1. $$\begin{vmatrix} -1 & 0 & -2 & -7 \\ 1 & 2 & k & 0 \\ 0 & -1 & 0 & 1 \\ 3 & 0 & 2 & 5 \end{vmatrix} = (-1)(-1)^5 \begin{vmatrix} -1 & -2 & -7 \\ 1 & k & 0 \\ 3 & 2 & 5 \end{vmatrix} + (-1)^7 \begin{vmatrix} -1 & 0 & -2 \\ 1 & 2 & k \\ 3 & 0 & 2 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} -2 & -7 \\ 2 & 5 \end{vmatrix} + k(-1)^2 \begin{vmatrix} -1 & -7 \\ 3 & 5 \end{vmatrix} - 2(-1)^2 \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} = -4 + 16k - 8 = -12 + 16k$$

Solve $-12 + 16k = 0 \implies k = 3/4$

2. (a) Since

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a free column (two, in fact), $\mathrm{null}(A) = \mathrm{null}(A - 0 \cdot I)$ is nontrivial. Thus, 0 is an eigenvalue of A.

(b) And because $\mathrm{rref}(A)$ has pivots in columns 1 and 3, it follows that $\{\langle 1, 1, 0, 2\rangle, \langle -1, 0, 1, -1\rangle\}$ (columns 1 and 3 from A itself) form a basis for $\mathrm{col}(A)$.

3. (a) False. It is a subspace of $\mathbb{R}^k$.
(b) True. $k = \ell$ forces A to be square.
(c) False. The statement could be fixed by replacing "dependent" w/ "independent."
(d) True. The condition ensures RREF of A doesn't have a row of zeros.
(e) True. A free column lies in the span of its preceding columns.
(f) True.

4. Call the given matrix A. Eigenvectors corresponding to $\lambda = -2$ are in $\mathrm{null}(A + 2I)$. So, we solve $(A + 2I)\vec{v} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 6 & -6 & -6 & 0 \\ 6 & -6 & -6 & 0 \\ -3 & 3 & 3 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \underbrace{0 \quad\quad 0}_{v_2,\, v_3 \text{ free}} & & 0 \end{array}\right] \qquad v_1 - v_2 - v_3 = 0$$

Eigenvectors look like

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 + v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, basis vectors for the eigenspace E_{-2} are $\langle 1,1,0 \rangle$ and $\langle 1,0,1 \rangle$.

5. $$0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 13 \\ -1 & 1-\lambda \end{vmatrix} = (-5-\lambda)(1-\lambda) + 13$$

$$= \lambda^2 + 4\lambda + 8 \implies \lambda = \tfrac{1}{2}\left(-4 \pm \sqrt{16 - 32}\right)$$

$$= \tfrac{1}{2}(-4 \pm 4i) = -2 \pm 2i.$$

eigenvalues are $-2-2i$ and $-2+2i$.

6. The system in matrix form is

$$\underbrace{\begin{bmatrix} 3 & -2 & 4 & 0 \\ 1 & 2 & -4 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}}_{\vec{b}}$$

To solve, we build the augmented matrix

$$\begin{array}{c} \left[\begin{array}{cccc|c} 3 & -2 & 4 & 0 & 0 \\ 1 & 2 & -4 & 0 & -1 \\ 0 & 0 & 3 & 1 & 3 \end{array}\right] \\ \begin{array}{cccc} x & y & z & w \end{array} \end{array} \xhookrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & 2/3 & 13/8 \\ 0 & 0 & 1 & 1/3 & 1 \end{array}\right]$$

w is free

Solutions satisfy

$$x = -1/4$$
$$y = 13/8 - 2/3\, w$$
$$z = 1 - \tfrac{1}{3} w$$

with w free (anything in $\mathbb{R}$)

$$\implies \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1/4 \\ 13/8 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

with $w \in \mathbb{R}$.