Math 251, Mon 19-Oct-2020 -- Mon 19-Oct-2020
Discrete Mathematics
Fall 2020

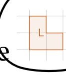
Monday, October 19th 2020

Due:: PS08

Monday, October 19th 2020

Wk 8, Mo
Topic:: Strong induction
Read:: Rosen 5.2
HW:: PS09 due Mon.

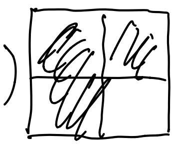
Strong Induction and the Well-Ordering Principle

7. One can tile an $2^n \times 2^n$ checkerboard with one space removed using tiles shaped like 

Where $P(n)$ is this statement

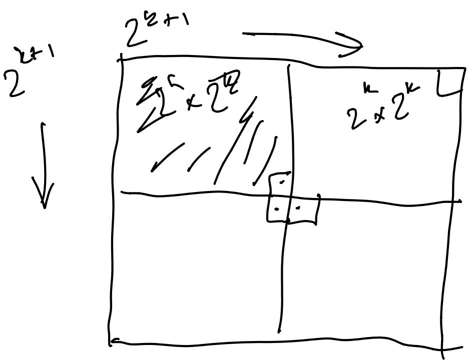
$n=1$

$P(k)$ (any $2^k \times 2^k$ grid can be tiled as claimed)

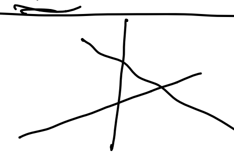


$- P(1)$

empty corner in base case



8. **Induction misused**. Let $P(n)$ be the statement "Any collection of $n \geqslant 2$ distinct lines in the plane, no two of which are parallel, shares a common point."



$\{2, 3, 4, \dots\}$

The following is an attempt to prove $\forall n \in \mathbb{Z}^+, P(n)$:
Base case: $P(2)$ says 2 non-parallel lines in the plane have a common point. This seems true enough without requiring proof.
Inductive step: We assume $P(k)$ is true for some integer $k \geqslant 2$. The case $P(k+1)$ has us considering $(k+1)$ non-parallel lines in the plane: $\{\ell_1, \ell_2, \dots, \ell_k, \ell_{k+1}\}$. Now the collection $\{\ell_1, \ell_2, \dots, \ell_k\}$ has k different non-parallel lines so by the induction hypothesis, this collection has a common point, call it P_1. As well, the induction hypothesis applies to the collection $\{\ell_2, \ell_3, \dots, \ell_k, \ell_{k+1}\}$, so these lines have a common point, call it P_2. But two points in a plane uniquely determine a line, and since no two lines found in both collections can be the same, it must be that points P_1 and P_2 are really the same point. Thus, our original collection $\{\ell_1, \ell_2, \dots, \ell_k, \ell_{k+1}\}$ shares a commont point, showing that $P(k+1)$ holds.

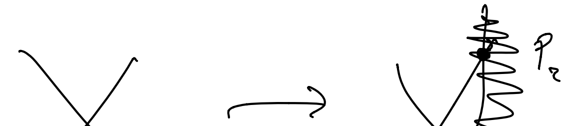
$P(2)$

Thus, by induction, $P(n)$ holds for all $n = 2, 3, 4, \dots$.

Assume $P(k)$ holds. Prove $P(k) \Rightarrow P(k+1)$

Consider a collection of lines $\{\ell_1, \ell_2, \ell_3, \dots, \ell_k, \ell_{k+1}\}$ in plane, nonparallel.
By the induction hyp. (IH) the collection $\{\ell_1, \dots, \ell_k\}$ has a point common to each of $\ell_1, \dots, \ell_k$ — call it P_1. Similarly, the collection $\{\ell_2, \dots, \ell_k, \ell_{k+1}\}$, again by IH. this collection has a point in common, P_2. So lines ℓ_2 and ℓ_k both contain points P_1 and P_2, so either ℓ_2 and ℓ_k are the same line (not possible) or $P_1 = P_2$.

