MATH 162: Calculus II
Framework for Thurs., Apr. 19
Polar Coordinates

Today's Goal: To understand the use of polar coordinates for specifying locations on the plane.

Important Note: In conjunction with this framework, you should look over Section 9.1 of your text.

Coordinate Systems for the Plane

1. **Rectangular coordinates** Coordinate pair (x, y) indicates how far one must travel in two perpendicular directions to arrive at point.

2. **Polar coordinates** Coordinate pair (r, θ) indicates *signed distance* and *bearing*

   - The r value (signed distance) is written first, followed by θ.
   - The *bearing* is an angle with the positive horizontal axis, with positive angles taken in the counterclockwise direction.
   - Any polar coordinate pair (r, θ) with $r = 0$ specifies the origin.
   - Each point has infinitely many specifications. For instance,

     $$\cdots = (-1, -\pi) = (1, 0) = (-1, \pi) = (1, 2\pi) = \ldots.$$

     In particular, for any point other than the origin, there are infinitely-many representations (r, θ) in polar coordinates with $r > 0$, and infinitely-many with $r < 0$.

Relationships between Rectangular/Polar Coordinates

- **Rectangular to polar**: If a point (x, y) is specified in rectangular coordinates, it has a corresponding polar representation (r, θ) determined by

  $$r = \sqrt{x^2 + y^2}, \qquad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}.$$

- **Polar to rectangular**: If a point (r, θ) is specified in rectangular coordinates, it has a corresponding polar representation (x, y) determined by

  $$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

When Polar Coordinates Are Useful

Using the "rectangular to polar" conversion above, equations in x and y (and the curves that correspond to them) may be expressed as polar equations (equations involving polar coordinates). Often (though not always), what was a simple equation in rectangular coordinates is uglier in polar coordinates. Nevertheless, even when this is the case, integration of particular functions over particular regions is sometimes more easily carried out in polar form than in rectangular. (See examples of this in the framework for Apr. 23.)

Examples of curves in both forms:

1. **Circles centered at the origin**.

   Rectangular form: $x^2 + y^2 \;=\; a^2$ Polar form: $r \;=\; \pm a$

2. **Circles centered on coordinate axis with point of tangency at the origin**.

   Rectangular: $x^2 + (y-a)^2 \;=\; a^2$ Polar: $r \;=\; \pm 2a\sin\theta$

3. **Certain ellipses**.

   Rectangular: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \;=\; 1$ Polar: $r = \dfrac{b}{\sqrt{1-\epsilon^2\cos^2\theta}}$, where $\epsilon = \sqrt{1-\dfrac{b^2}{a^2}}$

4. **Lines through the origin**.

   Rectangular: $y \;=\; mx$ Polar: $\theta = c$, where $c = \arctan m$

5. **Horizontal and vertical lines**.

   Rectangular: $y \;=\; b$ Polar: $r \;=\; b\csc\theta$

Polar Functions

For equations in rectangular coordinates, we often express y as a function of x (i.e., treat x as independent) when this is possible. Similarly, when it is possible to make θ the independent variable (i.e., when a polar equation may be solved for r), we tend to prefer doing so, writing $r = f(\theta)$. The polar forms in examples 1–3 and 5 above were expressed this way.

Examples of families of polar curves of some interest:

Note: If you place your graphing calculator in *polar* mode, you should be able to plot any polar curve written in the form $r = f(\theta)$. See these and other polar curves at this link.

1. **Cardioids**: $r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$
2. **Lemniscates**: $r^2 = a^2\cos(2\theta)$, $r^2 = a^2\sin(2\theta)$, etc.
3. **Limaçons**: $r = a + b\cos\theta$ or $r = a + b\sin\theta$
4. **Rose curves**: $r = a\cos(b\theta)$ or $r = a\sin(b\theta)$
5. **Spirals**: $r = a\theta$, $r = e^{a\theta}$, etc.