1. We are going to need an eigenvector to go with $\lambda = -3$. To get it, we look for a basis of the null $(\mathbf{A} + 3\mathbf{I})$:

$$\mathbf{A} + 3\mathbf{I} \;=\; \begin{bmatrix} -10 & 2 & 14 \\ -4 & 6 & 3 \\ -10 & 2 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this, we glean that there is one basis eigenvector, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, v_3 can be taken as *free*, and we must have $v_1 = (3/2)v_3$, $v_2 = (1/2)v_3$; $\mathbf{v} = \langle 3, 1, 2 \rangle$ is such a (basis) eigenvector, and the solution this eigenpair generates is $e^{-3t}\mathbf{v}$.

To get the solutions arising from the nonreal eigenpairs, we must identify

$$\alpha = 2, \;\; \beta = 1, \;\; \mathbf{u} = \langle 2, 1, 2 \rangle, \qquad \text{and} \qquad \mathbf{w} = \langle 0, 1, 0 \rangle.$$

The corresponding solutions are

$$e^{2t}\left(\cos t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2e^{2t}\cos t \\ e^{2t}(\cos t - \sin t) \\ 2e^{2t}\cos t \end{bmatrix} \quad \text{and} \quad e^{2t}\left(\sin t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2e^{2t}\sin t \\ e^{2t}(\cos t + \sin t) \\ 2e^{2t}\sin t \end{bmatrix}.$$

Using our three solutions to build the fundamental matrix, we have general solution

$$\mathbf{x}(t) \;=\; \begin{bmatrix} 3e^{-3t} & 2e^{2t}\cos t & 2e^{2t}\sin t \\ e^{-3t} & e^{2t}(\cos t - \sin t) & e^{2t}(\cos t + \sin t) \\ 2e^{-3t} & 2e^{2t}\cos t & 2e^{2t}\sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \;=\; \mathbf{\Phi}(t)\mathbf{c}.$$

Now, we seek to satisfy the IC:

$$\begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix} \;=\; \mathbf{x}(0) \;=\; \mathbf{\Phi}(0)\mathbf{c} \;=\; \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 3 & 2 & 0 & 8 \\ 1 & 1 & 1 & 5 \\ 2 & 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

giving us that $c_1 = 4$, $c_2 = -2$, $c_3 = 3$. Our solution, then, is

$$\mathbf{x}(t) \;=\; \mathbf{\Phi}(t)\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \;=\; \begin{bmatrix} 12e^{-3t} - 4e^{2t}\cos t + 6e^{2t}\sin t \\ 4e^{-3t} + e^{2t}\cos t + 5e^{2t}\sin t \\ 8e^{-3t} - 4e^{2t}\cos t + 6e^{2t}\sin t \end{bmatrix}$$

2. (a) The eigenvalues are found by solving

$$0 = \begin{vmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda)+1 = \lambda^2+4\lambda+4 = (\lambda+2)^2,$$

showing $\lambda = -2$ to have algebraic multiplicity 2. Solving for null $(\mathbf{A}+2\mathbf{I})$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since there is just one free column, the geometric multiplicity is 1, and $\lambda = -2$ is degenerate; a basis vector of its eigenspace is $\mathbf{v} = \langle 1, 1\rangle$. So, along with $e^{-2t}\mathbf{v}$, we seek a second solution of the form $e^{-2t}(\mathbf{w}+t\mathbf{v})$, where $\mathbf{w}$ solves $(\mathbf{A}+2\mathbf{I})\mathbf{w} = \mathbf{v}$:

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

We can use any vector $\mathbf{w} = \langle w_1, w_2\rangle$ for which $w_1 - w_2 = -1$; $\mathbf{w} = \langle 0, 1\rangle$ is such a vector. Thus, a fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & te^{-2t} \\ e^{-2t} & (1+t)e^{-2t} \end{bmatrix}.$$

(b) Since the eigenvalues are real and both positive, the equilibrium at the origin is an **unstable node**.

