1\. (a) $\det\left(\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\right) = 1$, so this matrix has rank 2, and the columns are a basis for $\mathbb{R}^2$.

(b)
$$\vec{x} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

(c) $\vec{b} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $[\vec{b}]_{\mathcal{B}_1} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$.

(d) M is the matrix of $C_{\mathcal{B}_2} \circ id \circ C_{\mathcal{B}_1}^{-1}$

$$\Rightarrow M = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -14 & 9 \\ 25 & -16 \end{bmatrix}$$

2\. The vectors $\vec{w}_1 = \langle -2, 1, -5 \rangle$, $\vec{w}_2 = \langle -2, 1, 1 \rangle$ and $\vec{w}_3 = \langle 1, 2, 0 \rangle$ are eigenvectors and mutually orthogonal already. So A is orthogonally diagonalizable. We obtain P by first turning these vectors into unit vectors:

$$\vec{u}_1 = \frac{1}{\sqrt{(-2)^2 + 1^2 + (-5)^2}} \vec{w}_1 = \frac{1}{\sqrt{30}} \langle -2, 1, -5 \rangle.$$

$$\vec{u}_2 = \frac{1}{\sqrt{(-2)^2 + 1^2 + 1^2}} \vec{w}_2 = \frac{1}{\sqrt{6}} \langle -2, 1, 1 \rangle$$

$$\vec{u}_3 = \frac{1}{\sqrt{1^2 + 2^2}} \langle 1, 2, 0 \rangle = \frac{1}{\sqrt{5}} \langle 1, 2, 0 \rangle$$

So,

$$P = \begin{bmatrix} -2/\sqrt{30} & -2/\sqrt{6} & 1/\sqrt{5} \\ 1/\sqrt{30} & 1/\sqrt{6} & 2/\sqrt{5} \\ -5/\sqrt{30} & 1/\sqrt{6} & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3\. (a), (c), (h)

4\. $\det(xI - A) = \begin{vmatrix} x-2 & -5 \\ 1 & x \end{vmatrix} = x(x-2) + 5 = x^2 - 2x + 5$

$$\Rightarrow x = \frac{2}{2} \pm \frac{\sqrt{4 - 4(1)(5)}}{2} = 1 \pm \frac{4\sqrt{-1}}{2} = 1 \pm 2i$$

5. (a) $E_{-2} = \text{null}(-2I - A)$ and

$$-2I - A = \begin{bmatrix} -4 & 2 & -2 \\ 8 & -4 & 4 \\ 4 & -2 & 2 \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 = 0, \text{ or } x_1 = \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ x_2 = s,\ x_3 = t \text{ are free} \end{array}$$

eigenvectors corresponding to $\lambda = -2$ satisfy

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \frac{1}{2}s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{2}t \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

So, a basis of E_{-2}: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

(b) In part (a), we learned $\lambda = -2$ has $GM = 2$, matching its algebraic multiplicity. Since the characteristic polynomial of A is degree 3, it can have only 3 roots: $\lambda = -2$ (twice) and $\lambda = 4$ (necessarily once). So, $AM = GM$ for this last eigenvalue, too. And since no eigenvalue is degenerate (i.e., with $GM < AM$), A is diagonalizable.

7. Since $A\vec{x}_1 = A\vec{x}_2$, we can subtract all to one side:

$$A\vec{x}_1 - A\vec{x}_2 = \vec{0} \quad \text{or} \quad A(\vec{x}_1 - \vec{x}_2) = \vec{0}.$$

But this, by definition, says $\vec{x}_1 - \vec{x}_2 \in \text{null}(A)$.