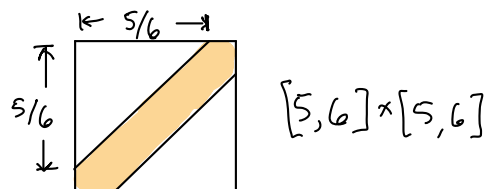


3.55 (a) $f_{X,Y}(x,y) = 1$ for $(x,y) \in [5,6] \times [5,6]$.

(b) $\Pr((X,Y) \in [5,5.5] \times [5,5.5]) = 1/4$.

(c) For them to arrive within 10 minutes of each other, (X,Y) must be a point in the orange shaded region, which has area (= probability)

$$1 - (2)(\frac{1}{2})(5/6)^2 = \frac{11}{36}.$$



3.57 (a) Let $A = \{(x,y) \mid x^2 + y^2 \leq R^2\}$. For the pdf to be a constant k , we require

$$1 = k \cdot \text{Area}(A) = \pi R^2 k \Rightarrow k = \frac{1}{\pi R^2}.$$

Thus

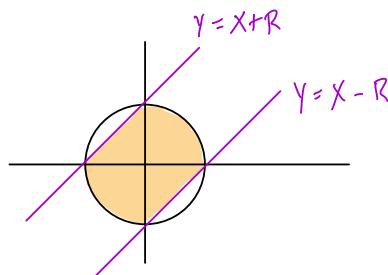
$$f(x,y) = \begin{cases} \frac{1}{\pi R^2}, & (x,y) \in A, \\ 0, & \text{otherwise} \end{cases}$$

(b) Let $H = \{(x,y) \mid x^2 + y^2 \leq \frac{R}{2}\}$. $\Pr((x,y) \in H) = \frac{\text{Area}(H)}{\text{Area}(A)} = \frac{\pi(R/2)^2}{\pi R^2} = \frac{1}{4}.$

(c) $|X - Y| \leq R \Leftrightarrow X - R \leq Y \leq X + R$

The desired probability corresponds to the area of the orange shaded region:

$$\left(\frac{1}{2} \pi R^2 + R^2\right) \cdot \frac{1}{\pi R^2} = \frac{1}{2} + \frac{1}{\pi}.$$



(d) Let $x \in [-R, R]$. Then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy = \frac{2}{\pi R^2} \sqrt{R^2-x^2}$$

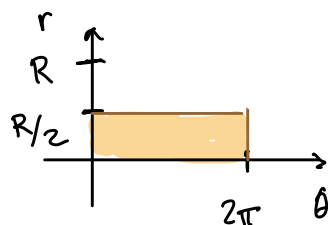
(e) X and Y are not independent. By symmetry, $f_Y(y) = \frac{2}{\pi R^2} \sqrt{R^2-y^2}$ for $y \in [-R, R]$.

But $f_X(x)f_Y(y) = \frac{4}{\pi^2 R^4} \sqrt{(R^2-x^2)(R^2-y^2)} \neq \frac{1}{\pi R^2}.$

3.58 This gives a different distribution. Under this approach,

$$\Pr((x,y) \in H) = \frac{1}{2\pi R} \cdot \text{Area}(\text{shaded region of } r\theta\text{-plane})$$

$$= \frac{1}{2\pi R} \cdot (2\pi)(\frac{1}{2}R) = \frac{1}{2}$$



a different result than in part (b) of Exercise 3.57.

3.63 $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$, so $M_X(t) = e^{e^t \lambda_1 - \lambda_1}$, and $M_Y(t) = e^{e^t \lambda_2 - \lambda_2}$.

By independence of X, Y ,

$$\begin{aligned} M_{X+Y}(t) &= M_X(t) M_Y(t) = e^{e^t \lambda_1 - \lambda_1} \cdot e^{e^t \lambda_2 - \lambda_2} \\ &= e^{e^t \lambda_1 + e^t \lambda_2 - (\lambda_1 + \lambda_2)} = e^{e^t (\lambda_1 + \lambda_2) - (\lambda_1 + \lambda_2)}. \end{aligned}$$

