$\star$16 (a) The homogeneous solution is

$$x_{\mathrm{h}}(t) = c_1 e^{2t}.$$

Consequently, the particular solution will be of the form

$$x_{\mathrm{p}}(t) = (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4)e^{5t} + a_5 e^{-2t}\cos(3t) + a_6 e^{-2t}\sin(3t).$$

(b) The homogeneous solution is

$$x_{\mathrm{h}}(t) = c_1 e^{3t}.$$

Consequently, the particular solution will be of the form

$$x_{\mathrm{p}}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + (a_7 + a_8 t + a_9 t^2)e^{3t}\cos(2t) + (a_{10} + a_{11} t + a_{12} t^2)e^{3t}\sin(2t).$$

(c) The homogeneous solution is

$$x_{\mathrm{h}}(t) = c_1 e^{-t}.$$

Consequently, the particular solution will be of the form

$$x_{\mathrm{p}}(t) = t(a_0 + a_1 t + a_2 t^2 + a_3 t^3)e^{-t}.$$

$\star$17 (b) Setting

$$x_1 = y, \quad x_2 = y',$$

gives the system

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ t^2 - e^{3t} \end{pmatrix}.$$

(e) Setting

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'',$$

gives the system

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tan(t) & -5 & e^{-t} \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 12\cos(4t) \end{pmatrix}.$$

(f) Setting

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad x_4 = y''',$$

gives the system

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -13 & 7 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^3 + 3t - 9\sin(2t) \end{pmatrix}.$$

$\underline{\star 18}$ Let $y_1(t) = u_1(t)$, $y_2(t) = u_1'(t)$, $y_3(t) = u_2(t)$, and $y_4(t) = u_2'(t)$. (Other definitions are possible, such as setting $y_1 = u_1$, $y_2 = u_2$, $y_3 = u_1'$ and $y_4 = u_2'$; such definitions do not alter the answer substantively, but do affect the layout of the matrix **A** and vector $\mathbf{b}(t)$.) Then

$$
\begin{aligned}
\frac{d}{dt}\mathbf{y} &= \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} u_1' \\ u_1'' \\ u_2' \\ u_2'' \end{bmatrix} = \begin{bmatrix} y_2 \\ \frac{k_2}{m_1}(u_2 - u_1) + \frac{c_2}{m_1}(u_2' - u_1') - \frac{k_1}{m_1}u_1 - \frac{c_1}{m_1}u_1' \\ y_4 \\ \frac{f(t)}{m_2} - \frac{k_2}{m_2}(u_2 - u_1) - \frac{c_2}{m_2}(u_2' - u_1') \end{bmatrix} \\
&= \begin{bmatrix} y_2 \\ -\frac{k_1+k_2}{m_1}u_1 - \frac{c_1+c_2}{m_1}u_1' + \frac{k_2}{m_1}u_2 + \frac{c_2}{m_1}u_2' \\ y_4 \\ \frac{k_2}{m_2}u_1 + \frac{c_2}{m_2}u_1' - \frac{k_2}{m_2}u_2 - \frac{c_2}{m_2}u_2' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix} \\
&= \begin{bmatrix} y_2 \\ -\frac{k_1+k_2}{m_1}y_1 - \frac{c_1+c_2}{m_1}y_2 + \frac{k_2}{m_1}y_3 + \frac{c_2}{m_1}y_4 \\ y_4 \\ \frac{k_2}{m_2}y_1 + \frac{c_2}{m_2}y_2 - \frac{k_2}{m_2}y_3 - \frac{c_2}{m_2}y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix} = \mathbf{A}\mathbf{y} + \mathbf{b}(t),
\end{aligned}
$$

with

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f(t)}{m_2} \end{bmatrix}.
$$

$\underline{\star 19}$ The particular solution must be found via variation of parameters. The homogeneous solution is

$$
x_{\mathrm{h}}(t) = \mathbf{\Phi}(t)\boldsymbol{c}, \quad \mathbf{\Phi}(t) = \begin{pmatrix} \cos(3t) & \sin(3t) \\ 2\cos(3t) - 4\sin(3t) & 2\sin(3t) + 4\cos(3t) \end{pmatrix}.
$$

