MATH 231, Worksheet
Date: May 8, 2020

Find the inverse Laplace transform for each function.

1. $F(s) = \dfrac{e^{-\pi s/2}}{s^2+9}$
2. $F(s) = \dfrac{1}{s^2(s^2+4)}$
3. $F(s) = \dfrac{s}{s^2+6s+11}$
4. $F(s) = e^{-s}\dfrac{s}{s^2+6s+11}$
5. $F(s) = \dfrac{1}{(s-1)^3} + \dfrac{1}{s^2+2s-8}$
6. $F(s) = e^{-2s}\dfrac{1}{(s-1)^3} + e^{-s}\dfrac{1}{s^2+2s-8}$

Overriding theme in course, made possible by a focus on linear problems

1. Take a homogeneous version of the problem at hand
   - solve it
   - soln. often contains "freedoms"
   - variously called: Null space, homogeneous soln., span of basis solutions
2. Find one soln. to the given problem: particular soln.

Full soln. is sum of results from 1 and 2.

Even Ch.5 is in this vein.

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \; y'(0) = y_1$$

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{f(t)\}$$

$$a(s^2Y - sy_0 - y_1) + b(sY - y_0) + cY = F$$

$$(as^2 + bs + c)Y = F(s) + asy_0 + ay_1 + by_0$$

$$Y(s) = \frac{F(s)}{as^2 + bs + c} + \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\{H(s)F(s)\}}_{\substack{\text{soln. to} \\ ay''+by'+cy=0, \\ \text{zero ICs}}} + \underbrace{\mathcal{L}^{-1}\{H(s)(asy_0 + ay_1 + by_0)\}}_{\substack{\text{soln. to} \\ ay''+by'+cy=0,\ y(0)=y_0, \\ y'(0)=y_1}}$$

particular

homog. soln.
freedoms disappeared
because of ICs

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1. $$\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s} \cdot \frac{1}{s^2+9}\right\} = u\left(t - \frac{\pi}{2}\right) \cdot \frac{1}{3}\sin\left(3\left(t - \pi/2\right)\right).$$

exponential
on s-side

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{1}{3}\sin(3t)$$

$$\boxed{\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as} \cdot \mathcal{L}\{f(t)\}}$$

s-side product

$$\mathcal{L}^{-1}\left\{ \mathcal{L}\{\delta(t-\pi/2)\} \cdot \mathcal{L}\{\tfrac{1}{3}\sin(3t)\} \right\}$$

$$= \delta(t-\pi/2) * \tfrac{1}{3}\sin(3t) \quad \text{(by convolution theorem)}$$

$$= \int_0^t \underbrace{\delta(w-\pi/2)}_{\text{activated when } w=\pi/2} \tfrac{1}{3}\sin(3(t-w))\,dw$$

$$= \begin{cases} 0, & t < \pi/2 \\ \tfrac{1}{3}\sin(3(t-\pi/2)), & t \geq \pi/2 \end{cases}$$

2. $\mathcal{L}^{-1}\left\{ \dfrac{1}{s^2(s^2+4)} \right\}$

comes from $A \cdot 1$ (t-side)

$B \cdot t$

One start $\dfrac{1}{s^2(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4}$

$$= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$= \frac{A\cancel{s}}{s^{\cancel{2}}} + \frac{B}{s^2} + \frac{Cs}{s^2+4} + \frac{D}{s^2+4}$$

t side: cosine

t side: sine

Another: See individually

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$$

Use convolution

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \cdot \frac{1}{s^2+4} \right\} = \underline{t} * \frac{1}{2}\sin(2t)$$

