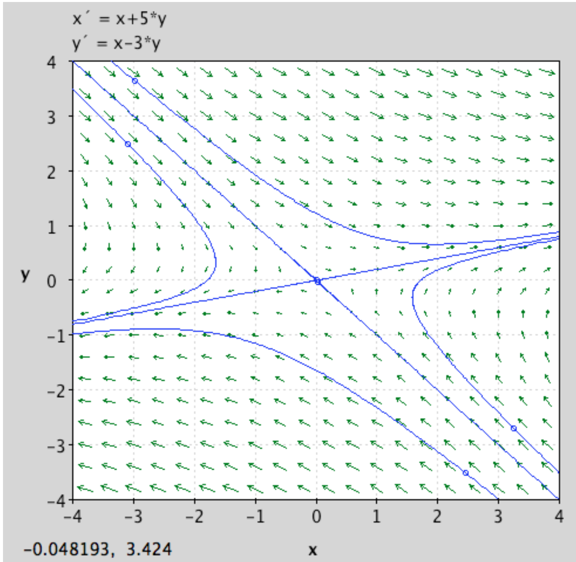
1. Origin is a **saddle point**

   Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

   $$\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}.$$

   The direction field and phase portrait is pictured at right.
   Analyzing this matrix, we find it has eigenpairs

eigenvalue	basis eigenvector(s)
2	(5, 1)
-4	(-1, 1)

   yielding general solution

   $$\mathbf{x}(t) \;=\; c_1 e^{2t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

   The eigenvectors explain the straight lines through the origin. These lines are, in fact, four separate trajectories: one arising when $c_1 = 0, c_2 > 0$; a second when $c_1 = 0, c_2 < 0$ (these two tend toward the origin as $t \to \infty$ because of the sign of the eigenvalue (-4)); a third when $c_1 > 0, c_2 = 0$; a fourth when $c_1 < 0, c_2 = 0$ (these two tend toward the origin as $t \to -\infty$).

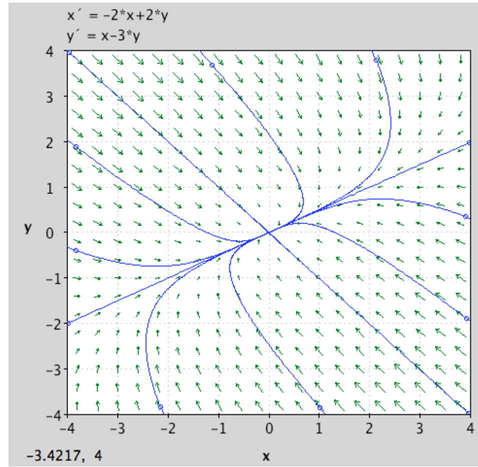
   A saddle point occurs whenever the eigenvalues of the 2-by-2 matrix $\mathbf{A}$ are real and of opposite sign. When you sketch a phase portrait, your drawing should include arrows on trajectories indicating direction of flow for increasing time. Make sure you are able to identify *eight* trajectories on the picture here, and know the appropriate orientation (arrow directions) on all eight.

2. Origin is a **node**

   The term **node** is applied to all situations in which both eigenvalues are real and of the same sign. But there are several kinds of nodes.

Node: Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}.$$



This matrix has eigenpairs

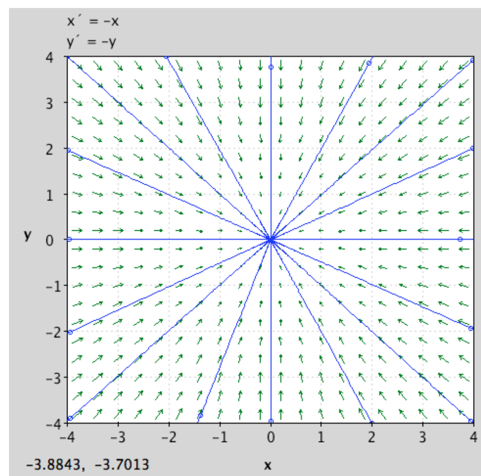
eigenvalue	basis eigenvector(s)
-1	(2, 1)
-4	(-1, 1)

yielding general solution

$$\mathbf{x}(t) \;=\; c_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Proper Node: Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

