Stat 343, Thu 22-Oct-2020 -- Thu 22-Oct-2020

Probability and Statistics

Fall 2020

Thursday, October 22nd 2020

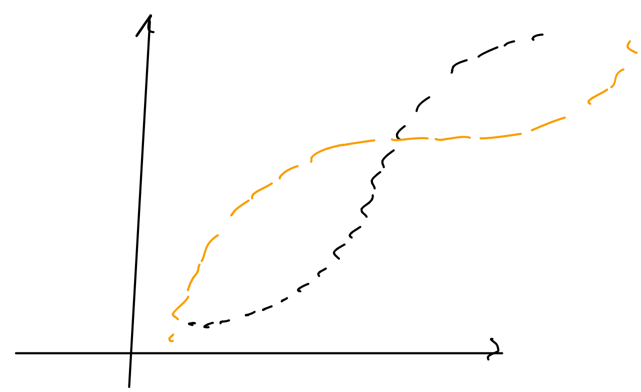
Wk 8, Th

Topic:: Joint normal distributions

Read:: FASt 3.8

at start: explore normal quantile plots for several distribution types

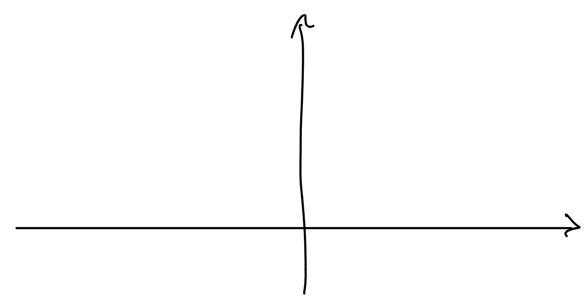
```
gf_qq(~rexp(1000,2))
gf_qq(~rbeta(1000,8,2))
gf_qq(~rt(1000,2))
```



Working with vectors, matrices in R

 - mosaic commands:

```
    dot(u, v)
    vlength(u)
```



iid = independent & identically distributed

Sums of iid random variables

Lemma 1: Suppose $X_1, \ldots, X_n$ are i.i.d. and that each $\mathrm{E}(X_i) = \mu$, each $\mathrm{Var}(X_i) = \sigma^2$. Let $S = X_1 + X_2 + \cdots + X_n$, and $\overline{X} = \frac{1}{n}S$. Then

(i) $\mathrm{E}(S) = n\mu$ and $\mathrm{Var}(S) = n\sigma^2$.
(ii) $\mathrm{E}(\overline{X}) = \mu$ and $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

$E(X_1 + X_2 + \cdots + X_n) = \sum E(X_i) = n\mu$

$Var(\frac{1}{n}X) = \frac{1}{n^2} Var(X) = \frac{1}{n^2} \cdot n\sigma^2$

$Var(X_1 + \cdots + X_n) = \sum Var(X_i) = n\sigma^2$

Lemma 2: Suppose $\mathbf{X} \overset{\text{i.i.d.}}{\sim} \mathsf{Norm}(\mu, \sigma)$, and define S, $\overline{X}$ as in the previous lemma. Then

(i) $S \sim \mathsf{Norm}(n\mu, \sigma\sqrt{n})$, and
(ii) $\overline{X} \sim \mathsf{Norm}(\mu, \sigma/\sqrt{n})$.

Extra: sum/average are normal once again.

Prove this, using Theorems 3.8.10 (p. 189) and 3.3.6 (p. 155).

Question: What if the components X_i of $\mathbf{X}$ have different means μ_i and standard deviations σ_i?

