Stat 343, Mon 19-Oct-2020 -- Mon 19-Oct-2020

Monday, October 19th 2020

Wk 8, Mo
Topic:: Joint continuous distributions
Read:: FASt 3.8

$\underline{X_1}, \underline{X_2}, \underline{X_3}$ all normal, independent, μ_i, σ_i

$$X_1 + X_2 + X_3 \overset{?}{\sim} \text{Norm}\left(\mu_1 + \mu_2 + \mu_3, \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}\right)$$

Yes

Joint Distributions for Continuous r.v.s

Definition 1: Let $f\colon \mathbb{R}^k \to \mathbb{R}$ be a nonnegative function for which the multiple integral

$$\int_{\mathbb{R}^k} f(\mathbf{x})\, d\mathbf{x} \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_k)\; dx_k \cdots dx_2\, dx_1 \;=\; 1.$$

Then f is called a **probability density function**. A continuous **random vector** $\mathbf{X} = \langle X_1, X_2, \ldots, X_k\rangle$ whose components X_i are continuous random variables is said to have (joint) pdf f if, for subsets A of $\mathbb{R}^k$,

$$P(\mathbf{X} \in A) \;=\; \int_A f(\mathbf{x})\, d\mathbf{x}.$$

We write $f_{\mathbf{X}}(\mathbf{x})$, or $f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$.

The joint cdf of $\mathbf{X}$, denoted by $F_{\mathbf{X}}(\mathbf{x})$ is evaluated as

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}) &= P(X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k) \\ &= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_k} f(t_1, t_2, \ldots, t_k)\, dt_k \cdots dt_2\, dt_1. \end{aligned}$$

Likewise, the pdf can be obtained from the cdf via differentiation:

$$f(\mathbf{x}) \;=\; f(x_1, x_2, \ldots, x_k) \;=\; \frac{\partial}{\partial x_k}\cdots\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}F(x_1, x_2, \ldots, x_k).$$

We will generally consider bivariate distributions for random vectors $\mathbf{X} = (X_1, X_2)$, but most results carry over naturally to multivariate distributions.

Definition 2: Suppose X, Y are jointly distributed continuous r.v.s with pdf $f_{X,Y}(x, y)$. Define the **marginal density function** for X (resp. Y) to be

$$f_X(x) \;=\; \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \qquad (\text{and } \; f_Y(y) \;=\; \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx).$$

Given the marginal distributions, we can define conditional ones via the ratio

$$f_{X|Y=y}(x) \;=\; f_{X|Y}(x \mid y) \;:=\; \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

of joint to marginal pdf. Here, $f_{X|Y=y}(x)$ is the **conditional distribution** of X given $Y = y$.

If

$$f_{X,Y}(x, y) = f_X(x) f_Y(y),$$

then X, Y are **independent**.

Example 1:

Let $f(x,y) = \left\{ \begin{array}{ll} kxy^2, & 0 \leq x \leq y \leq 1, \\ 0 & \text{otherwise} \end{array} \right\}.$

(a) Find the value of k for which f is a densit function.

(b) Determine if X and Y are independent.

■

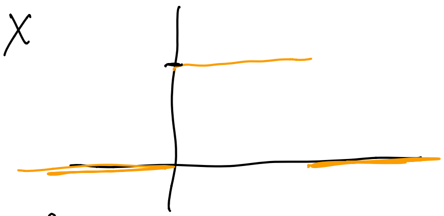
Lemma 1 (Lemma 3.8.7, p. 186 in FASt): If X, Y are independent continuous r.v.s, then for each x, y,

(i) $f_X(x) = f_{X|Y}(x|y)$, and
(ii) $f_Y(y) = f_{Y|X}(y|x)$.

Example 2:

support of pdf

pdf for X



Suppose $X, Y \sim \text{Unif}(0, 1)$ are independent.

(a) What is the joint pdf $f_{X,Y}(x, y)$?

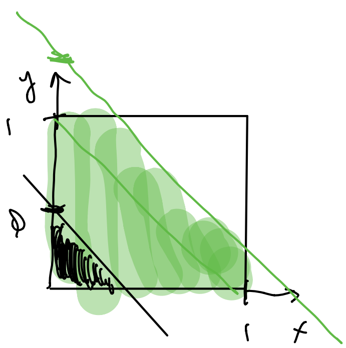
$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases} 1, & 0 \le x, y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

cdf $F_S(s)$

(b) Let $S = X + Y$. What are the cdf and pdf of S?

$$= P(S \le s) = \begin{cases} 0, & s < 0 \text{ or } s > 2 \\ P(X+Y \le s) = P(Y \le s - X) = \int f_Y(y) \end{cases}$$

■



$$= \begin{cases} 0, & s < 0 \text{ or } s > 2 \\ \frac{1}{2} s^2 & 0 \le s < 1 \\ 1 - \frac{1}{2}(2-s)^2, & 1 \le s \le 2 \end{cases}$$

$$f_S(s) = \begin{cases} 0, \\ s, & 0 < s < 1 \\ 2 - s \end{cases}$$

$$+\cancel{\tfrac{1}{2}}\cdot \cancel{2}(2-s)\cdot(+1)$$

Lemma 2 (Lemma 3.8.8, p. 188 in FASt): Let X, Y be independent r.v.s, t and s transformations. Then $t(X)$, $s(Y)$ are independent.