This is the mgf for another Poisson r.v. with parameter $\lambda_1 + \lambda_2$. Thus

$$X + Y \sim \text{Pois}(\lambda_1 + \lambda_2).$$

3.67 Each $X_i \sim \text{Gamma}(\alpha, \lambda)$. By independence,

$$M_S(t) = \prod_{i=1}^n \left(\frac{\lambda}{\lambda - t} \right)^\alpha = \left(\frac{\lambda}{\lambda - t} \right)^{n\alpha} \Rightarrow S \sim \text{Gamma}(n\alpha, \lambda).$$

$$M_{\bar{X}}(t) = M_{\frac{1}{n}S}(t) = M_S\left(\frac{1}{n}t\right) = \left(\frac{\lambda}{\lambda - t/n} \right)^{n\alpha} = \left(\frac{n\lambda}{n\lambda - t} \right)^{n\alpha} \Rightarrow \bar{X} \sim \text{Gamma}(n\alpha, n\lambda).$$

4.1 Let $\bar{X} = \frac{1}{n} \sum X_i$ be the first sample moment about the origin (a.k.a. the sample mean). The population mean for $\text{Binom}(1, \pi)$ is $1 \cdot \pi = \pi$. Our estimate is $\hat{\pi} = \bar{X}$.

4.4 For $X \sim \text{NBinom}$, $E(X) = \frac{\Delta}{\pi} - \Delta$. So, we set

$$\frac{\Delta}{\hat{\pi}} - \Delta = \bar{X} \Rightarrow \hat{\pi} = \frac{\Delta}{\Delta + \bar{X}}.$$

4.7 favstats reveals $\bar{X} \doteq 0.6091$, $s \doteq 0.248$, $n = 134 \Rightarrow v = 0.06105$.

From the formulas:

$$\hat{\alpha} = \bar{X} \left(\frac{\bar{X}(1-\bar{X})}{v} - 1 \right) \doteq 1.7665$$

$$\hat{\beta} = (1-\bar{X}) \left(\frac{\bar{X}(1-\bar{X})}{v} - 1 \right) \doteq 1.1337$$

The beta distribution using these shape parameters gives a very poor fit to the data. By filtering out the players with $\text{FTPct} = 0$, the new parameter estimates from remaining players are

$$\hat{\alpha} = 4.824, \quad \hat{\beta} = 2.387,$$

and the fit is vastly improved.

4.9 Take $\bar{x} = \frac{1}{n} \sum x_i$ and $v = \frac{1}{n} \sum (x_i - \bar{x})^2$.

For an exponential distribution, $\hat{\lambda} = \frac{1}{\bar{x}} \doteq 0.00763$

For a gamma distribution,

$$\frac{\hat{\alpha}}{\hat{\lambda}} = \bar{x} \Rightarrow \hat{\alpha} = \hat{\lambda} \bar{x}$$

$$\frac{\hat{\alpha}}{\hat{\lambda}^2} = v \Rightarrow \frac{\hat{\lambda} \bar{x}}{\hat{\lambda}^2} = \frac{\bar{x}}{\hat{\lambda}} = v$$

$$\Rightarrow \hat{\lambda} = \frac{\bar{x}}{v} \doteq 1.858 \quad \text{and} \quad \hat{\alpha} = \hat{\lambda} \bar{x} = \frac{\bar{x}^2}{v} \doteq 0.0141.$$

4.14 We know $SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$.

Thus, $Pr(|\bar{X} - \mu| < 3) = \text{pnorm}(3, 0, 2) - \text{pnorm}(-3, 0, 2) \doteq 0.866$.

4.16 (a) The 10 different SRS, along w/ resulting sample means:

Sample	\bar{x}	sample	\bar{x}
1, 6	3.5	6, 8	7
1, 6	3.5	6, 9	7.5
1, 8	4.5	6, 8	7
1, 9	5	6, 9	7.5
6, 6	6	8, 9	8.5

$$\Rightarrow \mu_{\bar{x}} = (3.5 + 7 + 7.5)(2/10) + (4.5 + 5 + 6 + 8.5)(1/10) = 6.$$

$$\begin{aligned} \text{Var}(\bar{x}) &= (3.5^2 + 7^2 + 7.5^2)(2/10) + (4.5^2 + 5^2 + 6^2 + 8.5^2)(1/10) - \mu_{\bar{x}}^2 \\ &= 2.85. \end{aligned}$$

