Geometric series in loans

Suppose Jan borrows \$10,000 at an annual interest rate of 12% compounded monthly (so, every month her balance increases 1 percent). One month later, Jan makes a payment of \$250, and does so repeatedly in ensuing months. Here are some questions.

1. Assuming Jan regularly pays on the first day of the new month, and that a_n represents the amount Jan owes after n months, find a_0, a_1, a_2, and a general expression (explicit formula) for a_n.
2. How much does Jan owe after making the 30^{th} payment?
3. Jan has just come into some extra money, and instead of making the 31^{st} payment the next month, Jan pays off the balance of the loan. If this payoff happens on the first day of month 31, how much has Jan actually paid to get free of this debt?

Answers:

1. a_0 is the amount owed on the day the loan is made, so $a_0 = 10000$. After one month, the balance rises because of interest, but falls because a payment is made:

$$a_1 = a_0(1 + 0.01) - 250 = a_0(1.01) - 250 = 9,850.$$

To get a_2, there is the option of simply using the figure of \$9,850 in place of a_0 in the prior calculation. But sometimes patterns are more easily discovered using symbols.

$$\begin{aligned}
a_2 &= a_1(1.01) - 250 = [a_0(1.01) - 250(1.01) - 250 = a_0(1.01)^2 - 250(1 + 1.01),\\
a_3 &= a_2(1.01) - 250 = [a_0(1.01)^2 - 250(1 + 1.01)](1.01) - 250 = a_0(1.01)^3 - 250(1 + 1.01 + 1.01^2)\\
a_4 &= a_3(1.01) - 250 = \cdots = = a_0(1.01)^4 - 250(1 + 1.01 + 1.01^2 + 1.01^3)\\
&\vdots\\
a_n &= a_{n-1}(1.01) - 250 = \cdots = = a_0(1.01)^n - 250\overbrace{(1 + 1.01 + 1.01^2 + \cdots + 1.01^{n-1})}^{\text{finite sum of geometric series}}\\
&= a_0(1.01)^n - (250)\frac{(1.01)^n - 1}{(1.01 - 1)} = a_0(1.01)^n - (25000)[(1.01)^n - 1].
\end{aligned}$$

2. We can use the formula for a_n to compute a_{30}:

$$a_{30} = (10000)(1.01)^{30} - (25000)[(1.01)^{30} - 1] = \$4,782.27.$$

3. By the time Jan makes the final payment, another month's interest has been assessed, so the balance has risen to

$$(4782.27)(1.01) = 4830.09.$$

Jan pays this amount, over and above the 30 monthly payments of \$250. All told, then, Jan pays

$$(30)(250) + 4830.09 = \$12,330.09.$$