

# MATH 231, Inverse Laplace Transform Worksheet

Find the inverse Laplace transform for each function.

$$1. F(s) = \frac{e^{-\pi s/2}}{s^2 + 9}$$

**Answer:** First, we deal with  $(s^2 + 9)^{-1}$ ; that is,  $F(s)$  without its exponential factor. We have

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 9} \right\} = \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 9} \right\} = \frac{1}{3} \sin(3t).$$

The exponential factor in  $F(s)$  calls for use of the **2<sup>nd</sup> shifting rule**, yielding

$$\mathcal{L}^{-1} \{F(s)\} = \frac{1}{3} u \left( t - \frac{\pi}{2} \right) \sin \left( 3 \left( t - \frac{\pi}{2} \right) \right).$$

$$2. F(s) = \frac{1}{s^2(s^2 + 4)}$$

**Answer:** The partial fraction expansion for  $F(s)$  is

$$\frac{1}{s^2(s^2 + 4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4}.$$

Multiplying through by the common denominator yields the equation

$$\begin{aligned} 1 &= As(s^2 + 4) + B(s^2 + 4) + (Cs + D)s^2 \\ &= (A + C)s^3 + (B + D)s^2 + 4As + 4B. \end{aligned}$$

Since this must hold for all real  $s$ , the coefficients of the various powers of  $s$  must be equal:

$$\begin{array}{rclcl} s^3: & 0 & = & A + C & C & = & 0 \\ s^2: & 0 & = & B + D & D & = & -1/4 \\ s^1: & 0 & = & 4A & A & = & 0 \\ s^0: & 1 & = & 4B & B & = & 1/4 \end{array} \Rightarrow$$

Thus,

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2(s^2 + 4)} \right\} = \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} - \frac{1}{8} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} = \frac{1}{4} t - \frac{1}{8} \sin(2t).$$

$$3. F(s) = \frac{s}{s^2 + 6s + 11}$$

**Answer:** First, note that  $s^2 + 6s + 11$  is an irreducible quadratic, having nonreal roots. Thus, we complete the square:

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 6s + 11} \right\} &= \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 6s + 9) + 2} \right\} = \mathcal{L}^{-1} \left\{ \frac{(s + 3) - 3}{(s + 3)^2 + 2} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2} \Big|_{s \mapsto s+3} - \frac{3}{s^2 + 2} \Big|_{s \mapsto s+3} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2} \Big|_{s \mapsto s+3} \right\} - \frac{3}{\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{2}}{s^2 + 2} \Big|_{s \mapsto s+3} \right\} \\ &= e^{-3t} \cos(\sqrt{2}t) - \frac{3}{\sqrt{2}} e^{-3t} \sin(\sqrt{2}t). \end{aligned}$$

$$4. F(s) = e^{-s} \frac{s}{s^2 + 6s + 11}$$

**Answer:** Building on our answer to the previous exercise, we have

$$\mathcal{L}^{-1} \left\{ e^{-s} \frac{s}{s^2 + 6s + 11} \right\} = u(t-1) \left[ e^{-3(t-1)} \cos(\sqrt{2}(t-1)) - \frac{3}{\sqrt{2}} e^{-3(t-1)} \sin(\sqrt{2}(t-1)) \right].$$

$$5. F(s) = \frac{1}{(s-1)^3} + \frac{1}{s^2 + 2s - 8}$$

**Answer:** We first perform partial fraction decomposition on the second term. [Note that such a decomposition of the first term *cannot* make it simpler.]

$$\frac{1}{s^2 + 2s - 8} = \frac{1}{(s-2)(s+4)} = \frac{A}{s-2} + \frac{B}{s+4}.$$

We find that  $A = 1/6$  and  $B = -1/6$ . Thus,

$$\begin{aligned} \mathcal{L}^{-1} \{F(s)\} &= \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^3} + \frac{1}{6} \cdot \frac{1}{s-2} - \frac{1}{6} \cdot \frac{1}{s+4} \right\} \\ &= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^3} \Big|_{s \rightarrow s-1} \right\} + \frac{1}{6} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} - \frac{1}{6} \mathcal{L}^{-1} \left\{ \frac{1}{s+4} \right\} \\ &= \frac{1}{2} t^2 e^t + \frac{1}{6} e^{2t} - \frac{1}{6} e^{-4t}, \end{aligned}$$

where, in the final step, we have used the **1<sup>st</sup> shifting rule** on the first of the three Laplace transforms.

$$6. F(s) = e^{-2s} \frac{1}{(s-1)^3} + e^{-s} \frac{1}{s^2 + 2s - 8}$$

**Answer:** First, if we look carefully back at the work of the previous problem, we note that

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^3} \right\} = \frac{1}{2} t^2 e^t \quad \text{and} \quad \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2s - 8} \right\} = \frac{1}{6} (e^{2t} - e^{-4t}).$$

By the **2<sup>nd</sup> shifting rule**, we obtain

$$\mathcal{L}^{-1} \{F(s)\} = \frac{1}{2} u(t-2) (t-2)^2 e^{t-2} + \frac{1}{6} u(t-1) (e^{2(t-1)} - e^{-4(t-1)}).$$