$a + ib$
$c + id$ $\Big\}$ product

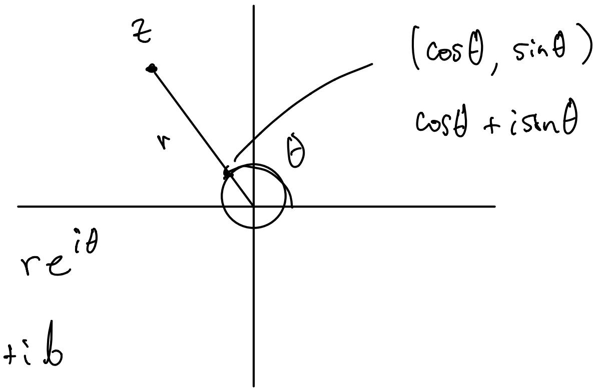
$$(a+ib)(c+id)$$
$$= ac + ibc + iad + i^2 bd$$
$$= \underbrace{(ac - bd)}_{Re} + i\underbrace{(ad + bc)}_{Im}$$

$i = \sqrt{-1}$

$i^2 = -1$

Take

$\theta = \arg.(a+ib)$



$$a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

polar form of $z = a + ib$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Euler's Formula

Maclaurin series

$$\begin{cases} e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(-1)^n \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{cases}$$

$x \rightsquigarrow i\theta$

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots$$
$$= 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 - \frac{1}{6!}\theta^6 - \frac{1}{7!}i\theta^7 + \cdots$$
$$= \underbrace{\left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \right)}_{} + i\underbrace{\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)}_{}$$

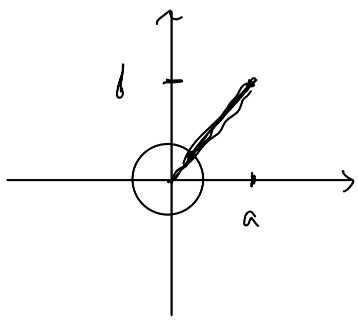
$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{Euler}$$

$$\underline{\cos\theta} - \underline{i\sin\theta} = \overline{\cos\theta + i\sin\theta}$$

$$= \underline{\cos(-\theta)} + i\,\underline{\sin(-\theta)} = e^{-i\theta}$$

$$\overline{e^{i\theta}} = e^{-i\theta}$$

$\underline{a+ib}$



modulus

$$|a+ib| = \sqrt{a^2+b^2}$$

$$= \sqrt{(a+ib)(a-ib)}$$

$$= \sqrt{z\bar{z}}$$

Vectors:

$\underline{\vec{x} \in \mathbb{R}^n}$ impossible to plot if $\underline{n \geq 4}$

$$\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle \qquad (\text{each } x_i \in \mathbb{R})$$

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$$

each component requires 2 axes

$\vec{z} \in \mathbb{C}^n$ is $\vec{z} = \langle z_1, z_2, \ldots, z_n \rangle$

$$= \langle a_1 + ib_1, a_2 + ib_2, \ldots, a_n + ib_n \rangle$$

impossible to plot if $\underline{n \geq 2}$

$$|\vec{z}| = \sqrt{\vec{z} \cdot \vec{z}} \geq 0 \quad \text{w/ } = 0 \text{ only when } \vec{z} = \vec{0}$$

$$\vec{z}_1 = \langle 2+3i,\ 4-7i \rangle$$

$$\vec{z}_1 \cdot \vec{z}_1 = \langle 2+3i,\ 4-7i \rangle \cdot \langle 2+3i,\ 4-7i \rangle$$

$$= (2+3i)\overline{(2+3i)} + (4-7i)\overline{(4-7i)}$$

$$= (2+3i)(2-3i) + (4-7i)(4+7i)$$

$$= (4+9) + i(6-6) + (16+49) + i(-28+28)$$

$$= 13 + 65 = \underline{\underline{78}}$$

For vectors $\vec{z}, \vec{w} \in \mathbb{C}^n$ we define

$$\vec{z} \cdot \vec{w} = \sum_{k=1}^{n} z_k \overline{w_k} \qquad \vec{z} \cdot \vec{w} = \overline{\vec{w} \cdot \vec{z}}$$

When we go to inner product of functions

$\sin\left(\frac{m\pi x}{l}\right)$ $\cos\left(\frac{n\pi x}{l}\right)$ are orthogonal when

$$\langle f, g \rangle = \int_0^l f(x)\, g(x)\, dx$$

When, f g may possible be complex-valued, must refine

$$\langle f, g \rangle = \int_0^l f(x)\, \overline{g(x)}\, dx$$

Consider functions $\ldots, e^{-2\pi\, 3ix/\ell}, e^{-2\pi\, 2ix/\ell}, \ldots, 1, e^{2\pi ix/\ell}, \ldots$

Take a pair $\underline{e^{2\pi imx/\ell}}$, $\underline{e^{2\pi inx/\ell}}$

their inner product, assuming $m \neq n$

$$\langle e^{2\pi imx/\ell}, e^{2\pi inx/\ell} \rangle = \int_0^\ell \underline{e}^{2\pi imx/\ell} \cdot \underline{e}^{-2\pi inx/\ell}\, dx$$

$$= \int_0^\ell e^{2\pi i(m-n)x/\ell}\, dx$$

$$= \frac{\ell}{2\pi i(m-n)} e^{2\pi i(m-n)x/\ell} \Big|_0^\ell$$

$$= \frac{\ell}{2\pi i(m-n)} \left(e^{2\pi i(m-n)} - e^0 \right) = 0.$$

but $e^{2\pi i(m-n)} \overset{\text{Euler's}}{=} \cos(2\pi(m-n)) + i \sin(2\pi(m-n))$