(b) From Coro. 4.3.3, we have

$$E(\bar{X}) = \mu = (1+6+6+8+9)(1/5) = 6,$$

a match with part (a). Furthermore,

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n} \cdot \frac{N-n}{N-1} = \frac{\sigma^2}{2} \cdot \frac{5-2}{5-1} = \frac{3}{8} \sigma^2,$$

where $\sigma^2 = (1^2+6^2+6^2+8^2+9^2)(1/5) - 6^2 = 7.6$. So,

$$\text{Var}(\bar{X}) = 3/8 \sigma^2 = 2.85, \quad \text{also matching part (a).}$$

(c) We may treat an iid sample as if we were rolling 5-sided dice, yielding pairings:

$$\begin{array}{ccccc} (1,1) & (1,6) & (1,6) & (1,8) & (1,9) \\ (6,1) & (6,6) & (6,6) & (6,8) & (6,9) \\ (6,1) & (6,6) & (6,6) & (6,8) & (6,9) \\ (8,1) & (8,6) & (8,6) & (8,8) & (8,9) \\ (9,1) & (9,6) & (9,6) & (9,8) & (9,9) \end{array}$$

The 5-by-5 table of means corresponds directly to these pairings

1	3.5	3.5	4.5	5
3.5	6	6	7	7.5
3.5	6	6	7	7.5
4.5	7	7	8	8.5
5	7.5	7.5	8.5	9

So,

$$\mu = (1+8+9)(1/25) + (4.5+5+8.5)(2/25) + (3.5+6+7+7.5)(4/25) = 6,$$

and

$$\begin{aligned} \sigma^2 &= (1^2+8^2+9^2)(1/25) + (4.5^2+5^2+8.5^2)(2/25) + (3.5^2+6^2+7^2+7.5^2)(4/25) - 6^2 \\ &= 3.8. \end{aligned}$$

$$4.39 \quad (a) \quad |\vec{u}_i|^2 = \left(\frac{1}{\sqrt{n}}\right)^2 + \dots + \left(\frac{1}{\sqrt{n}}\right)^2 = n \cdot \left(\frac{1}{n}\right) = 1 \quad \Rightarrow \quad |\vec{u}_i| = 1.$$

For $i = 2, 3, \dots, n$,

$$\begin{aligned} |\vec{u}_i|^2 &= \frac{1}{i(i-1)} \left[(i-1)^2 + \sum_{j=1}^{i-1} 1^2 \right] = \frac{1}{i(i-1)} [(i-1)^2 + (i-1)] \\ &= \frac{i-1}{i(i-1)} [(i-1) + 1] = \frac{1}{i} (i) = 1 \quad \Rightarrow \quad |\vec{u}_i| = 1. \end{aligned}$$

Moreover, for $1 < i < j$,

$$\vec{u}_i \cdot \vec{u}_j = 1 - i + \sum_{m=1}^{i-1} 1 = (1-i) + (i-1) = 0.$$

It is more transparent that each $\vec{u}_i \cdot \vec{u}_i = 0$, $i > 1$.

(b) Let $\vec{x} = \langle x_1, x_2, \dots, x_n \rangle$. Then,

$$\vec{x} \cdot \vec{u}_1 = \frac{1}{\sqrt{n}} (x_1 + x_2 + \dots + x_n) = \sqrt{n} \cdot \frac{1}{n} \sum x_i = \bar{x} \sqrt{n}.$$

(c) With \vec{x} as in part (b),

$$(\vec{x} \cdot \vec{u}_1) \vec{u}_1 = (\bar{x} \sqrt{n}) \cdot \frac{1}{\sqrt{n}} \langle 1, 1, \dots, 1 \rangle = \langle \bar{x}, \bar{x}, \dots, \bar{x} \rangle.$$

(d) For $\vec{x} = \langle 3, 4, 4, 7, 7 \rangle$, $\bar{x} = \frac{1}{5}(25) = 5$, and

$$\vec{v} = \langle 3, 4, 4, 7, 7 \rangle - \langle 5, 5, 5, 5, 5 \rangle = \langle -2, -1, -1, 2, 2 \rangle.$$