X, Y ind. $\quad \underbrace{X^2, Y^3}$

Theorem 1 (Theorem 3.8.9, p. 188 in FASt): Let X, Y be r.v.s. Then

(i) $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$.

(ii) $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$, if X, Y are independent.

(iii) $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$, with $\mathrm{Cov}(X, Y) = 0$ when X, Y are independent.

proof?
No, it's 2:02.

Theorem 2 (Theorem 3.8.10, p. 189 in FASt): Let M_X, M_Y be moment generating functions, defined on an interval containing 0, for independent r.v.s X, Y. Then $M_{X+Y}(t) = M_X(t)M_Y(t)$, with M_{X+Y} defined on the intersection of intervals of definition for M_X, M_Y.

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right) = E\left(e^{tX} \cdot e^{t \cdot Y}\right) = E\left(e^{tX}\right) \cdot E\left(e^{tY}\right)$$

$$= M_X(t) \cdot M_Y(t)$$

Example 3:

(a) The sum of two independent normal r.v.s is another normal r.v. $X_2 \sim \text{Norm}(\mu_2, \sigma_2)$
Note: For $X_1 \sim \text{Norm}(\mu_1, \sigma_1)$, $M_X(t) = e^{\mu_1 t + \sigma_1^2 t^2/2}$.

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = e^{\mu_1 t + \sigma_1^2 t^2/2} \cdot e^{\mu_2 t + \sigma_2^2 t^2/2}$$

$$= e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2} \Rightarrow X_1 + X_2 \sim \text{Norm}\left(\mu_1+\mu_2, \sqrt{\sigma_1^2+\sigma_2^2}\right)$$

(b) The sum of two independent binomial r.v.s with the same probability π of success is another binomial r.v.
Note: For $X_1 \sim \text{Binom}(m, \pi)$, $M_X(t) = (\pi e^t + 1 - \pi)^m$. $X_2 \sim \text{Binom}(n, \pi)$

$X_1 + X_2$ has $M_{X_1+X_2}(t) = M_{X_1}(t)\, M_{X_2}(t) = (\pi e^t + 1 - \pi)^m (\pi e^t + 1 - \pi)^n$ ■

$$= (\pi e + 1 - \pi)^{m+n}$$

Definition 3: Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random vector with joint pdf $f_{\mathbf{X}}(\mathbf{x})$. Suppose there is a single density function $f(x)$ that serves as the pdf for the marginal distribution for each X_j, so that

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{X}}(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n} f(x_i).$$

Then the r.v.s $X_1, \ldots, X_n$ are said to be **independent and identically distributed**, or i.i.d..

In particular, if the X_j are independent with each $X_j \sim \mathsf{Exp}(\lambda)$, we will denote this by $\mathbf{X} \overset{\text{i.i.d.}}{\sim} \mathsf{Exp}(\lambda)$.

Lemma 3: Suppose $X_1, \ldots, X_n$ are i.i.d. and that each $\mathrm{E}(X_i) = \mu$, each $\mathrm{Var}(X_i) = \sigma^2$. Let $S = X_1 + X_2 + \cdots + X_n$, and $\overline{X} = \frac{1}{n}S$. Then

(i) $\mathrm{E}(S) = n\mu$ and $\mathrm{Var}(S) = n\sigma^2$.
(ii) $\mathrm{E}(\overline{X}) = \mu$ and $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

Lemma 4: Suppose $\mathbf{X} \overset{\text{i.i.d.}}{\sim} \mathsf{Norm}(\mu, \sigma)$, and define S, $\overline{X}$ as in the previous lemma. Then

(i) $S \sim \mathsf{Norm}(n\mu, \sigma\sqrt{n})$, and
(ii) $\overline{X} \sim \mathsf{Norm}(\mu, \sigma/\sqrt{n})$.

Proof: By induction on Theorem 2, we have that

$$M_S(t) \;=\; \prod_{j=1}^{n} M_{X_j}(t) \;=\; \prod_{j=1}^{n} e^{\mu t + \sigma^2 t^2/2} \;=\; e^{n\mu t + n\sigma^2 t^2/2},$$

which is the mgf for a normal r.v. with distribution $\mathsf{Norm}(n\mu, n\sigma^2)$. This proves (i).

For (ii), Theorem 3.3.6 (p. 133) gives that $M_{\overline{X}}(t) = M_{X/n}(t) = M_X(t/n) = e^{\mu t + (\sigma^2/n)t^2/2}$, which is the mgf for an r.v. distributed as $\mathsf{Norm}(\mu, \sigma^2/n)$. $\square$

Question: What if the components X_i of $\mathbf{X}$ have different means μ_i and standard deviations σ_i?