3. (a) This problem is separable. We have

$$\begin{aligned} \frac{dy}{dt} = 2ty^2 \quad &\Rightarrow \quad \int -y^{-2}\,dy = -\int 2t\,dt \\ &\Rightarrow \quad y^{-1} = C - t^2 \\ &\Rightarrow \quad y(t) = \frac{1}{C-t^2} \qquad \text{(general solution)} \end{aligned}$$

(b) The problem is linear and nonhomogeneous, with $a(t) = 2t$, and $f(t) = 12t^3e^{t^2}$. The homogeneous solution is $C\varphi(t)$, where $\varphi(t) = e^{\int 2t\,dt} = e^{t^2}$. the variation of parameters formula gives

$$y_p(t) = e^{t^2}\int \frac{12t^3e^{t^2}}{e^{t^2}}\,dt = e^{t^2}(3t^4).$$

So, the general solution is $\quad y(t) = y_h(t) + y_p(t) = ce^{t^2} + 3t^4e^{t^2}.$

4. Whether you do this by Cramer's Rule or actually inverting the matrix, you will need

$$\left|\mathbf{\Phi}(t)\right| = e^{4t}\cdot(1+2t) - 2te^{4t} = e^{4t}.$$

Inverting $\mathbf{\Phi}(t)$, we have

$$\begin{aligned} \mathbf{\Phi}(t)^{-1}\mathbf{f}(t) &= \frac{1}{e^{4t}}\begin{bmatrix} (1+2t)e^{2t} & -te^{2t} \\ -2e^{2t} & e^{2t} \end{bmatrix}\begin{bmatrix} 2t \\ e^{-t} \end{bmatrix} = \begin{bmatrix} (1+2t)e^{-2t} & -te^{-2t} \\ -2e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} 2t \\ e^{-t} \end{bmatrix} \\ &= 2t\begin{bmatrix} (1+2t)e^{-2t} \\ -2e^{-2t} \end{bmatrix} + e^{-t}\begin{bmatrix} -te^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 2t(1+2t)e^{-2t} - te^{-3t} \\ -4te^{-2t} + e^{-3t} \end{bmatrix} \end{aligned}$$

5. Salt flows in at a rate

$$(\text{concentration}) \cdot (\text{flow rate}) = (15)(30).$$

Whatever amount of salt $y(t)$ is in the tank at time t, the outflow takes the same form as product of concentration and flow rate, but with concentration $y/300$. Taken together, our initial value problem is

$$\frac{dy}{dt} = (15)(30) - \left(\frac{y}{300}\right)(30) = 450 - \frac{1}{10}y, \qquad y(0) = 8500.$$

6. (a) The DE is in normal form $y' = g(x, y)$, with $g(x, y) = x + x\sqrt{y} + y$. This $g(x, y)$, as well as its partial $\partial g/\partial y = \frac{x}{2\sqrt{y}} + 2y$, are continuous throughout the open region in the xy-plane where $y > 0$. In fact, we can take that entire right half-plane $\{(x, y) \mid y > 0\}$ as our open rectangle enclosing $(x_0, y_0) = (2, 1)$ in which g, $\partial g/\partial y$ are continuous. Thus, the IVP has a unique solution.

   (b) We have $x_0 = 2$, $y_0 = 1$, $g(x, y) = x + x\sqrt{y} + y$ (as in part (a)). Since $h = 0.25$, it requires 4 steps/iterations to reach $x = 3$.

$$\begin{array}{l|l}
y_1 = y_0 + hg(x_0, y_0) = 1 + (0.25)(2 + 2\sqrt{1} + 1) = 2.25 & x_1 = x_0 + h = 2.25 \\
y_2 = y_1 + hg(x_1, y_1) = 2.25 + (0.25)(2.25 + 2.25\sqrt{2.25} + 2.25) = 4.2188 & x_2 = x_1 + h = 2.5 \\
y_3 = y_2 + hg(x_2, y_2) = 4.2188 + (0.25)[2.5 + 2.5\sqrt{4.2188} + 4.2188] = 7.1822 & x_3 = x_2 + h = 2.75 \\
y_4 = y_3 + hg(x_3, y_3) = 7.1822 + (0.25)[2.75 + 2.75\sqrt{7.1822} + 7.1822] = 11.5077 & x_4 = x_3 + h = 3.0
\end{array}$$

   So, $y(3) \approx 11.508$.