Form B Solutions

1. (a) We must subtract multiples of 15 from (-71) until our result (remainder) satisfies $0 \le r < 15$: This entails subtracting $q = -5$ multiples of 15: $-71 - (-5)(15) = 4 = r$.

   (b) We note that $5^2 = 25 \equiv -1 \pmod{13}$, and so

$$5^{302} = (5^2)^{151} \equiv (-1)^{151} = -1 \equiv 12 \pmod{13}.$$

   Thus, $5^{302} \bmod 13 = 12$.

2. (a) The arrival of the extra person offers $n-1$ new pairings/handshakes, the new person with the other $n-1$ people.

   (b) $h_n = h_{n-1} + (n-1)$.

   (c) The recurrence relation of part (b) is linear, but not homogeneous.

3. Since $\sum_{i=1}^{9} ix_i = (1)(0)+(2)(8)+(3)(7)+(4)(6)+(5)(2)+(6)(0)+(7)(3)+(8)(2)+(9)(1)+10x_{10} = 117+10x_{10} \equiv 7 - x_{10} \pmod{11}$, we need $7 - x_{10} \equiv 0 \pmod{11}$. Thus, $x_{10} = 7$.

4. (a) We have

$$\begin{aligned} 3114 &= 1(2106) + 1008 \qquad (1)\\ 2106 &= 2(1008) + 90 \qquad (2)\\ 1008 &= 11(90) + 18 \qquad (3)\\ 90 &= 5(18) + 0 \end{aligned}$$

   So, $\gcd(3114, 2106) = 18$.

   (b) We rearrange equations (1)–(3) above to say

$$\begin{aligned} 1008 &= 3114 - 2106 \qquad (4)\\ 90 &= 2106 - 2(1008) \qquad (5)\\ 18 &= 1008 - 11(90). \qquad (6) \end{aligned}$$

   Then, we insert (5) into (6) to obtain

$$18 = 1008 - 11[2106 - 2(1008)] = 23(1008) - 11(2106),$$

   and finally insert (4) into that expression to get

$$18 = 23[3114 - 2106] - 11(2106) = 23(3114) - 34(2106).$$

   Thus, we make take $s = 23$ and $t = -34$.

5. (a) Since $77 = (7)(11)$, with prime factors, we have

$$\varphi(77) = \varphi(7)\varphi(11) = (6)(10) = 60.$$

   (b) Euler's Theorem states that

$$a^{\varphi(n)} \equiv 1 \pmod{n},$$

   when a and n are relatively prime (i.e., when $\gcd(a, n) = 1$). Here $a = 77$ and $n = 6479$ share the common factor 11, so they are *not* relatively prime. Euler's Theorem does not apply in our setting.

6. The additive inverse of 4 $\pmod{16}$ is 12, and the multiplicative inverse of 3 $\pmod{16}$ is 11, prompting us to add 12 to both sides and then multiply by 11. The effect on our two equivalent quantities:

$$\begin{aligned} 11[(3x+4)+12] \equiv 11(2+12) \pmod{16} \quad &\Rightarrow \quad 11(3x+16) \equiv 11(14) \pmod{16} \\ &\Rightarrow \quad 11(3x+0) \equiv 154 \pmod{16} \\ &\Rightarrow \quad 33x \equiv 10 \pmod{16} \\ &\Rightarrow \quad 1x \equiv 10 \pmod{16}. \end{aligned}$$

The solution is $x = 10$.

7.

for 5 pts: We may apply the Master Theorem, taking $a = 5$, $b = 3$, $c = 2$ and $d = 0$. Since $a > b^d$ (i.e, $5 > 1$) we have that $f(n)$ is $O(n^{\log_3 5})$.

