Math 251, Wed 23-Sep-2020 -- Wed 23-Sep-2020
Discrete Mathematics
Fall 2020

Wednesday, September 23rd 2020

Due:: PS05 due at 6 pm

Wednesday, September 23rd 2020

Wk 4, We
Topic:: Functions
Read:: Rosen 2.3
HW:: WW functions due Sat.
HW:: PS06 due Mon.

If we set $n = \lfloor x \rfloor$ then

- $n \in \mathbb{Z}$
- $n \leq x < n+1$

$$\forall x \in \mathbb{R}\ \forall m \in \mathbb{Z}\ \left(\lfloor x + m \rfloor = \lfloor x \rfloor + m \right)$$

Here

$$n \leq x < n+1 \quad \rightarrow \quad n+m \leq x+m < n+m+1$$

So

$$\lfloor x+m \rfloor = n+m$$
$$= \lfloor x \rfloor + m.$$

$\blacksquare$

$f(x) = 3x+1$ valid inputs $\mathbb{R}$

$g(x) = \sqrt{x}$

$(g \circ f)(x) = \sqrt{3x+1}$ — restrict, from original domain $\mathbb{R}$ of f, so to using x values for which $3x+1 \geq 0$.

Some special funcitons

χ χ χ chi

Identity function ι: $A \to B$ requires $\boxed{A \subseteq B}$.

Indicator functions. Given a set $A \subseteq \mathbb{R}$, the indicator function on the set A is defined as

$$\boxed{\chi_A(x) \ := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}}$$

- evaluating an indicator function

$A = \{\text{alphabetical lower case letters}\} \subseteq U = \{\text{all symbols}\}$

$\chi_A(t) = 1$ $\qquad$ $\chi_A(!) = 0$

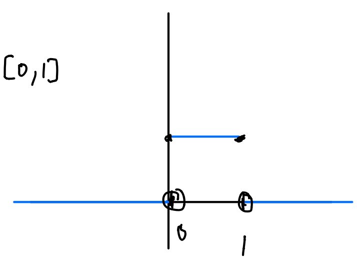
$A \subseteq B$

$\to \forall x \ (\chi_B(x) = 1$

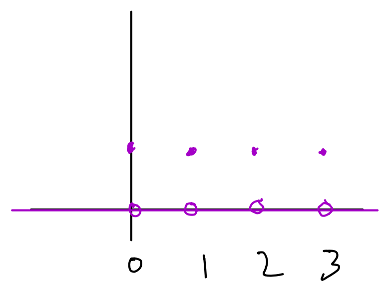
$\to \chi_A(x) = 1)$

- graph of an indicator function

$\chi_{[0,1]}$



$\chi_{\mathbb{N}}$



The floor/ceiling functions.

$\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$

$\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$

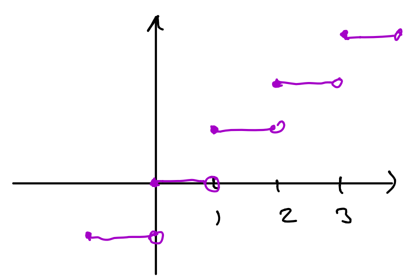
- how defined
- True or false?
  1. $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\ (\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor)$ False: counterexample

     $18 = \lfloor 18.1 \rfloor = \lfloor 12.6 + 5.5 \rfloor \neq \lfloor 12.6 \rfloor + \lfloor 5.5 \rfloor$

     $= 12 + 5 = 17$
  2. $\forall x \in \mathbb{R}\, \forall m \in \mathbb{Z}\ (\lfloor x + m \rfloor = \lfloor x \rfloor + m)$ True
  3. $\forall x \in \mathbb{R}\ (\lfloor -x \rfloor = -\lfloor x \rfloor)$ False: $x = 0.5$ is a counterexample

     change $-\lceil x \rceil$ makes statement true
  4. $\forall x \in \mathbb{R}\ (\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 0.5 \rfloor)$ True.
  5. $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ is a bijection.

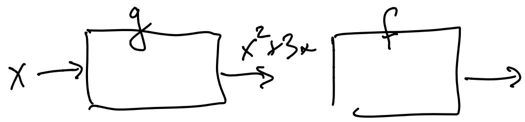
     surjective? Yes
     injective? No } Not a bijection



Composing functions

- meaning of $(f \circ g)(x)$
  - practice

$$f(x) = 2x - 1, \quad g(x) = x^2 + 3x \quad \Big\} \quad (f \circ g)(x) = f(g(x)) = 2(x^2+3x) - 1$$

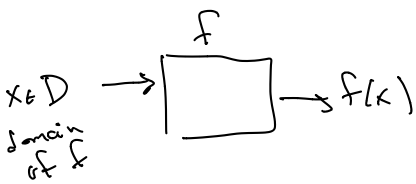


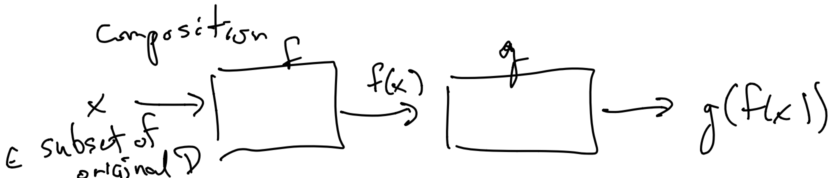
  - difference from $(g \circ f)(x)$

$$(g \circ f)(x) = g(f(x)) = (2x-1)^2 + 3(2x-1).$$

$$\cos(x^2+1) = (f \circ g)(x) \text{ for some pair } \quad f(x) = \cos x, \quad g(x) = x^2 + 1$$

  - compare domain of $g \circ f$ with domain of f





  - recursion: $(f \circ f)(x)$, $(f \circ f \circ f)(x)$

```
function factorial( int n )
  # check that input n is positive
  if n > 1
    return n * factorial(n - 1)
  else
    return 1
```

input $n \in \mathbb{Z}^+$

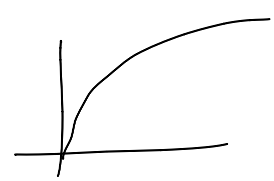
$$\text{factorial}(5) = 5 \cdot \text{factorial}(4) = 5 \cdot 4 \cdot \text{factorial}(3) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

- bijections, invertibility, composition with the inverse
  - examples of bijections

Is bijective: $f(x) = 3x + 1$ maps $\mathbb{R} \to \mathbb{R}$ (surjective)

not bijective: $f(x) = \sqrt{x}$ as a fn. $\mathbb{R} \to \mathbb{R}$

but it is bijection as fn. $\mathbb{R} \to [0, \infty)$



  - When $f\colon A \to B$ is a bijection, the sets A, B are in **one-to-one correspondence**. Say $|A| = |B|$.

    Examples of sets in/not in 1-to-1 correspondence:

    1. {alphabet} and {days of the week}

2. $\mathbb{N}$ and { positive odd integers }

3. $\mathbb{N}$ and $\mathbb{Z}$

4. $\mathbb{N}$ and $\mathbb{Q}$

5. $\mathbb{N}$ and $[0,1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$

So $\mathbb{N}$ and $[0,1$ are infinite sets, but $|\mathbb{N}| \neq |[0,1]|$. Write $|\mathbb{N}| = \aleph_0$, and $|\mathbb{R}| = \mathfrak{c}$.

- Question: If f, g are bijections, is $f \circ g$?
- When $f\colon A \to B$ is a bijection, $f^{-1}\colon B \to A$ exists (as a function)