Examples of recurrences

1. $a_n = 5a_{n-1} - 6a_{n-2}$, $\quad a_0 = 2,\ a_1 = 3$

linear, 2nd degree, const.-coeff., homogeneous

Our approach $\quad a_n = r^n$

$$r^n = 5r^{n-1} - 6r^{n-2}$$

$$r^{n-2}(r^2 - 5r + 6) = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0 \longrightarrow r = 2, 3$$

$$a_n = \alpha_1 2^n + \alpha_2 3^n \quad (*)$$

Choose α_1, α_2 in order to satisfy IC

ICs
$$\boxed{\begin{aligned} 2 &= a_0 \\ 3 &= a_1 \end{aligned}} \begin{aligned} &= \alpha_1 \cdot 2^0 + \alpha_2 \cdot 3^0 = \alpha_1 + \alpha_2 \\ &= \alpha_1 2^1 + \alpha_2 3^1 = 2\alpha_1 + 3\alpha_2 \end{aligned}$$

2 eqns in α_1, α_2

$$\alpha_1 + \alpha_2 = 2 \longrightarrow \alpha_1 = 2 - \alpha_2$$

$$2\alpha_1 + 3\alpha_2 = 3$$

subst. into 2nd eqn.

$$2(2-\alpha_2) + 3\alpha_2 = 3$$

$$4 + \alpha_2 = 3 \longrightarrow \alpha_2 = -1$$

$$\alpha_1 = 3$$

Our recurrences soln. (also satisfying the ICs) is

$$\boxed{a_n = 3(2^n) - 3^n}$$

2. $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = 1$, $a_1 = 3$

linear, 2^{nd} degree, const coeff., homog.

Take $a_n = r^n$

$$r^n - 4r^{n-1} + 4r^{n-2} = 0$$

$$r^{n-2}(r^2 - 4r + 4) = 0$$

char. eqn. $r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0 \longrightarrow$ roots (repeated) $r = 2$.

Have seq.

2^n: $1, 2, 2^2, 2^3, \ldots$

$2^n \longleftarrow$ usually 2^{nd} seq. is different (different root)

Repeated root makes it impossible to combine

Bad news $\alpha_1 \cdot 2^n + \alpha_2 \cdot 2^n$ to satisfy the ICs.

Good News

It's as simple as adding n as a factor to the 2^{nd} sequence

1^{st} seq.

	$n=0$	$n=1$	$n=2$	$n=3$	
2^n:	1,	2,	2^2,	2^3,	...

2^{nd} seq

$n2^n$: 0, 2, 8, 24, ...

$\uparrow$

added factor of n

- a trick to produce a 2^{nd} sequence which also satisfies the recurrence (not necessarily the ICs)
- a different seq. than the first
- only applies if there is a repeated root underneath

Now use just as in other examples mixture

$$a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n \quad (\star)$$

Still must solve for α_1, α_2

$$1 = a_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 0$$
$$3 = a_1 = \alpha_1 \cdot 2 + \alpha_2 \cdot 1 \cdot 2^1$$

2 eqns. in α_1, α_2

$$\boxed{\begin{aligned} 1 &= \alpha_1 \\ 3 &= 2\alpha_1 + 2\alpha_2 \end{aligned}}$$

get $\alpha_1 = 1$

$\alpha_2 = 1/2$

Soln. is (★) rewritten w/ these alphas:

$$a_n = 2^n + \frac{1}{2} \cdot n \cdot 2^n = 2^n\left(1 + \frac{n}{2}\right).$$

8.3: Divide-and-conquer algorithms

A. Binary search algorithm

Have sorted list of size n

Divide list in 2 halves

make 2 comparisons

- key w/ the element at end of 1^{st} sublist (tells which sublist might contain key)
- see if sublist has 2 or more elements in it

Have list (still sorted, half size of original) to search

Say, for a list of size n,

$f(n)$ = # of comparisons to search a sorted list of size n using this alg. for your key

If n is even

$$f(n) = 2 + f(n/2)$$

divide-and-conquer recurrence relation

B. Fast-integer multiplication

Idea: Say I'm stuck using 4-digit decimal nos.

$$d = (d_3 d_2 d_1 d_0)_{10} = d_3 \times 10^3 + d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0$$

Task: multiply two such nos. d and $\delta = (\delta_3 \delta_2 \delta_1 \delta_0)$

Define $f(n) \triangleq$ # of operations required to multiply n-digit decimal nos.