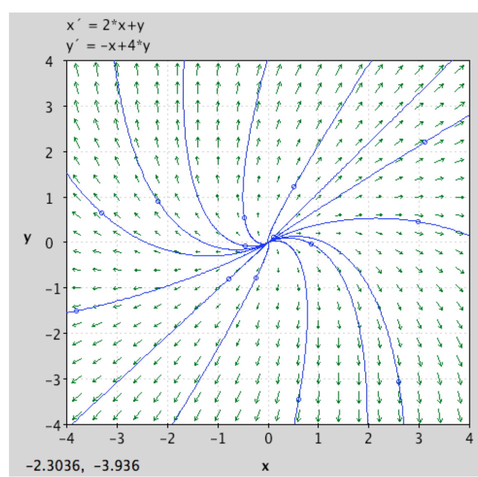


Obviously the eigenvalue (-1) has AM = 2. It is easily shown that GM = 2, and a basis of eigenvectors is $\{(1,0),(0,1)\}$, yielding general solution

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Improper/Degenerate Node: Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}.$$



The feature of $\mathbf{A}$ which may be used to identify a **degenerate node** is that it has a real eigenvalue (here, it is 3) with AM = 2, but GM = 1. In this instance a basis eigenvector is $\mathbf{v} = (1,1)$, and one solution of $(\mathbf{A} - 3\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$ is $\boldsymbol{\eta} = (-1,0)$, yielding general solution

$$\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right].$$

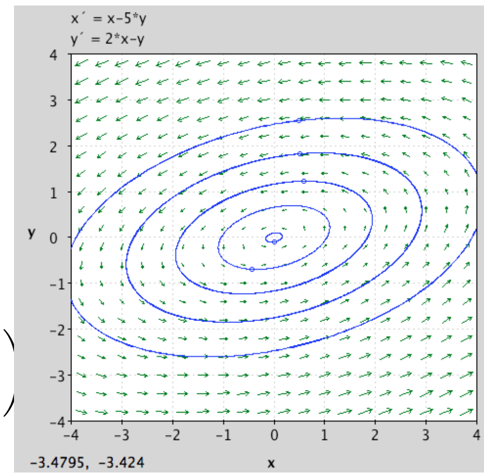
For solution trajectories with $c_2 = 0$, we get two rays emanating from the origin along the direction of the line parallel to the vector $(1,1)$. For those with $c_2 \neq 0$, notice that, as $t \to \pm\infty$, the relative influence of the two vectors in the sum

$$(-1,0) + t(1,1)$$

will be heavily tilted toward the eigenvector $(1,1)$. This means that, for $|t|$ large, trajectories should be more and more parallel to the vector $(1,1)$ as $t \to \pm\infty$, but during some intermediate range of t-values, it has to turn 180°.

3. Origin is a **center**

   Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where



$$\mathbf{A} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix}.$$

   Here, the eigenvalues are $\pm 3i$, and to the eigenvalue $3i$ there is a corresponding eigenvector $(-5, -1+3i)$. This yields general soln

$$\mathbf{x}(t) \;=\; c_1 \begin{pmatrix} -5\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} + c_2 \begin{pmatrix} -5\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}$$

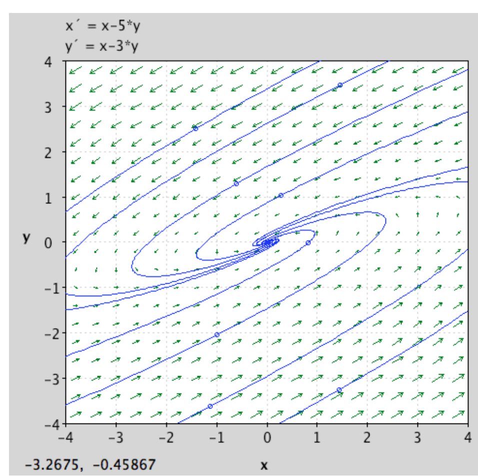
   Clearly there is a periodic nature to these solutions, explaining the closed loop trajectories. To determine orientation (direction of "flow" as t increases), take a test point, say, $(1,0)$. At this point, we have rate of change

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

   showing that, when we are at the point $(1,0)$, flow is upward to the right. Once you draw an arrow to this effect, orientation along any trajectory is the same.

