Ex.) IVP

$$y'' + 4y' + 5y = 8\cos t, \qquad y(0) = 0,\ y'(0) = 1$$

Spring model? $\quad mu'' + \gamma u' + ku = f(t), \quad m, \gamma, k$ all pos.

under-, critically, overdamped?

nonreal roots — repeated roots — real (but neg.) roots

homog. version

$$y'' + 4y' + 5y = 0 \qquad r^2 + 4r + 5 = 0$$

$$r = \frac{-4}{2} \pm \frac{1}{2}\sqrt{16 - 20}$$

$$= \underline{-2} \pm i$$

$$\Longrightarrow e^{-2t}\cos t,\ e^{-2t}\sin t$$

$$y_h(t) = \underline{c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t} \quad (\text{transient})$$

$$\left\{\begin{aligned} y_p(t) &= A\cos t + B\sin t \\ y_p' &= -A\sin t + B\cos t \\ y_p'' &= -A\cos t - B\sin t \end{aligned}\right.$$

No terms like those in $y_h$, so OK

LHS of DE w/ this proposal:

$$y'' + 4y' + 5y = -A\cos t - B\sin t + 4(-A\sin t + B\cos t) + 5(A\cos t + B\sin t)$$

$$= \cos t(4A + 4B) + \sin t(-4A + 4B)$$

$$\overset{\text{after}}{\underset{\text{target}}{=}} 8\cos t + 0\sin t$$

Equate coeffs:

$$\left.\begin{aligned} \cos t: \quad & 4A + 4B = 8 \\ \sin t: \quad & -4A + 4B = 0 \end{aligned}\right\} \quad A = 1 = B.$$

Have $\quad \underline{y_p(t) = \cos t + \sin t}$

$y_p(0) = 1 + 0 = 1$ doesn't satisfy zero ICs, so right to wait to find $c_1, c_2$.

gen'l soln.

$$\underline{y(t)} = y_h + y_p = \underline{c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t + \cos t + \sin t}$$

$$\underline{y'(t)} = \underline{-2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t - \sin t + \cos t}$$

ICs:

$$0 = y(0) = c_1 \cdot 1 + c_2 \cdot 0 + 1 + 0 \quad \text{or} \quad c_1 + 1 = 0$$

$$1 = y'(0) = -2c_1 \cdot 1 - c_1 \cdot 0 - 2c_2 \cdot 0 + c_2 \cdot 1 - 0 + 1 \quad \text{or} \quad -2c_1 + c_2 + 1 = 1$$

2 eqns. in unknowns $c_1, c_2$

$$\begin{aligned} c_1 + 1 &= 0 & \qquad c_1 &= -1 \\ -2c_1 + c_2 + 1 &= 1 & \qquad -2c_1 + c_2 &= 0 \end{aligned}$$

in matrix form

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{aligned} c_1 &= -1 \\ c_2 &= -2 \end{aligned}$$

Our soln. to IVP

$$\boxed{y(t) = \underbrace{-e^{-2t} \cos t - 2e^{-2t} \sin t}_{\text{transient}} + \underbrace{\cos t + \sin t}_{\text{steady state}}}$$

Building blocks for homog. soln. $e^{-2t} \cos t$, $e^{-2t} \sin t$

Wronskian is the determinant of matrix

$$\begin{bmatrix} e^{-2t}\cos t & e^{-2t}\sin t \\ -2e^{-2t}\cos t - e^{-2t}\sin t & -2e^{-2t}\sin t + e^{-2t}\cos t \end{bmatrix}$$

Insist that Wronskian $\neq 0$

Ex.) 2nd one like it

$$y'' + 6y' + 9y = 2e^{-3t}, \qquad y(0) = 2, \; y'(0) = -1$$

homog. $y'' + 6y' + 9y = 0$ has char. eqn. $\lambda^2 + 6\lambda + 9 = 0$

repeated root $\lambda = -3$ (critically damped)

$$y_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

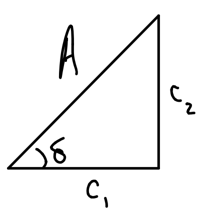
Propose

$$y_p = At^2 e^{-3t} \quad \text{will work}$$

Details for you:

IVP Soln. $y(t) = y_h + y_p = (t^2 + 5t + 2)\, e^{-3t}$

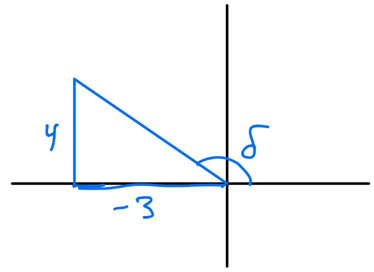
From Ch.4: Putting expression like $c_1\cos(\omega t) + c_2 \sin(\omega t)$ into a single cosine expression $A\cos(\omega t - \delta)$




$$A = \sqrt{c_1^2 + c_2^2}, \quad \cos\delta = \frac{c_1}{A}, \quad \sin\delta = \frac{c_2}{A}$$

Ex.

$$-3\cos(2t) + 4\sin(2t)$$




$$c_1 = -3, \quad c_2 = 4$$

$$A = \sqrt{(-3)^2 + 4^2} = 5, \quad \cos\delta = \frac{-3}{5}, \quad \sin\delta = \frac{4}{5}$$

$$\delta = \arccos\left(-3/5\right)$$

Ans.

$$5\cos\left(2t - \arccos(-3/5)\right)$$

Nonhomog. 1st-order system (Variation of params)

$$\frac{d}{dt}\vec{x}' = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\vec{x} + \begin{bmatrix} \sin t \\ e^{-2t} \end{bmatrix}$$

First handle homog. problem

$$\vec{x}' = A\vec{x}$$

Soln.

$$\vec{x}_h(t) = \Phi(t)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\begin{bmatrix} \sin t \\ e^{-2t} \end{bmatrix} dt$$