The Dot Product

Definition: For vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, we define the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ to be

$$\mathbf{u} \cdot \mathbf{v} \ := \ u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Notes:

- The dot product $\mathbf{u} \cdot \mathbf{v}$ is a scalar (number), not another vector.
- Properties
  1. $\mathbf{u} \cdot \mathbf{v} \ = \ \mathbf{v} \cdot \mathbf{u}$
  2. $c(\mathbf{u} \cdot \mathbf{v}) \ = \ (c\mathbf{u}) \cdot \mathbf{v} \ = \ \mathbf{u} \cdot (c\mathbf{v})$.
  3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \ = \ \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4. $\mathbf{0} \cdot \mathbf{v} \ = \ 0$
  5. $\mathbf{v} \cdot \mathbf{v} \ = \ \|\mathbf{v}\|^2$

Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the angle θ between them satisfies

$$\cos\theta \ = \ \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

Note that, when $\theta = \pi/2$, the numerator on the right-hand side must be zero. This motivates the following definition.

Orthogonality

Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be *orthogonal* (or *perpendicular*) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example: The zero vector $\mathbf{0}$ is orthogonal to every other vector. In 2D, the vectors $\langle a, b \rangle$ and $\langle -b, a \rangle$ are orthogonal, since

$$\langle a, b \rangle \cdot \langle -b, a \rangle \ = \ (a)(-b) + (b)(a) \ = \ 0.$$

Projections

Scalar component of u in the direction of v: $\|\mathbf{u}\| \cos\theta \;=\; \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}$.

Vector projection of u onto v:

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} \;:=\; \left(\begin{array}{c} \text{scalar component of } \mathbf{u} \\ \text{in direction of } \mathbf{v} \end{array} \right) \left(\begin{array}{c} \text{direction} \\ \text{of } \mathbf{u} \end{array} \right) \;=\; \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \right) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \;=\; \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\, \mathbf{v}.$$

Work

The *work* done by a constant force $\mathbf{F}$ acting through a displacement vector $\mathbf{D} = \overrightarrow{PQ}$ is given by

$$W \;=\; \mathbf{F} \cdot \mathbf{D} \;=\; \|\mathbf{F}\| \|\mathbf{D}\| \cos\theta,$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{D}$.