2.106 (a)
$$f_{B|W=w}(x) = \frac{\binom{3}{x}\binom{5}{3-w-x}}{\binom{8}{3-w}} = \text{dhyper}(x, 3, 5, 3-w).$$

(b)
$$f_{R|W=w}(x) = \frac{\binom{5}{x}\binom{3}{3-w-x}}{\binom{8}{3-w}} = \text{dhyper}(x, 5, 3, 3-w).$$

2.109 The 1st prize appears with the 1st kids' meal.

The 2nd prize appears with $1 + X_1$ more meals, where $X_1 \sim \text{Geom}(9/10)$.

The 3rd prize appears with $1 + X_2$ more meals, where $X_2 \sim \text{Geom}(8/10)$.

$\vdots$

The 9th prize appears with $1 + X_8$ more meals, where $X_8 \sim \text{Geom}(2/10)$.

There are $X_9 \sim \text{Geom}(1/10)$ more unsuccessful meals before obtaining the last prize.

So, $$X = 9 + \sum_{i=1}^{9} X_i = \sum_{i=1}^{9}(1 + X_i)$$

i. the X_i are independent

ii. $E(1+X_i) = 1 + \frac{1}{\pi_i} - 1 = \frac{1}{\pi_i} \qquad \left(\pi_i = \frac{10-i}{10}\right)$

iii. $V(X_i) = \frac{1-\pi_i}{\pi_i^2}$

Thus, $$E(X) = \sum_{i=1}^{9} E(1+X_i) = \sum_{i=1}^{9}\frac{1}{\pi_i} = \frac{7129}{252} \doteq 28.29$$

and $$Var(X) = \sum_{i=1}^{9} Var(1+X_i) = \sum_{i=1}^{9} Var(X_i) = \frac{7981633}{63504} \doteq 125.687.$$

3.44 (a) This data comes from a distribution whose center is more spread out than a normal distribution and whose tails are less so.

(b) This data comes from a distribution that is negatively skewed.

(c) This data comes from a distribution that is positively skewed.

(d) This data comes from a distribution whose extremes are more so than in a a normal distribution.

3.56 Let T_i = lifetime of the i^{th} lightbulb. The cdf of T:

$$F_T(t) = \Pr(T \le t) = [\Pr(T_i \le t)]^{10} = \left(1 - e^{-t/100}\right)^{10}, \quad \text{for } t \ge 0.$$

3.65 $X \sim \text{Gamma}(\alpha_1, \lambda)$, $Y \sim \text{Gamma}(\alpha_2, \lambda)$, so $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1}$, and $M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2}$.

X, Y independent means

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1 + \alpha_2},$$

revealing that $X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \lambda)$.

3.66 $X \sim \text{Gamma}(\alpha, \lambda_1)$, $Y \sim \text{Gamma}(\alpha, \lambda_2)$, so $M_X(t) = \left(\frac{\lambda_1}{\lambda_1 - t}\right)^{\alpha}$, and $M_Y(t) = \left(\frac{\lambda_2}{\lambda_2 - t}\right)^{\alpha}$.

X, Y independent means

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = \left(\frac{\lambda_1 \lambda_2}{(\lambda_1 - t)(\lambda_2 - t)}\right)^{\alpha}.$$

This is not the mgf of a gamma r.v.

3.67 Each $X_i \sim \text{Gamma}(\alpha, \lambda)$. By independence,

$$M_S(t) = \prod_{i=1}^{n} \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} = \left(\frac{\lambda}{\lambda - t}\right)^{n\alpha} \Rightarrow S \sim \text{Gamma}(n\alpha, \lambda).$$

$$M_{\bar{X}}(t) = M_{\frac{1}{n}S}(t) = M_S\left(\tfrac{1}{n}t\right) = \left(\frac{\lambda}{\lambda - t/n}\right)^{n\alpha} = \left(\frac{n\lambda}{n\lambda - t}\right)^{n\alpha} \Rightarrow \bar{X} \sim \text{Gamma}(n\alpha, n\lambda).$$

4.13 (a) $$E(\bar{X}_w) = E\left(\sum_{i=1}^{n} w_i X_i\right) = \sum_{i=1}^{n} w_i E(X_i)$$

$$= \mu \sum w_i$$

Since we want $E(\bar{X}_w) = \mu$, we must have $\sum_{i=1}^{n} w_i = 1$.

(b) $$\text{Var}(\bar{X}_w) = \text{Var}\left(\sum w_i X_i\right) = \sum \text{Var}(w_i X_i)$$

$$= \sum w_i^2\, \text{Var}(X_i) = \sigma^2 \sum w_i^2 \quad \left(= |\vec{w}|^2 \sigma^2\right)$$

(c) We assume the condition $\sum w_i = 1$ from part (a) is met.

When $n = 2$, this means $w_2 = 1 - w_1 = w$. Then

$$\sum w_i^2 = w_1^2 + w_2^2 = w^2 + (1 - w)^2 =: f(w).$$

$f'(w) = 2w - 2(1-w) = 4w - 2$, and $f'(w) = 0 \Rightarrow w = 1/2$.