Error in the proof: $P(2) \not\Rightarrow P(3)$



3 methods used proof which equivalent
- Mathematical induction
- Strong mathematical induction
- Well-Ordering Principle

Strong Induction and the Well-Ordering Principle

Mathematical induction can be expressed as the rule of inference

$$(P(a) \wedge (P(k) \rightarrow P(k+1))) \rightarrow \forall n \geqslant a, P(n).$$

Upon reflection, the portion $P(k) \rightarrow P(k+1)$, what we call the inductive step, is not the only thing that, coupled with the basis step which leads to the conclusion $\forall n P(n)$. Equally valid would be the conditional statement (containing a stronger hypothesis)

$$(P(i) \text{ is true for integers } a \leqslant i \leqslant k) \rightarrow P(k+1).$$

This leads to the following generalization of mathematical induction.

Definition 1 (Principle of Strong Mathematical Induction): Let $P(n)$ be a property that is defined for integers n, and let a, b be fixed integers with $a \leqslant b$. Suppose the following statements are true:

1. $P(a), P(a+1), \ldots, P(b)$ are all true (**basis step**).
2. For any integer $k \geqslant b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true (**inductive step**).

hold for $P(a), P(a+1), \ldots, P(k) \longrightarrow P(k+1)$

Then the statement "for all integers $n \geqslant a$, $P(n)$" is true.

The supposition that $P(i)$ is true for all integers i from a through k in number 2 above is called the **inductive hypothesis**.

To prove this is a valid rule of inference we rely on the **Well-Ordering principle**.

Definition 2 (Well-Ordering Principle): Suppose $A \subseteq \mathbb{N}$. Then A has a *smallest element*. That is, $\exists a \in A$ such that $\forall b \in A, (a \leqslant b)$.

Note that the set {positive real numbers} does not have a smallest element, but that this does not violate the well-ordering principle.

Generally speaking, anything provable via one of i) mathematical induction, ii) strong mathematical induction, or iii) the well-ordering principle, is provable with the other two. This is because all three statements are logically equivalent. However, sometimes one approach is easier than another. Some examples of statements and proof methods include:

$94 = 2 \cdot 47$

$885 = 3 \cdot 5 \cdot 59$

$$\begin{array}{r} 59 \\ 5\,\overline{)295} \\ 3\,\overline{)885} \end{array} \qquad 885 = 3 \cdot 295$$

Fundamental Theorem of Arithmetic

1. Every integer $n \geqslant 2$ is a prime or can be written as the product of primes (use strong mathematical induction).

Base step: $P(2)$ is prime or product of primes ✓

Strong Induction step: Assume $P(2), P(3), P(4), \ldots, P(k)$ for some $k \geq 2$. Must show $P(2) \wedge P(3) \wedge \cdots \wedge P(k) \to P(k+1)$.

$P(k+1)$ says: $k+1$ is prime or the product of primes.

If it is prime, we are done.
Suppose it isn't prime. Then $k+1 = mn$ for positive integers with m, n between 2 and k (inclusive).

$P(m), P(n)$ both hold by the induction hypothesis. So both m, n are primes or products of primes, so $k+1 = \left(\text{primes producing } m\right)\left(\text{primes producing } n\right)$

2. For any $n \geqslant 8$, n cents can be obtained using 3¢ and 5¢ coins (use strong mathematical induction)..

Base case $P(8), P(9), P(10)$

$8¢ = 3¢ + 5¢$
$9¢ = 3(3¢)$
$10¢ = 2(5¢)$

$11¢ = ¢8 + ¢3$
$12¢$
$13¢$

Induction step: assume $P(8), P(9), P(10), \ldots, P(k+2)$ for some $k \geq 8$

Show $P(8) \wedge P(9) \wedge P(10) \wedge \cdots \wedge P(k+2) \to P(k+3)$.

The case $P(k+3)$ is proved by adding ¢3 to the case $P(k)$.
So induction step is complete, and $\forall n \geq 8$, $P(n)$ holds by strong induction.

5. A simple polygon with $n \geqslant 3$ sides can be trtiangulated into $n-2$ triangles (use strong mathematical induction, and the fact that every simple polygon with at least four sides has an interior diagonal).

6. Given a strictly decreasing sequence of *positive* integers $r_1, r_2, r_3, \ldots$ (so $r_{i+1} < r_i$ for each i), the sequence terminates (use the well-ordering principle).

$\underbrace{\underline{111}, \underline{93}, \underline{85}, \underline{72}, \underline{66}, \underline{\quad}}$

rules I'm following
- only using pos. ints.
- every new term is < predecessor ($a_{n+1} < a_n$)

By well-ordering principle, this sequence terminates.