

Linear 1st-order homogeneous systems with nonreal eigenvalues

There are certain things we build on:

- **Euler's Formula:** Given a real number θ , and $i = \sqrt{-1}$, it says $e^{i\theta} = \cos \theta + i \sin \theta$. A corollary to it is that $e^{-i\theta} = \cos \theta - i \sin \theta$, making $e^{i\theta}$ and $e^{-i\theta}$ complex conjugates. For an explanation of why this amazing formula holds, and secondarily to justify in part your study of Maclaurin series in MATH 172, watch

<https://drive.google.com/file/d/1a7x1QIdNYGis6np3V9rXkq8xYh0wE3yD/view?usp=sharing>

Here are that video's finished notes

<http://scofield.site/courses/m231/coronaDays/complexEvalStuff/eulersFormula.jpg>

- When a matrix \mathbf{A} has real entries by a nonreal eigenvalue $\alpha + i\beta$, where α, β are real numbers, there will be at least one corresponding eigenvector $\mathbf{u} + i\mathbf{v}$, where \mathbf{u}, \mathbf{v} have real entries. Correspondingly, the complex conjugate $\alpha - i\beta$ is also an eigenvalue of \mathbf{A} , and has $\mathbf{u} - i\mathbf{v}$ as an eigenvector. For example, if

$$-3 + 2i \quad \text{is an eigenvalue with eigenvector} \quad \begin{bmatrix} 2 - 3i \\ 1 - i \\ 3i \end{bmatrix},$$

then we can identify

$$\alpha = -3, \quad \beta = 2, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix},$$

and conclude that

$$\alpha - i\beta = -3 - 2i \quad \text{is also an eigenvalue with eigenvector} \quad \mathbf{u} - i\mathbf{v} = \begin{bmatrix} 2 + 3i \\ 1 + i \\ -3i \end{bmatrix}.$$

- We have demonstrated and made use of the fact that, if the matrix \mathbf{A} has eigenpair (λ, \mathbf{v}) , then $e^{\lambda t}\mathbf{v}$ is a solution of the homogeneous linear 1st-order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. But it is not possible to make physical sense of such a solution

$$e^{(\alpha+i\beta)t}(\mathbf{u} + i\mathbf{v}) \quad \text{and its counterpart} \quad e^{(\alpha-i\beta)t}(\mathbf{u} - i\mathbf{v}),$$

when we are talking about nonreal eigenpairs of \mathbf{A} . In

<https://drive.google.com/file/d/1e1JcAEDA807WxB6JigSH3NYf5YgcY90C/view?usp=sharing>

I justify why it is reasonable and valid to trade out those nonreal solutions for these *real* substitutes:

$$e^{\alpha t} [\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}] \quad \text{and} \quad e^{\alpha t} [\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}].$$

Here are that video's finished notes

<http://scofield.site/courses/m231/coronaDays/complexEvalStuff/tradeNonrealSolnsForReal.jpg>

Some examples:

$$1. \frac{d}{dt}\mathbf{x} = \begin{bmatrix} -21 & -30 & -32 \\ -4 & -7 & -7 \\ 24 & 30 & 35 \end{bmatrix}$$

The details of solving this system are worked out in four videos:

https://drive.google.com/file/d/1xyn44wYgDn6_wlVggm1Z7-zYeQ_wwbjo/view?usp=sharing

<https://drive.google.com/file/d/1Jh6WfH3gxzkwmDnYT74a510SeZZ7idFM/view?usp=sharing>

https://drive.google.com/file/d/1s5bfHJw6M0g0004CUApkWrQq_BSQJiNY/view?usp=sharing

<https://drive.google.com/file/d/1f0stxjsy8leV-mo3vGAcItDQZCxiXY7t/view?usp=sharing>

These videos will be played during class. Here are the three pages of end notes:

- page 1: http://scofield.site/courses/m231/coronaDays/complexEvalStuff/A3by3with2complexEvals_p1.jpg
- page 2: http://scofield.site/courses/m231/coronaDays/complexEvalStuff/A3by3with2complexEvals_p2.jpg
- page 3: http://scofield.site/courses/m231/coronaDays/complexEvalStuff/A3by3with2complexEvals_p3.jpg

$$2. \frac{d}{dt}\mathbf{x} = \begin{bmatrix} -5 & -10 \\ 5 & 9 \end{bmatrix}$$

A video that finds the solution to this system is found at

https://drive.google.com/file/d/17bM_j4Zoc8kCpXKnQRxurSaLlm_kdptR/view?usp=sharing

For a snapshot of the resulting notes, consult page 3 above.