Stat 343, Fri 9-Oct-2020 -- Fri 9-Oct-2020 Probability and Statistics Fall 2020

If A B ind. courts

then A cal B are, the.

$$Var(aX) = a^2 Var(X)$$

Friday, October 09th 2020

There is a Corresponding power survey contains

Wk 6, Fr

Topic:: Moment generating functions (mgf)

To each Sequence a_0, a_1, a_2 .

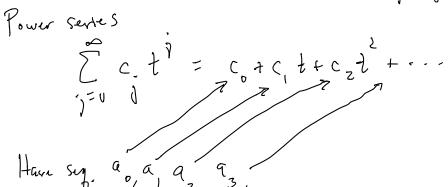
There is a Corresponding power survey contains

a $a_1 + a_2 + a_3 + a_4 + a_4 + a_5 + a$

Generating Functions

Power series

Notes and reminders:



- The appearance of any power series centered at t = a is . . .
- Radius and interval of convergence
- Any function f that is differentiable to arbitrary order at a point t = a has a formal power series at t = a, called its **Taylor series**:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

There are things we would like to be true, but are not generally:

- It is *not* generally true that the domain (interval of convergence) of the power series is the same as that for *f*.
- When t is in both domains, the value to which the power series converges need not be the same as f(t).

Despite the uncertainties of such facts, there are some pairings of functions with their Taylor series about which we have a good understanding of when they are equal:

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k, \quad -1 < t < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad -\infty < t < \infty$$

• Term-by-term differentiation

Generating functions

Want to generate a seguence Given r.v. X, $\mu_1, \mu_2, \mu_3, \mu_4, \dots$

Definition 1: Let $(a_k)_{k=0}^{\infty}$ be a sequence. The (formal) power series

$$A(t) = \sum_{k=0}^{\infty} a_k t^k$$

is called the **ordinary generating function** for (a_k) . Another power series,

$$B(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k,$$

is called the **exponential generating function** for (a_k) .

Example 1: Generating functions for exponential sequences

For $r \neq 0$, let $a_k = r^k$, k = 0,1,2,... Then generating functions for (a_k) have well-known closed-form expressions. In particular, the ordinary generating function for (a_k) is

$$A(t) = 1 + rt + (rt)^{2} + (rt)^{3} + \dots + (rt)^{k} + \dots = \frac{1}{1 - rt}$$

and the exponential generating function is

$$B(t) = 1 + rt + \frac{(rt)^2}{2!} + \dots + \frac{(rt)^k}{k!} + \dots = e^{rt}.$$

As these generating functions are power series centered at 0,

$$A(t) = \sum_{k=0}^{\infty} a_k t^k$$
 and $B(t) = \sum_{k=0}^{\infty} b_k t^k$,

with each $b_k = a_k/k!$, we have the relationship between coefficients and derivatives of A(t):

$$a_k = \frac{A^{(k)}(0)}{k!}$$
 and $b_k = \frac{A^{(k)}(0)}{(k!)^2}$, $k = 0, 1, 2, ...$

Similarly,

$$b_k = \frac{B^{(k)}(0)}{k!}$$
 and $a_k = B^{(k)}(0)$, $k = 0, 1, 2, ...$

Moment generating functions

Definition 2: Let X be a discrete or continuous random variable, and define the function $M_X(t) := \underbrace{\operatorname{E}(e^{tX})}$. When this function exists for some nontrivial interval $|t| \leq b$ about the origin, we call it the **moment generating function** (or **mgf**) of X.

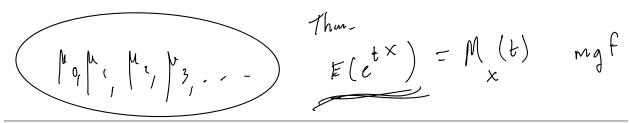
Why is M_X called by this name? • Case: X is discrete.