4. Origin is a **spiral point**

   Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where



$$\mathbf{A} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}.$$

   This matrix has eigenvalue $\lambda = -1 + i$ with corresponding eigenvector $(5, 2-i)$, yielding general soln

$$\mathbf{x}(t) \;=\; c_1 e^{-t} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

   If it weren't for the presence of the factor e^{-t}, one would expect another *center*—trajectories forming closed loops. But, because of the exponential decay scaling factor, we have trajectories that spiral inward (note how you would add arrows to indicate orientation) toward the origin.

Stability diagram

For 2-by-2 matrix $\mathbf{A} = (a_{ij})$, let us define

$$\begin{aligned} \tau &= \text{trace}(\mathbf{A}) = a_{11} + a_{12}, \\ \Delta &= \det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Notice that the characteristic polynomial of $\mathbf{A}$, in this 2-by-2 case, is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{12})\lambda + a_{11}a_{22} - a_{21}a_{12} = \lambda^2 - \tau\lambda + \Delta,$$

which has roots (the eigenvalues of $\mathbf{A}$)

$$\lambda_{1,2} = \frac{\tau}{2} \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta}.$$

Saddle points arise when are the two eigenvalues are nonzero real numbers of opposite sign, and this occurs precisely when

$$\sqrt{\tau^2 - 4\Delta} > |\tau| \qquad \Leftrightarrow \qquad \Delta < 0.$$

Nodes arise when these eigenvalues are distinct, but of the same sign, and this occurs precisely when the expression under the radical

$$0 < \tau^2 - 4\Delta < \tau^2 \qquad \Leftrightarrow \qquad 0 < \Delta < \tau^2/4.$$

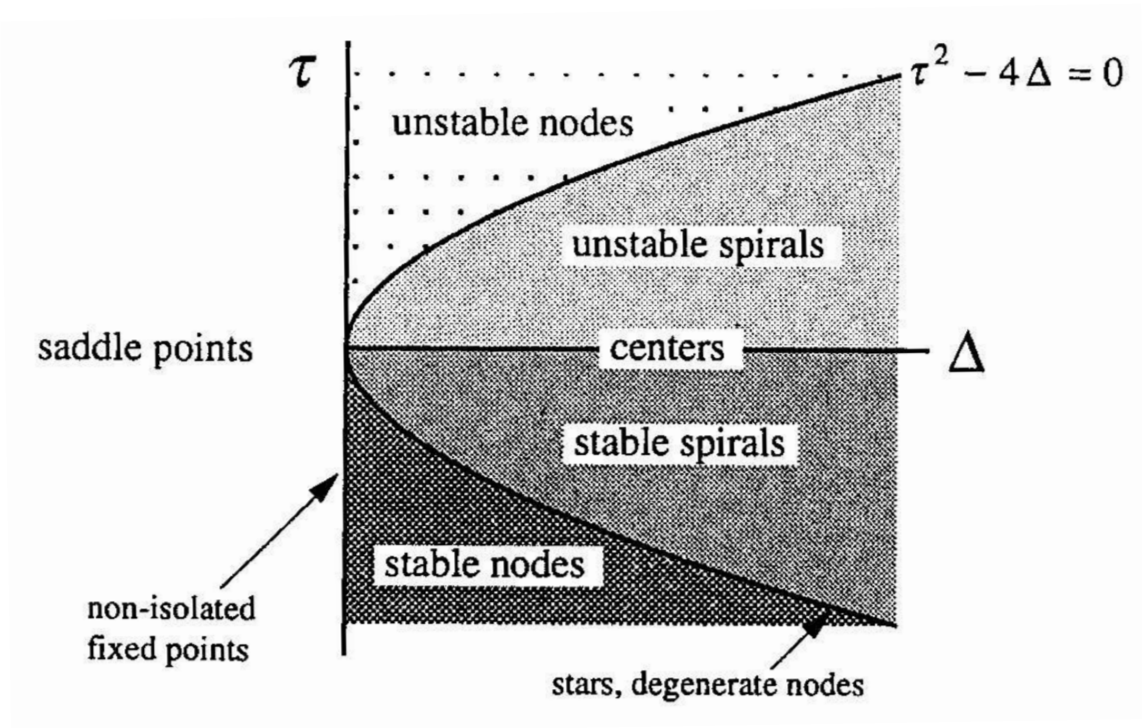
Proper and **improper nodes** arise when there is a repeated, nonzero eigenvalue, and this occurs precisely when

