

MATH 231: Differential Equations with Linear Algebra

Hand-Checked Assignment #5, due date: Wed., Apr. 21, 2021

Write up, carefully and legibly, your solutions to the following problems. While you do not need to present one problem per page, please do not put problems side-by-side (i.e., no two-column format). To submit your work it must be

- scanned (all pages) to a single .pdf file (one multi-page file containing all graded problems).
- submitted to <https://www.gradescope.com> as **hc05**.

★35 For $\omega \neq \omega_0$, consider the undamped, forced spring differential equation with *resting* initial conditions:

$$mu'' + ku = F_0 \cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0.$$

(a) Show that the solution of this IVP is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} [\cos(\omega t) - \cos(\omega_0 t)].$$

(b) Solve the system of two equations in the two unknowns A, B

$$A + B = \omega_0 t, \quad A - B = \omega t.$$

Then use the trigonometric identities

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta, \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

to rewrite the factor $\cos(\omega t) - \cos(\omega_0 t)$ as $2 \sin A \sin B$.

(c) Take $m = 1, F_0 = 1, \omega_0 = 441$ and $\omega = 439$ in the solution to part (a). Graph the functions

$$y = \sin(440t) \quad \text{and} \quad y = \sin t,$$

and use them to explain the look of the graph of the solution to part (a).

★36 Consider the undamped, forced spring differential equation

$$mu'' + ku = F_0 \cos(\omega_0 t),$$

where the periodic forcing function has frequency matching $\omega_0 = \sqrt{k/m}$.

- (a) Explain why, after the method of Undetermined Coefficients, you would propose the form

$$u_p(t) = At \cos(\omega_0 t) + Bt \sin(\omega_0 t)$$

for a particular solution rather than

$$u_p(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

- (b) Use the proposed u_p from part (a) to find the general solution $u_h(t) + u_p(t)$ of the differential equation. As a spoiler, you're shooting for

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

- (c) In class we investigated the undamped, forced spring vibrations under the assumption of a periodic forcing function whose frequency ω is *different* than ω_0 . In that case,

$$u_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Explain how the qualitative differences in the forms of $u_p(t)$ imply that an **undamped** mass-spring system might not break when forced at frequency $\omega \neq \omega_0$, but assuredly will break if $\omega = \omega_0$.

★37 [Part (f) below is optional.] Consider the 2nd order linear nonhomogeneous DE

$$y'' + 2by' + \omega_0^2 y = f_0 \cos(\omega t), \quad (1)$$

where b , ω_0 , and f_0 are all positive constants. We have seen that the DE (1) can be a model for a damped mass-spring system or an RLC circuit. In this problem, we investigate the influence of the forcing frequency on amplitude and phase shift.

- (a) Guessing (i.e., using the Method of Undetermined Coefficients) the form $y_p(t) = A \cos(\omega t) + B \sin(\omega t)$ for the steady-state solution, show that

$$A = \frac{(\omega_0^2 - \omega^2)f_0}{\Delta} \quad \text{and} \quad B = \frac{2b\omega f_0}{\Delta},$$

where $\Delta = (\omega_0^2 - \omega^2)^2 + 4b^2\omega^2$.

- (b) Determine formulas for R and ϕ so that $y_p(t) = R \cos(\omega t - \phi)$.
 (c) Take $f_0 = 1$, $b = 1$, and $\omega_0 = 15$. Plot R and ϕ as functions of the forcing frequency ω .
 (d) Repeat the previous part, but now with $f_0 = 1$, $b = 8$, and $\omega_0 = 20$.

- (e) Using the formula for R you found in part (b) (no specific values for f_0 , b nor ω_0) and calculus, find the value of ω which maximizes amplitude R .
- (f) What information do the phase functions $\phi(\omega)$ plotted in parts (c) and (d) tell? Give a physical interpretation that presumes the transient part of the solution has died out.

★38 Consider the 2nd-order system

$$\begin{aligned}\frac{d^2 u_1}{dt^2} &= \frac{k_2}{m_1}(u_2 - u_1) + \frac{c_2}{m_1}\left(\frac{du_2}{dt} - \frac{du_1}{dt}\right) - \frac{k_1}{m_1}u_1 - \frac{c_1}{m_1}\frac{du_1}{dt}, \\ \frac{d^2 u_2}{dt^2} &= \frac{f(t)}{m_2} - \frac{k_2}{m_2}(u_2 - u_1) - \frac{c_2}{m_2}\left(\frac{du_2}{dt} - \frac{du_1}{dt}\right).\end{aligned}$$

which may (it has been proposed by someone on the internet, but I have not verified it) accurately model the mass-spring assembly with dashpots (shock absorbers) pictured at the website

http://upload.wikimedia.org/wikipedia/commons/f/fd/Mass-Spring-Damper_%282_body_system%29.svg

It seems the displacements from equilibrium positions of masses m_1 , m_2 are denoted by u_1 , u_2 respectively. The spring constants are k_1 , k_2 , and c_1 , c_2 are the damping constants associated with the dashpots. There appears, as well, to be an external force $f(t)$ applied to the mass on the right.

Convert this model (whether it models what it purports to or not) to a first order linear system of DEs in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(t)$, specifying the entries of \mathbf{A} and $\mathbf{b}(t)$.

★39 From ODELA Section 4.2, pp. 215–217, do Exercise 4.2.6.