$\underline{\star 23}$ By exchanging rows 1 and 3, $\mathbf{H}$ gets to RREF:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Writing an element of null $(\mathbf{H})$ as $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, we see that we can take x_3, x_5, x_6 and x_7 as "free" variables (free to take on either of the values 0 or 1), while

$$\begin{aligned} x_1 &= x_3 + x_5 + x_7 \\ x_2 &= x_3 + x_6 + x_7 \ , \\ x_4 &= x_5 + x_6 + x_7 \end{aligned}$$

so vectors in the null space take the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, one possible basis is

$$\{(1,1,1,0,0,0,0), (1,0,0,1,1,0,0), (0,1,0,1,0,1,0), (1,1,0,1,0,0,1)\}.$$

Some notes:

- That this collection spans null $(\mathbf{H})$ is clear from the solution process. That it is linearly independent perhaps calls for forming a 7-by-4 matrix with these as the columns, reducing to echelon form and seeing that that echelon form has no free columns. I don't necessarily expect students will do this.
- As is pretty much always the case, there are other bases for the same subspace—we use a different basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ for null $(\mathbf{H})$ in part (b). While the one I've given above is the most likely basis for students to find, there are yet others. The easiest way to check a strange-looking answer is to make sure the proposed collection contains 4 vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ all from $\mathbb{Z}_2^7$, check that each one is in null $(\mathbf{H})$ (i.e., that $\mathbf{H}\mathbf{v}_j = \mathbf{0}$ for $j = 1, 2, 3, 4$), and that they are linearly independent.

$\underline{\star 24}$ (a) The transmitted $\mathbf{v}$ is given by

$$\mathbf{v} \;=\; 1\begin{bmatrix}1\\0\\0\\0\\0\\1\\1\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\\0\\1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\\0\\1\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\0\\1\\1\\1\\1\end{bmatrix} \;=\; \begin{bmatrix}1\\1\\0\\1\\0\\0\\1\end{bmatrix}.$$

(b) We have

$$\mathbf{H}\tilde{\mathbf{v}} \;=\; \begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}\begin{bmatrix}1\\0\\1\\1\\1\\0\\0\end{bmatrix} \;=\; \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Since $\mathbf{H}\tilde{\mathbf{v}}$ is not the zero vector, $\tilde{\mathbf{v}}$ is corrupted. If only corrupted in a single entry, it must be the the 3$^{\text{rd}}$ entry (as $\mathbf{H}\tilde{\mathbf{v}}$ equals the 3$^{\text{rd}}$ column of $\mathbf{H}$). Thus, the originally-intended 7-bit word is $\mathbf{v} = (1,0,0,1,1,0,0)$, from which we extract the 4-bit word $(1,0,0,1)$, (or 1001).

(c) Sadly, the use of the Hamming (7,4) scheme for detecting and correcting errors breaks down if two (or more) bits from a 7-bit transmitted word are corrupt. To see this, notice that if the 7-bit word $\mathbf{v} = (1,0,0,1,1,0,0)$ is corrupted to $\tilde{\mathbf{v}} = (1,1,0,1,0,0,0)$ (*two* altered bits), then we will, indeed, *detect* an error (it is still the case, with this $\tilde{\mathbf{v}}$, that $\tilde{\mathbf{v}} \notin \text{null}(\mathbf{H})$), but that our process for correction would make us think that the 7$^{\text{th}}$ bit alone was faulty (not the pair of 2$^{\text{nd}}$ and 5$^{\text{th}}$ bits). With *three* altered bits we might not even detect the error!

$\underline{\star 25}$ We use the notation $\mathrm{m_a}(\lambda)$ to denote the algebraic multiplicity of the eigenvalue λ.

(b) $n = 9$ with $\mathrm{m_a}(0) = 2$, $\mathrm{m_a}(1) = 4$, $\mathrm{m_a}(-3) = 1$, $\mathrm{m_a}(2+\mathrm{i}3) = 1$, $\mathrm{m_a}(2-\mathrm{i}3) = 1$. The matrix is not invertible.

(c) $n = 8$ with $\mathrm{m_a}(-2) = 3$, $\mathrm{m_a}(-5) = 1$, $\mathrm{m_a}(-3+\mathrm{i}4) = 2$, $\mathrm{m_a}(-3-\mathrm{i}4) = 2$. The matrix is invertible.

