Example 1: A 2^{nd} order nonhomogeneous linear DE using Variation of Parameters

We mean to solve the DE

$$y'' + 4y' + 3y \;=\; (t+2)e^{2t}.$$

We first find the homogeneous solution. Because the characteristic polynomial is $\lambda^2 + 4\lambda + 3$, having roots $\lambda = -3,\ -1$, we have the fundamental set of solutions $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-t}$, yielding homogeneous solution

$$y_h(t) \;=\; c_1 y_1(t) + c_2 y_2(t) \;=\; c_1 e^{-3t} + c_2 e^{-t}.$$

The expression $(t+2)e^{2t}$ that makes our DE nonhomogeneous can be characterized as a (1^{st} degree polynomial)(exponential), which means we could use the method of undetermined coefficients to find a particular solution. Nevertheless, our goal here is to demonstrate the use of variation of parameters. When we first encountered such problems in Chapter 4, we had the following background from Chapter 3:

- Every higher order DE (or system of DEs) can be converted to a 1^{st} order system through the introduction of new variables: $x_1 = y, x_2 = y', x_3 = y''$, etc. When we start with an n^{th} order *linear* DE

  $$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y \;=\; g(t), \tag{1}$$

  this conversion process leads to a *linear* first order system

  $$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix} = \mathbf{A}\mathbf{x} + \mathbf{f}(t). \tag{2}$$

- The variation of parameters formula for a particular solution $\mathbf{x}_p(t)$ of the 1^{st} order linear system (2) is $\mathbf{x}_p(t) \;=\; \Phi(t)\int^t \Phi^{-1}(w)\mathbf{f}(w)\,dw$.

Together, these imply that we may take our particular solution $y_p(t)$ to be the same as the first component of the vector $\mathbf{x}_p(t)$.

In the present setting we build our fundamental matrix from the fundamental set of solutions

$$\Phi(t) \;=\; \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \;=\; \begin{bmatrix} e^{-3t} & e^{-t} \\ -3e^{-3t} & -e^{-t} \end{bmatrix}.$$

The product

$$\begin{bmatrix} v_1'(w) \\ v_2'(w) \end{bmatrix} \;=\; \Phi^{-1}(w)\mathbf{f}(w) \;=\; \Phi^{-1}(t)\begin{bmatrix} 0 \\ (w+2)e^{2w} \end{bmatrix},$$

may be found in one of a couple ways, with the one discussed most often being Cramer's Rule:

$$v_1'(w) \;=\; \frac{\begin{vmatrix} 0 & e^{-w} \\ (w+2)e^{2w} & -e^{-w} \end{vmatrix}}{\begin{vmatrix} e^{-3w} & e^{-w} \\ -3e^{-3w} & -e^{-w} \end{vmatrix}} \;=\; \frac{-(w+2)e^{w}}{2e^{-4w}} \;=\; -\frac{1}{2}(w+2)e^{5w}.$$

Using integration by parts and taking the constant of integration to be 0, we have

$$v_1(t) \;=\; \int^t v_1'(w)\,dw \;=\; \int^t -\frac{1}{2}(w+2)e^{5w}\,dw \;=\; -\frac{1}{50}(5t-1)e^{5t} - \frac{1}{5}e^{5t} \;=\; -\left(\frac{1}{10}t + \frac{9}{50}\right)e^{5t}.$$

Similarly,

$$\begin{aligned} v_2(t) \;&=\; \int^t \frac{\begin{vmatrix} e^{-3w} & 0 \\ -3e^{-3w} & (w+2)e^{2w} \end{vmatrix}}{\begin{vmatrix} e^{-3w} & e^{-w} \\ -3e^{-3w} & -e^{-w} \end{vmatrix}}\,dw \;=\; \int^t \frac{(w+2)e^{-w}}{2e^{-4w}} \;=\; \int^t \frac{1}{2}(w+2)e^{3w}\,dw \\ &=\; \frac{1}{18}(3t-1)e^{3t} + \frac{1}{3}e^{3t} \;=\; \left(\frac{1}{6}t + \frac{5}{18}\right)e^{3t}. \end{aligned}$$

Finally,

$$\begin{aligned} y_p(t) \;&=\; \text{the first entry of } \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \;=\; v_1(t)y_1(t) + v_2(t)y_2(t) \\ &=\; -\left(\frac{1}{10}t + \frac{9}{50}\right)(e^{5t})(e^{-3t}) + \left(\frac{1}{6}t + \frac{5}{18}\right)(e^{3t})(e^{-t}) \\ &=\; -\left(\frac{1}{10}t + \frac{9}{50}\right)e^{2t} + \left(\frac{1}{6}t + \frac{5}{18}\right)e^{2t} \;=\; \left(\frac{1}{15}t + \frac{22}{225}\right)e^{2t}. \end{aligned}$$

Putting things together, our general solution is

$$y(t) \;=\; y_h(t) + y_p(t) \;=\; c_1e^{-3t} + c_2e^{-t} + \left(\frac{1}{15}t + \frac{22}{225}\right)e^{2t}.$$

■

What, in the above, is always true? Among other things, you can count on the following:

- the n^{th} order DE (1) will have a fundamental set of solutions, $y_1(t), y_2(t), \ldots, y_n(t)$ which is used to build the matrix

$$\Phi(t) \;=\; \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}.$$

- The formula for a particular solution (one of many) is

  $$y_p(t) \;=\; v_1(t)y_1(t) + v_2(t)y_2(t) + \cdots + v_n(t)y_n(t),$$

  where the *derivatives* $v_1', \ldots, v_n'$ of the functions $v_1(t), \ldots, v_n(t)$ come from

  $$\begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} \;=\; \Phi(t) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix},$$

  and may be found one-by-one using Cramer's rule.

As we have mentioned, there are infinitely many particular solutions. The one we found above, namely

$$y_p(t) \;=\; \left(\frac{1}{15}t + \frac{22}{225}\right)e^{2t},$$

satisfies $y_p(0) = 22/225$ and $y_p'(0) = 59/225$, so it is not the same particular solution we find via the Laplace transform when we enforce that $y_p(0) = 0$ and $y_p'(0) = 0$.