1. Your signature.

2. (a) We can use the given eigenvectors to learn corresponding eigenvalues, bypassing the usual process of finding the zeros of the characteristic polynomial:

$$\begin{bmatrix}4 & 4\\ 6 & 2\end{bmatrix}\begin{bmatrix}2\\ -3\end{bmatrix} = 2\begin{bmatrix}4\\ 6\end{bmatrix} - 3\begin{bmatrix}4\\ 2\end{bmatrix} = \begin{bmatrix}-4\\ 6\end{bmatrix} = (-2)\begin{bmatrix}2\\ -3\end{bmatrix} \quad\Rightarrow\quad \lambda = -2.$$

$$\begin{bmatrix}4 & 4\\ 6 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}4\\ 6\end{bmatrix} + \begin{bmatrix}4\\ 2\end{bmatrix} = \begin{bmatrix}8\\ 8\end{bmatrix} = 8\begin{bmatrix}1\\ 1\end{bmatrix} \quad\Rightarrow\quad \lambda = 8.$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1e^{-2t}\begin{bmatrix}2\\ -3\end{bmatrix} + c_2e^{8t}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}2e^{-2t} & e^{8t}\\ -3e^{-2t} & e^{8t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}.$$

(b) Here, I take $\alpha = -6$, $\beta = 3$, $\mathbf{u} = \begin{bmatrix}-1\\ 5\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}-3\\ 0\end{bmatrix}$, so that the given eigenpair is $\alpha + i\beta$, $\mathbf{u} + i\mathbf{v}$. The general solution, then, is

$$\begin{aligned}
\mathbf{x}(t) &= c_1e^{\alpha t}[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}] + c_2e^{\alpha t}[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}]\\
&= c_1e^{-6t}\left(\cos(3t)\begin{bmatrix}-1\\ 5\end{bmatrix} - \sin(3t)\begin{bmatrix}-3\\ 0\end{bmatrix}\right) + c_2e^{-6t}\left(\sin(3t)\begin{bmatrix}-1\\ 5\end{bmatrix} + \cos(3t)\begin{bmatrix}-3\\ 0\end{bmatrix}\right)\\
&= e^{-6t}\left(c_1\begin{bmatrix}-\cos(3t) + 3\sin(3t)\\ 5\cos(3t)\end{bmatrix} + c_2\begin{bmatrix}-\sin(3t) - 3\cos(3t)\\ 5\sin t\end{bmatrix}\right).
\end{aligned}$$

3. (a) saddle: for a typical picture, see this link

   (b) spiral sink: for a typical picture, see this link

4. We will first find eigenvectors corresponding to $\lambda = 2$, in null$(\mathbf{A} - 2\mathbf{I})$. Starting with the the matrix $\mathbf{A} - 2\mathbf{I}$:

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ 0 & -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix} \quad\Rightarrow\quad \begin{array}{l} v_1 = 0\\ v_2 + v_3 = 0\\ v_3 \text{ is free}\end{array}$$

With one free variable, the geometric multiplicity of $\lambda = 2$ is 1. A basis eigenvector is $\mathbf{v} = \langle 0, 1, -1\rangle$. We have two eigenpairs, and we need a third. We look for one of the form $e^{2t}(\mathbf{u} + t\mathbf{v})$ with $\mathbf{u}$ a generalized eigenvector satisfying $(\mathbf{A} - 2\mathbf{I})\mathbf{u} = \mathbf{v}$.

$$[\mathbf{A} - 2\mathbf{I}\,|\,\mathbf{v}] = \left[\begin{array}{rrrr}-1 & 1 & 1 & 0\\ 2 & -1 & -1 & 1\\ 0 & -1 & -1 & -1\end{array}\right] \sim \left[\begin{array}{rrrr}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right] \quad\Rightarrow\quad \begin{array}{l} u_1 = 1\\ u_2 + u_3 = 1\\ u_3 \text{ is free}\end{array}$$

