MATH 162: Calculus II

Framework for Mon., Mar. 26

Quadric Surfaces

Today's Goal: To review how equations in three variables are graphed, and to identify special graphs known as quadric surfaces.

What we already know: A 2nd-order polynomial in x and y takes the general form

$$p(x,y) \;=\; Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Such a polynomial is called a *quadratic polynomial* (in x and y).

Some special cases, usually treated in high school classes:

- **Case** $B = C = 0,\ A \neq 0$: $\quad p(x,y) \;=\; Ax^2 + Dx + Ey + F$
  - The level sets of p are parabolas.
  - By symmetry of argument, the level sets are parabolas opening sideways when $A = B = 0$ and $C \neq 0$.
- **Case** $B^2 < 4AC$:
  - Some level sets of p are ellipses.
  - When $B = 0$ and $A = C$, the ellipses are actually circles.
- **Case** $AC < 0$: Almost all level sets of p are hyperbolas.

Quadric Surfaces

Similar to the above, a quadratic polynomial in x, y and z is a 2nd-order polynomial having general form

$$p(x,y,z) \;=\; Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 + Gx + Hy + Iz + J. \tag{1}$$

- The graph of p would require 4 dimensions, but the level sets of p are *surfaces* in 3D.
- The solutions of the equation $p(x,y,z) = k$ (k a fixed number) coincide with the k-level surface for the quadratic function p.

> **Definition**: For a quadratic polynomial p in the form (1), the set of points (x,y,z) which satisfy the level surface equation $p(x,y,z) = k$ is called a *quadric surface.*

- When the coefficient $F = 0$ and at least one of D, E or I is nonzero, the level surface equation may be manipulated algebraically to solve for z as a function of x and y. In these cases, what we learned about graphing functions of 2 variables still applies.

  Example: $9x^2 - y^2 - 4z = 0$ (hyperbolic paraboloid)

  By symmetry of argument, the level surface equation $p(x, y, z) = k$ can be written as a function if

  - $A = 0$ and at least one of B, D or G is nonzero, in which case x may be written as a function of y and z, or
  - $C = 0$ and any one of B, E or H is nonzero, in which case y may be written as a function of x and z.

- Even when the equation $p(x, y, z) = k$ cannot be re-written as a function of two variables, a good way to get an idea of the graph of the level surface is to consider cross-sections:

  - slices by planes parallel to the xy-plane are the result of setting $z = z_0$.
  - slices by planes parallel to the xz-plane are the result of setting $y = y_0$.
  - slices by planes parallel to the yz-plane are the result of setting $x = x_0$.

  Examples:

  $9x^2 + y^2 - 4z^2 = 1$ (an hyperboloid in one sheet)

  $9x^2 + y^2 - 4z^2 = 0$ (an elliptic cone)

  $9x^2 - y^2 - 4z^2 = 1$ (an hyperboloid in two sheets)

  $9x^2 + y^2 + 4z^2 = 1$ (an ellipsoid)