Planar Systems

The homogeneous linear 1^{st}-order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

- has an equilibrium point at $\mathbf{x}_0$ if $\left.\frac{d\mathbf{x}}{dt}\right|_{\mathbf{x}=\mathbf{x}_0} = \mathbf{0}$. That is, if $\mathbf{x}_0$ is in the null space of $\mathbf{A}$. Note that the origin $\mathbf{0}$ is always an equilibrium point, and if $\mathbf{A}$ is nonsingular, it is the only one.
- is **planar** if it has exactly two dependent variables—i.e., $\mathbf{x} = \langle x_1, x_2 \rangle$. Necessarily, $\mathbf{A}$ is a 2-by-2 matrix. In a planar system, the equilibrium at the origin is described as
  - a **globally asymptotically stable node** if the eigenvalues of $\mathbf{A}$ are *real* and $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - an **unstable node** if the eigenvalues of $\mathbf{A}$ are *real* and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - an **unstable saddle point** if the eigenvalues of $\mathbf{A}$ are *real* and their product $\lambda_1\lambda_2 < 0$.
  - a **globally asymptotically stable spiral point** if the eigenvalues of $\mathbf{A}$ are *nonreal* and $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - an **unstable spiral point** if the eigenvalues of $\mathbf{A}$ are *nonreal* and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions regardless of initial condition.
  - a **stable center** if the eigenvalues of $\mathbf{A}$ are *nonreal* and the origin is neither an unstable spiral point nor a globally asymptotically stable spiral point.

A coordinate frame that gives an axis to each of the dependent variables but and no others (no axis for the independent variable t) is called **phase space** (the **phase plane**, for planar systems). A **phase portrait** is a sketch of trajectories/solutions in phase space.

Today's Work We will be considering the homogeneous planar systems numbered below. Each one of you should log into `https://b.socrative.com/login/student/`, identifying yourself by name using the convention "lastFirst" (Pat Walsh would log in as "walshPat"), and going to room "SCOFIELD3894". Give answers to the questions you find there. It is my intention that you use these two apps (familiar from yesterday) to ease the workload.
`http://scofield.site/teaching/demos/eigenstuff.html` and
`http://scofield.site/teaching/demos/PhasePortrait2D.html`
for finding eigenpairs and plotting phase portraits, respectively.

1. $\mathbf{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\mathbf{x}$
2. $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\mathbf{x}$
3. $\mathbf{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}\mathbf{x}$
4. $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\mathbf{x}$
5. $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\mathbf{x}$
6. $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{x}$

Solutions to the systems on the previous page, look like:

1. The first system, says
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or} \qquad \begin{array}{l} x' = x - 5y \\ y' = x - 3y \end{array}.$$
That's how you decide what belongs in the phase portrait app. Trajectories appear, possibly, to spiral in toward the origin.
The other app reports eigenpairs, one pair being
$$-1 + i \ \text{ with } \begin{bmatrix} 1 \\ 0.4 - 0.2i \end{bmatrix}, \qquad \text{yielding} \qquad \alpha = -1,\ \beta = 1,\ \mathbf{u} = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix},\ \mathbf{v} = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}.$$
You do not need to know the other eigenpair to identify these. For your teacher, the knowledge that eigenvalues are nonreal and α is negative is enough to conclude you have a *globally asymptotically stable spiral point*, meaning I am able to answer Question 1 in Socrative and move on. Here is how I know.

You are not asked to do so, but I get my insights above through practice on the types of solutions we discussed Wed. In particular, we saw that complex eigenpairs lead to *real* solutions
$$e^{\alpha t}\left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right] = e^{-t}\left(\cos t \begin{bmatrix} 1 \\ 0.4 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}\right) = \begin{bmatrix} e^{-t}\cos t \\ e^{-t}(0.4\cos t + 0.2\sin t) \end{bmatrix},$$
and
$$e^{\alpha t}\left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right] = e^{-t}\left(\sin t \begin{bmatrix} 1 \\ 0.4 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}\right) = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}(0.4\sin t - 0.2\cos t) \end{bmatrix}.$$
You can see these solutions have a factor that is periodic, but another factor which dies off exponentially. This should confirm the sense we had from the phase portrait app that trajectories spiral inward. The general solution is made up of linear combinations of these two
$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{bmatrix} e^{-t}\cos t \\ e^{-t}[0.4\cos t + 0.2\sin t] \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\sin t \\ e^{-t}[0.4\sin t - 0.2\cos t] \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ e^{-t}[0.4\cos t + 0.2\sin t] & e^{-t}[0.4\sin t - 0.2\cos t] \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}. \end{aligned}$$

2. The phase portrait app gives us the impression of trajectories as orbits around the origin, not converging toward it either as $t \to \infty$ nor as $t \to -\infty$. The eigenvalue app reports one of the eigenpairs as
$$i \ \text{ with } \begin{bmatrix} 1 \\ 0.4 - 0.2i \end{bmatrix} \qquad \text{leading to the identifications} \qquad \alpha = 0,\ \beta = 1,\ \mathbf{u} = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix},\ \mathbf{v} = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}.$$

Just knowing we have nonreal eigenvalues and $\alpha = 0$ is enough to identify the origin as a *stable center*.

