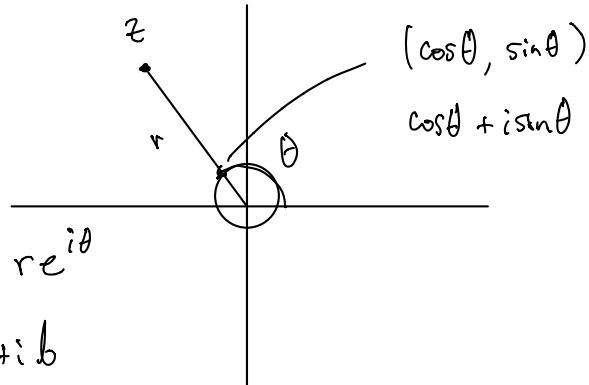


$$\begin{array}{c}
 \left. \begin{array}{c} a+ib \\ c+id \end{array} \right\} \text{product} \\
 \overbrace{(a+ib)(c+id)}^{\substack{= ac + ibc + iad + i^2 bd}} \\
 = \underbrace{(ac - bd)}_{\text{Re}} + i \underbrace{(ad + bc)}_{\text{Im}}
 \end{array}$$

$i = \sqrt{-1}$   
 $i^2 = -1$

Take

$$\theta = \arg(a+ib)$$



$$a+ib = r (\cos \theta + i \sin \theta) = re^{i\theta}$$

polar form of  $z = a+ib$

$$\boxed{e^{i\theta} = \cos \theta + i \sin \theta} \quad \text{Euler's Formula}$$

MacLaurin series

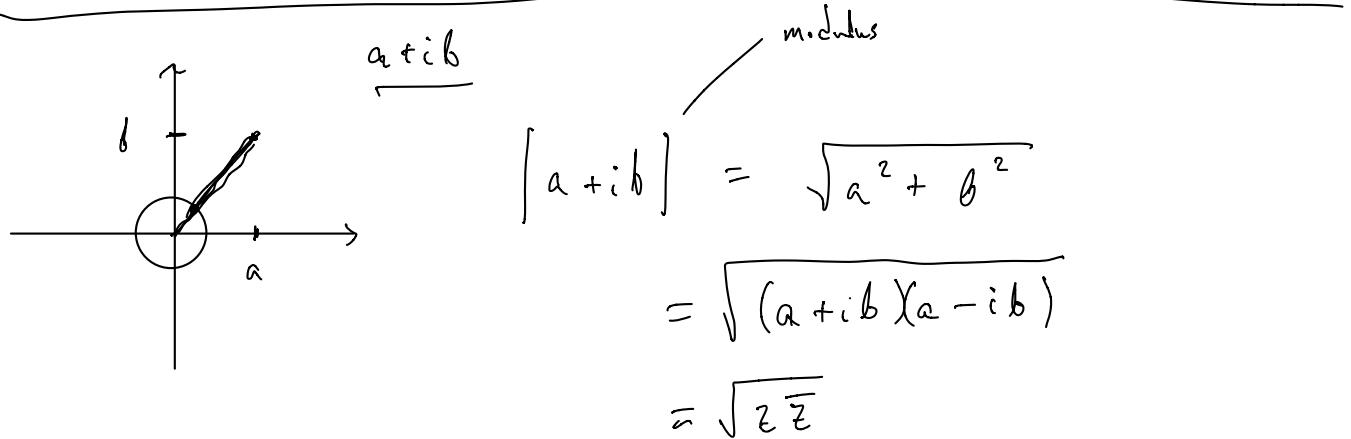
$$\left\{
 \begin{array}{l}
 e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
 \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} (-1)^n \\
 \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
 \end{array}
 \right.$$

$$x \hookrightarrow i\theta$$

$$\begin{aligned}
 e^{i\theta} &= 1 + i\theta + \frac{1}{2!} (i\theta)^2 + \frac{1}{3!} (i\theta)^3 + \dots \\
 &= 1 + i\theta - \frac{1}{2!} \theta^2 - \frac{1}{3!} i\theta^3 + \frac{1}{4!} \theta^4 + \frac{1}{5!} i\theta^5 - \frac{1}{6!} \theta^6 - \frac{1}{7!} i\theta^7 + \dots \\
 &= \underbrace{\left( 1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \frac{1}{6!} \theta^6 + \dots \right)}_{\text{Real part}} + i \underbrace{\left( \theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right)}_{\text{Imaginary part}}
 \end{aligned}$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{Euler}$$

$$\begin{aligned} \frac{\cos \theta - i \sin \theta}{\cos \theta + i \sin \theta} &= \frac{\overline{\cos \theta + i \sin \theta}}{\cos \theta + i \sin \theta} \\ &= \underbrace{\cos(-\theta)}_{\text{modulus}} + i \underbrace{\sin(-\theta)}_{\text{modulus}} = e^{-i\theta} \\ \overline{e^{i\theta}} &= e^{-i\theta} \end{aligned}$$



