If A, B ind. events then A^c and B are, too.

$$\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$$

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Stat 343, Fri 9-Oct-2020 -- Fri 9-Oct-2020
Probability and Statistics
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To each sequence $a_0, a_1, a_2, \ldots$ there is a corresponding power series centered at 0

$$a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{j=0}^{\infty} a_j t^j.$$

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Friday, October 09th 2020
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Wk 6, Fr
Topic:: Moment generating functions (mgf)
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Power series

$$\sum_{j=0}^{\infty} c_j t^j = c_0 + c_1 t + c_2 t^2 + \cdots$$

Have seq. $a_0, a_1, a_2, a_3,$

Generating Functions

Power series

Notes and reminders:

- The appearance of any power series centered at $t = a$ is ...
- Radius and interval of convergence
- Any function f that is differentiable to arbitrary order at a point $t = a$ has a formal power series at $t = a$, called its **Taylor series**:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

  There are things we would like to be true, but are not generally:

  - It is *not* generally true that the domain (interval of convergence) of the power series is the same as that for f.
  - When t is in both domains, the value to which the power series converges need not be the same as $f(t)$.

  Despite the uncertainties of such facts, there are some pairings of functions with their Taylor series about which we have a good understanding of when they are equal:

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k, \qquad -1 < t < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad -\infty < t < \infty$$

- Term-by-term differentiation

Generating functions

Want to generate a sequence

Given r.v. X, $\mu_1, \mu_2, \mu_3, \mu_4, \ldots$

Definition 1: Let $(a_k)_{k=0}^{\infty}$ be a sequence. The (formal) power series

$$A(t) \ = \ \sum_{k=0}^{\infty} a_k t^k$$

is called the **ordinary generating function** for (a_k). Another power series,

$$B(t) \ = \ \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k,$$

is called the **exponential generating function** for (a_k).

Example 1: Generating functions for exponential sequences

For $r \neq 0$, let $a_k = r^k$, $k = 0, 1, 2, \ldots$. Then generating functions for (a_k) have well-known closed-form expressions. In particular, the ordinary generating function for (a_k) is

$$A(t) \ = \ 1 + rt + (rt)^2 + (rt)^3 + \cdots + (rt)^k + \cdots \ = \ \frac{1}{1 - rt},$$

and the exponential generating function is

$$B(t) \ = \ 1 + rt + \frac{(rt)^2}{2!} + \cdots + \frac{(rt)^k}{k!} + \cdots \ = \ e^{rt}.$$

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As these generating functions are power series centered at 0,

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \qquad \text{and} \qquad B(t) = \sum_{k=0}^{\infty} b_k t^k,$$

with each $b_k = a_k/k!$, we have the relationship between coefficients and derivatives of $A(t)$:

$$a_k \ = \ \frac{A^{(k)}(0)}{k!} \qquad \text{and} \qquad b_k \ = \ \frac{A^{(k)}(0)}{(k!)^2}, \qquad k = 0, 1, 2, \ldots.$$

Similarly,

$$b_k \ = \ \frac{B^{(k)}(0)}{k!} \qquad \text{and} \qquad a_k \ = \ B^{(k)}(0), \qquad k = 0, 1, 2, \ldots.$$

Moment generating functions

Definition 2: Let X be a discrete or continuous random variable, and define the function $M_X(t) := \mathrm{E}\left(e^{tX}\right)$. When this function exists for some nontrivial interval $|t| \leq b$ about the origin, we call it the **moment generating function** (or **mgf**) of X.

$(\mu_0, \mu_1, \mu_2, \ldots)$

Why is M_X called by this name? (X might be Pois, Binom, DUnif)

- **Case:** X **is discrete**. So it has a pmf $f_X(x)$

$$E(e^{tX}) = \sum_x e^{tx} f_X(x) = \sum_x f_X(x) \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} = \sum_{j=0}^{\infty} \sum_x \frac{(tx)^j}{j!} f_X(x)$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \underbrace{\sum_x x^j f_X(x)}_{\mu_j} = \sum_{j=0}^{\infty} \frac{\mu_j}{j!} t^j$$

- **Case:** X **is continuous**.

