MATH 162: Calculus II

Framework for Mon., Feb. 26

Convergence of Taylor Series

Today's Goal: To determine if a function equals its power series.

In a remark from the last class, it was stated that, while a certain function f may allow the construction of a Taylor series about $x = a$ with positive radius of convergence, one may not assume this Taylor series converges to f. In our "favorite Taylor series" (see the framework for that day), however, the convergence of the MacLaurin series for $(1-x)^{-1}$, $\arctan x$ and $\ln(1+x)$ to their respective functions throughout their intervals of convergence has already been established. What has yet to be established is whether the MacLaurin series for e^x, $\cos x$ and $\sin x$ converge to their respective functions.

The Remainder

Suppose f has $(n+1)$ derivatives throughout an interval I around $x = a$. Under these conditions, we can write down the n^{th}-order Taylor polynomial for f about $x = a$:

$$P_{n,a}(x) \;=\; f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

Here the subscript a has been added to indicate that this polynomial is about $x = a$. The discrepancy between the function and its Taylor polynomial is called the *remainder* term:

$$R_{n,a}(x) \;:=\; f(x) - P_{n,a}(x).$$

Theorem (Lagrange): Suppose f, $P_{n,a}$ and $R_{n,a}$ are as described above, and that x (fixed) is a number in the interval I. Then there is a number t between a and x such that

$$R_{n,a}(x) \;=\; \frac{f^{(n+1)}(t)}{(n+1)!}\,(x-a)^{n+1}.$$

Example: We can use Lagrange's theorem to show that $\sin x$ is equal to its MacLaurin series for every real number x. For any (fixed) x, the theorem guarantees the existence of a number t between 0 and x such that

$$|R_{n,0}(x)| \;=\; \frac{|\sin^{(n+1)}(t)|}{(n+1)!}\,|x|^{n+1} \;\leq\; \frac{|x|^{n+1}}{(n+1)!} \;\to\; 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$\lim_{n\to\infty} P_{n,0}(x) \;=\; \lim_{n\to\infty}\left[\sin x - R_{n,0}(x)\right] \;=\; \sin x - \lim_{n\to\infty} R_{n,0}(x) \;=\; \sin x,$$

which says that the sequence of partial sums of the MacLaurin series for the sine function converges to sine at x. Since we did not assume anything special about the x involved in this calculation, the result holds for any real x.

A similar type of argument may be used to establish that the MacLaurin series for e^x converges to e^x for all real x, and that the MacLaurin series for $\cos x$ converges to $\cos x$ for all real x.

Example (a weird function): Let f be defined by the formula

$$f(x) := \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It can be shown that $f^{(n)}(0) = 0$ for all integer $n \geq 0$. The MacLaurin series for f is thus

$$\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0,$$

the zero function (not even an infinite series, so of course it converges for all x). A graph of f appears in Figure 8.14 on p. 558 of the text. It may not be obvious from the picture, but while $f(0) = 0$, for all other choices of x, $f(x) > 0$. Hence, the only place its MacLaurin series equals f is at $x = 0$.