$\vec{X} = \langle 3, 5, 1 \rangle$

$\vec{u}_1 = \langle 1, 1, 0 \rangle$ and $\vec{u}_2 = \langle 1, -1, 0 \rangle$

So $$\text{proj}(\vec{x} \to \vec{u}_1) = \frac{\vec{u}_1 \circ \vec{X}}{|\vec{u}_1|^2} \vec{u}_1 = \frac{8}{2} \vec{u}_1$$

$$= \langle 4, 4, 0 \rangle$$

$$\text{proj}(\vec{x} \to \vec{u}_2) = \frac{-2}{2} \vec{u}_2 = \langle -1, 1, 0 \rangle$$

Take $\vec{v}_1 = \langle 1, 1, 0 \rangle$ and $\vec{v}_2 = \langle 1, 0, 0 \rangle$. Then

$$\text{proj}(\vec{x} \to \vec{v}_1) = \frac{\vec{x} \circ \vec{v}_1}{|\vec{v}_1|^2} \vec{v}_1 = \langle 4, 4, 0 \rangle$$

$$\text{proj}(\vec{x} \to \vec{v}_2) = \frac{3}{1} \vec{v}_2 = \langle 3, 0, 0 \rangle.$$

Now $\vec{u}_1, \vec{u}_2$ are linearly independent, so their span is a plane $\mathcal{U}$ in $\mathbb{R}^3$. The vector $\vec{x}$ may be decomposed into the sum $\vec{u} + \vec{w}$ of a vector $\vec{u} \in \mathcal{U}$ and a $\vec{w} \in \mathbb{R}^3$ perpendicular to the plane $\mathcal{U}$, making $\vec{u} = \text{proj}(\vec{x} \to \mathcal{U})$ and $\vec{w}$ the shortest vector from $\mathcal{U}$ to $\vec{x}$.

Clearly the plane $\mathcal{U}$ is what we call the xy-plane inside $\mathbb{R}^3$, and both $\{\vec{u}_1, \vec{u}_2\}$, $\{\vec{v}_1, \vec{v}_2\}$ can serve as bases for $\mathcal{U}$.

One might <u>hope</u> that, having the basis $\{\vec{v}_1, \vec{v}_2\}$ of $\mathcal{U}$, one might obtain $\vec{u} = \operatorname{proj}(\vec{x} \longrightarrow \mathcal{U})$ by adding the projections onto individual basis elements. If - worked generally, then using the basis $\{\vec{v}_1, \vec{v}_2\}$, we would have

$$\begin{aligned} \vec{u} &= \operatorname{proj}(\vec{x} \to \vec{v}_1) + \operatorname{proj}(\vec{x} \to \vec{v}_2) \\ &= \langle 4, 4, 0 \rangle + \langle 3, 0, 0 \rangle \\ &= \langle 7, 4, 0 \rangle, \end{aligned}$$

and we could get the perpendicular vector $\vec{w}$ via subtraction: since $\vec{x} = \vec{u} + \vec{w}$,

$$\begin{aligned} \vec{w} = \vec{x} - \vec{u} &= \langle 3, 5, 1 \rangle - \langle 7, 4, 0 \rangle \\ &= \langle -4, 1, 1 \rangle. \end{aligned}$$

We have broken $\vec{x}$ into parts to be sure,

$$\vec{x} = \langle 7, 4, 0 \rangle + \langle -4, 1, 1 \rangle,$$

but this is not the unique splitting of $\vec{x}$ into its projection onto $\mathcal{U}$ and an orthogonal part, which is clear since the dot product

$$\langle 7, 4, 0 \rangle \cdot \langle -4, 1, 1 \rangle = -24$$

is not 0.

However, had we built our potential $\vec{u}$ from the projections onto the other bases elements, calling

$$\begin{aligned} \vec{u} &= \text{proj}(\vec{x} \to \vec{u}_1) + \text{proj}(\vec{x} \to \vec{u}_2) \\ &= \langle 4, 4, 0 \rangle + \langle -1, 1, 0 \rangle \\ &= \langle 3, 5, 0 \rangle \end{aligned}$$

then set

$$\begin{aligned} \vec{w} = \vec{x} - \vec{u} &= \langle 3, 5, 1 \rangle - \langle 3, 5, 0 \rangle \\ &= \langle 0, 0, 1 \rangle, \end{aligned}$$

this time not only is

$$\vec{x} = \vec{u} + \vec{w},$$

but $\vec{u} \cdot \vec{w} = \langle 3, 5, 0 \rangle \cdot \langle 0, 0, 1 \rangle = 0$.

making it an orthogonal decomposition, as we sought, with

$$\langle 3, 5, 0 \rangle = \text{proj}(\vec{x} \to U),$$

and

$$\langle 0, 0, 1 \rangle = \text{proj}(\vec{x} \to U^{\perp}).$$

Q: So, why does it work to add the projections onto basis vectors in order to yield the projection onto their span in the case of basis $\{\vec{u}_1, \vec{u}_2\}$, but not for basis $\{\vec{v}_1, \vec{v}_2\}$?

A: The basis elements $\{\vec{u}_1, \vec{u}_2\}$ are orthogonal to each other, but those in $\{\vec{v}_1, \vec{v}_2\}$ are not.