Moment generating functions for joint distributions

r.v. $\vec{X} = \langle X_1, X_2, \ldots, X_n \rangle$
r.v.s

Definition 1: Let $\mathbf{X}$ be the random vector with jointly distributed component r.v.s $X_1, X_2, \ldots, X_n$. We define the **expected value** of $\mathbf{X}$ to be the vector whose components are the expected values of the individual X_is. That is,

$$\mathrm{E}(\mathbf{X}) := \langle \mathrm{E}(X_1), \mathrm{E}(X_2), \ldots, \mathrm{E}(X_n) \rangle .$$

$E(\vec{X})$

Definition 2: For random vector $\mathbf{X}$, whose components are jointly distributed r.v.s, define the **moment generating function** (mgf) for $\mathbf{X}$ to be

$$M_{\mathbf{X}}(\mathbf{t}) = \mathrm{E}(e^{\mathbf{t} \cdot \mathbf{X}}) = \mathrm{E}(e^{t_1 X_1 + t_2 X_2 + \cdots + t_n X_n}).$$

vector

$M_{\langle X_1, X_2, \ldots, X_n \rangle}(\langle t_1, t_2, \ldots, t_n \rangle)$

Example 1:

Suppose $\mathbf{X} \overset{\text{i.i.d.}}{\sim} \text{Norm}(\mu, \sigma)$. $\vec{X} = \langle X_1, X_2 \rangle$ w/ both $X_i \sim \text{Norm}(\mu, \sigma)$

(a) Compute $M_{\mathbf{X}}(\mathbf{t})$.

$$M_{\vec{X}}(\vec{t}) := E\left(e^{\vec{t} \cdot \vec{X}}\right) = E\left(e^{\langle t_1, t_2 \rangle \cdot \langle X_1, X_2 \rangle}\right)$$

$$= E\left(e^{t_1 X_1 + t_2 X_2}\right) = E\left(e^{t_1 X_1} \cdot e^{t_2 X_2}\right) = E\left(e^{t_1 X_1}\right) \cdot E\left(e^{t_2 X_2}\right)$$

ind. vars since X_1, X_2 are

$$= M_{X_1}(t_1)\, M_{X_2}(t_2) = e^{\mu t_1 + \sigma^2 t_1^2/2}\, e^{\mu t_2 + \sigma^2 t_2^2/2}$$

(b) Recover $M_{X_1}(t)$ from the answer to (a).

$$= e^{\mu(t_1 + t_2) + \sigma^2 (t_1^2 + t^2)/2}.$$

Get individual component's mgf — that of X_1, by setting $t_1 = t$, $t_2 = 0$:

$$M_{\vec{X}}(\vec{t}) = e^{\mu(t_1 + t_2) + \sigma^2 (t_1^2 + t^2)/2}$$

$$M_{X_1}(t) = e^{\mu(t + 0) + \sigma^2 (t^2 + 0)/2} = e^{\mu t + \sigma^2 t^2/2}.$$ ■

New Example 2

~~**Example 2**~~:

$\vec{\mu} \circ \vec{t} + \frac{1}{2} \vec{t} \circ \Sigma \vec{t}$ 2x2

Suppose $M_{\mathbf{X}}(\mathbf{t}) = e^{\mu \cdot \mathbf{t} + \mathbf{t} \cdot \Sigma \mathbf{t}}$, where $\vec{\mu} = \langle 3, -1 \rangle$, $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1.5 \\ 1.5 & 4 \end{bmatrix}$. $\vec{X} = \langle X_1, X_2 \rangle$

2x1

(a) Find the moment generating functions for the components of **X**.

$$\vec{\mu} \circ \vec{t} + \frac{1}{2}\vec{t} \circ (\Sigma \vec{t}) = \langle 3, -1 \rangle \circ \langle t_1, t_2 \rangle + \frac{1}{2} \langle t_1, t_2 \rangle \circ \begin{bmatrix} 1 & 1.5 \\ 1.5 & 4 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$= \frac{1}{2}(t_1^2 + 3t_1t_2 + 4t_2^2) + 3t_1 - t_2$$

$$\boxed{M_{\vec{X}}(\vec{t}) = e^{\frac{1}{2}t_1^2 + 3/2\, t_1 t_2 + 2t_2^2 + 3t_1 - t_2}}$$

