Predicates

A **predicate**, or **propositional function**, is a statement which accepts inputs from a **domain** (also known as a **universe of discourse**) and, for each (set of) inputs, the output is a proposition (i.e., has a truth value).

Examples

- $P(x)$ denotes the statement "x is a city in Michigan," and the domain is names of places. $P(\text{Detroit})$ is True; $P(\text{Philadelphia})$ is False.
- $C(x, y)$ denotes the statement "$y = x^2 - 1$", and the domain is (for instance) the set of coordinate-pairs of real-numbers. $C(1, 1)$ is False, while $C(2, 3)$ is True.
- $A(x, y)$ denotes the statement "The word x contains the letter y," and the input pairs (x, y) should include a word x, and a letter y of the alphabet. $A(\text{cloud}, \text{u})$ is True.

Statements involving logical operators, such as $\neg P(x)$, $P(x) \wedge Q(x) \to R(x)$, etc., have the same meaning as for propositions. A predicate $P(x_1, x_2, \ldots, x_n)$ requiring n inputs might be called an n**-ary predicate**.

Quantifiers. We indicate the

- **universal quantifier** using the symbol $\forall$, which is read aloud as "for all" or "for every." If $P(x)$ is the statement "x is mortal," and the domain is *human beings*, then $\forall x P(x)$ can be read as the proposition "for all human beings x, x is mortal," or more simply, "every human being is mortal."
  If we take D to be the set of numbers $\{1, 2, 3, 4, 5\}$, is the proposition $\forall x \in D(x^2 \geqslant x)$ True?
  We can use the universal quantifier on more than one variable: $\forall x \forall y(xy = yx)$, with both x, y being real numbers (domain).
- **existential quantifier** using the symbol $\exists$, which is read aloud as "there exists" or "some." So, $\exists x(x^2 = 2)$ asserts (probably with the understood domain of real numbers) that some number, when squared, yields the value 2.
  Try interpreting the statement $\forall a_0 \forall a_1 \forall a_2 \forall a_3((a_0 \neq 0) \to \exists x(a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0))$.
- **uniqueness quantifier** using the symbol $\exists!$, which is read aloud as "there exists a unique" or "there is precisely one." So, $\exists! x(x \text{ is omniscient, omnipresent and omnipotent})$ can be interpreted as saying "there is one and only one all-powerful God."
  Can you interpret this statement: $\forall x((x \neq 0) \to \exists! y(xy = 1))$?

Quantifiers take precedence over logical operators. Thus

$$\forall x P(x) \wedge Q(x) \qquad \text{means} \qquad (\forall x P(x)) \wedge Q(x), \qquad \text{not} \qquad \forall x(P(x) \wedge Q(x)).$$

The latter is logically equivalent to $\forall x P(x) \wedge \forall x Q(x)$.

When a quantifier is used with a variable, we say that variable is **bound**. If a variable has no quantifier nor is set to a particular value, then we say that variable is **free**.

Negation of universal quantifiers. One generic-looking statement using the universal quantifier is $\forall x P(x)$, read as "for all x, $P(x)$ holds True." This statement is false if there is a single instance of a value, say $x = x_0$ in the domain, called a **counterexample**, for which $P(x_0)$ is False. That is, the negation $\neg\forall x P(x)$ can be written using the existential quantifier as $\exists x \neg P(x)$.

On the other hand, a generic statement using the existential quantifier might be $\exists x P(x)$, "some x exists for which $P(x)$ holds True." The negation of that would be that "no x exists for which $P(x)$ holds" or, equivalently, "for all x, it is not the case that $P(x)$ holds," a statement which employs the universal quantifier. Thus $\neg\exists x P(x) \equiv \forall x \neg P(x)$.