Vectors:

$\vec{x} \in \mathbb{R}^n$  impossible to plot if  $n \geq 4$

$$\vec{x} = \langle x_1, x_2, \dots, x_n \rangle \quad (\text{each } x_i \in \mathbb{R})$$

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$$

$$\vec{z} \in \mathbb{C}^n \quad \text{is} \quad \vec{z} = \langle z_1, z_2, \dots, z_n \rangle$$

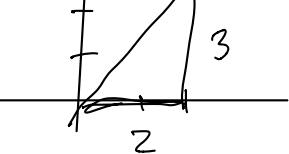
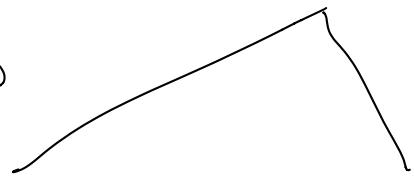
$$= \langle a_1 + ib_1, a_2 + ib_2, \dots, a_n + ib_n \rangle$$

impossible to plot if  $n \geq 2$

$$|\vec{z}| = \sqrt{\vec{z} \cdot \vec{z}} \geq 0 \quad w/ = 0 \text{ only when } \vec{z} = \vec{0}$$

+ 2+3i

$$\vec{z}_1 = \langle 2+3i, 4-7i \rangle$$



$$\begin{aligned}
 \vec{z}_1 \cdot \vec{z}_1 &= \langle 2+3i, 4-7i \rangle \cdot \langle 2+3i, 4-7i \rangle \\
 &= (2+3i)(\overline{2+3i}) + (4-7i)(\overline{4-7i}) \\
 &= (2+3i)(2-3i) + (4-7i)(4+7i) \\
 &= (4+9) + i(6-6) + (16+49) + i(-28+28) \\
 &= 13 + 65 = \underline{\underline{78}}
 \end{aligned}$$

For vectors  $\vec{z}, \vec{w} \in \mathbb{C}^n$  we define

$$\vec{z} \cdot \vec{w} = \sum_{k=1}^n z_k \overline{w_k} \quad \vec{z} \cdot \vec{w} = \overline{\vec{w} \cdot \vec{z}}$$

When we go to inner product of functions

$\sin\left(\frac{m\pi x}{l}\right)$      $\cos\left(\frac{n\pi x}{l}\right)$  are orthogonal when

$$\langle f, g \rangle = \int_0^l f(x) \overline{g(x)} dx$$

When,  $f, g$  may possibly be complex-valued, must refine

$$\langle f, g \rangle = \int_0^l f(x) \overline{g(x)} dx$$

Consider functions  $\dots, e^{-2\pi i \frac{3x}{l}}, e^{-2\pi i \frac{2x}{l}}, \dots, 1, e^{\frac{2\pi i x}{l}}, \dots$

Take a pair  $e^{2\pi i m x/l}, e^{2\pi i n x/l}$

their inner product, assuming

$m \neq n$

$$\langle e^{2\pi i m x/l}, e^{2\pi i n x/l} \rangle = \int_0^l e^{2\pi i m x/l} \cdot e^{-2\pi i n x/l} dx$$

$$= \int_0^l e^{2\pi i (m-n) x/l} dx$$

$$= \frac{l}{2\pi i (m-n)} e^{2\pi i (m-n) x/l} \Big|_0^l$$

$$= \frac{l}{2\pi i (m-n)} \left( e^{2\pi i (m-n)} - e^0 \right) = 0.$$

but

$$\begin{aligned} e^{2\pi i (m-n)} & \stackrel{\text{Euler's}}{=} \cos(2\pi(m-n)) + i \sin(2\pi(m-n)) \\ & = 1 \end{aligned}$$

A good exercise (?)

$$\underbrace{\langle e^{2\pi i m x/l}, e^{2\pi i n x/l} \rangle}_{\text{same fn. in both slots}} = l \quad (\text{details omitted})$$

same fn. in both  
slots

## Fourier Series

before based on this list of mutually orthog fn.

