Warmup Exercises

1. Solve (find the general solution to) $\frac{d\vec{x}}{dt} = \begin{bmatrix} 4 & 6 & 6 \\ -9 & -17 & -18 \\ 6 & 12 & 13 \end{bmatrix} \vec{x}$.

2. Same task, but for $\frac{d\vec{x}}{dt} = \begin{bmatrix} -7 & -12 & -1 \\ 6 & 11 & 1 \\ -2 & -4 & 3 \end{bmatrix} \vec{x}$, given that one eigenpair of the matrix is $\lambda = 4+i$, with e-vector $\begin{bmatrix} 1+i \\ -1-i \\ 2 \end{bmatrix}$.

1. $A = \begin{bmatrix} 4 & 6 & 6 \\ -9 & -17 & -18 \\ 6 & 12 & 13 \end{bmatrix}$

e-vals: Solve

$$0 = \begin{vmatrix} 4-\lambda & 6 & 6 \\ -9 & -17-\lambda & -18 \\ 6 & 12 & 13-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} -17-\lambda & -18 \\ 12 & 13-\lambda \end{vmatrix} - 6\begin{vmatrix} -9 & -18 \\ 6 & 13-\lambda \end{vmatrix} + 6\begin{vmatrix} -9 & -17-\lambda \\ 6 & 12 \end{vmatrix}$$

$$= (4-\lambda)(-221+4\lambda+\lambda^2+216) - 6(-117+9\lambda+108) + 6(-108+102+6\lambda)$$

$$= (4-\lambda)(\lambda^2+4\lambda-5) - 6(9\lambda-9) + 6(6\lambda-6)$$

$$= (4-\lambda)(\lambda+5)(\lambda-1) - 54(\lambda-1) + 36(\lambda-1)$$

$$= (\lambda-1)\left[(4-\lambda)(\lambda+5) - 54 + 36\right] = (\lambda-1)(-\lambda^2-\lambda+20-18)$$

$$= -(\lambda-1)(\lambda^2+\lambda-2) = -(\lambda-1)(\lambda+2)(\lambda-1) = -(\lambda+2)(\lambda-1)^2$$

$\lambda = -2$ $(\underline{AM=1})$, $\lambda = 1$ $(AM = 2)$.

E-vectors, $\lambda = -2$

1 free col, so GM = 1

$$A + 2I = \begin{bmatrix} 6 & 6 & 6 \\ -9 & -15 & -18 \\ 6 & 12 & 15 \end{bmatrix} \hookrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} v_1 - \frac{1}{2}v_3 = 0 \\ v_2 + \frac{3}{2}v_3 = 0 \end{array}$$

v_3 free

e-vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1/2\, v_3 \\ -3/2\, v_3 \\ v_3 \end{bmatrix} = \frac{1}{2}v_3 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \Rightarrow$ basis e-vec. $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

$\lambda = 1$ $(AM = 2)$

$$A - I = \begin{bmatrix} 3 & 6 & 6 \\ -9 & -18 & -18 \\ 6 & 12 & 12 \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad v_1 = -2v_2 - 2v_3$$

v_2 v_3 free (GM = 2)

basis e-vectors

e-vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2v_2 - 2v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Individual solns:
$$e^{-2t}\begin{bmatrix}1\\-3\\2\end{bmatrix},\quad e^{t}\begin{bmatrix}-2\\1\\0\end{bmatrix},\quad e^{t}\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

gen'l soln.
$$\vec{x}_h(t) = c_1e^{-2t}\begin{bmatrix}1\\-3\\2\end{bmatrix} + c_2e^{t}\begin{bmatrix}-2\\1\\0\end{bmatrix} + c_3e^{t}\begin{bmatrix}-2\\0\\1\end{bmatrix} = \underbrace{\begin{bmatrix}e^{-2t} & -2e^{t} & -2e^{t}\\ -3e^{-2t} & e^{t} & 0\\ 2e^{-2t} & 0 & e^{t}\end{bmatrix}}_{\Phi(t)}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

