
Wednesday, October 25th 2023

Topic:: Midterm #2

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Definitions: linear combination, span, linear dependence and independence, basis, subspace, dimension, linear transformation, rank, column space, row space, null space, nullity, orthogonal complement, orthogonal basis, projection

Skills

- Given two points $P(x_1, x_2, \ldots, x_n)$ and $Q(y_1, y_2, \ldots, y_n)$,
  - determine the vector $\overrightarrow{PQ}$
  - write a parametrization of the line through P and Q
  - find the distance from P to Q (it equals the length of $\overrightarrow{PQ}$)
  - express $\vec{v} = \overrightarrow{PQ}$ as the product of its length and direction
  - write the equation of the hyperplane through P orthogonal to $\vec{v}$
- Given vectors $\vec{u}, \vec{v}$
  - calculate $\vec{u} \cdot \vec{v}$, and determine if they are orthogonal
  - find the angle $\theta \in [0, \pi]$ between $\vec{u}, \vec{v}$
  - depict their sum $\vec{u} + \vec{v}$ (drawing a parallelogram?)
  - depict their difference $\vec{u} - \vec{v}$
  - calculate and depict $\operatorname{proj}_{\vec{u}} \vec{v}$
  - calculate and depict $\vec{v} - \operatorname{proj}_{\vec{u}} \vec{v}$
- Given a collection of vectors $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ taken from $\mathbb{R}^n$, and denoting $W = \operatorname{span}(S)$,
  - determine if S is linearly independent
  - determine $\dim(W)$
  - determine if a given vector $\vec{v}$ is in $\operatorname{span}(S)$ and, when the answer is yes, write $\vec{v}$ as a linear combination of vectors in S
  - find an orthogonal basis for $\operatorname{span}(S)$ (Gram-Schmidt process)
  - for a given $\vec{v} \in \mathbb{R}^n$, find $\operatorname{proj}_W \vec{v}$
  - find a basis for $W^\perp$
- Given an m-by-n matrix A,
  - determine the rank and nullity of A
  - determine a basis of $\operatorname{col}(A)$, $\operatorname{row}(A)$, $\operatorname{null}(A)$
- Miscellaneous
  - Given a *description* of a collection V of vectors in $\mathbb{R}^n$, determine if V is a *subspace* of $\mathbb{R}^n$.
  - Given a function $T \colon \mathbb{R}^n \to \mathbb{R}^m$
    - determine (and be able to demonstrate) whether T is *linear*

- in cases where T is linear, find an m-by-n matrix A such that $T(\vec{x}) = A\vec{x}$ whenever $\vec{x} \in \mathbb{R}^n$

Some highlighted results to be familiar with

- Cauchy-Schwarz Inequality
- Triangle Inequality
- Linear transformations of the plane ($\mathbb{R}^2$)
  - $T(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}$ rotates the plane counterclockwise through angle θ
  - $T(\vec{x}) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \vec{x}$

    dilates the plane if $c > 1$
    leaves everything in place if $c = 1$
    contracts if $0 < c < 1$
    takes everything to the zero vector if $c = 0$
    and reflects through the origin if $c < 0$
  - $T(\vec{x}) = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \vec{x}$ multiplies one coordinate of $\vec{x}$ by c, the other by d