★31 (a) The task, for each solution $\mathbf{x}_j(t)$ in the fundamental set, is to show that the left- and right-hand sides of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ evaluate to the same expression.

For the case of the system with matrix $\mathbf{A}$ in number 2:

Let's label the three solutions of the fundamental set

$$\mathbf{x}_1(t) = \begin{bmatrix} 3e^{-t} \\ 21e^{-t} \\ 8e^{-t} \end{bmatrix}, \qquad \mathbf{x}_2(t) = \begin{bmatrix} -3e^{4t} \\ -e^{4t} \\ 2e^{4t} \end{bmatrix}, \qquad \mathbf{x}_3(t) = \begin{bmatrix} (-3t-2)e^{4t} \\ -te^{4t} \\ 2te^{4t} \end{bmatrix}.$$

Then

$$\frac{d}{dt}\mathbf{x}_1(t) = \begin{bmatrix} -3e^{-t} \\ -21e^{-t} \\ -8e^{-t} \end{bmatrix}, \quad \text{and} \quad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 5.5 & -1.5 & 1.5 \\ 0.5 & -0.5 & -1.5 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3e^{-t} \\ 21e^{-t} \\ 8e^{-t} \end{bmatrix} = \begin{bmatrix} -3e^{-t} \\ -21e^{-t} \\ -8e^{-t} \end{bmatrix}.$$

Similarly,

$$\frac{d}{dt}\mathbf{x}_2(t) = \begin{bmatrix} -12e^{-t} \\ -4e^{-t} \\ 8e^{-t} \end{bmatrix}, \quad \text{and} \quad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 5.5 & -1.5 & 1.5 \\ 0.5 & -0.5 & -1.5 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3e^{-t} \\ -e^{-t} \\ 2e^{-t} \end{bmatrix} = \begin{bmatrix} -12e^{-t} \\ -4e^{-t} \\ 8e^{-t} \end{bmatrix}.$$

Finally,

$$\frac{d}{dt}\mathbf{x}_3(t) = \begin{bmatrix} -3e^{4t} \\ -e^{4t} \\ 2e^{4t} \end{bmatrix} + \begin{bmatrix} 4(-3t-2)e^{4t} \\ -4te^{4t} \\ 8te^{4t} \end{bmatrix} = \begin{bmatrix} (-12t-11)e^{4t} \\ (-4t-1)e^{4t} \\ (8t+2)e^{4t} \end{bmatrix},$$

while

$$\mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 5.5 & -1.5 & 1.5 \\ 0.5 & -0.5 & -1.5 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} (-3t-2)e^{-t} \\ -te^{-t} \\ 2te^{-t} \end{bmatrix} = \begin{bmatrix} (-12t-11)e^{4t} \\ (-4t-1)e^{4t} \\ (8t+2)e^{4t} \end{bmatrix}.$$

For the case of the system with matrix $\mathbf{A}$ in number 3:

Here, we might label the three solutions of the fundamental set

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \qquad \mathbf{x}_2(t) = \begin{bmatrix} (t-3)e^{-t} \\ (t+2)e^{-t} \\ -te^{-t} \end{bmatrix}, \qquad \mathbf{x}_3(t) = \begin{bmatrix} (t^2/2-3t+5)e^{-t} \\ (t^2/2+2t-3)e^{-t} \\ (-t^2/2)e^{-t} \end{bmatrix}.$$

Then

$$\frac{d}{dt}\mathbf{x}_1(t) = \begin{bmatrix} -e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix}, \quad \text{and} \quad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} -4 & -4 & -7 \\ 7 & 10 & 18 \\ -3 & -5 & -9 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix},$$

$$\frac{d}{dt}\mathbf{x}_2(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \\ -e^{-t} \end{bmatrix} + \begin{bmatrix} -(t-3)e^{-t} \\ -(t+2)e^{-t} \\ te^{-t} \end{bmatrix} = \begin{bmatrix} (-t+4)e^{-t} \\ (-t-1)e^{-t} \\ (t-1)e^{-t} \end{bmatrix}$$

while

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} -4 & -4 & -7 \\ 7 & 10 & 18 \\ -3 & -5 & -9 \end{bmatrix} \begin{bmatrix} (t-3)e^{-t} \\ (t+2)e^{-t} \\ -te^{-t} \end{bmatrix} = \begin{bmatrix} (-t+4)e^{-t} \\ (-t-1)e^{-t} \\ (t-1)e^{-t} \end{bmatrix},$$

and

$$\frac{d}{dt}\mathbf{x}_3(t) = \begin{bmatrix} (t-3)e^{-t} \\ (t+2)e^{-t} \\ -te^{-t} \end{bmatrix} + \begin{bmatrix} -(t^2/2-3t+5)e^{-t} \\ -(t^2/2+2t-3)e^{-t} \\ (t^2/2)e^{-t} \end{bmatrix} = \begin{bmatrix} (-t^2/2+4t-8)e^{-t} \\ (-t^2/2-t+5)e^{-t} \\ (t^2/2-t)e^{-t} \end{bmatrix},$$

while

$$\mathbf{A}\mathbf{x}_3 = \begin{bmatrix} -4 & -4 & -7 \\ 7 & 10 & 18 \\ -3 & -5 & -9 \end{bmatrix} \begin{bmatrix} (t^2/2-3t+5)e^{-t} \\ (t^2/2+2t-3)e^{-t} \\ (-t^2/2)e^{-t} \end{bmatrix} = \begin{bmatrix} (-t^2/2+4t-8)e^{-t} \\ (-t^2/2-t+5)e^{-t} \\ (t^2/2-t)e^{-t} \end{bmatrix}.$$

