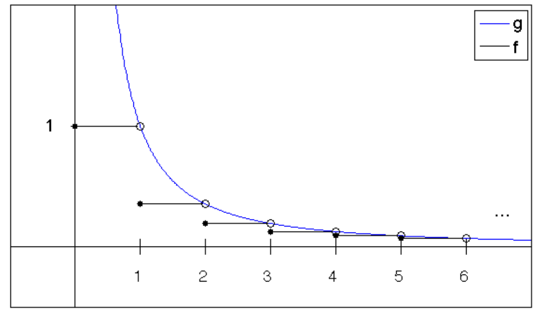
MATH 162: Calculus II

Framework for Tues., Feb. 13

Introduction to Series

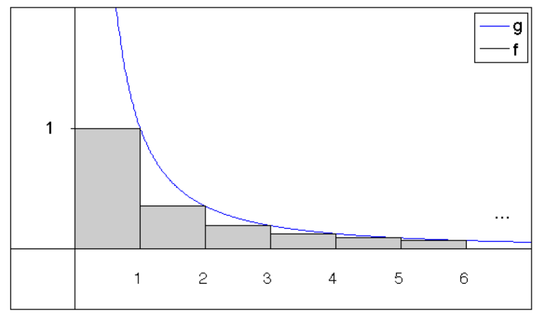


Example: Application of the direct comparison test

Suppose $g(x) = x^{-p}$, with $p > 0$ (so g has the general shape of the blue curve for $x > 0$), and f is the step function pictured in black.

- By the direct comparison test, if $p > 1$ then

$$\begin{aligned}\int_1^{\infty} f(x)\,dx &= 2^{-p} + 3^{-p} + 4^{-p} + \cdots + n^{-p} + \cdots \\ &= \sum_{n=2}^{\infty} n^{-p}\end{aligned}$$

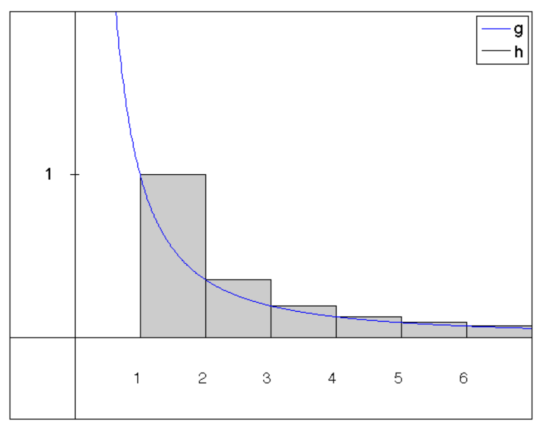


converges. And, since it is the case that

$$\sum_{n=1}^{\infty} n^{-p} = 1 + \sum_{n=2}^{\infty} n^{-p},$$

$\sum_{n=1}^{\infty} n^{-p}$ converges as well when $p > 1$.

- To conclude $\sum_{n=1}^{\infty} n^{-p}$ diverges for $p \leq 1$, we must deal with a function like f that stays above g. The function $h(x) = f(x-1)$ will do (see at right). The improper integral



$$\begin{aligned}\int_1^{\infty} h(x)\,dx &= 1^{-p} + 2^{-p} + 3^{-p} + \cdots \\ &= \sum_{n=1}^{\infty} n^{-p}\end{aligned}$$

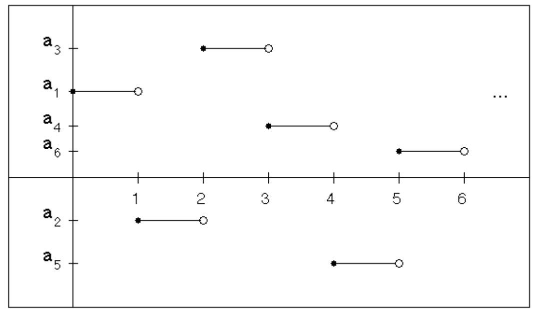
diverges since $\int_1^{\infty} x^{-p}\,dx$ diverges for $p \leq 1$.

Infinite Series

- An infinite sum of numbers: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$.
- Can be thought of as an improper integral

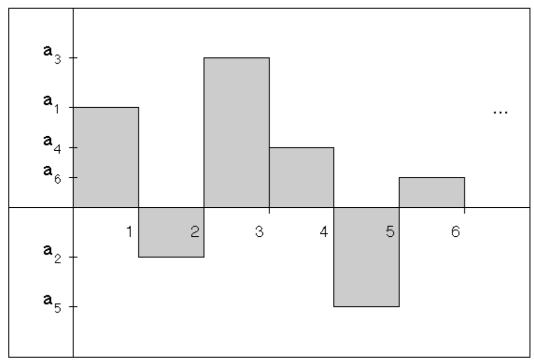
  Define $f(x) := \begin{cases} a_1, & 0 \le x < 1, \\ a_2, & 1 \le x < 2, \\ \vdots & \vdots \\ a_n, & n-1 \le x < n, \\ \vdots & \vdots \end{cases}$

  (See graph of step fn. at right.)



  Then our given series may be expressed as an improper integral of f:

$$\sum_{n=1}^{\infty} a_n = \int_0^{\infty} f(x)\, dx.$$



- Will be said to *converge* or *diverge*. As with other improper integrals, convergence requires the existence of a limit of "proper sums" (actually called *partial sums*). Define

$$\begin{aligned} s_1 &:= a_1, \\ s_2 &:= a_1 + a_2, \\ s_3 &:= a_1 + a_2 + a_3, \\ &\vdots \\ s_n &:= a_1 + a_2 + \cdots + a_n, \\ &\vdots \end{aligned}$$

  The series $\sum_{n=1}^{\infty} a_n$ converges precisely when the *sequence* s_n converges to some real number limit.

Examples:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \qquad \text{diverges.}$$

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \qquad \text{converges.}$$