# MATH 231: Differential Equations with Linear Algebra

## Hand-Checked Assignment #1, due date: Mon., Mar. 2, 2020

$\star$1 Which of the following matrices are guaranteed to equal $(\mathbf{A}+\mathbf{B})^2$?

$$(\mathbf{B}+\mathbf{A})^2\,, \quad \mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2\,, \quad \mathbf{A}(\mathbf{A}+\mathbf{B})+\mathbf{B}(\mathbf{A}+\mathbf{B})\,, \quad (\mathbf{A}+\mathbf{B})(\mathbf{B}+\mathbf{A})\,, \quad \mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2\,.$$

For each one you choose, provide a justification.

$\star$2 Suppose $\mathbf{A}$ is a square matrix that commutes with every other square matrix of the same size as $\mathbf{A}$ (i.e., $\mathbf{AB} = \mathbf{BA}$ for every matrix $\mathbf{B}$).

(a) Consider the special case in which $\mathbf{A}$ is a 2-by-2 matrix, and $\mathbf{AB} = \mathbf{BA}$ for all 2-by-2 matrices $\mathbf{B}$. As there are no exceptional matrices $\mathbf{B}$, we note particularly that

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad \mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Use these two instances to deduce that $a = d$ and $b = c = 0$—that is, if $\mathbf{AB} = \mathbf{BA}$ for even just these two choices of $\mathbf{B}$, then $\mathbf{A}$ is a multiple of the identity matrix.

(b) Will such an $\mathbf{A}$ (as the one from part (a), which was chosen so as to commute with $\mathbf{B}_1$ and $\mathbf{B}_2$) *really* commute with all other choices of 2-by-2 matrices $\mathbf{B}$? Demonstrate the truth of your response.

(c) For general $n$, make a conjecture about the type of $n$-by-$n$ matrix $\mathbf{A}$ that will commute with all others. Then provide evidence in the $n = 3$ case that your answer is correct.

$\star$3 (a) If $\mathbf{A}$ is nonsingular (invertible) and $\mathbf{AB} = \mathbf{AC}$, show (using just one algebraic operation) that $\mathbf{B} = \mathbf{C}$.

(b) When $\mathbf{A}$ is singular, "cancellation" (as in the previous part) is not possible. Show this in the case of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. That is, find examples of matrices $\mathbf{B}$ and $\mathbf{C}$ (i.e., give their entries as numbers) so that $\mathbf{AB} = \mathbf{AC}$ but $\mathbf{B} \neq \mathbf{C}$.

$\star$4 Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 2 & 1 & 3 & 3 \\ 2 & 1 & 2 & -1 & 4 \\ 1 & -3 & 1 & 1 & 7 \\ 2 & 0 & 1 & -2 & 2 \end{array}\right].$$

(a) Write down the corresponding linear system of 4 (algebraic) equations in variables $x_1$, $x_2$, $x_3$ and $x_4$ that corresponds to this augmented matrix.

(b) Carry out the following sequence of **elementary row operations** (EROs) in the given order, writing the new form of the augmented matrix after each step.

i. ERO1: swap rows 1 and 3; i.e., $\mathbf{r}_1 \leftrightarrow \mathbf{r}_3$

ii. ERO3: add $(-2)$ multiples of row 1 to row 2; that is, $(-2)\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2$

iii. ERO3: add $(-2)$ multiples of row 1 to row 4; $(-2)\mathbf{r}_1 + \mathbf{r}_4 \to \mathbf{r}_4$

iv. ERO1: swap rows 2 and 3; $\mathbf{r}_2 \leftrightarrow \mathbf{r}_3$

v. ERO3: add $(-7/2)$ multiples of row 2 to row 3; $(-7/2)\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3$

vi. ERO3: add $(-3)$ multiples of row 2 to row 4; $(-3)\mathbf{r}_2 + \mathbf{r}_4 \to \mathbf{r}_4$

vii. ERO3: add $(-8/7)$ multiples of row 3 to row 4; $(-8/7)\mathbf{r}_3 + \mathbf{r}_4 \to \mathbf{r}_4$

What you should have after the 7 steps is

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 0 & 0 & 17/7 & 17/7 \end{array}\right].$$

[Note: While a given matrix has many echelon forms, you should get this particular one if you followed the sequence of EROs given above.]

