Work independently or in groups to answer these questions:

1. Write down the Maclaurin series for

$$\sin x, \qquad \cos x, \qquad e^x, \qquad \arctan x, \qquad \ln(1+x).$$

Answer:
These appear (along with some others) on p. 571 of the textbook.

$$\begin{aligned}
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}, \qquad -\infty < x < \infty \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}, \qquad -\infty < x < \infty \\
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad -\infty < x < \infty \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}, \qquad -1 \leqslant x \leqslant 1 \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}x^n, \qquad -1 < x \leqslant 1
\end{aligned}$$

2. What are the radii and intervals of convergence for the series you listed in the previous problem? What tool(s) do you use to determine these if you forget them?

Answer:
The intervals of convergence were given above, following each series. The radii of convergence are $R = +\infty$ for $\sin x$, $\cos x$ and e^x, and $R = 1$ for $\arctan x$ and $\ln(1+x)$. One can determine intervals of convergence (and from those, center and radius) by use of the Ratio Test (though you learn the fate of the series at endpoints—and, correspondingly, whether endpoints are included in the interval—using one of the tests learned in Sections 10.1-10.4).

3. Consider the (somewhat-unspecified) power series $\displaystyle\sum_{n=0}^{\infty} a_n(3x+4)^n$.

   (a) What is the center of this series?

   Answer:
   One way to determine the center is to write the series in canonical form

$\sum_{n=0}^{\infty}(\text{coefficient})(x-c)^n$. Doing so here, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(3x+4)^n &= a_0 + a_1(3x+4) + a_2(3x+4)^2 + a_3(3x+4)^3 + \cdots \\ &= a_0 + a_1[3(x+4/3)] + a_2[3(x+4/3)]^2 + a_3[3(x+4/3)]^3 + \cdots \\ &= a_0 + a_1[3(x-(-4/3))] + a_2[3(x-(-4/3))]^2 + a_3[3(x-(-4/3))]^3 + \cdots \\ &= a_0 + \underbrace{3a_1}_{\text{coeff.}}\underbrace{(x-(-4/3))}_{(x-c)} + \underbrace{9a_2}_{\text{coeff.}}\underbrace{(x-(-4/3))^2}_{(x-c)^2} + \underbrace{27a_3}_{\text{coeff.}}\underbrace{(x-(-4/3))^3}_{(x-c)^3} + \cdots\end{aligned}$$

This reveals $(-4/3)$ as the center. Another (simpler?) way is to find which x value reduces the series to the sum

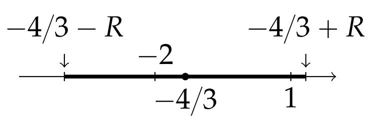
$$a_0 + a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + \cdots .$$

That is, the center is the x-value which makes all the terms with $(3x+4)^n$, $n > 1$ vanish.

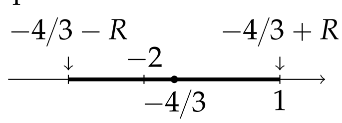
(b) Suppose the series converges when $x = 1$. What can be said about the fate of the series when $x = -2$? Why?

Answer:

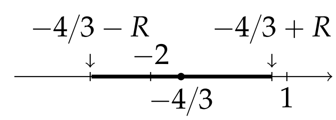
We have already found the series center is at $(-4/3)$. And, despite the fact we do not know the actual radius of convergence, we know it extends out from this center far enough to reach $x = 1$. So, of the three possibilities depicted below, only the left-hand and middle ones remain possible when we know the series converges at $x = 1$.



$x = 1$ in the interior



$x = 1$ on the boundary



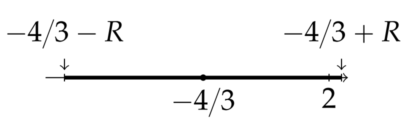
$x = 1$ on the exterior

In both those cases, $x = -2$ is inside the interval of convergence, which means the series converges at $x = -2$.