Notice:

$$d = (d_3 \times 10^1 + d_2 \times 10^0) \times 10^2 + (d_1 \times 10^1 + d_0 \times 10^0)$$

$$= (d_3 d_2)_{10} \times 10^2 + (d_1 d_0)_{10}$$

$$\left(\begin{aligned} 2791 &= 27 \times 10^2 + 91 \\ &= (27) \times 10^2 + (91) \end{aligned}\right.$$

$$= D_1 \times 10^2 + D_0$$

Can do the same w/ δ: Define $\Delta_1 = (\delta_3 \delta_2)_{10}$, $\Delta_0 = (\delta_1 \delta_0)_{10}$

so $\delta = \Delta_1 \times 10^2 + \Delta_0$

Intent: Multiply

$$d\delta = (D_1 \cdot 10^2 + D_0)(\Delta_1 \cdot 10^2 + \Delta_0)$$

$$= D_1 \Delta_1 \cdot 10^4 + 10^2 (D_1 \Delta_0 + D_0 \Delta_1) + D_0 \Delta_0$$

$= $ same expression as last $+ 10^2 D_1 \Delta_1 - 10^2 D_1 \Delta_1 + 10^2 D_0 \Delta_0 - 10^2 D_0 \Delta_0$

you verify $= (10^4 + 10^2) D_1 \Delta_1 + 10^2 (D_1 - D_0)(\Delta_0 - \Delta_1) + (10^2 + 1) D_0 \Delta_0$

Note: Each of $D_1, D_0, \Delta_1, \Delta_0$ are 2-digit nos.)

If I multiply $d\delta$ in the manner prescribed here

I replace a single 4-digit multiplication problem

(w/ $f(4)$ operations) by 3 multiplications of

2 digit nos w/ $3 \cdot f(2)$ operations

also picking up 4 operations (add/subtracts)

So
$$f(4) = 3f(2) + 4$$

More generally

$$f(n) = 3f(n/2) + 4$$

Examples: Solving linear recurrences

1. $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 2$, $a_1 = 3$

Substitute $a_n = r^n$

	$n=0$	$n=1$	$n=2$	
2^n:	1,	2,	4,	...
3^n:	1,	3,	9,	...

$r^n = 5r^{n-1} - 6r^{n-2}$

$r^{n-2}(r^2 - 5r + 6) = 0$

characteristic eqn: $r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0 \longrightarrow r = 2, 3$

Need a mixture

$a_n = \alpha_1 2^n + \alpha_2 3^n \quad (\star)$

Get α_1, α_2 applying ICs

ICs

$2 = a_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 3^0 = \alpha_1 + \alpha_2$

$3 = a_1 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 3^1 = 2\alpha_1 + 3\alpha_2$

To finish, must solve 2 eqns. in unknowns α_1, α_2

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 2 \\ 2\alpha_1 + 3\alpha_2 &= 3 \end{aligned}\right\} \longrightarrow \begin{aligned} \alpha_1 &= 3 \\ \alpha_2 &= -1 \end{aligned}$$

Rewrite $(\star)$ as a final soln. using these

$$\boxed{a_n = 3 \cdot 2^n - 3^n}$$

2. (contains a new element)

$$a_n = 4a_{n-1} - 4a_{n-2}, \qquad a_0 = 1, \; a_1 = 3$$

Insert $a_n = r^n$

$$r^n - 4r^{n-1} + 4r^{n-2} = 0$$

$$r^{n-2}(r^2 - 4r + 4) = 0$$

char. eqn. $r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0 \longrightarrow$ repeated root $r = 2$

(New!)

In other examples

		$n=0$	$n=1$	$n=2$	$n=3$	...
root 1	r_1 :	1	r_1	r_1^2	r_1^3	...
root 2	r_2 :	1	r_2	r_2^2	r_2^3	...

