Consider the 2nd order linear nonhomogeneous DE

$$y'' + 2by' + \omega_0^2 y = f_0 \cos(\omega t), \tag{1}$$

where b, ω_0, and f_0 are all positive constants. We have seen that the DE (1) can be a model for a damped mass-spring system or an RLC circuit. In this problem, we investigate the influence of the forcing frequency on amplitude and phase shift.

1. Guessing (i.e., using the Method of Undetermined Coefficients) the form $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$ for the steady-state solution, show that

$$A = \frac{(\omega_0^2 - \omega^2)f_0}{\Delta} \qquad \text{and} \qquad B = \frac{2b\omega f_0}{\Delta},$$

where $\Delta = (\omega_0^2 - \omega^2)^2 + 4b^2\omega^2$.

2. Determine formulas for R and ϕ so that $y_p(t) = R\cos(\omega t - \phi)$.

3. Take $f_0 = 1$, $b = 1$, and $\omega_0 = 15$. Plot R and ϕ as functions of the forcing frequency ω.

4. Repeat the previous part, but now with $f_0 = 1$, $b = 8$, and $\omega_0 = 20$.

5. Using the formula for R you found in part (b) (no specific falues for f_0, b nor ω_0) and calculus, find the value of ω which maximizes amplitude R.

6. What information do the phase functions $\phi(\omega)$ plotted in parts (c) and (d) tell? Give a physical interpretation that presumes the transient part of the solution has died out.

Answer:

1. Using the method of undetermined coefficients, we propose the form of $y_p(t)$ to be

$$
\begin{aligned}
y_p &= A\cos(\omega t) + B\sin(\omega t), &\text{so that} \\
y_p' &= -A\omega\sin(\omega t) + B\omega\cos(\omega t), &\text{and} \\
y_p'' &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t).
\end{aligned}
$$

We require y_p satisfy the nonhomogeneous DE, so

$$
\begin{aligned}
f_0\cos(\omega t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) + 2b[-A\omega\sin(\omega t) + B\omega\cos(\omega t)] \\
&\quad + \omega_0^2[A\cos(\omega t) + B\sin(\omega t)] \\
&= (A\omega_0^2 + 2bB\omega - A\omega^2)\cos(\omega t) + (B\omega_0^2 - 2bA\omega - B\omega^2)\sin(\omega t).
\end{aligned}
$$

Equating coefficients yields

$$
\left.
\begin{aligned}
(\omega_0^2 - \omega^2)A + 2b\omega B &= f_0 \\
-2b\omega A + (\omega_0^2 - \omega^2)B &= 0
\end{aligned}
\right\}
\quad \Rightarrow \quad
\begin{bmatrix} \omega_0^2 - \omega^2 & 2b\omega \\ -2b\omega & \omega_0^2 - \omega^2 \end{bmatrix}
\begin{bmatrix} A \\ B \end{bmatrix}
=
\begin{bmatrix} f_0 \\ 0 \end{bmatrix}.
$$

Thus, by Cramer's rule,

$$A \;=\; \frac{\begin{vmatrix} f_0 & 2b\omega \\ 0 & \omega_0^2 - \omega^2 \end{vmatrix}}{\Delta} \;=\; \frac{(\omega_0^2 - \omega^2)f_0}{\Delta}, \qquad \text{and}$$

$$B \;=\; \frac{\begin{vmatrix} \omega_0^2 - \omega^2 & f_0 \\ -2b\omega & 0 \end{vmatrix}}{\Delta} \;=\; \frac{2b\omega f_0}{\Delta},$$

with $\Delta = (\omega_0^2 - \omega^2)^2 + 4b^2\omega^2$. Thus,

$$y_p(t) \;=\; \frac{(\omega_0^2 - \omega^2)f_0}{\Delta}\cos(\omega t) + \frac{2b\omega f_0}{\Delta}\sin(\omega t).$$
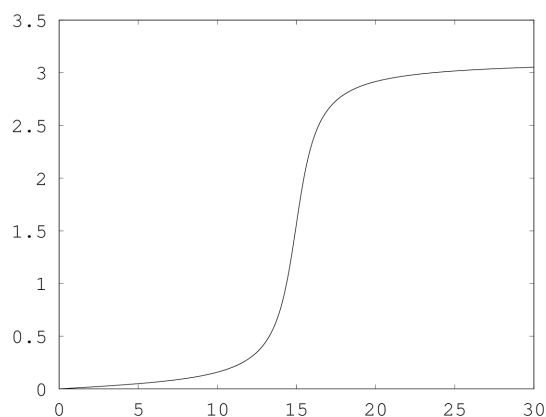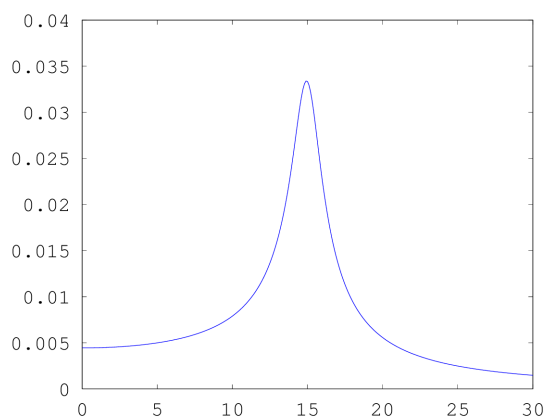
2. We have $y_p(t) = R\cos(\omega t - \phi)$, with