Going further, but without all the details I included above, the form of the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t \\ 0.4\cos t + 0.2\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ 0.4\sin t - 0.2\cos t \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ 0.4\cos t + 0.2\sin t & 0.4\sin t - 0.2\cos t \end{bmatrix} \mathbf{c}.$$

We do not have the exponential growth/decay function multiplying the periodic parts, so trajectories are truly orbits, characteristic of a stable center.

3. This time the eigenpairs are reported to be

$$4 \text{ with } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad 2 \text{ with } \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

which gives rise to basis solutions

$$e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix}.$$

The general solution is formed from linear combinations of these two:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix} = \begin{bmatrix} {}^{4t} & e^{2t} \\ e^{4t} & 3e^{2t} \end{bmatrix} \mathbf{c}$$

When $c_1 = c_2 = 0$ you have the equilibrium solution at the origin, but for any other choice of constants solutions run away from the origin as $t \to \infty$ (or toward it as $t \to -\infty$). The origin is an *unstable node*.

4. The eigenvalue app reports one of the eigenpairs as

$$1{+}2i \text{ with } \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \qquad \text{leading to the identifications} \qquad \alpha = 1,\ \ \beta = 2,\ \ \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \ \mathbf{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Just knowing we have nonreal eigenvalues and $\alpha > 0$ is enough to identify the origin as an *unstable spiral point*.

You are not asked to do so, but I'll go ahead and write the general solution, again using the real solutions from formulas

$$e^{\alpha t}\left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right] \qquad \text{and} \qquad e^{\alpha t}\left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right].$$

I skip some details, but with our $\alpha, \beta, \mathbf{u}, \mathbf{v}$ inserted, our general solution is a linear combination of them:

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{bmatrix} e^t\cos(2t) \\ e^t[\cos(2t) + \sin(2t)] \end{bmatrix} + c_2 \begin{bmatrix} e^t\sin(2t) \\ e^t[\sin(2t) - \cos(2t)] \end{bmatrix} \\ &= \begin{bmatrix} e^t\cos(2t) & e^t\sin(2t) \\ e^t[\cos(2t) + \sin(2t)] & e^{-t}[\sin(2t) - \cos(2t)] \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}. \end{aligned}$$

5. The eigenpairs here are 2 with $\langle 2, 1\rangle$ (Note: any scalar multiple of an eigenvector is an eigenvector) and -1 with $\langle 1, 2\rangle$. Since eigenvalues are real and of opposite sign, the origin is a *saddle point*.

Using the basis solutions derived from these eigenpairs, we get general solution

$$\mathbf{x}(t) = c_1 e^{2t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2e^{2t} & e^{2t}\\ e^{-t} & 2e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}.$$

6. The eigenpairs here are -1 with $\langle 1, 1\rangle$ and -3 with $\langle 1, -1\rangle$. Since the solutions built from these eigenpairs will both seek the origin as $t \to \infty$, so will all solutions, making the origin a *globally asymptotically stable node*.

Here, the general solution is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}e^{-t} & e^{-3t}\\ e^{-t} & -e^{-3t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}.$$

7. It is the solutions with sines and cosines—the spiral points and centers, which have no straight-line trajectories. The nodes and saddles do.

8. Our general solution for the saddle system is

$$\mathbf{x}(t) = c_1 e^{2t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\2\end{bmatrix} = \mathbf{\Phi}(t)\mathbf{c}.$$

Initial condition and choices for c_1, c_2 go hand-in-hand. We see, however, that if both c_1 and c_2 are nonzero, then both basis solutions hold sway, with one dominating when $t < 0$ and the other when $t > 0$. If we take $c_1 = 0$ and $c_2 = 1$, however, then our solution with that choice is

$$\mathbf{x}(t) = e^{-t}\begin{bmatrix}1\\2\end{bmatrix} \qquad \text{which satisfies IC} \qquad \mathbf{x}(0) = \begin{bmatrix}1\\2\end{bmatrix}.$$

As $t \to \infty$, this solution's path is straight to the origin along the line of the vector $\langle 1, 2\rangle$.

9. Having seen how to get a straight-line trajector in Number 8, it makes sense to do the same thing in 9. Here, since both solutions come from positive eigenvalues, it doesn't matter which of c_1, c_2 is set to zero. I'll take $c_1 = 1$ and $c_2 = 0$, so that we have solution

$$\mathbf{x}(t) = e^{4t}\begin{bmatrix}1\\1\end{bmatrix} \qquad \text{which satisfies IC} \qquad \mathbf{x}(0) = \begin{bmatrix}1\\1\end{bmatrix}.$$

10. You can, for example, pick the vector $\langle 1, 0\rangle$ facing due east of the origin, and find that

$$\left.\begin{bmatrix}x'\\y'\end{bmatrix}\right|_{\langle x,y\rangle=\langle 1,0\rangle} = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}, \qquad \text{so that} \qquad \left.y'\right|_{\langle x,y\rangle=\langle 1,0\rangle} \text{ is positive (=1).}$$

Thus, at the moment a trajectory reaches the point $(1, 0)$, it is rising, and must be going counterclockwise around the origin.
