1. (a) qnorm(0.75, 80, 5)

(b) A random sample (iid or SRS) $X_1, \dots, X_{20}$ has total weight

$S = X_1 + \cdots + X_n \sim \text{Norm}((20)(80), 5\sqrt{20}) = \text{Norm}(1600, 10\sqrt{2})$.

The command:

1 − pnorm(1700, 1600, 10 * sqrt(2))

It also works to think in terms of sample mean $\bar{X} \sim \text{Norm}(80, \frac{5}{\sqrt{20}})$.

Overloading occurs precisely when $\bar{X} > 1700/20 = 85$. So this also works:

1 − pnorm(85, 80, 5/sqrt(20)).

2. $M_X'(t) = 2e^{2t}(1-t^2)^{-1} - e^{2t}(1-t^2)^{-2}(-2t) = 2e^{2t}(1-t^2)^{-1} + 2te^{2t}(1-t^2)^{-2}$

$$\Longrightarrow \boxed{E(X) = \mu_1 = M_X'(0) = 2.}$$

$$\begin{aligned} M_X''(t) &= 4e^{2t}(1-t^2)^{-1} - 2e^{2t}(1-t^2)^{-2}(-2t) \\ &\quad + 2e^{2t}(1-t^2)^{-2} + 4te^{2t}(1-t^2)^{-2} - 4te^{2t}(1-t^2)^{-3}(-2t) \\ &= 4e^{2t}(1-t^2)^{-1} + 2e^{2t}(1-t^2)^{-2} + 8te^{2t}(1-t^2)^{-2} + 8t^2e^{2t}(1-t^2)^{-3} \end{aligned}$$

$$\Longrightarrow E(X^2) = \mu_2 = M_X''(0) = 6$$

$$\boxed{\text{Var}(X) = E(X^2) - [E(X)]^2 = 6 - (2)^2 = 2}$$

3. We know

$$\begin{aligned} M_S(t) &= E(e^{tS}) = E(e^{t(X_1+\cdots+X_n)}) = E(e^{tX_1 + tX_2 + \cdots + tX_n}) \\ &= E(e^{tX_1} \cdots e^{tX_n}) = E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_n}) \\ &= M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t). \end{aligned}$$

True because the X_i are independent

This holds for each of letters (a) − (e). As to specifics,

(a) $M_S(t) = [\pi(e^t - 1) + 1][\pi(e^t - 1) + 1] \cdots [\pi(e^t - 1) + 1]$ (n factors, all identical)

$= (\pi e^t + 1 - \pi)^n$,

which shows $S \sim \text{Binom}(n, \pi)$.

(b) $M_S(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \cdots \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} = \left[\left(\frac{\lambda}{\lambda - t}\right)^{\alpha}\right]^n = \left(\frac{\lambda}{\lambda - t}\right)^{n\alpha},$

which shows $S \sim \text{Gamma}(n\alpha, \lambda)$.

(c) $M_S(t) = \left(\frac{\lambda}{\lambda - t}\right) \cdots \left(\frac{\lambda}{\lambda - t}\right) = \left(\frac{\lambda}{\lambda - t}\right)^n,$

which shows $S \sim \text{Gamma}(n, \lambda)$.

(d) $M_S(t) = \left(e^{\lambda e^t - \lambda}\right)^n = e^{n\lambda e^t - n\lambda},$ showing $S \sim \text{Pois}(n\lambda)$.

(e) $M_S(t) = \left(e^{\mu t + \sigma^2 t^2/2}\right)^n = e^{n\mu t + n\sigma^2 t^2/2} = e^{(n\mu)t + (\sigma\sqrt{n})^2 t/2},$

revealing that $S \sim \text{Norm}(n\mu, \sigma\sqrt{n})$.

