The example not done in today's class

Today I wrote this example integral on the board

$$\int \tan^2 x \sec^3 x \, dx,$$

and a little into the process of working it out, decided to set it aside. It seemed necessary to have done this one first:

$$\int \sec^3 x \, dx \;=\; \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C.$$

Now, we can return to the "postponed" one. Note I will both use the *result* of $\int \sec^3 x \, dx$, and also the similar approach: to integrate by parts, and use a Pythagorean trig identity. Write

$$I = \int \tan^2 x \sec^3 x \, dx.$$

Then

$$\begin{aligned} I &= \int (\sec^2 x - 1) \sec^3 x \, dx \;=\; \int \sec^5 x \, dx - \int \sec^3 x \, dx \\ &= \int \sec^3 x \sec^2 x \, dx - \int \sec^3 x \, dx \qquad \text{(preparing first integral to do by parts)} \\ &= \left(\sec^3 x \tan x - \int 3 \sec^3 x \tan^2 x \, dx \right) - \int \sec^3 x \, dx \qquad (\text{with } u = \sec^3 x,\, dv = \sec^2 x \, dx) \\ &= \sec^3 x \tan x - 3I - \int \sec^3 x \, dx. \end{aligned}$$

Since the desired integral I appeared on the right-hand side, I will add $3I$ to both sides, so that

$$I \;=\; \sec^3 x \tan x - 3I - \int \sec^3 x \, dx$$

becomes

$$4I \;=\; \sec^3 x \tan x - \int \sec^3 x \, dx \;=\; \sec^3 x \tan x - \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

Dividing by 4, we get

$$I \;=\; \frac{1}{4} \sec^3 x \tan x - \frac{1}{8} \sec x \tan x + \frac{1}{8} \ln|\sec x + \tan x| + C.$$