Math 231, Thu 18-Feb-2021 -- Thu 18-Feb-2021 Differential Equations and Linear Algebra Spring 2020

Thursday, February 18th 2021

Due:: WW LAConcepts1.03-1.08 due at 11 pm

Other calendar items

Thursday, February 18th 2021

Wk 3, Th

Topic:: Determinants
Topic:: Cramer's rule

Read:: ODELA 1.9-1.10

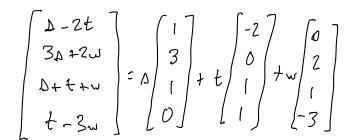
https://pad.disroot.org/p/m231-17feb2021

Exercise:

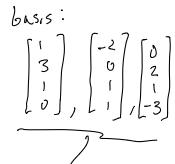
- (a) Find a basis for the collection of vectors <s 2t, 3s + 2w, s + t + w, t-3w>
- (b) We called what we found a basis, which presumes this collection is a subspace of something. What larger space does it reside in? How do we know it is a subspace?
- (c) Can you write a matrix A whose column space corresponds to this collection of vectors?
- (d) Can our basis be "enhanced" in order to create a basis for R⁴?
- Q2: Suppose b is a nonzero vector, and Ax = b is consistent. Do the solutions of Ax = b form a subspace of R^n ?
 - Q3: (If there is time)

 Visit the website https://pad.disroot.org/p/m231-17feb2021

 and write, as a class, things we can conclude in each setting.



1,t,weR



```
To consider
```

```
- linear independence of functions on an interval
    1, \sin^2 x, \cos^2 x are L.D.
    specification of interval is important!
      Example: x and |x| on (0, \inf y) vs. (-\inf y, \inf y)
    Test:
      Form n-by-n matrix, fns along top row, derivs. up to order (n-1) down.
      If at some t\in I, A(t) has no free col., then fns are L.I. on I.
Determinants
 - 2-by-2: | a b |
            \mid c \mid d \mid = ad - bc
  indicates a singular matrix when zero (parallel lines)
 - What a determinant can determine
 notation |A|, det(A)
 - extending to other square matrices using cofactor expansion
    recursive definition
    same result whether you expand along one row/col or another
    may lead you to choose row/col with most zeros
 - Theorem: Determinant of a triangular A is product of its diagonal elements.
 - Theorem: Determinants and EROs. If B arises from A due to
     a row swap, then |B| = -|A|.
     a row of B is r times a row of A, then |B| = r|A|.
     a row of B is one row of A plus r times another row of A, then |B| = |A|.
 - Theorem: For n-by-n matrix A, TFAE:
     A is nonsingular
     det(A) \ne 0.
     cols of A form a basis for R<sup>n</sup>
     RREF(A) = I
     Every b of R<sup>n</sup> is in col(A)
     null(A) is trivial
     rank(A) = n
     nullity(A) = 0
 - Cramer's Rule
```

Cramer's Rule

Cramer's rule provides a method for solving a system of linear algebraic equations for which the associated matrix problem $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a coefficient matrix which is *nonsingular*. It is of no use if this criterion is not met and, considering the effectiveness of algorithms we have learned already for solving such a system (inversion of the matrix \mathbf{A} , and Gaussian elimination, specifically), it is not clear why we need yet another method. Nevertheless, it is a tool (some) people use, and should be recognized/understood by you when you run across it. We will describe the method, but not explain why it works, as this would require a better understanding of determinants than our time affords.

So, let us assume the n-by-n matrix \mathbf{A} is nonsingular, that \mathbf{b} is a known vector in \mathbb{R}^n , and that we wish to solve the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ for an unknown (unique) vector $\mathbf{x} \in \mathbb{R}^n$. Cramer's rule requires the construction of matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, where each \mathbf{A}_j , $1 \le j \le n$ is built from the original \mathbf{A} and \mathbf{b} . These are constructed as follows: the jth column of \mathbf{A} is replaced by \mathbf{b} to form \mathbf{A}_j .

