Recent DE systems material practice

The purpose of these exercises is to give you some practice with recent material in Chapter 3, in a group/table setting, in place of assigning homework. For each problem,

- begin by discussing with classmates at your table what sort of process leads to solutions of the system, and what sort of form those solutions take.
- solve it, and compare notes with other students at your table, so as to see how closely your answers agree.

1. Find the general solution to the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\mathbf{x}$. How would you classify the equilibrium at the origin?

2. Solve the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\mathbf{x}$, with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

3. Solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the matrix

$$\mathbf{A} = \begin{bmatrix} -4 & -2 & 2 & 3 \\ 0 & -11 & 6 & 10 \\ -8 & -1 & 5 & 2 \\ 3 & -10 & 6 & 8 \end{bmatrix}$$

has just one real eigenvalue (-3), and also has a nonreal eigenvalue $2+i$. As further information, note that the matrix

$$\mathbf{A} - (2+i)\mathbf{I} \qquad \text{has RREF} \qquad \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -(3+i)/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Find the general solution to $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & 3 \\ -6 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -e^t \\ -3e^t \end{bmatrix}$.

Solutions

1. The general solution

$$\mathbf{x}(t) = c_1 e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -e^{3t} & e^{3t}(\frac{1}{2} - t) \\ e^{3t} & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Note that there is flexibility in the appearance of this answer. In particular, if, instead of using eigenvector $\langle -1, 1\rangle$ we had used $\langle 2, -2\rangle$, or any other nonzero scalar multiple, the answer would be just as valid. Moreover, the vector I took as $\langle 1/2, 0\rangle$ is replaceable by any other vector $\mathbf{u}\,\langle u_1, u_2\rangle$ whose components sum to 1/2.

Since both eigenvalues are real and positive, the equilibrium at the origin is an **unstable node**.

2. The general solution before applying the initial condition is

$$\mathbf{x}(t) \;=\; c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right).$$

The initial condition requires

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \;=\; \mathbf{x}(0) \;=\; \begin{bmatrix} -3 & -3t-1 \\ 1 & t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \Rightarrow \qquad c_1 = 4, \;\; c_2 = -14.$$

Thus, the solution of the IVP is

$$\mathbf{x}(t) \;=\; 4 \begin{bmatrix} -3 \\ 1 \end{bmatrix} - 14 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right) \;=\; \begin{bmatrix} 2+42t \\ 4-14t \end{bmatrix}.$$

3. The eigenvalue (-3) must contribute two linearly independent solutions. Only one comes from an eigenpair (since its GM = 1), and the other is of the form $e^{-3t}(\mathbf{u} + t\mathbf{v})$. These are

$$e^{-3t} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad e^{-3t} \left(\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right) \;=\; e^{-3t} \begin{bmatrix} t-1 \\ 2t-1 \\ t-1 \\ t \end{bmatrix}.$$

An eigenvector that goes with eigenvalue $2+i$ is

$$\begin{bmatrix} 1/3 \\ 1 \\ 1/2 - 1/6i \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} 1/3 \\ 1 \\ 1/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1/6 \\ 0 \end{bmatrix} \;=\; \mathbf{p} + i\mathbf{q}.$$

Coupled with $\alpha = 2$ and $\beta = 1$, this gives us two more solutions

$$e^{2t} \left(\cos t \begin{bmatrix} 1/3 \\ 1 \\ 1/2 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 0 \\ 1/6 \\ 0 \end{bmatrix} \right) \;=\; e^{2t} \begin{bmatrix} 1/3\cos t \\ \cos t \\ 1/2\cos t - 1/6\sin t \\ \cos t \end{bmatrix},$$

$$e^{2t} \left(\sin t \begin{bmatrix} 1/3 \\ 1 \\ 1/2 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 0 \\ 1/6 \\ 0 \end{bmatrix} \right) \;=\; e^{2t} \begin{bmatrix} 1/3\sin t \\ \sin t \\ 1/2\sin t + 1/6\cos t \\ \sin t \end{bmatrix}.$$

