Sequences

A **sequence** $a(n)$, or a_n, is a function whose inputs include some smallest integer n_0 and all the integers following it: $n_0+1, n_0+2, \ldots$. For most of the sequences we encounter, $n_0 = 0$ or $n_0 = 1$.

For functions f accepting real numbers x, calculus has led us to consider related functions:

- **derivative function**: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
- **antiderivative function**: the signed-area-up-to-x function with starting point b $F(x) = \int_b^x f(t)\, dt$

The analogs to these for a sequence $a : a_0, a_1, a_2, \ldots$

- **sequence d_n of adjacent differences**: $d_n = a_n - a_{n-1}$, for $n = 1, 2, \ldots$
  For example, for a starter sequence $a_n = 2^n$, we have

  $$d_1 = a_1 - a_0 = 2 - 1 = 1,$$
  $$d_2 = a_2 - a_1 = 2^2 - 2 = 3,$$

  and generally $d_n = a_n - a_{n-1} = 2^n - 2^{n-1} = 2^{n-1}$.
- **sequence s_n of partial sums**: $s_n = \sum_{j=0}^n a_j = a_0 + a_1 + \cdots + a_n,\ \ n = 0, 1, 2, \ldots$.
  For example, for a starter sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ (the Fibonacci numbers, for those who have encountered them before),

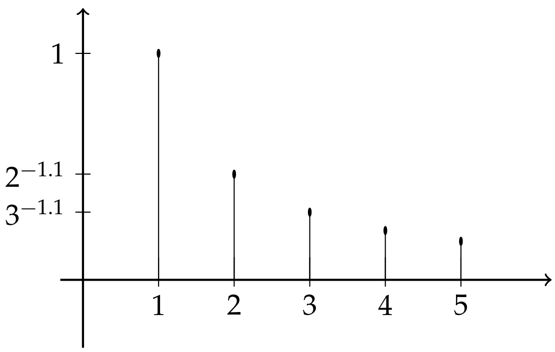
  $$s_0 = 1,$$
  $$s_1 = 1 + 1 = 2,$$
  $$s_2 = 1 + 1 + 2 = 4,$$
  $$s_3 = 1 + 1 + 2 + 3 = 6, \text{ etc.}$$

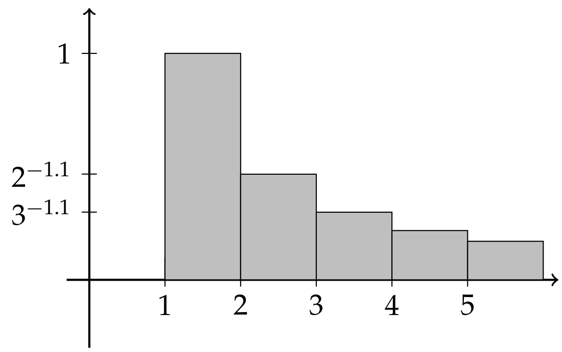
The "geometry" of sums of terms in a sequence

Let's take a particular sequence, $a_n = 1/n^{1.1}$; that is, a_0 doesn't make sense, but $a_1 = 1$, $a_2 = 2^{-1.1}$, $a_3 = 3^{-1.1}$, etc. It's associated sequence of partial sums also starts at $n = 1$:

$$s_1 = 1, \quad s_2 = 1 + \frac{1}{2^{1.1}}, \qquad s_3 = 1 + \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}}, \quad \text{etc.}$$

The terms of this new sequence s_n can be visualized in relation to the original sequence a_n either as the sum of heights of points on the graph of a_n (left) or sums of areas of rectangles (right).





Though the pictures displayed are specifically drawn for the example where $a_n = n^{-1.1}$, similar views are possible for any base sequence a_n, even one with negative values (which corresond to rectangles contributing negative area).