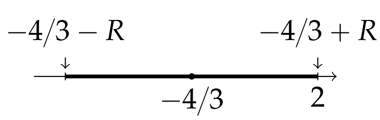
(c) Suppose the series diverges when $x = 2$. What can be said about the fate of the series when $x = 3$? Why?

Answer:

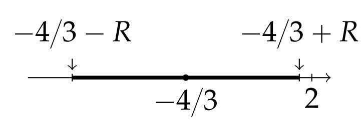
Of the three possibilities—$x = 2$ is inside, on the boundary of, or outside the interval of convergence (all depicted below)—the fact that the series diverges at $x = 2$ means that only the two on the right are possible.



$x = 2$ in the interior



$x = 2$ on the boundary



$x = 2$ on the exterior

In both those cases, $x = 3$ is outside the interval of convergence, which means the series diverges at $x = 3$.

(d) Suppose the series converges when $x = 1$ *and* diverges when $x = 2$. What can be said about the radius of convergence?

Answer:

Combining the reasoning of parts (b) and (c), we have that

R is large enough to reach from $x = -4/3$ to $x = 1$, so $R \geqslant 1 - (-4/3) = 7/3$, and

R is, at most, big enough to reach from $x = -4/3$ to $x = 2$, so $R \leqslant 2 - (-4/3) = 10/3$.

Thus, $7/3 \leqslant R \leqslant 10/3$.

4. In the last week we used the equation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1,$$

as a prototype for writing power series for other similar-looking functions (i.e., functions expressed as $(1 - \text{something})^{-1}$). Taking your series expressions in Problem 1 as additions to the list of prototypes, write series expansions for the specified function, centered at the c

(a) $\sin(3x),\ \ c = 0$ (b) $\ln(2+3x),\ \ c = -1/3$ (c) $\dfrac{(x-2)^2}{1+3x},\ \ c = 2$

(d) $e^{x^2},\ \ c = 0$ (e) $\displaystyle\int_0^x e^{-t^2}\,dt,\ \ c = 0$ (f) $e^x,\ \ c = 5$

Answer:

(a) $$\sin(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(3x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+1} = 3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \cdots$$

(b) $$\ln(2+3x) = \ln(1+(1+3x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(1+3x)^n = (1+3x) - \frac{1}{2}(1+3x)^2 + \frac{1}{3}(1+3x)^3 - \cdots$$

(c) $$\frac{(x-2)^2}{1+3x} = \frac{(x-2)^2}{7} \cdot \frac{1}{1-[-3/7(x-2)]} = \frac{(x-2)^2}{7}\sum_{n=0}^{\infty}[-\frac{3}{7}(x-2)]^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{7^{n+1}}(x-2)^{n+2}$$

(d) $$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

(e) Arguing as above, we can first write

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} = 1 - t^2 + \frac{t^4}{2!} - \frac{x^6}{3!} + \cdots$$

Thus,

$$\int_0^x e^{-t^2}\,dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}\right) dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\int_0^x t^{2n}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} t^{2n+1}\Big|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$$

(f) $$e^x = e^5 \cdot e^{x-5} = e^5 \sum_{n=0}^{\infty} \frac{(x-5)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^5}{n!}(x-5)^n$$

5. Determine radii of convergence for the series in the previous problem.

   Answer:

   (a) In the Maclaurin series for $\sin x$, convergence occurs for $-\infty < x < \infty$. Replacing x with $3x$ now means we require $-\infty < 3x < \infty$, and since dividing ∞ by 3 could only yield the result ∞, this means our interval of convergence is $-\infty < x < \infty$. Thus $R = +\infty$

   (b) In the Maclaurin series for $\ln(1+x)$, convergence requires $-1 < x \leqslant 1$. Replacing x with $(1+3x)$ now means we require $-1 < 1+3x \leqslant 1$. After algebraic manipulations (with the goal that the middle of our inequality be simply x), we obtain interval of convergence $-2/3 < x \leqslant 0$. Thus $2R = 0 - (-2/3)$, making $R = 1/3$.

