1. (a) Echelon form is not unique, so the answer (to part (a)) that follows is not the only correct one. For all correct answers, however, the 4^{th} column will be free.

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ -2 & 4 & -3 & -4 \\ 0 & 0 & 3 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \overset{2r_1 + r_2 \to r_2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \overset{r_2 \leftrightarrow r_3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\overset{r_2 \leftrightarrow r_4}{\sim} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 3 & 6 \end{bmatrix} \overset{3r_3 + r_4 \to r_4}{\sim} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Augmenting A with the zero vector is a scenario where the Gaussian elimination we have already performed is easily adapted:

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 0 \\ -2 & 4 & -3 & -4 & 0 \\ 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 - 2x_2 + x_3 + x_4 = 0 \\ \Rightarrow x_2 + 2x_3 + 3x_4 = 0 \\ \Rightarrow -x_3 - 2x_4 = 0 \\ \end{array}$$

So, $x_4 = t$ is free, and using backward substitution,

$$x_3 = -2x_4 = -2t$$

$$x_2 = -2x_3 - 3x_4 = -2(-2t) - 3t = t$$

$$x_1 = 2x_2 - x_3 - x_4 = 2t - (-2t) - t = 3t$$

We have solutions of the homogeneous system $A\vec{x} = \vec{0}$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

2. The augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -4 \\ 0 & 1 & 2 & 3 & 1 \end{array}\right]$ is already in RREF. We see $x_3 = s$ and $x_4 = t$ are free variables, and solving for basic variables x_1, x_2:

$$x_1 = s - 2t - 4 \quad \text{and} \quad x_2 = -2s - 3t + 1.$$

So, solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - 2t - 4 \\ -2s - 3t + 1 \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

for s, t any reals.

3. Since

$$\begin{bmatrix} -2 & 4 & 3 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 2 \\ 2 & -1 \end{bmatrix} = \Bigg[\overbrace{3\begin{bmatrix} -2 \\ -4 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}}^{\text{linear comb.} \to 1^{st} \text{ col.}} \quad \overbrace{-2\begin{bmatrix} -2 \\ -4 \end{bmatrix} + 2\begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}}^{\text{linear comb.} \to 2^{nd} \text{ col.}}\Bigg]$$

$$= \begin{bmatrix} 4 & 9 \\ -11 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 4 \\ 3 & 14 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -11 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 14 & 9 \end{bmatrix}$$

4. (a) $|AB| = \det(A)\det(B) = (-2)(4) = -8.$

(b) $|-2B| = (-2)^3 \det(B) = (-8)(4) = -32.$

(c) $|B^T| = |B| = 4.$

(d) $|A^{-1}| = 1/|A| = -\frac{1}{2}.$

(e) $|B^3| = \det(B)^3 = 64.$

5. $$\left[\begin{array}{cc|c} -3 & -5 & h \\ -6 & k & -6 \end{array}\right] \overset{-2r_1 + r_2 \to r_2}{\sim} \left[\begin{array}{cc|c} -3 & -5 & h \\ 0 & k+10 & -6-2h \end{array}\right]$$

So that there is the potential of infinitely many solutions, we require

$$k + 10 = 0 \quad \Rightarrow \quad \boxed{k = -10.}$$

Infinitely many solutions arises from having a free column, so long as the system is consistent, which additionally means we require

$$-6 - 2h = 0 \quad \Rightarrow \quad \boxed{h = -3}$$

6. $A(-2\vec{u}-2\vec{v}) = -2A(\vec{u}+\vec{v})$

$$= (-2)\left(\begin{bmatrix}2\\-1\\-2\end{bmatrix}+\begin{bmatrix}-5\\1\\4\end{bmatrix}\right) = (-2)\begin{bmatrix}-3\\0\\2\end{bmatrix} = \begin{bmatrix}6\\0\\-4\end{bmatrix}.$$

