Counting

Multiplication principle

Based on this fact: $|A \times B| = |A|\,|B|$, when both A and B are finite sets, and $A \times B$ is their Cartesian product.

Examples:

1. The number of ways to distribute a red, a blue, and a yellow ball amongst 7 bins.
2. The number of ways to distribute 20 non-leap-year birthdates so that no two people have the same one.

1. R B Y

$(1, 5, 3)$

$(3, 2, 2)$

7 options each $7^3 = 343$

2. $(\underset{365 \text{ opts}}{\_\_}, \underset{364 \text{ opts.}}{\_\_}, \underset{363 \text{ opt}}{\_\_}, \ldots, \underset{346 \text{ opts}}{\_\_})$

$(365)(364)\cdots(346) = \dfrac{365!}{345!}$

$\dfrac{365!}{(365-20)!}$

Bijection principle ~~principle~~

A function $f\colon A \to B$ is

- injective (one-to-one) if $f(x_1) = f(x_2)$ only when $x_1 = x_2$.
- surjective (onto) if for each $y \in B$ there is $x \in A$ with $f(x) = y$.
- bijective if both injective and surjective.

Note: If there is a function $f\colon A \to B$ that is bijective, then $|A| = |B|$.

Examples:

1. The numbers between 15 and 200 that are divisible by 6.
2. The number of ways write the number 15 as a sum $x + y + z$ of positive integers x, y, and z.

$f: \{1, 2, 3, \ldots, 31\} \xrightarrow{\text{bij.}} \{18, 24, \ldots, 198\}$

$\boxed{f(n) = 6(n+2)}$

$f(1) = 18$

$f(31) = 198$

$(1, 1, 13)$

$15 = 1 + 1 + 13$

$= 1 + 2 + 12$ $(1, 2, 12)$

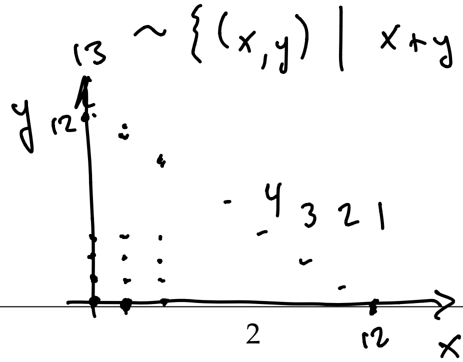
$= 1 + 12 + 2$ $(1, 12, 2)$

takes a while

$A = \{(x, y, z) \mid x + y + z = 15, \ x, y, z \text{ pos. ints}\}$

$\sim \{(x, y, z) \mid x + y + z = 12, \ x, y, z \text{ nonneg. ints.}\}$

$\sim \{(x, y) \mid x + y \le 12, \ x, y \text{ nonneg. ints.}\}$



$1 + 2 + 3 + \cdots + 13$

Complement rule

$|E| + |E^c| = |S|$

If the sample space S is finite and $E \subset S$, then $|E| = |S| - |E^c|$.

Incidentally, a corollary to this is that, if $P(E)$ represents the probability of event E, then

$$P(E) = 1 - P(E^c).$$

Example: Under the assumption that a year has 365 days (i.e., ignoring leap years) and that all days are equally likely, find the probability that, out of 20 randomly-selected people, no two have the same birthdate.

$P(\overbrace{\text{two or more of the 20 people have same birthday}}^{E})$

$= 1 - P(E^c)$

$P(E^c) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{346}{365} = \left(\frac{365!}{345!}\right) \cdot \frac{1}{365^{20}}$

Counting combinations

Binomial coefficients $\binom{n}{k}$ or ${}_nC_k$.

${}_nC_k = \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!}$

RRR red balls distributed to 7 bins, no bin gets two

$\left(\text{7 choices for 1st ball}\right)\left(\text{6 choices 2nd ball}\right)\left(\text{5 choices 3rd}\right) / 3!$

$= \frac{7 \cdot 6 \cdot 5}{3!} = \frac{7!}{(7-3)!\,3!}$

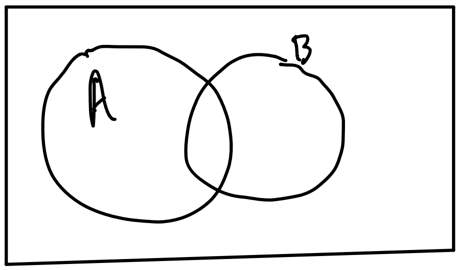
R, R, R

(1, 4, 5)

(4, 1, 5)

Inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Corollary

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

regular deck

$$P(2 \text{ hearts or } 2 \text{ face cards})$$

$$= P(2 \text{ hearts}) + P(2 \text{ face cards}) - P(2 \text{ heart face cards})$$

$$= \underbrace{\left(\frac{1}{4}\right)\left(\frac{12}{51}\right)}_{\text{conditional probability}} + \left(\frac{12}{52}\right)\left(\frac{11}{51}\right) - \left(\frac{3}{52}\right)\left(\frac{2}{51}\right)$$