1. (a) $\mathcal{L}\{f(t)\} = \mathcal{L}\{(2t+4)[1-u(t-3)]\} = \mathcal{L}\{2t+4\} - \mathcal{L}\{u(t-3)(2t+4)\}$

$$= \frac{2}{s^2} + \frac{4}{s} - \mathcal{L}\{u(t-3)[2(t-3)+10]\}$$

$$= \frac{2}{s^2} + \frac{4}{s} - \mathcal{L}\left\{u(t-3)\left(2t+10\Big|_{t\mapsto t-3}\right)\right\}$$

$$= \frac{2}{s^2} + \frac{4}{s} - e^{-3s}\mathcal{L}\{2t+10\} = \frac{2}{s^2} + \frac{4}{s} - e^{-3s}\left(\frac{2}{s^2}+\frac{10}{s}\right).$$

(b) $\mathcal{L}\{e^{-3t} * (2t^2-1)\} = \mathcal{L}\{e^{-3t}\}\cdot\mathcal{L}\{2t^2-1\} = \frac{1}{s+3}\cdot\left(2\cdot\frac{2!}{s^3} - \frac{1}{s}\right).$

(c) $\frac{3}{s(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$

$$\Rightarrow \quad 3 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1)$$

@ $s=0$: $\quad 3 = 3A \quad \Rightarrow A = 1$

@ $s=-1$: $\quad 3 = -2B \quad \Rightarrow B = -3/2$

@ $s=-3$: $\quad 3 = 6C \quad \Rightarrow C = 1/2$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{3}{s(s^2+4s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

By a shifting rule,

$$\mathcal{L}^{-1}\left\{\frac{3e^{-2s}}{s(s^2+4s+3)}\right\} = u(t-2)\cdot\left[1 - \frac{3}{2}e^{-(t-2)} + \frac{1}{2}e^{-3(t-2)}\right].$$

2. Because of the zero ICs, after Laplace transforms applied to both sides we have

$$s^2Y + 2sY + 5Y = \mathcal{L}\{f(t)\} \quad \Rightarrow \quad Y(s) = \mathcal{L}\{f(t)\}\cdot\frac{1}{s^2+2s+5}$$

Now $\quad h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}\cdot\frac{1}{2} = \frac{1}{2}e^{-t}\sin(2t).$

By the Convolution Theorem,

$$y(t) = (h*f)(t) = \frac{1}{2}\int_0^t f(w)\,e^{-(t-w)}\sin(2(t-w))\,dw.$$

3. (a) $y'' + 4y = 0$ has characteristic equation $r^2 + 4 = 0 \Rightarrow r = \pm 2i$ with roots of the form $\alpha \pm \beta i$, $\alpha = 0$, $\beta =$, our general solution is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

(b) Here the characteristic equation is $0 = r^2 + 4r + 4 = (r+2)^3$, giving repeated root $r = -2$. So, the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

4. (a) The characteristic eqn. is

$$r^2 + 2r + 2 = 0$$

which has nonreal roots $r = -1 \pm i$. This is characteristic of underdamping.

(b) The homogeneous soln. $y_h(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$ is built from basis solutions containing exponential decay functions, which die off (very quickly) as $t \to \infty$. The steady state of

$$y(t) = y_h(t) + y_p(t),$$

the part that does not die off, is contained in $y_p(t)$.

(c) The forcing term $85\sin(3t)$ dictates we propose

$$y_p(t) = A\cos(3t) + B\sin(3t) \Rightarrow y_p' = -3A\sin(3t) + 3B\cos(3t)$$

$$y_p'' = -9A\cos(3t) - 9B\sin(3t)$$

Inserting this into the DE,

$$\begin{aligned} LHS &= -9A\cos(3t) - 9B\sin(3t) + 2\left[-3A\sin(3t) + 3B\cos(3t)\right] \\ &\quad + 2\left[A\cos(3t) + B\sin(3t)\right] \\ &= (-7A + 6B)\cos(3t) + (-7B - 6A)\sin(3t) \end{aligned}$$

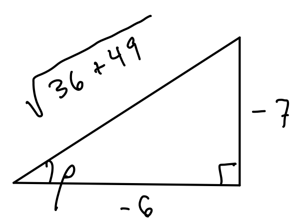
To equal the RHS $(85\sin(3t))$, we need

$$\begin{bmatrix} -7 & 6 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 85 \end{bmatrix} \Rightarrow A = \frac{\begin{vmatrix} 0 & 6 \\ 85 & -7 \end{vmatrix}}{\begin{vmatrix} -7 & 6 \\ -6 & -7 \end{vmatrix}} = -6, \quad B = \frac{\begin{vmatrix} -7 & 0 \\ -6 & 85 \end{vmatrix}}{\begin{vmatrix} -7 & 6 \\ -6 & -7 \end{vmatrix}} = -7.$$

Our particular soln., then, is

$$y_p(t) = -6\cos(3t) - 7\sin(3t).$$

(d) $A = \sqrt{(-6)^2 + (-7)^2} = \sqrt{85}$



5. After dividing by t^2 to get a coefficient 1 for y'', our $g(t) = \dfrac{3t^2-1}{t^2} = 3 - t^{-2}$.

And,
$$W = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1-2 = -3$$

So
$$u_1 = -\int \frac{t^{-1}(3-t^{-2})}{(-3)}\,dt = \frac{1}{3}\int (3t^{-1} - t^{-3})\,dt = \ln|t| + \frac{1}{6}t^{-2}$$

and
$$u_2 = \int \frac{t^2(3-t^{-2})}{(-3)}\,dt = -\frac{1}{3}\int (3t^2-1)\,dt = \frac{1}{3}t - \frac{1}{3}t^3$$

Thus,
$$y_p = u_1y_1 + u_2y_2 = \frac{1}{6} + t^2\ln|t| + \frac{1}{3} - \frac{1}{3}t^2$$
$$= \frac{1}{2} - \frac{1}{3}t^2 + t^2\ln|t|.$$