MATH 162: Calculus II

Framework for Tues., Mar. 27

More about Lines and Planes in Space

Today's Goal: To review how lines and planes in space are represented, and use these notions to derive some useful formulas and algorithms involving points, lines and planes.

Lines and Planes

We have derived the following representations.

- **Lines**. The line trough point $P = (x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

  component form: $$x = x_0 + v_1 t, \quad y = y_0 + v_2 t, \quad z = z_0 + v_3 t, \qquad -\infty < t < \infty,$$

  vector form: $$\mathbf{r}(t) = (x_0 + v_1 t)\mathbf{i} + (y_0 + v_2 t)\mathbf{j}(z_0 + v_3 t)\mathbf{k}, \qquad -\infty < t < \infty.$$

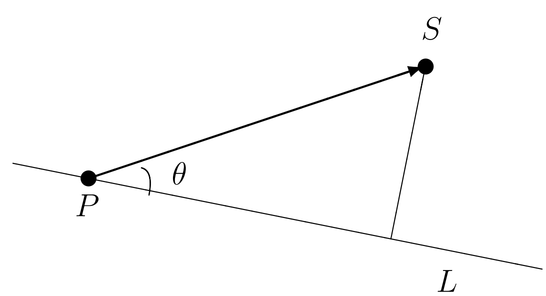
- **Planes**. The plane trough point $P = (x_0, y_0, z_0)$ perpendicular to $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

  $$\mathbf{n} \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0, \qquad \text{or} \qquad ax + by + cz = d,$$

  where $d = ax_0 + by_0 + cz_0$.

Formulas and Algorithms for Lines and Planes

- **Distance from a point S to a line L.**

  Keys to a formula:

  1. Our distance is $|\overrightarrow{PS}| \sin\theta$, where P is any point on line L.
  2. For two vectors $\mathbf{u}$ and $\mathbf{v}$, $\quad |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$.

  From these we get

  $$|\overrightarrow{PS}| \sin\theta = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|},$$

  where $\mathbf{v}$ is any vector parallel to line L.

- **Distance from a point S to a plane** containing the point P with normal vector $\mathbf{n}$.

  Keys to a formula:

  1. Our distance is $|\overrightarrow{PS}||\cos\theta|$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{n}$.
  2. If θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.

  Thus, we get
  $$|\overrightarrow{PS}||\cos\theta| = \frac{|\overrightarrow{PS}\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

- **Angle between two planes**.

  Definition: The *angle between planes* is taken to be the angle $\theta \in [0, \pi/2]$ between normal vectors to the planes.

  By this definition, if $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors to the two planes, then the angle between the planes is
  $$\theta = \begin{cases} \arccos\left(\dfrac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right), & \text{if } \mathbf{n}_1\cdot\mathbf{n}_2 \geq 0, \\[2ex] \pi - \arccos\left(\dfrac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right), & \text{if } \mathbf{n}_1\cdot\mathbf{n}_2 < 0. \end{cases}$$

- **Line of intersection between two non-parallel planes**.

  It should not be difficult to find a point on the desired line. If the two planes have equations $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$, then it is quite likely the line of intersection will eventually pass through a point P where the x-coordinate is zero. Assuming this is so, we may do the usually steps of solving the simultaneous equations in 2 unknowns
  $$\begin{aligned} b_1y + c_1z &= d_1 \\ b_2y + c_2z &= d_2 \end{aligned}$$
  for the corresponding y and z coordinates of this point. (If the solution process fails to yield corresponding y and z coordinates, one can instead look for the point P for which the y or, alternatively, the z-coordinate is zero.)

  Once a point P on our line of intersection is found, we next need a vector that is parallel to our line. Such a vector would be perpendicular to normal vectors to both planes, and so could be any multiple of
  $$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$