## Systems of DEs

In class yesterday I offered several examples of how systems of DEs arise in applications, namely

1. A **predator-prey** system. For $x(t)$, $y(t)$ representing the sizes of a prey and predator population, resepectively,
$$\begin{aligned} dx/dt &= ax - bxy \\ dy/dt &= -cy + dxy \end{aligned} \tag{1}$$
We said system (1) is in **normal form**.

2. The **two-body** problem. With the Earth at the origin and a satellite at position $(x(t), y(t))$ relative to the Earth, we used the inverse square law and Newton's 2nd Law to derive the system
$$\begin{aligned} d^2x/dt^2 &= -\frac{GMx}{(x^2+y^2)^{3/2}} \\ d^2y/dt^2 &= -\frac{GMy}{(x^2+y^2)^{3/2}} \end{aligned} \tag{2}$$
System (2) is not in normal form, mainly because it is $2^{\text{nd}}$ order. We turned it into a $1^{\text{st}}$-order system by introducing two more dependent variables, $u$ and $v$, new names for $dx/dt$, $dy/dt$, thereby arriving at the normal form
$$\frac{d}{dt}\begin{bmatrix} x \\ u \\ y \\ v \end{bmatrix} = \begin{bmatrix} u \\ -\frac{GMx}{(x^2+y^2)^{3/2}} \\ v \\ -\frac{GMy}{(x^2+y^2)^{3/2}} \end{bmatrix}.$$

Watch https://drive.google.com/file/d/1-4WJrIPYuyF88kFcEnrS6Dc24wW6QCHf/view?usp=sharing to see examples of other higher-order DEs converted to 1st-order systems.

## Linear first-order systems of DEs

In both the pendulum and RLC circuit examples of the video, we were able to take a $2^{\text{nd}}$-order DE to a first-order system in normal form. For the RLC circuit DE, the resulting normal form looks like
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}(t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{f}(t), \tag{3}$$
where $\mathbf{A}(t)$ is a (constant) 2-by-2 matrix, and $\mathbf{f}(t)$ is a 2-by-1 vector. (3) is a system because it has multiple (at least two) dependent variables, but those have been grouped together into a single entity/vector $\mathbf{x}$, so that (3) has the form
$$\frac{d}{dt}\mathbf{x} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t). \tag{4}$$

The vector **x** containing the dependent variables appears, along with its first derivative, in two locations, the very same locations as $y$, $y'$ in the $1^{\text{st}}$-order linear problems we studied in Chapter 2:

$$y' = a(t)y + f(t).$$

Systems with normal forms like (4) are called **$1^{\text{st}}$ order linear systems**. It is not possible to put the converted pendulum DE into this form, which means it is **nonlinear**.

**The tank-mixing problem of Section 3.1.1**. Kapitula derives a $1^{\text{st}}$-order system for a two-tank mixing problem. At one stage in the derivation (bottom of p. 135) he has

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{100}x_1 + \frac{3}{200}x_2 + 12 \\ \frac{1}{100}x_1 - \frac{8}{200}x_2 + 35 \end{bmatrix}.$$

I wish to point out that, while one would be *technically correct* in writing this as

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{3}{200}x_2 - \frac{7}{100}x_1 + 12 \\ -\frac{8}{200}x_2 + \frac{1}{100}x_1 + 35 \end{bmatrix} = x_2\begin{bmatrix} \frac{3}{200} \\ -\frac{8}{200} \end{bmatrix} + x_1\begin{bmatrix} -\frac{7}{100} \\ \frac{1}{100} \end{bmatrix} + \begin{bmatrix} 12 \\ 35 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{200} & -\frac{7}{100} \\ -\frac{8}{200} & \frac{1}{100} \end{bmatrix}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 12 \\ 35 \end{bmatrix},\end{aligned}$$

this does not get us to the form (4), as the vector being differentiated on the left

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{is not the same as the one that appears on the right} \qquad \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

It serves you well to pay attention to this detail as you write a linear system in matrix form. When we make sure it is the same **x** in both places, as in the middle of p. 136, we wind up with a different matrix (the columns are swapped).

**Another multi-tank problem**. In Exercise 3.1.3, p. 142, the details are given for a closed system involving 3 tanks. Let us define $x_1(t)$, $x_2(t)$, $x_3(t)$ to be the amounts of salt in Tanks $A$, $B$, $C$ at time $t$, respectively. With these definitions, you should verify that the appropriate mixture model can be written as

$$\mathbf{x}' = \begin{bmatrix} -\frac{8}{350} & \frac{8}{100} & 0 \\ \frac{3}{350} & -\frac{8}{100} & \frac{5}{400} \\ \frac{5}{350} & 0 & -\frac{5}{400} \end{bmatrix}\mathbf{x} + \mathbf{0},$$

making it a **homogeneous** 1st-order linear system. As the vector **x** represents a single entity enclosing the three unknown quantities

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad \text{it can also represent their three initial values} \qquad \mathbf{x}(0) = \begin{bmatrix} 600 \\ 1700 \\ 3200 \end{bmatrix}.$$