$= \underline{1}$

A good exercise (?)

$$\underbrace{\langle e^{2\pi imx/\ell}, e^{2\pi imx/\ell} \rangle}_{\text{same fn. in both slots}} = \ell \quad \text{(details omitted)}$$

Fourier series

before based on this list of mutually orthog fn.

$$1, \cos\left(\frac{2\pi x}{\ell}\right), \sin\left(\frac{2\pi x}{\ell}\right), \cos\left(\frac{4\pi x}{\ell}\right), \sin\left(\frac{4\pi x}{\ell}\right), \ldots$$

now could use, instead

$$\begin{array}{ccccccc} & n=-2 & n=-1 & n=0 & n=1 & n=2 & \ldots \\ \ldots, & & e^{-2\pi i x/\ell}, & 1, & \underbrace{e^{2\pi i x/\ell}}, & e^{4\pi i x/\ell}, & e^{6\pi i x/\ell}, \ldots \end{array}$$

$e^{-2\pi i x/\ell} = \cos\left(\frac{2\pi x}{\ell}\right) - i\sin\left(\frac{2\pi x}{\ell}\right)$

$e^{2\pi i x/\ell} = \cos\left(\frac{2\pi x}{\ell}\right) + i\sin\left(\frac{2\pi x}{\ell}\right)$

Complex F.S.

replace

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi n x}{\ell}\right) + b_n\sin\left(\frac{2n\pi x}{\ell}\right)\right]$$

w/

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/\ell}$$

and get the c_n much like the a_n, b_n

For a given function $f(x)$ (periodic, period $=\ell$)

$$c_n = \frac{\langle f, e^{2\pi i n \cdot/\ell}\rangle}{\langle e^{2\pi i n\cdot/\ell}, e^{2\pi i n\cdot/\ell}\rangle} = \frac{1}{\ell}\int_0^{\ell} f(x)\, e^{-2\pi i n x/\ell}\,dx$$

Full complex F.S.

$$\cdots + c_{-4}e^{-8\pi i x/\ell} + \boxed{c_{-3}e^{-6\pi i x/\ell} + c_{-2}e^{-4\pi i x/\ell} + \cdots + c_{3}e^{6\pi i x/\ell}} + c_{4}e^{8\pi i x/\ell} + \cdots$$

If truncated to $N=3$ "harmonics", we include only those boxed.

So, generally, a truncated complex F.S. looks like

$$\sum_{k=-N}^{N} c_k e^{2\pi i k x/\ell}$$

still computing the c_k's according to formula above.

It can be shown the coeffs. c_k for complex F.S. bear a relationship to the coeffs. a_k, b_k from real (trigonometric, original) F.S.

$$c_n = \begin{cases} \frac{1}{2}(a_{-n} + i b_{-n}), & n < 0 \\ 1/2\, a_0, & n = 0 \\ \frac{1}{2}(a_n - i b_n), & n > 0 \end{cases}$$

So, in particular, choose a case like $n = 2, -2$ (2^{nd} harmonic)

$$\left.\begin{aligned} c_{-2} &= \tfrac{1}{2}(a_2 + i b_2) \\ \text{while} \quad c_2 &= \tfrac{1}{2}(a_2 - i b_2) \end{aligned}\right\} \text{complex conjugates}$$

Assignment

1. Find a polar reprentation $z = re^{i\theta}$ for

   (a) $z = -3 + 4i$

   (b) $z = \sqrt{5} - \sqrt{11}\, i$

   Give θ in radians (decimal approximations OK).

2. Find $\vec{z} \cdot \vec{w}$ for the given $\vec{z}, \vec{w}$.

   (a) $\vec{z} = \langle 3i, -4, 2-i \rangle$, $\vec{w} = \langle 2, 1+2i, 3-i \rangle$

   (b) $\vec{z} = \langle 2, 1+2i, 3-i \rangle$, $\vec{w} = \langle 3i, -4, 2-i \rangle$

   (c) $\vec{z} = \langle 3-2i, 1+i, 4+3i \rangle$, and $\vec{w} = \vec{z}$.

3. If $\varphi_n(x)$ is defined as $e^{2\pi i n x \ell}$, show that

$$\langle \varphi_n(x), \varphi_n(x) \rangle = \ell .$$

4. In Octave, take the period 2 function I defined in class

```
> f = @(x) mod(x,2).^2 .* (mod(x,2)<1)
            + (mod(x,2) >= 1)
```

Plot it and overlay the truncated trigonometric Fourier seres w/ 5 harmonics:

$$\frac{a_0}{2} + \sum_{k=1}^{5} \left[a_k \cos\left(\frac{2\pi k x}{\ell}\right) + b_k \sin\left(\frac{2\pi k x}{\ell}\right) \right].$$

Then use the truncatedExpFS.m script to overlay the real part of the truncated complex exponential Fourier series

$$\sum_{k=-5}^{5} c_k e^{2\pi i k x/\ell}.$$

To get the real part I imagine somethin likey

```
plot(xs, real(truncatedExpFS(xs, f, ...)))
```