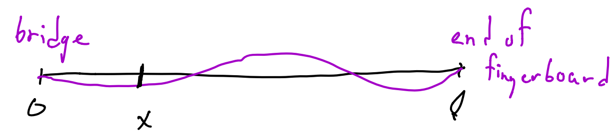
The Wave Equation

In Section 3.2 of the Benson text, he provides a derivation, based on physical principles, of the 1-dimensional wave equation for strings. This is a partial differential equation, where we assume a string residing in an interval of length ℓ can be displaced transversally only. If we denote its displacement at position x, $0 \leq x \leq \ell$, and time $t \geq 0$ by $y(t,x)$, then this function $y(t,x)$ is modeled as satisfying

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < \ell,\ t > 0. \tag{1}$$

Partial derivatives



When a function $y = y(t,x)$ depends on two variables t and x, we can think about differentiating with respect to one variable only, treating the other variable as constant. This is called **partial differentiation**:

$$y_t = \frac{\partial y}{\partial t} := \lim_{h\to 0} \frac{y(t+h,x) - y(t,x)}{h}, \qquad \frac{df}{dx} \qquad \frac{d^2 f}{dt^2}$$

defines the partial derivative of y with respect to t (assuming the limit in the definition exists). If you repeat partial differentiation with respect to t on $\partial y/\partial t$, you get the second partial derivative with respect to t, $\frac{\partial^2 y}{\partial t^2}$, also denoted as u_{tt}. Naturally, y_x and y_{xx} denote the first and second partial derivatives of y with respect to x.

$$y_x = \frac{\partial y}{\partial x} = \lim_{h\to 0} \frac{y(t, x+h) - y(t,x)}{h}$$

Adding conditions to the wave equation

A **differential equation** is an equation that can be solved for an unknown function, and at least one derivative of that function appears in the equation. As such, even the equation

$$y' = 2 \Rightarrow y(x) = \int 2\,dx = 2x + C \qquad y' = 2, \quad y(0) = 1 \tag{2}$$

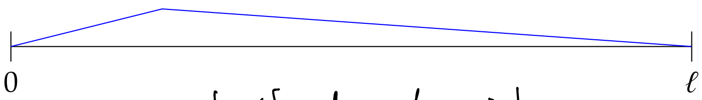
is a differential equation. It is clear that $y(x) = 2x$ has derivative equal to 2, making it a solution of the DE (2). But it is far from the only solution! Differential equations do not, generally speaking, have unique solutions without additional conditions, and the wave equation (1) is no exception.

One determines these added conditions from the physical problem under consideration. In the instance of a plucked string on a violin, it is very natural to consider the problem

$$\frac{\partial^2 y}{\partial t^2} = y_{tt} \qquad y_{tt} = c^2 y_{xx} \quad \text{subject to} \quad \begin{cases} \text{BCs:} & y(t,0) = 0 \text{ and } y(t,\ell) = 0 \\ \text{ICs:} & y(0,x) = f(x) \quad \text{initial profile} \\ & y_t(0,x) = 0 \end{cases} \tag{3}$$

where $u_t(0,x) = 0$ says that all along the string, the instantaneous rate of change of displacement is zero. The function $f(t)$ gives the initial profile, all along the string, of its displacement, which for $t = 0$, the moment the plucked string is released, might look like this graph.

Solution $y(t,x) = $ formula



velocity at each point x, prior to release is 0.

Figure 1: Profile $f(x)$ of plucked string at time $t = 0$

Solving the wave equation

Let us solve the wave equation where its ends cannot be displaced from zero, and its initial displacement and velocity profiles are $f(x)$, $g(x)$ respectively.

$$y_{tt} = c^2 y_{xx} \qquad \text{subject to} \quad \left\{ \begin{array}{ll} \text{BCs:} & y(t,0) = 0 \text{ and } y(t,\ell) = 0 \\ \text{ICs:} & y(0,x) = f(x) \\ & y_t(0,x) = g(x) \end{array} \right\} \tag{4}$$

Fourier came up with the idea that our unknown function $y(t,x)$ might be *separable* into t and x factors:

$$y(t,x) = \underline{q(t)p(x)}.$$

$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left[q(t)\,p(x)\right] = p(x)\frac{dq}{dt} = p\dot{q}$

Following his lead, we can make a series of deductions, including

$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(p(x)q(t)) = q(t)\,p'(x)$

A. There is equality of ratios

$$\frac{\ddot{q}(t)}{c^2 q(t)} = \frac{p''(x)}{p(x)}. \tag{5}$$

$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}(p\dot{q}) = p\ddot{q}$

Since the ratio on the left is a function of t, the ratio on the right is a function of x, and the equality must hold for $0 < x < \ell$ and $t > 0$, the only possibility is that both sides equal a constant.

B. Denoting as λ the constant that both sides of (5) must equal, we arrive at two separate differential equations, one for the unknown function $p(x)$, the other for $q(t)$:

solved after E $\rightarrow$ (i) $p'' = \lambda p$

(ii) $\ddot{q} = \underline{\lambda} c^2 q$

C. Assuming $y(t,x) = q(t)p(x)$ is not *trivial* (i.e., not always zero), the boundary conditions $u(t,0) = u(t,\ell) = 0$ imply that $p(0) = p(\ell) = 0$. This provides additional conditions on the differential equation (i), namely

$$p'' = \lambda p \qquad \text{is subject to} \qquad p(0) = 0, \;\; p(\ell) = 0.$$

more challenging than others

D. The problem in part C is solvable only if the constant λ is a <u>real</u>, <u>negative number</u>. So, we can rewrite this problem as

can write $\lambda = -\omega^2$

$$p'' + \omega^2 p = 0, \qquad \text{subject to} \qquad p(0) = 0, \;\; p(\ell) = 0.$$

