Some observations about series

We consider a series $\sum_{j=1}^{\infty} a_j$ whose underlying sequence is $a_1, a_2, a_3, \ldots$. (The index used for the initial value, be it 0 as in a_0, 1 as in a_1, or any other choice, is merely a matter of labeling.)

- Among other analogies already discussed between integration and series, there is a "substitution" process for series. That is, just as the substitution $u = x + 4$ changes the appearance of the integral
$$\int_1^{\infty} \frac{dx}{x} \qquad \text{to} \qquad \int_5^{\infty} \frac{du}{u-4},$$
so one can substitute $k = j + 4$ and change the appearance of the series
$$\sum_{j=1}^{\infty} \frac{1}{j} \qquad \text{to} \qquad \sum_{k=5}^{\infty} \frac{1}{k-4}.$$

- Just as one can evaluate an integral $\int_b^{\infty} f(x)\,dx$ in "pieces" by picking an intermediate point $x = c$ and writing
$$\int_b^{\infty} f(x)\,dx \;=\; \int_b^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$
so we can similarly split a series by picking some integer N and writing
$$\sum_{j=1}^{\infty} a_j \;=\; \sum_{j=1}^{N} a_j + \sum_{j=N+1}^{\infty} a_j \;=\; a_1 + a_2 + \cdots + a_N + \sum_{j=N+1}^{\infty} a_j.$$
For any such splitting, the behavior (convergence or divergence) of the full series $\sum_{j=1}^{\infty} a_j$ is linked to the behavior of the *series tail* $\sum_{j=N+1}^{\infty} a_j$. That is,

Theorem 1: The series $\sum_{j=1}^{\infty} a_j$ converges if and only if its tail end $\sum_{j=N+1}^{\infty} a_j$ converges no matter the choice of starting point $N + 1$ for that tail.

- It may be the case that, following some slot/index N, the terms of the sequence a_{N+1}, a_{N+2}, $a_{N+3}, \ldots$ are all positive. The result is that, from that slot on, the terms in the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ are strictly increasing—that is,
$$s_N < s_{N+1} = s_N + a_{N+1} < s_{N+2} = s_N + a_{N+1} + a_{N+2} < \cdots,$$
and since
$$\sum_{j=1}^{\infty} a_j \;=\; \lim_j s_j,$$

there are only two options: either $\sum_{j=1}^{\infty} a_j$ converges, or it diverges to $(+\infty)$.

Similarly, if beyond some index N the terms $a_{N+1}, a_{N+2}, \ldots$ are all negative, then the only options are that $\sum_{j=1}^{\infty} a_j$ converges, or it diverges to $(-\infty)$.

There are series which diverge, but neither to $(+\infty)$ nor $(-\infty)$. Given the last two sentences, these series must have infinitely-many terms that are positive as well as infinitely-many which are negative. One example is

$$\sum_{j=0}^{\infty}(-1)^j \;=\; 1-1+1-1+\cdots\,,$$

whose partial sums are

$$s_0 = 1,\;\; s_1 = 0,\;\; s_2 = 1,\;\; \text{etc., with } s_{\text{even}} = 1 \text{ and } s_{\text{odd}} = 0.$$

Since $\lim_n s_n$ does not exist, the series diverges (but neither to $(+\infty)$ nor to $(-\infty)$).