Reflections on 1st-order homogeneous linear systems

Some of the 1st-order linear homogeneous systems of DEs we have encountered in the past several days, and their general solutions:

1. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\mathbf{x}$, has solution $\mathbf{x}_h(t) = \begin{bmatrix} -e^{3t} & e^{6t} \\ e^{3t} & 2e^{6t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}$.

2. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} -3 & 0 & 3 \\ -12 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x}$, has solution $\mathbf{x}_h(t) = \begin{bmatrix} e^{-3t} & 3e^{-2t} & 0 \\ 3e^{-3t} & 12e^{-2t} & e^{t} \\ 0 & e^{-2t} & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}$.

3. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} -21 & -30 & -32 \\ -4 & -7 & -7 \\ 24 & 30 & 35 \end{bmatrix}\mathbf{x}$, solved by

$$\mathbf{x}_h(t) = \begin{bmatrix} -11e^{3t} & -10e^{2t}\cos(3t) & -10e^{2t}\sin(3t) \\ -4e^{3t} & -3e^{2t}\cos(3t) - e^{2t}\sin(3t) & -3e^{2t}\sin(3t) + e^{2t}\cos(3t) \\ 12e^{3t} & 10e^{2t}\cos(3t) & 10e^{2t}\sin(3t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}.$$

4. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} -5 & -10 \\ 5 & 9 \end{bmatrix}\mathbf{x}$, has $\mathbf{x}_h(t) = \begin{bmatrix} -7e^{2t}\cos t - e^{2t}\sin t & -7e^{2t}\sin t + e^{2t}|cost \\ 5e^{2t}\cos t & 5e^{2t}\sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}$.

In each case, I am calling my formula for $\mathbf{x}_h(t)$ a **general solution**, referring to the matrix $\mathbf{\Phi}(t)$ as a **fundamental matrix solution**. In what sense are these solutions *general* or *fundamental*? These names start to make sense when we add to our DEs an initial condition.

Example 1:

Consider the IVP associated with System 1 above:

$$\frac{d}{dt}\mathbf{x} = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\mathbf{x}, \qquad \text{subject to } \mathbf{x}(t_0) = \mathbf{b},$$

where $\mathbf{b}$ is a specified vector. To satisfy this IC, we need $\mathbf{b} = \mathbf{x}_h(t_0) = \mathbf{\Phi}(t_0)\mathbf{c}$.
Given what we learn from the Existence/Uniqueness Theorem of Section 3.3, a general solution should be such that a unique vector $\mathbf{c}$ exists for every choice of vector $\mathbf{b}$. That imposes the requirements that $\mathbf{\Phi}(t_0)$ be 1) square and have 2) nonzero determinant. We see

$$|\mathbf{\Phi}(t)| = \begin{vmatrix} -e^{3t} & e^{6t} \\ e^{3t} & 2e^{6t} \end{vmatrix} - 2e^{9t} - e^{9t} = -3e^{9t},$$

which is nonzero *for all t*.

■

The determinant $W(t) = |\mathbf{\Phi}(t)|$ is called the **Wronksian**. The phenomenon we just witnessed in the case of System 1 is addressed by a theorem of Abel.

Theorem 1: Suppose the columns of a square matrix $\mathbf{\Phi}(t)$ all satisfy the homogeneous linear 1st-order system of DEs $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$ for a given matrix $\mathbf{A}(t)$. Then $W(t)$ is either the constant zero function for $\alpha < t < \beta$, or $W(t)$ is never zero there.

When the matrix $\mathbf{A}$ is constant, the interval of solution will be $-\infty < t < \infty$. To be sure you have the general solution, one should check that $\mathbf{\Phi}(t)$ has a nonzero Wronskian.

Exercise: Check that we have the general solution for Systems 2–4 above.

One result offers a blanket conclusion about the Wronskian being nonzero.

Theorem 2: Let $\mathbf{A}$ be an n-by-n real matrix.

(i) Eigenvectors corresponding to distinct eigenvalues are linearly independent.

(ii) [Corollary to (i)]. Suppose $\mathbf{A}$ has eigenvalue λ_1 with corresponding eigenvector $\mathbf{v}_1$, $\ldots$, λ_n with corresponding eigenvector $\mathbf{v}_n$, and every eigenvalue is **simple** (algebraic multiplicity is 1). Then the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is nonsingular.

(iii) [Corollary to (ii)]. If $\mathbf{A}$ has eigenpairs $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$, where each eigenvalue is simple, then the matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & \cdots & e^{\lambda_n t}\mathbf{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix},$$

has a nonzero Wronskian and is a fundamental matrix solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Further exercises

1. Consider the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$.

   (a) If the vector $\mathbf{x}$ has components $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then write $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has a system:

   $$x' = \underline{\hspace{3cm}} \qquad y' = \underline{\hspace{3cm}}$$

(b) Use the app at http://scofield.site/teaching/demos/eigenstuff.html to find the eigenvalues and corresponding basis eigenvectors. Then write a fundamental matrix $\mathbf{\Phi}(t)$.

(c) For the same matrix $\mathbf{A}$, solve the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(d) Use the app at http://scofield.site/teaching/demos/PhasePortrait2D.html to plot the solution in the phase plane (no t-axis, on axes for dependent vars) for the solution you just found. Explain why the trajectory looks the way it does.

(e) Repeat parts (c) and (d) this time with the initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

(f) Repeat parts (c) and (d) this time with the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

2. Consider the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

(a) Use the app at http://scofield.site/teaching/demos/eigenstuff.html to find the eigenvalues and corresponding basis eigenvectors. Then write a fundamental matrix $\mathbf{\Phi}(t)$.

(b) For the same matrix $\mathbf{A}$, solve the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

(c) Use the app at http://scofield.site/teaching/demos/PhasePortrait2D.html to plot the solution in the phase plane for the solution you just found. Explain why the trajectory looks the way it does.

(d) Repeat parts (b) and (c) this time with the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

(e) Repeat parts (b) and (c) this time with the initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

3. Consider the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Use the app at http://scofield.site/teaching/demos/eigenstuff.html to find the eigenvalues and corresponding basis eigenvectors. Then write a matrix $\mathbf{\Phi}(t)$.

(b) How do you know your answer to part (a) is a fundamental matrix?