This is the location of the global minimum of f and corresponds to

$$w_1 = w_2 = 1/2.$$

4.14 We know $SE_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2.$

Thus, $\Pr(|\bar{X} - \mu| < 3) = \text{pnorm}(3,0,2) - \text{pnorm}(-3,0,2)$

$\doteq 0.866.$

4.16 (a) The 10 different SRS, along w/ resulting sample means:

Sample	$\bar{X}$	Sample	$\bar{X}$
1, 6	3.5	6, 8	7
1, 6	3.5	6, 9	7.5
1, 8	4.5	6, 8	7
1, 9	5	6, 9	7.5
6, 6	6	8, 9	8.5

$$\Rightarrow \mu_{\bar{X}} = (3.5 + 7 + 7.5)(2/10) + (4.5 + 5 + 6 + 8.5)(1/10) = 6.$$

$$Var(\bar{X}) = (3.5^2 + 7^2 + 7.5^2)(2/10) + (4.5^2 + 5^2 + 6^2 + 8.5^2)(1/10) - \mu_{\bar{X}}^2$$

$$= 2.85.$$

(b) From Coro. 4.3.3, we have

$$E(\bar{X}) = \mu = (1 + 6 + 6 + 8 + 9)(1/5) = 6,$$

a match with part (a). Furthermore,

$$Var(\bar{X}) = \frac{\sigma^2}{n} \cdot \frac{N-n}{N-1} = \frac{\sigma^2}{2} \cdot \frac{5-2}{5-1} = \frac{3}{8}\sigma^2,$$

where $\sigma^2 = (1^2 + 6^2 + 6^2 + 8^2 + 9^2)(1/5) - 6^2 = 7.6.$ So,

$$Var(\bar{X}) = 3/8\, \sigma^2 = 2.85,$$ also matching part (a).

(c) We may treat an iid sample as if we were rolling 5-sided dice, yielding pairings:

$$\begin{array}{ccccc}
(1,1) & (1,6) & (1,6) & (1,8) & (1,9) \\
(6,1) & (6,6) & (6,6) & (6,8) & (6,9) \\
(6,1) & (6,6) & (6,6) & (6,8) & (6,9) \\
(8,1) & (8,6) & (8,6) & (8,8) & (8,9) \\
(9,1) & (9,6) & (9,6) & (9,8) & (9,9)
\end{array}$$

The 5-by-5 table of means corresponds directly to these pairings

$$\begin{array}{ccccc}
1 & 3.5 & 3.5 & 4.5 & 5 \\
3.5 & 6 & 6 & 7 & 7.5 \\
3.5 & 6 & 6 & 7 & 7.5 \\
4.5 & 7 & 7 & 8 & 8.5 \\
5 & 7.5 & 7.5 & 8.5 & 9
\end{array}$$

So,

$$\mu = (1+8+9)(1/25) + (4.5+5+8.5)(2/25) + (3.5+6+7+7.5)(4/25) = 6,$$

and

$$\sigma^2 = (1^2+8^2+9^2)(1/25) + (4.5^2+5^2+8.5^2)(2/25) + (3.5^2+6^2+7^2+7.5^2)(4/25) - 6^2$$

$$= 3.8.$$

4.40 (a) $\bar{x} = (3+4+5+8)/4 = 5$

$$s^2 = \frac{1}{3}\left[(3-5)^2 + (4-5)^2 + (5-5)^2 + (8-5)^2\right] = \frac{14}{3}.$$

(b) $\vec{p}_1 = \langle 5,5,5,5\rangle$, as determined in Exercise 4.39.

$$\vec{p}_2 = \frac{1}{\sqrt{2}}(3-4)\vec{u}_2 = \langle -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\rangle$$

$$\vec{p}_3 = \frac{1}{\sqrt{6}}(3+4-10)\vec{u}_3 = \langle -\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0\rangle$$

$$\vec{p}_4 = \frac{1}{\sqrt{12}}(3+4+5-24)\vec{u}_4 = \langle -1,-1,-1,3\rangle$$

and $\sum \vec{p}_i = \langle 3,4,5,8\rangle$ as predicted.

(c) $l_1 = |\vec{p}_1| = \sqrt{4(5^2)} = 10$

$l_2 = |\vec{p}_2| = \sqrt{2(1/2)^2} = 1/\sqrt{2}$

$l_3 = |\vec{p}_3| = \sqrt{2(\frac{1}{2})^2 + 1^2} = \sqrt{3/2}$

$l_4 = |\vec{p}_4| = \sqrt{3(-1)^2 + 3^2} = 2\sqrt{3}$

(d) $\sum_{i=2}^{4} l_i^2 = \frac{1}{2} + \frac{3}{2} + 12 = 14 = 3\left(\frac{14}{3}\right) = 3s^2.$