$$R \;=\; \sqrt{\left(\frac{(\omega_0^2 - \omega^2)f_0}{\Delta}\right)^2 + \left(\frac{2b\omega f_0}{\Delta}\right)^2} \;=\; \frac{f_0}{\Delta}\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} \;=\; \frac{f_0}{\sqrt{\Delta}},$$

$$\cos\phi \;=\; \frac{(\omega_0^2 - \omega^2)f_0/\Delta}{R} \;=\; \frac{\omega_0^2 - \omega^2}{\sqrt{\Delta}},$$

$$\sin\phi \;=\; \frac{2b\omega f_0/\Delta}{R} \;=\; \frac{2b\omega}{\sqrt{\Delta}}.$$

3. The amplitude curve $R = R(\omega)$ is on the left, while the phase curve $\phi = \phi(\omega)$ is on the right.

The phase curve pictured is one I obtained using the arccosine function, specifically

$$\phi(\omega) \;=\; \arccos\left((15^2 - \omega^2)/\sqrt{((15^2 - \omega^2)^2 + 4\omega^2)}\right).$$

I think it presents the truest picture, but students have not been told (by me) that this expression is to be preferred over the use of arcsine or arctangent

$$\phi(\omega) \;=\; \arcsin\left(2\omega/\sqrt{((15^2 - \omega^2)^2 + 4\omega^2)}\right) \qquad \text{or} \qquad \phi(\omega) \;=\; \arctan\left(2\omega/(15^2 - \omega^2)\right).$$

Note to Grader: You should plot both of these to see just how different their pictures are. I'll not mention this again, but it will affect answers in part (d), and perhaps in (f).

4. The amplitude curve $R = R(\omega)$ is on the left, while the phase curve $\phi = \phi(\omega)$ is on the right.
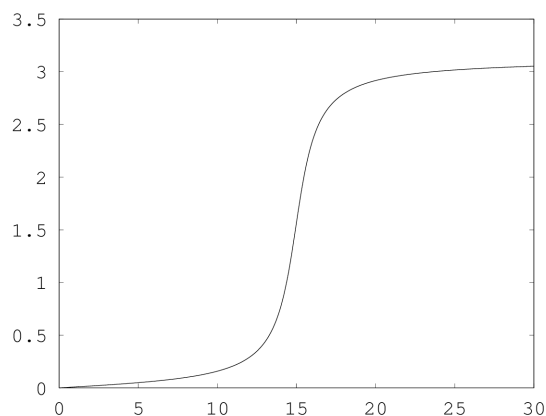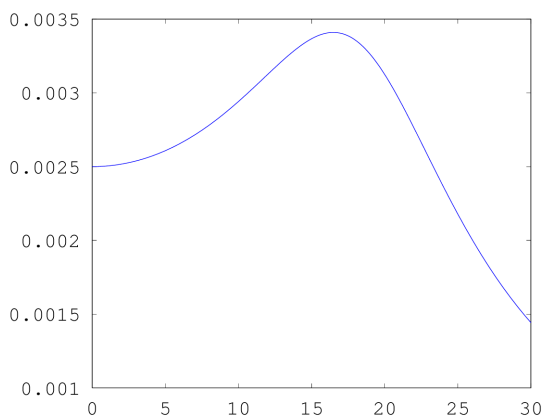
5. The derivative,

$$\frac{dR}{d\omega} = \frac{d}{d\omega} \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}} = \frac{[2\omega(\omega_0^2 - \omega^2) - 4b^2\omega]f_0}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{3/2}},$$

has a denominator which cannot be zero. Setting the numerator equal to zero, we get

$$\omega\left[\omega^2 + 2b^2 - \omega_0^2\right] = 0 \qquad \Rightarrow \qquad \omega = \pm\sqrt{\omega_0^2 - 2b^2},$$

where we have tossed out the possibility that $\omega = 0$. We may also toss out the negative ω value, leaving only

$$\omega = \sqrt{\omega_0^2 - 2b^2}.$$

When this formula is applied to the amplitude curves in parts (c) and (d) above, we get that the peak amplitudes occur at $\omega \doteq 14.933$ (part (c), when $\omega_0 = 15$ and $b = 1$) and $\omega \doteq 16.492$ (part (d), when $\omega_0 = 20$ and $b = 8$).

6. For fixed values of the constants (including ω), one can plot the graph of the forcing function $f_0 \cos(\omega t)$ and superimpose a plot of the solution $y(t) = R\cos(\omega t - \phi)$. The two will be periodic curves having the same period. They will likely have different amplitudes, and reach their peaks at different times. The lag between peaks is ϕ. The plots of $\phi(\omega)$ indicate how this lag varies as one tweaks the frequency ω of the forcing function.