Exercise 4.1

(a) Using the pdf associated with the r.v. X, we have

$$\Pr\left(X \geq \frac{1}{2}\right) = \int_{1/2}^{\infty} f(x)\,dx = \int_{1/2}^{1} 2x\,dx = x^2\Big|_{1/2}^{1} = 1 - \frac{1}{4} = \frac{3}{4}.$$

One might also compute this using the computational power of R:

```
f = makeFun(2 * x * (x >= 0 & x <= 1) ~ x)
integrate(f, 1/2, 1)

0.75 with absolute error < 8.3e-15
```

matching our value of this integral above.

(b) Here, the conditional probability is

$$\Pr\left(X \geq \frac{1}{2} \,\Big|\, X \geq \frac{1}{4}\right) = \frac{\Pr\left(X \geq \frac{1}{2} \text{ and } X \geq \frac{1}{4}\right)}{\Pr\left(X \geq \frac{1}{4}\right)} = \frac{\Pr\left(X \geq \frac{1}{2}\right)}{\Pr\left(X \geq \frac{1}{4}\right)} = \frac{3/4}{15/16} = \frac{4}{5}.$$

Exercise 4.2

Let

$$g(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ (2-x)^2, & 1 \leq x \leq 2 \end{cases}$$

We compute

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)\,dx &= \int_0^1 x^2\,dx + \int_1^2 (4 - 4x + x^2)\,dx = \frac{1}{3}x^3\Big|_0^1 + \left[4x - 2x^2 + \frac{1}{3}x^3\right]_1^2 \\ &= \frac{1}{3} + \left(8 - 8 + \frac{8}{3}\right) - \left(4 - 2 + \frac{1}{3}\right) = \frac{2}{3}. \end{aligned}$$

The same can be achieved through the code

```
g = makeFun(x^2 * (x >= 0 & x <= 1) + (2 - x)^2 * (x > 1 & x <= 2) ~ x)
integrate(g, 0, 2)

0.6666667 with absolute error < 7.4e-15
```

Then, for $f(x) = Cg(x)$ to enclose an area of 1, we require $C = \frac{3}{2}$.

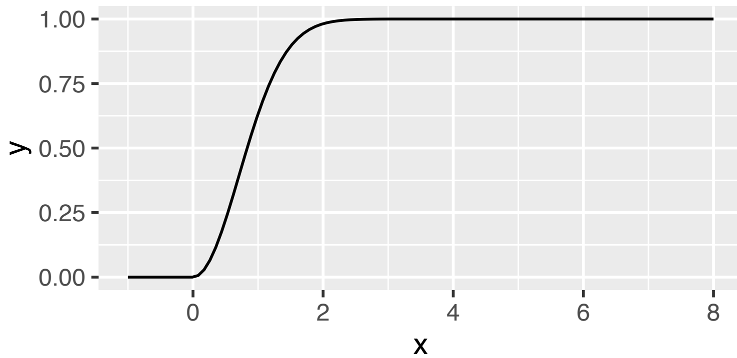
Exercise 4.3

(a) The given function h cannot be a pdf, nor a cdf, on account of it producing negative values/output.

(b) The function h does not satisfy $\int_{-\infty}^{\infty} h(x)\,dx = 1$, so h cannot be a pdf. Nor is h an increasing (nondecreasing) function, ruling out that it is a cdf.

(c) Plotting the function h,

```
h = makeFun((1 - exp(-x^2)) * (x >= 0) ~ x)
gf_fun(h(x) ~ x, xlim = c(-1, 8))
```



we see that h increases from the value 0 to limit 1 (as $x \to \infty$). This h is a cdf. (It is, therefore, not a pdf, as Exercise 4.5 will determine.)