Here I take $u_3 = 0$, which makes $\mathbf{u} = \langle 1, 1, 0\rangle$. The general solution is

$$\mathbf{x}(t) = c_1e^{-t}\begin{bmatrix}-3\\ 4\\ 2\end{bmatrix} + c_2e^{2t}\begin{bmatrix}0\\ 1\\ -1\end{bmatrix} + c_2e^{2t}\left(\begin{bmatrix}1\\ 1\\ 0\end{bmatrix} + t\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}\right) = \begin{bmatrix}-3e^{-t} & 0 & e^{2t}\\ 4e^{-t} & e^{2t} & (1+t)e^{2t}\\ 2e^{-t} & -e^{2t} & -te^{2t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}.$$

5. (a) This problem is linear, 1st-order, nonhomogeneous. Putting it in normal form, we have

$$y' = -\frac{2}{t}y + t - 1 + \frac{1}{t} \qquad \Rightarrow \qquad a(t) = -\frac{2}{t}, \quad f(t) = t - 1 + \frac{1}{t}.$$

We have

$$\phi(t) = e^{-\int (2/t)\,dt} = e^{-2\ln|t|} = e^{\ln(t^{-2})} = t^{-2} \qquad \Rightarrow \qquad y_h(t) = Ct^{-2}.$$

This problem does not fit the usual criteria for the method of undetermined coefficients, so we use the variation of parameters formula to find y_p:

$$y_p(t) = \phi(t)\int \frac{f(t)}{\phi(t)}\,dt = t^{-2}\int t^2\left(t - 1 + \frac{1}{t}\right)dt = t^{-2}\int \left(t^3 - t^2 + t\right)dt = t^{-2}\left(\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2\right),$$

taking $C = 0$ in the integral. The general solution is

$$y(t) = y_h(t) + y_p(t) = Ct^{-2} + \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2}.$$

(b) The DE is separable.

$$4x^3\,dx = (-2t+3)\,dt \qquad \Rightarrow \qquad \int 4x^3\,dy = \int (-2t+3)\,dt \qquad \Rightarrow \qquad x^4 = -t^2 + 3t + C$$

$$x(0) = -\frac{1}{\sqrt{2}} \qquad \Rightarrow \qquad C = \frac{1}{4} \qquad \Rightarrow \qquad x^4 = -t^2 + 3t + \frac{1}{4}$$

Need explicit solution $\Rightarrow$ $x(t) = -\sqrt[4]{-t^2 + 3t + \frac{1}{4}}.$ (neg. root so IC is met)

6. The homogeneous version of this (linear) problem is

$$y' = 2y \qquad \text{which has solution} \qquad y_h(t) = Ce^{\int 2\,dt} = Ce^{2t}.$$

We note that the nonhomogeneous term is $f(t) = (3-t)e^{2t}$, a product best described as: (1st-degree polynomial)(exponential). It would seem natural to propose a particular solution also in that form

$$y_p(t) = (At+B)e^{2t} = Ate^{2t} + Be^{2t},$$

but the second of these terms would be of no use, being exactly like $y_h(t)$ in form. We have learned to remedy this by introducing an extra factor t in our proposal:

$$y_p(t) = (At+B)te^{2t} = (At^2+Bt)e^{2t}.$$

We can rearrange the DE to have $y' - 2y$ on the left-hand side and the target function $f(t) = (3-t)e^{2t}$ on the right. Using this and $y_p' = (2At+B)e^{2t} + 2(At^2+Bt)e^{2t}$, we have

$$y_p' - 2y_p = (2At+B)e^{2t} + 2(At^2+Bt)e^{2t} - 2(At^2+Bt)e^{2t} = (2At+B)e^{-3t}.$$

As the target function is $(-t+3)e^{2t}$, we see we get a match if we choose A and B so that

$$2A = -1 \quad \text{and} \quad B = 3, \qquad \text{so that} \qquad y_p(t) = \left(-\frac{1}{2}t^2 + 3t\right)e^{2t}.$$