MATH 162: Calculus II

Framework for Fri., Feb. 16

Absolute and Conditional Convergence

p-Series Results Revisited

- Results we have shown: The series whose terms are all positive

$$\sum_{n=1}^{\infty} n^{-p} \;=\; 1 + 2^{-p} + 3^{-p} + 4^{-p} + \cdots \tag{1}$$

converges for $p > 1$, and diverges for $p \leq 1$. The series with alternating signs

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-p} \;=\; 1 - 2^{-p} + 3^{-p} - 4^{-p} + \cdots \tag{2}$$

converges for $p > 0$, and diverges for $p \leq 0$.

- The above results apply narrowly—only to series in the forms (1) and (2) respectively. Thus, nothing we have learned tells us whether

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$$

converges.

- The "borderline" case of (1), the one with $p = 1$,

$$\sum_{n=1}^{\infty} n^{-1} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \tag{3}$$

is divergent, and has been named the *harmonic series*.

- When the convergence/divergence of a series $\sum a_n$ is known, then the convergence/divergence of certain modified forms of that series can be known as well. In particular,

  - Any nonzero multiple of a series that converges (resp. diverges) will also converge (resp. diverge). Thus,

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \cdots \;=\; \frac{1}{3}\sum_{n=1}^{\infty} n^{-1},$$

  diverges, being a multiple of the harmonic series (3).

– Suppose $\sum a_n$ is a series whose convergence/divergence is known. Any series which has the same "tail" as that of $\sum a_n$ will converge (resp. diverge) based on what $\sum a_n$ does. For instance, since we know

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1/2} \;=\; 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

converges, we can conclude

$$\sum_{n=4}^{\infty}(-1)^{n-1}n^{-1/2} \;=\; -\frac{1}{2} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

and

$$b_1 + b_2 + \cdots + b_{50} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

converge as well. Here $b_1, \ldots, b_{50}$ is any arbitrary list of 50 numbers. The important thing is not the values of these b_j's, but that there are only finitely many (in this case, 50) of them.

Absolute and Conditional Convergence

Definition: Let $\sum a_n$ be a convergent series. If the corresponding series $\sum |a_n|$ in which every term has been made positive diverges, then the original series $\sum a_n$ is said to be *conditionally convergent*.

Example: The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^{-1} = 1 - 1/2 + 1/3 - 1/4 + \cdots$ is conditionally convergent.

Definition: Let $\sum a_n$ be a given series (i.e., one for which the values of the terms a_j are known). If the corresponding series $\sum |a_n|$ with all positive terms converges, then the original series $\sum a_n$ is said to be *absolutely convergent*.

Theorem (Absolute Convergence Test): All absolutely convergent series are convergent.

Example: The series

$$\frac{11}{3} + \frac{11}{6} - \frac{11}{12} + \frac{11}{24} - \frac{11}{48} - \frac{11}{96} - \frac{11}{192} + \cdots$$

is absolutely convergent, since

$$\frac{11}{3} + \frac{11}{6} + \frac{11}{12} + \frac{11}{24} + \frac{11}{48} + \cdots \;=\; \sum_{n=0}^{\infty}\left(\frac{11}{3}\right)\left(\frac{1}{2}\right)^n$$

converges (being geometric, with $r = 1/2$). By the absolute convergence test, the original series converges as well.