$\underline{\star 26}$ Perhaps the easiest way to do this is to note that, if $y(x) \to 2/3$ as $x \to \infty$, then $y'(x)$ is simultaneously going to zero. If we set y' equal to zero we get the equation $ay+b = 0$, which implies that, should y' ever reach zero, the solution reaches $y = -b/a$. So, we should choose a, b so that the ratio $-b/a$ is $2/3$; $a = -3$, $b = 2$ works, corresponding to the DE $y' = 3y - 2$, but there are other choices. Unfortunately, $a = 3$, $b = -2$ *does not work*, as its solution goes to $2/3$ as $x \to -\infty$.

A hammer-it-out approach would involve solving the DE outright. It is both *linear* and *separable*. Capitalizing on the latter, we solve:

$$\begin{aligned}
\frac{1}{ay+b}\frac{dy}{dx} = 1 \qquad &\Rightarrow \qquad \int \frac{1}{ay+b}\,dy = \int dx \\
&\Rightarrow \qquad \frac{1}{a}\ln|ay+b| = t + C \\
&\Rightarrow \qquad \ln|ay+b| = at + \tilde{C} \\
&\Rightarrow \qquad |ay+b| = e^{C}\cdot e^{at} \\
&\Rightarrow \qquad ay+b = \tilde{C}e^{at} \\
&\Rightarrow \qquad y(t) = Ce^{at} - \frac{b}{a}.
\end{aligned}$$

The Ce^{at} part in the solution will decay exponentially to 0 if and only if $a < 0$. Given this, the overall solution will go to $(-b/a)$, which we want to be $2/3$. So, as long as a, b are chosen so that

- this ratio is $2/3$, and
- $a < 0$

then the resulting DE $y' = ay + b$ will have solutions that behave as requested.

$\star$27 The answer requires construction of the matrix $\Phi(t)$.

(a) We have

$$\Phi(t) = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix} \quad\rightsquigarrow\quad \Phi(0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}.$$

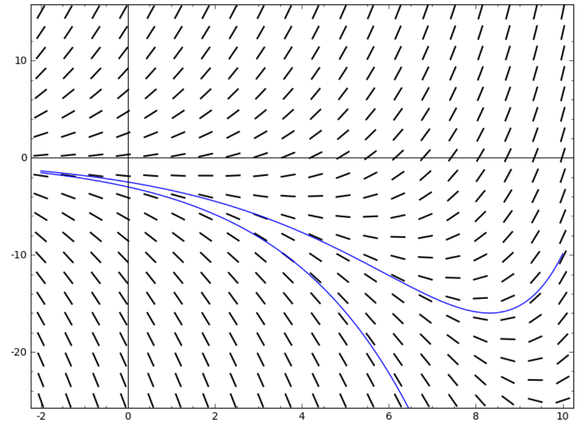
Since $\Phi(0) \xrightarrow{\text{RREF}} \boldsymbol{I}_3$, the functions are linearly independent.

(b) We have

$$\Phi(t) = \begin{pmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{pmatrix} \quad\rightsquigarrow\quad \Phi(0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since $\Phi(0) \xrightarrow{\text{RREF}} \boldsymbol{I}_3$, the functions are linearly independent.

$\star$28 (a) A direction field, along with a couple solutions (not required), appears at right. The appearance is that solutions grow without bound as $t \to \infty$, though some appear to go to $+\infty$ and others to $-\infty$, depending on the choice of a.



(b) While some estimates may be more precise than others, reasonable values should be between (-2) and (-4).

(c) To solve the (linear) DE $y' - \frac{1}{2}y = \frac{1}{2}\exp(t/3)$, we first solve $y' = \frac{1}{2}y$ to get $\Phi(t) = \exp(t/2)$, a basis for all solutions to the homogeneous problem. Then, using the variation of parameters formula,

$$y_p(t) \;=\; e^{t/2}\int \frac{e^{t/3}}{2e^{t/2}}\,dt \;=\; \frac{1}{2}e^{t/2}\int e^{-t/6}\,dt \;=\; (-6)\frac{1}{2}e^{t/2}\cdot e^{-t/6} \;=\; -3e^{t/3}.$$

Thus, the general solution is $y(t) = y_h(t) + y_p(t) = Ce^{t/2} - 3e^{t/3}$. Applying the IC $y(0) = a$, we have

$$a \;=\; C - 3 \qquad \Rightarrow \qquad C = a + 3.$$

Thus,

$$y(t) \;=\; (a+3)e^{t/2} - 3e^{t/3}.$$

We note that $e^{t/2}$ is a faster-growing exponential than $e^{t/3}$, and when its coefficient $(a+3)$ is opposite in sign to that (-3) of $e^{t/3}$, solutions will go to $+\infty$ as $t \to \infty$; otherwise, they will go to $-\infty$. So, $a_0 = -3$.

(d) When $a = -3$, the solution is $y(t) = -3e^{t/3}$, which goes to $-\infty$ as $t \to \infty$.