E. The linear combinations of $\cos(\omega t)$, $\sin(\omega t)$

$$p(x) \;=\; A\cos(\omega x) + B\sin(\omega x)$$

all satisfy $p'' + \omega^2 p = 0$. However, to be of this form and to satisfy $p(0) = 0$ requires $A = 0$. As well, for it to solve $p(\ell) = 0$, we need $\omega = \frac{n\pi}{\ell}$ for some $n = 1, 2, 3, \ldots$. Thus, the problem in part C is solvable nontrivially precisely when λ_n is one of the special values

$$\lambda_n = -\frac{n^2\pi^2}{\ell^2}, \qquad n = 1, 2, 3, \ldots,$$

Summary:
- (i) solvable only if $\lambda = \lambda_n$
- form $p_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$ one for each λ_n

in which case the corresponding solution $p_n(x)$ is a scalar multiple of $\sin(n\pi x/\ell)$.

F. Turning to the differential equation (ii) of part (b), the counterpart $q_n(t)$ to $p_n(x)$ must satisfy

$$\ddot{q}_n + \frac{n^2\pi^2c^2}{\ell^2}q = 0,$$

something the linear combinations of $\cos(n\pi ct/\ell)$, $\sin(n\pi ct/\ell)$

$$q_n(t) \;=\; a_n\cos\left(\frac{n\pi ct}{\ell}\right) + b_n\sin\left(\frac{n\pi ct}{\ell}\right)$$

do.

Summarizing steps A–F, we have discovered that, for each $n = 1, 2, 3, \ldots$, the combination

$$q_n(t)p_n(x) \;=\; \left[a_n\cos\left(\frac{n\pi ct}{\ell}\right) + b_n\sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right)$$

satisfies the partial differential equation $u_{tt} = c^2u_{xx}$ along with the two boundary conditions of (4). We have not yet addressed the ICs.

Fourier deduced that sums, even possibly infinite sums, would also solve the PDE and BCs, and could perhaps be induced to satisfy the ICs, too. That is, we look to find a solution of the form

$$u(t,x) \;=\; \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi ct}{\ell}\right) + b_n\sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right). \tag{6}$$

Thus,

G. The expression for the t-partial derivative of $u(t,x)$ is plausibly

$$u_t(t,x) \;=\; \sum_{n=1}^{\infty}\frac{n\pi c}{\ell}\left[-a_n\sin\left(\frac{n\pi ct}{\ell}\right) + b_n\cos\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right),$$

and, thus,

$$f(x) \;=\; \sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{\ell}\right), \qquad \text{and} \qquad g(x) \;=\; \sum_{n=1}^{\infty}\frac{n\pi c}{\ell}b_n\sin\left(\frac{n\pi x}{\ell}\right).$$

$f(x) = u(0,x)$

H. Using $\langle f, g\rangle = \int_0^\ell f(x)g(x)\,dx$ as inner product for functions defined on $[0,\ell]$, we obtain formulas

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \qquad \text{and} \qquad b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\sin\left(\frac{n\pi x}{\ell}\right)dx.$$

I. The spatial part—the way the solution $u(t,x)$ varies in *space*—of the individual terms in the solution (6) is dictated by the factors $p_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$. The only shapes, spatially, which occur are ones with period/wavelengths 2ℓ, ℓ, $2\ell/3$, $2\ell/4$, ...

Assignment

1. **Read** or skim Sections 3.1–3.2 of the Benson book, ”Music: A Mathematical Offering”. Visit Room SCOFIELD3894 at socrative.com (you may want to be there already when you begin the reading), by which time a different quiz should be active. Answer the reading questions.

2. There are conclusions labeled with letters—A, B, C, etc.—drawn during the solution of the wave equation. Together with members of your team, explain 6 of these conclusions. To clarify, you are being asked only to explain the conclusion described in the letter, assuming all the steps that come before.

3. One of the .m files I supplied is called `sqWave.m`. Display the command in this file by typing it:

```
> type sqWave
```

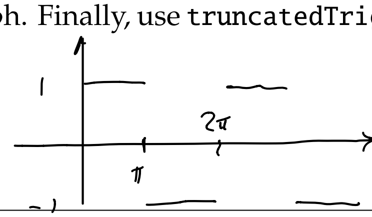
(You can do this with other .m files in your working directory if you so choose.) You can see, on p. 63 in the Benson text, the function being implemented here. Plot, for instance, the version with $\rho = .1$ and $T = 0.5$ using commands like this:

```
> xs = 0:.01:2;
> plot(xs, sqWave(xs, 0.1, 0.5))
```

Study the definition for `sqWave` well enough to write a definition for the function that appears on p. 35.

```
> g = @(x) ...
```

Then use the `fourierTrigCoeffs.m` function to verify the statements about Fourier coefficients a_n, b_n which appear also on p. 35 just below the graph. Finally, use `truncatedTrigFS.m` to produce plots such as those on p. 36.



$p_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$

4. A string of length ℓ on a violin has fixed ends, $u(t,0) = u(t,\ell) = 0.$ In the solution of the wave equation, this translated to boundary conditions $p(0) = p(\ell) = 0$ for the differential equation

$$p''(x) = \lambda p(x),$$

and led to spatial modes $p_n(x) = \sin(n\pi x/\ell)$.

As Professor Kung explained in his first lecture, an aerophone instrument has columns of air which vibrate according to the wave equation, but may have different boundary conditions. Suppose, in place of the fixed-end conditions, we had one end fixed and the other with a zero derivative:

$$\underline{u(t,0) = 0} \qquad \text{and} \qquad \underline{u_x(t,\ell) = 0.}$$

Work out what the resulting spatial modes $p_n(x)$ should be in this case.

$p_n(x) = ?$