(c) While part (b) yields an echelon form for the original augmented matrix, it is not in **reduced row echelon form** (RREF). Describe (using notation akin to the instructions given to you in part (b)) a sequence of EROs which, starting from the echelon form above, takes the matrix to RREF. Give both your sequence of EROs, and the contents of the matrix after each step.

(d) Write, in vector form, the solution of the system of equations in part (a).

★5 Suppose there is a town which perenially follows these rules:

- The number of households always stays fixed at 10000.
- Every year 30 percent of households currently subscribing to the local newspaper cancel their subscriptions.
- Every year 20 percent of households not receiving the local newspaper subscribe to it.

(a) Suppose one year, there are 8000 households taking the paper. According to the data above, these numbers will change the next year. The total of subscribers will be

$$(0.7)(8000) + (0.2)(2000) = 6000 ,$$

and the total of nonsubscribers will be

$$(0.3)(8000) + (0.8)(2000) \;=\; 4000\,.$$

If we create a 2-vector whose first component is the number of subscribers and whose 2nd component is the number of nonsubscribers, then the initial vector is $(8000, 2000)$, and the vector one year later is

$$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}.$$

What is the long-term outlook for newspaper subscription numbers?

(b) Does your answer above change if the initial subscription numbers are changed to 9000 subscribing households? Explain.

<u>★6</u> (a) Suppose $\mathbf{A}$ is an $m$-by-4 matrix. Find a matrix $\mathbf{P}$ (you should determine appropriate dimensions for $\mathbf{P}$, as well as specify its entries) so that $\mathbf{AP}$ has the same entries as $\mathbf{A}$ but the $1^{\text{st}}$, $2^{\text{nd}}$, $3^{\text{rd}}$ and $4^{\text{th}}$ *columns* of $\mathbf{AP}$ are the $2^{\text{nd}}$, $4^{\text{th}}$, $3^{\text{rd}}$ and $1^{\text{st}}$ columns of $\mathbf{A}$ respectively. Such a matrix $\mathbf{P}$ is called a **permutation matrix**.

(b) Suppose $\mathbf{A}$ is a 4-by-$n$ matrix. Find a matrix $\mathbf{P}$ so that $\mathbf{PA}$ has the same entries as $\mathbf{A}$ but the $1^{\text{st}}$, $2^{\text{nd}}$, $3^{\text{rd}}$ and $4^{\text{th}}$ rows of $\mathbf{PA}$ are the $2^{\text{nd}}$, $4^{\text{th}}$, $3^{\text{rd}}$ and $1^{\text{st}}$ rows of $\mathbf{A}$ respectively.

(c) Suppose $\mathbf{A}$ is an $m$-by-3 matrix. Find a matrix $\mathbf{B}$ so that $\mathbf{AB}$ again has 3 columns, the first of which is the sum of all three columns of $\mathbf{A}$, the $2^{\text{nd}}$ is the difference of the $1^{\text{st}}$ and $3^{\text{rd}}$ columns of $\mathbf{A}$ (column 1 - column 3), and the $3^{\text{rd}}$ column is 3 times the $1^{\text{st}}$ column of $\mathbf{A}$.

<u>★7</u> **A Basis for the Null Space of the 3-by-7 Hamming Matrix**. Consider the set $\mathbb{Z}_2^n$. The objects in this set are $n$-by-1 matrices (in that respect they are like the objects in $\mathbb{R}^n$), with entries that are *all zeros or ones*; each object in $\mathbb{Z}_2^n$ can be thought of as an $n$-bit binary word.

We wish to define what it means to *add* objects in $\mathbb{Z}_2^n$, and how to multiply these objects by a reduced list of scalars—namely 0 and 1. When we add vectors from $\mathbb{Z}_2^n$, we do so componentwise (as in $\mathbb{R}^n$), but with each sum calculated mod 2.[1] Scalar multiplication is done mod 2 as well. For instance, in $\mathbb{Z}_2^3$ we have

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

[1]Modular arithmetic is the type of *integer* arithmetic we use with clocks. For a standard clock, the *modulus* is 12, resulting in statements like "It is now 8 o'clock; in 7 hours it will be 3 o'clock" (i.e., "8 + 7 = 3"). In mod 2 arithmetic, the modulus is 2, and we act as if the only numbers on our "clock" are 0 and 1.

