

1. Here

$$\mathbf{B} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} -4 & 4 \\ -4 & 3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} -4 & 16 \\ -6 & 14 \\ -9 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -12 \\ 8 & -9 \\ 12 & 10 \end{bmatrix}.$$

2. There is more than one correct answer. Here is one such sequence:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(1/2)\mathbf{r}_3 \rightarrow \mathbf{r}_3} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-5\mathbf{r}_3 + \mathbf{r}_2 \rightarrow \mathbf{r}_2} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2\mathbf{r}_2 + \mathbf{r}_1 \rightarrow \mathbf{r}_1} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. The precondition to a system of n equations in n unknowns being *inconsistent* is that the matrix be singular. So, we calculate the determinant of the matrix (I'm expanding in cofactors along the first row)

$$\begin{aligned} \begin{vmatrix} 6 & 6 & 5 \\ -7 & 5 & k \\ -8 & 16 & -5 \end{vmatrix} &= 6(-1)^2 \begin{vmatrix} 5 & k \\ 16 & -5 \end{vmatrix} + 6(-1)^3 \begin{vmatrix} -7 & k \\ -8 & -5 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} -7 & 5 \\ -8 & 16 \end{vmatrix} \\ &= 6(-25 - 16k) - 6(35 + 8k) + 5(-112 + 40) \\ &= -150 - 96k - 210 - 48k - 560 + 200 = -720 - 144k. \end{aligned}$$

Solving to make this determinant zero, we have $144k = -720$, or $k = -5$. To ensure the system is consistent, we must have $k \neq -5$.

4. (a) We form a matrix whose columns are the given vectors and take it to RREF:

$$\begin{bmatrix} 2 & -4 & 0 & 8 \\ 1 & -2 & 0 & 4 \\ -3 & 7 & 1 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that only two of the four vectors are linearly independent, as there are just 2 pivot columns. Thus, \mathcal{W} is a 2-dimensional subspace of \mathbb{R}^3 , a **plane**.

(b) We keep linearly independent columns of the matrix above as a basis. One option is the first two columns: \mathbf{v}_1 and \mathbf{v}_2 .

5. (a) The system has augmented matrix

$$\begin{bmatrix} 4 & -5 & 4 & 3 & 2 \\ -1 & 1 & 2 & 3 & 2 \\ 3 & -4 & 6 & 6 & 4 \\ -2 & 2 & 4 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -14 & -18 & -12 \\ 0 & 1 & -12 & -15 & -10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 3 and 4—or variables x_3, x_4 —are free. Rows 1 and 2 of RREF say

$$x_1 = 14x_3 + 18x_4 - 12 \quad \text{and} \quad x_2 = 12x_3 + 15x_4 - 10.$$

So, solutions of the system take the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -12 \\ -10 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 14 \\ 12 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 18 \\ 15 \\ 0 \\ 1 \end{bmatrix}, \quad \text{with } x_3, x_4 \in \mathbb{R}.$$

- (b) The matrix is the same as in part (a), so its null space is revealed in the answer to part (a) as the part with freedoms. A basis for the null space is

$$\begin{bmatrix} 14 \\ 12 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 18 \\ 15 \\ 0 \\ 1 \end{bmatrix},$$

6. We have

$$0 = \begin{vmatrix} 3-\lambda & -6 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) + 18 = \lambda^2 - 5\lambda + 24,$$

$$\Rightarrow \lambda = \frac{1}{2} \left(5 \pm \sqrt{25 - (4)(24)} \right) = \frac{5}{2} \pm i \frac{\sqrt{71}}{2}.$$

7. If we call the given matrix \mathbf{A} , then using the given eigenvalue λ , the problem "solve $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ " has augmented matrix

$$[\mathbf{A} - 2\mathbf{I} \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 2 & 0 & -2 & 0 & 0 \\ -4 & 0 & 4 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

RREF has two free columns (so that is the dimension of the eigenspace), and we take x_2 and x_3 , components of an eigenvector, as free, leading to eigenvectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

span the eigenspace. They are also linearly independent, making this collection a *basis*.