Mathematical Induction

$= \{1, 2, 3, \ldots\}$

- It is a technique for proving a statement $\forall n \in \mathbb{Z}^+ P(n)$.
- Can be adapted to prove the correctness of some algorithms.
- As a rule of inference, it is

$$(P(1) \wedge \forall k\,(P(k) \to P(k+1))) \to \forall n\, P(n).$$

$P(1)$ is called the **basis step**, $P(k) \to P(k+1)$ is called the **inductive step**, and the assumption that the hypothesis $P(k)$ of the inductive step holds is called the **inductive hypothesis**.

Induction is not helpful in discovering in discovering new mathematical statements which are true. Once a pattern or truth has been conjectured, however, induction can often establish that it is true.

Examples:

1. $\sum_{j=1}^{n}(2j-1) \;=\; 1+3+5+\cdots+(2n-1) \;=\; ?.$

2. For all positive integers, $23^n - 1$ is divisible by 11.

$\{1, 2, 3, 4, \ldots\}$

$\forall n \in \mathbb{Z}^+ P(n)$ where $P(n): n < 2^n$

$P(5): 5 < 2^5 = 32$

$P(1): 1 < 2^1 = 2$

Basis: $P(1)$

$P(k): k < 2^k$

$P(k+1): k+1 < 2^{k+1}$

3. For all positive integers, $n < 2^n$.

Induction step: $P(k) \to P(k+1)$

$k < 2^k$ by induction hyp.

Add 1 to both sides

$k+1 < 2^k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

destination

$k!$

$(k+1)! = (k+1)(k)(k-1)(\cdots)(1)$
$= (k+1)k!$

$\forall n \in \{4,5,6,7,\ldots\},\ P(n)$

4. For all $n \in \mathbb{N} - \{0,1,2,3\}$, $\boxed{2^n < n!}$

basis step: $P(4)$ $2^4 < 4!$ ✓

induction step: Can assume $P(k)$ holds for some $k \in \{4,5,6,\ldots\}$

So $2^k < k!$ ($P(k)$ states this, we assume it's true)

Must show $P(k+1)$: $2^{k+1} < (k+1)!$

$2^k < k!$ Mult. both sides by 2:

destination

$2^{k+1} < 2 \cdot k! < (k+1)k! = (k+1)!$

So, $P(k+1)$ holds

Thus $\forall n \in \{4,5,6,\ldots\}$, $P(n)$ is true by math induction

5. If B is a set with $|B| = n$, then $|\mathcal{P}(B)| = 2^n$, for all $n \in \mathbb{N}$.

$P(n)$

Base case: $n=0$, only $B = \emptyset$. So $\mathcal{P}(\emptyset) = \{\emptyset\}$

and $|\mathcal{P}(B)| = 1 = 2^0$. ✓

Induction step: Assume, for some $k \in \mathbb{N}$, $P(k)$ holds.

Now let B be a set $|B| = k+1$. Break off one element from B: That is, let $b \in B$ and write $B = \underbrace{(B - \{b\})}_{\text{call } B_1} \cup \{b\}$

Note: $|B_1| = k$

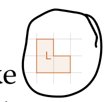
Note also: Every subset of B_1 can be used to generate 2 subsets of B

subset of B_1 → leave as is / → add to it b } both subsets of B

$|\mathcal{P}(B)| = 2|\mathcal{P}(B_1)|$

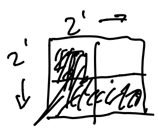
$= 2 \cdot 2^k$ (by the I.H.) $= 2^{k+1}$ completing the Induction step.

6. Show that $3n^3 + 2n + 7 \leqslant 4n^3$ for $n = 3, 4, 5, \ldots$.

7. One can tile an $2^n \times 2^n$ checkerboard with one space removed using tiles shaped like 

$\forall n \in \{1, 2, 3, 4, \ldots\} = \mathbb{Z}^+$

Basis step: $P(1)$



Induction step: Can assume $P(k)$ holds for some $k \in \mathbb{Z}^+$.

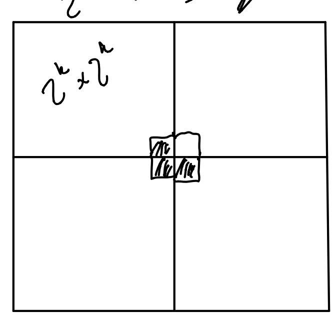
Take a $2^{k+1} \times 2^{k+1}$ grid/checkerboard

$k+1$

$2^{k+1} \rightarrow$

$2^{k+1} \downarrow$

$2^k \times 2^k$



If we can assume, in $P(k)$, that the empty square can be placed in the corner, then each of the 4 subgrids, $2^k \times 2^k$ in size, can be tiled w/ the empty square adjoining those in the other subgrids. We finish it off w/ one tile as shown.

8. **Induction misused**. Let $P(n)$ be the statement "Any collection of $n \geqslant 2$ distinct lines in the plane, no two of which are parallel, shares a common point.

   The following is an attempt to prove $\forall n \in \mathbb{Z}^+, P(n)$:
   Base case: $P(2)$ says 2 non-parallel lines in the plane have a common point. This seems true enough without requiring proof.
   Inductive step: We assume $P(k)$ is true for some integer $k \geqslant 2$. The case $P(k+1)$ has us considering $(k+1)$ non-parallel lines in the plane: $\{\ell_1, \ell_2, \ldots, \ell_k, \ell_{k+1}\}$. Now the collection $\{\ell_1, \ell_2, \ldots, \ell_k\}$ has k non-parallel lines so by the induction hypothesis, this collection has a common point, call it P_1. As well, the induction hypothesis applies to the collection $\{\ell_2, \ell_3, \ldots, \ell_k, \ell_{k+1}\}$, so these lines have a common point, call it P_2. But two points in a plane uniquely determine a line, and since no two lines found in both collections can be the same, it must be that points P_1 and P_2 are really the same point. Thus, our original collection $\{\ell_1, \ell_2, \ldots, \ell_k, \ell_{k+1}\}$ shares a commont point, showing that $P(k+1)$ holds.
   Thus, by induction, $P(n)$ holds for all $n = 2, 3, 4, \ldots$.

Ex. $B = \{1, 2, 3\}$

$b = 3$

Let $B_1 = \{1, 2\}$

Subsets of B_1	Subsets of B
$\{\}$	$\{\}$
	$\{3\}$
$\{1\}$	$\{1\}$
	$\{1, 3\}$
$\{2\}$	$\{2\}$
	$\{2, 3\}$
$\{1, 2\}$	$\{1, 2\}$
	$\{1, 2, 3\}$

$$|\mathcal{P}(B)| = 2|\mathcal{P}(B_1)|$$