The general solution is any linear combination of the four solutions we found, which may also be written as $\mathbf{\Phi}(t)\mathbf{c}$:

$$\mathbf{x}(t) = \begin{bmatrix} e^{-3t} & (t-1)e^{-3t} & (1/3)e^{2t}\cos t & (1/3)e^{2t}\sin t \\ 2e^{-3t} & (2t-1)e^{-3t} & e^{2t}\cos t & e^{2t}\sin t \\ e^{-3t} & (t-1)e^{-3t} & (1/6)e^{2t}(3\cos t - \sin t) & (1/6)e^{2t}(3\sin t + \cos t) \\ e^{-3t} & te^{-3t} & e^{2t}\cos t & e^{2t}\sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

4. The matrix multiplying $\mathbf{x}$ in the DE has a nonreal eigenvalue $1+3i$ with corresponding basis eigenvector $\langle 1+i, -2\rangle = \langle 1, -2\rangle + i\langle 1, 0\rangle$. Thus, the complementary homogeneous problem has a fundamental set of solutions comprised of

$$e^t\left(\cos(3t)\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \sin(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad e^t\left[\sin(3t)\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \cos(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

The fundamental matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^t[\cos(3t) - \sin(3t)] & e^t[\cos(3t) + \sin(3t)] \\ -2e^t\cos(3t) & -2e^t\sin(3t) \end{bmatrix}$$

has inverse

$$\mathbf{\Phi}^{-1}(t) = \frac{1}{2e^{2t}}\begin{bmatrix} -2e^t\sin(3t) & -e^t[\cos(3t) + \sin(3t)] \\ 2e^t\cos(3t) & e^t[\cos(3t) - \sin(3t)] \end{bmatrix} = \begin{bmatrix} -e^{-t}\sin(3t) & -\frac{1}{2}e^{-t}[\cos(3t) + \sin(3t)] \\ e^{-t}\cos(3t) & \frac{1}{2}e^{-t}[\cos(3t) - \sin(3t)] \end{bmatrix}.$$

Thus,

$$\begin{aligned} \mathbf{\Phi}^{-1}(t)\mathbf{f}(t) &= \begin{bmatrix} -e^{-t}\sin(3t) & -\frac{1}{2}e^{-t}[\cos(3t) + \sin(3t)] \\ e^{-t}\cos(3t) & \frac{1}{2}e^{-t}[\cos(3t) - \sin(3t)] \end{bmatrix}\begin{bmatrix} -e^t \\ -3e^t \end{bmatrix} \\ &= \begin{bmatrix} \sin(3t) + \frac{3}{2}[\cos(3t) + \sin(3t)] \\ -\cos(3t) - \frac{3}{2}[\cos(3t) - \sin(3t)] \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\cos(3t) + \frac{5}{2}\sin(3t) \\ -\frac{5}{2}\cos(3t) + \frac{3}{2}\sin(3t) \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} \mathbf{x}_p(t) &= \mathbf{\Phi}(t)\int \begin{bmatrix} \frac{3}{2}\cos(3t) + \frac{5}{2}\sin(3t) \\ -\frac{5}{2}\cos(3t) + \frac{3}{2}\sin(3t) \end{bmatrix} dt \\ &= \begin{bmatrix} e^t[\cos(3t) - \sin(3t)] & e^t[\cos(3t) + \sin(3t)] \\ -2e^t\cos(3t) & -2e^t\sin(3t) \end{bmatrix}\begin{bmatrix} \frac{1}{2}\sin(3t) - \frac{5}{6}\cos(3t) \\ -\frac{5}{6}\sin(3t) - \frac{1}{2}\cos(3t) \end{bmatrix} \\ &= e^t\begin{bmatrix} -4/3 \\ 5/3 \end{bmatrix} \end{aligned}$$

The general solution, then, is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = c_1e^t\begin{bmatrix} \cos(3t) - \sin(3t) \\ -2\cos(3t) \end{bmatrix} + c_2e^t\begin{bmatrix} \cos(3t) + \sin(3t) \\ -2\sin(3t) \end{bmatrix} + e^t\begin{bmatrix} -4/3 \\ 5/3 \end{bmatrix}$$