MATH 162: Calculus II

Framework for Fri., Mar. 16

Vector Functions and Integral Calculus

Today's Goal: To understand how to integrate vector functions.

Indefinite integrals

Recall: For scalar functions $f(t)$,

- A function F is called an antiderivative of f on the interval I if $F'(t) = f(t)$ at each point $t \in I$.
- Given any antiderivative F of f and any constant C, $F(t) + C$ is also an antiderivative of f.
- The indefinite integral
$$\int f(t)\,dt$$
stands for the set of all antiderivatives of f.

For a vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

- An antiderivative of $\mathbf{r}(t)$ on an interval I is another vector function $\mathbf{R}(t)$ for which $\mathbf{R}'(t) = \mathbf{r}(t)$ at each $t \in I$.
- Finding an antiderivative of $\mathbf{r}(t)$ comes down to finding antiderivatives for its component functions. That is, if F, G and H are antiderivatives of f, g and h on the interval I, then
$$\mathbf{R}(t) \;=\; F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}$$
is an antiderivative of $\mathbf{r}(t)$.
- Given any antiderivative $\mathbf{R}(t)$ of $\mathbf{r}(t)$ and any constant vector $\mathbf{C}$, $\mathbf{R}(t) + \mathbf{C}$ is also an antiderivative of $\mathbf{r}(t)$.
- The symbol
$$\int \mathbf{R}(t)\,dt$$
stands for the set of all antiderivatives of $\mathbf{r}$.

Example: $\displaystyle\int \left[\left(\frac{1}{1+t^2}\right)\mathbf{i} + (\sin t \cos t)\,\mathbf{j} + \left(\frac{t}{\sqrt{1+3t^2}}\right)\mathbf{k}\right] dt$

Definite Integrals

Definition: Suppose that the component functions of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are all integrable over the interval $[a, b]$ (true, say, if each of f, g and h are continuous over that interval). Then we say the vector function $\mathbf{r}(t)$ is *integrable over* $[a, b]$, and define its integral to be

$$\int_a^b \mathbf{r}(t)\,dt \ := \ \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

Since the fundamental theorem of calculus holds for the components of $\mathbf{r}(t)$, it holds for $\mathbf{r}(t)$ as well. Here we state just part II.

Theorem: If $\mathbf{r}(t)$ is continuous at each point of the interval $[a, b]$ and if $\mathbf{R}$ is any antiderivative of $\mathbf{r}$ on $[a, b]$, then

$$\int_a^b \mathbf{r}(t)\,dt \ = \ \mathbf{R}(b) - \mathbf{R}(a).$$

Application: Projectile motion

For projectiles near enough to sea level, we think of them as having constant acceleration $\mathbf{a}(t) = -g\mathbf{k}$, where g has the value 9.8 m/s^2 or 32 ft/s^2. Since velocity is an antiderivative of acceleration, we may write

$$\int_0^t \mathbf{a}\,d\tau \ = \ -gt\mathbf{k} \ = \ \mathbf{v}(t) - \mathbf{v}(0),$$

or, abbreviating $\mathbf{v}(0)$ by $\mathbf{v}_0$,

$$\mathbf{v}(t) \ = \ \mathbf{v}_0 - gt\mathbf{k}.$$

The position $\mathbf{r}(t)$ is an antiderivative of velocity, so

$$\int_0^t \mathbf{v}(\tau)\,d\tau \ = \ t\mathbf{v}_0 - \frac{1}{2}gt^2\mathbf{k} \ = \ \mathbf{r}(t) - \mathbf{r}(0),$$

or

$$\mathbf{r}(t) \ = \ \mathbf{r}_0 + t\mathbf{v}_0 - \frac{1}{2}gt^2\mathbf{k},$$

where $\mathbf{r}_0 := \mathbf{r}(0)$ is the initial position.