A comparison of QR factorizations

Yesterday in class we used the Gram-Schmidt algorithm to find an orthonormal basis for $W = \operatorname{span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\})$, where

$$\mathbf{a}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}.$$

We obtained the orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$

$$\mathbf{q}_1 = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}, \qquad \mathbf{q}_2 = \frac{1}{\sqrt{12}}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}, \doteq \begin{bmatrix}-0.86603\\0.28868\\0.28868\\0.28868\end{bmatrix}, \qquad \mathbf{q}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}0\\-2\\1\\1\end{bmatrix}, \doteq \begin{bmatrix}0\\-0.81650\\0.40825\\0.40825\end{bmatrix}.$$

We concluded that the matrix $\mathbf{A}$ whose columns are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ of the original basis of W, has a factorization $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$, having columns $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, is the same shape as $\mathbf{A}$, and $\mathbf{R}$ is square, nonsingular:

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}1/2 & -3/\sqrt{12} & 0\\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6}\\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6}\\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6}\end{bmatrix}\begin{bmatrix}\mathbf{a}_1^T\mathbf{q}_1 & \mathbf{a}_2^T\mathbf{q}_1 & \mathbf{a}_3^T\mathbf{q}_1\\ 0 & \mathbf{a}_2^T\mathbf{q}_2 & \mathbf{a}_3^T\mathbf{q}_2\\ 0 & 0 & \mathbf{a}_3^T\mathbf{q}_3\end{bmatrix}$$

$$\doteq \begin{bmatrix}0.5 & -0.86603 & 0\\ 0.5 & 0.28868 & -0.81650\\ 0.5 & 0.28868 & 0.40825\\ 0.5 & 0.28868 & 0.40825\end{bmatrix}\begin{bmatrix}2.0 & 1.5 & 1.0\\ 0 & 0.86604 & 0.57736\\ 0 & 0 & 0.8165\end{bmatrix}.$$

Octave has a `qr()` function. Let us compare its output with our work above.

```
octave:928> [que, arr] = qr([1 0 0; 1 1 0; 1 1 1; 1 1 1])
que =

  -0.50000   0.86603   0.00000   0.00000
  -0.50000  -0.28868   0.81650  -0.00000
  -0.50000  -0.28868  -0.40825  -0.70711
  -0.50000  -0.28868  -0.40825   0.70711

arr =

  -2.00000  -1.50000  -1.00000
   0.00000  -0.86603  -0.57735
   0.00000   0.00000  -0.81650
   0.00000   0.00000   0.00000
```