This time

$2:\quad 1 \quad 2 \quad 2^2 \quad 2^3 \quad \ldots$

identical $\longrightarrow$ $2:\quad 1 \quad 2 \quad 2^2 \quad 2^3$

Bad news: Repeated root $\longrightarrow$ impossible to combine

$$\alpha_1 \Big(\underbrace{\text{first}}_{r_1^n} \Big) + \alpha_2 \left(r_2^n \right)$$

and satisfy the ICs

Good news: ready replacement for that 2^{nd} sequence

— i.e. repeated

$$2^n : 2^0, 2^1, 2^2 \ldots$$

Replace w/

$$n 2^n : 0 \cdot 2^0, 1 \cdot 2^1, 2 \cdot 2^2, 3 \cdot 2^3, \ldots$$

Note: This sort of replacement for a soln. seq. does not work in general, but it does when a repeated root underlies things.

Plan of action:

$$a_n = \alpha_1 2^n + \alpha_2 n 2^n \quad (\cancel{A})$$

choosing α_1, α_2 via ICs as before.

ICs

$$1 = a_0 = \alpha_1 + \alpha_2 \cdot 0 \longrightarrow \alpha_1 = 1$$

$$3 = a_1 = \qquad 2\alpha_1 + 2\alpha_2 = 3$$

get $\alpha_1 = 1, \alpha_2 = 1/2$

Soln.

$$a_n = 2^n + \frac{1}{2} n \cdot 2^n = 2^n \left(1 + \frac{n}{2}\right)$$

8.3 Divide-and-conquer algorithms.

A. Binary search

Sorted list of size n

Write: $f(n)$ = # of comparisons required in searching this list of size n for a key using binary search.

$$\underbrace{f(n) = 2 + f(n/2)}_{\text{divide-and-conq. recurrence relation}} \qquad \text{if } n \text{ is even}$$

$$f_n = 2 + f_{n/2}$$

B. Fast-integer multiplication

Restrained to 4-digit decimal nos

$$d = (d_3 d_2 d_1 d_0)_{10} = d_3 \times 10^3 + d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0$$

$$= \underbrace{(d_3 d_2)_{10}}_{D_1} \times 10^2 + \overbrace{(d_1 d_0)_{10}}^{D_0}$$

like saying $4279 = (42) \times 10^2 + 79$

$$= D_1 \times 10^2 + D_0.$$

Task: Multiply 2 4-digit nos. $d = (d_3 d_2 d_1 d_0)_{10}$ and $\delta = (\delta_3 \delta_2 \delta_1 \delta_0)_{10}$

Similarly,
$$\delta = \Delta_1 \times 10^2 + \Delta_0$$

$$d\delta = (D_1 \cdot 10^2 + D_0)(\Delta_1 \cdot 10^2 + \Delta_0)$$

Note: $D_1, D_0, \Delta_1, \Delta_0$ are all 2-digit nos.

$$= D_1 \Delta_1 \cdot 10^4 + 10^2 (D_1 \Delta_0 + D_0 \Delta_1) + D_0 \Delta_0$$

$$= \text{same as above} + 10^2 D_1 \Delta_1 - 10^2 D_1 \Delta_1 + 10^2 D_0 \Delta_0 - 10^2 D_0 \Delta_0$$

you verify $$= (10^4 + 10^2) D_1 \Delta_1 + 10^2 (D_1 - D_0)(\Delta_0 - \Delta_1) + (10^2 + 1) D_0 \Delta_0$$

Why?!!!!

Let $f(n)$ = # of operations required to multiply 2 n-digit decimal nos.

Our work shows

$$f(4) = 3 f(2) + \underbrace{4}_{\text{adds/subtracts}}$$

More generally, if n is even

$$f(n) = 3 f(n/2) + 4$$ divide-and-conq. recurrence equ.

$$(10^2 + 1) \cdot 59 = (5900) + 59$$

$$= 5959$$

$$(2^4 + 2^2 + 1) \cdot (10)_2$$

$$= (100000)_2 + (1000)_2 + (10)_2$$

$$= (01010)_2$$