Divide and Conquer

Suppose $f(n)$ is the count of operations required, using a certain algorithm, to perform a task of size n (n is a measure on the input to the algorithm). If f satisfies a recurrence relation of the form

$$f(n) \;=\; af(n/b) + g(n), \tag{1}$$

with a, $b > 0$, called a **divide-and-conquer** recurrence relation, then the algorithm is said to be a **divide-and-conquer** algorithm.

Example 1:

1. **Binary search**. Take $f(n)$ to be the number of comparisons required to find a search key in an ordered list of length n using the binary search algorithm. (See Section 2.1). Then $f(n) = f(n/2) + 2$.
2. **Fast integer multiplication**. Let $f(n)$ be the count of bit operations required to multiply two $(2n)$-bit integers. Let a, b be two such integers with binary representations
$$a = (a_{2n-1}\dots a_2a_1a_0)_2 \qquad \text{and} \qquad b = (b_{2n-1}\dots b_2b_1b_0)_2,$$
and write $a = A_0 + 2^nA_1$, $a = B_0 + 2^nB_1$, so that each of A_0, A_1, B_0, B_1 are n-bit numbers; note that
$$A_0 = (a_{n-1}\dots a_2a_1a_0)_2 \qquad \text{and} \qquad A_1 = (a_{2n-1}\dots a_{n+2}a_{n+1}a_n)_2,$$
with similar relationships between the binary represenions for B_0, B_1 and b. By writing
$$\begin{aligned} ab &= (A_0 + 2^nA_1)(B_0 + 2^nB_1) \;=\; 2^{2n}A_1B_1 + 2^n(A_0B_1 + A_1B_0) + A_0B_0 \\ &= (2^{2n} + 2^n)A_1B_1 - 2^nA_1B_1 + 2^n(A_0B_1 + A_1B_0) - 2^nA_0B_0 + (2^n + 1)A_0B_0 \\ &= (2^{2n} + 2^n)A_1B_1 + 2^n(A_1 - A_0)(B_0 - B_1) + (2^n + 1)A_0B_0 \end{aligned}$$
and interpreting multiplications like 2^kC as a *sliding* of bits k places to the left (rather than actual multiplication), we see that the problem of multiplying two $(2n)$-bit integers a and b has been replaced with three multiplications involving n-bit integers, along with several slidings, subtractions and additions, the count of which is proportional to n. Thus,
$$f(2n) \;=\; 3f(n) + Cn.$$
3. Consider the number of comparisons required to sort a list of n items via the *merge sort* algorithm described in Section 3.5 (Rosen, 7$^{\text{th}}$ ed.). This algorithm, for even n, divides the list into two lists of size $n/2$ and, once the two sub-lists are sorted, requires fewer than n comparisons to merge the two sorted sub-lists into one complete (and sorted) list. Thus, the number of comparisons used by the algorithm on a list of size n is less than $M(n)$, a function which satisfies the divide-and-conquer relation
$$M(n) \;=\; 2M(n/2) + n.$$

■

Some relevant details

Logarithms. Write $r = log_b x$ when $b^r = x$. Said another way, $\log_b x$ returns the number r for which $b^r = x$. Some properties that arise from this idea:

