4.1 If $\lambda = 0$, then (1) becomes $X'' = 0$

$$\Rightarrow X(x) = ax + b, \quad \text{where } a, b \text{ are arbitrary constants.}$$

To satisfy

$$0 = X(0) = a \cdot 0 + b \quad \text{requires} \quad b = 0.$$

To satisfy

$$0 = X(\pi) = a \cdot \pi \quad \text{requires} \quad a = 0$$

So, the only solution is trivially, $X(x) = 0$.

4.3 Along a fixed characteristic $x(t) = ct + \xi$, $\tilde{u}(t) = u(x(t), t)$ satisfies

$$\tilde{u}' = -\lambda \tilde{u} \quad \Rightarrow \quad \tilde{u}' + \lambda u = 0$$

$$\tilde{u}' e^{\lambda t} + \lambda e^{\lambda t} u = 0$$

or $$\frac{d}{dt}\left(\tilde{u} e^{\lambda t}\right) = 0.$$

For $x > ct$, characteristics $x(t) = ct + \xi$ emanate from $(\xi, 0)$, when $t = 0$.

Thus, it makes sense to use $t = 0$ as the lower limit in an integral:

$$\int_0^t \frac{d}{d\tau}\left(\tilde{u} e^{\lambda \tau}\right) d\tau = \int_0^t 0 \cdot d\tau$$

$$\tilde{u}(t) e^{\lambda t} - \tilde{u}(0) = 0$$

$$0 = u(x,t) e^{\lambda t} - u(\xi, 0) = u(x,t) e^{\lambda t} \quad \Rightarrow u(x,t) = 0 \text{ when } x > ct.$$

For $x < ct$, characteristics emanate from $t = \frac{-\xi}{c}$ (where $x(t) = 0$). Making this the lower limit of integration gives us

$$x = ct + \xi$$
$$-\xi = ct - x$$
$$-\xi/c = t - x/c$$

$$\int_{-\xi/c}^t \frac{d}{d\tau}\left(\tilde{u} e^{\lambda \tau}\right) d\tau = \int_{-\xi/c}^t 0 \cdot d\tau$$

$$\tilde{u}(t) e^{\lambda t} - \tilde{u}(-\xi/c) e^{-\lambda \xi/c} = 0$$

Now, $\tilde{u}(-\xi/c) = u(x(-\xi/c), -\xi/c) = u(0, -\xi/c) = g(-\xi/c) = g(t - x/c)$.

So, $$\tilde{u}(t) = g(t - x/c) e^{-\lambda \xi/c} \cdot e^{-\lambda t} = g(t - x/c) e^{-\lambda/c(\xi + ct)} = g\left(t - \frac{x}{c}\right) e^{-\lambda x/c}$$

$$\Rightarrow \boxed{u(x,t) = \begin{cases} 0, & \text{if } x > ct \\ g(t - \frac{x}{c})\, e^{-\lambda x/c}, & \text{if } x < ct \end{cases}}$$

4.4 Following the hint,

$$E'(t) = \frac{d}{dt}\int_0^\ell [u(x,t)]^2\,dx = \int_0^\ell \frac{d}{dt}[u(x,t)]^2\,dx = \int_0^\ell 2uu_t\,dx$$

$$= 2k\int_0^\ell uu_{xx}\,dx \qquad (\text{since } u(x,t) \text{ solves the PDE for } 0<x<\ell)$$

$$= 2k\Big[u(x,t)u_x(x,t)\Big]_0^\ell - 2k\int_0^\ell (u_x)^2\,dx \qquad \text{after integrating by parts}$$

$$= 0 - 2k\int_0^\ell (u_x)^2\,dx \le 0$$

Thus, for $t \ge 0$,

$$\int_0^\ell [u(x,t)]^2\,dx = E(t) \le E(0) = \int_0^\ell [u_0(x)]^2\,dx$$

4.5 Set $\quad w(x,t) = u(x,t) - \frac{\ell - x}{\ell}g(t) - \frac{x}{\ell}h(t).$

Then $\quad w_{xx} = u_{xx}$, and

$$w_t = u_t - \frac{\ell - x}{\ell}g'(t) - \frac{x}{\ell}h'(t)$$

$$= ku_{xx} - \frac{\ell - x}{\ell}g'(t) - \frac{x}{\ell}h'(t),$$

or w satisfies the nonhomogeneous heat pde

$$w_t = kw_{xx} - \frac{\ell - x}{\ell}g'(t) - \frac{x}{\ell}h'(t)$$

Moreover, since u satisfies the conditions (BCs + IC) of its problem,

$$\left.\begin{aligned} w(0,t) &= u(0,t) - \frac{\ell - 0}{\ell}g(t) - \frac{0}{\ell}h(t). \\ &= g(t) - g(t) = 0, \\ w(\ell,t) &= u(\ell,t) - \frac{\ell - \ell}{\ell}g(t) - \frac{\ell}{\ell}h(t). \\ &= h(t) - h(t) = 0 \end{aligned}\right\} \text{ w satisfies homogeous BCs}$$

$$w(x,0) = u(x,0) - \frac{l-x}{l}g(0) - \frac{x}{l}h(0).$$
$$= u_0(x) - \frac{l-x}{l}g(0) - \frac{x}{l}h(0).$$

This last expression says w satisfies the same IC as u, though altered by a linear function of x.