MATH 162: Calculus II

Framework for Thurs., Apr. 26–Fri., Apr. 27

Moments, Centers of Mass

Today's Goal: To learn to apply the skill of setting up and evaluating triple integrals in the context of finding centers of mass.

Important Note: In conjunction with this framework, you should look over Section 13.6 of your text.

Point Masses along a Line (1D)

- Assume n masses $m_1, \ldots, m_n$ sit at locations $x_1, \ldots, x_n$ on a number line.
- Call the location $\bar{x}$ about which the total torque is zero. That is, $\bar{x}$ is the location to place a fulcrum so that

  $$\sum_{k=1}^{n} m_k(x_k - \bar{x}) \;=\; 0.$$

  Solving for $\bar{x}$, we get

  $$\bar{x} \;=\; \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k} \;=:\; \frac{\text{1st moment about } x=0}{\text{total mass}} \tag{1}$$

Continuous Mass along a Line (1D; more instructive than practical)

When

- mass is distributed throughout a continuous body (instead of being concentrated at finitely-many distinct positions),
- $\rho(x)$ gives the mass density (mass per unit length) inside the interval $a \le x \le b$,

then the total mass is

$$\int_a^b \rho(x)\,dx.$$

The continuous analog to the numerator of (**??**) comes from the following definition:

Definition: If $\rho(x)$ is the mass density (in mass per unit *length*) of a substance contained in a region $a \le x \le b$ of 1D space, then the *first moment* about $x = 0$ is given by

$$\int_a^b x\rho(x)\,dx.$$

Using this, the corresponding center of mass is

$$\bar{x} \;=\; \frac{\int_a^b x\rho(x)\,dx}{\int_q^b \rho(x)\,dx} \tag{2}$$

Centers of Mass in 3D

Definition: If $\rho(x, y, z)$ is the mass density (in mass per unit volume) of a substance contained in a region D of 3D space, then the *first moment* about the yz-plane ($x = 0$) is given by

$$M_{yz} \;:=\; \iiint\limits_D x\rho(x, y, z)\,dV.$$

Similarly, the first moments about the xz and xy-planes are

$$M_{xz} \;:=\; \iiint\limits_D y\rho(x, y, z)\,dV \quad \text{and} \quad M_{xy} \;:=\; \iiint\limits_D z\rho(x, y, z)\,dV$$

respectively.

The corresponding *center of mass* will reside at a point $(\bar{x}, \bar{y}, \bar{z})$ in space. If we denote the total mass in the region D by

$$M \;:=\; \iiint\limits_D \rho(x, y, z)\,dV,$$

then the **coordinates of the center of mass** are given by

$$\bar{x} \;=\; \frac{M_{yz}}{M}, \quad \bar{y} \;=\; \frac{M_{xz}}{M}, \quad \text{and} \quad \bar{z} \;=\; \frac{M_{xy}}{M}. \tag{3}$$

Remarks:

- One can develop formulas for the position $(\bar{x}, \bar{y})$ of the center of mass in two dimensions. In that case it is assumed $\rho(x, y)$ gives mass density in mass per unit area, and the first moments $M_y = \iint_R x\rho(x, y)\,dA$ and $M_x = \iint_R y\rho(x, y)\,dA$ are *about the x-axis and y-axis* respectively.

- When there is constant mass density $\rho(x, y, z) = \delta$ throughout the region D, then another name for the center of mass is the *centroid*. In this case,

$$\bar{x} \;=\; \frac{M_{yz}}{M} \;=\; \frac{\delta \iiint\limits_D x\,dV}{\iiint\limits_D \delta\,dV} \;=\; \frac{\iiint\limits_D x\,dV}{\iiint\limits_D dV} \;=\; \text{avg. } x\text{-value in } D.$$

  Similar statements may be made about the other coordinates $\bar{y}$ and $\bar{z}$ of the centroid.