Definition 1: Suppose a is a number in an open interval I on which two functions f, g are defined. We say f and g **agree to order** n **at** $x = a$ precisely when

$$f(a) = g(a), \quad f'(a) = g'(a), \quad f''(a) = g''(a), \quad \ldots, \quad f^{(n)}(a) = g^{(n)}(a).$$

Initial questions to investigate:

1. To what order n do the functions $f(x) = \sin x$ and $g(x) = x$ agree at $x = 0$?
2. If g is to be a 2^{nd}-degree polynomial (i.e., $g(x) = ax^2 + bx + c$), determine its coefficients so that it agrees to order 2 with $f(x) = \sin x$ at $x = 0$. Is the answer a surprise?
3. If g is to be a 2^{nd}-degree polynomial (i.e., $g(x) = ax^2 + bx + c$), determine its coefficients so that it agrees to order 2 with $f(x) = e^x$ at $x = 0$.
4. If g is to be a 2^{nd}-degree polynomial (i.e., $g(x) = ax^2 + bx + c$), determine its coefficients so that it agrees to order 2 with $f(x) = \frac{1}{1-x}$ at $x = 2$.
5. If g is to be a 3^{rd}-degree polynomial, determine an expression for g so that it agrees to order 3 with $f(x) = x^2 + 3x - 7$ at $x = 2$.

Some facts: Assume that f is continuous and has continuous $1^{\text{st}}, 2^{\text{nd}}, \ldots, n^{\text{th}}$ derivatives in an open interval containing $x = a$.

- There is exactly one polynomial function of degree at most n which agrees with f to order n at $x = a$. This polynomial is called n^{th} **Taylor polynomial of** f **centered at** $x = a$ and, in context where the function f and location a are understood, it will usually be denoted $T_n(x)$.
- There is this recipe/formula for the n^{th} Taylor polynomail of f centered at $x = a$:
$$T_n(x) \;=\; \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \;=\; \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$
- By inspection, for a given f and location a,
$$\begin{aligned}
T_1(x) &= T_0(x) + f'(a)(x-a) \\
T_2(x) &= T_1(x) + \frac{f''(a)}{2!}(x-a)^2 \\
T_3(x) &= T_2(x) + \frac{f'''(a)}{3!}(x-a)^3 \\
&\vdots \\
T_n(x) &= T_{n-1}(x) + \frac{f(n)(a)}{n!}(x-a)^n.
\end{aligned}$$