$t-w \to w \qquad w \to t-w$

$$= \int_0^t (t-w)\,\frac{1}{2}\sin(2w)\,dw$$

$$= \int_0^t w \cdot \frac{1}{2}\sin\big(2(t-w)\big)\,dw$$

3. $$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+6s+11} \right\}$$

roots of $s^2+6s+11=0$ $\qquad s = \frac{-6}{2} \pm \frac{1}{2}\sqrt{36-44}$

nonreal

$\Longrightarrow s^2+6s+11$ irreducible

(no partial fractions)

$$\frac{s}{s^2+6s+11} = \frac{s}{s^2+6s+9+2} = \frac{s+3-3}{(s+3)^2+2}$$

$$= \frac{s+3}{(s+3)^2+2} - \underbrace{\frac{\sqrt{2}}{(s+3)^2+2}} \cdot \frac{3}{\sqrt{2}}$$

comes from $e^{at}\cos(bt)$ w/ $a=-3$, $b=\sqrt{2}$

comes from $e^{-3t}\sin(\sqrt{2}\,t)$

Answer: $\mathcal{L}^{-1}\{\ \} = e^{-3t}\cos(\sqrt{2}\,t) - \frac{3}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}\,t).$

4. $\mathcal{L}^{-1}\left\{ e^{-s} \cdot \frac{s}{s^2+6s+11} \right\}$

Start: $\mathcal{L}^{-1}\left\{ \frac{s}{s^2+6s+11} \right\}$ = from # 3

Presence of exponential e^{-1s} on s-side

Ans. $$u(t-1) \cdot e^{-3(t-1)}\left[\cos(\sqrt{2}(t-1)) - \frac{3}{\sqrt{2}}\sin(\sqrt{2}(t-1))\right]$$

5. $\mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^3} + \frac{1}{s^2+2s-8} \right\}$

6. $\mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{1}{(s-1)^3} + e^{-s} \cdot \frac{1}{s^2+2s-8} \right\}$

#5

$\frac{1}{(s-1)^3}$ like entry for $t^n e^{at} \overset{\mathcal{L}}{\curvearrowright} \frac{n!}{(s-a)^{n+1}}$

So $t^2 e^t \curvearrowright \frac{2}{(s-1)^3}$

$$\mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^3} \right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{2}{(s-1)^3} \right\} = \frac{1}{2} t^2 e^t$$

$\frac{1}{s^2+2s-8} = \frac{1}{(s+4)(s-2)}$ partial fractions is possible

So is convolution

mult. on s-side

$$\mathcal{L}^{-1}\left\{ \frac{1}{s+4} \cdot \frac{1}{s-2} \right\} = \mathcal{L}^{-1}\left\{ \mathcal{L}\{e^{-4t}\} \cdot \mathcal{L}\{e^{2t}\} \right\}$$

$$= e^{-4t} * e^{2t} = \int_0^t e^{-4w} e^{2(t-w)}\, dw$$

$$= \int_0^t e^{-4w} \cdot e^{2t-2w}\, dw = \int_0^t e^{2t-6w}\, dw$$

$$= \int_0^t e^{2t} \cdot e^{-6w}\, dw = e^{2t} \int_0^t e^{-6w}\, dw$$

t like constant
for w-integral

$$= e^{2t} \left[-\frac{1}{6} e^{-6w} \right]_0^t = e^{2t} \cdot \left[-\frac{1}{6} e^{-6t} - \left(-\frac{1}{6}\right) \right]$$

$$= \frac{1}{6} e^{2t} \left(1 - e^{-6t}\right) = \boxed{\frac{1}{6}\left(e^{2t} - e^{-4t}\right)}$$

So, putting these together

$$\mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^3} + \frac{1}{s^2+2s-8} \right\} = \frac{1}{2} t^2 e^t + \frac{1}{6}\left(e^{2t} - e^{-4t}\right).$$

#6

$$\mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{1}{(s-1)^3} + e^{-s} \cdot \frac{1}{s^2+2s-8} \right\}$$

$$= \mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{1}{(s-1)^3} \right\} + \mathcal{L}^{-1}\left\{ e^{-s} \cdot \frac{1}{s^2+2s-8} \right\}$$

Entry in table $\mathcal{L}\{ u(t-a) f(t-a) \} = \underline{e^{-as} \cdot \mathcal{L}\{f(t)\}}$ can help

So, for 1st term

$$\mathcal{L}^{-1}\left\{ e^{-2s} \cdot \underbrace{\frac{1}{(s-1)^3}}_{=\mathcal{L}\left\{\frac{1}{2}t^2e^t\right\}} \right\} = u(t-2)\cdot\frac{1}{2}(t-2)^2 e^{t-2}$$

