For r.v. X, $M_X(t) = E(e^{tX})$

■

Theorem 1: For X a discrete or continuous r.v., a, b constants,

(i) $M_{aX}(t) = M_X(at)$.

(ii) $M_{X+b}(t) = e^{bt} M_X(t)$.

(iii) $M_{aX+b}(t) = e^{bt} M_X(at)$.

$$M_{aX}(t) = E\left(e^{t(aX)}\right) = E\left(e^{(at)X}\right) = M_X(at)$$

$$M_{X+b}(t) = E\left(e^{t(X+b)}\right) = E\left(\underbrace{e^{bt}}_{\text{const.}} \cdot e^{tX}\right) = e^{bt} \cdot E(e^{tX}) = e^{bt} M_X(t)$$

$$\mu_2' = \mu_2 - \mu_1^2 = E(X^2) - [E(X)]^2$$

Two applications of mgfs:

- Since $M_x(t)$ is the exponential generating function of the sequence (μ_k) of k^{th} moments, we have, in general, $\mu_k = M_X^{(k)}(0)$.
- mgfs are unique, providing a signature for recognizing a random variable, as specified in this theorem:

Theorem 2: Let X, Y be r.v.s with mgfs M_X, M_Y. Then X, Y are **identically distributed** (i.e., they have the same cdfs) if and only if $M_X(t) = M_Y(t)$ for all t in some nontrivial interval containing 0.

Another Useful Continuous Distribution Family

- must be nonnegative
- must have finite val. $\int_{-\infty}^{\infty}$

Example 6:

Show that the function $g(x) = e^{-x^2/2}$ is a **kernel function**, and determine a scalar such that $ke^{-x^2/2}$ is a density function.

$$T^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}\cdot e^{-y^2/2}\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dy\,dx \qquad r^2 = x^2+y^2$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\cdot dr\,d\theta = \cdots = 2\pi$$

So $T = \sqrt{2\pi}$.

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

We define $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, which is now shown to be a pdf.
Note: Though $P(X \leq x) = \int_{-\infty}^{x} \phi(x)\,dx$, in practice one cannot apply techniques such as those above to evaluate this integral.

$$\Phi(x) = P(X \leq x) = \int_{-\infty}^{x} \phi(u)\,du$$

■

Definition 3: Let ϕ (resp. Φ) be the pdf (resp. cdf) of a continuous random variable Z. We denote this by $Z \sim \mathsf{Norm}(0,1)$. Say that Z has a **standard normal distribution**, writing $Z \sim \mathsf{Norm}(0,1)$, if its pdf is $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$. Write

$$\Phi(z) := \int_{-\infty}^{z} \phi(u)\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}\,du$$

for the cdf of X.

Example 7:

What is the mgf for $Z \sim \mathsf{Norm}(0,1)$?

$$-\frac{1}{2}(z^2-2tz)$$

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz}\cdot\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\left(-\frac{z^2}{2}+tz\right)}\,dz$$

■

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t^2}\cdot e^{-t^2}e^{-\frac{1}{2}(z^2-2tz)}\,dz = \frac{e^{t^2}}{\sqrt{2\pi}}\cdot\underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2}\,dz}_{=\sqrt{2\pi}} = e^{t^2}$$

$Z \sim \text{Norm}(0, 1)$

X has mean $E(X)$ and var. $Var(X)$

Set $Y = b + aX$

$\rightarrow E(Y) = b + aE(X)$

$Var(Y) = a^2 E(X)$

> **Theorem 3:** For $Z \sim \mathsf{Norm}(0,1)$,
>
> (i) $\mathrm{E}(Z) = 0$, and
>
> (ii) $\mathrm{Var}(Z) = 1$ (so the standard deviation of Z is 1).

Proof: Left as an exercise (Exercise 3.21). □

> **Definition 4:** A continuous random variable X has a **normal distribution** with parameters μ, σ if $-\infty < \mu < \infty$, $\sigma > 0$, and $X = \mu + \sigma Z$, where $Z \sim \mathsf{Norm}(0,1)$. In this case, we write $X \sim \mathsf{Norm}(\mu,\sigma)$.

Use

- Lemma 3.2.6 to find the mean, variance for $X \sim \mathsf{Norm}(\mu,\sigma)$.
- The cdf method to find the pdf for $X \sim \mathsf{Norm}(\mu,\sigma)$.

  Here's a start:

$$F(x) \;=\; P(X \le x) \;=\; P\left(\frac{X-\mu}{\sigma} \;<\; \frac{x-\mu}{\sigma}\right) \;=\; \Phi\left(\frac{x-\mu}{\sigma}\right).$$

  Now obtain $f_X(x)$ by differentiating.

- Theorem 3.3.6 (p. 133) to find the mgf for $X \sim \mathsf{Norm}(\mu,\sigma)$.

Use CDF method w/ $X = \mu + \sigma Z$ w/ $Z \sim \text{Norm}(0,1)$.

CDF of X $\rightarrow P(X \le x) = P(\mu + \sigma Z \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$

$$= \Phi_Z\left(\frac{x-\mu}{\sigma}\right) \leftarrow \text{CDF for } Z$$

pdf of X: $\frac{d}{dx} \Phi_Z\left(\frac{x-\mu}{\sigma}\right) = \cdots = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$

Say $X \sim \text{Norm}(\mu, \sigma)$.

Ex.) Pick random person. $IQ \sim \text{Norm}(\underset{\mu}{100}, \underset{\sigma}{15})$ Let X = person's IQ.

$$P(X > 115) = 1 - P(X \le 115)$$

$$= 1 - P\left(Z \le \frac{115-100}{15}\right) = 1 - P(Z \le 1)$$

$$= 1 - \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$