Laplace transforms involving $\underline{H}(t-c) = u_c(t) = \mathcal{U}(t-c)$

today I'll use

Have (last entry on table, p. 242)

$$\mathcal{L}\{\mathcal{U}(t-c)\, f(t-c)\} = e^{-cs}\,\mathcal{L}\{f(t)\}$$

Practice

① $\mathcal{L}\{\mathcal{U}(t-\pi/2)\sin(t-\pi/2)\} = e^{-\pi/2\, s}\cdot\dfrac{1}{s^2+1}$

Same shift to right

Know $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

② $\mathcal{L}\{\mathcal{U}(t-2)\cdot(t-2)\} = e^{-2s}\cdot\dfrac{1}{s^2}$

$\mathcal{L}$

Shift by 2, for latter factor, its $\underline{t}$ which is shifted to obtain $t-2$.

③ $\mathcal{L}\{\mathcal{U}(t-2)(t+3)\} = e^{-2s}\cdot\mathcal{L}\{t+5\} = e^{-2s}\left(\mathcal{L}\{t\}+\mathcal{L}\{5\}\right)$

Shift right 2

$= \left(\dfrac{1}{s^2} + 5\cdot\dfrac{1}{s}\right)e^{-2s}.$

What starting fn., when shifted right 2, gives $t+3$?

$$t+3 = \underbrace{t+5}_{\text{starting fn.}} - 2 = t+5\,\Big|_{t\mapsto t-2}$$

① An alternate way for finding starting fn.: Taylor series

Taylor series of $f(t)$ centered at 'c'

$$f(c) + \frac{f'(c)}{1!}(t-c) + \frac{f''(c)}{2!}(t-c)^2 + \frac{f'''(c)}{3!}(t-c)^3 + \cdots$$

Here, my shift (in Heaviside $\mathcal{U}(t-2)$) is 2, center Taylor series at 2.

fn. $g(t) = t+3$

$g'(t) = 1$

$g''(t) = \underline{0} = g'''(t) = \cdots$

So $t+3$ has Taylor series centered at 2:

$$g(2) + \frac{g'(2)}{1!}(t-2) + \underbrace{\frac{g''(2)}{2!}(t-2)^2 + \cdots}_{\text{zero}}$$

$$= 5 + \frac{1}{1}(t-2) + 0$$

$$= 5 + (t-2) = (5+t)\Big|_{t \mapsto t-2}$$

starter fn.

② Shift your given fn. 2 to the left

given fn. (alongside $\mathcal{U}(t-2)$) was $t+3$

shift it left 2: replace t by $t+2$

$$(t+2)+3 = \underline{t+5}$$

Ex.

$$\mathcal{L}\{\mathcal{U}(t-3)\cdot(2t^2+5t-4)\} = e^{-3s}\cdot\mathcal{L}\{2t^2+17t+29\}$$

shift right 3

what starting fn. shifted right 3 produces this?

$$= \boxed{e^{-3s}\left(\frac{4}{s^3} + \frac{17}{s^2} + \frac{29}{s}\right)}$$

replace $t \mapsto t+3$ (left shift 3)

$$2(t+3)^2 + 5(t+3) - 4 = 2(t^2+6t+9) + 5t + 11$$

$$= \underline{2t^2 + 17t + 29}$$

Topic: How does solving DEs (IVPs) lead to L.T.?

Connection is from results such as

Given $F(s) = \mathcal{L}\{f(t)\}$

What is $\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt = \cdots$

$$= s \boxed{\int_0^\infty f(t)\, e^{-st}\, dt} - f(0)$$

$$= s\, F(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

?

Ex.)

$$y' + 3y = 8, \qquad y(0) = 4$$

Take L.T. of both sides

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{8\}$$

$$\underbrace{\mathcal{L}\{y'\}} + 3\,\underbrace{\mathcal{L}\{y\}} = 8\,\mathcal{L}\{1\}$$

$$\overbrace{sY - y(0)} + 3Y = \frac{8}{s}$$

$$sY + 3Y = 4 + \frac{8}{s} \qquad \text{(applying the IC)}$$

Solve for Y

$$(s+3)\,Y = 4 + \frac{8}{s} \implies Y = \frac{4}{s+3} + \frac{8}{s(s+3)}$$

Get $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{8}{s(s+3)}\right\}$

like $\frac{1}{s-a}$

$$= 4e^{-3t} + \frac{8}{3} - \frac{8}{3}e^{-3t} = \boxed{\frac{8}{3} + \frac{4}{3}e^{-3t}}$$

Use partial fractions

$$\frac{8}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \quad \times s(s+3) \quad \Rightarrow 8 = A(s+3) + Bs$$

@ $s=0$: $\quad 8 = 3A + 0 \quad \Rightarrow \quad A = 8/3$

@ $s=-3$: $\quad 8 = 0 - 3B \quad \Rightarrow \quad B = -8/3$

$$\mathcal{L}^{-1}\left\{\frac{8}{s(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{8/3}{s} - \frac{8/3}{s+3}\right\}$$

$$= \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{8}{3}\cdot 1 - \frac{8}{3}e^{-3t}$$

Ex] $\quad y'' + by' + cy = f(t), \quad y(0) = y_0, \; y'(0) = y_1$

Take L.T.

$$\mathcal{L}\{y''\} + b\,\mathcal{L}\{y'\} + c\,\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\underline{s^2Y} - sy(0) - y'(0) + b\left[\underline{sY} - y(0)\right] + \underline{cY} = F$$

$$s^2Y + bsY + cY - sy_0 - y_1 - by_0 = F$$

Do algebra to get Y

$$\underbrace{(s^2+bs+c)}Y = F(s) + sy_0 + by_0 + y_1$$

char. poly.

$$Y(s) = \boxed{\frac{1}{s^2+bs+c}} F(s) + \frac{1}{s^2+bs+c}(sy_0+by_0+y_1)$$

$$y(t) = \mathcal{L}^{-1}\{ \qquad \} + \mathcal{L}^{-1}\{ \qquad \}$$

called the <u>transfer</u> <u>fn</u>, denoted $H(s)$

Observe: Two related problems

① $y'' + by' + cy = f(t), \quad y(0)=0, \ y'(0)=0$

Repeat my calculations, get

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+bs+c} \cdot F(s) \right\}$$

② $y'' + by' + cy = 0, \quad y(0) = y_0, \ y'(0) = y_1$

Same calculations

Solve like we did in Ch. 4, perhaps

that is, get this answer w/out using L.T

$$y(t) = \mathcal{L}^{-1}\{ H(t)\cdot(sy_0+by_0+y_1) \}$$