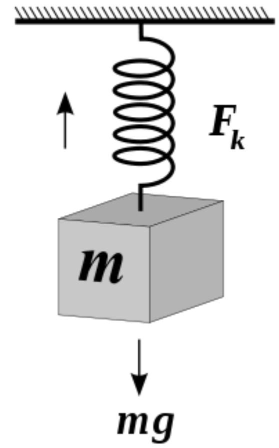
Spring–mass assembly (2nd Order DE Models)



Suppose the spring, whose natural length is ℓ, is stretched to the new length $(\ell + L)$ by a mass m. At equilibrium

$$mg = kL,$$

where k is the *spring constant*. If $u(t)$ represents a further displacement from equilibrium, then by Newton's 2nd Law

$$\begin{aligned} m\frac{d^2u}{dt^2} &= mg + F_k + F_r + F_e \\ &= mg - k(L+u) - \gamma\frac{du}{dt} + F_e \\ &= -ku - \gamma\frac{du}{dt} + F_e, \end{aligned}$$

or

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F_e. \tag{1}$$

Undamped, unforced vibrations

Suppose $F_r = 0$ (so no damping), and $F_e = 0$. Our DE (1) is

$$m\frac{d^2u}{dt^2} + ku = 0, \qquad \text{or} \qquad \frac{d^2u}{dt^2} + \omega_0^2 u = 0,$$

with $\omega_0 = \sqrt{k/m}$. The idealized assembly that follows this model is said to experience **simple harmonic motion**, having general solution

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

Note: An expression such as this can always be put in the form

$$u(t) = R\cos(\omega_0 t - \delta),$$

where

$$R = \sqrt{A^2+B^2}, \qquad \cos\delta = \frac{A}{R}, \qquad \text{and} \qquad \sin\delta = \frac{B}{R}. \tag{2}$$

This may be justified using the trigonometric identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

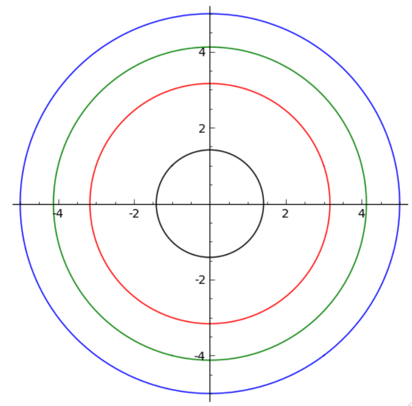
as this means

$$R\cos(\beta t - \delta) = R\cos\delta\cos(\beta t) + R\sin\delta\sin(\beta t).$$

Here, R is called the **amplitude**, while δ is the **phase shift**; ω_0 is called the **natural frequency**.

One may plot the solution (for various choices of c_1, c_2) in the tu-plane, of course. A less-familiar way to plot solutions is parametrically in the uu'-plane (called the **phase plane**), and we do so below (using Sage) for each combination of choices $c_1 = -3, 1$ and $c_2 = -1, 4$, with $\omega_0 = 1$ (fixed).

```
ts = linspace(0, pi, 300);
plot(ts, cos(ts)+sin(ts), 'r')
hold on
 plot(ts, -3*cos(ts)+sin(ts), 'b')
 plot(ts, cos(ts)+4*sin(ts), 'k')
 plot(ts, 3*cos(ts)+sin(ts), 'g')
hold off
```



Forced, undamped vibrations

Assume the external force has the form $F_e = F_0\cos(\omega t)$, where F_0 is a nonzero constant. Our DE (1) in this case is

$$mu'' + ku = F_0\cos(\omega t), \tag{3}$$

which (still) has complementary solution

$$u_c(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t). \tag{4}$$

It makes sense to guess a particular solution of (3) to be

$$u_p(t) \;=\; A\cos(\omega t) + B\sin(\omega t).$$

Note:

- Such a guess *does not work* if $\omega = \omega_0$, as we would proposing $u_p(t)$ identical to $u_c(t)$! In that case, the modified guess
$$u_p(t) \;=\; [A\cos(\omega t) + B\sin(\omega t)]t. \tag{5}$$
does work. We will discuss the resulting solution in the "Resonance" section below.
- When $\omega \neq \omega_0$, the (original) guess *will* work, but one can argue that the simpler guess
$$u_p(t) \;=\; A\cos(\omega t)$$
should work. (No product rule nor u' term implies no sine term.) Plugging and solving for A, one gets general solution
$$u(t) \;=\; c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t). \tag{6}$$

