1. We are going to need an eigenvector to go with $\lambda = 2$. To get it, we look for a basis of the null $(\mathbf{A} - 2\mathbf{I})$:

$$\mathbf{A} - 2\mathbf{I} \;=\; \begin{bmatrix} 6 & -6 & -6 \\ 0 & -6 & 3 \\ 6 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this, we glean that there is one basis eigenvector, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, v_3 can be taken as *free*, and we must have $v_1 = (3/2)v_3$, $v_2 = (1/2)v_3$; $\mathbf{v} = \langle 3, 1, 2 \rangle$ is such a (basis) eigenvector, and the solution this eigenpair generates is $e^{2t}\mathbf{v}$. To get the solutions arising from the nonreal eigenpairs, we must identify

$$\alpha = -1, \;\; \beta = 3, \;\; \mathbf{u} = \langle 2, 1, 2 \rangle, \qquad \text{and} \qquad \mathbf{w} = \langle 0, -1, 0 \rangle.$$

The corresponding solutions are

$$e^{-t}\left(\cos(3t) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2e^{-t}\cos(3t) \\ e^{-t}[\cos(3t) + \sin(3t)] \\ 2e^{-t}\cos(3t) \end{bmatrix} \quad \text{and} \quad e^{-t}\left(\sin(3t) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2e^{-t}\sin(3t) \\ e^{-t}[\sin(3t) - \cos(3t)] \\ 2e^{-t}\sin(3t) \end{bmatrix}.$$

Using our three solutions to build the fundamental matrix, we have general solution

$$\mathbf{x}(t) \;=\; \begin{bmatrix} 3e^{2t} & 2e^{-t}\cos(3t) & 2e^{-t}\sin(3t) \\ e^{2t} & e^{-t}[\cos(3t) + \sin(3t)] & e^{-t}[\sin(3t) - \cos(3t)] \\ 2e^{2t} & 2e^{-t}\cos(3t) & 2e^{-t}\sin(3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \;=\; \mathbf{\Phi}(t)\mathbf{c}.$$

Now, we seek to satisfy the IC:

$$\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} \;=\; \mathbf{x}(0) \;=\; \mathbf{\Phi}(0)\mathbf{c} \;=\; \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 3 & 2 & 0 & 4 \\ 1 & 1 & -1 & -4 \\ 2 & 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix},$$

giving us that $c_1 = 2$, $c_2 = -1$, $c_3 = 5$. Our solution, then, is

$$\mathbf{x}(t) \;=\; \mathbf{\Phi}(t) \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \;=\; \begin{bmatrix} 6e^{2t} - 2e^{-t}\cos(3t) + 10e^{-t}\sin(3t) \\ 2e^{2t} - 6e^{-t}\cos(3t) + 4e^{-t}\sin(3t) \\ 4e^{2t} - 2e^{-t}\cos(3t) + 10e^{-t}\sin(3t) \end{bmatrix}$$

2. (a) The eigenvalues are found by solving

$$0 = \begin{vmatrix} 7-\lambda & 16 \\ -1 & -1-\lambda \end{vmatrix} = (7-\lambda)(-1-\lambda)+16 = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2,$$

showing $\lambda = 3$ to have algebraic multiplicity 2. Solving for null $(\mathbf{A} - 3\mathbf{I})$

$$\left[\begin{array}{cc|c} 4 & 16 & 0 \\ -1 & -4 & 0 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since there is just one free column, the geometric multiplicity is 1, and $\lambda = 3$ is degenerate; a basis vector of its eigenspace is $\mathbf{v} = \langle 4, -1 \rangle$. So, along with $e^{3t}\mathbf{v}$, we seek a second solution of the form $e^{3t}(\mathbf{w} + t\mathbf{v})$, where $\mathbf{w}$ solves $(\mathbf{A} - 3\mathbf{I})\mathbf{w} = \mathbf{v}$:

$$\left[\begin{array}{cc|c} 4 & 16 & 4 \\ -1 & -4 & -1 \end{array}\right] \quad \text{which has RREF} \quad \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

We can use any vector $\mathbf{w} = \langle w_1, w_2 \rangle$ for which $w_1 + 4w_2 = 1$; $\mathbf{w} = \langle 1, 0 \rangle$ is such a vector. Thus, a fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{bmatrix} 4e^{3t} & (1+4t)e^{3t} \\ -e^{3t} & -te^{3t} \end{bmatrix}.$$

(b) Since the eigenvalues are real and both positive, the equilibrium at the origin is an **unstable node**.

