Test: Fri. Oct. 9

Covers Ch.1: 1.1 - 1.5

Ch.2: 2.1 - 2.5 (ish)

Math 251, Wed 30-Sep-2020 -- Wed 30-Sep-2020
Discrete Mathematics
Fall 2020

From 2.5, some things you are to know:

- meaning of "countable / uncountable" set
- which sets have cardinality aleph-nought
- why are we sure some infinite cardinalities are larger?

A: No bijection exists between $\mathbb{N}$ and $[0,1]$

A2: No bijection exists between any set A and its $\mathcal{P}(A)$ (power set).

Wednesday, September 30th 2020

Wk 5, We
Topic:: Sequences and sums
Read:: Rosen 2.4
HW:: WW SequencesAndSeries due Sat.
HW:: PS07 due Mon.

Sequences

Definition 1: A **sequence** is a function $a\colon A \to B$ for which $A \subseteq \mathbb{Z}$.

- Most commonly, $A = \mathbb{N}$ or $\mathbb{Z}^+$, the integers beginning with either 0 or 1.
- Since the domain of a includes only integers, you can talk about $a(2)$, $a(1000)$, etc., but not $a(2.3)$.
- Usually a subscript notation is adopted, a_n instead of $a(n)$, but both refer to the same thing, the value of the sequence for input n.
- The specification of inputs is somewhat arbitrary, less important than the outputs themselves. Different ways of naming the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

include

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad \ldots$ valid

equally valid: $a_5 = 1$

$a_6 = 1/2$

$a_7 = 1/3$

$a_8 = 1/4$

$\vdots$

85, 81, 77, __, __, __

terms of arithmetic seq

2. $\sum_{k=1}^{25}(5k+3) = 8 + 13 + 18 + 23 + 28 + \cdots + 128 = S$?

$128 + 123 + 118 + 113 + 108 + \cdots + 8 = S$

$136 + 136 + 136 + 136 + 136 + \cdots + 136 = 2S$

General case of summing terms in an arithmetic sequence

Recall: Arithmetic seq. has

- a_0, d, governing recurrence relation $a_n = a_{n-1} + d$
- $a_n = a_0 + dn$ (explicit formula)

common ratio

$160(\frac{1}{2}) + 160(\frac{1}{2})^2 + \cdots + 160(\frac{1}{2})^{25} + 160(\frac{1}{2})^{26} = \frac{1}{2}S$

3. $\sum_{m=0}^{25} 160\left(\frac{1}{2}\right)^m = 160 + 160(\frac{1}{2}) + 160(\frac{1}{2})^2 + \cdots + 160(\frac{1}{2})^{25} = S$

$160(\frac{1}{2})^{26} - 160 = \frac{1}{2}S - S$

General case of summing terms in an geometric sequence

Every geom. seq.: starting term a_0, common ratio r, recurrence rel. $a_n = r a_{n-1}$

$\Longrightarrow a_n = a_0 r^n$ closed formula

Now summing: (up to term w/ subscript n)

$a_0 + a_1 + a_2 + \cdots + a_n = a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^n$

Other general formulas (see p. 166)

$$\sum_{j=1}^{n} j^2 = 1 + 4 + 9 + 16 + \cdots + n^2 = S$$

Can use to

$$\sum_{j=1}^{n}(3j - 2j^2) = \sum_{j=1}^{n} 3j - \sum_{j=1}^{n} 2j^2 = 3\sum_{j=1}^{n} j - 2\sum_{j=1}^{n} j^2$$

Say you borrow \$10,000 (a_0) at 12% interest, compounded monthly. If at the end of each month you pay \$250,

Each month, 1% interest applied

- how much will you owe after one month? $a_1 = (10000)(1.01) - 250 = 1.01 a_0 - 250$
- how much will you owe after two months? $a_2 = 1.01 a_1 - 250$
- Write a recursion formula for the amount owed after *n* months.

$$\boxed{a_n = 1.01 a_{n-1} - 250}$$

- Write a closed formula for the amount owed after *n* months.

$$a_n = 1.01\, a_{n-1} - 250 = 1.01\left(1.01\, a_{n-2} - 250\right) - 250$$

$$= 1.01\left(1.01\left(1.01\, a_{n-3} - 250\right) - 250\right) - 250$$

$$= 1.01\left(\cdots \left(1.01\left(1.01\, a_0 - 250\right) - 250\right) \cdots \right) - 250$$

$$= (1.01)^n a_0 - 250\left(\underbrace{1 + (1.01) + (1.01)^2 + \cdots + (1.01)^{n-1}}_{\text{geometric series}}\right)$$

$$S = a_0 + a_1 + a_2 + \cdots + a_n \quad \text{(terms from arithmetic seq.)}$$

$$S = a_n + a_{n-1} + a_{n-2} + \cdots + a_0$$

first, last terms

$$2S = \underbrace{(a_0 + a_n) + (a_0 + a_n) + \cdots + (a_0 + a_n)}_{\#\text{ of terms } = n+1} = (n+1)(a_0 + a_n)$$

$$2S = (n+1)(a_0 + a_n)$$

$$S = (n+1)\left(\frac{a_0 + a_n}{2}\right)$$

i.e. our sum is the product of

(# of terms added)(avg. of 1st, last ones added)

$$160\left(\frac{1}{2}\right)^{26} - 160 = -\frac{1}{2} S$$

mult by (-2)

$$S = -2(160) \cdot \left[\left(\frac{1}{2}\right)^{26} - 1\right]$$

logged in today:
Jacob
Noah
Richmond
Brian L.
Daniel S?
Oscar?

Generally

$$\sum_{j=0}^{n} a_0 r^j = a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^n = S$$

$$a_0 r + a_0 r^2 + \quad + a_0 r^n + a_0 r^{n+1} = rS$$

subtract

$$a_0 - a_0 r^{n+1} = S - rS$$

$$a_0\left(1 - r^{n+1}\right) = (1-r)S$$

$$\Longrightarrow \boxed{S = a_0 \frac{1-r^{n+1}}{1-r}}$$

(sum of finitely many terms in geom. seq.)

$$\xrightarrow[n\to\infty]{} a_0 \cdot \frac{1}{1-r}$$

when $|r| < 1$