Copy B

1. $y' = f(t, y)$ with $f(t,y) = t^2 + \sqrt{y}$. From the IC, $t_0 = 1$, $y_0 = 2$.

$$y_1 = y_0 + hf(t_0, y_0) = 2 + (0.25)\left(1^2 + \sqrt{2}\right) \doteq 2.6036$$

$$t_1 = t_0 + h = 1.25$$

$$y_2 = y_1 + hf(t_1, y_1) = 2.6036 + (0.25)\left(1.25^2 + \sqrt{2.6036}\right) \doteq 3.3976$$

$$t_2 = t_1 + h = 1.5$$

$$y_3 = y_2 + hf(t_2, y_2) = 3.3976 + (0.25)\left(1.5^2 + \sqrt{3.3976}\right) \doteq 4.4209$$

$$t_3 = t_2 + h = 1.75$$

$$y_4 = y_3 + hf(t_3, y_3) = 4.4209 + (0.25)\left(1.75^2 + \sqrt{4.4209}\right) \doteq 5.7122$$

$$t_4 = t_3 + h = 2.0$$

$y(2) \approx 5.7122$.

2. (a) $\alpha = -6$, $\beta = 3$, $\vec{u} = \langle -1, 5 \rangle$, $\vec{w} = \langle -3, 0 \rangle$. So the general soln. is

$$\vec{x}(t) = c_1 e^{-6t}\left(\cos(3t)\begin{bmatrix}-1\\5\end{bmatrix} - \sin(3t)\begin{bmatrix}-3\\0\end{bmatrix}\right) + c_2 e^{-6t}\left(\sin(3t)\begin{bmatrix}-1\\5\end{bmatrix} + \cos(3t)\begin{bmatrix}-3\\0\end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{-6t}[3\sin(3t) - \cos(3t)] & -e^{-6t}[3\cos(3t) + \sin(3t)] \\ 5e^{-6t}\cos(3t) & 5e^{-6t}\sin(3t) \end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

(b) It's easiest to learn the e-values through the relation $A\vec{v} = \lambda\vec{v}$:

$$\begin{bmatrix}-4 & -3\\-6 & -1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}2\\-4\end{bmatrix} = 2\begin{bmatrix}1\\-2\end{bmatrix} \Longrightarrow \lambda = 2 \text{ for e-vector } \begin{bmatrix}1\\-2\end{bmatrix}$$

$$\begin{bmatrix}-4 & -3\\-6 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-7\\-7\end{bmatrix} = -7\begin{bmatrix}1\\1\end{bmatrix} \Longrightarrow \lambda = -7 \text{ for e-vector } \begin{bmatrix}1\\1\end{bmatrix}$$

So,

$$\vec{x}(t) = c_1 e^{2t}\begin{bmatrix}1\\-2\end{bmatrix} + c_2 e^{-7t}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}e^{2t} & e^{-7t}\\-2e^{2t} & e^{-7t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}.$$

3. (a) Since $\alpha = -5 < 0$, solutions are origin-seeking (the origin is stable). Since the e-values are nonreal (w/ $\alpha \neq 0$), the origin is a spiral point. Since $a_{21} = -5 < 0$, trajectories spiral clockwise. These lead to Figure A.

(b) Since the eigenvalues are real but of opposite sign, the origin is an (unstable) saddle. The straight-line trajectories are in the directions of the eigenvectors. See Figure B.

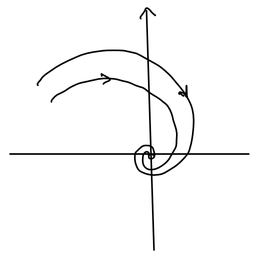


Figure A

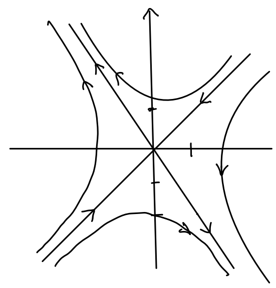


Figure B

4. (a) This DE is first-order linear. It's normal form is

$$y' = -\frac{2}{t}y + t - 1 + \frac{1}{t}, \quad \text{making } a(t) = -\frac{2}{t}, \quad f(t) = t - 1 + \frac{1}{t}.$$

So, the homogeneous soln. is $x_h(t) = C\varphi(t)$, where

$$\varphi(t) = e^{\int -2t^{-1}dt} = e^{-2\ln|t|} = e^{\ln t^{-2}} = t^{-2}.$$

Using variation of parameters,

$$x_p(t) = \varphi(t)\int \frac{f(t)}{\varphi(t)}\,dt = t^{-2}\int (t^3 - t^2 + t)\,dt = t^{-2}\left(\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2\right)$$

The soln.:

$$x(t) = x_h(t) + x_p(t) = Ct^{-2} + \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2}.$$

(b) This DE is nonlinear, but separable.

$$\frac{dx}{dt} = \frac{-2t+3}{4x^3} \implies \int 4x^3\,dx = \int (-2t+3)\,dt$$

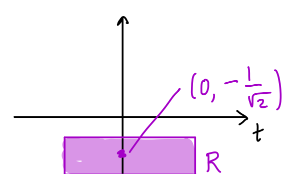
$$\implies x^4 = -t^2 + 3t + C$$

Explicit expressions for x might be either $x(t) = \pm\sqrt[4]{-t^2+3t+C}$.
But, for the IC to be satisfied, we require the negative 4th root, and $C = \frac{1}{4}$:

$$x(t) = -\sqrt[4]{-t^2+3t+\frac{1}{4}}.$$

5. Here $x' = g(t,x)$, with $g(t,x) = \dfrac{-2t+3}{4x^3}$.

The partial derivative $\dfrac{\partial g}{\partial x} = \dfrac{6t-9}{4x^4}$.



Both g and $\partial g/\partial x$ are continuous except at $x=0$, so we can draw a box/rectangle R around the point $(t_0, x_0) = (0, -1/\sqrt{2})$ throughout which both g, $\partial g/\partial x$ are continuous. By the Fundamental Theorem on Existence and Uniqueness, the IVP in 4(b) has exactly one solution.

6. Letting $x_1 = y$, $x_2 = y'$, $x_3 = y''$, we have

$x_1' = x_2$ and $x_2' = x_3$ naturally from our definitions, and

$y''' = 2y'' - 3ty' + 4y + e^{5t}$ becomes $x_3' = 2x_3 - 3t\,x_2 + 4x_1 + e^{5t}$

So,
$$\frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 4x_1 - 3t x_2 + 2x_3 + e^{5t} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -3t & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ e^{5t} \end{bmatrix}.$$

The IC becomes
$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$