3. (a) This problem is separable. We have

$$\begin{aligned} \frac{dy}{dt} = 2ty^2 \quad &\Rightarrow \quad \int -y^{-2}\,dy = -\int 2t\,dt \\ &\Rightarrow \quad y^{-1} = C - t^2 \\ &\Rightarrow \quad y(t) = \frac{1}{C - t^2} \qquad \text{(general solution)} \end{aligned}$$

(b) The problem is linear and nonhomogeneous, with $a(t) = 2t$, and $f(t) = 12t^3e^{t^2}$. The homogeneous solution is $C\varphi(t)$, where $\varphi(t) = e^{\int 2t\,dt} = e^{t^2}$. the variation of parameters formula gives

$$y_p(t) = e^{t^2} \int \frac{12t^3 e^{t^2}}{e^{t^2}}\,dt = e^{t^2}(3t^4).$$

So, the general solution is $\quad y(t) = y_h(t) + y_p(t) = ce^{t^2} + 3t^4e^{t^2}.$

4. Whether you do this by Cramer's Rule or actually inverting the matrix, you will need

$$\left|\mathbf{\Phi}(t)\right| = 2te^{4t} - e^{4t}\cdot(1+2t) = -e^{4t}.$$

Inverting $\mathbf{\Phi}(t)$, we have

$$\begin{aligned} \mathbf{\Phi}(t)^{-1}\mathbf{f}(t) &= \frac{1}{-e^{4t}} \begin{bmatrix} 2e^{2t} & -e^{2t} \\ -(1+2t)e^{2t} & te^{2t} \end{bmatrix} \begin{bmatrix} e^{3t} \\ 3t^2 \end{bmatrix} = \begin{bmatrix} -2e^{-2t} & e^{-2t} \\ (1+2t)e^{-2t} & -te^{-2t} \end{bmatrix} \begin{bmatrix} e^{3t} \\ 3t^2 \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} -2e^{-2t} \\ (1+2t)e^{-2t} \end{bmatrix} + 3t^2 \begin{bmatrix} e^{-2t} \\ -te^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^t + 3t^2e^{-2t} \\ (1+2t)e^t - 3t^3e^{-2t} \end{bmatrix} \end{aligned}$$

5. Salt flows in at a rate

$$\text{(concentration)} \cdot \text{(flow rate)} = (18)(22).$$

Whatever amount of salt $y(t)$ is in the tank at time t, the outflow takes the same form as product of concentration and flow rate, but with concentration $y/200$. Taken together, our initial value problem is

$$\frac{dy}{dt} \;=\; (18)(22) - \left(\frac{y}{200}\right)(22) \;=\; 396 - \frac{11}{100}y, \qquad y(0) = 6000.$$

6. (a) The DE is in normal form $y' = g(x, y)$, with $g(x, y) = x^2 - xy + y^2$. This $g(x, y)$, as well as its partial $\partial g/\partial y = -x + 2y$, are continuous throughout the xy-plane. In fact, we can take that entire plane as our open rectangle enclosing $(x_0, y_0) = (1, 1)$ in which g, $\partial g/\partial y$ are continuous. Thus, the IVP has a unique solution.

   (b) We have $x_0 = 1$, $y_0 = 1$, $g(x, y) = x^2 - xy + y^2$ (as in part (a)). Since $h = 0.5$, it requires 4 steps/iterations to reach $x = 3$.

$$\begin{array}{l|l}
y_1 = y_0 + hg(x_0, y_0) = 1 + (0.5)(1^2 - 1^2 + 1^2) = 1.5 & x_1 = x_0 + h = 1.5 \\
y_2 = y_1 + hg(x_1, y_1) = 1.5 + (0.5)(1.5^2 - 1.5^2 + 1.5^2) = 2.625 & x_2 = x_1 + h = 2.0 \\
y_3 = y_2 + hg(x_2, y_2) = 2.625 + (0.5)[2^2 - (2)(2.625) + 2.625^2] = 5.4453 & x_3 = x_2 + h = 2.5 \\
y_4 = y_3 + hg(x_3, y_3) = 5.4453 + (0.5)[2.5^2 - (2.5)(5.4453) + 5.4453^2] = 16.589 & x_4 = x_3 + h = 3.0
\end{array}$$

   So, $y(3) \approx 16.589$.