# MATH 162: Calculus II

## Framework for Fri., Feb. 9

## Improper Integrals

Thus far in MATH 161/162, definite integrals $\int_a^b f(x)\,dx$ have:

- been over regions of integration which were finite in length (i.e., $a \neq -\infty$ and $b \neq \infty$)
- involved integrands $f$ which are finite througout the region of integration

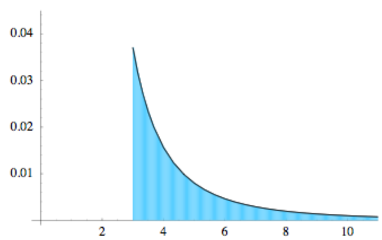
Q: How would we make sense of definite (*improper*, as they are called) integrals that violate one or both of these assumptions?

A: As limits (or sums of limits), when they exist, of definite integrals.

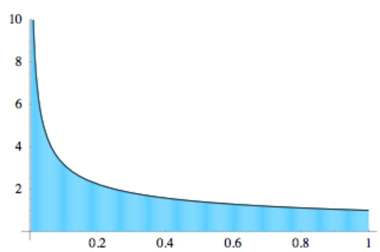
- When all of the limits involved exist, the integral is said to *converge*.
- When even one of the limits involved does not exist, the integral is said to *diverge*.

Examples:

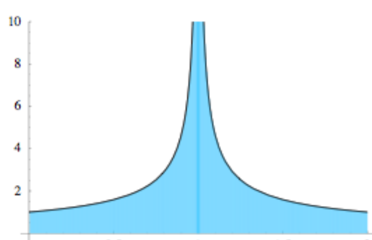
$$\int_3^\infty \frac{dx}{x^3}$$




$$\int_0^1 \frac{dx}{\sqrt{x}}$$




$$\int_0^2 \frac{dx}{(x-1)^{2/3}}$$




$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$$

Evaluating them:

$$\int_{-\infty}^{0} e^x \, dx$$

$$\int_{0}^{1} \ln x \, dx$$

$$\int_{1}^{\infty} \frac{dx}{x^p}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Even when an improper integral cannot be evaluated exactly, one might be able to determine if it converges or not. One of several possible theorems which address this issue:

**Theorem (Direct Comparison Test)**: Suppose $f$, $g$ satisfy $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

(i) $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

(ii) $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.