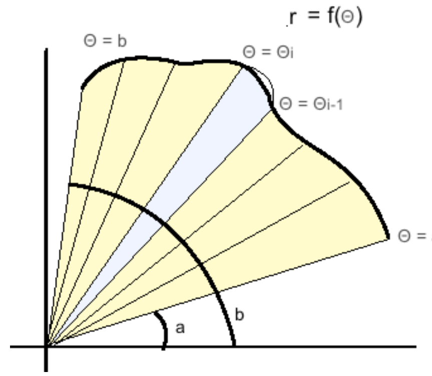
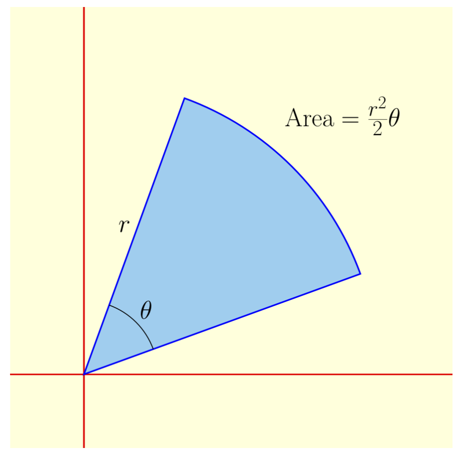
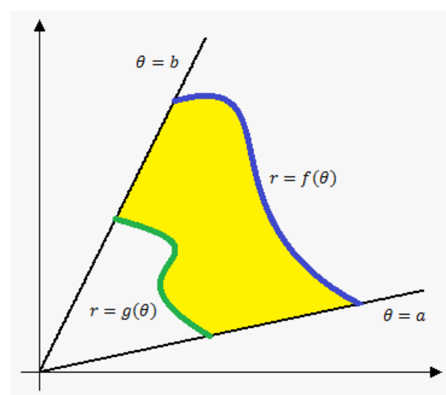
Polar area

The typical polar function is one which specifies $r = f(\theta)$ (θ as independent, r as dependent). The sorts of regions whose areas we might naturally compute this way are ones like those depicted in the top picture at right. As $r > 0$ reflects a distance from a point back to the origin (not the x-axis), slices look like wedges out of a near-circular region. We need to know how to compute areas of true wedges (taken from true circles), such as those depicted in the second picture at right. Its area satisfies a proportion:







$$\frac{\text{Area(wedge)}}{\text{Area(full circle)}} = \frac{\text{measure(central angle)}}{\text{measure(angle for one rotation)}},$$

or

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \qquad \Rightarrow \qquad A = \frac{1}{2} r^2 \theta.$$

Using this, a typical nearly-wedge-shaped slice in the top figure would have area approximately equal to

$$\frac{1}{2}[f(\theta_i)]^2(\theta_i - \theta_{i-1}) \;=\; \frac{1}{2}[f(\theta)]^2 \Delta\theta,$$

and an approximation to the full area could be obtained via the sum

$$\sum_{i=1}^{n} \frac{1}{2}\,[f(\theta_i)]^2 \Delta\theta.$$

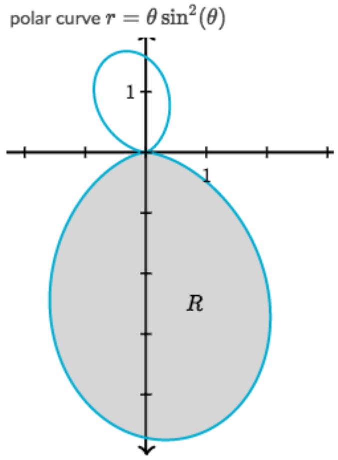
The approximation improves as $\Delta\theta \to 0$, giving the actual area as

$$\int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta.$$

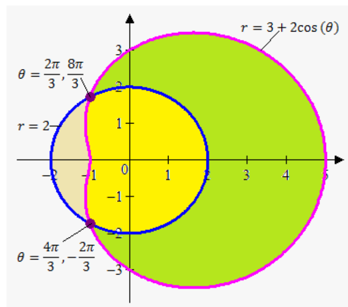
Adapting this to the computation of area for a region between two polar curves (see the bottom figure), we have

$$\text{Area of shaded region} \;=\; \int_a^b \frac{1}{2}\left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta.$$

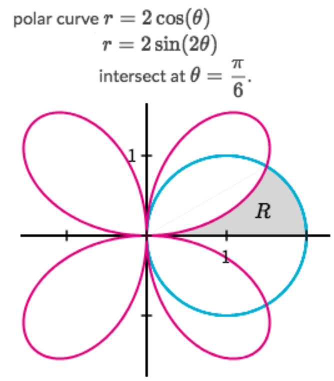
Examples of areas of polar regions



1.



2.



3.

Slopes and lengths along polar arcs

Key idea: Combine polar-to-rectangular conversion with polar functions to get a parametrization. That is, insert $r = f(\theta)$ into $x = r\cos\theta$, $y = r\sin\theta$.

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta \end{aligned}\right.$$

- **Slope**. As with other parametrized curves, slope is found by taking the ratio of the change in y (derivative of y) to the change in x (derivative of x). Thus,

$$\text{slope at } \theta \;=\; \frac{dy/d\theta}{dx/d\theta} \;=\; \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

- **Arc length**. From Section 11.2 we have the arc length of for a parametric curve coming from the formula $L = \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta$. Inserting these derivatives and simplifying yields

$$\begin{aligned} L &= \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{[f'(\theta)\sin\theta + f(\theta)\cos\theta]^2 + [f'(\theta)\cos\theta - f(\theta)\sin\theta]^2}\, d\theta \;=\; \cdots \\ &= \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta. \end{aligned}$$