Solutions

1. (a) The direction of $\boldsymbol{v}$ is

$$\boldsymbol{u} \;=\; \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \;=\; \frac{1}{\sqrt{1+4+4}}(\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}) \;=\; \frac{1}{3}\hat{\boldsymbol{\imath}} - \frac{2}{3}\hat{\boldsymbol{\jmath}} - \frac{2}{3}\hat{\boldsymbol{k}}.$$

In general, $\nabla f = \left(y - \dfrac{3y}{(z-x)^2}\right)\hat{\boldsymbol{\imath}} + \left(x - \dfrac{3}{z-x}\right)\hat{\boldsymbol{\jmath}} + \dfrac{3y}{(z-x)^2}\hat{\boldsymbol{k}}$, so $\nabla f(1,-1,2) = 2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$. Thus,

$$D_{\boldsymbol{u}} f(1,-1,2) \;=\; (2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}})\circ\left(\frac{1}{3}\hat{\boldsymbol{\imath}} - \frac{2}{3}\hat{\boldsymbol{\jmath}} - \frac{2}{3}\hat{\boldsymbol{k}}\right) \;=\; 4.$$

(b) We know ∇f points in the direction of maximum increase. We already have ∇f at the point $(1,-1,2)$, so what we need to do is find its direction:

$$\frac{\langle 2,-2,-3\rangle}{\|\langle 2,-2,-3\rangle\|} \;=\; \frac{1}{\sqrt{4+4+9}}\langle 2,-2,-3\rangle \;=\; \frac{2}{\sqrt{17}}\hat{\boldsymbol{\imath}} - \frac{2}{\sqrt{17}}\hat{\boldsymbol{\jmath}} - \frac{3}{\sqrt{17}}\hat{\boldsymbol{k}}.$$

2. $$\frac{\partial w}{\partial v} \;=\; \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}.$$

3. (a) We have

$$\frac{\partial f}{\partial x} = 3x^2 - 12y$$
$$\frac{\partial f}{\partial y} = -12x + 24y^2$$

Setting $f_y = 0$ gives $-12x + 24y^2 = 0$, or $x = 2y^2$. Setting $f_x = 0$ and substituting $(2y^2)$ for x gives

$$3(2y^2)^2 - 12y = 0 \qquad\Rightarrow\qquad 12y(y^3-1) = 0 \qquad\Rightarrow\qquad y = 0, 1.$$

Since $x = 2y^2$, the option $y = 0$ is paired with $x = 0$, confirming that $(0,0)$ is a critical point. The option $y = 1$ is paired with $x = 2$, giving another critical point at $(2,1)$.

(b) The 2nd partial derivatives are

$$f_{xx} = 6x, \qquad f_{xy} = -12, \qquad f_{yy} = 48y,$$

so the relevant determinant is

$$D(x,y) \;=\; \begin{vmatrix} 6x & -12 \\ -12 & 48y \end{vmatrix} \;=\; 288xy - 144.$$

At $(0,0)$: $D(0,0) = -144 < 0$, which means there is a saddle point there.
At $(2,1)$: $D(2,1) = 576 - 144 = 432 > 0$ and $f_{xx}(2,1) = 12 > 0$, which means there is a local minimum there.

4. The projection (shadow region) of our object in the xy-plane has a boundary coming from the intersection of $z = 12$ and $z = 4(x^2+y^2) = 4r^2$. Setting these equal, we have

$$12 = 4r^2 \qquad\Rightarrow\qquad r = \sqrt{3}.$$

Thus, our first moment is

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{4r^2}^{12} Krz\,dz\,dr\,d\theta.$$

Other orders are possible, such as

$$M_{xy} = \int_0^{2\pi}\int_0^{12}\int_0^{\sqrt{z}/2} Krz\,dr\,dz\,d\theta.$$

5. Recognizing the region D is the spherical box

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/2, \quad 1 \le \rho \le 2,$$

is crucial. Using spherical coordinates, our integral is

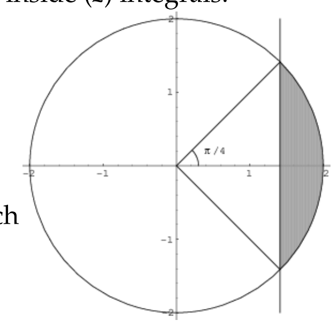
$$\begin{aligned}
\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, dV &= \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi\, e^{\rho^3}\, d\rho\, d\phi\, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left(\int_1^2 3\rho^2 e^{\rho^3}\, d\rho\right) d\phi\, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left(\int_1^8 e^u\, du\right) d\phi\, d\theta \qquad (\text{substituting } u = \rho^3) \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \Big[e^u\Big]_1^8\, d\phi\, d\theta \;=\; \frac{1}{3}(e^8 - e)\int_0^{2\pi}\left(\int_0^{\pi/2} \sin\phi\, d\phi\right) d\theta \\
&= \frac{1}{3}(e^8 - e)\int_0^{2\pi}\Big[-\cos\phi\Big]_0^{\pi/2}\, d\theta \;=\; \frac{1}{3}(e^8 - e)\int_0^{2\pi} d\theta \;=\; \frac{2\pi}{3}(e^8 - e).
\end{aligned}$$

In contrast, the integral is more difficult to set up in cylindrical coordinates where, for instance, the order $dV = r\, dz\, dr\, d\theta$ is the sum of integrals:

$$\int_0^{2\pi}\int_0^1\int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} re^{(r^2+z^2)^{3/2}}\, dz\, dr\, d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{4-r^2}} re^{(r^2+z^2)^{3/2}}\, dz\, dr\, d\theta.$$

Once set up, good luck finding an antiderivative for the inside (z) integrals.

6. In polar form we have

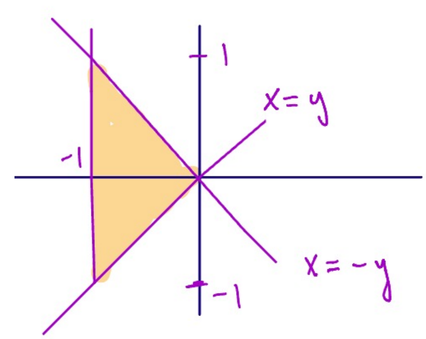


$$\int_{-\pi/4}^{\pi/4}\int_{\sqrt{2}\sec\theta}^{2} r^2\cos\theta\, dr\, d\theta$$

It is possible to integrate in the opposite order, in which case the integral looks this way:

$$\int_{\sqrt{2}}^{2}\int_{-\arccos(\sqrt{2}/r)}^{\arccos(\sqrt{2}/r)} r^2\cos(\theta)\, d\theta\, dr.$$

7. The region is depicted at right. There are straight-line boundaries at $x = y$ and $x = -y$. The same result, with order of integration reversed, is obtained via the integral



$$\int_{-1}^{0}\int_{x}^{-x} e^{1-x^2}\, dy\, dx.$$