3.1 $\quad \tilde{u}' + \tilde{u} = e^{x(t)+2t}$ reveals $a = -1$. So, multiply through by e^t.

$$\tilde{u}' e^t + \tilde{u} e^t = e^{t+\xi+3t} \quad \text{or} \quad \frac{d}{dt}\left(\tilde{u} e^t\right) = e^{4t+\xi}$$

$$\int_0^t \frac{d}{d\tau}\left(\tilde{u} e^\tau\right) d\tau = \int_0^t e^{4\tau+\xi}\, d\tau$$

$$\tilde{u}(t) e^t - \tilde{u}(0) = \frac{1}{4} e^{4\tau+\xi}\Big|_0^t = \frac{1}{4}\left(e^{4t+\xi} - e^{\xi}\right)$$

$$\tilde{u}(t) e^t = \varphi(\xi) + \frac{1}{4}\left(e^{4t+x-t} - e^{x-t}\right)$$

$$\boxed{u(x,t) = \varphi(x-t)e^{-t} + \frac{1}{4}\left(e^{x+2t} - e^{x-2t}\right)}$$

3.2 If $m \neq n$, then

$$\int_0^l \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_0^l \left[\cos\left((m-n)\frac{\pi x}{l}\right) - \cos\left((m+n)\frac{\pi x}{l}\right)\right]dx$$

$$= \frac{l}{2\pi}\left[\frac{1}{m+n}\sin\left((m-n)\frac{\pi x}{l}\right) - \frac{1}{m+n}\sin\left((m+n)\frac{\pi x}{l}\right)\right]_0^l$$

$$= \frac{l}{2\pi}\left[\frac{1}{m+n}\sin\left((m-n)\pi\right) - \frac{1}{m+n}\sin\left((m+n)\pi\right)\right]$$

$$= 0.$$

3.3 (a) $$\frac{\partial}{\partial x}u(r,\theta) = u_r\frac{\partial r}{\partial x} + u_\theta\frac{\partial \theta}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x)u_r - \frac{y}{x^2+y^2}u_\theta$$

$$= \frac{r\cos\theta}{r}u_r + \frac{r\sin\theta}{r^2}u_\theta = \left(\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)u.$$

(b) $$\frac{\partial}{\partial y}u(r,\theta) = u_r\frac{\partial r}{\partial y} + u_\theta\frac{\partial \theta}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y)u_r + \frac{x}{x^2+y^2}u_\theta$$

$$= \frac{r\sin\theta}{r}u_r + \frac{r\cos\theta}{r^2}u_\theta = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)u$$

$$\frac{\partial^2}{\partial y^2} u(r,\theta) = \sin\theta \frac{\partial}{\partial r}\left(\sin\theta\ u_r + \frac{\cos\theta}{r} u_\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\ u_r + \frac{\cos\theta}{r} u_\theta\right)$$

$$= \sin^2\theta\ u_{rr} + \frac{\cos\theta\sin\theta}{r}\left(u_{\theta r} - \frac{1}{r}u_\theta\right) + \frac{\cos^2\theta}{r}u_r + \frac{\cos\theta\sin\theta}{r}u_{r\theta} - \frac{\cos\theta\sin\theta}{r^2}u_\theta + \frac{\cos^2\theta}{r^2}u_{\theta\theta}$$

(c)

$$\Delta u = \left(\cos^2\theta + \sin^2\theta\right)u_{rr} + (2-2)\frac{\sin\theta\cos\theta}{r}u_{r\theta} + \frac{1}{r^2}\left(\cos^2\theta + \sin^2\theta\right)u_{\theta\theta} + (2-2)\frac{\cos\theta\sin\theta}{r^2}u_\theta$$

$$+ \frac{1}{r}\left(\cos^2\theta + \sin^2\theta\right)u_r$$

$$= \boxed{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}}$$