$X_1 \sim Norm(3, 1)$

$X_2 \sim Norm(-1, 2)$

$$M_{X_1}(t) = M_{\vec{X}}(\langle t, 0 \rangle) = e^{3t + \frac{1}{2}t^2} \leftarrow \text{mgf for } Norm(3, 1)$$

$$e^{-t + \frac{4}{2}t^2}$$

(b) Compute partial derivatives up to 2nd order.

$$M_{normal} = e^{\mu t + \sigma^2 t^2/2}$$

■

$$\frac{\partial}{\partial t_1}\left(e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1t_2 + 2t^2 + 3t_1 - t_2}\right) = \boxed{e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1t_2 + 2t^2 + 3t_1 - t_2} \cdot \left(t_1 + \frac{3}{2}t_2 + 3\right)}$$

$$= \frac{\partial M_{\vec{X}}(t_1, t_2)}{\partial t_1}$$

Insert $(0,0)$ for (t_1, t_2) $\quad \dfrac{\partial M_X(0,0)}{\partial t_1} = 3 \overset{?}{=} E(X_1)$

$$E(X_2) = \frac{\partial M_X(0,0)}{\partial t_2} = \cdots = -1$$

$$E(X_1^2) = \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_1} M_X(0,0)\right)$$

$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)$

$$E(X_1 X_2) = \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_2} M_X\right)\Big|_{(0,0)}$$

$$Var(X_1) = E(X^2) - [E(X)]^2 = 5 - (-1)^2 = 4$$

$\vec{t} \circ \Sigma \vec{t}$

$$\langle t_1, t_2 \rangle \cdot \begin{bmatrix} 1 & 1.5 \\ 1.5 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}_{2\times 1} = \langle t_1, t_2 \rangle \cdot \left(t_1 \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} + t_2 \begin{bmatrix} 1.5 \\ 4 \end{bmatrix} \right)$$

$$= \langle t_1, t_2 \rangle \cdot \begin{bmatrix} t_1 + 1.5 t_2 \\ 1.5 t_1 + 4 t_2 \end{bmatrix}_{2\times 1} = t_1 (t_1 + 1.5 t_2) + t_2 (1.5 t_1 + 4 t_2)$$

$$= t_1^2 + 3 t_1 t_2 + 4 t_2^2$$

$$\vec{\mu} \cdot \vec{t} + \frac{1}{2} \vec{t} \cdot \Sigma \vec{t} = 3t_1 - t_2 + \frac{1}{2} t_1^2 + \frac{3}{2} t_1 t_2 + 2 t_2^2$$

$$\Rightarrow M_{\vec{X}}(\vec{t}) = e^{\vec{\mu} \cdot \vec{t} + \frac{1}{2} \vec{t} \cdot \Sigma \vec{t}}$$

$$= e^{\frac{1}{2} t_1^2 + \frac{3}{2} t_1 t_2 + 2 t_2^2 + 3t_1 - t_2}$$

For X_1, $\quad M_{X_1}(t) = M_{\vec{X}}(\langle t, 0 \rangle) = e^{3t + t^2/2} \quad \Rightarrow \quad X_1 \sim \text{Norm}(3, 1)$

and X_2, $\quad M_{X_2}(t) = M_{\vec{X}}(\langle 0, t \rangle) = e^{-t + 2t^2} = e^{-t + 4t^2/2}$

$$\Rightarrow X_2 \sim \text{Norm}(-1, 2).$$

$\Rightarrow$ the variables X_1, X_2 are non-independent jointly distributed normal r.v.s.

