MATH 162: Calculus II

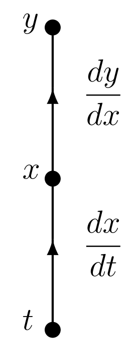
Framework for Thurs., Mar. 8

Chain Rules

Today's Goal: To extend the chain rule to functions of multiple variables.

Chain rule, single (independent) variable case

Setting: y is a function of x, while x is a function of t.
More explicitly, $y = y(x)$, and $x = x(t)$ (so $y = y(x(t))$).



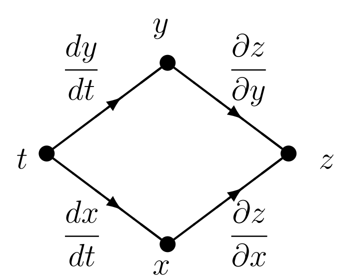
Chain Rule: $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Note here that

- y is the (final) dependent variable.
- t is the independent variable.
- x is an intermediate variable.

Many Multivariate Chain Rules

Setting 1: $z = f(x, y)$, with $x = x(t)$, $y = y(t)$



Chain Rule: $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Setting 2: $w = f(x, y, z)$, with $x = x(t)$, $y = y(t)$, $z = z(t)$

Chain Rule: $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Setting 3: $z = f(x, y)$, with $x = x(u, v)$, $y = y(u, v)$,

Chain Rules: $$\begin{cases} \dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} \end{cases}$$

Another Look at Implicit Differentiation

Many problems from MATH 161 in which implicit differentation was used involved equations which could be put in the form $F(x, y) = 0$. Assuming that this equation defines y implicitly as a function of x (an assumption that is generally true), then by the chain rule

$$\frac{dF}{dx} \;=\; \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} \;=\; F_x + F_y\,\frac{dy}{dx}.$$

This is the x-derivative of one side of the equation $F(x, y) = 0$. The x-derivative of the other side is, naturally, 0. Thus, we have

$$F_x + F_y\,\frac{dy}{dx} \;=\; 0 \qquad \Rightarrow \qquad \frac{dy}{dx} \;=\; -\frac{F_x}{F_y}.$$