MATH 162: Calculus II

Framework for Mon., Jan. 29

Review of Differentiation and Integration

Differentiation

Definition of derivative $f'(x)$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{or} \qquad \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Differentiation rules:

1. **Sum/Difference rule**: If f, g are differentiable at x_0, then
$$(f \pm g)'(x_0) \;=\; f'(x_0) \pm g'(x_0).$$

2. **Product rule**: If f, g are differentiable at x_0, then
$$(fg)'(x_0) \;=\; f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

3. **Quotient rule**: If f, g are differentiable at x_0, and $g(x_0) \neq 0$, then
$$\left(\frac{f}{g}\right)'(x_0) \;=\; \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}.$$

4. **Chain rule**: If g is differentiable at x_0, and f is differentiable at $g(x_0)$, then
$$(f \circ g)'(x_0) \;=\; f'(g(x_0))g'(x_0).$$

   This rule may also be expressed as
$$\left.\frac{dy}{dx}\right|_{x=x_0} \;=\; \left(\left.\frac{dy}{du}\right|_{u=u(x_0)}\right)\left(\left.\frac{du}{dx}\right|_{x=x_0}\right).$$

   Implicit differentiation is a consequence of the chain rule. For instance, if y is really dependent upon x (i.e., $y = y(x)$), and if $u = y^3$, then
$$\frac{d}{dx}(y^3) \;=\; \frac{du}{dx} \;=\; \frac{du}{dy}\frac{dy}{dx} \;=\; \frac{d}{dy}(y^3)y'(x) \;=\; 3y^2y'.$$

   Practice: Find
$$\frac{d}{dx}\left(\frac{x}{y}\right), \qquad \frac{d}{dx}(x^2\sqrt{y}), \qquad \text{and} \qquad \frac{d}{dx}[y\cos(xy)].$$

Integration

The definite integral

- the area problem
- Riemann sums
- definition

Fundamental Theorem of Calculus:

I: Suppose f is continuous on $[a, b]$. Then the function given by $F(x) := \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b), with derivative

$$F'(x) \;=\; \frac{d}{dx}\int_a^x f(t)\,dt \;=\; f(x).$$

II: Suppose that $F(x)$ is continuous on the interval $[a, b]$ and that $F'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a).$$

Remarks:

- Part I says there is always a formal antiderivative on (a, b) to continuous f. A vertical shift of one antiderivative results in another antiderivative (so, if one exists, infinitely many do). But if an antiderivative is to pass through a particular point (an *initial value problem*), there is often just one satisfying this additional criterion.
- Part II indicates the definite integral is equal to the total change in any (and all) antiderivatives.

The *average value of f over* $[a, b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx,$$

when this integral exists.

Integration by substitution:

- Counterpart to the chain rule (Q: What rules for integration correspond to the other differentiation rules?)
- Examples:
  1. $\displaystyle\int e^{3x}\,dx$

2. $\displaystyle\int_0^5 \frac{dx}{2x+1}$

3. $\displaystyle\int_0^{\sqrt{\pi}/2} 2x\cos(x^2)\,dx$

4. $\displaystyle\int \frac{\ln x}{x}\,dx$

5. $\displaystyle\int \frac{dx}{1+(x-3)^2}$

6. $\displaystyle\int \frac{dx}{x\sqrt{4x^2-1}}$

7. $\displaystyle\int \cos(3x)\sin(3x)\,dx$

8. $\displaystyle\int \frac{\arctan(2x)}{1+4x^2}\,dx$

9. $\displaystyle\int \tan^m x\sec^2 x\,dx$

10. $\displaystyle\int \tan x\,dx$ (worth extra practice)

11. $\displaystyle\int \sec x\,dx$ (worth extra practice)