Beats

Consider the problem

$$mu'' + ku \;=\; F_0 \cos(\omega t), \qquad u(0) = u'(0) = 0. \tag{7}$$

General solution of (7) is given above in (6). Only now apply ICs to get coefficients:

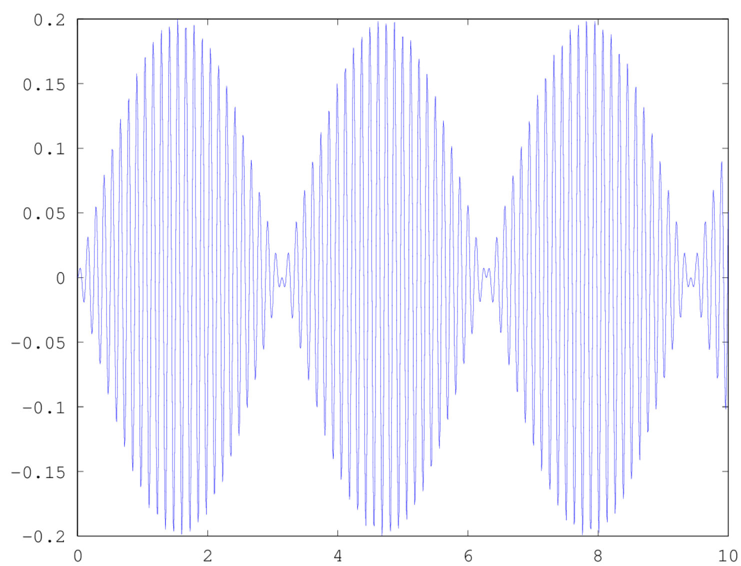
$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \qquad c_2 = 0.$$

Thus, the unique solution of (7) is

$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)}[\cos(\omega t) - \cos(\omega_0 t)] \;=\; \ldots \text{(see Trench, pp. 274-75, for details)} \\ &= \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \underbrace{\sin\left(\frac{(\omega_0 - \omega)t}{2}\right)}_{\text{large period when } \omega_0 - \omega \text{ is small}} \right] \underbrace{\sin\left(\frac{(\omega_0 + \omega)t}{2}\right)}_{\text{small period when } \omega_0 + \omega \text{ is "large"}} . \end{aligned}$$

The plot at right depicts the solution in the case where

$m = 1$
$F_0 = 20$
$\omega_0 = 51$ (so $k = m/\omega_0^2$ is fixed, too)
$\omega = 49$



Resonance

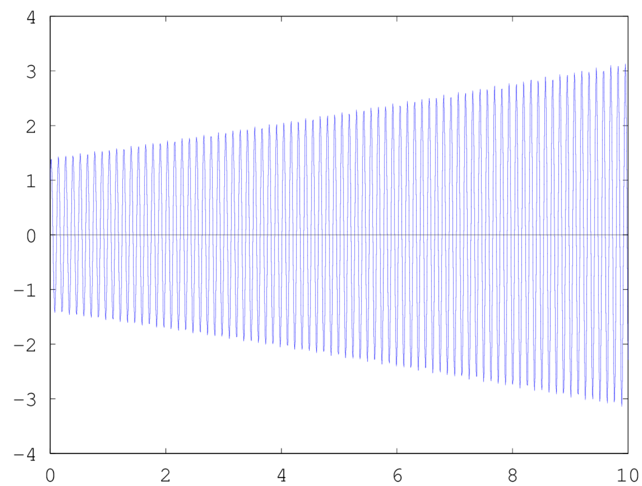
When $\omega = \omega_0$, the proposed solution (5) of (3) leads to general solution

$$u(t) \;=\; c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0}\, t \sin(\omega_0 t).$$

This solution grows without bound as $t \to \infty$, a phenomenon known as **resonance**. Such a scenario is not physically realizable (damping is always present), but interesting all the same.

