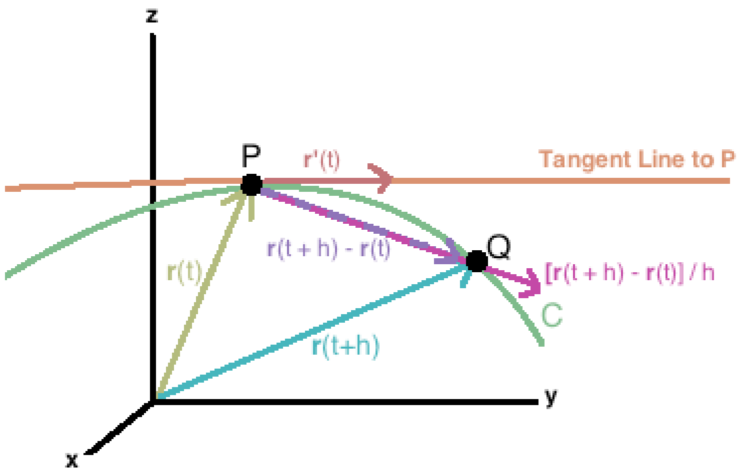
Differentiation of vector functions

Definition: A vector function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is *differentiable at* t if the limit

$$\mathbf{r}'(t) \; := \; \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

exists.

The visual understanding of this limit is just like the one taught in 1st-semester calculus:



As h approach zero, the vector [r(t+h) - r(t)] / h approaches the vector **r**'(t), a vector that lies on the tangent line of the point P. We call this vector, **r**'(t), the **tangent vector** to the curve C at defined by the vector equation **r**(t) at P.

Notes:

- The **(principal) unit tangent vector** is given by $\quad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.
- Equivalently (to the above definition), $\mathbf{r}(t)$ is differentiable at a given t-value precisely when each of its component functions $x(t)$, $y(t)$ and $z(t)$ are differentiable there. When this is so, we have
$$\frac{d\mathbf{r}}{dt} \; = \; x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$
- If a vector function $\mathbf{r}(t)$ is differentiable, then the derivative $\mathbf{r}'(t)$ is itself another vector function, which may be differentiable as well. If so, then
$$\frac{d^2\mathbf{r}}{d^2t} \; = \; x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}.$$
- If the position vector function $\mathbf{r}(t)$ is differentiable, then $d\mathbf{r}/dt$ is the corresponding *velocity* vector function. What we call **speed** is actually the length $\|d\mathbf{r}/dt\|$ of the velocity function.

  If $d\mathbf{r}/dt$ is differentiable, then we call $d^2\mathbf{r}/dt^2$ the **acceleration** vector function.

- Properties: All are extensions of rules you learned in 1^{st}-semester calculus

1. $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$ (constant function rule)

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$ (constant multiple rule)

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ (product of scalar and vector fn.)

4. $\dfrac{d}{dt}\left[\dfrac{\mathbf{u}(t)}{f(t)}\right] = \dfrac{f'(t)\mathbf{u}(t) - f(t)\mathbf{u}'(t)}{[f(t)]^2}$ (quotient of vector and scalar fn.)

5. $\dfrac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$ (sum and difference rules)

6. $\dfrac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ (dot product rule)

7. $\dfrac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ (cross product rule)

8. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (chain rule)

Integration of vector functions

Assume $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\hat{k}$. Then the **indefinite integral**

$$\int \mathbf{r}(t)\, dt$$

can be found by, in turn, finding antiderivatives $F(t)$ of $f(t)$, $G(t)$ of $g(t)$, and $H(t)$ of $h(t)$. Then all antiderivatives of $\mathbf{r}(t)$ are characterized by

$$[F(t) + C_1]\mathbf{i} + [G(t) + C_2]\mathbf{j} + [H(t) + C_3]\mathbf{k} = \langle F(t), G(t), H(t)\rangle + \langle C_1, C_2, C_3\rangle,$$

where the vector $\mathbf{C} = \langle C_1, C_2, C_3\rangle$ is in the role of an (arbitary) constant vector of integration.

Definite integration occurs over a typically finite interval $a \leqslant t \leqslant b$:

$$\begin{aligned}\int_a^b \mathbf{r}(t)\, dt &= \left(\int_a^b f(t)\, dt\right)\mathbf{i} + \left(\int_a^b g(t)\, dt\right)\mathbf{j} + \left(\int_a^b h(t)\, dt\right)\mathbf{k} \\ &= (F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k})\Big|_a^b \\ &= [F(b) - F(a)]\mathbf{i} + [G(b) - G(a)]\mathbf{j} + [H(b) - H(a)]\mathbf{k}.\end{aligned}$$

Practice

1. Find the derivative of $\mathbf{r}(t) = \left\langle e^{-t}, \sin(3t), 10\sqrt{t} \right\rangle$. What is the tangent vector when $t = \pi/2$?
2. Find the expression for the unit tangent vector $\mathbf{T}(t)$ to $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$.
3. Given
$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} - t^4\mathbf{k} \qquad \text{and} \qquad \mathbf{s}(t) = \sin(t)\mathbf{i} + e^t\mathbf{j} + \cos(t)\mathbf{k},$$
find
    (a) $\mathbf{r}''(t)$
    (b) $\mathbf{r}'(t^2)$
    (c) $\frac{d}{dt}[t^2\mathbf{s}(t)]$
    (d) $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{s}(t)]$
4. Suppose a particle moves with acceleration $\mathbf{a}(t) = -5\cos(t)\mathbf{i} - 5\sin(t)\mathbf{j}$.
    (a) Find, generally, the velocity $\mathbf{v}(t)$ and position $\mathbf{r}(t)$.
    (b) What refinements on the previous answers are possible when we have the additional information that $\mathbf{v}(0) = \langle 9, 2\rangle$ and $\mathbf{r}(0) = \langle 5, 0\rangle$?