To write $y''' + 2y'' - 7y' - 2y = \cos(7t)$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 2$ as a first order (linear) system, we set new dependent vars

$$x_1 = y \qquad x_2 = y' \qquad x_3 = y''$$

So that

① $x_1' = \frac{d}{dt} y = y' = x_2$

② $x_2' = \frac{d}{dt} y' = y'' = x_3$

and the original DE becomes

③ $x_3' + 2x_3 - 7x_2 - 2x_1 = \cos(7t)$

Arranging these equations ①–③ in vector form (and solving for derivatives)

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 2x_1 + 7x_2 - 2x_3 + \cos(7t) \end{bmatrix} = x_1 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(7t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(7t) \end{bmatrix}, \quad \text{or} \quad \vec{x}' = A\vec{x} + \vec{f}(t).$$

$y' = a(t)\,y + f(t)$

And, since the original was an IVP, the ICs translate into an initial value for the vector $\vec{x}$:

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

It is an odd thing to see someone assemble the dep. vars. into $\vec{x}$ with x_3 on top. If you do so, then the system becomes

$$\begin{bmatrix} x_3' \\ x_2' \\ x_1' \end{bmatrix} = \begin{bmatrix} -2 & 7 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} \cos(7t) \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Derivative of this vector, so requires

To solve $y'' + 4y' + 4y = 5e^{-t}$, $y(0) = 1$, $y'(0) = -1$

we can first solve the homogeneous problem:

$$y'' + 4y' + 4y = 0 \qquad \curvearrowright \qquad r^2 + 4r + 4 = 0$$

$$\text{or } (r+2)^2 = 0 \quad \Rightarrow r = -2 \text{ is repeated root}$$

e^{-2t} and te^{-2t} both solve,

and so do all linear combinations: $y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}$.

For undetermined coeffs., the natural proposal is

$$y_p(t) = Ae^{-t} \quad \Rightarrow \quad y_p' = -Ae^{-t}, \quad y_p'' = Ae^{-t}$$

So,

$$y_p'' + 4y_p' + 4y_p = Ae^{-t} - 4Ae^{-t} + 4Ae^{-t} = Ae^{-t} \overset{\text{target}}{=} 5e^{-t}$$

Thus we take $A = 5$. So

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + 5e^{-t} \quad \Rightarrow \quad 1 = y(0) = c_1 + 5$$

$$y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t} - 5e^{-t}$$

$$\Rightarrow -1 = y'(0) = -2c_1 + c_2 - 5$$

We get $c_1 = -4$, and $c_2 = -4$. So, the solution is

$$\boxed{y(t) = -4e^{-2t} - 4te^{-2t} + 5e^{-t}}$$

To solve $y'' + 2y' + 2y = 3t^2 + t - 4$,

the homogeneous problem $y'' + 2y' + 2y = 0$ leads to characteristic equation $r^2 + 2r + 2 = 0$: $\quad r = \frac{-2}{2} \pm \frac{1}{2}\sqrt{4-8} = -1 \pm i$

So, basis solns are $e^{-t}\cos t$ and $e^{-t}\sin t$

$$\Rightarrow \quad y_h(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

The target fn. (nonhomogeneous term) is a 2^{nd} degree polynomial, so we propose $y_p(t) = At^2 + Bt + C$. Note that no term in this proposal appears in y_h.

$$y_p'' + 2y_p' + 2y_p = (2A) + 2(2At + B) + 2(At^2 + Bt + C)$$

$$= 2At^2 + (4A + 2B)t + 2A + 2B + 2C$$

$$\overset{\text{enforce}}{\underset{\text{this}}{=}} 3t^2 + t - 4$$

So

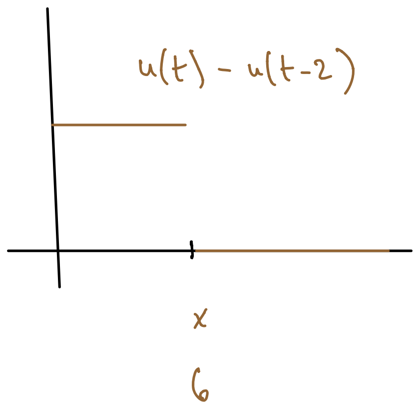
term type	LHS		RHS		
t^2:	$2A$	$=$	3	$\Rightarrow$	$A = 3/2$
t:	$4A + 2B$	$=$	1	$\Rightarrow$	$B = -5/2$
const:	$2A + 2B + 2C$	$=$	-4	$\Rightarrow$	$C = -1$

