

Polar area

The typical polar function is one which specifies $r = f(\theta)$ (θ as independent, r as dependent). The sorts of regions whose areas we might naturally compute this way are ones like those depicted in the top picture at right. As $r > 0$ reflects a distance from a point back to the origin (not the x -axis), slices look like wedges out of a near-circular region. We need to know how to compute areas of true wedges (taken from true circles), such as those depicted in the second picture at right. Its area satisfies a proportion:

$$\frac{\text{Area(wedge)}}{\text{Area(full circle)}} = \frac{\text{measure(central angle)}}{\text{measure(angle for one rotation)'}}$$

or

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \quad \Rightarrow \quad A = \frac{1}{2} r^2 \theta.$$

Using this, a typical nearly-wedge-shaped slice in the top figure would have area approximately equal to

$$\frac{1}{2} [f(\theta_i)]^2 (\theta_i - \theta_{i-1}) = \frac{1}{2} [f(\theta)]^2 \Delta\theta,$$

and an approximation to the full area could be obtained via the sum

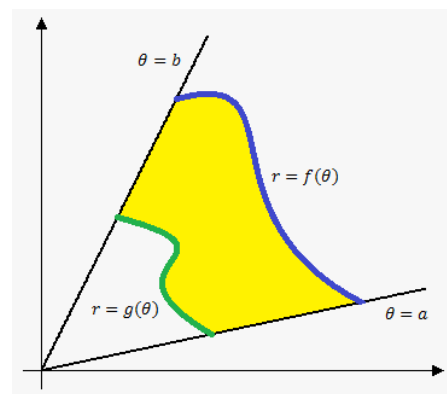
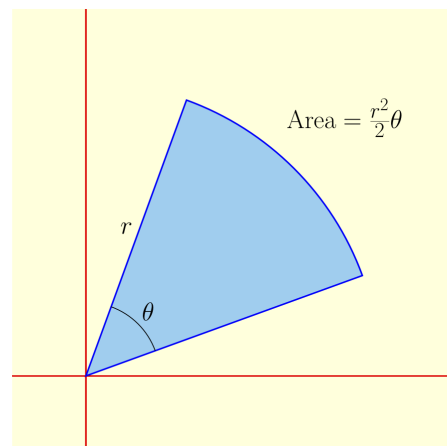
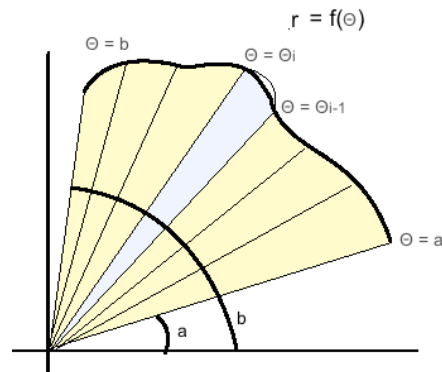
$$\sum_{i=1}^n \frac{1}{2} [f(\theta_i)]^2 \Delta\theta.$$

The approximation improves as $\Delta\theta \rightarrow 0$, giving the actual area as

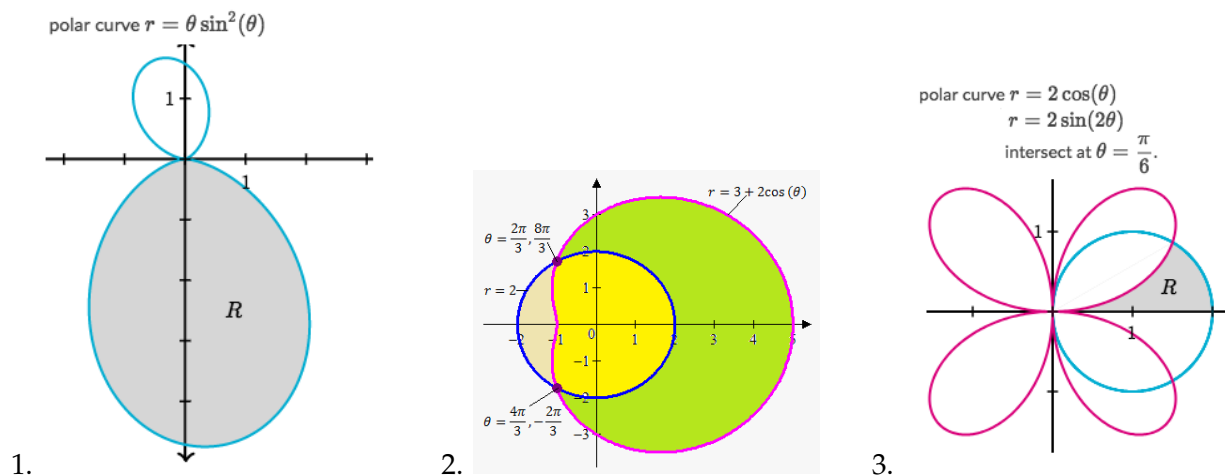
$$\int_a^b \frac{1}{2} [f(\theta)]^2 d\theta.$$

Adapting this to the computation of area for a region between two polar curves (see the bottom figure), we have

$$\text{Area of shaded region} = \int_a^b \frac{1}{2} ([f(\theta)]^2 - [g(\theta)]^2) d\theta.$$



Examples of areas of polar regions



Slopes and lengths along polar arcs

Key idea: Combine polar-to-rectangular conversion with polar functions to get a parametrization. That is, insert $r = f(\theta)$ into $x = r \cos \theta$, $y = r \sin \theta$. Then find $dy/dx = (dy/d\theta)/(dx/d\theta)$:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \Rightarrow \begin{cases} x = f(\theta) \cos \theta \\ y = f(\theta) \sin \theta \end{cases}$$

- **Slope.** As with other parametrized curves, slope is found by taking the ratio of the change in y (derivative of y) to the change in x (derivative of x). Thus,

$$\text{slope at } \theta = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

- **Arc length.** From Section 11.2 we have the arc length of for a parametric curve coming from the formula $L = \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta$. Inserting these derivatives and simplifying yields

$$\begin{aligned} L &= \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{[f'(\theta) \sin \theta + f(\theta) \cos \theta]^2 + [f'(\theta) \cos \theta - f(\theta) \sin \theta]^2} d\theta = \dots \\ &= \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta. \end{aligned}$$