The Cross Product

For vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$, we define the *cross product* of $\mathbf{u}$ and $\mathbf{v}$ using 3-by-3 determinants:

$$\begin{aligned} \mathbf{u} \times \mathbf{b} \;&:=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \;=\; \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k} \\ &=\; (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}. \end{aligned}$$

Notes:

- It is the case that $(\mathbf{u} \times \mathbf{v})$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. More specifically,

  Fact: For vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$,
  $$\mathbf{u} \times \mathbf{v} := (\|\mathbf{u}\| \|\mathbf{v}\| \sin\theta)\, \mathbf{n},$$
  where $\mathbf{n}$ is a **unit vector** orthogonal to the plane containing $\mathbf{u}, \mathbf{v}$, and for which the vector triple $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ form a right-hand system.

- If the angle θ between $\mathbf{u}, \mathbf{v}$ is zero (i.e., $\mathbf{u}, \mathbf{v}$ are parallel), $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
- Properties
  1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$, given any two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$
  2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$
  3. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
  4. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$, for all vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$, all numbers r, s
  5. $\mathbf{u} \times \mathbf{0} = \mathbf{0}$, for any vector $\mathbf{u}$ in $\mathbb{R}^3$
  6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$, for any vector $\mathbf{u}$ in $\mathbb{R}^3$
  7. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$
- The cross product is not associative! This means that, in general, it is *not* true that
  $$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \qquad \text{and} \qquad \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$
  are equal.

Applications

- $\mathbf{r} \times \mathbf{F}$ is the *torque* vector resulting from a force $\mathbf{F}$ applied at the end of a lever arm $\mathbf{r}$.
- $\|\mathbf{u} \times \mathbf{v}\|$ (the length of the cross product $\mathbf{u} \times \mathbf{v}$) is the area of a parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.
- $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ (the absolute value of the scalar $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$) is the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
- Finding normal vectors to planes.

  Example: The vectors $\mathbf{u} = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle -2, 3, 1 \rangle$

  - are not parallel,
  - so they determine a family of parallel planes.

  Find a unit vector that is normal to all of these planes.

  Answer: Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are normal/perpendicular to all of these planes. The requirement that we produce a unit vector means there are only two correct answers, $\mathbf{n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{5\sqrt{3}}, \frac{7}{5\sqrt{3}} \right\rangle$ or $(-\mathbf{n})$.