Thus, $y_p(t) = \frac{3}{2}t^2 - \frac{5}{2}t - 1$, and the general soln. is

$$y(t) = y_h(t) + y_p(t) = \boxed{c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{3}{2}t^2 - \frac{5}{2}t - 1}$$

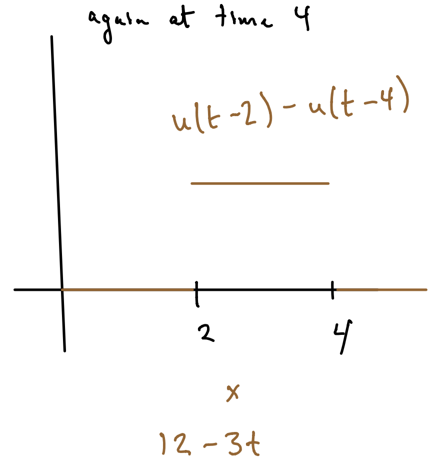
To write
$$f(t) = \begin{cases} 6, & 0 < t < 2 \\ -3t+12, & 2 \le t < 4 \\ 0, & t \ge 4 \end{cases}$$
we note that

6 is switched on at the start, but switched off at time 2



+

$(12-3t)$ is switched on at time 2 and off again at time 4



So,
$$\boxed{f(t) = 6\left[u(t) - u(t-2)\right] + (12-3t)\left[u(t-2) - u(t-4)\right]}$$
$$= 6u(t) + (6-3t)\,u(t-2) - (12-3t)\,u(t-4)$$

To find the Laplace transform of $(3t^2+2t-1)\,u(t-3)$, it is possible to use the definition directly:
$$\mathcal{L}\left\{(3t^2+2t-1)\,u(t-3)\right\} = \int_0^\infty (3t^2+2t-1)\,u(t-3)\,e^{-st}\,dt$$
$$= \underbrace{\int_0^3 \cancel{(3t^2+2t-1)\cdot(0)}\,e^{-st}\,dt}_{=0} + \int_3^\infty (3t^2+2t-1)\cdot(1)\,e^{-st}\,dt$$

but the latter integral will be calculated only after integrating by parts twice.

A better (?) way: Use that $\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

To do so, we see $u(t-3)$ has a shift right 3.

Must find the (pre-shift) fn. which, after being shifted right 3 units, becomes $3t^2+2t-1$.

To get it, shift the (post-shift) fn. left 3 units:

$$3t^2+2t-1\Big|_{t\mapsto t+3} = 3(t+3)^2 + 2(t+3) - 1$$

$$= 3(t^2+6t+9) + 2t + 5$$

$$= 3t^2 + 20t + 32.$$

So,

$$\mathcal{L}\{u(t-3)\,(3t^2+2t-1)\} = e^{-3s}\cdot\mathcal{L}\{3t^2+20t+32\}$$

$$= \boxed{e^{-3s}\left(3\cdot\frac{2}{s^3} + 20\cdot\frac{1}{s^2} + 32\cdot\frac{1}{s}\right)}$$

To get the inverse L.T. of $\frac{2s-3}{s^2+6s+13}$, we complete the square in the denominator:

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+6s+9+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{(s+3)^2+4}\right\}$$

then manufacture the same shift in the numer.

$$= \mathcal{L}^{-1}\left\{\frac{2(s+3)-6-3}{(s+3)^2+4}\right\}$$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} - \frac{9}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}$$

$$= \boxed{2e^{-3t}\cos(2t) - \frac{9}{2}e^{-3t}\sin(2t)}.$$

For $\mathcal{L}^{-1}\left\{ e^{-3s} \cdot \frac{4}{s^2+6s+5} \right\}$, we first work on $\frac{4}{s^2+6s+5}$:

The denominator is reducible, factoring to

$$\frac{4}{s^2+6s+5} = \frac{4}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$$

or

$$4 = A(s+1) + B(s+5) \implies A = -1, \ B = 1$$

Now we can write

$$\mathcal{L}^{-1}\left\{ \frac{4}{s^2+6s+5} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s+5} \right\} = e^{-t} - e^{-5t}.$$

Returning to the original, with its exponential,

$$\mathcal{L}^{-1}\left\{ e^{-3s} \cdot \frac{4}{s^2+6s+5} \right\} = \boxed{u(t-3) \cdot \left[e^{-(t-3)} - e^{-5(t-3)} \right]}.$$