Solutions

1. (a) The direction of $\boldsymbol{v}$ is

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{4+4+1}}(2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = \frac{2}{3}\hat{\boldsymbol{\imath}} - \frac{2}{3}\hat{\boldsymbol{\jmath}} - \frac{1}{3}\hat{\boldsymbol{k}}.$$

In general, $\nabla f = \left(y + \dfrac{3}{y-z}\right)\hat{\boldsymbol{\imath}} + \left(x - \dfrac{3x}{(y-z)^2}\right)\hat{\boldsymbol{\jmath}} + \dfrac{3x}{(y-z)^2}\hat{\boldsymbol{k}}$, so $\nabla f(-1,1,2) = -2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$. Thus,

$$D_{\boldsymbol{u}} f(1,-1,2) = (-2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}})\circ\left(\frac{2}{3}\hat{\boldsymbol{\imath}} - \frac{2}{3}\hat{\boldsymbol{\jmath}} - \frac{1}{3}\hat{\boldsymbol{k}}\right) = -\frac{5}{3}.$$

(b) We know ∇f points in the direction of maximum increase. We already have ∇f at the point $(1,-1,2)$, so what we need to do is find its direction:

$$\frac{\langle -2,2,-3\rangle}{\|\langle -2,2,-3\rangle\|} = \frac{1}{\sqrt{4+4+9}}\langle -2,2,-3\rangle = \frac{-2}{\sqrt{17}}\hat{\boldsymbol{\imath}} + \frac{2}{\sqrt{17}}\hat{\boldsymbol{\jmath}} - \frac{3}{\sqrt{17}}\hat{\boldsymbol{k}}.$$

2. $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

3. (a) We have

$$\frac{\partial f}{\partial x} = 24x^2 + 12y$$
$$\frac{\partial f}{\partial y} = 12x + 3y^2$$

Setting $f_x = 0$ gives $24x^2 + 12y = 0$, or $y = -2x^2$. Setting $f_y = 0$ and substituting $(-2x^2)$ for y gives

$$12x + 3(-2x^2)^2 = 0 \qquad \Rightarrow \qquad 12x(1+x^3) = 0 \qquad \Rightarrow \qquad x = 0, -1.$$

Since $y = -2x^2$, the option $x = 0$ is paired with $y = 0$, confirming that $(0,0)$ is a critical point. The option $x = -1$ is paired with $y = -2$, giving another critical point at $(-1,-2)$.

(b) The 2[nd] partial derivatives are

$$f_{xx} = 48x, \qquad f_{xy} = 12, \qquad f_{yy} = 6y,$$

so the relevant determinant is

$$D(x,y) = \begin{vmatrix} 48x & 12 \\ 12 & 6y \end{vmatrix} = 288xy - 144.$$

At $(0,0)$: $D(0,0) = -144 < 0$, which means there is a saddle point there.
At $(-1,-2)$: $D(-1,-2) = 576 - 144 = 432 > 0$ and $f_{xx}(-1,-2) = -48 < 0$, which means there is a local maximum there.

4. The projection (shadow region) of our object in the xy-plane has a boundary coming from the intersection of $z = 8$ and $z = 4(x^2+y^2) = 4r^2$. Setting these equal, we have

$$8 = 4r^2 \qquad \Rightarrow \qquad r = \sqrt{2}.$$

Thus, our first moment is

$$M_{xz} = \iiint_D Ky\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{4r^2}^{8} Kr^2\sin\theta\,dz\,dr\,d\theta.$$

Other orders are possible, such as

$$M_{xz} = \int_0^{2\pi}\int_0^{8}\int_0^{\sqrt{z}/2} Kr^2\sin\theta\,dr\,dz\,d\theta.$$

5. Recognizing the region D is the spherical box

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/2, \quad 1 \le \rho \le 2,$$

is crucial. Using spherical coordinates, our integral is

$$\begin{aligned}
\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, dV &= \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi\, e^{\rho^3}\, d\rho\, d\phi\, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left(\int_1^2 3\rho^2 e^{\rho^3}\, d\rho\right) d\phi\, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left(\int_1^8 e^u\, du\right) d\phi\, d\theta \qquad \text{(substituting } u = \rho^3\text{)} \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \Big[e^u\Big]_1^8\, d\phi\, d\theta \;=\; \frac{1}{3}(e^8 - e)\int_0^{2\pi}\left(\int_0^{\pi/2} \sin\phi\, d\phi\right) d\theta \\
&= \frac{1}{3}(e^8 - e)\int_0^{2\pi}\Big[-\cos\phi\Big]_0^{\pi/2}\, d\theta \;=\; \frac{1}{3}(e^8 - e)\int_0^{2\pi} d\theta \;=\; \frac{2\pi}{3}(e^8 - e).
\end{aligned}$$

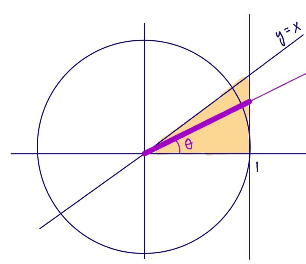
In contrast, the integral is more difficult to set up in cylindrical coordinates where, for instance, the order $dV = r\, dz\, dr\, d\theta$ is the sum of integrals:

$$\int_0^{2\pi}\int_0^1\int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} re^{(r^2+z^2)^{3/2}}\, dz\, dr\, d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{4-r^2}} re^{(r^2+z^2)^{3/2}}\, dz\, dr\, d\theta.$$

Once set up, good luck finding an antiderivative for the inside (z) integrals.

6. In polar form we have

$$\int_0^{\pi/4}\int_0^{\sec\theta} r\sin(r^2)\, dr\, d\theta$$



In D (orange region), $0 \le \theta \le \pi/4$. At fixed θ, $0 \le r \le r_{max}$, where $\cos\theta = \frac{1}{r_{max}} \Rightarrow r_{max} = \sec\theta$

It is possible to integrate in the opposite order, but it requires the sum of integrals:

$$\int_0^1\int_0^{\pi/4} r\sin(r^2)\, d\theta\, dr + \int_1^{\sqrt{2}}\int_{\arccos(1/r)}^{\pi/4} r\sin(r^2)\, d\theta\, dr$$

7. The region is depicted at right. The equation of the two boundaries are $y = x - 1$ (the straight line) and $y^2 - 2x = 6$ (the parabola). The same result, with order of integration reversed, is obtained via the integral

$$\int_{-2}^4\int_{-3+y^2/2}^{y+1} f(x, y)\, dx\, dy.$$

$y = \pm\sqrt{2x+6}$

$\Longrightarrow y^2 = 2x + 6$

$\Longrightarrow x = \frac{1}{2}y^2 - 3$