   (c) In the Maclaurin series for $(1-x)^{-1}$, convergence requires $-1 < x < 1$. Replacing x with $(-3/7)(x-2)$ now means we require $-1 < (-3/7)(x-2) < 1$. When solved, the equivalent inequality is $-1/3 < x < 13/3$. Thus $2R = (13/3) - (-1/3)$, making $R = 7/3$.

   (d) In the Maclaurin series for e^x, convergence happens in the interval $-\infty < x < \infty$. Replacing x with x^2 now means we require $-\infty < x^2 < \infty$, and since $\sqrt{\infty}$ must be ∞, this means our interval of convergence is $-\infty < x < \infty$. Thus $R = +\infty$

   (e) Arguing as above, the interval of convergence for the series giving e^{-t^2} must be $(-\infty, \infty)$, which has radius of convergence $R = +\infty$. Since integrating a series does not change its radius of convergence, $R = +\infty$ applies to our result, too.

   (f) In the Maclaurin series for e^x, convergence happens in the interval $-\infty < x < \infty$. Replacing x with $x-5$ now means we require $-\infty < x-5 < \infty$, or $-\infty < x < \infty$. Thus $R = +\infty$

6. Why is the approximation $\sin x \approx x$ sometimes used if $|x|$ is small?

   Answer:
   This is because $\sin x \;=\; x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + \cdots,$

   and when $|x|$ is small, all the terms involving x^3 (or any higher power) are *much* smaller than the x^1-term, making them fairly insignificant.

7. Consider the series expression for $\ln(1+x)$ from the first problem.

   (a) Is the Maclaurin series for $\ln(1+x)$ of any use for finding $\ln(1/2)$? How about $\ln(3)$? Why or why not? **Answer**:
   $\ln(1/2) = \ln(1 + (-1/2))$, which means that, to use the Maclaurin series for $\ln(1+x)$ as a means of evaluation, we must plug in $x = 1/2$. In Problem 1 we see that $x = 1/2$ is in the interval of convergence, so this series is useful for calculating $\ln(1/2)$.

   On the other hand, $\ln(3) = \ln(1+2)$, which means that, to use the Maclaurin series for $\ln(1+x)$ as a means of evaluation, we would plug in $x = 2$, a number at which the series diverges. So this series is *not* useful for calculating $\ln(3)$.

(b) Recall that logarithms have the property $\log_b \dfrac{f(x)}{g(x)} = \log_b f(x) - \log_b g(x)$. Use series expressions for $\ln(1+x)$ and for $\ln(1-x)$ to write a (combined) Maclaurin series for $\ln\left(\dfrac{1+x}{1-x}\right)$.

Answer:

Subtracting series (both centered at 0)

$$\begin{aligned}\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\ \ln(1-x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots,\end{aligned}$$

we see that even powers of x cancel each other out, and obtain

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\right) = \sum_{n=0}^{\infty} \frac{2}{2n+1}x^{2n+1}.$$

i. Without concerning yourself with endpoints, what is the interval of convergence for this combined series?

Answer:

Since both the series for $\ln(1+x)$ and $\ln(1-x)$ were centered at 0 (there are Taylor series for these same functions centered at other $c \neq 0$, but we worked with Maclaurin series above), and since both had radius of convergence $R = 1$, these two facets are preserved for the resulting series obtained by subtracting the one from the other. That is, the series for $\ln((1+x)/(1-x))$ converges for all $|x| < 1$ and diverges for $|x| > 1$.

ii. How might it be used to determine $\ln(3)$?

Answer:

There is an x-value for which $(1+x)/(1-x) = 3$:

$$\frac{1+x}{1-x} = 3 \qquad \Rightarrow \qquad x = \frac{1}{2},$$

and this number is on the interior of the interval of convergence "found" in part i. Thus,

$$\ln(3) = \ln\left(\frac{1+(1/2)}{1-(1/2)}\right) = 2\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \frac{1}{7}\left(\frac{1}{2}\right)^7 + \cdots\right].$$