1. $b^{\log_b x} = x$, akin to saying the number of ounces in a 32-ounce jar is 32.
2. $\log_b(xy) = \log_b x + \log_b y$, since
$$b^{\log_b x + \log_b y} \;=\; b^{\log_b x} \cdot b^{\log_b y} \;=\; xy.$$
3. $\log_b(x/y) = \log_b x - \log_b y$, demonstrated similarly.
4. $\log_b(x^r) = r \log_b x$, since
$$b^{r \log_b x} \;=\; (b^{\log_b x})^r \;=\; x^r.$$
5. $\log_a x = \log_b x / \log_b a$, since
$$b^{(\log_a x)(\log_b a)} \;=\; (b^{\log_b a})^{\log_a x} \;=\; a^{\log_a x} \;=\; x.$$
Thus, $(\log_a x)(\log_b a)$ is the exponent to which, when b is raised, yields x—i.e., it equals $\log_b x$.
6. For positive real numbers a, b, and c,
$$a^{\log_b c} \;=\; c^{\log_b a}.$$
This is true because
$$\log_a \left(c^{\log_b a}\right) \;=\; \left(\log_b a\right)\left(\log_a c\right) \;=\; \log_b c,$$
by Property 5 above. This means that $\log_b c$ is the power to which you must raise a in order to produce $c^{\log_b a}$.
7. $O(\log_b n)$ is independent of base b. That is, if a is any other base, and if $|f(n)| \leqslant C|\log_b n|$ (the meaning of $O(\log_b n)$), then by Property 5 above,
$$|f(n)| \;\leqslant\; C|\log_b n| \;=\; \frac{C}{|\log_a b|}|\log_a n| \;=\; \tilde{C}|\log_a n|,$$
which shows f is $O(\log_a n)$ as well. Convention, then, is to write $O(\log n)$ without reference to a particular base b.

Question: For an integer n, how many stages of dividing into b parts, then subdividing those parts into b parts, and so on, may be carried out before all constituent parts are of size 1?

Answer: We can develop some intuition by investigating the number of ways to divide an integer by 2. The numbers $5, 6, 7$, and 8 each require 3 stages. The numbers $9, 10, 11, 12, 13, 14, 15$, and 16 require 4 stages. In general the integers $2^{k-1} < n \leqslant 2^k$ all require $k = \log_2 2^k = \lceil \log_2 \rceil n$ stages.

Speaking generally, if an integer n satisfies $b^{k-1} < n \leqslant b^k$ and, at each stage, is to be divided into b parts, then it requires $k = \log_b b^k = \lceil \log_b \rceil n$ stages.

Important theorems

When f satisfies the divide-and-conquer relation (1) and $n = b^k$, we have

$$\begin{aligned} f(n) &= af(n/b) + g(n) \;=\; a\left(af(n/b^2) + g(n/b)\right) + g(n) \\ &= a^2 f(n/b^2) + ag(n/b) + g(n) \\ &= a^3 f(n/b^3) + a^2 g(n/b^2) + ag(n/b) + g(n) \;=\; \cdots \\ &= a^k f(n/b^k) + \sum_{j=0}^{k-1} a^j g(n/b^j). \end{aligned}$$

In the special case where $g(n) = c$ (a constant), this becomes

$$f(n) \;=\; a^k f(n/b^k) + c\sum_{j=0}^{k-1} a^j \;=\; a^k f(n/b^k) + \frac{c(a^k-1)}{a-1}. \tag{2}$$

This gives rise to the following theorem.

Theorem 1: Suppose f is an increasing function which satisfies the recurrence relation

$$f(n) = af(n/b) + c,$$

whenever n is an integer divisible by (integer) $b > 1$. Suppose $a \geqslant 1$ and $c > 0$. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}), & \text{if } a > 1, \\ O(\log n), & \text{if } a = 1. \end{cases}$$

In addition, when $n = b^k$ for integer $k > 0$, we have

$$f(n) \;=\; \left(f(1) + \frac{c}{a-1}\right) n^{\log_b a} - \frac{c}{a-1}.$$

Proof: **Case**: $n = b^k$ (so $k = \log_b n$).
If $a = 1$, then Equation (2) says

$$f(n) \;=\; f(1) + ck \;=\; f(1) + c\log_b n,$$

showing f is $O(\log n)$.
Now suppose $a > 1$. Equation (2) says

$$f(n) \;=\; a^k f(1) + \frac{c(a^k-1)}{a-1} \;=\; a^{\log_b n}\left(f(1) + \frac{c}{a-1}\right) - \frac{c}{a-1} \;=\; n^{\log_b a}\left(f(1) + \frac{c}{a-1}\right) - \frac{c}{a-1}.$$