Example 1: Construction of A_1 , A_2 , A_3 when A is 3-by-3

Suppose $\mathbf{A} = (a_{ij})$ is a 3-by-3 matrix, and $\mathbf{b} = (b_i)$, then

$$\mathbf{A}_{1} = \begin{pmatrix} b_{1} & a_{12} & a_{13} \\ b_{2} & a_{22} & a_{23} \\ b_{3} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{A}_{2} = \begin{pmatrix} a_{11} & b_{1} & a_{13} \\ a_{21} & b_{2} & a_{23} \\ a_{31} & b_{3} & a_{33} \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_{3} = \begin{pmatrix} a_{11} & a_{12} & b_{1} \\ a_{21} & a_{22} & b_{2} \\ a_{31} & a_{32} & b_{3} \end{pmatrix}.$$

Armed with these \mathbf{A}_j , $1 \le j \le n$, the solution vector $\mathbf{x} = (x_1, \dots, x_n)$ has its j^{th} component given by

$$x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}, \qquad j = 1, 2, \dots, n. \tag{1}$$

It should be clear from this formula why it is necessary that **A** be nonsingular.

Example 2:

Use Cramer's rule to solve the system of equations

$$x + 3y + z - w = -9$$

$$2x + y - 3z + 2w = 51$$

$$x + 4y + 2w = 31$$

$$-x + y + z - 3w = -43$$

3

Here, **A** and **b** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -9 \\ 51 \\ 31 \\ -43 \end{pmatrix}, \quad \text{so} \quad |\mathbf{A}| = \begin{vmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{vmatrix} = -46.$$

Thus,

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|} \begin{vmatrix} -9 & 3 & 1 & -1 \\ 51 & 1 & -3 & 2 \\ 31 & 4 & 0 & 2 \\ -43 & 1 & 1 & -3 \end{vmatrix} = \frac{-230}{-46} = 5,$$

$$y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 1 & -9 & 1 & -1 \\ 2 & 51 & -3 & 2 \\ 1 & 31 & 0 & 2 \\ -1 & -43 & 1 & -3 \end{vmatrix} = \frac{-46}{-46} = 1,$$

$$z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 1 & 3 & -9 & -1 \\ 2 & 1 & 51 & 2 \\ 1 & 4 & 31 & 2 \\ -1 & 1 & -43 & -3 \end{vmatrix} = \frac{276}{-46} = -6,$$

$$w = \frac{|\mathbf{A}_4|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 1 & 3 & 1 & -9 \\ 2 & 1 & -3 & 51 \\ 1 & 4 & 0 & 31 \\ -1 & 1 & 1 & -43 \end{vmatrix} = \frac{-506}{-46} = 11,$$

yielding the solution $\mathbf{x} = (x, y, z, w) = (5, 1, -6, 11)$.

Determinants

) eterminants . Know how for a 2-by-2
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
: $det = ad-bc$

· Notation Jet (A) /A/

· Use: determines whether a metrix singular or not 1 determinant >> Singular = no inversa nonzero determinant (> nonsingalor = invertible

. det (A) makes sense only for square matrices

Example:

4x4 metrix A det.

No have, above, "expensed our det(A) in correctors along 4th new"

that is,

$$det(A) = a_{11} + a_{12} + a_{12} + a_{13} + a_{14} + a_{15}$$
Surprising (?)

Expansion if det(A) in correctors along

any row or

any column

always gives the same final number/result.

Criven this, expensely along col. 3

$$det(A) = a_{15} C_{15} + a_{23} C_{23} + a_{35} C_{33} + a_{35} C_{34}$$

$$= 0 \left(-1\right) \left(-1\right)$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = (3)(-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 0 + (-1)(-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$=(3)(1)(3)-(-3)=12$$

$$0$$
 riginal $det(A) = (-4)(4) - 12 = -28$

$$A = \begin{bmatrix} -5 & 3 \\ \sim 6 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -2 & 5 \end{bmatrix}$$

Which
$$\vec{b} \in \mathbb{R}^3$$
 make the vector ega. $A\vec{x} = \vec{b}$ consistent
D which \vec{b} can be written as linear combination of $\begin{bmatrix} 2\\1\\-2 \end{bmatrix}$, $\begin{bmatrix} 4\\5\\ \end{bmatrix}$ — that is, so $\vec{b} = x$, $\begin{bmatrix} 2\\1\\-2 \end{bmatrix} + x_2 \begin{bmatrix} 4\\5\\ \end{bmatrix}$

3) Describe the Col(A).

All are asking the some thing, using different language.

$$A = \begin{bmatrix} 4 & 2 & 1 & -2 & -1 \\ 2 & 1 & 3 & -2 & 3 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Col(A) is spanned by $\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -7 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$