(b) The requirement that $(\mathbf{A}+3\mathbf{I})\mathbf{u} = \mathbf{v}$ leads to the augmented matrix

$$\left[\begin{array}{ccc|c} 8 & 2 & -12 & 2 \\ -4 & 2 & 12 & -2 \\ 0 & -2 & -4 & 1 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

revealing that $(\mathbf{A}+3\mathbf{I})\mathbf{u} = \mathbf{v}$ has no solution. Hence, there just isn't a solution of the form $(t\mathbf{v}+\mathbf{u})e^{-3t}$, where $\mathbf{v}$, $\mathbf{u}$ satisfy the stated equations, to be found.

(c) The requirement that $(\mathbf{A}-4\mathbf{I})\mathbf{w} = \mathbf{u}$ leads to the augmented matrix

$$\left[\begin{array}{ccc|c} 1.5 & -1.5 & 1.5 & -2 \\ 0.5 & -4.5 & -1.5 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

showing that $(\mathbf{A}-4\mathbf{I})\mathbf{w} = \mathbf{u}$ has no solution. Hence, there just isn't a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of the proposed form.

(d) We will look for for a solution of the form

- $(t\mathbf{v}+\mathbf{u})e^{\lambda t}$ only when λ is an eigenvalue with GM=1 and AM ≥ 2,
- $\left(\frac{1}{2!}t^2\mathbf{v}+t\mathbf{u}+\mathbf{w}\right)e^{\lambda t}$ only when λ is an eigenvalue with GM=1 and AM ≥ 3,
- $\left(\frac{1}{3!}t^3\mathbf{v}+\frac{1}{2!}t^2\mathbf{u}+t\mathbf{w}+\mathbf{z}\right)e^{\lambda t}$ only when λ is an eigenvalue with GM=1 and AM ≥ 4.

$\star$32 (a) The (approximate) homogeneous solution is

$$\begin{aligned}\mathbf{x}_h(t) &= c_1 e^{-0.0448t}\begin{bmatrix}1\\-0.6895\\-0.0871\end{bmatrix} + c_2 e^{-0.02t}\begin{bmatrix}1\\1.2963\\-0.1948\end{bmatrix} + c_3 e^{-0.0000306t}\begin{bmatrix}1\\0.3893\\892.56\end{bmatrix}\\ &= \begin{bmatrix}e^{-0.0448t} & e^{-0.02t} & e^{-0.0000306t}\\ -0.6895e^{-0.0448t} & 1.2963e^{-0.02t} & 0.3893e^{-0.0000306t}\\ -0.0871e^{-0.0448t} & -0.1948e^{-0.02t} & 892.56e^{-0.0000306t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}.\end{aligned}$$

(b) All of the eigenvalues of $\mathbf{A}$ are negative, which means that, as $t \to \infty$, the three fundamental solutions all go to $\mathbf{0}$. Thus, each component of $\mathbf{x}_h(t)$ representing, respectively, the amount of lead in the bloodstream, body tissue, and bone, goes to 0.

(c) Writing, as we usually do, the matrix of part (a) as $\Phi(t)$, we must solve

$$\begin{bmatrix}50\\0\\0\end{bmatrix} = \Phi(0)\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ -0.6895 & 1.2963 & 0.3893\\ -0.0871 & -0.1948 & 892.56\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}.$$

Using Gaussian elimination, we get approximate values $c_1 = 32.64$, $c_2 = 1.736$, $c_3 = 0.00697$. The 3$^{\text{rd}}$ (bone) component of the solution $\mathbf{x}_h(t)$, then, is

$$\begin{aligned}x_3(t) &\doteq (32.64)(-0.0871)e^{-0.0448t} + (1.736)(-0.1948)e^{-0.02t} + (0.00697)(892.56)e^{-0.0000306t}\\ &\doteq -2.85e^{-0.0448t} - 0.338e^{-0.02t} + 6.22e^{-0.0000306t}\end{aligned}$$

The peak value of the function $x_3(t)$, approximately 6.176, occurs around $t = 187$, i.e., after 187 days. The approximate time when the value of $x_3(t)$ returns to 0.5 is $t = 82383$ days, or about 225 years.