8. $|A| = (2)(-2)-(-2)(3) = -4+6 = 2$

Since $|A| \neq 0$, A^{-1} exists. To find it,

$$\left[\underbrace{\begin{matrix}2 & 3\\-2 & -2\end{matrix}}_{A}\,\middle|\,\underbrace{\begin{matrix}1 & 0\\0 & 1\end{matrix}}_{I}\right] \overset{r_1+r_2\to r_2}{\sim} \left[\begin{matrix}2 & 3\\0 & 1\end{matrix}\,\middle|\,\begin{matrix}1 & 0\\1 & 1\end{matrix}\right]$$

$$\overset{r_1-3r_2\to r_1}{\sim} \left[\begin{matrix}2 & 0\\0 & 1\end{matrix}\,\middle|\,\begin{matrix}-2 & -3\\1 & 1\end{matrix}\right] \overset{\frac{1}{2}r_1\to r_1}{\sim} \left[\overbrace{\begin{matrix}1 & 0\\0 & 1\end{matrix}}^{I}\,\middle|\,\overbrace{\begin{matrix}-1 & -3/2\\1 & 1\end{matrix}}^{A^{-1}}\right]$$

We get $A^{-1} = \begin{bmatrix}-1 & -3/2\\1 & 1\end{bmatrix}$

9. Finding $|A|$ via Laplace expansion along the 4^{th} column:

$$|A| = 0 + (-1)(-1)^5\begin{vmatrix}0 & -1 & 3\\-1 & 2 & 1\\3 & 1 & -2\end{vmatrix} + 0 + (-1)(-1)^7\begin{vmatrix}2 & 4 & 1\\0 & -1 & 3\\3 & 1 & -2\end{vmatrix} = -22+37 = \boxed{15}$$

$$\begin{vmatrix}0 & -1 & 3\\-1 & 2 & 1\\3 & 1 & -2\end{vmatrix} = 0 + (-1)(-1)^3\begin{vmatrix}-1 & 1\\3 & -2\end{vmatrix} + (3)(-1)^4\begin{vmatrix}-1 & 2\\3 & 1\end{vmatrix} = -1-21 = -22$$

$$\begin{vmatrix}2 & 4 & 1\\0 & -1 & 3\\3 & 1 & -2\end{vmatrix} = (2)(-1)^2\begin{vmatrix}-1 & 3\\1 & -2\end{vmatrix} + 0 + (3)(-1)^4\begin{vmatrix}4 & 1\\-1 & 3\end{vmatrix} = (2)(-1)+(3)(13) = 37$$

Or, finding $|A|$ using GE:

$$\begin{vmatrix}2 & 4 & 1 & -1\\0 & -1 & 3 & 0\\-1 & 2 & 1 & -1\\3 & 1 & -2 & 0\end{vmatrix} = (-1)\underbrace{\begin{vmatrix}-1 & 2 & 1 & -1\\0 & -1 & 3 & 0\\2 & 4 & 1 & -1\\3 & 1 & -2 & 0\end{vmatrix}}_{\text{single row swap}} = (-1)^2\underbrace{\begin{vmatrix}-1 & 2 & 1 & -1\\0 & 1 & -3 & 0\\2 & 4 & 1 & -1\\3 & 1 & -2 & 0\end{vmatrix}}_{\text{multiplied } r_2 \text{ by } (-1)} = (-1)^2\underbrace{\begin{vmatrix}-1 & 2 & 1 & -1\\0 & 1 & -3 & 0\\0 & 8 & 3 & -3\\0 & 7 & 1 & -3\end{vmatrix}}_{\substack{2r_1+r_3\to r_3\\3r_1+r_4\to r_4}}$$

$$= (-1)^2 \begin{vmatrix} -1 & 2 & 1 & -1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 7 & 1 & -3 \end{vmatrix} = (-1)^2 \begin{vmatrix} -1 & 2 & 1 & -1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 22 & -3 \end{vmatrix} = (-1)^2(5) \begin{vmatrix} -1 & 2 & 1 & -1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 22 & -3 \end{vmatrix}$$

$r_3 - r_4 \to r_3$

$r_3 - r_2 \to r_3$
$r_4 - 7r_2 \to r_4$

$\frac{1}{5} r_3 \to r_3$

$$= (-1)^2(5) \begin{vmatrix} -1 & 2 & 1 & -1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = (-1)^2(5) \cdot (-1)(1)(1)(-3) = \underline{15} \quad \left(\text{same as above}\right)$$

$r_4 - 22r_3 \to r_4$