1. The characteristic equation: $r^2 - 2r + 17 = 0$

$$\Rightarrow \quad r = \frac{2}{2} \pm \frac{\sqrt{4-(4)(17)}}{2} = 1 \pm 4i$$

$\Rightarrow$ the homogeneous DE has independent solns. $y_1 = e^t \cos(4t)$, $y_2 = e^t \sin(4t)$

$\Rightarrow$ $y_h(t) = c_1 e^t \cos(4t) + c_2 e^t \sin(4t)$ (No overlap with $22\cos(5t) + 34\sin(5t)$)

For particular soln. pose the form

$$y_p(t) = A\cos(5t) + B\sin(5t) \quad \Rightarrow \quad \begin{cases} y_p' = -5A\sin(5t) + 5B\cos(5t) \\ y_p'' = -25A\cos(5t) - 25B\sin(5t) \end{cases}$$

So

$$\begin{aligned} y_p'' - 2y_p' + 17y_p &= -25A\cos(5t) - 25B\sin(5t) \\ &\quad - 2[-5A\sin(5t) + 5B\cos(5t)] \\ &\quad + 17[A\cos(5t) + B\sin(5t)] \\ &= \underbrace{(-8A - 10B)}_{\text{must } = 22}\cos(5t) + \underbrace{(10A - 8B)}_{\text{must } = 34}\sin(5t) \end{aligned}$$

$$\begin{bmatrix} -8 & -10 \\ 10 & -8 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 22 \\ 34 \end{bmatrix} \Rightarrow A = \frac{\begin{vmatrix} 22 & -10 \\ 34 & -8 \end{vmatrix}}{\begin{vmatrix} -8 & -10 \\ 10 & -8 \end{vmatrix}} = 1, \quad B = \frac{\begin{vmatrix} -8 & 22 \\ 10 & 34 \end{vmatrix}}{164} = -3$$

Together, $y(t) = y_h(t) + y_p(t) = c_1 e^t \cos(4t) + c_2 e^t \sin(4t) + \cos(5t) - 3\sin(5t)$

$$y'(t) = c_1 e^t \cos(4t) - 4c_1 e^t \sin(4t) + c_2 e^t \sin(4t) + 4c_2 e^t \cos(4t) - 5\sin(5t) - 15\cos(5t)$$

The ICs: $-2 = y(0) = c_1 + 1 \Rightarrow c_1 = -3$

$-14 = y'(0) = c_1 + 4c_2 - 15 \Rightarrow c_2 = 1$

$$\Rightarrow \boxed{y(t) = -3e^t\cos(4t) + e^t\sin(4t) + \cos(5t) - 3\sin(5t)}$$

2. (a) Since $\mathcal{L}[t^2 - 3t] = \mathcal{L}[t^2] - 3\mathcal{L}[t] = \dfrac{2}{s^3} - \dfrac{3}{s^2}$

and $\mathcal{L}[5e^{-2t}] = 5\mathcal{L}[e^{-2t}] = \dfrac{5}{s+2}$

we have $\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)] = \boxed{\left(\dfrac{2}{s^3} - \dfrac{3}{s^2}\right) \cdot \dfrac{5}{s+2}}$

(b) From the definition,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st}(3t+2)\,dt$$

$dv = e^{-st}dt$ | $u = 3t+2$
$v = -\frac{1}{s}e^{-st}$ | $du = 3\,dt$

$$\overset{\text{by parts}}{=} -\frac{1}{s}e^{-st}(3t+2)\Big|_0^4 + \frac{3}{s}\int_0^4 e^{-st}\,dt$$

$$= -\frac{14}{s}e^{-4s} + \frac{2}{s} - \frac{3}{s^2}\left[e^{-st}\right]_0^4 = \boxed{\frac{2}{s} - \frac{14}{s}e^{-4s} + \frac{3}{s^2}\left(1 - e^{-4s}\right)}$$