At right, we depict the solution in the case where

$$
\begin{aligned}
c_1 &= 1\\
c_2 &= 1\\
m &= 1\\
F_0 &= 20\\
\omega_0 &= 50
\end{aligned}
$$



Damped vibrations

We first consider the homogeneous (unforced) problem

$$mu'' + \gamma u' + ku \;=\; 0. \tag{8}$$

Since the coefficients m, γ and k are all positive, the roots of our characteristic equation

$$mx^2 + \gamma x + k \;=\; 0, \qquad \text{given by} \qquad r_{1,2} = \frac{1}{2m}\left(-\gamma \pm \sqrt{\gamma^2 - 4mk}\right),$$

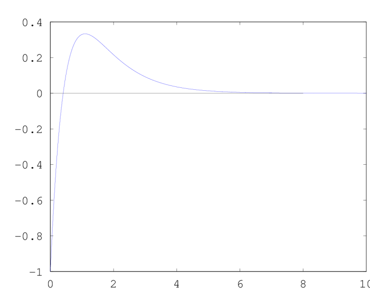
are

- real, with $r_1 \neq r_2$ and both $r_{1,2} < 0$, yielding general solution

  $$u_c(t) \;=\; c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

  This case, occuring when $\gamma > 2\sqrt{mk}$, is called **overdamping**.

Plot with $c_1 = 2$, $c_2 = -3$, $r_1 = -1$, $r_2 = -2$:

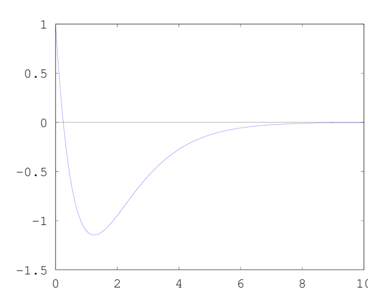


- real, with $r_1 = r_2 = -\gamma/(2m)$, yielding general solution

  $$u_c(t) \;=\; c_1 e^{-\gamma t/2m} + c_2 t e^{-\gamma t/2m}.$$

  This case, occurring when $\gamma = 2\sqrt{mk}$, is called **critical damping**.

Plot with $c_1 = 1$, $c_2 = -4$, $r_1 = r_2 = -1$:

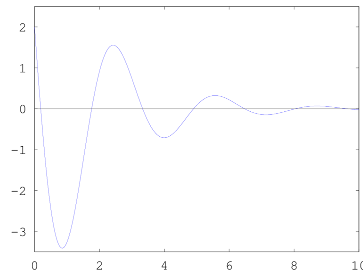


- nonreal $r_{1,2} = \alpha \pm i\beta$, with $\alpha = -\gamma/(2m)$ and $\beta = \sqrt{4mk - \gamma^2}/(2m)$, yielding general solution

  $$u_c(t) \;=\; c_1 e^{-\gamma t/(2m)} \cos(\beta t) + c_2 e^{-\gamma t/(2m)} \sin(\beta t).$$

  This case, occuring when $\gamma < 2\sqrt{mk}$, is called **underdamping**.

Plot with $c_1 = 2$, $c_2 = -5$, $\alpha = -1/2$, $\beta = 2$:



Regardless of which situation we have, solutions die off exponentially as $t \to \infty$.

The implications of this on the forced (nonhomogeneous) DE

$$mu'' + \gamma u' + ku \;=\; F_0 \cos(\omega t),$$

are two-fold:

1. The general solution will contain a **transient** part $u_c(t)$ (coming from one of the three cases above) and a **steady-state** part $u_p(t)$.

2. What we pose for a particular solution (*a la* the method of undetermined coefficients) is just

   $$u_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{9}$$

   One can solve for A, B to get

   $$A \;=\; \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2} \qquad \text{and} \qquad B \;=\; \frac{F_0\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}. \tag{10}$$

   Using the relations (2) on our expressions (10), we obtain particular solution

   $$u_p(t) \;=\; \frac{F_0}{\Delta}\cos(\omega t - \delta),$$

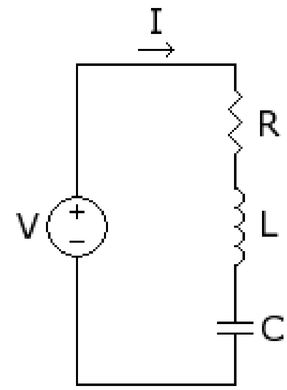
   with

   $$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta}, \quad \text{and} \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

   This $u_p(t)$ represents the steady-state of the motions of our spring–mass assembly, showing that our assembly eventually settles into a delayed/rescaled version of the forcing term $F_0\cos(\omega t)$.

Electric circuits

Consider the electric circuit pictured at right. Here L, R, C are constants representing the **inductance**, **resistance** and **capacitance**, respectively. Let $Q(t)$ represent the total charge on the capacitor at time t, and $E(t)$ be the *impressed voltage*. One may reason (see the text, pp. 201–202 for some details) that the governing DE model is



$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} \;=\; E(t).$$

The main point I wish to make is that the "features" we studied for a spring–mass assembly have analogues in the case of a simple RLC series circuit.