MATH 231, Worksheet
Date: May 8, 2020

Find the inverse Laplace transform for each function.

1. $F(s) = \dfrac{e^{-\pi s/2}}{s^2+9}$
2. $F(s) = \dfrac{1}{s^2(s^2+4)}$
3. $F(s) = \dfrac{s}{s^2+6s+11}$
4. $F(s) = e^{-s}\dfrac{s}{s^2+6s+11}$
5. $F(s) = \dfrac{1}{(s-1)^3} + \dfrac{1}{s^2+2s-8}$
6. $F(s) = e^{-2s}\dfrac{1}{(s-1)^3} + e^{-s}\dfrac{1}{s^2+2s-8}$

There is an overriding theme in the course, made possible by our focusing on linear problems:

Problem presented

1. Take homogeneous version
   - Solve it – called Null space, homogeneous soln.
     Span of a basis of solns.
   - Often freedoms in the soln.
2. Find particular soln.

Original is solved by sum of solns. found in 1 and 2.

Theme holds even in Ch. 5.

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0, \; y'(0) = y_1$$

$$\mathcal{L}\{\text{LHS}\} = \mathcal{L}\{\text{RHS}\}$$

$$a\,\mathcal{L}\{y''\} + b\,\mathcal{L}\{y'\} + c\,\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$a(s^2Y - sy_0 - y_1) + b(sY - y_0) + cY = F$$

$$(as^2 + bs + c)Y = asy_0 + ay_1 + by_0 + F$$

$$Y = \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c} + F(s)\cdot\frac{1}{as^2+bs+c}$$

$$\Rightarrow y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{asy_0 + ay_1 + by_0}{as^2+bs+c}\right\}}_{\text{soln. to}} + \underbrace{\mathcal{L}^{-1}\{F(s)H(s)\}}_{\text{soln. to}}$$

soln. to

$ay'' + by' + cy = 0$

$y(0) = y_0,\ y'(0) = y_1$

homogeneous soln. where freedoms have disappeared due to applying ICs

soln. to

$ay'' + by' + cy = f(t)$

w/ zero ICs

(like a particular soln.)

Certainly can solve another way, using Ch. 4 methods

May not be able to solve w/out Laplace transform

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1. $$\mathcal{L}^{-1}\left\{\underbrace{e^{-\frac{\pi}{2}s}}\cdot\frac{1}{s^2+9}\right\} = \boxed{u(t-\pi/2)\,\tfrac{1}{3}\sin(3(t-\pi/2))}$$

exponential on s-side

$\Rightarrow \mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$ applies

First $$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\cdot\frac{1}{3} = \frac{1}{3}\sin(3t)$$

Another approach arise from Convolution Thm

s-side multiplication

$$\mathcal{L}^{-1}\left\{ e^{-\frac{\pi}{2}s} \cdot \frac{1}{s^2+9} \right\} = \mathcal{L}^{-1}\left\{ \mathcal{L}\{\delta(t-\pi/2)\} \cdot \mathcal{L}\left\{\tfrac{1}{3}\sin(3t)\right\} \right\}$$

$$= \delta\left(t-\tfrac{\pi}{2}\right) * \tfrac{1}{3}\sin(3t)$$

$$= \int_0^t \underbrace{\delta\left(w-\tfrac{\pi}{2}\right)}_{\text{activated when } w=\pi/2} \tfrac{1}{3}\sin\left(3(t-w)\right)\,dw$$

$$= \boxed{\begin{cases} 0, & t < \pi/2 \\ \frac{1}{3}\sin\left(3\left(t-\frac{\pi}{2}\right)\right), & t > \pi/2 \end{cases}}$$

2. $\mathcal{L}^{-1}\left\{ \dfrac{1}{s^2(s^2+4)} \right\}$

$$\frac{1}{s^2(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$= \frac{A\cancel{s}}{s^{\cancel{2}}} + \frac{B}{s^2} + \frac{Cs}{s^2+4} + \frac{D}{s^2+4}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

How we teach the splitting via partial fractions

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = A\cdot\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + B\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + C\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$+ D\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

An option? Convolution?

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \underline{t}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \underline{\frac{1}{2}\sin(2t)}$$

we can assert

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+4}\right\} = \underset{t-w}{t} * \underset{w}{\tfrac{1}{2}\sin(2t)}$$

mult. on s-side

$$= \int_0^t (t-w)\cdot\frac{1}{2}\sin(2w)\,dw$$

3. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+6s+11}\right\}$

Since $s^2+6s+11=0$ has roots $s = \frac{-6}{2} \pm \frac{1}{2}\underbrace{\sqrt{36-44}}_{\text{nonreal}}$

say our denominator is $\underbrace{\text{irreducible}}$.

Complete the square

$$\frac{s}{s^2+6s+11} = \frac{s}{s^2+6s+9+2} = \frac{s}{(s+3)^2+2}$$

like Table entry for $e^{at}\cos(bt)$

w/ $a=-3$, $b=\sqrt{2}$

but numerator isn't what we need: should be $s+3$

Manufacture the numer. we want

$$\dots = \frac{s+3}{(s+3)^2+2} - \frac{\sqrt{2}}{(s+3)^2+2} \cdot \frac{3}{\sqrt{2}}$$

comes from

Ans. $e^{-3t}\cos(\sqrt{2}\,t) - \frac{3}{\sqrt{2}} e^{-3t}\sin(\sqrt{2}\,t)$

4. $\mathcal{L}^{-1}\left\{\underbrace{e^{-s}}_{\text{Exponential: On } t \text{ side expect shift and Heaviside term}} \cdot \underbrace{\frac{s}{s^2+6s+11}}_{\#3 \text{ we found } \mathcal{L}^{-1}\{\text{this}\}}\right\} = u(t-1)e^{-3(t-1)}\left[\cos\left(\sqrt{2}(t-1)\right) - \frac{3}{\sqrt{2}}\sin\left(\sqrt{2}(t-1)\right)\right]$

Exponential: On t side

expect shift and Heaviside term

#3 we found $\mathcal{L}^{-1}\{\text{this}\}$