Or, using the table entry $\mathcal{L}[f(t-a)\,\mathcal{U}(t-a)] = e^{-sa}\mathcal{L}[f(t)]$:

$$f(t) = [1 - \mathcal{U}(t-4)](3t+2) = 3t+2 - \mathcal{U}(t-4)[3(t-4)+14]$$

$$= 3t+2 - \mathcal{U}(t-4)\cdot\left(3t+14\big|_{t\mapsto t-4}\right)$$

$$\Rightarrow F(s) = 3\mathcal{L}[t] + 2\mathcal{L}[1] - e^{-4s}\cdot\mathcal{L}[3t+14] = \boxed{\frac{3}{s^2} + \frac{2}{s} - e^{-4s}\left(\frac{3}{s^2} + \frac{14}{s}\right)}$$

(c) Using partial fractions

$$\frac{2s+1}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} \Rightarrow 2s+1 = A(s^2+2s+5) + Bs^2 + Cs$$

$$= \underbrace{(A+B)}_{\text{must } = 0}s^2 + \underbrace{(2A+C)}_{\text{must } = 2}s + \underbrace{5A}_{\text{must } = 1}$$

$5A = 1 \Rightarrow A = 1/5$

$A+B = 0 \Rightarrow B = -1/5$

$4A + C = 2 \Rightarrow C = 8/5$

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s(s^2+2s+5)}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{-1/5\,s + 8/5}{(s+1)^2+4}\right] = \frac{1}{5} + \mathcal{L}^{-1}\left[\frac{-1/5\,(s+1) + 9/5}{(s+1)^2+4}\right]$$

$$= \frac{1}{5} - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] + \frac{9}{10}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right] = \boxed{\frac{1}{5} - \frac{1}{5}e^{-t}\cos(2t) + \frac{9}{10}e^{-t}\sin(2t)}$$

3. (a) $$H(s) = \frac{1}{s^2+4s+3}$$

(b) $$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+4s+3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+3}\cdot\frac{1}{s+1}\right]$$ (can also write as $\frac{A}{s+3} + \frac{B}{s+1}$ using partial fractions)

$$= \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] * \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-3t} * e^{-t}$$

$$= \int_0^t e^{-3w}e^{-(t-w)}\,dw = e^{-t}\int_0^t e^{-2w}\,dw = -\frac{1}{2}e^{-t}\left[e^{-2w}\right]_0^t = -\frac{1}{2}e^{-t}\left(e^{-2t}-1\right)$$

$$= \boxed{\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}}$$

(c) The characteristic equation, $r^2+4r+3=0$, has 2 distinct real (and negative) roots. So, it is <u>overdamped</u>.

(d) $$y(t) = (f * h)(t) = \int_0^t f(t-w)\, h(w)\, dw$$

$$= \frac{1}{2}\int_0^t f(t-w)\left(e^{-w} - e^{-3w}\right) dw$$

4. Here, $y'' - \frac{6}{t^2} y = 5t^2 - 3t^{-2} = f(t)$,

and $$W = \begin{vmatrix} t^3 & t^{-2} \\ 3t^2 & -2t^{-3} \end{vmatrix} = -2-3 = -5.$$

$$y_p(t) = t^3 \int \frac{1}{5}\left(5t^2 - 3t^{-2}\right) t^{-2}\, dt - t^{-2}\int \frac{1}{5}\left(5t^2-3t^{-2}\right) t^3\, dt$$

$$= t^3 \int \left(1 - \frac{3}{5}t^{-4}\right) dt - t^{-2}\int\left(t^5 - \frac{3}{5}t\right) dt$$

$$= t^3\left(t + \frac{1}{5}t^{-3}\right) - t^{-2}\left(\frac{1}{6}t^6 - \frac{3}{10}t^2\right)$$

$$= t^4 + \frac{1}{5} - \frac{1}{6}t^4 + \frac{3}{10} = \boxed{\frac{5}{6}t^4 + \frac{1}{2}}$$