$$1, \cos\left(\frac{2\pi x}{l}\right), \sin\left(\frac{2\pi x}{l}\right), \cos\left(\frac{4\pi x}{l}\right), \sin\left(\frac{4\pi x}{l}\right), \dots$$

can instead use

$$\left\{ e^{2n\pi i x/l} \right\}_{n=-\infty}^{\infty} : \dots, e^{-2\pi i x/l}, 1, e^{2\pi i x/l}, e^{4\pi i x/l}, e^{6\pi i x/l}, \dots$$

$$= \cos\left(\frac{2\pi x}{l}\right) - i \sin\left(\frac{2\pi x}{l}\right)$$

$$= \cos\left(\frac{2\pi x}{l}\right) + i \sin\left(\frac{2\pi x}{l}\right)$$

## Complex F.S.

replaces

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi n x}{l}\right) + b_n \sin\left(\frac{2\pi n x}{l}\right) \right]$$

with

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/l}$$

and get the  $c_n$  much like the  $a_n, b_n$

For a given function  $f(x)$  (periodic, period =  $l$ )

$$c_n = \frac{\langle f, e^{2\pi i n x/l} \rangle}{\langle e^{2\pi i n x/l}, e^{2\pi i n x/l} \rangle} = \frac{1}{l} \int_0^l f(x) e^{-2\pi i n x/l} dx$$

Full complex F.S.

$$\cdots + C_{-4} e^{-8\pi i x/l} + \boxed{C_{-3} e^{-6\pi i x/l} + C_{-2} e^{-4\pi i x/l} + \cdots + C_3 e^{6\pi i x/l} + C_4 e^{8\pi i x/l}}$$

If truncated to  $N=3$  "harmonics", we include only those boxed.

So, generally, a truncated complex F.S. looks like

$$\sum_{k=-N}^N C_k e^{2\pi i k x/l}$$

still computing the  $C_k$ 's according to formula above.

It can be shown the coeffs.  $C_k$  for complex F.S.

bear a relationship to the coeffs.  $a_n, b_n$  from real (trigonometric, original) F.S.

$$C_n = \begin{cases} \frac{1}{2}(a_{-n} + i b_{-n}), & n < 0 \\ \frac{1}{2}a_0, & n = 0 \\ \frac{1}{2}(a_n - i b_n), & n > 0 \end{cases}$$

So, in particular, choose a case like  $n = 2, -2$  ( $2^{\text{nd}}$  harmonic)

$$C_{-2} = \frac{1}{2}(a_2 + i b_2) \quad \left. \right\} \text{complex conjugates}$$

which

$$C_2 = \frac{1}{2}(a_2 - i b_2)$$

# Assignment

1. Find a polar representation  $z = re^{i\theta}$  for

$$(a) z = -3 + 4i$$

$$(b) z = \sqrt{5} - \sqrt{11}i$$

Give  $\theta$  in radians (decimal approximations OK).

2. Find  $\vec{z} \cdot \vec{w}$  for the given  $\vec{z}, \vec{w}$ .

$$(a) \vec{z} = \langle 3i, -4, 2-i \rangle, \vec{w} = \langle 2, 1+2i, 3-i \rangle$$

$$(b) \vec{z} = \langle 2, 1+2i, 3-i \rangle, \vec{w} = \langle 3i, -4, 2-i \rangle$$

$$(c) \vec{z} = \langle 3-2i, 1+i, 4+3i \rangle, \text{ and } \vec{w} = \vec{z}.$$

3. If  $\phi_n(x)$  is defined as  $e^{\frac{2\pi i n x}{l}}$ , show that

$$\langle \phi_n(x), \phi_m(x) \rangle = l.$$

4. In Octave, take the period 2 function I defined in class

$$f = @ (x) \bmod (x, 2) .12 .* (\bmod (x, 2) < 1) \\ + (\bmod (x, 2) \geq 1)$$

Plot it and overlay the truncated trigonometric Fourier series w/

5 harmonics:

$$\frac{a_0}{2} + \sum_{k=1}^5 \left[ a_k \cos\left(\frac{2\pi k x}{l}\right) + b_k \sin\left(\frac{2\pi k x}{l}\right) \right].$$

Then use the truncatedExpFS.m script to overlay the real part of the truncated complex exponential Fourier series

$$\sum_{k=-5}^5 c_k e^{\frac{2\pi i k x}{l}}$$

To get the real part I imagine something like

`plot(xs, real(truncatedExpFS(xs, f, ...)))`