2. $A = \begin{bmatrix}-7 & -12 & -1\\ 6 & 11 & 1\\ -2 & -4 & 3\end{bmatrix}$

Know $\lambda = 4 \pm i$ are e-vals.
so both $[\lambda-(4+i)]$ and $[\lambda-(4-i)]$
are factors of char. poly. $\det(A-\lambda I)$.

$$\begin{vmatrix}-7-\lambda & -12 & -1\\ 6 & 11-\lambda & 1\\ -2 & -4 & 3-\lambda\end{vmatrix} = -(\lambda+7)\begin{vmatrix}11-\lambda & 1\\ -4 & 3-\lambda\end{vmatrix} + 12\begin{vmatrix}6 & 1\\ -2 & 3-\lambda\end{vmatrix} - 1\begin{vmatrix}6 & 11-\lambda\\ -2 & -4\end{vmatrix}$$

$$= -(\lambda+7)(33-14\lambda+\lambda^2+4) + 12(18-6\lambda+2) - (-24+22-2\lambda)$$

$$= -(\lambda+7)(\lambda^2-14\lambda+37) + 240-72\lambda + 2 + 2\lambda$$

$$= -(\lambda^3-7\lambda^2-61\lambda+259) + 242-70\lambda$$

$$= -\lambda^3+7\lambda^2-9\lambda-17 = (\lambda^2-8\lambda+17)(\underline{\text{something}})$$

The product $(\lambda-4-i)(\lambda-4+i) = \lambda^2-4\lambda+\underline{\underline{i\lambda}}-4\lambda+16\underline{-4i}-\underline{\underline{i\lambda}}\underline{+4i}-i^2$

$$= \lambda^2-8\lambda+17$$

is also a factor.

Use long division

$$\begin{array}{r} -\lambda - 1 \;=\; -1(\lambda+1) \\ \lambda^2 - 8\lambda + 17 \;\big)\; -\lambda^3 + 7\lambda^2 - 9\lambda - 17 \\ -\left(-\lambda^3 + 8\lambda^2 - 17\lambda\right) \\ \hline -\lambda^2 + 8\lambda - 17 \\ -\left(-\lambda^2 + 8\lambda - 17\right) \\ \hline 0 \end{array}$$

$$\det(A - \lambda I) = \left[\lambda - (4+i)\right]\left[\lambda - (4-i)\right](-1)(\lambda+1)$$

$\Rightarrow$ e-vals: $4 \pm i$, -1

Knew: $4+i$ goes with $\begin{bmatrix} 1+i \\ -1-i \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{\vec{u}} + i \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{v}} = \vec{u} + i\vec{v}$

$\alpha + i\beta$
with $\alpha = 4$, $\beta = 1$

Get solns.

$$e^{4t}\left(\cos t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = e^{4t} \begin{bmatrix} \cos t - \sin t \\ \sin t - \cos t \\ 2\cos t \end{bmatrix}$$

$$e^{4t}\left(\sin t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \cos t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = e^{4t} \begin{bmatrix} \cos t + \sin t \\ -\cos t - \sin t \\ 2\sin t \end{bmatrix}$$

E-vectors corresp. to $\lambda = -1$, must find basis for null$(\underline{A + I})$

$$\begin{bmatrix} -6 & -12 & -1 \\ 6 & 12 & 1 \\ -2 & -4 & 4 \end{bmatrix} \hookrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} v_1 + 2v_2 = 0 \\ v_3 = 0 \end{array}$$

v_2 free

e-vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2v_2 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$ basis e-vec. $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{x}_h(t) = c_1 e^{4t} \begin{bmatrix} \cos t - \sin t \\ \sin t - \cos t \\ 2\cos t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \cos t + \sin t \\ -\cos t - \sin t \\ 2\sin t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} e^{4t}(\cos t - \sin t) & e^{4t}(\cos t + \sin t) & -2e^{-t} \\ e^{4t}(\sin t - \cos t) & e^{4t}(-\cos t - \sin t) & e^{-t} \\ 2e^{4t}\cos t & 2e^{4t}\sin t & 0 \end{bmatrix}}_{\Phi(t)} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$