Solutions

1. Writing the solution $\mathbf{x} = \langle x_1, x_2, x_3, x_4\rangle$, we have x_3, x_4 free. We may set $x_3 = s$ and $x_4 = s$, where s, t are arbitrary reals. The rows from RREF then give us

$$x_1 - s + t = 1 \qquad \text{and} \qquad x_2 + 3s - 2t = 4,$$

or solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1+s-t \\ 4-3s+2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{where } s, t \text{ are any real nos.}$$

2. The task is to find a basis for $\mathrm{Null}(\mathbf{A} - \lambda\mathbf{I})$, with $\lambda = 2$. We go to RREF:

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 27 & 64 & 44 \\ -13 & -31 & -21 \\ -1 & -2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, eigenvectors have components which satisfy equations

$$x_1 = -4x_3 \qquad \text{and} \qquad x_2 = x_3,$$

which means eigenvectors look like

$$\begin{bmatrix} -4x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{a line in } \mathbb{R}^3 \text{ with basis vector} \qquad \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}.$$

3. (a) One sequence of EROs that takes the given matrix $\mathbf{B}$ to echelon form is as follows:

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{\sim}{(1/3)\mathbf{r}_3 \to \mathbf{r}_3} \quad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{\sim}{(1/2)\mathbf{r}_2 \to \mathbf{r}_2} \quad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & -1/2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{(-2)\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2} \quad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{\sim}{(-1)\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1} \quad \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\underset{\sim}{(-3)\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1} \quad \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The *rank* of a matrix is the number of linearly independent columns it has, determined by counting its pivot columns. In the instance here, $\mathrm{rank}(\mathbf{A}) = 3$.

(c) Since columns 1, 2 and 4 are pivot columns, those form a basis of the column space. That is, we have basis consisting of $\langle 1, 2, 0, 0\rangle$, $\langle 3, 6, 2, 2\rangle$, and $\langle 1, 5, 4, 7\rangle$.

(d) This is false, because that would require the column space to be all of $\mathbb{R}^4$. The column space is, in fact, a 3-dimensional subspace of $\mathbb{R}^4$.

4. A straightforward matrix-vector product calculation shows that $\mathbf{Av} = -\mathbf{v}$, , where $\mathbf{v} = \langle 18, -5, -2, 2 \rangle$. Thus, $\mathbf{v}$ is an eigenvector corresponding to eigenvalue (-1).

5. (a) When the matrix $\mathbf{A}$ has real-number entries, then its characteristic polynomial has real-number coefficients, which then means eigenvalues and eigenvectors come in complex-conjugate pairs. Since our matrix has nonreal eigenpair

$$\lambda_1 = -4 + 2i, \qquad \mathbf{v}_1 = \begin{bmatrix} 2-i \\ 3 \\ -5+3i \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix},$$

we know it has, as well, the related eigenpair

$$\lambda_2 = -4 - 2i, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} - i \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+i \\ 3 \\ -5-3i \end{bmatrix}.$$

(b) The matrix has eigenvectors with 3 components, so $\mathbf{A}$ must itself be 3-by-3, and hence has a total of 3 eigenvalues, counting algebraic multiplicities. We were given one, deduced a second, and since there is only one more, it must be a real number, different from the first two. Thus, all three have algebraic multiplicity 1.

6. (a) The solution of the nonhomogeneous problem can always be seen as the sum of two parts, a particular solution added with vectors in the nullspace solving the homogeneous problem: $\mathbf{x}_p + \mathbf{x}_h$. So, we discover what vectors are in the nullspace of $\mathbf{A}$ by breaking apart the solution of the nonhomogeneous problem:

$$\begin{bmatrix} 4s_1 - 2s_2 + 3 \\ -s_1 - 2s_2 + 1 \\ 3s_2 - 8 \\ 2 - s_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -8 \\ 2 \end{bmatrix} + s_1 \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix} + s_2 \begin{bmatrix} -2 \\ -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -8 \\ 2 \end{bmatrix} + \text{span}\left(\left\{ \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 3 \\ 0 \end{bmatrix} \right\}\right).$$

The null space is the part with the free variables. A basis for it consists of the two linearly independent vectors

$$\begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

(b) This is false, since $\mathbf{A}$ has a nontrivial null space, which means it has free columns.

7. The characteristic equation is

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} \\ &= (6-\lambda)(-1-\lambda) - 8 = \lambda^2 - 5\lambda - 14 = (\lambda - 7)(\lambda + 2). \end{aligned}$$

Thus, eigenvalues are $\lambda = -2,\ 7$.