1. By exchanging rows 1 and 3, $\mathbf{H}$ gets to RREF:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Writing an element of null $(\mathbf{H})$ as $\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \rangle$, we see that we can take x_3, x_5, x_6 and x_7 as "free" variables (free to take on either of the values 0 or 1), while

$$\begin{aligned} x_1 &= x_3 + x_5 + x_7 \\ x_2 &= x_3 + x_6 + x_7 \ , \\ x_4 &= x_5 + x_6 + x_7 \end{aligned}$$

so vectors in the null space take the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, one possible basis is

$$\{\langle 1,1,1,0,0,0,0 \rangle, \langle 1,0,0,1,1,0,0 \rangle, \langle 0,1,0,1,0,1,0 \rangle, \langle 1,1,0,1,0,0,1 \rangle\}.$$

Some notes:

- That this collection spans null $(\mathbf{H})$ is clear from the solution process. That it is linearly independent perhaps calls for forming a 7-by-4 matrix with these as the columns, reducing to echelon form and seeing that that echelon form has no free columns. I don't necessarily expect students will do this.
- As is pretty much always the case, there are other bases for the same subspace—we use a different basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ for null $(\mathbf{H})$ in a follow-up problem. While the one I've given above is the most likely basis for students to find, there are yet others. The easiest way to check a strange-looking answer is to make sure the proposed collection contains 4 vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ all from $\mathbb{Z}_2^7$, check that each one is in null $(\mathbf{H})$ (i.e., that $\mathbf{H}\mathbf{v}_j = \mathbf{0}$ for $j = 1, 2, 3, 4$), and that they are linearly independent.

2. Perhaps the easiest way to do this is to note that, if $y(x) \to 2/3$ as $x \to \infty$, then $y'(x)$ is simultaneously going to zero. If we set y' equal to zero we get the equation $ay + b = 0$,

which implies that, should y' ever reach zero, the solution reaches $y = -b/a$. So, we should choose a, b so that the ratio $-b/a$ is 2/3; $a = -3$, $b = 2$ works, corresponding to the DE $y' = 3y - 2$, but there are other choices. Unfortunately, $a = 3$, $b = -2$ *does not work*, as its solution goes to 2/3 as $x \to -\infty$.

A hammer-it-out approach would involve solving the DE outright. It is both *linear* and *separable*. Capitalizing on the latter, we solve:

$$\begin{aligned} \frac{1}{ay+b}\frac{dy}{dx} \;=\; 1 \qquad &\Rightarrow \qquad \int \frac{1}{ay+b}\,dy \;=\; \int dx \\ &\Rightarrow \qquad \frac{1}{a}\ln|ay+b| \;=\; t + C \\ &\Rightarrow \qquad \ln|ay+b| \;=\; at + \tilde{C} \\ &\Rightarrow \qquad |ay+b| \;=\; e^{C}\cdot e^{at} \\ &\Rightarrow \qquad ay + b \;=\; \tilde{C}e^{at} \\ &\Rightarrow \qquad y(t) \;=\; Ce^{at} - \frac{b}{a}. \end{aligned}$$

The Ce^{at} part in the solution will decay exponentially to 0 if and only if $a < 0$. Given this, the overall solution will go to $(-b/a)$, which we want to be 2/3. So, as long as a, b are chosen so that

- this ratio is 2/3, and
- $a < 0$

then the resulting DE $y' = ay + b$ will have solutions that behave as requested.

3. (a) A model fitting the description is $mv' = mg - \gamma v^2$, where γ is a constant of proportionality.

   (b) If we tend to a *limiting velocity*, then $v' \approx 0$. Setting $v' = 0$ in our DE model, we get

$$mg - \gamma v^2 \;=\; 0 \qquad \Rightarrow \qquad v \;=\; \sqrt{\frac{mg}{\gamma}}.$$

   (c) We solve $49 = \sqrt{\dfrac{98}{\gamma}}$, obtaining $\gamma = \dfrac{2}{49}$ kg/m.

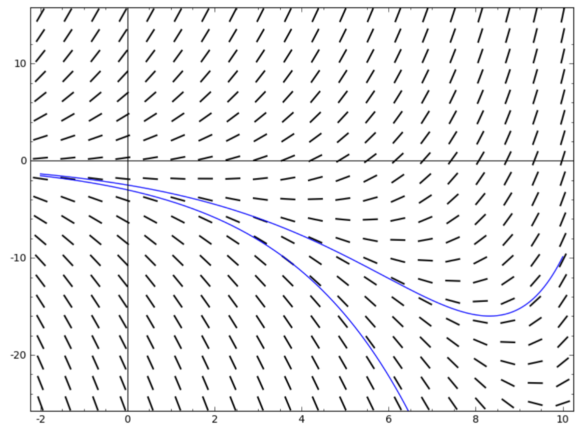
   (d) This is a first order nonlinear, separable DE. It can be solved (though students have not been asked to do so) to get an implicit solution