for 10 pts: Here,

$$\begin{aligned} f(3^k) &= 5f(3^{k-1}) + 2 \;=\; 5[5f(3^{k-2}) + 2] + 2 \;=\; 5^2 f(3^{k-2}) + (5)(2) + 2 \\ &= 5^2[5f(3^{k-3}) + 2] + (5)(2) + 2 \;=\; 5^3 f(3^{k-3}) + (5^2)(2) + (5)(2) + 2 \\ &= 5^3 f(3^{k-3}) + 2[5^2 + 5 + 1] \;=\; \cdots \;=\; 5^k f(1) + 2[5^{k-1} + 5^{k-2} + \cdots + 5^2 + 5 + 1] \\ &= 2 \cdot 5^k + 2[5^{k-1} + 5^{k-2} + \cdots + 5^2 + 5 + 1] \;=\; 2[5^k + 5^{k-1} + 5^{k-2} + \cdots + 5^2 + 5 + 1] \\ &= 2 \cdot \frac{5^{k+1} - 1}{5 - 1} \;=\; \frac{1}{2}(5^{k+1} - 1). \end{aligned}$$

for 8 pts: We have

$$\begin{aligned} a_n &= 3a_{n-1} + 5 \;=\; 3[3a_{n-2} + 5] + 5 \;=\; 3^2 a_{n-2} + (3)(5) + 5 \\ &= 3^2[3a_{n-3} + 5] + (3)(5) + 5 \;=\; 3^3 a_{n-3} + (3^2)(5) + (3)(5) + 5 \\ &= 3^3 a_{n-3} + 5[3^2 + 3 + 1] \;=\; \cdots \;=\; 3^n a_0 + 5[3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1] \\ &= 3^n a_0 + 5\frac{3^n - 1}{3 - 1} \;=\; 4 \cdot 3^n + \frac{5}{2}(3^n - 1) \;=\; \frac{9}{2}3^n - \frac{5}{2}. \end{aligned}$$

8.
- In the first option, we assume $a \mid b$ and $b \mid c$. By definition, this means $\exists k_1 \in \mathbb{Z}$ and $\exists k_2 \in \mathbb{Z}$ such that $ak_1 = b$ and $bk_2 = c$. Thus,

  $$c \;=\; bk_2 \;=\; (ak_1)k_2 \;=\; a(k_1k_2).$$

  Since the product k_1k_2 of integers k_1, k_2 is an integer, this says that $a \mid c$.
- The given congruences, $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ mean, by definition, that $m \mid a - b$ and $m \mid b - c$—that is, $\exists k_1 \in \mathbb{Z}$ and $\exists k_2 \in \mathbb{Z}$ such that $mk_1 = a - b$ and $mk_2 = b - c$. We must show that $m \mid a - c$. But,

  $$a - c \;=\; (a - b) + (b - c) \;=\; mk_1 + mk_2 \;=\; m(k_1 + k_2).$$

  Since the sum $k_1 + k_2$ of integers k_1, k_2 is an integer, this shows that $m \mid a - c$.

9. Our recurrence relation is linear, homogeneous, with constant coefficients. For solving these, we assume solutions exist of the from $a_n = r^n$. Substituting this into the recurrence relation turns

$$a_n \;=\; 4a_{n-1} - 4a_{n-2} \qquad \text{into} \qquad r^n \;=\; 4r^{n-1} - 4r^{n-2}, \;\text{ or }\; r^{n-2}(r^2 - 4r + 4) \;=\; 0.$$

We are looking for nontrivial solutions, thereby ruling out $r = 0$, and solve the quadratic equation

$$r^2 - 4r + 4 \;=\; (r-2)^2 \;=\; 0,$$

arriving at the repeated root $r = 2$. It is true, the sequence

$$2^n : \; 1, \; 2, \; 2^2, \; 2^3, \; \ldots$$

satisfies the recurrence relation, but it does not satisfy the initial conditions. As in the past, we know a repeated root also generates a related sequence, in this case

$$n2^n : \; 0 \cdot 0, \; 1 \cdot 2, \; 2 \cdot 2^2, \; 3 \cdot 2^3, \; \ldots$$

which also satisfies the recurrence relation, but not the initial values. We now seek a linear combination,

$$a_n \; = \; \alpha 2^n + \beta n 2^n,$$

with constants α and β to be determined by applying the known initial values:

$$\left.\begin{array}{rcrcr} 3 & = & a_0 & = & \alpha \cdot 2^0 + \beta \cdot 0 \\ 10 & = & a_1 & = & \alpha \cdot 2 + \beta \cdot 2 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rcl} \alpha & = & 3 \\ \beta & = & 2 \end{array}\right.$$

Thus, $a_n = 3 \cdot 2^n + 2 \cdot n2^n = (3 + 2n)2^n$.