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \ldots$$

$$= \sum_{j=0}^{\infty} \frac{\mu_j}{j!} t^j.$$

In either case, if the sequence $(\mu_k)_{k=1}^{\infty}$ of k^{th} moments of X about the origin exists, then $M_X(t)$ is the exponential generating function for the sequence of (μ_k).

$\mu_0, \mu_1, \mu_2, \mu_3, \ldots$

Then $E(e^{tX}) = M_X(t)$ mgf

Example 2: Moment generating function for $X \sim \mathsf{Pois}(\lambda)$ $\Longrightarrow$ pmf $f_X(x) = e^{-\lambda}\dfrac{\lambda^x}{x!}$

To calculate mgf

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \cdot e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\underbrace{\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}}_{e^{\lambda e^t}}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda e^t - \lambda}$$

$$= e^{-\lambda(1-e^t)}$$

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Example 3: Moment generating function for $X \sim \mathsf{Binom}(n, \pi)$

Here,

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{n} e^{tx}\binom{n}{x}\pi^x(1-\pi)^{n-x} = \sum_{x=0}^{n}\binom{n}{x}(e^t\pi)^x(1-\pi)^{n-x} \\ &= (\pi e^t + 1 - \pi)^n. \end{aligned}$$

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Example 4: Moment generating function for $X \sim \mathsf{Unif}(a, b)$

Since X is a continuous random variable, we have

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a}\,dx = \left.\frac{e^{tx}}{t(b-a)}\right|_a^b = \frac{e^{bt}-e^{at}}{t(b-a)}.$$

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Example 5: Moment generating function for $X \sim \mathsf{Exp}(\lambda)$

Here,

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \lim_{A\to\infty}\int_0^A \lambda e^{(t-\lambda)x}\,dx = \lim_{A\to\infty}\left.\frac{\lambda}{t-\lambda}e^{(t-\lambda)x}\right|_0^A \\ &= \lim_{A\to\infty}\frac{\lambda}{t-\lambda}(e^{A(t-\lambda)}-1) = \frac{\lambda}{\lambda-t} = \frac{1}{1-\frac{t}{\lambda}}, \end{aligned}$$

if $t < \lambda$ (so, in particular, the limit exists for $|t| < \lambda$).

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Theorem 1: For X a discrete or continuous r.v., a, b constants,

(i) $M_{aX}(t) = M_X(at)$.

(ii) $M_{X+b}(t) = e^{bt}M_X(t)$.

(iii) $M_{aX+b}(t) = e^{bt}M_X(at)$.

Two applications of mgfs:

- Since $M_x(t)$ is the exponential generating function of the sequence (μ_k) of k^{th} moments, we have, in general, $\mu_k = M_X^{(k)}(0)$.
- mgfs are unique, providing a signature for recognizing a random variable, as specified in this theorem:

Theorem 2: Let X, Y be r.v.s with mgfs M_X, M_Y. Then X, Y are **identically distributed** (i.e., they have the same cdfs) if and only if $M_X(t) = M_Y(t)$ for all t in some nontrivial interval containing 0.

Say have seq.

$$1, 1, 1, 1, 1, 1, \ldots$$

Corresp. power series

$$\underbrace{\underline{1} + 1\cdot t + 1\cdot t^2 + 1\cdot t^3 + \cdots = \sum_{j=0}^{\infty} t^j}_{\text{geometric series}}$$

$$= \frac{1}{1-t}$$ ← somehow a generator of orig. seq.

Since $f(t) = \dfrac{1}{1-t}$ produces Maclaurin series

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} t^j$$

$$f(0) = 1$$

$$f'(t) = \frac{d}{dt}(1-t)^{-1} = -1(1-t)^{-2}(-1)$$

$$= \frac{1}{(1-t)^2}$$

$$\Rightarrow f'(0) = 1$$

$$f''(t) = -2(1-t)^{-3}(-1) = \frac{2!}{(1-t)^3}$$

$$\Rightarrow \frac{f''(0)}{2!} = 1$$

If we had gone with the exponential generating fn approach

$1, 1, 1, 1, 1, \ldots$

Corresp. series (modified) $\quad \frac{1}{0!} + \frac{1}{1!}t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$

$$= e^t$$