(d) The definition shows h can only produce values ≥ 0. Integrating h

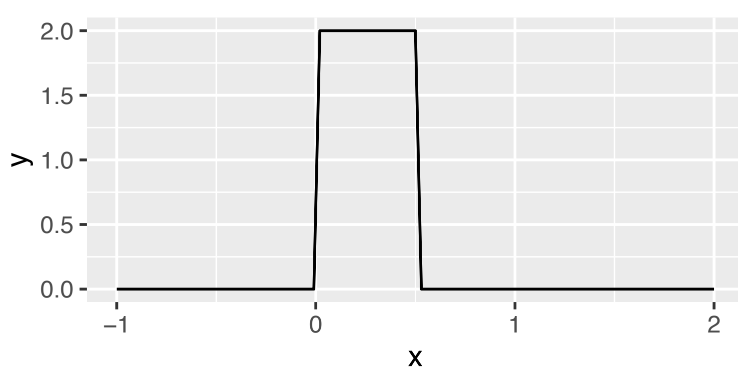
```
h = makeFun(2 * x * exp(-x^2) * (x >= 0) ~ x)
integrate(h, -1, Inf)

1 with absolute error < 6.8e-06
```

we see h is a pdf (and hence not a cdf).

Exercise 4.4

For the first part, the pdf of the uniform r.v. $X \sim \mathsf{Unif}(0, 0.5)$ has value $f(x) = 2$ for each $0 \leq x \leq 0.5$.

```
gf_fun(dunif(x, 0, 0.5) ~ x, xlim = c(-1, 2))
```



There is no pdf for which $f(x) > 1$ for all real x, as such an f would satisfy $\int_{-\infty}^{\infty} f(x)\,dx = \infty$.

Exercise 4.5

It is not possible for a function to be both a pdf and a cdf. For, if $f(x)$ were a pdf, then the requirement that

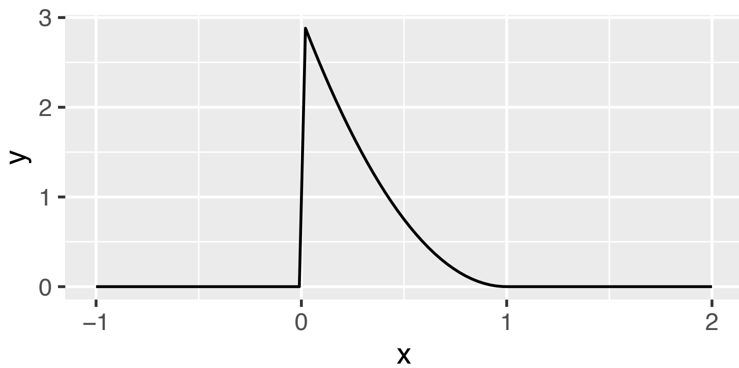
$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad \text{implies} \qquad \lim_{x\to\infty} f(x) = 0.$$

On the other hand, if f were a cdf, it would satisfy $\lim_{x\to\infty} f(x) = 1$. No f can have both of these limits as $x \to \infty$.

Exercise 4.6

(a) The given function f is never negative, and has integral 1, as the code displays:

```
f = makeFun(3 * (1 - x)^2 * (x >= 0 & x <= 1) ~ x)
gf_fun(f(x) ~ x, xlim = c(-1, 2))
integrate(f, 0, 1)

1 with absolute error < 1.1e-14
```



(b) To get the mean and variance, we can compute $E(X)$ and $E(X^2)$:

```
xf = makeFun(3 * x * (1 - x)^2 * (x >= 0 & x <= 1) ~ x)
xxf = makeFun(3 * x^2 * (1 - x)^2 * (x >= 0 & x <= 1) ~ x)
```

The expected value is 0.25:

```
value(integrate(xf, 0, 1))

[1] 0.25
```

and the variance is 0.0375:

```
value(integrate(xxf, 0, 1)) - value(integrate(xf, 0, 1))^2

[1] 0.0375
```

(c) The desired probability comes from $\int_{-\infty}^{1/2} f(x)\,dx$. That is,

```
integrate(f, 0, 0.5)

0.875 with absolute error < 9.7e-15
```

or 0.875.

(d) Here, the conditional probability is

$$\Pr\left(X \leq \frac{1}{2} \,\middle|\, X \geq \frac{1}{4}\right) = \frac{\Pr\left(\frac{1}{4} \leq X \leq \frac{1}{2}\right)}{\Pr\left(X \geq \frac{1}{4}\right)}.$$

Thus, we can obtain the answer by taking the ratio

```
value(integrate(f, 0.25, 0.5))/value(integrate(f, 0.25, 1))

[1] 0.7037037
```

So, this conditional probability equals 0.7037.

Exercise 4.7

We have

$$\mathrm{Var}(2X+1) = 2^2 \cdot \mathrm{Var}(X) = (4)(3) = 12.$$