$$\frac{5}{2}\ln|v + 49| - \frac{5}{2}\ln|v - 49| = t + C,$$

which can be further manipulated to obtain

$$\begin{aligned}
\ln\left(\left|\frac{v+49}{v-49}\right|^{5/2}\right) = t + C \qquad &\Rightarrow \qquad \left|\frac{v+49}{v-49}\right|^{5/2} = e^C e^t \\
&\Rightarrow \qquad \frac{v+49}{v-49} = Ce^{2t/5} \\
&\Rightarrow \qquad v(t) = 49 \cdot \frac{Ce^{2t/5}+1}{Ce^{2t/5}-1}.
\end{aligned}$$

4. (a) A direction field, along with a couple solutions (not required), appears at right. The appearance is that solutions grow without bound as $t \to \infty$, though some appear to go to $+\infty$ and others to $-\infty$, depending on the choice of a.



(b) While some estimates may be more precise than others, reasonable values should be between (-2) and (-4).

(c) To solve the (linear) DE $y' - \frac{1}{2}y = \frac{1}{2}\exp(t/3)$, we first solve $y' = \frac{1}{2}y$ to get $\Phi(t) = \exp(t/2)$, a basis for all solutions to the homogeneous problem. Then, using the variation of parameters formula,

$$y_p(t) = e^{t/2}\int \frac{e^{t/3}}{2e^{t/2}}\,dt = \frac{1}{2}e^{t/2}\int e^{-t/6}\,dt = (-6)\frac{1}{2}e^{t/2}\cdot e^{-t/6} = -3e^{t/3}.$$

Thus, the general solution is $y(t) = y_h(t) + y_p(t) = Ce^{t/2} - 3e^{t/3}$. Applying the IC $y(0) = a$, we have

$$a = C - 3 \qquad \Rightarrow \qquad C = a + 3.$$

Thus,

$$y(t) = (a+3)e^{t/2} - 3e^{t/3}.$$

We note that $e^{t/2}$ is a faster-growing exponential than $e^{t/3}$, and when its coefficient $(a+3)$ is opposite in sign to that (-3) of $e^{t/3}$, solutions will go to $+\infty$ as $t \to \infty$; otherwise, they will go to $-\infty$. So, $a_0 = -3$.

(d) When $a = -3$, the solution is $y(t) = -3e^{t/3}$, which goes to $-\infty$ as $t \to \infty$.

5.

2.4.67 $y' + 2y = xe^{-2x} + x^3$

With integrating factor $e^{\int 2\,dx} = e^{2x}$, we obtain homogeneous Soln

$$y_h(x) = C/e^{2x} = Ce^{-2x}.$$

The nonhomogeneous term in the DE is a sum consisting of

x^3, a 3rd degree polynomial $\longrightarrow$ y_p should contain $Ax^3 + Bx^2 + Cx + D$.

xe^{-2x}, a product of an exponential and a 1st-degree polynomial

$\longrightarrow$ propose y_p contain $(Ex + F)e^{-2x} = Exe^{-2x} + Fe^{-2x}$.

But, glancing at y_h, we see that Fe^{-2x} repeats (a term in) y_h. So, we throw in an extra x: $(Ex + F)xe^{-2x} = Ex^2e^{-2x} + Fxe^{-2x}$.

So, proposing

$$y_p(x) = Ax^3 + Bx^2 + Cx + D + Ex^2e^{-2x} + Fxe^{-2x},$$

we have left-hand side

$$\begin{aligned} y_p' + 2y_p &= 3Ax^2 + 2Bx + C + 2Exe^{-2x} - 2Ex^2e^{-2x} + Fe^{-2x} - 2Fxe^{-2x} \\ &\quad + 2Ax^3 + 2Bx^2 + 2Cx + 2D + 2Ex^2e^{-2x} + 2Fxe^{-2x} \\ &= 2Ax^3 + (3A + 2B)x^2 + (2B + 2C)x + (C + 2D) + 2Exe^{-2x} + Fe^{-2x} \end{aligned}$$

Equating coefficients of various terms with those of the target (RHS) function

term	LHS	RHS
x^3	$2A$	1
x^2	$3A + 2B$	0
x^1	$2B + 2C$	0
x^0	$C + 2D$	0
xe^{-2x}	$2E$	1
e^{-2x}	F	0

leads to the matrix problem

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

which, when solved using Gaussian Elimination, yields

$A = 1/2$
$B = -3/4$
$C = 3/4$
$D = -3/8$
$E = 1/2$
$F = 0$

Thus, our general solution is $y_h + y_p$, or

$$Ce^{-2x} + \frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8} + \frac{1}{2}x^2e^{-2x}$$