$$\frac{\partial}{\partial t_1} M_{\vec{X}}(\vec{t}) = \frac{\partial}{\partial t_1} \left(e^{\frac{1}{2} t_1^2 + \frac{3}{2} t_1 t_2 + 2 t_2^2 + 3t_1 - t_2} \right)$$

$$= \left(t_1 + \frac{3}{2} t_2 + 3 \right) e^{\frac{1}{2} t_1^2 + \frac{3}{2} t_1 t_2 + 2 t_2^2 + 3t_1 - t_2}$$

$$\Rightarrow \left. \frac{\partial}{\partial t_1} M_{\vec{X}}(\vec{t}) \right|_{\vec{t} = \langle 0, 0 \rangle} = 3 \cdot 1 = 3$$

$$\frac{\partial^2}{\partial t_1^2} M_{\vec{X}}(\vec{t}) = \frac{\partial}{\partial t_1} \left[\left(t_1 + \frac{3}{2} t_2 + 3 \right) e^{\frac{1}{2} t_1^2 + \frac{3}{2} t_1 t_2 + 2 t_2^2 + 3t_1 - t_2} \right]$$

$$= e^{\frac{1}{2} t_1^2 + \frac{3}{2} t_1 t_2 + 2 t_2^2 + 3t_1 - t_2} \cdot \left[1 + \left(t_1 + \frac{3}{2} t_2 + 3 \right)^2 \right]$$

$$\Rightarrow \left. \frac{\partial^2}{\partial t_1^2} M_{\vec{X}}(\vec{t}) \right|_{\vec{t} = \langle 0, 0 \rangle} = 10$$

$$\frac{\partial^2}{\partial t_2 \partial t_1} M_X(\vec{t}) = \frac{\partial}{\partial t_2}\left[\left(t_1 + \frac{3}{2}t_2 + 3\right) e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1 t_2 + 2t_2^2 + 3t_1 - t_2}\right]$$

$$= e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1 t_2 + 2t_2^2 + 3t_1 - t_2} \left[\frac{3}{2} + \left(t_1 + \frac{3}{2}t_2 + 3\right)\left(\frac{3}{2}t_1 + 4t_2 - 1\right)\right]$$

$$\Rightarrow \left.\frac{\partial^2}{\partial t_2 \partial t_1} M_{\vec{X}}(\vec{t})\right|_{\vec{t} = \langle 0,0\rangle} = \left(\frac{3}{2} - 3\right) = -\frac{3}{2}.$$

$$\frac{\partial}{\partial t_2} M_{\vec{X}}(\vec{t}) = \frac{\partial}{\partial t_2}\left(e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1 t_2 + 2t_2^2 + 3t_1 - t_2}\right)$$

$$= \left(\frac{3}{2}t_1 + 4t_2 - 1\right) e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1 t_2 + 2t_2^2 + 3t_1 - t_2}$$

$$\Rightarrow \left.\frac{\partial}{\partial t_2} M_{\vec{X}}(\vec{t})\right|_{\vec{t} = \langle 0,0\rangle} = -1$$

$$\frac{\partial^2}{\partial t_1 \partial t_2} M_X(\vec{t}) = \frac{\partial^2}{\partial t_2 \partial t_1} M_X(\vec{t}) \quad \text{(result from MATH 271)}$$

$$\Rightarrow \left.\frac{\partial^2}{\partial t_1 \partial t_2} M_X(\vec{t})\right|_{t = \langle 0,0\rangle} = -\frac{3}{2} \quad \text{again}$$

$$\frac{\partial^2}{\partial t_2^2} M_{\vec{X}}(\vec{t}) = \frac{\partial}{\partial t_2}\left[\left(\frac{3}{2}t_1 + 4t_2 - 1\right) e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1 t_2 + 2t_2^2 + 3t_1 - t_2}\right]$$

$$= e^{\frac{1}{2}t_1^2 + \frac{3}{2}t_1 t_2 + 2t_2^2 + 3t_1 - t_2} \cdot \left[4 + \left(\frac{3}{2}t_1 + 4t_2 - 1\right)^2\right]$$

$$\Rightarrow \left.\frac{\partial^2}{\partial t_2^2} M_{\vec{X}}(\vec{t})\right|_{\vec{t} = \langle 0,0\rangle} = 5$$