Infinite Series

For some base sequence a_n, the infinite series $\sum_{j=0}^{n} a_j$ is too much to wrap one's mind around. Just as we treated an improper integral (also difficult to wrap one's mind around)

$$\int_b^\infty f(x)\,dx \qquad \text{as the limit of proper integrals} \qquad \lim_{N\to\infty}\int_b^N f(x)\,dx,$$

so we treat the **infinite series** as a limit of (finite) sums—that is, we think of

$$\sum_{j=0}^{\infty} a_j \qquad \text{as} \qquad \lim_{n\to\infty}\sum_{j=0}^{n} a_j \;=\; \lim_{n\to\infty}(a_0 + a_1 + \cdots + a_n) \;=\; \lim_{n\to\infty} s_n.$$

We say the series $\sum_{j=0}^{\infty} a_j$ **converges** to s if its sequence of partial sums has limit $\lim_n s_n = s$. If the sequence of partial sums s_n diverges (i.e., does not have a limit), then we say the series $\sum_{j=0}^{\infty} a_j$ **diverges**.

Talking about whether a series converges or not is like talking about the existence of God—much easier to agree on the meaning and significance of the question than it is to give evidence that proves an answer. Take this example, for instance:

Example 1:

Let $a_n = n^{-1}$. The first few partial sums are

$s_1 = a_1 = 1.$
$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = 1.5.$
$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{3} \doteq 1.833.$
$s_4 = \sum_{j=1}^{4} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \doteq 2.083.$
…(skipping a bit)

$s_{10} = \sum_{j=1}^{10} \frac{1}{j} \doteq 2.929.$
$s_{100} = \sum_{j=1}^{100} \frac{1}{j} \doteq 5.187.$
$s_{1000} = \sum_{j=1}^{1000} \frac{1}{j} \doteq 7.486.$

Can we tell if the sequence s_n has a limit based on this evidence? Not conclusively.

■

Section 10.1 laid out many tools for determining if a sequence converges. The problem is that, when deciding whether the infinite series $\sum_j a_j$ converges, the analogy with improper integrals calls on us to consider the associated sequence s_n of partial sums, and despite the fact we may have a nice (explicit) formula for a_n, that doesn't necessarily translate into an explicit formula for s_n. There are *two* cases where explicit formulas for s_n exist:

1. **Geometric series**. The underlying (base) sequence $a_n = a_0 r^n$ is geometric:

$$a_0,\ a_0 r,\ a_0 r^2,\ a_0 r^3,\ \ldots.$$

It can be shown that

$$\sum_{j=0}^{n-1} a_0 r^j = a_0(1 + r + r^2 + \cdots + r^{n-1}) = \frac{a_0(1-r^n)}{1-r} \qquad \text{or} \qquad \sum_{j=0}^{n} a_0 r^j = \frac{a_0(1-r^{n+1})}{1-r}.$$

2. **Telescoping series**. This is a case where the vast majority of "in-between" terms in s_n cancel out, leaving a simpler formula than $s_n = \sum_j^n a_j$. Examples include:

(a) $\displaystyle\sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1}\right).$ For this series,

$$\begin{aligned} s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n}. \end{aligned}$$

(b) $\displaystyle\sum_{j=1}^{\infty} \left(\frac{1}{2j-1} - \frac{1}{2j+3}\right).$ This time,

$$\begin{aligned} s_n &= \sum_{j=1}^{n} \left(\frac{1}{2j-1} - \frac{1}{2j+3}\right) = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+3}\right) \\ &= 1 + \frac{1}{3} - \frac{1}{2n+1} - \frac{1}{2n+3}. \end{aligned}$$

Examples of infinite series

1. Determine if the series converges/diverges. For the convergent ones, find, if possible, the series sum.
   (a) $\displaystyle\sum_{j=1}^{\infty}\left(5\frac{2^j}{3^{j+1}}-\frac{1}{2^j}\right)$.
   (b) $\displaystyle\sum_{j=0}^{\infty}\frac{3^{2j}}{4^{j+1}}$.
   (c) $\displaystyle\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{j+1}\right)$.
   (d) $\displaystyle\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j+3}\right)$.
2. $\displaystyle\sum_{j=1}^{\infty}\frac{1}{j2^j}=\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{4}\left(\frac{1}{2}\right)^4+\cdots$ Compare with a geometric series.
3. $\displaystyle\sum_{j=0}^{\infty}(-1)^j$.
4. Suppose a superball is dropped from a height of 10 feet. On each impact, the ball rebounds to 80% of its previous height. How far, in total, does the ball travel?
5. Determine the length of the Koch curve.