1. $$\mathbf{B} = \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 & 2 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -2 & 1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 22 & 1 \\ -6 & 16 \end{bmatrix} = \begin{bmatrix} -19 & -2 \\ 8 & -10 \end{bmatrix}.$$

2. The solution of the nonhomogeneous problem can always be seen as the sum of two parts, a particular solution added with vectors in the nullspace: $\mathbf{x}_p + \mathbf{x}_n$. So, we discover what vectors are in the nullspace of $\mathbf{A}$ by breaking apart the solution of the nonhomogeneous problem:

$$\begin{bmatrix} 3+t-2s \\ s+4 \\ 3s+4t-1 \\ 2t-5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ -5 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ -5 \end{bmatrix} + \operatorname{span}\left(\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix} \right\} \right).$$

The null space is the part with the free variables. A basis for it consists of the two linearly independent vectors

$$\begin{bmatrix} -2 \\ 1 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}.$$

3. When the matrix $\mathbf{A}$ has real-number entries, then its characteristic polynomial has real-number coefficients, which then means eigenvalues and eigenvectors come in complex-conjugate pairs. Since our matrix has nonreal eigenpair

$$\lambda_1 = -3-2i, \qquad \mathbf{v}_1 = \begin{bmatrix} 4-2i \\ -i \\ -1+3i \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} + i \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix},$$

we know it has, as well, the related eigenpair

$$\lambda_2 = -3+2i, \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} - i \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+2i \\ i \\ -1-3i \end{bmatrix}.$$

4. We solve the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$:

$$0 = \begin{vmatrix} 3-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) + 2 = \lambda^2 - 7\lambda + 14.$$

By the quadratic formula, there are two nonreal roots

$$\lambda = \frac{1}{2}\left(7 \pm \sqrt{49-(4)(14)}\right) = \frac{7}{2} \pm i\frac{\sqrt{7}}{2}.$$

These are the eigenvalues of $\mathbf{A}$.

5. One sequence of EROs that takes the given matrix $\mathbf{B}$ to echelon form is as follows:

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 6 & -4 & 5 \\ 0 & 2 & -1 & 4 \\ 0 & 2 & -1 & 7 \end{bmatrix} \quad \begin{matrix} \mathbf{r}_2 - 2\mathbf{r}_1 \to \mathbf{r}_2 \\ \sim \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 2 & -1 & 7 \end{bmatrix} \quad \begin{matrix} \mathbf{r}_2 \leftrightarrow \mathbf{r}_4 \\ \sim \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 2 & -1 & 7 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{matrix} \mathbf{r}_3 - \mathbf{r}_2 \to \mathbf{r}_3 \\ \sim \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \begin{matrix} \mathbf{r}_3 + \mathbf{r}_4 \to \mathbf{r}_4 \\ \sim \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is not *the* right answer, only *one* right one, as echelon form is not unique, and even if we continued until arriving at RREF, which is unique, the number ways to get there are numerous.

6. (a) The task here is easier than if we were asked to find the eigenvalues outright. We only need compute $\det(\mathbf{A} - 2\mathbf{I})$ and show, so long as things go like the instructions suggest, that the result is 0. Expanding in cofactors along the first column, we have

$$\begin{aligned} |\mathbf{A} - 2\mathbf{I}| &= \begin{vmatrix} -15 & -10 & -40 \\ -5 & -10 & -20 \\ 5 & 5 & 15 \end{vmatrix} \\ &= (-15)(-1)^2 \begin{vmatrix} -10 & -20 \\ 5 & 15 \end{vmatrix} + (-5)(-1)^3 \begin{vmatrix} -10 & -40 \\ 5 & 15 \end{vmatrix} + (5)(-1)^4 \begin{vmatrix} -10 & -40 \\ -10 & -20 \end{vmatrix} \\ &= (-15)\Big[(-10)(15) - (5)(-20)\Big] + (5)\Big[(-10)(15) - (5)(-40)\Big] + (5)\Big[(-10)(-20) - (-10)(-40)\Big] \\ &= (-15)(-150 + 100) + (5)(-150 + 200) + (5)(200 - 400) \\ &= (-15)(-50) + (5)(50) + (5)(-200) \;=\; 750 + 250 - 1000 \;=\; 0. \end{aligned}$$

That is, $\lambda = 2$ is one of the solutions to $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, making it an eigenvalue of $\mathbf{A}$.

(b) This part requires we find a basis for $\text{Null}(\mathbf{A} - \lambda\mathbf{I})$, with $\lambda = -3$. We go to RREF:

$$\mathbf{A} - (-3)\mathbf{I} = \begin{bmatrix} -10 & -10 & -40 \\ -5 & -5 & -20 \\ 5 & 5 & 20 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, there are two free variables, x_2 and x_3, and the components of eigenvectors $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$ satisfy the equation $x_1 = -x_2 - 4x_3$, so they look like

$$\mathbf{x} = \begin{bmatrix} -x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{a plane in } \mathbb{R}^3 \text{ with basis vectors} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}.$$

7. There are, in fact, infinitely many solutions. If we use one ERO on the given augmented matrix—namely, subtract two multiples of row 2 from row 1—we wind up with RREF having top two rows

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 11 & -7 \\ 0 & 0 & 1 & -5 & 4 \end{array}\right].$$

In the resulting equations, we may take $x_2 = s$ and $x_4 = t$ to be free. The pivot variables x_1, x_3 must satisfy

$$x_1 = -7 - 3x_2 - 11x_4 = -7 - 3s - 11t \qquad \text{and} \qquad x_3 = 4 + 5x_4 = 4 + 5t.$$

Writing solutions in vector notation $\mathbf{x} = \langle x_1, x_2, x_3, x_4 \rangle$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 - 3s - 11t \\ s \\ 4 + 5t \\ t \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -11 \\ 0 \\ 5 \\ 1 \end{bmatrix}, \qquad \text{where } s, t \text{ are any real nos.}$$