Note that, when operations are performed mod 2, an $m$-by-$n$ matrix times a vector in $\mathbb{Z}_2^n$ produces a vector in $\mathbb{Z}_2^m$. For instance

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 1 \\ 7 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and is equivalent to} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Consider the matrix

$$\mathbf{H} := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

An easy way to remember this matrix, known as the **Hamming Matrix**, is through noting that beginning from its left column you have, in sequence, the 3-bit binary representations of the integers 1 through 7. Find a basis for $\operatorname{null}(\mathbf{H})$, where the matrix product $\mathbf{Hx}$ is to be interpreted mod 2 as described above.

A couple of observations may be helpful. First, if you had a 2-by-5 matrix with entries from $\mathbb{Z}_2$ such as this one

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

the next step in Gaussian elimination would be to zero out the rest of column 2 under the pivot. You can do this by adding row 1 to row 2—that is:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

At this point, this 2-by-5 matrix has reached echelon form (not quite RREF, yet).

Secondly (but related), in $\mathbb{Z}_2$ each of the two possible numbers (0 and 1) are their own additive inverses. That is,

$$0 + 0 = 0 \qquad \text{and} \qquad 1 + 1 = 0.$$

This means that, when you have a variable $x$ that represents a number in $\mathbb{Z}_2$, then $x + x = 0$. So, if you have a $\mathbb{Z}_2$ equation which says

$$x_1 + x_3 + x_4 = 0,$$

you can *add* $x_3$ and $x_4$ to both sides to get

$$x_1 = x_3 + x_4.$$

Bizarre, yet kinda cool, too.

$\star$8 **Error-Correcting Codes: The Hamming (7,4) Code**. In this problem, we wish to look at a method for transmitting the 16 possible 4-bit **binary words**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0000 | 0001 | 0010 | 0011 | 0100 | 0101 | 0110 | 0111 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

in such a way that if, for whatever reason (perhaps electrostatic interference), some digit is reversed in transmission (a 0 becomes a 1 or vice versa), then the error is *both* detected and corrected.

In the previous problem, you found a basis for the null space of the Hamming matrix

$$\mathbf{H} := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

In most instances, a vector space has numerous different bases, and $\operatorname{null}(\mathbf{H})$ is no exception. Though it is likely a different collection of basis vectors than you found in the previous problem, going forward we will make use of the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$, where $\mathbf{u}_1 = (1,0,0,0,0,1,1)$, $\mathbf{u}_2 = (0,1,0,0,1,0,1)$, $\mathbf{u}_3 = (0,0,1,0,1,1,0)$, and $\mathbf{u}_4 = (0,0,0,1,1,1,1)$.

**Transmitting a 4-bit Word**

Let $(c_1, c_2, c_3, c_4)$ be a 4-bit word (i.e., each $c_i$ is 0 or 1), one we wish to transmit. We could do so as is, but if an error occurred in transmission, there would be no automatic indicator of this. Instead, we use the values $c_1, \ldots, c_4$ to generate a 7-bit word which will be the one we transmit. This 7-bit word is a linear combination (mod 2) of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$. To be precise, instead of the original 4-bit word, we transmit the 7-bit word

$$\mathbf{v} := c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 = (c_1, c_2, c_3, c_4, c_2 + c_3 + c_4, c_1 + c_3 + c_4, c_1 + c_2 + c_4).$$

This $\mathbf{v}$ is in both $\mathbb{Z}_2^7$ and $\operatorname{null}(\mathbf{H})$. (Do you see why it is an element of the latter?)

(a) Suppose we wish to transmit the 4-bit binary word 1101. What 7-bit word corresponding to this one will actually be transmitted?

