1. Your signature.

2. (a) We can use the given eigenvectors to learn corresponding eigenvalues, bypassing the usual process of finding the zeros of the characteristic polynomial:

$$\begin{bmatrix} -4 & -3 \\ -6 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Rightarrow \quad \lambda = 2.$$

$$\begin{bmatrix} -4 & -3 \\ -6 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -7 \end{bmatrix} = (-7)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda = -7.$$

Thus, the general solution is

$$\mathbf{x}(t) \;=\; c_1 e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} e^{2t} & e^{-7t} \\ -2e^{2t} & e^{-7t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

(b) Here, I take $\alpha = 2$, $\beta = 3$, $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, so that the given eigenpair is $\alpha + i\beta$, $\mathbf{u} + i\mathbf{v}$. The general solution, then, is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\alpha t}[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}] + c_2 e^{\alpha t}[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}] \\ &= c_1 e^{2t}\left(\cos(3t)\begin{bmatrix} 7 \\ -1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin(3t)\begin{bmatrix} 7 \\ -1 \end{bmatrix} + \cos(3t)\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) \\ &= e^{2t}\left(c_1\begin{bmatrix} 7\cos(3t) \\ 2\sin(3t) - \cos(3t) \end{bmatrix} + c_2\begin{bmatrix} 7\sin(3t) \\ -\sin(3t) - 2\cos(3t) \end{bmatrix}\right). \end{aligned}$$

3. (a) saddle: for a typical picture, see this link

   (b) unstable spiral (source): for a typical picture, see this link

4. We will first find eigenvectors corresponding to $\lambda = 2$, in null$(\mathbf{A} - 2\mathbf{I})$. Starting with the the matrix $\mathbf{A} - 2\mathbf{I}$:

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} v_1 = 0 \\ v_2 + v_3 = 0 \\ v_3 \text{ is free} \end{array}$$

With one free variable, the geometric multiplicity of $\lambda = 2$ is 1. A basis eigenvector is $\mathbf{v} = \langle 0, 1, -1 \rangle$. We have two eigenpairs, and we need a third. We look for one of the form $e^{2t}(\mathbf{u} + t\mathbf{v})$ with $\mathbf{u}$ a generalized eigenvector satisfying $(\mathbf{A} - 2\mathbf{I})\mathbf{u} = \mathbf{v}$.

$$[\mathbf{A} - 2\mathbf{I} \,|\, \mathbf{v}] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} u_1 = 1 \\ u_2 + u_3 = 1 \\ u_3 \text{ is free} \end{array}$$

Here I take $u_3 = 0$, which makes $\mathbf{u} = \langle 1, 1, 0 \rangle$. The general solution is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3e^{-t} & 0 & e^{2t} \\ 4e^{-t} & e^{2t} & (1+t)e^{2t} \\ 2e^{-t} & -e^{2t} & -te^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

5. (a) This problem is linear, 1st-order, nonhomogeneous. Putting it in normal form, we have

$$y' = -\frac{2}{t}y + \frac{\sin t}{t^2} \qquad \Rightarrow \qquad a(t) = -\frac{2}{t}, \quad f(t) = \frac{\sin t}{t^2}.$$

We have

$$\phi(t) = e^{-\int (2/t)\,dt} = e^{-2\ln|t|} = e^{\ln(t^{-2})} = t^{-2} \qquad \Rightarrow \qquad y_h(t) = Ct^{-2}.$$

This problem does not fit the usual criteria for the method of undetermined coefficients, so we use the variation of parameters formula to find y_p:

$$y_p(t) = \phi(t)\int \frac{f(t)}{\phi(t)}\,dt = t^{-2}\int t^2\left(\frac{\sin t}{t^2}\right)dt = t^{-2}\int \sin t\,dt = -t^{-2}\cos t,$$

taking $C = 0$ in the integral. The general solution is

$$y(t) = y_h(t) + y_p(t) = Ct^{-2} - \frac{\cos t}{t^2}.$$

(b) The DE is separable.

$$4y^3\,dy = (t^3 + t)\,dt \qquad \Rightarrow \qquad \int 4y^3\,dy = \int (t^3 + t)\,dt \qquad \Rightarrow \qquad y^4 = \frac{1}{4}t^4 + \frac{1}{2}t^2 + C$$

$$y(0) = -\frac{1}{\sqrt{2}} \qquad \Rightarrow \qquad C = \frac{1}{4} \qquad \Rightarrow \qquad y^4 = \frac{1}{4}t^4 + \frac{1}{2}t^2 + \frac{1}{4}$$

Need explicit solution $\quad \Rightarrow \quad y(t) = -\sqrt[4]{\frac{1}{4}t^4 + \frac{1}{2}t^2 + \frac{1}{4}}.$ (neg. root so IC is met)

6. The homogeneous version of this (linear) problem is

$$y' = -3y \qquad \text{which has solution} \qquad y_h(t) = Ce^{\int -3\,dt} = Ce^{-3t}.$$

We note that the nonhomogeneous term is $f(t) = (2 + t)e^{-3t}$, a product best described as: (1st-degree polynomial)(exponential). It would seem natural to propose a particular solution also in that form

$$y_p(t) = (At + B)e^{-3t} = Ate^{-3t} + Be^{-3t},$$

but the second of these terms would be of no use, being exactly like $y_h(t)$ in form. We have learned to remedy this by introducing an extra factor t in our proposal:

$$y_p(t) = (At + B)te^{-3t} = (At^2 + Bt)e^{-3t}.$$

We can rearrange the DE to have $y' + 3y$ on the left-hand side and the target function $f(t) = (2 + t)e^{-3t}$ on the right. Using this and $y_p' = (2At + B)e^{-3t} - 3(At^2 + Bt)e^{-3t}$, we have

$$y_p' + 3y_p = (2At + B)e^{-3t} - 3(At^2 + Bt)e^{-3t} + 3(At^2 + Bt)e^{-3t} = (2At + B)e^{-3t}.$$

As the target function is $(t + 2)e^{-3t}$, we see we get a match if we choose A and B so that

$$2A = 1 \quad \text{and} \quad B = 2, \qquad \text{so that} \qquad y_p(t) = \left(\frac{1}{2}t^2 + 2t\right)e^{-3t}.$$