Implications

Biconditionals

We write $p \leftrightarrow q$ to mean

$$(p \to q) \wedge (q \to p).$$

The first half, $p \to q$, may be read as "p only if q," while the second half, $q \to p$, may be read as "p if q." Thus, $p \leftrightarrow q$ is often read aloud as "p if and only if q," or "p iff q," more succinctly. Biconditionals have the truth table

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

or simply

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

.

Tautologies and contradictions

A compound proposition formed from propositional variables p, q, etc. which is True regardless of the values of these variables, is called a **tautology**. If the negation of a compound proposition is a tautology, then the proposition itself is called a **contradiction**.

A very simple example of a tautology is $p \vee \neg p$.

Logical equivalence

We say two compound propositions P, Q are **logically equivalent** pecisely in the case that $P \leftrightarrow Q$ is a tautology. One way to establish logical equivalence is to write out truth tables for both P, Q in terms of the same base set of propositional variables p, q, etc. If every combination of truth values for the propositional variables leads to a truth value for P that is mirrored in the truth value of Q, then they are logically equivalent, and we write $P \equiv Q$.

Some important logical equivalences:

- DeMorgan's Laws:
  $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- Identity Laws:
  $p \wedge T \equiv p$
  $p \vee F \equiv p$

- Associative Laws:
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Commutative Laws:
  $p \wedge q \equiv q \wedge p$
  $p \vee q \equiv q \vee p$
- Distributive Laws:
  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

There are others, such as $p \to q \equiv \neg p \vee q$. See Tables 6-8 on pp. 27–28.

Example 1: Contrapositives, converses and inverses

Let p, q be propositions, and consider the implication $p \to q$. There are three related implications with names as noted:

$q \to p$, known as the **converse** of $p \to q$.
$\neg q \to \neg p$, known as the **contrapositive** of $p \to q$.
$\neg p \to \neg q$, known as the **inverse**.

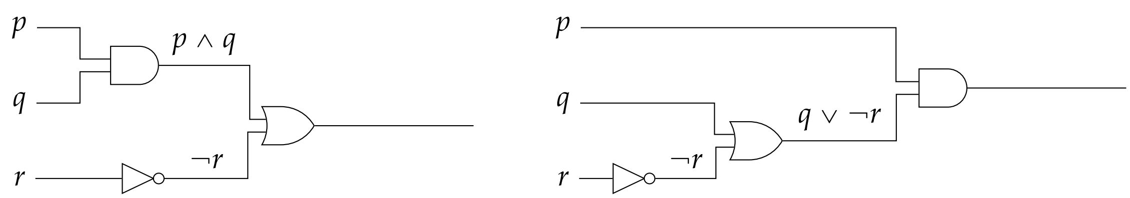
Use a truth table to demonstrate that $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology, thereby establishing the logical equivalence of an implication statement and its contrapositive. (Note that this also establishes the logical equivalence of the converse and inverse statements, not to the original implication, but to each other.) Then show that an implication is *not* logically equivalent to its converse.

p	q	$\neg q$	$\neg p$	$p \to q$	$\neg q \to \neg p$	$q \to p$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	F
F	F	T	T	T	T	T

■

Example 2: Order of operations for logical operators

A circuit diagram naturally indicates the order in which operations are performed.



(a) Label correctly the output of these circuits.

(b) Use the truth table to demonstrate that the order in which logical operators is performed matters.

p	q	r	$p \wedge q$	$q \vee \neg r$	$(p \wedge q) \vee \neg r$	$p \wedge (q \vee \neg r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	F	F
F	F	F	F	T	T	F

■

Since order of operations matter with logical operators (much as with the arithmetic operators $+$, $-$, $\times$, $\div$, ˆ of mathematics), you should read and learn the content of the section "Precedence of Logical Operators" on p. 11 of the textbook.

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p  q  (p or q)  (p or q) and p
T  T     T             T
T  F     T             T
F  T     T             F
F  F     F             F
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