Thus,

i	\vec{u}_i	$\vec{x} \cdot \vec{u}_i$	$\vec{v} \cdot \vec{u}_i$	$ \text{proj}(\vec{x} \rightarrow \vec{u}_i) $	$ \text{proj}(\vec{v} \rightarrow \vec{u}_i) $
1	$\frac{1}{\sqrt{5}} \langle 1, 1, 1, 1, 1 \rangle$	$5^{3/2}$	0	$5^{3/2}$	0
2	$\frac{1}{\sqrt{2}} \langle 1, -1, 0, 0, 0 \rangle$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
3	$\frac{1}{\sqrt{6}} \langle 1, 1, -2, 0, 0 \rangle$	$-1/\sqrt{6}$	$-1/\sqrt{6}$	$1/\sqrt{6}$	$1/\sqrt{6}$
4	$\frac{1}{\sqrt{12}} \langle 1, 1, 1, -3, 0 \rangle$	$-10/\sqrt{12}$	$-10/\sqrt{12}$	$10/\sqrt{12}$	$10/\sqrt{12}$
5	$\frac{1}{\sqrt{20}} \langle 1, 1, 1, 1, -4 \rangle$	$-10/\sqrt{20}$	$-10/\sqrt{20}$	$10/\sqrt{20}$	$10/\sqrt{20}$

For $i \geq 2$, $\vec{x} \cdot \vec{u}_i = \vec{v} \cdot \vec{u}_i$. This is true for other $\vec{x} \in \mathbb{R}^5$, too, since

$$\vec{v} \cdot \vec{u}_i = (\vec{x} - \bar{x} \vec{u}_1) \cdot \vec{u}_i = \vec{x} \cdot \vec{u}_i - \bar{x} \cdot \vec{u}_i = \vec{x} \cdot \vec{u}_i - 0$$

since $\bar{x} \vec{u}_1$, being parallel to \vec{u}_1 , is orthogonal to each \vec{u}_i with $i \geq 2$.

4.40 (a) $\bar{x} = (3+4+5+8)/4 = 5$

$$s^2 = \frac{1}{3} \left[(3-5)^2 + (4-5)^2 + (5-5)^2 + (8-5)^2 \right] = \frac{14}{3}.$$

(b) $\vec{p}_1 = \langle 5, 5, 5, 5 \rangle$, as determined in Exercise 4.39.

$$\vec{p}_2 = \frac{1}{\sqrt{2}} (3-4) \vec{u}_2 = \langle -\frac{1}{2}, \frac{1}{2}, 0, 0 \rangle$$

$$\vec{p}_3 = \frac{1}{\sqrt{6}} (3+4-10) \vec{u}_3 = \langle -\frac{1}{2}, -\frac{1}{2}, 1, 0 \rangle$$

$$\vec{p}_4 = \frac{1}{\sqrt{12}} (3+4+5-24) \vec{u}_4 = \langle -1, -1, -1, 3 \rangle$$

and $\sum \vec{p}_i = \langle 3, 4, 5, 8 \rangle$ as predicted.

$$(c) \quad l_1 = |\vec{p}_1| = \sqrt{4(5^2)} = 10$$

$$l_2 = |\vec{p}_2| = \sqrt{2\left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

$$l_3 = |\vec{p}_3| = \sqrt{2\left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{3/2}$$

$$l_4 = |\vec{p}_4| = \sqrt{3(-1)^2 + 3^2} = 2\sqrt{3}$$

$$(d) \quad \sum_{i=2}^4 l_i^2 = \frac{1}{2} + \frac{3}{2} + 12 = 14 = 3\left(\frac{14}{3}\right) = 3s^2.$$

$$C.10 \quad \text{proj}(\langle 1, 2, 3 \rangle \rightarrow \langle 1, 1, 1 \rangle) = \frac{\langle 1, 2, 3 \rangle \cdot \langle 1, 1, 1 \rangle}{|\langle 1, 1, 1 \rangle|^2} \langle 1, 1, 1 \rangle = \langle 2, 2, 2 \rangle$$

This is the closest vector to $\langle 1, 2, 3 \rangle$ in $W = \text{span}(\langle 1, 1, 1 \rangle)$, so

$$\langle 1, 2, 3 \rangle - \text{proj}(\langle 1, 2, 3 \rangle \rightarrow \langle 1, 1, 1 \rangle) = \langle -1, 0, 1 \rangle$$

is perpendicular/orthogonal to $\langle 1, 1, 1 \rangle$, and

$$\text{span}\{\langle 1, 1, 1 \rangle, \langle 1, 2, 3 \rangle\} = \text{span}\{\langle 1, 1, 1 \rangle, \langle -1, 0, 1 \rangle\}.$$

To get these orthogonal vectors to be of length 1, we divide by their lengths:

$$\vec{u}_1 = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle.$$