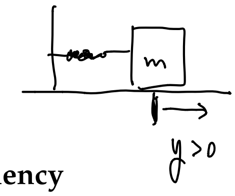
Local model for vibrations

Newton's 2nd Law $\quad m\frac{d^2y}{dt^2} = \sum \text{forces}$

$my'' = -ky$

Premise: To understand mechanisms of sound transmission, it is useful to understand springs



An unforced, undamped spring, Newton's 2nd law leads to

$$m\frac{d^2y}{dt^2} + ky = 0.$$

Solutions are linear combinations of sines, cosines at a particular **natural angular frequency** $\omega_0 = \sqrt{k/m}$:

$$y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

$\frac{1}{2L}\sqrt{T/\rho}$

Might provide an external force to excite the spring. In our model, let us assume it is $F_0\cos(\omega t)$ (periodic).

Aside: About angular frequencies, frequencies and period

Angular freq $\quad$ Freq. $\quad$ How many cycles per unit time $\quad$ (Hertz = cycles/sec.)

$\omega = 2\pi\nu$

$T = \frac{1}{\nu} = \frac{2\pi}{\omega}$

New differential equation:

$$m\frac{d^2y}{dt^2} + ky = F_0\cos(\omega t).$$

external force

- solvable (take MATH 231)
  see https://www.geogebra.org/m/QhmhszMs
- consequences of ω (excitation frequency)
  - being quite different from ω_0 (natural frequency)
  - approaching ω_0: **beats**
  - equalling ω_0: **resonance**

Cochlea

Imagine the consequence of having many "springs" in your ear, all with different natural frequencies, and a vibrating wave form exciting them.

Fourier Series

$l > 0$

Suppose $f(t)$ is a periodic function with period ℓ. For functions $f, g\colon [0, \ell] \to \mathbb{R}$, define an inner product $\langle f, g\rangle$ in this manner:

$$\langle f, g\rangle = \int_0^{\ell} f(x)g(x)\,dx. \tag{1}$$

If $=0$, say $f \perp g$

vector $\vec{f} = [f_1\ f_2\ \dots\ f_n]$

$\vec{g} = [g_1\ g_2\ \cdots\ g_n]$ $\quad \vec{f}\cdot\vec{g} = f_1g_1 + f_2g_2 + \cdots + f_ng_n = \sum f_j g_j$

Facts consistent with yesterday's homework: When m, n are integers, $\varphi_m\ \psi_n$

1. $\left\langle \underbrace{\cos\left(\frac{2\pi m\cdot}{\ell}\right)}_{\varphi_m}, \underbrace{\cos\left(\frac{2\pi n\cdot}{\ell}\right)}_{\varphi_n}\right\rangle = \int_0^\ell \cos\left(\frac{2\pi mx}{\ell}\right)\cos\left(\frac{2\pi nx}{\ell}\right) dx = \left\{\begin{array}{ll} 0, & \text{if } m \neq n \\ \ell, & \text{if } m = n = 0 \\ \ell/2, & \text{if } m = n \neq 0 \end{array}\right\}$, whenever $m, n \geq 0$.

   Take special note of the case $m = n = 0$. $\quad \langle 1, 1\rangle = \int_0^\ell 1\cdot 1\, dx = x\Big|_0^\ell = \ell$

2. $\left\langle \underbrace{\sin\left(\frac{2\pi m\cdot}{\ell}\right)}_{\psi_m}, \underbrace{\sin\left(\frac{2\pi n\cdot}{\ell}\right)}_{\psi_n}\right\rangle = \int_0^\ell \sin\left(\frac{2\pi mx}{\ell}\right)\sin\left(\frac{2\pi nx}{\ell}\right) dx = \left\{\begin{array}{ll} 0, & \text{if } m \neq n \\ \ell/2, & \text{if } m = n \end{array}\right\}$, whenever $m, n \geq 1$.

3. $\left\langle \underbrace{\cos\left(\frac{2\pi m\cdot}{\ell}\right)}_{\varphi_m}, \underbrace{\sin\left(\frac{2\pi n\cdot}{\ell}\right)}_{\psi_n}\right\rangle = \int_0^\ell \cos\left(\frac{2\pi mx}{\ell}\right)\sin\left(\frac{2\pi nx}{\ell}\right) dx = 0$, whenever $m \geq 0, n \geq 1$.

For any periodic function f with period ℓ, set

$$a_m = \frac{2}{\ell}\left\langle f, \cos\left(\frac{2m\pi\cdot}{\ell}\right)\right\rangle = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{2\pi mx}{\ell}\right) dx, \quad m = 0, 1, 2, \ldots, \text{ and}$$

$$b_m = \frac{2}{\ell}\left\langle f, \sin\left(\frac{2m\pi\cdot}{\ell}\right)\right\rangle = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{2\pi mx}{\ell}\right) dx, \quad m = 1, 2, \ldots.$$

Then consider the infinite (Fourier) series

$$\boxed{\frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m\cos\left(\frac{2\pi mx}{\ell}\right) + b_m\sin\left(\frac{2\pi mx}{\ell}\right)\right].} \tag{2}$$

linear combs. of φ_n, ψ_m

Scalar multipliers for $\cos(0)$: a_0; $\cos(2\pi mx/\ell)$: a_m; $\sin(2\pi mx/\ell)$: b_m

Some Octave details:

- `fourierTrigCoeffs.m` computes, for a specified function f and ℓ, the coefficients $a_0, a_1, \ldots, a_k$, $b_1, b_2, \ldots, b_k$.

- `truncatedTrigFS.m` evaluates the function

$$\underbrace{\frac{a_0}{2} + \sum_{m=1}^{N}\left[a_m\cos\left(\frac{2\pi mx}{\ell}\right) + b_m\sin\left(\frac{2\pi mx}{\ell}\right)\right]}$$

  at specified input values.

Think of

$$\boxed{a_1\cos\left(\frac{2\pi x}{\ell}\right) + b_1\sin\left(\frac{2\pi x}{\ell}\right),}$$

case w/ $m=1$

as the fundamental (1st harmonic),

$$a_2 \cos\left(\frac{4\pi x}{\ell}\right) + b_2 \sin\left(\frac{4\pi x}{\ell}\right), \quad m = 2 \quad \left(2^{nd}\ \cancel{\text{overtone}}\ \text{harmonic}\right)$$

as the first overtone (2nd harmonic),

$$a_3 \cos\left(\frac{6\pi x}{\ell}\right) + b_3 \sin\left(\frac{6\pi x}{\ell}\right), \quad m = 2 \quad 3^{rd}\ \text{harmonic}$$

as the second overtone (3rd harmonic), and so on.

Say "plane" spanned by $\{\varphi_m, \psi_n\}$ $\begin{matrix} m = 0, 1, \ldots \\ n = 1, 2, \ldots \end{matrix}$ — call it P

hyperplane

$\text{Proj}_P f$ = form of linear comb. of φ_m, ψ_n's

Q What coeffs to use? A: Same idea as yesterdays' H.W.

— find projections of f onto individual φ_m, ψ_n's

Sum them up