MATH 162: Calculus II

Framework for Tues., May. 1

Integration in Spherical Coordinates

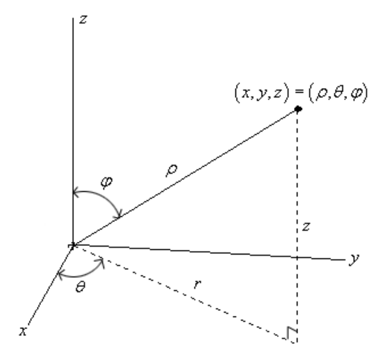
Today's Goal: To learn to set up and evaluate triple integrals in spherical coordinates.

Important Note: In conjunction with this framework, you should look over Section 13.7 of your text.

Simple Equations in Spherical Coordinates and Their Graphs

- $\rho = \rho_0$ (a constant) corresponds to a sphere of radius ρ_0.
- $\phi = \phi_0$ corresponds to a cone with vertex at the origin and the z-axis as axis of symmetry.
- $\theta = \theta_0$ corresponds to a half-plane with z-axis as the terminal edge.

Changing (x, y, z) to (ρ, ϕ, θ)



Recall that we have the following relationships:

$$
\begin{aligned}
x &= \rho \sin\phi \cos\theta, \\
y &= \rho \sin\phi \sin\theta, \\
z &= \rho \cos\phi.
\end{aligned}
$$

Thus, the equation (in rectangular coordinates)

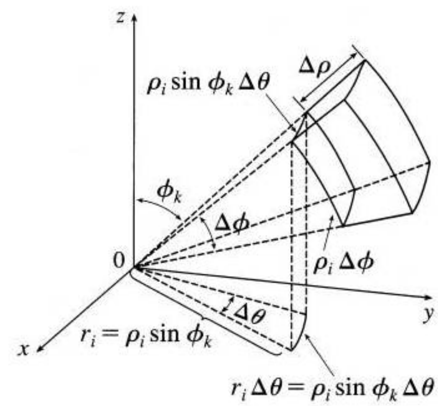
$$(x-2)^2 + y^2 + z^2 = 4$$

for a sphere of radius 2 centered at the point $(x, y, z) = (2, 0, 0)$ may be rewritten as

$$\rho = 2\left(\sin\phi\cos\theta + \sqrt{\sin^2\phi\cos^2\theta + 1}\right).$$

(Try verifying this.)

Volume Element dV in Spherical Coordinates



Pictured at right is a typical "volume element" ΔV at a spherical point (ρ, ϕ, θ) corresponding to small changes $\Delta\rho$, $\Delta\phi$ and $\Delta\theta$ in the spherical variables. Its sides, as can be verified using trigonometry, have approximate measures $\Delta\rho$, $(\rho\Delta\phi)$ and $(\rho \sin\phi\Delta\theta)$. Thus

$$dV \;=\; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

As a result

$$\iiint\limits_D f(x,y,z)\,dV \;=\; \iiint\limits_D \rho^2 \sin\phi \, f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,d\rho\,d\phi\,d\theta,$$

Examples:

1. Evaluate $\iiint_D 16z\,dV$, where D is the upper half of the sphere $x^2+y^2+z^2=1$.
2. Find the volume of the smaller section cut from a solid ball of radius a by the plane $z=1$.