Inference for μ

Yesterday: Chi-square dists

If $X \sim t(n)$

$E(X) = 0$

$Var(X) = \frac{n}{n-2}$

Definition 1: Suppose we have independent random variables $Z \sim \mathsf{Norm}(0,1)$ and $V \sim \mathsf{Chisq}(n)$. The distribution of

$$\frac{Z}{\sqrt{V/n}}$$

is called the *t*-**distribution** with n **degrees of freedom**. We write $\frac{Z}{\sqrt{V/n}} \sim t(n)$.

Definition 2: Suppose we have independent random variables $U \sim \mathsf{Chisq}(m)$ and $V \sim \mathsf{Chisq}(n)$. The distribution of

$$\frac{U/m}{V/n}$$

is called the *F*-**distribution** with m and n degrees of freedom. We write $\frac{U/m}{V/n} \sim F(m,n)$.

- Knowing $\mathbf{X} = \langle X_1, \ldots, X_n \rangle$ is an i.i.d.sample or SRS from a population with mean μ and variance σ^2, coupled with a reasonable sample size n means, by the Central Limit Theorem, that

$$\overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j \sim \mathsf{Norm}(\mu, \frac{\sigma}{\sqrt{n}}, \qquad \text{or} \qquad Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \mathsf{Norm}(0,1).$$

not-so-useful, since σ is unknown

- But, we do not have σ, and must estimate it. It is natural to use the altered statistic

$$T = \frac{\overline{X}-\mu}{S/\sqrt{n}}.$$

Know

• numerator has a Norm(0, 1) dist.

• denominator

The denominator $S/\sqrt{n}$ is an estimator of the standard error.

Note that

$$T = \frac{\overline{X}-\mu}{S/\sqrt{n}} = \frac{\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)}{S/\sigma}.$$

By these facts

- the numerator from above

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \mathsf{Norm}(0,1)$$

holds at least approximately, by the Central Limit Thm.

- the denominator from above S/σ has a square

$$\frac{S^2}{\sigma^2} = \frac{1}{n-1} \cdot \frac{(n-1)S^2}{\sigma^2}$$

which is a rescaled (by factor $(n-1)^{-1}$) r.v. with a $\mathsf{Chisq}(n-1)$ distribution.

So the statistic

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1).$$

Our new test statistic

- doesn't require knowledge of μ, σ
  only uses sample values
- has a known distribution: $\underline{t(n-1)}$.

Examples

1. Say you sample from a population w/ unknown mean μ, s.d. σ. Your SRS produces a mean $\overline{x} = 22.7$
$$S = 3.8$$
w/ sample size $n = 20$

Q: What strength of evidence against
$$H_0: \mu = 20 \qquad H_a: \mu \neq 20 \ ?$$

Past: $$z = \frac{\bar{x} - (\text{proposed } \mu)}{\sigma/\sqrt{n}} \underset{a}{\sim} \text{Norm}(0,1)$$
on the authority of CLT

Now

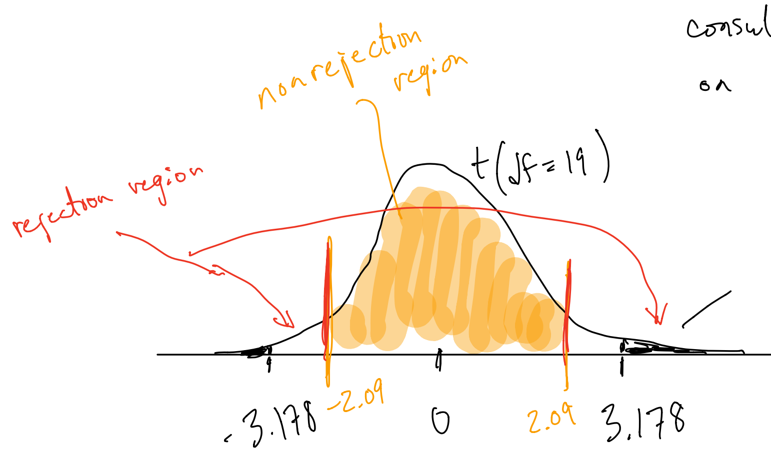
$$t = \frac{\bar{x} - (\text{proposed } \mu)}{s/\sqrt{n}} \sim t(n-1)$$

on what authority?

$$= \frac{\text{r.v. w/ Norm}(0,1)}{\sqrt{V/n-1}} \quad V \sim \text{chisq.}(n-1)$$

$$t = \frac{22.7 - 20}{3.8/\sqrt{20}} = 3.178$$

consult areas in tails on $t(19)$



area given by $2 * pt(-3.178, 19)$

$= 0.0049$

Q: What t-score would at the threshold separating

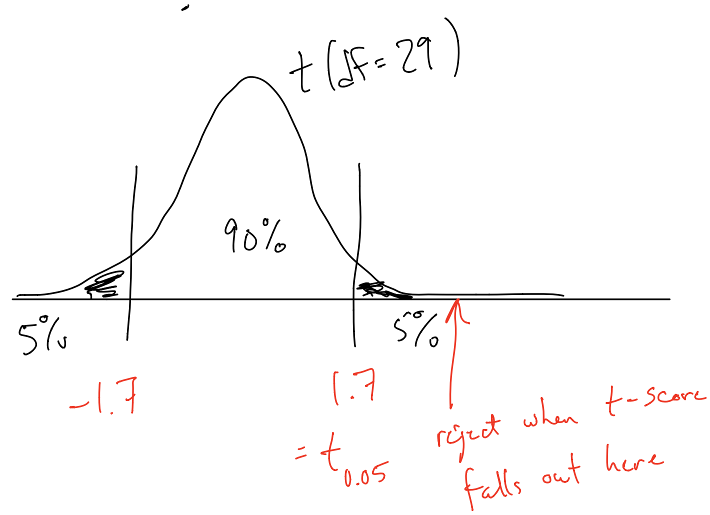
97.5% of area (on the left)

from 2.5% of area ?

A: $2.09 = qt(0.975, df = 19) = t_{0.025}$

Note: This $t_{0.025}$ and its negative are boundaries between rejection region and non-rejection region for hypothesis tests w/ $\alpha = 0.05$.

Q: Boundaries for nonrejection region if $\underline{\alpha = 0.1}$ and sample size is $n = 30$?



Ex.) Say I want to use same data to construct a 90% CI for μ

sample data: $\bar{x} = 22.7$, $s = 3.8$, $\underline{n = 20}$

t-score that serves boundary for nonrejection region

$t_{0.05} = qt(.95, df = 19)$

$= 1.73$

pt. est. $\pm$ (margin of error)

$$\bar{x} \pm (1.73)\left(s/\sqrt{n} \right)$$

threshhold t estimate of $SE_{\bar{x}}$

$t_{0.05}$ $\sigma/\sqrt{n}$

$$22.7 \pm (1.73)\left(3.8/\sqrt{20}\right)$$

or $(21.23, 24.17)$ is a 90% CI for μ.