MATH 162: Calculus II
Framework for Tues., Mar. 6
Differentiability

Today's Goal: To understand the relationship between partial derivatives and continuity.

The Mixed Partial Derivatives

We have learned that the partial derivative f_x at (x_0, y_0) may be interpreted geometrically as providing the slope at the point $(x_0, y_0, f(x_0, y_0))$ along the curve that results from slicing the surface $z = f(x, y)$ with the plane $y = y_0$. If one thinks of the x-axis as "facing east", then what we are talking about is akin to standing on a patch of (possibly) hilly ground and asking what slope you would immediately experience heading eastward from your current position. Now imagine moving northward (i.e., in the direction of the positive y-axis), but still determining eastward slopes. The rate at which those eastward slopes changed as you moved northward is precisely what $f_{xy} = \partial/\partial y(f_x)$ provides.

One might ask the following question: Suppose I mark a particular spot on this hypothetical terrain. Then I cross over the mark twice. The first time, I do so heading northward, noting the rate of change of eastward-facing slopes as I cross (that is, f_{xy}). The second time, I do so heading eastward, noting the rate at which northward-facing slopes change as I cross (i.e., f_{yx}). Should these two rates of change be equal? There does not seem to be a particular reason why they should be, but experimenting with various formulas $f(x, y)$ we find, nevertheless, that they often are.

Example: $f(x, y) = \cos(x^2 y)$

This phenomenon has much to do with our natural inclination to choose "nice" functions. In general, f_{xy} and f_{yx} are not equal. But, under the conditions of the following theorem, they are.

Theorem: (**The Mixed Derivative Theorem**, p. 26) If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region of the plane containing the point (x_0, y_0), and are all continuous at (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

Differentiability and Continuity

In MATH 161, we learn

- how to differentiate a function of a single variable

- at points of differentiability, the function
  - is also continuous.
  - looks (locally) like a straight line.

For functions of multiple variables, we have learned how to take partial derivatives, and what these partial derivatives represent. Unfortunately, existence of partial derivatives does not, by itself, imply continuity.

Example: For the function

$$f(x,y) := \begin{cases} 1, & \text{if } xy = 0, \\ 0, & \text{if } xy \neq 0, \end{cases}$$

the partial derivatives exist at $(0,0)$. However, f is not continuous at $(0,0)$. (The graph of this function is given on p. 725 of your text.)

We would like functions of multiple variables, like their single-variable counterparts, to be continuous whenever they are differentiable. In light of the previous example, we will require more of such a function than just "its partial derivatives exist" before we call it *differentiable*.

Definition: A function $z = f(x,y)$ is said to be *differentiable at* (x_0, y_0), a point in the domain of f, if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ both exist, and $\Delta z := f(x,y) - f(x_0,y_0)$ satisfies the equation

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{1}$$

where

$$\Delta x := x - x_0, \qquad \Delta y := y - y_0,$$

and ϵ_1, $\epsilon_2 \longrightarrow 0$ as $(x,y) \longrightarrow (x_0,y_0)$.

If we drop the terms in equation (1) that become more and more negligible as $(x,y) \to (x_0,y_0)$ (the ones involving ϵ_1 and ϵ_2), then we obtain the approximation

$$\Delta z \approx f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y, \tag{2}$$

or

$$f(x,y) - f(x_0,y_0) \approx f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0),$$

or

$$f(x,y) \approx f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + f(x_0,y_0).$$

The right-hand side of this last version of the approximation is in the form

$$Ax + By + C.$$

Later in the course, we shall see that this is one form of the equation of a plane. Thus, the definition says that $z = f(x, y)$ is differentiable at (x_0, y_0) if, locally speaking, the surface at the point looks like (is well-approximated by) the plane

$$f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

We know (from the example above) that existence of partial derivatives at the point (x_0, y_0) alone is not sufficient to guarantee that a function is differentiable there. However, the following theorem provides a stronger condition that guarantees it.

Theorem: Let f be be a function of 2 variables whose partial derivatives f_x and f_y are continuous throughout an open region R of the plane. Then f is differentiable at each point of R.

Given our notion of differentiability, we may prove this analog to the theorem from MATH 161 relating differentiability and continuity.

Theorem: If a function f of two variables is differentiable at (x_0, y_0), then f is continuous there.

Differential Notation and Linear Approximation

For functions of one variable $y = f(x)$, we sometimes write $dy = f'(x)dx$. What does this mean?

- dx is an independent variable (think of it like Δx)
- dy is a dependent variable, a function of both x and dx.
- This "differential notation" is another way of writing the linear approximation to f.

Now, for the function $z = f(x, y)$, we may analogously write

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Compare this to equation (2).

Example: The volume of a right circular cylinder is given by $v(r, h) = \pi r^2 h$. Thus

$$dv \;=\; 2\pi rh\, dr + \pi r^2\, dh.$$

Thus, if a cylinder of radius 2 in. and height 5 in. is deformed to a different cylinder, now of radius 1.98 in. and height 5.03 in., then the approximate change in volume is

$$2\pi(2)(5)(-0.02) + \pi(2)^2(0.03) \;=\; -0.8797$$

cubic inches. (The actual change is -0.8809 cubic inches.)