**Error Detection and Correction**

Suppose a 7-bit word $\tilde{\mathbf{v}}$ is received. It may be the same as the transmitted $\mathbf{v}$, or it may be a corrupted version of $\mathbf{v}$. Suppose that at most one binary digit of $\tilde{\mathbf{v}}$ is in error. Then the matrix product $\mathbf{H}\tilde{\mathbf{v}}$ tells us what we need to know. To see this, consider two cases:

- There are no errors (that is, $\tilde{\mathbf{v}} = \mathbf{v}$).
  In this case, $\mathbf{H}\tilde{\mathbf{v}} = \mathbf{H}\mathbf{v} = \mathbf{0}$, and the receiver, who takes this as an indication that the word arrived uncorrupted, throws out the final 3 bits and keeps the first 4 (entries) of $\mathbf{v}$ as the 4-bit word originally intended.
- There is an error in position $i$ (so $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{e}_i$, where $\mathbf{e}_i$ is a vector of zeros except in its $i^{\text{th}}$ position, where it has a 1).
  In this case, $\mathbf{H}\tilde{\mathbf{v}} = \mathbf{H}(\mathbf{v} + \mathbf{e}_i) = \mathbf{H}\mathbf{v} + \mathbf{H}\mathbf{e}_i = \mathbf{0} + \mathbf{H}\mathbf{e}_i = \mathbf{H}\mathbf{e}_i = i$th column of $\mathbf{H}$. Thus, $\mathbf{H}\tilde{\mathbf{v}} \neq \mathbf{0}$ in this case. Moreover, by inspecting which column of $\mathbf{H}$ is equal to $\mathbf{H}\tilde{\mathbf{v}}$, we learn which of $\tilde{\mathbf{v}}$'s digits is different from those of $\mathbf{v}$. The receiver may correct that bit in $\tilde{\mathbf{v}}$, and once again take the first 4 bits of this (newly-corrected) $\tilde{\mathbf{v}}$ as the intended word.

(b) For practice (i.e., **don't hand in this first bit of work**), take your answer from part (a)—call this $\mathbf{v}$—and switch/corrupt the $1^{\text{st}}$ entry (binary digit), calling this new 7-bit word $\tilde{\mathbf{v}}$. Calculate $\mathbf{H}\tilde{\mathbf{v}}$, and use the procedure outlined above to convince yourself that the corrupted bit can be detected and corrected. Repeat this several times, corrupting some other bit of $\mathbf{v}$ to form $\tilde{\mathbf{v}}$.

Now, for something **to write up**: Suppose that the 7-bit word $(1, 0, 1, 1, 1, 0, 0)$ is received. Assuming that this was originally a 4-bit word that was sent using the Hamming (7,4) error-correcting code, and assuming at most one binary digit becomes corrupted during transmission, what was the original 4-bit word?

(c) {This part is optional.} What happens if more than one bit of the 7-bit (transmitted) word is corrupted? Investigate this question, and see if the procedure outlined above can be relied upon to *detect* and, if so, *correct* two corrupted bits in the transmitted word. Report on your findings.

$\star$9 [This one for practice only, not to be handed in.] Determine which of the following is an echelon form.

(a) $\begin{bmatrix} 0 & 2 & 1 & 6 & 5 & -1 \\ 0 & 0 & 0 & 3 & 2 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 7 & 3 & -1 & -5 \\ -1 & 1 & 1 & 4 & 2 \\ 0 & 2 & 3 & 5 & 1 \\ 0 & 0 & -1 & -1 & 7 \\ 0 & 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 4 & 2 & 8 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 5 \end{bmatrix}$

$\star$10 For this problem, the matrices involved are

- augmented matrices corresponding to some system of linear algebraic equations, and
- already in echelon form (RREF, in fact).

Thus, *no Gaussian elimination is required* of you here. Your task is to write the solution(s) of the system of equations. Specifically, when no solutions exist, state this. When solution(s) exist, express them in the form $\mathbf{x}_p + \mathbf{x}_n$; that is, identify the portion of the solution that makes up the null space, along with the particular solution.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

[Compare with part (a).]

(c) $\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(e) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

As an example, for the system with augmented matrix (in RREF)

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

solutions are $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, with

$$\mathbf{x}_p = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_n = t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \qquad \text{where } t \text{ is any real number.}$$