1. (a) The Wronskian for the set is

$$W(t) = \det(\Phi(t)) = \begin{vmatrix} \mathrm{e}^{at} & t\mathrm{e}^{at} \\ a\mathrm{e}^{at} & (1+at)\mathrm{e}^{at} \end{vmatrix} = e^{2at} \neq 0.$$

Thus, $\Phi(t)$ invertible, and the functions used to build its columns are linearly independent.

(b) The Wronskian for the set is

$$W(t) = \det(\Phi(t)) = \begin{vmatrix} \mathrm{e}^{at} & t\mathrm{e}^{at} & t^2\mathrm{e}^{at} \\ a\mathrm{e}^{at} & (1+at)\mathrm{e}^{at} & (at^2+2t)\mathrm{e}^{at} \\ a^2\mathrm{e}^{at} & (a^2t+2a)\mathrm{e}^{at} & (a^2t^2+4at+2)\mathrm{e}^{at} \end{vmatrix} = 2e^{3at} \neq 0.$$

As in part (a), we reach the conclusion that $\Phi(t)$ invertible and the functions used to build its columns are linearly independent.

(c) The DE $y'' + 6y' + 9y = 0$ has repeated root $r = -3$ to its characteristic equation, making both e^{-3t} and te^{-3t} solutions. A 2^{nd}-order problem requires 2 solutions in a fundamental set of solutions, and these two fit the criteria that

- they both solve, and
- they have a nonzero Wronskian.

So, they form a fundamental set of solutions.

(d) The DE $y'' + 7y' + 12y = 0$ has roots $r = -4, -3$ to its characteristic equation, making e^{-3t} a solution of this homogeneous, linear, 2nd-order DE. One can insert te^{-3t} to verify that it does **not** solve this DE. Thus, the collection $\{e^{-3t},\ te^{-3t}\}$ does not comprise a fundamental set of solutions of the DE.

(e) So long as $r = a$ is a triple root of the characteristic equation associated with your 3rd-order linear homogeneous DE, then e^{at}, te^{at}, and t^2e^{at}, can be part of the fundamental solutions to your DE. It will only comprise a full set of fundamental solutions if the DE is 3^{rd} order.

2. (a) (i) We have

$$\Phi(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ \frac{d}{dt}e^{-t} & \frac{d}{dt}e^{5t} \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & 5e^{5t} \end{bmatrix}.$$

(ii) By Cramer's rule,

$$v_1'(t) = \frac{\begin{vmatrix} 0 & e^{5t} \\ 2te^{-4t} & 5e^{5t} \end{vmatrix}}{\begin{vmatrix} e^{-t} & e^{5t} \\ -e^{-t} & 5e^{5t} \end{vmatrix}} = \frac{-2te^{t}}{6e^{4t}} = -\frac{1}{3}te^{-3t},$$

and

$$v_2'(t) \;=\; \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & 2te^{-4t} \end{vmatrix}}{\begin{vmatrix} e^{-t} & e^{5t} \\ -e^{-t} & 5e^{5t} \end{vmatrix}} \;=\; \frac{2te^{-5t}}{6e^{4t}} \;=\; \frac{1}{3}te^{-9t},$$

(iii) Integrating by parts, we have

$$\begin{aligned} v_1(t) &= \int -\frac{1}{3}te^{-3t}\,dt \;=\; \frac{1}{9}te^{-3t} - \int \frac{1}{9}e^{-3t}\,dt \qquad (\text{with } u = t,\, dv = (-1/3)e^{-3t}\,dt) \\ &= \frac{1}{9}te^{-3t} + \frac{1}{27}e^{-3t} + K_1 \end{aligned}$$

Similarly, $v_2(t) = -\frac{1}{27}te^{-9t} - \frac{1}{243}e^{-9t} + K_2.$

(b) (i) We have

$$\Phi(t) \;=\; \begin{bmatrix} e^{-2t} & e^t & te^t \\ \frac{d}{dt}e^{-2t} & \frac{d}{dt}e^t & \frac{d}{dt}(te^t) \\ \frac{d^2}{dt^2}e^{-2t} & \frac{d^2}{dt^2}e^t & \frac{d^2}{dt^2}(te^t) \end{bmatrix} \;=\; \begin{bmatrix} e^{-2t} & e^t & te^t \\ -2e^{-2t} & e^t & (1+t)e^t \\ 4e^{-2t} & e^t & (2+t)e^t \end{bmatrix}.$$

Note that $\det(\Phi(t)) = 9$, used below.