$\star$33 First consider the homogeneous solution. Setting

$$\boldsymbol{A} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \boldsymbol{f}(t) = \sec(t)\begin{pmatrix}2\\3\end{pmatrix},$$

the eigenvalues and associated eigenvectors for $\boldsymbol{A}$ are given by

$$\lambda_1 = \mathrm{i},\ \boldsymbol{v}_1 = \begin{pmatrix}1\\0\end{pmatrix} + \mathrm{i}\begin{pmatrix}0\\1\end{pmatrix};\quad \lambda_2 = -\mathrm{i},\ \boldsymbol{v}_2 = \begin{pmatrix}1\\0\end{pmatrix} - \mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}.$$

A fundamental set of solutions to the homogeneous problem is provided by

$$\boldsymbol{x}_1(t) = \cos(t)\begin{pmatrix}1\\0\end{pmatrix} - \sin(t)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\cos(t)\\ -\sin(t)\end{pmatrix},$$

and

$$\boldsymbol{x}_2(t) = \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

The homogeneous solution is then given by

$$\boldsymbol{x}_\mathrm{h}(t) = c_1\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} = \boldsymbol{\Phi}(t)\boldsymbol{c}, \quad \boldsymbol{\Phi}(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

As for the particular solution, using variation of parameters we have

$$\boldsymbol{x}_\mathrm{p}(t) = \boldsymbol{\Phi}(t)\int^t \boldsymbol{\Phi}(s)^{-1}\boldsymbol{f}(s)\,\mathrm{d}s = \boldsymbol{\Phi}(t)\int^t \begin{pmatrix} 2-3\tan(s) \\ 3+2\tan(s) \end{pmatrix}\mathrm{d}s = \boldsymbol{\Phi}(t)\left[t\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \ln(\cos(t))\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right].$$

Upon simplification

$$\boldsymbol{x}_\mathrm{p}(t) = t\cos(t)\begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\sin(t)\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \cos(t)\ln(\cos(t))\begin{pmatrix} 3 \\ -2 \end{pmatrix} - \sin(t)\ln(\cos(t))\begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

so that the general solution is

$$\boldsymbol{x}(t) = c_1\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} + \\ t\cos(t)\begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\sin(t)\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \cos(t)\ln(\cos(t))\begin{pmatrix} 3 \\ -2 \end{pmatrix} - \sin(t)\ln(\cos(t))\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

★34 First consider the homogeneous solution. Setting

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix},$$

the eigenvalues and associated eigenvectors are given by

$$\lambda_1 = 1,\ \boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \quad \lambda_2 = 3,\ \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The homogeneous solution is then given by

$$\boldsymbol{x}_\mathrm{h}(t) = c_1\mathrm{e}^t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\mathrm{e}^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{\Phi}(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \boldsymbol{\Phi}(t) = \begin{pmatrix} 2e^t & 0 \\ e^t & e^{3t} \end{pmatrix}$$

We can find a particular solution using the variation of parameters formula. Here,

$$\boldsymbol{\Phi}^{-1}(t) = \frac{1}{2e^{4t}}\begin{pmatrix} e^{3t} & 0 \\ -e^t & 2e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-t} & 0 \\ -e^{-3t} & 2e^{-3t} \end{pmatrix},$$

and the nonhomogeneous term in the original DE is

$$\mathbf{f}(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix},$$

so

$$\begin{aligned}
\mathbf{x}_p(t) &= \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{f}(t)\,dt \;=\; \frac{1}{2}\mathbf{\Phi}(t)\int \begin{pmatrix} e^{-t} & 0 \\ -e^{-3t} & 2e^{-3t} \end{pmatrix}\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} dt \\
&= \frac{1}{2}\mathbf{\Phi}(t)\int \begin{pmatrix} e^{t} \\ -3e^{-t} \end{pmatrix} dt \;=\; \frac{1}{2}\mathbf{\Phi}(t)\begin{pmatrix} \int e^{t}\,dt \\ \int -3e^{-t}\,dt \end{pmatrix} \\
&= \frac{1}{2}\mathbf{\Phi}(t)\begin{pmatrix} e^{t} \\ 3e^{-t} \end{pmatrix} \qquad \text{(needing only } \textit{one} \text{ particular soln, may take constants from integrals to be 0)} \\
&= \frac{1}{2}\begin{pmatrix} 2e^{t} & 0 \\ e^{t} & e^{3t} \end{pmatrix}\begin{pmatrix} e^{t} \\ 3e^{-t} \end{pmatrix} \;=\; \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} \;=\; e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},
\end{aligned}$$

and the general solution is

$$\boldsymbol{x}(t) = c_1 \mathrm{e}^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{e}^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

As for the initial condition,

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \boldsymbol{x}(0) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad\rightsquigarrow\quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\boldsymbol{c} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad\rightsquigarrow\quad \boldsymbol{c} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

In conclusion, the solution to the IVP is

$$\boldsymbol{x}(t) = \frac{1}{2}\mathrm{e}^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{5}{2}\mathrm{e}^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{e}^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$