Stat 343, Tue 6-Oct-2020 -- Tue 6-Oct-2020
Probability and Statistics
Fall 2020

Tuesday, October 06th 2020

Wk 6, Tu
Topic:: Moments, power series
Read:: FASt 3.3

Mean/Variance of Continuous r.v.s

If X is a continuous random variable with pdf $f_X(x)$, we have defined its expected value and variance to be

$$\begin{aligned} \mu_X &= \mathrm{E}(X) &:= \int_{-\infty}^{\infty} x f_X(x)\,dx \\ \sigma_X^2 &= \mathrm{Var}(X) &:= \mathrm{E}((X-\mu_X)^2). \end{aligned}$$

The definition for $\mathrm{E}(X)$ is analogous to that for a discrete r.v.s, with an integral replacing a sum, and a pdf taking over the former role of the pmf.

Some other facts that we demonstrated for mean/variance of discrete r.v.s naturally hold, as well, for their continuous counterparts.

Facts about expected values and variances: If X is a continuous r.v. with pdf $f_X(x)$, then

- if $Y = t(X)$ (a transformation/function of X), then $\mathrm{E}(Y) = \int_{-\infty}^{\infty} t(x) f_X(x)\,dx$
- $\mathrm{E}(aX+b) = a\,\mathrm{E}(X) + b$ and $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$ (effect of linear operations on X).
- $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

Moments

$\mu_1 = E(X^1) =$ 1st moment about origin

$\mu_2 = E(X^2)$

Quantities like $\mathrm{E}(X^2)$ and $\mathrm{E}((X-\mu_X)^2)$ can be generalized.

> **Definition 1:** Let X be a random variable with expected value μ. We define
>
> (i) the k^{th} **moment about the origin** to be $\mu_k := \mathrm{E}(X^k)$, when this number is defined.
>
> (ii) the k^{th} **moment about the mean** to be $\mu'_k := \mathrm{E}((X-\mu)^k)$, when this number is defined.

Note that $E(X^2) - [E(X)]^2$

$$\boxed{\mu'_2 = \mu_2 - (\mu_1)^2}$$

- $\mu = \mu_1$ is the first moment about the origin.
- $\mathrm{Var}(X) = \mu_2 - (\mu_1)^2$, or the 2^{nd} moment about the mean mu'_2 is the difference of the 2^{nd} moment about the origin μ_2 and the square of the first moment about the origin.
- The definition of moments closely matches definitions given in calculus textbooks when studying centers of mass.
- For $k \geq 1$, if the k^{th} moment about the origin μ_k exists, then all lower moments, μ_i with $i \leq k$ exist as well. This is half of Lemma 3.3.2.
  The other half of the lemma asserts that, not only do the moments μ'_i about the mean exist for $i \leq k$, but there is a formula for μ'_k based on the various moments about the origin:
  $$\mu'_k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \mu_i \mu^{k-i}.$$

$\mu_0 = E(X^0) = 1$

The above formula applied to μ'_3, μ'_4:

$$\mu'_3 = \sum_{i=0}^{3} \binom{3}{i} (-1)^{3-i} \mu_i \mu^{3-i} = \underbrace{\binom{3}{0}}_{=1}(-1)^3 \underbrace{\mu_0}_{=1}\mu^3 + \binom{3}{1}(-1)^2 \mu_1 \mu^2$$
$$+ \binom{3}{2}(-1)^1 \mu_2 \mu + \underbrace{\binom{3}{3}}_{=1}(-1)^0 \mu_3 \underbrace{\mu^0}_{=1}$$

$$\mu'_3 = -\mu^3 + 3\mu^3 - 3\mu_2\mu + \mu_3$$

Uses of higher moments about the mean (see Definition 3.3.3)

- **coefficient of skewness** γ_1: symmetric distributions have $\gamma_1 = 0$

- **coefficient of kurtosis** γ_2: normal distributions have $\gamma_2 = 0$

Power Series

a-R a+R

a t

interval of convergence

When $a = 0$ MacLaurin

$$\sum_{n=0}^{\infty} c_n (t-a)^n = c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_n(t-a)^n + \cdots$$

"infinite polynomials"

Notes and reminders:

- The appearance of any power series centered at $t = a$ is ...
- Radius and interval of convergence
  R
- Any function f that is differentiable to arbitrary order at a point $t = a$ has a formal power series at $t = a$, called its **Taylor series**:

Take f: apply differentiation at 'a'

$f(a), f'(a), f''(a), \ldots$

$$\boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.}$$

Taylor Series at a

There are things we would like to be true, but are not generally:

  – It is *not* generally true that the domain (interval of convergence) of the power series is the same as that for f.
  – When t is in both domains, the value to which the power series converges need not be the same as $f(t)$.

Despite the uncertainties of such facts, there are some pairings of functions with their Taylor series about which we have a good understanding of when they are equal:

$f(t) = \frac{1}{1-t} \Rightarrow f(0) = 1$

$f'(t) = \frac{+1}{(1-t)^2} \Rightarrow f'(0) = 1$

$f''(t) = \frac{-2(-1)}{(1-t)^3} \Rightarrow f''(0) = 2$

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k, \qquad -1 < t < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad -\infty < t < \infty$$

- Term-by-term differentiation

If $f(t) = \sum_{n=0}^{\infty} c_n t^n$, then $f'(t) = \frac{d}{dt}\left(c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots\right)$

$= c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \cdots$

$= \sum_{n=1}^{\infty} n c_n t^{n-1}$