Linear 1^{st}-order homogeneous systems with nonreal eigenvalues

There are certain things we build on:

- **Euler's Formula**: Given a real number θ, and $i = \sqrt{-1}$, it says $e^{i\theta} = \cos\theta + i\sin\theta$.
  A corollary to it is that $e^{-i\theta} = \cos\theta - i\sin\theta$, making $e^{i\theta}$ and $e^{-i\theta}$ complex conjugates.
  For an explanation of why this amazing formula holds, and secondarily to justify in part your study of Maclaurin series in MATH 172, watch
  https://drive.google.com/file/d/1a7x1QIdNYGis6np3V9rXkq8xYh0wE3yD/view?usp=sharing
  Here are the finished notes from the video.

- When a matrix $\mathbf{A}$ has real entries by a nonreal eigenvalue $\alpha + i\beta$, where α, β are real numbers, there will be at least one corresponding eigenvector $\mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$, $\mathbf{v}$ have real entries. Correspondingly, the complex conjugate $\alpha - i\beta$ is also an eigenvalue of $\mathbf{A}$, and has $\mathbf{u} - i\mathbf{v}$ as an eigenvector. For example, if

$$-3 + 2i \qquad \text{is an eigenvalue with eigenvector} \qquad \begin{bmatrix} 2-3i \\ 1-i \\ 3i \end{bmatrix},$$

  then we can identify

$$\alpha = -3, \quad \beta = 2, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix},$$

  and conclude that

$$\alpha - i\beta = -3 - 2i \qquad \text{is also an eigenvalue with eigenvector} \qquad \mathbf{u} - i\mathbf{v} = \begin{bmatrix} 2+3i \\ 1+i \\ -3i \end{bmatrix}.$$

- We have demonstrated and made of the fact that, if the matrix $\mathbf{A}$ has eigenpair $(\lambda, \mathbf{v})$, then $e^{\lambda t}\mathbf{v}$ is a solution of the homogeneous linear 1^{st}-order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, But it is not possible to make physical sense of such a solution

$$e^{\alpha + i\beta t}(\mathbf{u} + i\mathbf{v}) \qquad \text{and its counterpart} \qquad e^{\alpha - i\beta t}(\mathbf{u} - i\mathbf{v}),$$

  when we are talking about nonreal eigenpairs of $\mathbf{A}$. In
  https://drive.google.com/file/d/1elJcAEDA807WxB6JigSH3NYf5YgcY9OC/view?usp=sharing
  I justify why it is reasonable and valid to trade out those nonreal solutions for these *real* substitutes:

$$e^{\alpha t}\left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right] \qquad \text{and} \qquad e^{\alpha t}\left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right].$$

  Here are the finished notes from that video.

Some examples:

1. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} -21 & -30 & -32 \\ -4 & -7 & -7 \\ 24 & 30 & 35 \end{bmatrix}$
   Videos will be played during class, but here are the three pages of end notes:
   - page 1
   - page 2
   - page 3
2. $\dfrac{d}{dt}\mathbf{x} = \begin{bmatrix} -5 & -10 \\ 5 & 9 \end{bmatrix}$
   For end notes, consult page 3 above.