Math 251, Wed 14-Oct-2020 -- Wed 14-Oct-2020
Discrete Mathematics
Fall 2020

Wednesday, October 14th 2020

Topic:: Induction
Read:: Rosen 5.1

$m \times n$ matrix is a table of nos. w/ m rows, n cols.

$$A_{n\times n} \cdot B_{n\times n} = C_{n\times n} = (c_{ij})$$

refer to the entry of A in row i, col j. by a_{ij}
" " " " " B " " " " " b_{ij}

From linear alg. you learn that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}}_{n \text{ mults, } n-1 \text{ adds}}$$

$\longrightarrow$ $2n-1$ flops

Entire matrix C has n^2 entries, each requiring $2n-1$ flops

$\longrightarrow$ alg. for matrix mult. is $n^2(2n-1) = O(n^3)$.

$P(k)$: $1+3+5+\cdots+\underline{(2k-1)} = k^2$ (can assume)

$P(k+1)$: $1+3+5+\cdots+(2k-1)+(2k+1) = (k+1)^2$ (must show)

Algorithmic Complexity

Basic idea: relate size n of input to, for instance

- time complexity (often assessed by number of steps)
  - worst-case analysis
  - average-case analysis
- space complexity
- terms like
  - linear complexity
  - quadratic complexity
  - polynomial complexity
  - exponential complexity

Algorithm:

1. Seek divisor of $n \in \mathbb{Z}^+$
   Similar to analysis of linear search algorithm
2. binary search

   $O(\log n)$

3. bubble sort

   list $n = 6$

   input list: 51, 39, 72, 18, 9, 44

   39, 51, 72, 18, 9, 44

   39, 51, 18, 72, 9, 44

   39, 51, 18, 9, 72, 44

   39, 51, 18, 9, 44, $\boxed{72}$

   5 comparisons

   # of comparisons
   pass 1: 5
   2: 4
   3
   2
   1
   15 comps.

   If input list size n
   total comparisons $\boxed{(n-1)+(n-2)+\cdots+1}$
   $= n(n-1)/2$

   $O(n^2)$

4. matrix multiplication

// Rosen Ch. 3

Ch. 4

$\forall n \in \mathbb{Z}^+,\ \underbrace{1+2+3+\cdots+n = \frac{n(n+1)}{2}}_{P(n)}$

Mathematical Induction

- It is a technique for proving a statement $\boxed{\forall n \in \mathbb{Z}^+\, P(n).}$
- Can be adapted to prove the correctness of some algorithms.
- As a rule of inference, it is

$$(\overbrace{P(1)}^{\text{basis step}} \wedge \forall k\, \overbrace{(\underbrace{P(k)}_{\text{inductive hypothesis}} \to P(k+1))}^{\text{inductive step}}) \to \forall n\, P(n).$$

$P(1)$ is called the **basis step**, $P(k) \to P(k+1)$ is called the **inductive step**, and the assumption that the hypothesis $P(k)$ of the inductive step holds is called the **inductive hypothesis**.

Induction is not helpful in discovering in discovering new mathematical statements which are true. Once a pattern or truth has been conjectured, however, induction can often establish that it is true.

Examples:

Claim: $\forall n \in \mathbb{Z}^+,\ P(n)$

1. $\boxed{\overbrace{\sum_{j=1}^{n}(2j - 1) \ = \ 1+3+5+\cdots+(2n - 1) \ = \ n^2}^{P(n)}}$

basis step

$P(1): \ 1 = 1^2$ ✓

inductive step: Can assume $P(k)$ holds for some $k \in \mathbb{Z}^+$. Must show $P(k+1)$ holds

$1+3+5+\cdots+(2k-1)+(2k+1) = [1+3+5+\cdots+(2k-1)] + \underline{2k+1}$

$= k^2 + 2k + 1 = (k+1)^2$ So, $\underline{\forall n \in \mathbb{Z}^+, P(n)}$.

2. For all positive integers, $23^n - 1$ is divisible by 11.

$P(n)$: $23^n - 1$ is divisible by 11. Prove: $\forall n \in \mathbb{Z}^+,\ P(n)$.

basis step: $P(1)$ says $23^1 - 1$ is divisible by 11.

inductive step: $P(k)$ assumed, show $\underline{P(k+1)}$, which says $23^{k+1} - 1$ is div. by 11.

Note $\underline{23^{k+1} - 1} = 23^{k+1} + 0 - 1 = 23^{k+1} - 23^k + \underbrace{23^k - 1}_{\text{is divisible by 11 I.H.}}$

$= \underbrace{23^k(23-1)}_{\text{div. by 11}} + \underbrace{23^k - 1}_{\text{div. by 11}}$, showing $P(k+1)$ holds.

Thus $\forall n,\ P(n)$ holds

3. For all positive integers, $n < 2^n$.