General Case. When n is not a power of b, there is an integer $k \geqslant 0$ such that $b^k < n < b^{k+1}$. We treat the case with $a > 1$ only. Because f is an increasing function,

$$f(n) \leqslant f(b^{k+1}) = C_1 a^{k+1} + C_2 = (C_1 a)a^k + C_2 = (C_1 a)a^{\log_b n} + C_2,$$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$. Hence, the result holds. □

The previous result is applicable to the binary search algorithm which, as we found, gives rise to the recurrence relation $f(n) = f(n/2) + 2$. To draw conclusions about the divide-and-conquer recurrence relations of fast integer multiplication and the merge sort, we need a more general theorem.

Theorem 2 (Master Theorem): Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d,$$

whenever $n = b^k$, where k is a positive integer, $a \geqslant 1$, b is an integer greater than 1, and $c > 0$, $d \geqslant 0$ are real numbers. Then

$$f(n) \text{ is } \begin{cases} O(n^d), & \text{if } a < b^d, \\ O(n^d \log n), & \text{if } a = b^d, \\ O(n^{\log_b a}), & \text{if } a > b^d. \end{cases}$$

Proof: If $a = b^d$ and $n = b^k$, then

$$\begin{aligned} f(n) &= af(n/b) + cn^d = a\left[af(n/b^2) + c\left(\frac{n}{b}\right)^d\right] + cn^d \\ &= a^2 f(n/b^2) + ac\left(\frac{n}{b}\right)^d + cn^d \\ &= a^3 f(n/b^3) + a^2 c\left(\frac{n}{b^2}\right)^d + ac\left(\frac{n}{b}\right)^d + cn^d = \cdots \\ &= a^k f(1) + cn^d \sum_{j=0}^{k-1}\left(\frac{a}{b^d}\right)^j = (b^d)^k f(1) + cn^d \sum_{j=0}^{k-1} 1 \\ &= f(1)n^d + ckn^d = f(1)n^d + cn^d \log_b n. \end{aligned}$$

Now, assume $k \geqslant 0$ is such that $b^k < n \leqslant b^{k+1}$. Because f is an increasing function, we have

$$
\begin{aligned}
f(n) \quad \leqslant \quad f(b^{k+1}) \quad &= \quad f(1)b^{(k+1)d} + c(k+1)b^{(k+1)d} \\
&= \quad f(1)b^d \cdot (b^k)^d + cb^d \cdot (b^k)^d + cb^d \cdot (b^k)^d k \\
&\leqslant \quad [f(1)+c]an^d + can^d \log_b n.
\end{aligned}
$$

Thus, in the case $a = b^d$, we have the desired result, as the $n^d \log n$ term above dominates the n^d term. □

Theorems from Rosen, 5th Ed., Section 8.3

Theorem 3: Suppose f is an increasing function which satisfies the recurrence relation

$$f(n) = af(n/b) + c,$$

whenever n is an integer divisible by (integer) $b > 1$. Suppose $a \geqslant 1$ and $c > 0$. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}), & \text{if } a > 1, \\ O(\log n), & \text{if } a = 1. \end{cases}$$

In addition, when $n = b^k$ for integer $k > 0$, we have

$$f(n) \;=\; \left(f(1) + \frac{c}{a-1}\right) n^{\log_b a} - \frac{c}{a-1}.$$

Theorem 4 (Master Theorem): Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d,$$

whenever $n = b^k$, where k is a positive integer, $a \geqslant 1$, b is an integer greater than 1, and $c > 0$, $d \geqslant 0$ are real numbers. Then

$$f(n) \text{ is } \begin{cases} O(n^d), & \text{if } a < b^d, \\ O(n^d \log n), & \text{if } a = b^d, \\ O(n^{\log_b a}), & \text{if } a > b^d. \end{cases}$$