1. (a) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & -2 & 4 & 2 \\ 2 & 3 & -1 & 1 \end{bmatrix} \underset{\sim}{-3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -8 & 7 & -4 \\ 2 & 3 & -1 & 1 \end{bmatrix}$

$\underset{\sim}{-2\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -8 & 7 & -4 \\ 0 & -1 & 1 & -3 \end{bmatrix}$

$\underset{\sim}{\mathbf{r}_2 \leftrightarrow \mathbf{r}_3,} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & -8 & 7 & -4 \end{bmatrix}$

$\underset{\sim}{-8\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & -1 & 20 \end{bmatrix}$

(b) $\underset{\sim}{-r_2 \to r_2} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 20 \end{bmatrix}$

$\underset{\sim}{-r_3 \to r_3} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -20 \end{bmatrix}$

$\underset{\sim}{r_3 + r_2 \to r_2} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & -17 \\ 0 & 0 & 1 & -20 \end{bmatrix}$

$\underset{\sim}{r_3 + r_1 \to r_1} \begin{bmatrix} 1 & 2 & 0 & -18 \\ 0 & 1 & 0 & -17 \\ 0 & 0 & 1 & -20 \end{bmatrix}$

$\underset{\sim}{-2r_2 + r_1 \to r_1} \begin{bmatrix} 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & -17 \\ 0 & 0 & 1 & -20 \end{bmatrix}$ RREF

2. B must be 2x2 for the sum on the left to make sense.

$$2B = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 6 & 2 \end{bmatrix} - \begin{bmatrix} -5 & 4 \\ -2 & -6 \end{bmatrix} = \begin{bmatrix} -3 & -8 \\ -8 & -2 \end{bmatrix} - \begin{bmatrix} -5 & 4 \\ -2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -12 \\ -6 & 4 \end{bmatrix}$$

$$\Rightarrow \quad B = \begin{bmatrix} 1 & -6 \\ -3 & 2 \end{bmatrix}$$

3. $$[A|\vec{b}] = \left[\begin{array}{ccccc|c} 2 & -3 & 2 & 1 & 1 & -4 \\ 4 & -6 & 1 & -4 & 0 & -22 \\ -2 & 3 & 1 & 5 & 2 & 19 \\ 2 & -3 & -1 & -5 & -3 & -20 \end{array}\right] \xhookrightarrow{\text{RREF}} \left[\begin{array}{ccccc|c} 1 & -3/2 & 0 & -3/2 & 0 & -13/2 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_1 &= \tfrac{3}{2}x_2 + \tfrac{3}{2}x_4 - \tfrac{13}{2} \\ x_3 &= -2x_4 + 4 \\ x_5 &= 1 \end{aligned}\right\} \Rightarrow \text{solns. } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3/2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -13/2 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.$$

4. $$\begin{vmatrix} 5-\lambda & 5 \\ -5 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) - (-25) = \lambda^2 - 4\lambda + 20$$

$$\lambda = \frac{4}{2} \pm \frac{\sqrt{16-80}}{2} = 2 \pm \frac{\sqrt{-64}}{2} = 2 \pm 4i$$

5. We solve $\left[A - (-2I)\right]\vec{v} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -3 & 6 & -3 & 0 \\ -3 & 6 & -3 & 0 \\ 3 & -6 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 = 2v_2 - v_3$$

So, corresponding eigenvectors take the form

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_2 - v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{basis: } \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

6. (a) $$\begin{bmatrix} 0 & 19 & 19 & -15 \\ -1 & 11 & 10 & -15 \\ -3 & 37 & 34 & -50 \\ 0 & 3 & 3 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ revealing that columns 1, 2, and 4 are linearly independent.

So, a basis for the column space is $$\begin{bmatrix} 0 \\ -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ 11 \\ 37 \\ 3 \end{bmatrix}, \begin{bmatrix} -15 \\ -15 \\ -50 \\ -3 \end{bmatrix}$$

(b) $4 - \text{rank}(A) = 4 - 3 = 1$ tells the dimension of $\text{null}(A)$.