MATH 162: Calculus II

Framework for Tues., Feb. 20

Introduction to Power Series

Definition: A function of x which takes the series form

$$\sum_{n=0}^{\infty} c_n(x-a)^n \;=\; c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \tag{1}$$

is called a *power series about* $x = a$. The number a is called the *center*, and the coefficients c_0, c_1, c_2, ... are constants.

Remarks:

- As for any function, a power series has a domain. The acceptable inputs x to a power function are those x for which the series converges.
- If it were not for the coefficients c_j (if, say, each $c_j = 1$), a power series would look geometric. Indeed, the geometric series

  $$\sum_{n=0}^{\infty} x^n \;=\; 1 + x + x^2 + x^3 + \cdots \tag{2}$$

  is a power series about $x = 0$, that is known to converge to $(1-x)^{-1}$ when $-1 < x < 1$ and to diverge when $|x| \geq 1$. So, the domain of power series (2) is $(-1, 1)$.

  More generally, for a special type of power series about $x = a$ with coefficients $c_j = \beta^j$ (i.e., whose coefficients are ascending powers of some fixed number β)

  $$\sum_{n=0}^{\infty} \beta^n(x-a)^n \;=\; 1 + \beta(x-a) + \beta^2(x-a)^2 + \beta^3(x-a)^3 + \cdots\,, \tag{3}$$

  we have that this series

  converges to $\dfrac{1}{1-\beta(x-a)}$ when $|\beta(x-a)| < 1$, that is, for $a - \dfrac{1}{|\beta|} < x < a + \dfrac{1}{|\beta|}$,

  and diverges when $|(x-a)| \geq \dfrac{1}{|\beta|}$.

  Thus, the domain of series (3) is $(a - 1/|\beta|, a + 1/|\beta|)$.

- In the most general case, where the coefficients c_j in (1) do not, in general, equal β^j for some number β, the determination of the domain usually requires

1. the use of the ratio test on the series $\sum_{n=0}^{\infty} |c_n(x-a)^n|$. That is, one looks at
$$\lim_{n\to\infty} \frac{|c_{n+1}||x-a|^{n+1}}{|c_n||x-a|^n} = \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)|x-a|.$$
If $\lim_{n\to\infty} |c_{n+1}|/|c_n|$ exists, and if we let
$$\rho = \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)|x-a|,$$
then part (i) of the ratio test imposes constraints on what values x may take. Specifically, we generally wind up with a number $R \geq 0$, called the *radius of convergence*, for which the series (1)

    converges if x is inside the open interval $(a-R, a+R)$, and

    diverges if $|x-a| > R$ (i.e., if x is outside the closed interval $[a-R, a+R]$).

In those cases where $\lim_{n\to\infty} |c_{n+1}|/|c_n| = 0$, the value of $R := +\infty$, and when this happens there is no need to proceed to step 2.
2. the determination (by some other means than the ratio test) of whether the series converges when $|x-a| = R$ (i.e., at the points $x = a \pm R$).

The upshot is that the domain of a power series whose radius of convergence R is nonzero is always an interval, an interval that has $x = a$ at its center and, in the case $R \neq +\infty$, may include one or both of its endpoints $x = a \pm R$. For this reason the domain of a power series is usually called its *interval of convergence*.

Example: Determine the interval of convergence for

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n3^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} n!x^n$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n 3^n}{n^{3/2}}$

Power Series Expressions for Some Fns. (building new series from known ones)

We know that, for $|x| < 1$, the fn. $f(x) = 1/(1-x)$ may be expressed as a power series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

Thus, we may express similar-looking fns. as power series:

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty} 3^n x^n, \qquad |3x| < 1 \;\Rightarrow\; -\frac{1}{3} < x < \frac{1}{3}$$

$$\frac{1}{2-x} = \frac{1}{2}\cdot\frac{1}{1-(x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)x^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}, \qquad \left|\frac{x}{2}\right| < 1 \;\Rightarrow\; -2 < x < 2.$$

$$\frac{x^3}{1-x} = x^3\cdot\frac{1}{1-x} = x^3\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3}, \qquad -1 < x < 1.$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}[(-1)x]^n = \sum_{n=0}^{\infty}(-1)^n x^n, \qquad |-x| < 1 \;\Rightarrow\; -1 < x < 1.$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n (x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}, \qquad |x^2| < 1 \;\Rightarrow\; -1 < x < 1.$$

All of the above are power series about $x = 0$. We show how the 2nd one (in the above list of 5) could also be written as a power series centered around $x = 1$:

$$\frac{1}{2-x} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty}(x-1)^n, \qquad |x-1| < 1 \;\Rightarrow\; 0 < x < 2.$$