Root and Ratio Tests

Example 1:

Consider the series

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + \frac{2^n}{n!} + \frac{2^{n+1}}{(n+1)!} + \cdots .$$

The ratio of two (general) consecutive terms in the sequence is

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = 2 \cdot \frac{n!}{(n+1)n!} = \frac{2}{n+1}.$$

The terms in the base sequence $a_0, a_1, a_2, \ldots$ do not form a geometric sequence as the ratio a_{n+1}/a_n is not constant. Nevertheless, for $n = 2$ and beyond ($n = 3, 4$, etc.), the fact that

$$\frac{a_{n+1}}{a_n} = \frac{2}{n+1} \quad \text{means} \quad \frac{a_3}{a_2} = \frac{2}{3}, \quad \frac{a_4}{a_3} = \frac{2}{4} < \frac{2}{3}, \quad \frac{a_5}{a_4} = \frac{2}{5} < \frac{2}{3}, \text{ etc.}$$

In other words, the terms beyond $n = 2$ in the series are positive, yet smaller than the terms of a geometric series with common ratio $r = 2/3$. Since that sort of geometric series converges, so does ours by the direct comparison test.

■

One could draw this same conclusion, even for series $\sum_n a_n$ that have positive and negative terms, whenever $\lim_n |a_{n+1}/a_n|$ exists and is less than 1. This is one of the major conclusions of the **ratio test**.

Theorem 1 (Ratio test): Suppose $(a_n)_{n=0}^{\infty}$ is a sequence for which the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists and equals ρ. If

- If $\rho < 1$, then the series $\sum_n a_n$ converges absolutely.
- If $\rho > 1$, then the series $\sum_n a_n$ diverges.
- If $\rho = 1$, then the test is inconclusive.

Example 2: More examples using the ratio test

1. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n + 7}{5^n}$ (convergent)
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ (divergent)
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (test inconclusive)
4. $\displaystyle\frac{1}{(2)(3+1)} + \frac{1 \cdot 3}{(2 \cdot 4)(3^2+1)} + \frac{1 \cdot 3 \cdot 5}{(2 \cdot 4 \cdot 6)(3^3+1)} + \cdots = \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{[2 \cdot 4 \cdot \cdots \cdot (2n)](3^n+1)}$

■

There are some series which are more easily handled by a similar test, called the **root test**.

Theorem 2 (Root test): Suppose $(a_n)_{n=0}^{\infty}$ is a sequence for which the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exists and equals L. If

- If $L < 1$, then the series $\sum_n a_n$ converges absolutely.
- If $L > 1$, then the series $\sum_n a_n$ diverges.
- If $L = 1$, then the test is inconclusive.

Example 3:

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ converges.

■