So if has a proof $f_X(x)$ $E(e^{tX}) = \sum_{x} e^{tx} f_X(x) = \sum_{x} f_X(x) \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} = \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} f_X(x)$ $= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{x} x^j f_X(x) = \sum_{j=0}^{\infty} \frac{h^j}{j!} \int_{\mathbb{R}^n} \frac{(tx)^n}{j!} dx$

• Case: *X* is continuous.

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_{x}(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{2} dx$$

In either case, if the sequence $(\mu_k)_{k=1}^{\infty}$ of k^{th} moments of X about the origin exists, then $M_X(t)$ is the exponential generating function for the sequence of (μ_k) .



Example 2: Moment generating function for $X \sim Pois(\lambda)$

$$\Rightarrow pmf f(x) = e^{\lambda} \frac{\lambda}{x!}$$

To calculate mgf
$$M(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot e^{-\lambda} \frac{\lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$-\lambda \quad \lambda e^{t} \quad \lambda e^{t} - \lambda$$

$$= e^{-\lambda} \cdot e^{\lambda e^{t}} = e^{\lambda e^{t} - \lambda}$$
$$-\lambda (1 - e^{t})$$

$$= -\lambda(1-e^t)$$

Example 3: Moment generating function for $X \sim \text{Binom}(n, \pi)$

Here,

$$M_X(t) = \sum_{x=0}^n e^{tx} \binom{n}{x} \pi^x (1-\pi)^{n-x} = \sum_{x=0}^n \binom{n}{x} (e^t \pi)^x (1-\pi)^{n-x}$$
$$= (\pi e^t + 1 - \pi)^n.$$

Example 4: Moment generating function for $X \sim \mathsf{Unif}(a,b)$

Since *X* is a continuous random variable, we have

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{e^{tx}}{t(b-a)} \Big|_a^b = \frac{e^{bt} - e^{at}}{t(b-a)}.$$

Example 5: Moment generating function for $X \sim \mathsf{Exp}(\lambda)$

Here,

$$\begin{split} M_X(t) &= \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx &= \lim_{A \to \infty} \int_0^A \lambda e^{(t-\lambda)x} \, dx &= \lim_{A \to \infty} \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^A \\ &= \lim_{A \to \infty} \frac{\lambda}{t-\lambda} (e^{A(t-\lambda)} - 1) &= \frac{\lambda}{\lambda-t} &= \frac{1}{1-\frac{t}{\lambda}}, \end{split}$$

if $t < \lambda$ (so, in particular, the limit exists for $|t| < \lambda$).

Theorem 1: For *X* a discrete or continuous r.v., *a*, *b* constants,

- (i) $M_{aX}(t) = M_X(at)$.
- (ii) $M_{X+b}(t) = e^{bt} M_X(t)$.
- (iii) $M_{aX+b}(t) = e^{bt} M_X(at)$.

Two applications of mgfs:

- Since $M_x(t)$ is the exponential generating function of the sequence (μ_k) of k^{th} moments, we have, in general, $\mu_k = M_X^{(k)}(0)$.
- mgfs are unique, providing a signature for recognizing a random variable, as specified in this theorem:

Theorem 2: Let X, Y be r.v.s with mgfs M_X , M_Y . Then X, Y are **identically distributed** (i.e., they have the same cdfs) if and only if $M_X(t) = M_Y(t)$ for all t in some nontrivial interval containing 0.

$$\frac{1+1.t+1.t^{3}+---}{5} = \frac{2}{5} = 0$$

Since
$$f(t) = \frac{1}{1-t}$$
 produces Maclanoin series

$$f'(t) = \frac{d}{dt} ((-t)^{-1}) = -1 ((-t)^{-2}(-1))$$

$$= \frac{1}{(1-t)^{2}}$$

$$\Rightarrow f(0) = 1$$

$$f''(t) = -2(1-t)^{3}(-1) = 2!$$

$$\Rightarrow \frac{f'(o)}{21} = 1$$

If we had gone with the exponential generating for approach

Corresp. Series
$$\frac{1}{\text{(modified)}} \frac{1}{\text{0!}} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{3!} + \dots$$