Stat 343, Thu 5-Nov-2020 -- Thu 5-Nov-2020

Probability and Statistics

Fall 2020

Thursday, November 05th 2020

Wk 10, Th

Topic:: Confidence intervals

Read:: FASt 4.5

Setting for inference

(1) There variable we investi~~age~~gate has a fixed population of interest.

(2) Our only means to obtain information about this population is through sampling.

(3) The parameters of the population are unknown to us.

(4) Sample data allows for us to produce numerical summaries (**statistics**).

The goal of **statistical inference** is information about the unknown parameters of the population (in the context of a distributional model) from known sample statistics.

Two general inference paradigms

1. **Hypothesis testing**. A statement, called a **null hypothesis**, is made about the parameter(s) of a distribution, a sample summary, called a **test statistic**, is used to compute a P-value.

   A P-value is the probability of obtaining, in a random sample from a population where the null hypothesis is true, a test statistic at least as extreme as the one in our sample. Assessing a P-value requires a model for the sampling distribution of the sample statistic under the null hypothesis. We call this model the **null distribution**.

   When the P-value is small (below the predefined significance level α), we may draw one of these conclusions:

   - The parameter is not as asserted in the null hypothesis.
   - The sample was not sufficiently random—not enough like an i.i.d. random sample or SRS.
   - Our null distribution (model of the sampling distribution) is inaccurate.
   - Our sample just happens to be one that produces one of the possible yet more rarely-occurring values of the test statistic when the null hypothesis is true. (This option is closely tied to Type I error.)

2. **Confidence interval**. The goal is to estimate the value of a parameter, not just with a single point estimate, but with an interval of numbers.

   Typically, we have an unbiased estimator of the desired parameter, and a sampling distribution we can assume is reasonably normal:

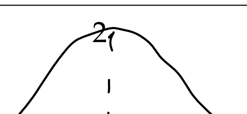
$\overline{X}$ for μ

sample proportion = $\hat{\pi}$ for π

π = probability of success = long term rel. freq.
= proportion of times success happens in the population

> **Definition 1:** For $\alpha \in [0,1]$, define $z_\alpha = \Phi^{-1}(\alpha)$. That is, given the standard normal variable $Z \sim \text{Norm}(0,1)$, take z_α to be the number satysfying
>
> $$\Pr(Z \geq z_\alpha) = \alpha,$$

$z_{0.025}$ = qnorm(0.975) = 1.96



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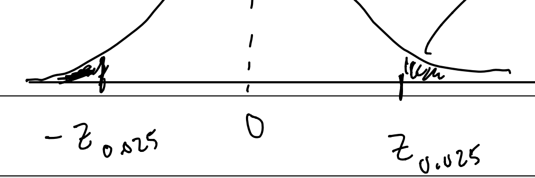


the α-quantile in a standard normal distribution.

Definition 2 (Defn. 4.5.1, p. 260): Let $\mathbf{X} = \langle X_1, \ldots, X_n \rangle$ be a random sample (an i.i.d. or SRS) of size n from a population with unknown mean μ and known variance σ^2. The approximate 100C% confidence interval for μ is

$$\overline{x} \pm \underbrace{(z_{\alpha/2})(\mathrm{SE}_{\overline{X}})}_{\text{margin of error}},$$

where $\alpha = 1 - C$.

$C = 95\%$ conf.

$\alpha = 1 - 0.95 = 0.05$

point estimate for μ

95% confidence

$\text{m.o.e.} = (z_{0.025})(\mathrm{SE}_{\overline{x}})$

Examples

Understanding confidence levels

We hope a confidence interval contains the desired paramater. We will not generally know that it does. The level of confidence corresponds to a **coverage rate**. In particular,

(1) The process used in its construction is random in that

- sampling is a random process,
- the sample statistic (estimator) built from samples is a random variable,
- the endpoints of the interval can be seen as random variables, too. We expect success, in the sense of covering the parameter, to be approximately equal that of the confidence level.

(2) The estimand/parameter is not random. It is unknown, but fixed. Any particular sample, particular statstics, or resulting confidence interval is also non-random. Since these are not random, you cannot talk about probabilities.

Example: Suppose we have a sample of size $n = 20$ from a population w/

- an unknown mean
- $\sigma = 2.5$

sample mean $\bar{x} = 4.7$

SE name for std. dev. of the sampling dist.

Construct a 95% CI for μ (the population mean).

$$\underset{\text{pt. est.}}{\bar{x}} \pm \underbrace{\left(z_{0.025}\right)\left(\sigma/\sqrt{n}\right)}_{\text{m.o.e.}}$$

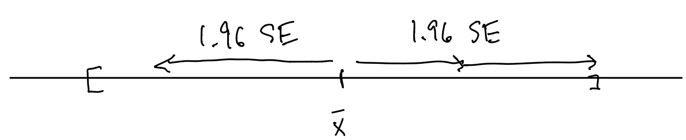
$\frac{\alpha}{2}$ w/ $\alpha = 1 - C$

Know: $\bar{x}$ (taken from an i.i.d. SRS) has an increasingly normal dist. w/ mean $= \mu$ (pop. mean)

sd $= \sigma/\sqrt{n}$ (pop. std. dev. $\div \sqrt{n}$)

95% CI

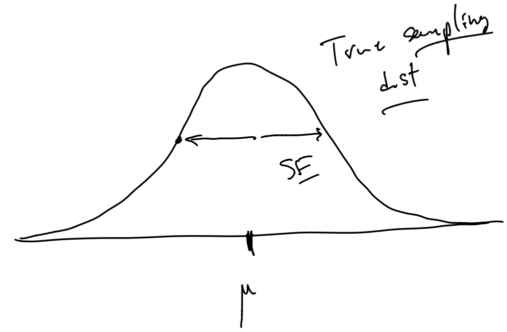
$$4.7 \pm (1.96)\frac{2.5}{\sqrt{20}}$$



pt. est.

Is this sensible?

- Gives an interval of possible values for μ (improvement over a simple pt. est. $\bar{x}$)
- If $\bar{X}$ has $\text{Norm}(\mu, \sigma/\sqrt{n})$ dist. then 95% of sample values $\bar{x}$ lie within 1.96 SE's of μ.



Interpretation:

What does 95% quantify?

Probability needs some random process to give it substance.

We are following a process that has a 95% (level C) success rate at capturing μ (population parameter).

Example

Consider: rolling a die (might be weighted)

$n = 100$ times to estimate true probability of a "6". (w/ 95% conf.)

Call this true probability π.

Sample data $\hat{\pi} = \frac{11}{100}$

Distribution (sampling) for $\hat{\pi} \sim \text{Binom}(100, \pi)$

We have at least 10 successes
10 failures
} rule of thumb for a normal approx.

So

$\text{Binom}(100, \pi)$ is reasonably like

$\text{Norm}\left(\pi, \sqrt{\frac{\pi(1-\pi)}{n}}\right)$

since $\hat{\pi}$ is unbiased est. of π

look this up in 2-3-ago notes

Do as before

use pt. estimate as in method of moments

$$\hat{\pi} \pm (1.96)\left(\sqrt{\frac{\pi(1-\pi)}{100}}\right)$$

point est.

again $z_{0.025}$