Ex.) "A mass of 5 kg stretches a spring 10 cm."

Use to obtain spring const. k

$$(5\ kg)(9.8\ m/s^2) = k(0.1\ m)$$

$$m \quad g \quad = k\,(\text{stretched dist})$$

$$\Rightarrow k = 490$$

"Its acted on by a external force of $\boxed{10 \sin(t/2)}$ Newton — nonhomog. term, external force

and moves in a medium that has viscous force of 2N when speed is 4 cm/s."

$$\text{damping force} = \gamma\ \text{speed}$$

$$2 = \gamma(.04) \Rightarrow \gamma = 50$$

$$\boxed{5u'' + 50u' + 490u = 10\sin(t/2)}$$

"Mass is set in motion from equilibrium w/ initial velocity 3 cm/s."

ICs: $\boxed{\begin{array}{l} u(0) = 0 \\ u'(0) = 0.03 \end{array}}$ (initial displacement)

To solve:

1. Solve homog. problem: dividing by 5

$$u'' + 10u' + 98u = 0 \quad \text{— linear, const. coeff.}$$

char. eqn.

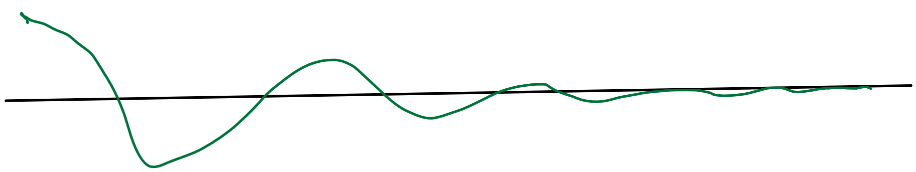
$$r^2 + 10r + 98 = 0 \implies r = -5 \pm \frac{1}{2}\sqrt{100 - 392}$$

$$= -5 \pm i\sqrt{73}$$

nonreal roots: $\alpha = -5, \ \beta = \sqrt{73}$

homog. soln.

$$u_h(t) = \underbrace{c_1 e^{-5t}\cos(\sqrt{73}\,t) + c_2 e^{-5t}\sin(\sqrt{73}\,t)}_{\text{all transient}}$$



RHS:

$$u'' + 10u' + 98u = \underbrace{2\sin(t/2)}_{(0\ \text{deg. poly})\cdot\ \text{sine}}$$

insert info DE

$$u_p(t) = A\cos(t/2) + B\sin(t/2)$$

compare w/ u_h to see if common terms

$$u_p' = -\frac{A}{2}\sin(t/2) + \frac{1}{2}B\cos(t/2)$$

$$u_p'' = -\frac{A}{4}\cos(t/2) - \frac{B}{4}\sin(t/2)$$

$$98\left(A\cos(t/2) + B\sin(t/2)\right) + 10\left(-\frac{1}{2}A\sin(t/2) + \frac{1}{2}B\cos(t/2)\right) + -\frac{1}{4}A\cos\left(\frac{t}{2}\right) - \frac{1}{4}B\sin\left(\frac{t}{2}\right)$$

$$= \cos(t/2)\cdot\left(98A + 5B - \frac{1}{4}A\right) + \sin\left(\frac{t}{2}\right)\left(98B - 5A - \frac{1}{4}B\right)$$

$$98 - \frac{1}{4} = \frac{391}{4}$$

$$= \left(\frac{391}{4}A + 5B\right)\cos(t/2) + \left(-5A + \frac{391}{4}B\right)\sin(t/2)$$

enforce

$$= 0 \cos(t/2) + 2 \sin(t/2)$$

Equate coeffs:

$$\begin{bmatrix} 391/4 & 5 \\ -5 & 391/4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -1/958 \\ 695/34057 \end{bmatrix}$$

$$u_p = -\frac{1}{958}\cos(t/2) + \frac{695}{34057}\sin(t/2)$$

$$u(t) = u_h + u_p$$

$$= c_1 e^{-5t}\cos(\sqrt{73}\,t) + c_2 e^{-5t}\sin(\sqrt{73}\,t) + \frac{-1}{958}\cos(t/2) + \frac{695}{34057}\sin(t/2)$$

Now use $u(0)=0$, $u'(0) = 0.03$ to get c_1 and c_2.

$\boxed{\text{Ex.}}$ $y'' + 8y' + 17y = e^{2t}$, $y(0) = 1$, $y'(0) = -1$

Split into 2

zeros nonhomog. term $\longrightarrow$ ① $y'' + 8y' + 17y = 0$, $y(0)=1$, $y'(0) = -1$

zeros IC $\longrightarrow$ ② $y'' + 8y' + 17y = e^{2t}$, $y(0) = 0$, $y'(0) = 0$

① Use Ch 4 ideas

$$r^2 + 8r + 17 = 0 \qquad r = -4 \pm \frac{1}{2}\sqrt{64 - 68}$$

$$= -4 \pm i \qquad (\alpha = -4,\ \beta = 1)$$

$$y_h(t) = \underline{c_1 e^{-4t}\cos t + c_2 e^{-4t}\sin t}$$

Apply ICs.

$$y_h'(t) = \underline{-4c_1 e^{-4t}\cos t - c_1 e^{-4t}\sin t - 4c_2 e^{-4t}\sin t + c_2 e^{-4t}\cos t}$$

$$\underline{1} = y(0) = c_1 \cdot 1 + c_2 \cdot 0 = \underline{c_1}$$

$$-1 = y'(0) = -4c_1 \cdot 1 - c_1 \cdot 0 - 4c_2 \cdot 0 + c_2 \cdot 1$$

$$c_2 = 3$$

Soln to ①

$$\boxed{y_1(t) = e^{-4t}\cos t + 3e^{-4t}\sin t}$$

② $y'' + 8y' + 17y = e^{2t}$ w/ 0 ICs.

$$\mathcal{L}\{\qquad\} = \mathcal{L}\{e^{2t}\}$$

$$\mathcal{L}\{y''\} + 8\mathcal{L}\{y'\} + 17\mathcal{L}\{y\} = \frac{1}{s-2}$$

$$s^2 Y - \cancel{s y(0)} - \cancel{y'(0)} + 8(sY - \cancel{y(0)}) + 17Y = \frac{1}{s-2}$$

both zero

$$(s^2 + 8s + 17)Y = \frac{1}{s-2}$$

$$Y(s) = \frac{1}{(s-2)(s^2+8s+17)}$$

$$y(t) = \mathcal{L}^{-1}\{ Y(s) \}$$

$$\frac{1}{(s-2)(s^2+8s+17)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+8s+17}$$

irreducible (has nonreal roots)

comes from Ae^{2t}

$$\mathcal{L}^{-1}\left\{ \frac{Bs+C}{s^2+8s+17} \right\}$$

$$\frac{Bs+C}{s^2+8s+17} = \frac{Bs+C}{(s^2+8s+16)+1} = \frac{Bs+C}{(s+4)^2+1}$$

to get same shift in s in numerator, manufacture it

$$= \frac{B(s+4)-4B+C}{(s+4)^2+1}$$

$$= \underline{B\frac{s+4}{(s+4)^2+1}} + \frac{C-4B}{(s+4)^2+1}$$

comes from

$$B \cdot e^{-4t}\cos t + (C-4B)e^{-4t}\sin t$$

soln. to ②

$$y_2(t) = \boxed{Ae^{2t} + Be^{-4t}\cos t + (C-4B)e^{-4t}\sin t}$$

Soln to original

$$y_1(t) + y_2(t)$$