1. (a) $\mathcal{L}\{f(t)\} = \mathcal{L}\{(3t-1)[1-u(t-2)]\} = \mathcal{L}\{3t-1\} - \mathcal{L}\{u(t-2)(3t-1)\}$

$$= \frac{3}{s^2} - \frac{1}{s} - \mathcal{L}\{u(t-2)[3(t-2)+5]\}$$

$$= \frac{3}{s^2} - \frac{1}{s} - \mathcal{L}\left\{u(t-2)\left(3t+5\Big|_{t\mapsto t-2}\right)\right\}$$

$$= \frac{3}{s^2} - \frac{1}{s} - e^{-2s}\mathcal{L}\{3t+5\} = \frac{3}{s^2} - \frac{1}{s} - e^{-2s}\left(\frac{3}{s^2}+\frac{5}{s}\right).$$

(b) $\mathcal{L}\{e^{-2t} * (4t^3+t)\} = \mathcal{L}\{e^{-2t}\}\cdot\mathcal{L}\{4t^3+t\} = \frac{1}{s+2}\cdot\left(4\frac{3!}{s^4}+\frac{1}{s^2}\right).$

(c) $$\frac{2}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$\Rightarrow 2 = A(s+1)(s+2) + Bs(s+1) + Cs(s+2)$$

@ $s=0$: $2 = 2A \Rightarrow A = 1$

@ $s=-1$: $2 = -C \Rightarrow C = -2$

@ $s=-2$: $2 = 2B \Rightarrow B = 1$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2}{s(s^2+3s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - 2e^{-t} + e^{-2t}.$$

By a shifting rule,

$$\mathcal{L}^{-1}\left\{\frac{2e^{-3s}}{s(s^2+3s+2)}\right\} = u(t-3)\cdot\left[1-2e^{-(t-3)}+e^{-2(t-3)}\right].$$

2. Because of the zero ICs, after Laplace transforms applied to both sides we have

$$s^2Y + 4sY + 5Y = \mathcal{L}\{f(t)\} \Rightarrow Y(s) = \mathcal{L}\{f(t)\}\cdot\frac{1}{s^2+4s+5}$$

Now $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$

By the Convolution Theorem,

$$y(t) = (h*f)(t) = \int_0^t f(w)\,e^{-2(t-w)}\sin(t-w)\,dw.$$

3. (a) $y'' + 9y = 0$ has characteristic equation $r^2 + 9 = 0 \Rightarrow r = \pm 3i$ with roots of the form $\alpha \pm \beta i$, $\alpha = 0$, $\beta = 3$, our general solution is

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

(b) Here the characteristic equation is $0 = r^2 + 6r + 9 = (r+3)^2$, giving repeated root $r = -3$. So, the general solution is

$$y(t) = c_1 e^{-3t} + c_2 t e^{-3t}.$$

4. (a) The characteristic eqn. is

$$r^2 + 3r + 2 = 0$$

which has distinct real roots, $r = -1, -2$. This is characteristic of overdamping.

(b) The homogeneous soln. $y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$ is built from exponential decay functions, which die off (very quickly) as $t \to \infty$. The steady state of

$$y(t) = y_h(t) + y_p(t),$$

the part that does not die off, is contained in $y_p(t)$.

(c) The forcing term $20\sin(2t)$ dictates we propose

$$y_p(t) = A\cos(2t) + B\sin(2t) \Rightarrow y_p' = -2A\sin(2t) + 2B\cos(2t)$$
$$y_p'' = -4A\cos(2t) - 4B\sin(2t)$$

Inserting this into the DE,

$$\begin{aligned} LHS &= -4A\cos(2t) - 4B\sin(2t) + 3\left[-2A\sin(2t) + 2B\cos(2t)\right] \\ &\quad + 2\left[A\cos(2t) + B\sin(2t)\right] \\ &= (-2A + 6B)\cos(2t) + (-2B - 6A)\sin(2t) \end{aligned}$$

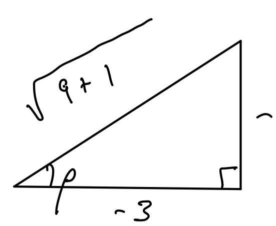
To equal the RHS $(20\sin(2t))$, we need

$$\begin{bmatrix} -2 & 6 \\ -6 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix} \Rightarrow A = \frac{\begin{vmatrix} 0 & 6 \\ 20 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 6 \\ -6 & -2 \end{vmatrix}} = -3, \quad B = \frac{\begin{vmatrix} -2 & 0 \\ -6 & 20 \end{vmatrix}}{\begin{vmatrix} -2 & 6 \\ -6 & -2 \end{vmatrix}} = -1.$$

Our particular soln., then, is

$$y_p(t) = -3\cos(2t) - \sin(2t).$$

(d) $A = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$.



5. After dividing by t^2 to get a coefficient 1 for y'', our $g(t) = 2t$. Next,

$$W = \begin{vmatrix} t & te^t \\ 1 & (1+t)e^t \end{vmatrix} = t^2e^t.$$

So

$$u_1 = -\int \frac{2t^2e^t}{t^2e^t}\,dt = -2t,$$

and

$$u_2 = \int \frac{2t^2}{t^2e^t}\,dt = 2\int e^{-t}\,dt = -2e^{-t}$$

Thus,

$$y_p = u_1y_1 + u_2y_2 = -2t^2 - 2t.$$