4. (a) We have, by definition,

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b) f_X(x)\, dx = a\int_{-\infty}^{\infty} x f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx$$

$f_X(x)$ is a density function, so this equals 1.

$$= a\,E(X) + b.$$

(b)
$$\text{Var}(aX+b) = E\left[\left((aX+b) - \mu_{aX+b}\right)^2\right] = E\left[(aX+b)^2 - 2\mu_{aX+b}\cdot(aX+b) + \mu_{aX+b}^2\right]$$

$$= E\left((aX+b)^2\right) - 2\mu_{aX+b}\, E(aX+b) + \mu_{aX+b}^2 \qquad \text{(by part (a))}$$

$$= E(a^2X^2 + 2abX + b^2) - 2\mu_{aX+b}^2 + \mu_{aX+b}^2$$

$$= a^2 E(X^2) + 2ab\,E(X) + b^2 - \mu_{aX+b}^2 \qquad \text{(also by part (a))}$$

$$= a^2 E(X^2) + 2ab\,\mu_X + b^2 - (a\mu_X + b)^2 \qquad \text{(part (a) again)}$$

$$= a^2 E(X^2) + 2ab\,\mu_X + b^2 - (a^2\mu_X^2 + 2ab\mu_X + b^2)$$

$$= a^2\left(E(X^2) - \mu_X^2\right)$$

$$= a^2\,\text{Var}(X).$$

5. (a) Since $\mu = \frac{1}{\pi} - 1$, we choose $\hat{\pi}$ so

$$\frac{1}{\hat{\pi}} - 1 = \bar{x} \implies \frac{1}{\hat{\pi}} = \bar{x} + 1 \implies \hat{\pi} = \frac{1}{\bar{x}+1}.$$

Given $\bar{x} = 32$, we have $\hat{\pi} = 1/33$.

(b) Gamma distributions require two parameters α, λ. Since $\mu = \frac{\alpha}{\lambda}$ and $\sigma^2 = \frac{\alpha}{\lambda^2}$, we require $\hat{\alpha}, \hat{\lambda}$ (estimates of these parameters) to satisfy

$$\frac{\hat{\alpha}}{\hat{\lambda}} = \bar{x} \quad \text{and} \quad \frac{\hat{\alpha}}{\hat{\lambda}^2} = v.$$

But

$$v = \frac{\hat{\alpha}}{\hat{\lambda}^2} = \left(\frac{\hat{\alpha}}{\hat{\lambda}}\right) \cdot \frac{1}{\hat{\lambda}} = \bar{x} \cdot \frac{1}{\hat{\lambda}} \implies \hat{\lambda} = \frac{\bar{x}}{v} = \frac{32}{12.5} = 2.56$$

$$\implies \hat{\alpha} = \bar{x}\hat{\lambda} = (32)(2.56) = 81.92.$$

(c) gf_qq(~ myData, distribution = qgamma, dparams = c(shape = 81.92, rate = 2.56))

6. (a) We require

$$\frac{1}{k} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = \int_0^1\int_0^1 x(y+y^2)\,dy\,dx = \int_0^1 x\left[\frac{1}{2}y^2 + \frac{1}{3}y^3\right]_0^1 dx$$

$$= \int_0^1 \left(\frac{1}{2}+\frac{1}{3}\right)x\,dx = \frac{5}{6}\int_0^1 x\,dx = \frac{5}{12}\left[x^2\right]_0^1 = \frac{5}{12}$$

So, $k = \frac{12}{5}$.

(b) If $x \in [0,1]$, then (by definition)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{12}{5}\int_0^1 xy(1+y)\,dy = \frac{12}{5}x\int_0^1 (y+y^2)\,dy$$

$$= \frac{12}{5}x\left[\frac{1}{2}y^2 + \frac{1}{3}y^3\right]_0^1 = \frac{12}{5}\left(\frac{1}{2}+\frac{1}{3}\right)x = 2x \cdot [\![x \in [0,1]]\!]$$

This is the marginal pdf. The conditional pdf, when $y \in [0,1]$, is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{12/5\; xy(1+y)}{2x} = \frac{6}{5}y(1+y) \cdot [\![y \in [0,1]]\!]$$