1. (a) Solutions take the form $t\langle -1,2,1,0\rangle$; that is, any scalar multiple of $\langle -1,2,1,0\rangle$ solves.

   (b) Solutions take the form $\langle 2-t, 3+2t, t, -1\rangle = \langle 2,3,0,-1\rangle + t\langle -1,2,1,0\rangle$.

   (c) Solutions take the form $\langle -s-3t, t, s, -2s, s\rangle = t\langle -3,1,0,0,0\rangle + s\langle -1,0,1,-2,1\rangle$. That is, any linear combination of the vectors $\langle -3,1,0,0,0\rangle$ and $\langle -1,0,1,-2,1\rangle$.

   (d) The final line of this matrix contains the statement $0 = 1$, and such nonsense means the system has no solution.

   (e) Solutions take the form $\langle 2+3t-s, t, -1-2s, s\rangle = \langle 2,0,-1,0\rangle + t\langle 3,1,0,0\rangle + s\langle -1,0,-2,1\rangle$.

2. (a) It is

$$\begin{aligned} 2x_2 + x_3 + 3x_4 &= 3, \\ 2x_1 + x_2 + 2x_3 - x_4 &= 4, \\ x_1 - 3x_2 + x_3 + x_4 &= 7, \\ 2x_1 + x_3 - 2x_4 &= 2. \end{aligned}$$

   (b) We have

$$\left[\begin{array}{cccc|c} 0 & 2 & 1 & 3 & 3 \\ 2 & 1 & 2 & -1 & 4 \\ 1 & -3 & 1 & 1 & 7 \\ 2 & 0 & 1 & -2 & 2 \end{array}\right] \quad \underset{\sim}{\mathbf{r}_1 \leftrightarrow \mathbf{r}_3} \quad \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 2 & 1 & 2 & -1 & 4 \\ 0 & 2 & 1 & 3 & 3 \\ 2 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\underset{\sim}{(-2)r_1 + \mathbf{r}_2 \to \mathbf{r}_2} \quad \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 7 & 0 & -3 & -10 \\ 0 & 2 & 1 & 3 & 3 \\ 2 & 0 & 1 & -2 & 2 \end{array}\right] \quad \underset{\sim}{(-2)r_1 + \mathbf{r}_4 \to \mathbf{r}_4} \quad \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 7 & 0 & -3 & -10 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 6 & -1 & -4 & -12 \end{array}\right]$$

$$\underset{\sim}{\mathbf{r}_2 \leftrightarrow \mathbf{r}_3} \quad \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 7 & 0 & -3 & -10 \\ 0 & 6 & -1 & -4 & -12 \end{array}\right] \quad \underset{\sim}{(-7/2)\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3} \quad \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 6 & -1 & -4 & -12 \end{array}\right]$$

$$\underset{\sim}{(-3)\mathbf{r}_2 + \mathbf{r}_4 \to \mathbf{r}_4} \quad \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 0 & -4 & -13 & -21 \end{array}\right] \quad \underset{\sim}{(-8/7)\mathbf{r}_3 + \mathbf{r}_4 \to \mathbf{r}_4} \quad \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 0 & 0 & 17/7 & 17/7 \end{array}\right]$$

   (c) There is, unfortunately, not just one sequence of EROs leading to RREF, though the end result must always be the same. Here is one sequence that produces the desired result.

   viii. ERO2: rescale row 2 by a factor of (1/2); i.e., $(1/2)\mathbf{r}_2 \to \mathbf{r}_2$

   ix. ERO2: rescale row 3 by a factor of $(-2/7)$; that is, $(-2/7)\mathbf{r}_3 \to \mathbf{r}_3$

   x. ERO2: rescale row 4 by a factor of (7/17); $(7/17)\mathbf{r}_4 \to \mathbf{r}_4$

   xi. ERO3: $\mathbf{r}_1 - \mathbf{r}_4 \to \mathbf{r}_1$

xii. ERO3: $\mathbf{r}_2 - (3/2)\mathbf{r}_4 \rightarrow \mathbf{r}_2$

xiii. ERO3: $\mathbf{r}_3 - (27/7)\mathbf{r}_4 \rightarrow \mathbf{r}_3$

xiv. ERO3: $\mathbf{r}_2 - (1/2)\mathbf{r}_3 \rightarrow \mathbf{r}_2$

xv. ERO3: $\mathbf{r}_1 - \mathbf{r}_3 \rightarrow \mathbf{r}_1$

xvi. ERO3: $\mathbf{r}_1 + 3\mathbf{r}_2 \rightarrow \mathbf{r}_1$

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & -3.5 & -13.5 & -20.5 \\ 0 & 0 & 0 & 17/7 & 17/7 \end{array}\right] \begin{array}{c} (1/2)\mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ (-2/7)\mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ \sim \\ (7/17)\mathbf{r}_4 \rightarrow \mathbf{r}_4 \end{array} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 7 \\ 0 & 1 & 0.5 & 1.5 & 1.5 \\ 0 & 0 & 1 & 27/7 & 41/7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} -\mathbf{r}_4 + \mathbf{r}_1 \rightarrow \mathbf{r}_1 \\ (-3/2)\mathbf{r}_4 + \mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ \sim \\ (-27/7)\mathbf{r}_4 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \end{array} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & 6 \\ 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \mathbf{r}_2 - (1/2)\mathbf{r}_3 \rightarrow \mathbf{r}_2 \\ \sim \\ \mathbf{r}_1 - \mathbf{r}_3 \rightarrow \mathbf{r}_1 \end{array} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \mathbf{r}_1 + 3\mathbf{r}_2 \rightarrow \mathbf{r}_1 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

(d) The (only) solution is $\mathbf{x} = (1, -1, 2, 1)$.