1. (a)

$$\begin{bmatrix} 3 & 1 & -3 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 2 & -3 \end{bmatrix} \overset{\mathbf{r}_1 \leftrightarrow \mathbf{r}_3}{\sim} \begin{bmatrix} 1 & 2 & 2 & -3 \\ 2 & 3 & 1 & -2 \\ 3 & 1 & -3 & 5 \end{bmatrix}$$

$$\overset{-2\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2}{\sim} \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & -1 & -3 & 4 \\ 3 & 1 & -3 & 5 \end{bmatrix}$$

$$\overset{-3\mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3,}{\sim} \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & -1 & -3 & 4 \\ 0 & -5 & -9 & 14 \end{bmatrix}$$

$$\overset{-5\mathbf{r}_2 + \mathbf{r}_3 \rightarrow \mathbf{r}_3.}{\sim} \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

(b)

$$\overset{-r_2 \Rightarrow r_2}{\sim} \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

$$\overset{(1/6)r_3 \rightarrow r_3}{\sim} \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overset{-3r_3 + r_2 \rightarrow r_2}{\sim} \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overset{-2r_3 + r_1 \rightarrow r_1}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overset{-2r_2 + r_1 \rightarrow r_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{RREF}$$

2. B must be 2×2 for the sum on the left to make sense.

$$2B = \begin{bmatrix} 2 & 1 & -2 \\ 1 & -3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -11 \\ -9 & -2 \end{bmatrix} - \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -10 \\ -6 & -6 \end{bmatrix}$$

$$\Rightarrow \quad B = \begin{bmatrix} -1 & -5 \\ -3 & -3 \end{bmatrix}$$

3.

$$[A|\vec{b}] = \left[\begin{array}{ccccc|c} 2 & -3 & 2 & 1 & 1 & -4 \\ 4 & -6 & 1 & -4 & 0 & -22 \\ -2 & 3 & 1 & 5 & 2 & 19 \\ 2 & -3 & -1 & -5 & -3 & -20 \end{array}\right] \overset{\text{RREF}}{\hookrightarrow} \left[\begin{array}{ccccc|c} 1 & -3/2 & 0 & -3/2 & 0 & -13/2 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_1 &= \tfrac{3}{2}x_2 + \tfrac{3}{2}x_4 - \tfrac{13}{2} \\ x_3 &= -2x_4 + 4 \\ x_5 &= 1 \end{aligned}\right\} \Rightarrow \text{solns.} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3/2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -13/2 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.$$

4. $$\begin{vmatrix} -2-\lambda & 2 \\ -20 & 10-\lambda \end{vmatrix} = (-2-\lambda)(10-\lambda) - (-40) = \lambda^2 - 8\lambda + 20$$

$$\lambda = \frac{8}{2} \pm \frac{\sqrt{64-80}}{2} = 4 \pm \frac{\sqrt{-16}}{2} = 4 \pm 2i$$

5. We solve $[A - (-2I)]\vec{v} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 3 & -3 & -3 & 0 \\ -6 & 6 & 6 & 0 \\ 3 & -3 & -3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 = v_2 + v_3$$

So, corresponding eigenvectors take the form

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 + v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{basis: } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

6. (a) $$\begin{bmatrix} 2 & 11 & 13 & 2 \\ 2 & 5 & 7 & 1 \\ -1 & -2 & -3 & -1 \\ -1 & -5 & -6 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ revealing that columns 1, 2, and 4 are linearly independent.

So, a basis for the column space is $$\begin{bmatrix} 2 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 11 \\ 5 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

(b) $4 - \text{rank}(A) = 4 - 3 = 1$ tells the dimension of $\text{null}(A)$.