Example 1: A Newton's Law of Cooling Problem

We solve the DE model

$$\frac{dq}{dt} = k(q - T_0), \qquad \text{subject to} \qquad q(0) = 60,\ q(10) = 70,\ q(20) = 76.$$

The goal here is to find the value of T_0, the ambient temperature. We separate variables to obtain

$$\begin{aligned}
\frac{1}{q - T_0}\frac{dq}{dt} = k \quad &\Rightarrow \quad \int \frac{dq}{q - T_0} = \int k\,dt \\
&\Rightarrow \quad \ln|q - T_0| = kt + C \\
&\Rightarrow \quad |q - T_0| = e^{kt+C} \\
&\Rightarrow \quad q - T_0 = Ce^{kt} \\
&\Rightarrow \quad q(t) = T_0 + Ce^{kt}, \qquad \text{our solution.} \\
q(0) = 60 \quad &\Rightarrow \quad T_0 + C = 60. \\
q(10) = 70 \quad &\Rightarrow \quad T_0 + Ce^{10k} = 70. \\
q(20) = 76 \quad &\Rightarrow \quad T_0 + Ce^{20k} = 76.
\end{aligned}$$

We have $T_0 = 60 - C$, so we may substitute this into the other two equations in our constants T_0, C, k:

$$\begin{aligned}
T_0 + Ce^{10k} = 70 \quad &\Rightarrow \quad Ce^{10k} - C = 10 \\
T_0 + Ce^{20k} = 76 \quad &\Rightarrow \quad Ce^{20k} - C = 16.
\end{aligned}$$

Subtracting the top equation (on the right) from the bottom one, we get

$$C(e^{20k} - e^{10k}) = 6 \qquad \Rightarrow \qquad C = \frac{6}{e^{20k} - e^{10k}}.$$

Using this value for C in the equation $Ce^{10k} - C = 10$, we get

$$\begin{aligned}
\frac{6e^{10k}}{e^{20k} - e^{10k}} - \frac{6}{e^{20k} - e^{10k}} = 10 \quad &\Rightarrow \quad 6e^{10k} - 6 = 10e^{20k} - 10e^{10k} \\
&\Rightarrow \quad 5(e^{10k})^2 - 8e^{10k} + 3 = 0 \\
&\Rightarrow \quad e^{10k} = \frac{8 \pm \sqrt{4}}{10} \doteq \frac{6}{5}, \frac{2}{5}.
\end{aligned}$$

Since we need for k to be negative, we may rule out e^{10k} equalling $6/5$. We solve

$$e^{10k} = \frac{2}{5} \quad \Rightarrow \quad k = \frac{1}{10}\ln\left(\frac{2}{5}\right).$$

Thus,

$$e^{10k} = \frac{2}{5}, \qquad \text{and} \qquad e^{20k} = (e^{10k})^2 = \frac{4}{25},$$

and

$$C = \frac{6}{e^{20k} - e^{10k}} = \frac{6}{4/25 - 2/5} = -25.$$

Thus, $T_0 = 85°$F.