MATH 162: Calculus II
Framework for Fri., Mar. 9
Vectors

Today's Goal: To understand vectors and be able to manipulate them.

Vectors in 2 and 3 Dimensions

> **Definition**: A *vector* is a directed line segment. If P and Q are points in $\mathbb{R}^2$ or $\mathbb{R}^3$, then the directed line segment from the *initial* point P to the *terminal* point Q is denoted $\overrightarrow{PQ}$.

- **Vector names**: bold-faced letters (usually lower-case) $\mathbf{v}$, or letters with arrows $\vec{v}$

- There are two things that distinguish a vector $\mathbf{v}$: its *length* and its *direction*. Thus, two directed line segments which are parallel, have the same length, and are oriented in the same direction (arrow pointing the same way) are considered to be the same (equal) even if their initial and terminal points are different.

- **Component form**: Given what was said above, any vector $\mathbf{v}$ may be moved rigidly so as to make its initial point be the origin. Writing (v_1, v_2, v_3) for the resulting terminal point, we then say that $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. This is called the *component form* of $\mathbf{v}$. The numbers v_1, v_2 and v_3 are the *components* of $\mathbf{v}$.

- **Equality of vectors**: Two vectors are considered equal when they are equal in each component.

- If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ then the *length* (or *magnitude*) of $\mathbf{v}$, denoted $|\mathbf{v}|$, is given by

$$|\mathbf{v}| \ := \ \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

  The only vector whose length is zero is the one whose components are all zero. We call this the *zero vector*, denoting it by $\mathbf{0}$. The other vectors (the ones with nonzero length) are collectively referred to as *nonzero* vectors.

Vector Operations

Vector Addition: If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, we define

$$\mathbf{u} + \mathbf{v} \ := \ \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$

Scalar Multiplication: If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and c is a real number (a *scalar*), then we define

$$c\mathbf{v} \ := \ \langle cv_1, cv_2, cv_3 \rangle.$$

Note that:

- Our definitions for vector addition and scalar multiplication are enough to give us the notion of *vector subtraction* as well, since we may think of $\mathbf{u} - \mathbf{v}$ as

$$\mathbf{u} + (-1)\mathbf{v} \;=\; \langle u_1, u_2, u_3 \rangle + \langle -v_1, -v_2, -v_3 \rangle \;=\; \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle .$$

- We make sense of an expression like $\mathbf{v}/c$ (i.e., dividing a vector by a scalar) by thinking of it as $(1/c)\mathbf{v}$ (i.e., the reciprocal of c multiplied by $\mathbf{v}$). For any nonzero vector $\mathbf{v}$, $\mathbf{v}/|\mathbf{v}|$ is a vector whose length is 1, called the *direction* of $\mathbf{v}$.

- No attempt has been made to define any type of multiplication (not yet) nor division (never!) between two vectors.

Unit Vectors

Any vector whose length is 1 is called a *unit vector*.

Example: For each $\mathbf{v} \neq \mathbf{0}$, the direction $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ of $\mathbf{v}$ is a unit vector. Thus, in 2D, the vector $\mathbf{v} = \langle -2, 5 \rangle$ has direction

$$\mathbf{d} \;=\; \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + 5^2}} \;=\; \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle .$$

We may then write $\mathbf{v}$ as a product of its magnitude times its direction

$$\mathbf{v} \;=\; \sqrt{29}\,\mathbf{d}.$$

Standard unit vectors (the ones parallel to the coordinate axes): $\mathbf{i} := \langle 1, 0, 0 \rangle$, $\mathbf{j} := \langle 0, 1, 0 \rangle$, and $\mathbf{k} := \langle 0, 0, 1 \rangle$.

Notice that, for $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, it is the case that

$$\mathbf{v} \;=\; v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$