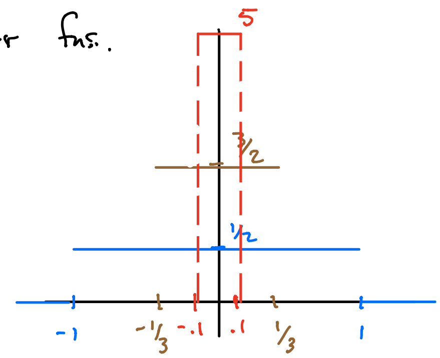
Unit impulse function, a.k.a. Dirac delta fn.: $\delta(t)$

Define by building it up from simpler fns.

For example

$$d_\beta(t) = \begin{cases} 0 & \text{if } |t| > \beta \\ \frac{1}{2\beta} & \text{if } |t| < \beta \end{cases}$$



We take

$$\delta(t) = \lim_{\beta \to 0} d_\beta(t)$$

blue: $d_1(t)$
brown: $d_{1/3}(t)$
red: $d_{0.1}(t)$

$\Bigg\}$ Total area under $d_\beta(t) = 1$

Properties

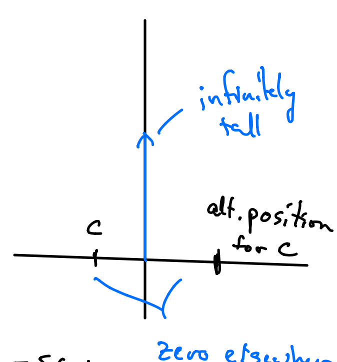
- $\delta(t) = \begin{cases} 0 & \text{for all } t \neq 0 \\ +\infty & \text{for } t = 0 \end{cases}$ Not really a fn., called a "generalized fn."
- Like each of the $d_\beta(t)$ used to build $\delta(t)$, the area under δ is 1. That is

$$\underline{1} = \int_{-\infty}^{\infty} \delta(t)\,dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt$$

- Note if we integrate up to 'c'

$$\int_{-\infty}^{c} \delta(t)\,dt = \begin{cases} 0, & c < 0 \\ 1, & c > 0 \end{cases}$$

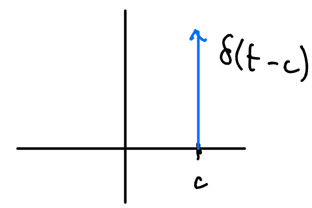
$$= \mathcal{U}(t) \quad \text{Heaviside unit step}$$



So in some generalized sense $\frac{d}{dt}\mathcal{U}(t) = \delta(t)$.

- Sifting property

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-c)\,dt = f(c)$$



infinite value because of shift by c

- $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$

$\left(\text{Compare w/ } \mathcal{L}\{\mathcal{U}(t-c)\} = \dfrac{e^{-cs}}{s} \right)$

and, so by letting $c \to 0^+$, get $\mathcal{L}\{\delta(t)\} = 1$.

An example of solving a DE containing δ:

$\boxed{\text{Ex.}}$ $y'' + 2y' + 2y = \delta(t-\pi)$, $\quad y(0) = 1,\ y'(0) = 0$.

Strategy: Solve 2 related problems

Our orig. problem is solved by the sum of solns. to ① and ②

① homog. version w/ attached ICs

$$y'' + 2y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 0$$

② nonhomog. (given) DE w/ zeroed ICs

$$y'' + 2y' + 2y = \delta(t-\pi), \quad y(0) = 0,\ y'(0) = 0.$$

Attacking ①, seems Ch. 4 methods are usable.

$$y'' + 2y' + 2y = 0 \quad \overset{\text{char. eqn.}}{\curvearrowright} \quad r^2 + 2r + 2 = 0$$

roots $\quad r = \dfrac{-2}{2(1)} \pm \dfrac{1}{2(1)}\sqrt{2^2 - 4(1)(2)} = -1 \pm i$

are nonreal w/ $\alpha = -1$, $\beta = 1$

$\Rightarrow$ basis solns. $\quad e^{-t}\cos t,\ e^{-t}\sin t$

general soln. to DE in ① is linear combs. of these

$$y_1(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

$$y_1'(t) = -c_1 e^{-t}\cos t - c_1 e^{-t}\sin t - c_2 e^{-t}\sin t + c_2 e^{-t}\cos t$$

Applying ICs:

$$1 = y_1(0) = c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = c_1$$

$$0 = y_1'(0) = -c_1\cdot 1 - c_1\cdot 0 - c_2\cdot 0 + c_2\cdot 1 = c_2 - c_1$$

$$\Rightarrow c_2 = 1$$

So $\quad y_1(t) = e^{-t}\cos t + e^{-t}\sin t \quad$ solves ①.

Attack ② using L.T.: take L.T. of both sides of DE

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t-\pi)\}$$

$$\underbrace{\mathcal{L}\{y''\}} + 2\,\underbrace{\mathcal{L}\{y'\}} + 2\mathcal{L}\{y\} = e^{-\pi s}$$

$$\overbrace{s^2 Y - s\underbrace{\cancel{y(0)}}_{=0} - \underbrace{\cancel{y'(0)}}_{=0}} + 2\overbrace{\left[sY - \underbrace{\cancel{y(0)}}_{=0}\right]} + 2Y = e^{-\pi s}$$

$$(s^2 + 2s + 2)Y = e^{-\pi s}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+2s+2}\cdot e^{-\pi s}$$

Need $\quad y_2(t) = \mathcal{L}^{-1}\{ \quad \underset{\text{this}}{\downarrow} \quad \}$

Use entry

coeff. in exponential identifies amt. of shift

$$\mathcal{L}\{\mathcal{U}(t-c)\,f(t-c)\} = e^{-cs}\cdot\mathcal{L}\{f(t)\}$$

which explains how exponential appears on s-side, and how to deal with it.

Task: to find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$

roots of s^2+2s+2 nonreal so complete the square

$$\frac{1}{s^2+2s+2} = \frac{1}{s^2+2s+1+1} = \frac{1}{(s+1)^2+1} = \frac{1}{[s-(-1)]^2+1}$$

Have entry in table

$$\frac{b}{(s-a)^2+b^2} \quad \text{comes from} \quad e^{at}\sin(bt)$$

So $$\frac{1}{s^2+2s+2} = \frac{1}{[s-(-1)]^2+(1)^2} \quad \text{comes from } e^{-t}\sin t.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = \underline{e^{-t}\sin t}$$

and

$$y_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\cdot e^{-\pi s}\right\} = \mathcal{U}(t-\pi)\cdot \underbrace{e^{-(t-\pi)}\sin(t-\pi)}_{\text{shifted version of what we found } \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} \text{ to be}}.$$

So, our original problem has soln.

$$y_1(t)+y_2(t) = \boxed{e^{-t}\cos t + e^{-t}\sin t + \mathcal{U}(t-\pi)e^{-(t-\pi)}\sin(t-\pi)}$$

Q: What, physically speaking, does $\delta(t)$ model?

A: Both IVPs

$$y''+by'+cy = 0, \quad y(0)=0, \quad y'(0)=1$$ — Unforced spring in equilibrium w/ velocity initially $=1$

and

$$y''+by'+cy = \delta(t), \quad y(0)=0, \quad y'(0)=0$$ — like a blow to a system

otherwise at
rest.

have the <u>same</u> <u>solution</u>.