MATH 162: Calculus II
Framework for Wed., Apr. 4
Unconstrained Optimization of Functions of 2 Variables

Today's Goal: To be able to locate and classify local extrema for functions of two variables.

Definition: Suppose the domain of $f(x, y)$ includes the point (a, b).

1. $f(a, b)$ is called a *local maximum* (or *relative maximum*) value of f if $f(a, b) \geq f(x, y)$ for all points from dom(f) contained in some open disk (an open disk of some positive, though perhaps quite small, radius) centered at (a, b).
2. $f(a, b)$ is called a *local minimum* (or *relative miminum*) value of f if $f(a, b) \leq f(x, y)$ for all points from dom(f) contained in some open disk centered at (a, b).

Remarks:

- As with functions of a single variable (think of the absolute value function), local extrema (maxima or minima) of functions f of two variables may occur at points where f is not differentiable.
- When an extremum occurs at an interior point (a, b) of dom(f) where f is differentiable, one would expect f to have a horizontal tangent plane there. The equation for the tangent plane to $z = f(x, y)$ at a point (x_0, y_0) where f is differentiable is
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) \;=\; 0,$$
or
$$z \;=\; z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$
while the equation of a horizontal plane (one parallel to the xy-plane is) $z =$ constant. We may, therefore, conclude:

Theorem: If $f(x, y)$ has a local extremum at an interior point (a, b) of dom(f), and if the partial derivatives of f exist there, then
$$f_x(a, b) = 0 \qquad \text{and} \qquad f_y(a, b) = 0.$$

This motivates the following definition.

Definition: Let f be a function of two variables. An interior point of dom(f) where

(i) both f_x and f_y are zero, or

(ii) at least one of f_x, f_y does not exist

is called a *critical point* of f.

Classifying Critical Points

Just as with functions of one variable, not all critical points of $f(x, y)$ correspond to a local extremum. On pp. 757–759 of the text, Figures 12.37 and 12.41 depict situations in which $(0, 0)$ is a critical point corresponding to an extremum; Figure 12.40 depicts situations in which $(0, 0)$ is the location of a saddle point.

Definition: Suppose $f(x, y)$ is a differentiable function with critical point (a, b). If every open disk centered at (a, b) contains both domain points (x, y) for which $f(x, y) > f(a, b)$ and domain points (x, y) for which $f(x, y) < f(a, b)$, then f is said to have a *saddle point* at (a, b).

With functions of a single variable, we have several tests (the First Derivative Test and the Second Derivative Test) for determining when a critical point corresponds to a local extremum. The following theorem provides a test for those critical points of type (i) for which f is twice continuously differentiable throughout a disk surrounding the critical point.

Theorem: Suppose that $f(x, y)$ and its first and 2nd partial derivatives are continuous throughout a disk centered at (a, b), and that $\vec{\nabla} f(a, b) = \mathbf{0}$. Let D be given by the following two-by-two determinant:

$$D(x, y) \ := \ \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{xy}(x, y) & f_{yy}(x, y) \end{vmatrix} \ = \ f_{xx} f_{yy} - f_{xy}^2.$$

Then

(i) f has a local maximum at (a, b) if $f_{xx}(a, b) < 0$ and $D(a, b) > 0$.

(ii) f has a local minimum at (a, b) if $f_{xx}(a, b) > 0$ and $D(a, b) > 0$.

(iii) f has a saddle point at (a, b) if $D(a, b) < 0$.

If $D(a, b) = 0$, or if $D(a, b) > 0$ and $f_{xx}(a, b) = 0$, then this test fails to classify the critical point (a, b).

Examples:

$$f(x, y) \ = \ x^3 y + 12x^2 - 8y$$

$$f(x, y) \ = \ \frac{x^2 y^2 - 8x + y}{xy}$$

$$f(x, y) \ = \ xy(1 - x - y)$$