MATH 162: Calculus II
Framework for Thurs., Feb. 15
Geometric Series and Series Introduction

Geometric Series

- Form of series under this classification

$$a + ar + ar^2 + \cdots + ar^n + \cdots \;=\; \sum_{n=0}^{\infty} ar^n,$$

  a, r nonzero constants

- Zeno's paradox about crossing a room
  - If L is length of room, then he is looking at adding up distances

$$L \cdot \left(\frac{1}{2}\right) + L \cdot \left(\frac{1}{2}\right)^2 L \cdot \left(\frac{1}{2}\right)^3 + \cdots \;=\; \sum_{n=0}^{\infty} ar^n,$$

    with $a = L/2$, $r = 1/2$.
  - Evidence that (some) geometric series converge

- Partial sums s_n
  - Define in customary way:

$$s_1 = a,\; s_2 = a + ar,\; s_3 = a + ar + ar^2, \quad \text{etc.}$$

  - nth partial sum has nice closed-form formula:

$$s_n \;=\; \begin{cases} \dfrac{a(1-r^n)}{1-r}, & \text{when } r \neq 1, \\ na, & \text{when } r = 1. \end{cases}$$

  - **Main Result**: Geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ converges to $\dfrac{a}{1-r}$ when $|r| < 1$, and diverges otherwise.

- Note the *divergence* when $|r| = 1$:

  $r = 1: \quad \displaystyle\sum_{n=0}^{\infty} a = a + a + \cdots + a + \cdots \qquad$ (divergent)

  $r = -1: \quad \displaystyle\sum_{n=0}^{\infty} a = a - a + a - a + a - a + \cdots \qquad$ (divergent)

Remarks concerning infinite series (general, not just geometric ones) $\sum_{n=1}^{\infty} a_n$:

1. Convergence relies on the partial sums $s_n := a_1 + \cdots + a_n$ approaching a limit as $n \to \infty$

2. Assessing the limit of partial sums directly requires a nice closed-form expression for s_n. Such an expression exists only in rare cases, such as the following examples we've already done

   Geometric series: $\displaystyle\sum_{n=0}^{\infty} ar^n$

   "Telescoping series": $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

3. When no closed-form expression for s_n is available, determining if limit exists is usually more difficult.

   Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}, \ p \geq 0$

4. Systematic tests can help

   - Some of the tests that have been developed ('*' indicate ones we will study)
     - *n**th-term test for divergence** (p. 519)
     - **integral test** (p. 525): formalization of the approach we used to determine which p-series $\sum_{n=1}^{\infty} n^{-p}$ converge/diverge
     - **direct comparison test** (p. 529): practically a restatement of the one of the same name for improper integrals
     - **limit comparison test** (p. 530): did not do comparable result for improper integrals
     - ***ratio test** (p. 533)
     - **root test** (p. 535)
     - **alternating series test** (p. 538): formalization of the approach we used to show $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges for $p \geq 0$
     - ***absolute convergence test** (p. 540)
   - Must be cognizant of
     - the situations in which a test may be applied
     - what conclusions may and may not be drawn from such tests