$\star$29 (a) This is $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (there is no $\mathbf{b}(t)$, as this system is homogeneous) with

$$\mathbf{A} \;=\; \begin{bmatrix} 6 & 4 & 4 \\ -7 & -2 & -1 \\ 7 & 4 & 3 \end{bmatrix}. \qquad \text{The initial condition is} \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}.$$

(b) One might put the full solution together in one vector before differentiating, to get

$$\mathbf{x}(t) \;=\; \begin{bmatrix} e^{6t} + 2e^{2t} \\ -e^{6t} - 4e^{2t} - e^{-t} \\ e^{6t} + 2e^{2t} + e^{-t} \end{bmatrix}.$$

Thus,

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 6e^{6t} + 4e^{2t} \\ -6e^{6t} - 8e^{2t} + e^{-t} \\ 6e^{6t} + 4e^{2t} - e^{-t} \end{bmatrix},$$

while

$$\begin{aligned}\mathbf{A}\mathbf{x}(t) &= \begin{bmatrix} 6 & 4 & 4 \\ -7 & -2 & -1 \\ 7 & 4 & 3 \end{bmatrix}\begin{bmatrix} e^{6t} + 2e^{2t} \\ -e^{6t} - 4e^{2t} - e^{-t} \\ e^{6t} + 2e^{2t} + e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} 6e^{6t} + 12e^{2t} - 4e^{6t} - 16e^{2t} - 4e^{-t} + 4e^{6t} + 8e^{2t} + 4e^{-t} \\ -7e^{6t} - 14e^{2t} + 2e^{6t} + 8e^{2t} + 2e^{-t} - e^{6t} - 2e^{2t} - e^{-t} \\ 7e^{6t} + 14e^{2t} - 4e^{6t} - 16e^{2t} - 4e^{-t} + 3e^{6t} + 6e^{2t} + 3e^{-t} \end{bmatrix} = \begin{bmatrix} 6e^{6t} + 4e^{2t} \\ -6e^{6t} - 8e^{2t} + e^{-t} \\ 6e^{6t} + 4e^{2t} - e^{-t} \end{bmatrix},\end{aligned}$$

showing that the two, indeed, are equal, when $\mathbf{x}(t)$ is as proposed. Moreover, for the proposed $\mathbf{x}$, $\mathbf{x}(0)$ satisfies the IC.

$\star$30 (a) One can show this matrix has the repeated eigenvalue $\lambda = -3$; its algebraic multiplicity is 2. To find corresponding eigenvectors, we solve

$$\left(\begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\mathbf{x} = \mathbf{0},$$

which has augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|} 1 & 1 \\ 0 & 0 \end{array}\right].$$

A basis for these eigenvectors consists of just one (since GM = 1), and can be taken as $\langle 1, -1\rangle$. The eigenvalue is degenerate, so along with the solution

$$\mathbf{x}_1(t) = e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

we expect a second solution in a fundamental set of solutions of the form

$$\mathbf{x}_2(t) = e^{-3t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \mathbf{u}\right),$$

where $\mathbf{u}$ satisfies $(\mathbf{A} + 3\mathbf{I})\mathbf{u} = \langle 1, -1\rangle$. To find such a $\mathbf{u}$, we work with the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

The second component of $\mathbf{u}$, u_2, is free, and we may take it to be 0, in which case $u_1 = 1$, and $\mathbf{u} = \langle 1, 0\rangle$. Thus, the general solution is

$$\mathbf{x}(t) = c_1e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} t+1 \\ -t \end{bmatrix} = e^{-3t}\begin{bmatrix} 1 & t+1 \\ -1 & -t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

(b) From software, we learn that the matrix of this problem has eigenpairs

$$\lambda = -2 \quad \text{with corresp. basis of eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix},$$

$$\lambda = 16 \qquad \text{with corresp. basis eigenvector} \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Continuing with my original eigenpairs, these give rise to general solution

$$\mathbf{x}(t) \;=\; c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + c_3 e^{16t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

(c) From software, we learn that the matrix of this problem has eigenpairs

$\lambda = 1$ with corresp. basis eigenvector $(1, -1, 2)$,
$\lambda = 4 + i$ with corresp. basis eigenvector $(1, 1, 2) + i(-1, 1, 0)$,
$\lambda = 4 - i$ with corresp. basis eigenvector $(1, 1, 2) - i(-1, 1, 0)$.