(ii) By Cramer's rule,

$$v_1'(t) \;=\; \frac{\begin{vmatrix} 0 & e^t & te^t \\ 0 & e^t & (1+t)e^t \\ e^{-t} & e^t & (2+t)e^t \end{vmatrix}}{\begin{vmatrix} e^{-2t} & e^t & te^t \\ -2e^{-2t} & e^t & (1+t)e^t \\ 4e^{-2t} & e^t & (2+t)e^t \end{vmatrix}} \;=\; \frac{e^t}{9},$$

$$v_2'(t) \;=\; \frac{\begin{vmatrix} e^{-2t} & 0 & te^t \\ -2e^{-2t} & 0 & (1+t)e^t \\ 4e^{-2t} & e^{-t} & (2+t)e^t \end{vmatrix}}{\det(\Phi(t))} \;=\; \frac{1}{9}(-3t-1)e^{-2t},$$

and

$$v_3'(t) \;=\; \frac{\begin{vmatrix} e^{-2t} & e^t & 0 \\ -2e^{-2t} & e^t & 0 \\ 4e^{-2t} & e^t & e^{-t} \end{vmatrix}}{\det(\Phi(t))} \;=\; \frac{1}{3}e^{-2t}.$$

(iii) We have

$$\begin{aligned} v_1(t) &= \int \frac{1}{9}e^t\,dt = \frac{1}{9}e^t + K_1, \\ v_2(t) &= \int -\frac{1}{9}(3t+1)e^{-2t}\,dt = \cdots = \frac{1}{36}(6t+5)e^{-2t} + K_2, \\ v_3(t) &= \int \frac{1}{3}e^{-2t}\,dt = -\frac{1}{6}e^{-2t} + K_3. \end{aligned}$$

where the expression for $v_2(t)$ is obtained integrating by parts.

(c) Since $v_1(t)e^{-t} + v_2(t)e^{5t}$ represents a particular solution when $f(t) = 2te^{-4t}$, we have that

$$c_1e^{-t} + c_2e^{5t} + \left(\frac{1}{9}te^{-3t} + \frac{1}{27}e^{-3t}\right)e^{-t} + \left(-\frac{1}{27}te^{-9t} - \frac{1}{243}e^{-9t}\right)e^{5t}$$

(what we obtain for $c_1e^{-t} + c_2e^{5t} + v_1(t)e^{-t} + v_2(t)e^{5t}$ when the constants K_1, K_2 of integration in the expressions for v_1, v_2 are taken to be zero) solves the nonhomogeneous linear 2^{nd} order DE

$$y'' - 4y' - 5y = 2te^{-4t}.$$

(d) Here, $v_1(t)e^{-2t} + v_2(t)e^t + v_3(t)te^t$ represents a particular solution when $f(t) = e^{-t}$, we have that

$$c_1e^{-2t} + c_2e^t + c_3te^t + \left(\frac{1}{9}e^t\right)e^{-2t} + \left(\frac{1}{36}(6t+5)e^{-2t}\right)e^t - \left(\frac{1}{6}e^{-2t}\right)te^t$$

solves the nonhomogeneous linear 2^{nd} order DE

$$y''' - 3y' + 2y = e^{-t}.$$

3. (a) The Wronskian

$$\begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r_2t} \end{vmatrix} = r_2e^{(r_1+r_2)t} - r_1e^{(r_1+r_2)t} = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0,$$

since $r_1 \neq r_2$. This means that anytime you have two solutions, arising from distinct real roots, to a 2^{nd}-order DE, those two solutions will form a fundamental set of solutions.

(b) Here we have

$$\begin{aligned} W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)) &= \begin{vmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t) & \alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t) \end{vmatrix} \\ &= \alpha e^{2\alpha t}\cos(\beta t)\sin(\beta t) + \beta e^{2\alpha t}\cos^2(\beta t) - \alpha e^{2\alpha t}\cos(\beta t)\sin(\beta t) + \beta e^{2\alpha t}\sin^2(\beta t) \\ &= \beta e^{2\alpha t}[\cos^2(\beta t) + \sin^2(\beta t)] = \beta e^{2\alpha t}, \end{aligned}$$

and the latter is nonzero. This means that anytime you have two solutions, arising from complex-conjugate roots, to a 2^{nd}-order DE, those two solutions will form a fundamental set of solutions.

4. The amplitude of $\frac{F_0}{\Delta}\cos(\omega t - \delta)$ is the ratio $F_0/\Delta(\omega)$, a constant over a function that varies with ω. When that function $\Delta(\omega)$ reaches a minimum, the ratio F_0/Δ will hit a maximum. So, we differentiate $\Delta(\omega)$:

$$\begin{aligned}
\Delta(\omega) &= \left(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)^{1/2} \\
\Rightarrow\quad \Delta'(\omega) &= \frac{1}{2}\left(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)^{-1/2}\left[-4m^2\omega(\omega_0^2 - \omega^2) + 2\gamma^2\omega\right] \\
&= \frac{2\gamma^2\omega - 4m^2\omega(\omega_0^2 - \omega^2)}{2\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}
\end{aligned}$$

To find critical points requires finding zeros of this derivative. But a fraction equals zero only when the numerator equals zero, so it suffices to set the numerator equal to 0.

$$2\gamma^2\omega - 4m^2\omega(\omega_0^2 - \omega^2) = 0.$$

Assuming $\omega \neq 0$, we divide it out of both terms and get

$$2\gamma^2 = 4m^2(\omega_0^2 - \omega^2) \qquad \Rightarrow \qquad \omega_0^2 - \omega^2 = \frac{2\gamma^2}{4m^2} \qquad \Rightarrow \qquad \omega^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}.$$