

Form B Solutions

1. (a) Both $f(x) = 3x^2 + 8x + 7$ and $g(x) = x^2$ are nonnegative and have graphs which are parabolas. The graph of f rises more steeply than the graph of g , owing to the leading coefficient of 3 in the x^2 -term. But, by taking $C = 5$, the rescaled $5g(x)$ will rise more steeply. In fact, if we take $k = 5$, then $|f(5)| < 5|g(5)|$, and they will not cross again any further along. So, $C = 5$ and $k = 5$ are witnesses to this Big-Oh relationship.
- (b) Since we require positive numbers C and k so that, for $x > k$,

$$C \cdot 1 \leq |f(x)|,$$

this means $|f(x)|$ (eventually) rises beyond some fixed positive value and stays above it forever afterward. Constant functions, polynomials, polylogs, exponential growth, and factorial functions (among others) all do this.

- (c) (i) and (iv) are true.
2. One can conclude (ii) and (iv).
3. In order of ascending complexity,

$$n - \sqrt{n}, \quad n \log n, \quad n^2 + \log n, \quad n^2 \log n, \quad (n-1)(n^2 + \log n), \quad n^8 - 2^n, \quad 2^n(\log n - 1), \quad 3^n$$

4. (a) $P(1)$ says $1 \cdot 1! = 2! - 1$.
- (b) $P(4)$ says $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! = 5! - 1$.
- (c) $P(k+1)$ says $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$.
- (d) The basis step is stated in the answer to part (a), and that equation holds.
For the induction step, we assume $P(k)$ holds for some $k \in \mathbb{Z}^+$ —that is,

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1.$$

Now, we throw in the extra term on the left-hand side required in $P(k+1)$:

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k!) + (k+1) \cdot (k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \quad (\text{induction hypothesis}) \\ &= (k+1)! + (k+1)(k+1)! - 1 \\ &= (k+1)![1 + (k+1)] - 1 \\ &= (k+1)!(k+2) - 1 \\ &= (k+2)! - 1. \end{aligned}$$

5. Using Form 1, a term like $2x^5$ requires 5 multiplications $(2)(1.5)(1.5)(1.5)(1.5)$, and a careful count shows the whole process requiring 20 flops. Horner's method (a la Form 2) evaluates $f(1.5)$ using 10 flops, half as many.
6. A linear search through an unsorted list of length n is, in the worst case, of complexity $O(n)$. To do the two-pronged approach described above requires complexity $O(\max(n \log n, \log n)) = O(n \log n)$, making it less efficient for a single search (that is, the linear search is more efficient).
7. The strings 101, 110101, and 111010101 are admitted into A on the 1st, 2nd and 3rd recursive steps, respectively.

8. (a) S has a smallest element, by the **Well-Ordering Principle**.
(b) The basis step is $P(5)$.
(c) In this instance, we would have $P(n)$ holds for $n = 5, 8, 11, 14, 17, \dots$
9. The problem in the proof is that it does not satisfy $P(0) \rightarrow P(1)$. That is, a crucial detail of the inductive step is to write $k + 1 = i + j$ with both i, j being numbers in $\{0, \dots, k\}$. That simply isn't possible when $k = 0$, as this would enforce $i = j = 0$, so that $i + j = 0$, not 1.