Identifying $\alpha = 4$, $\beta = 1$, $\mathbf{u} = (1, 1, 2)$ and $\mathbf{w} = (-1, 1, 0)$ in order to circumvent complex exponential solutions arising from the nonreal eigenpairs, we obtain general solution

$$\begin{aligned}
\mathbf{x}(t) \;&=\; c_1 e^{t} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 e^{4t} \left(\cos t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) + c_3 e^{4t} \left(\sin t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \cos t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&=\; c_1 \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{4t}(\cos t + \sin t) \\ e^{4t}(\cos t - \sin t) \\ 2e^{4t} \cos t \end{bmatrix} + c_3 \begin{bmatrix} e^{4t}(\sin t - \cos t) \\ e^{4t}(\sin t + \cos t) \\ 2e^{4t} \sin t \end{bmatrix}.
\end{aligned}$$

To satisfy the IC, we need

$$\begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix} \;=\; \mathbf{x}(0) \;=\; c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Solving this vector equation we obtain $c_1 = 1$, $c_2 = 2$, $c_3 = -1$. So, the unique solution to the IVP is

$$\mathbf{x}(t) = \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix} + 2\begin{bmatrix} e^{4t}(\cos t + \sin t) \\ e^{4t}(\cos t - \sin t) \\ 2e^{4t}\cos t \end{bmatrix} - \begin{bmatrix} e^{4t}(\sin t - \cos t) \\ e^{4t}(\sin t + \cos t) \\ 2e^{4t}\sin t \end{bmatrix}$$

$$= \begin{bmatrix} e^t + e^{4t}(3\cos t + \sin t) \\ -e^t + e^{4t}(\cos t - 3\sin t) \\ 2e^t + 2e^{4t}(\cos t + \sin t) \end{bmatrix}.$$

(d) The eigenvalues of the matrix are (-1) (AM=1) and 4 (AM=2). Since

$$\left[\begin{array}{ccc|c} 6.5 & -1.5 & 1.5 & 0 \\ 0.5 & 0.5 & -1.5 & 0 \\ -1 & -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3/8 & 0 \\ 0 & 1 & -21/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

eigenvectors corresponding to $\lambda = -1$ look like

$$\mathbf{v} = \begin{bmatrix} (3/8)v_3 \\ (21/8)v_3 \\ v_3 \end{bmatrix} = \frac{v_3}{8}\begin{bmatrix} 3 \\ 21 \\ 8 \end{bmatrix},$$

or scalar multiples of $\langle 3, 21, 8\rangle$. This gives us a first solution toward a fundamental set:

$$\mathbf{x}_1(t) = e^{-t}\begin{bmatrix} 3 \\ 21 \\ 8 \end{bmatrix}.$$

In order to find eigenvectors corresponding to $\lambda = 4$, we work with the augmented matrix

$$\left[\begin{array}{ccc|c} 1.5 & -1.5 & 1.5 & 0 \\ 0.5 & -4.5 & -1.5 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

There is just one free column, so $\lambda = 4$ is a degenerate eigenvalue. It's eigenspace is 1-dimensional, with basis eigenvector $\langle -3, -1, 2\rangle$. This gives us a second solution for a fundamental set,

$$\mathbf{x}_2(t) = e^{4t}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}.$$

We look for an extra solution of the form

$$\mathbf{x}_3(t) = e^{4t}\left(t\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} + \mathbf{u}\right), \qquad \text{where} \qquad (\mathbf{A} - 4\mathbf{I})\mathbf{u} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}.$$

The augmented matrix for used for finding $\mathbf{u}$ is

$$\left[\begin{array}{ccc|c} 1.5 & -1.5 & 1.5 & -3 \\ 0.5 & -4.5 & -1.5 & -1 \\ -1 & -1 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & -2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so if we take the free variable $u_3 = 0$, then $u_1 = -2$ and $u_2 = 0$, so $\mathbf{u} = \langle -2, 0, 0 \rangle$. Thus,

$$\mathbf{x}_3(t) \;=\; e^{4t}\left(t\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}\right) \;=\; e^{4t}\begin{bmatrix} -3t-2 \\ -t \\ 2t \end{bmatrix}.$$

Thus, the general solution is

$$\begin{aligned} \mathbf{x}(t) \;&=\; c_1e^{-t}\begin{bmatrix} 3 \\ 21 \\ 8 \end{bmatrix} + c_2e^{4t}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} + c_3e^{4t}\begin{bmatrix} -3t-2 \\ -t \\ 2t \end{bmatrix} \\ &=\; \begin{bmatrix} 3e^{-t} & -3e^{4t} & -(3t+2)e^{4t} \\ 21e^{-t} & -e^{4t} & -te^{4t} \\ 8e^{-t} & 2e^{4t} & 2te^{4t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \end{aligned}$$