Can do 2^{nd} term same way:

$$\mathcal{L}^{-1}\left\{ e^{-s} \cdot \underbrace{\frac{1}{s^2+2s-8}}_{=\mathcal{L}\left\{\frac{1}{6}\left(e^{2t}-e^{-4t}\right)\right\}} \right\} = u(t-1)\cdot\frac{1}{6}\left[e^{2(t-1)} - e^{-4(t-1)}\right]$$

But both could have been done using Conv. Thm. Just the 2^{nd}:

$$\mathcal{L}^{-1}\left\{ e^{-s}\cdot\frac{1}{s^2+2s-8} \right\} = \mathcal{L}^{-1}\left\{ \mathcal{L}\{\delta(t-1)\} \cdot \mathcal{L}\left\{ \frac{1}{6}\left(e^{2t}-e^{-4t}\right)\right\} \right\}$$

$$= \delta(t-1) * \frac{1}{6}\left(e^{2t}-e^{-4t}\right) = \int_0^t \underbrace{\delta(w-1)}_{\substack{\text{activated} \\ \text{when } w=1}} \frac{1}{6}\left(e^{2(t-w)} - e^{-4(t-w)}\right) dw$$

$$= \begin{cases} 0, & \text{if } t<1 \\ \frac{1}{6}\left[e^{2(t-1)} - e^{-4(t-1)}\right], & \text{if } t>1 \end{cases} = u(t-1)\cdot\frac{1}{6}\left[e^{2(t-1)} - e^{-4(t-1)}\right]$$

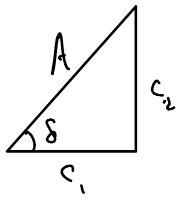
$$\mathcal{L}^{-1}\left\{ e^{-2s}\cdot\frac{1}{(s-1)^3} + e^{-s}\cdot\frac{1}{s^2+2s-8} \right\}$$

$$= \boxed{u(t-2)\cdot\frac{1}{2}(t-2)^2 e^{t-2} + u(t-1)\cdot\frac{1}{6}\left[e^{2(t-1)} - e^{-4(t-1)}\right]}$$

Another idea (forgotten on my list of review topics)

Putting together into one single term expressions like

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(\omega t - \delta)$$

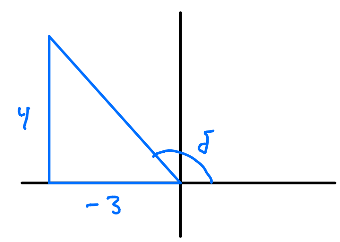


$$A = c_1^2 + c_2^2, \quad \boxed{\cos\delta = \frac{c_1}{A}, \quad \sin\delta = \frac{c_2}{A}}$$

$$\tan\delta = \frac{c_2}{c_1}, \text{ or } \delta = \arctan\left(\frac{c_2}{c_1}\right)$$

Ex.) Write as a single cosine expression

$$-3\cos(2t) + 4\sin(2t)$$



$$c_1 = -3, \quad c_2 = 4$$

$$A = \sqrt{(-3)^2 + 4^2} = 5$$

$$\cos\delta = \frac{-3}{5}, \quad \sin\delta = \frac{4}{5} \quad (\text{only happens in II})$$

$$\delta = \arccos(-3/5)$$

So

$$-3\cos(2t) + 4\sin(2t) = 5\cos(2t - \arccos(-3/5)).$$

Not on the last test, from Ch. 3: Nonhomog. 1^{st}-order system

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{F}(t)$$

like

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}\vec{x} + \begin{bmatrix} \sin t \\ e^{2t} \end{bmatrix}$$

start w/ solving homog. problem $\vec{x}' = A\vec{x}$

$$x_h(t) = \Phi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

then get

$$x_p(t) = \Phi(t) \cdot \int \Phi^{-1}(t)\,\vec{f}(t)\,dt$$

$$= \Phi(t) \cdot \int \Phi^{-1}(t) \begin{bmatrix} \sin t \\ e^{2t} \end{bmatrix} dt$$

usually long in calculations