usable by us above, namely

$$\mathbf{u} \;=\; \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \;=\; \begin{bmatrix} 5u_3 - 1 \\ 1 - 5u_3 \\ u_3 \end{bmatrix} \;=\; u_3 \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

We only need *one* vector satisfying this equation, and are allowed to fix any value we desire for the free variable u_3 as we make this choice. The easiest thing is to take $u_3 = 0$, which yields

$$\mathbf{u} \;=\; \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

(which is precisely the final column in RREF).

Finally, to obtain $\mathbf{w}$ we solve $(\mathbf{A} - 3\mathbf{I})\mathbf{w} = \mathbf{u}$, which has augmented matrix

$$\left[\begin{array}{ccc|c} 11 & 16 & 25 & -1 \\ -11 & -16 & -25 & 1 \\ 2 & 3 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

With w_3 being free, there are, again, infinitely many choices for $\mathbf{w}$, but we need only one, such as

$$\mathbf{w} \;=\; \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$$

(the one that arises from taking $w_3 = 0$). So, our second and third solutions are

$$\mathbf{x}_2(t) \;=\; \left(t \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) e^{3t} \;=\; \begin{bmatrix} 5t-1 \\ 1-5t \\ t \end{bmatrix} e^{3t},$$

and

$$\mathbf{x}_3(t) \;=\; \left(\frac{1}{2!} t^2 \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} \right) e^{3t} \;=\; \begin{bmatrix} (5/2)t^2 - t - 3 \\ (-5/2)t^2 + t + 2 \\ (1/2)t^2 \end{bmatrix} e^{3t}.$$

One can check that the matrix

$$\Phi(t) \;=\; e^{3t} \begin{bmatrix} 5 & 5t-1 & (5/2)t^2 - t - 3 \\ -5 & 1-5t & (-5/2)t^2 + t + 2 \\ 1 & t & (1/2)t^2 \end{bmatrix}$$

built from these three solutions has a nonzero determinant (easier to check this Wronskian at a specific t value such as $t = 0$, which Abel's Theorem says suffices), and hence the general solution is

$$\mathbf{x}(t) \;=\; e^{3t} \begin{bmatrix} 5 & 5t-1 & (5/2)t^2 - t - 3 \\ -5 & 1-5t & (-5/2)t^2 + t + 2 \\ 1 & t & (1/2)t^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

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