Strong Induction and the Well-Ordering Principle

Mathematical induction can be expressed as the rule of inference

$$(P(a) \wedge (P(k) \to P(k+1))) \to \forall n \geqslant a,\ P(n).$$

Upon reflection, the portion $P(k) \to P(k+1)$, what we call the inductive step, is not the only thing that, coupled with the basis step which leads to the conclusion $\forall n P(n)$. Equally valid would be the conditional statement (containing a stronger hypothesis)

$$(P(i) \text{ is true for integers } a \leqslant i \leqslant k) \to P(k+1).$$

This leads to the following generalization of mathematical induction.

Definition 1 (Principle of Strong Mathematical Induction): Let $P(n)$ be a property that is defined for integers n, and let a, b be fixed integers with $a \leqslant b$. Suppose the following statements are true:

1. $P(a), P(a+1), \ldots, P(b)$ are all true (**basis step**).
2. For any integer $k \geqslant b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true (**inductive step**).

Then the statement "for all integers $n \geqslant a$, $P(n)$" is true.

The supposition that $P(i)$ is true for all integers i from a through k in number 2 above is called the **inductive hypothesis**.

To prove this is a valid rule of inference we rely on the **Well-Ordering principle**.

Definition 2 (Well-Ordering Principle): Suppose $A \subseteq \mathbb{N}$. Then A has a *smallest element*. That is, $\exists a \in A$ such that $\forall b \in A$, $(a \leqslant b)$.

Note that the set {positive real numbers} does not have a smallest element, but that this does not violate the well-ordering principle.

Generally speaking, anything provable via one of i) mathematical induction, ii) strong mathematical induction, or iii) the well-ordering principle, is provable with the other two. This is because all three statements are logically equivalent. However, sometimes one approach is easier than another. Some examples of statements and proof methods include:

1. Every integer $n \geqslant 2$ is a prime or can be written as the product of primes (use strong mathematical induction).
2. For any $n \geqslant 8$, n cents can be obtained using 3¢ and 5¢ coins (use strong mathematical induction)..
3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the 2$^{\text{nd}}$-order linear recursion relation

$$a_n = 6a_{n-1} - 5a_{n-2}, \quad \text{for } n \geqslant 2, \quad \text{with } a_0 = 0,\ a_1 = 4.$$

   Take $P(n)$: $a_n = 5^n - 1$. Then $\forall n \in \mathbb{N}$, $P(n)$ (use strong mathematical induction).
4. Use strong mathematical induction to show the product of n numbers requires $n-1$ multiplications, regardless of grouping.
5. A simple polygon with $n \geqslant 3$ sides can be trtiangulated into $n-2$ triangles (use strong mathematical induction, and the fact that every simple polygon with at least four sides has an interior diagonal).
6. Given any integer n and any positive integer d, there exist integers q and r such that $n = dq + r$ and $0 \leqslant r < d$ (use the well-ordering principle).
7. Given a strictly decreasing sequence of *positive* integers $r_1, r_1, r_2, \ldots$ (so $r_{i+1} < r_i$ for each i), the sequence terminates (use the well-ordering principle).