Using

$$
\mathbf{\Phi}(t)^{-1} = \frac{1}{4}\begin{pmatrix} 2\sin(3t) + 4\cos(3t) & -\sin(3t) \\ -2\cos(3t) + 4\sin(3t) & \cos(3t) \end{pmatrix},
$$

we have

$$\boldsymbol{\Phi}(t)^{-1}\cdot\sin(3t)\begin{pmatrix}2\\3\end{pmatrix}=\frac{1}{4}\begin{pmatrix}8\cos(3t)\sin(3t)+\sin^2(3t)\\-\cos(3t)\sin(3t)+8\sin^2(3t)\end{pmatrix}.$$

Consequently,

$$\begin{aligned}x_{\mathrm{p}}(t)&=\boldsymbol{\Phi}(t)\int^t\boldsymbol{\Phi}(s)^{-1}\boldsymbol{f}(s)\,\mathrm{d}s=\boldsymbol{\Phi}(t)\begin{pmatrix}\sin^2(3t)/3-\sin(6t)/48+t/8\\-\sin^2(3t)/24-\sin(6t)/6+t\end{pmatrix}\\&=\boldsymbol{\Phi}(t)\left[\frac{1}{24}\sin^2(3t)\begin{pmatrix}8\\-1\end{pmatrix}-\frac{1}{48}\sin(6t)\begin{pmatrix}1\\8\end{pmatrix}+\frac{1}{8}t\begin{pmatrix}8\\1\end{pmatrix}\right].\end{aligned}$$

The interested student can perform (and simplify!) the last algebraically messy calculation.

$\star$20 (a) The (approximate) homogeneous solution is

$$\begin{aligned}\mathbf{x}_h(t)&=c_1e^{-0.0447t}\begin{bmatrix}1\\-0.69\\-0.09\end{bmatrix}+c_2e^{-0.02t}\begin{bmatrix}1\\1.3\\-0.19\end{bmatrix}+c_3e^{-0.0000306t}\begin{bmatrix}1\\0.39\\892.56\end{bmatrix}\\&=\begin{bmatrix}e^{-0.0447t}&e^{-0.02t}&e^{-0.0000306t}\\-0.69e^{-0.0447t}&1.3e^{-0.02t}&0.39e^{-0.0000306t}\\-0.09e^{-0.0447t}&-0.19e^{-0.02t}&892.56e^{-0.0000306t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}.\end{aligned}$$

(b) All of the eigenvalues of $\mathbf{A}$ are negative, which means that, as $t\to\infty$, the three fundamental solutions all go to $\mathbf{0}$. Thus, each component of $\mathbf{x}_h(t)$ representing, respectively, the amount of lead in the bloodstream, body tissue, and bone, goes to 0.

(c) Writing, as we usually do, the matrix of part (a) as $\Phi(t)$, we must solve

$$\begin{bmatrix}50\\0\\0\end{bmatrix}=\Phi(0)\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1&1&1\\-0.69&1.3&0.39\\-0.09&-0.19&892.56\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}.$$

Using Gaussian elimination, we get approximate values $c_1=32.7$, $c_2=17.3$, $c_3=0.007$. The 3$^{\text{rd}}$ (bone) component of the solution $\mathbf{x}_h(t)$, then, is

$$\begin{aligned}x_3(t)&\doteq(32.7)(-0.09)e^{-0.0447t}+(17.3)(-0.19)e^{-0.02t}+(0.007)(892.56)e^{-0.0000306t}\\&\doteq-2.94e^{-0.0447t}-3.29e^{-0.02t}+6.25e^{-0.0000306t}\end{aligned}$$

The peak value of the function $x_3(t)$, approximately 6.18, occurs around $t=292.5$, i.e., after 292 days. The approximate time when the value of $x_3(t)$ returns to 0.5 is $t=82540$ days, or about 226 years.