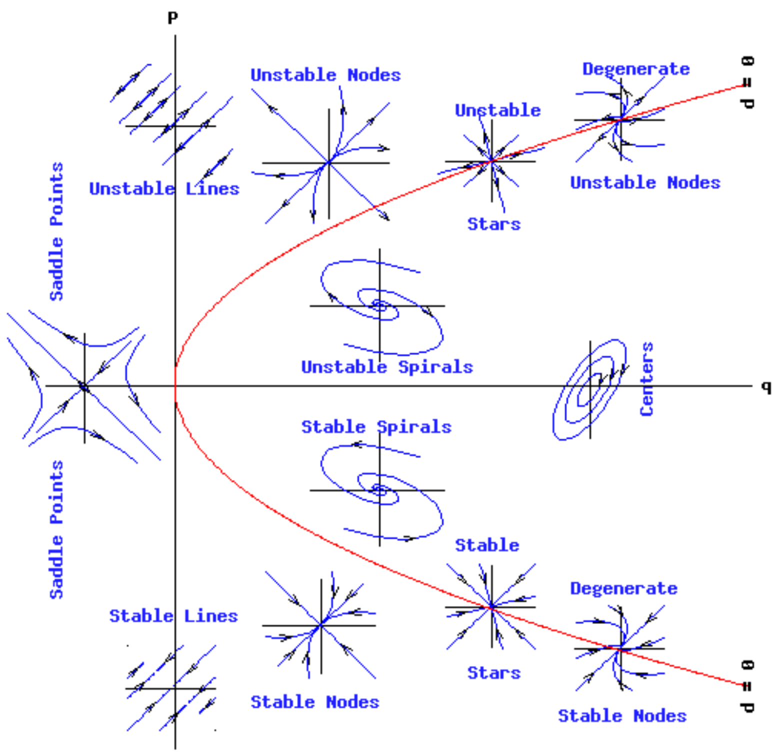
$$\tau \neq 0 \text{ and } \tau^2 - 4\Delta = 0 \qquad \Leftrightarrow \qquad 0 < \Delta = \frac{1}{4}\tau^2.$$

Spiral points arise when eigenvalues are complex $\alpha + i\beta$ with neither α nor β equal to 0; this occurs precisely when

$$\tau \neq 0 \text{ and } \tau^2 - 4\Delta < 0 \qquad \Leftrightarrow \qquad 0 < \frac{1}{4}\tau^2 < \Delta.$$

We gather all this information into the **stability diagram** below. Note that we are observing the $\Delta\tau$-plane here. An alternate version, one I found on the internet, which draws little characterization-diagrams for the various names, appears further down. It uses q and p for Δ and τ, respectively.


τ
$\tau^2 - 4\Delta = 0$
unstable nodes
unstable spirals
saddle points
centers
Δ
stable spirals
stable nodes
non-isolated
fixed points
stars, degenerate nodes


p
q
Unstable Nodes
Degenerate
Unstable
Unstable Nodes
Stars
Unstable Lines
Saddle Points
Unstable Spirals
Centers
Stable Spirals
Saddle Points
Stable
Degenerate
Stable Lines
Stable Nodes
Stars
Stable Nodes
d = 0
d = 0