

MATH 162: Calculus II  
Framework for Mon., Jan. 29  
Review of Differentiation and Integration

## Differentiation

Definition of derivative  $f'(x)$ :

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}.$$

Differentiation rules:

1. **Sum/Difference rule:** If  $f, g$  are differentiable at  $x_0$ , then

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0).$$

2. **Product rule:** If  $f, g$  are differentiable at  $x_0$ , then

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

3. **Quotient rule:** If  $f, g$  are differentiable at  $x_0$ , and  $g(x_0) \neq 0$ , then

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}.$$

4. **Chain rule:** If  $g$  is differentiable at  $x_0$ , and  $f$  is differentiable at  $g(x_0)$ , then

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

This rule may also be expressed as

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left.\left(\frac{dy}{du}\right)\right|_{u=u(x_0)} \left.\left(\frac{du}{dx}\right)\right|_{x=x_0}.$$

Implicit differentiation is a consequence of the chain rule. For instance, if  $y$  is really dependent upon  $x$  (i.e.,  $y = y(x)$ ), and if  $u = y^3$ , then

$$\frac{d}{dx}(y^3) = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = \frac{d}{dy}(y^3)y'(x) = 3y^2y'.$$

*Practice:* Find

$$\frac{d}{dx} \left( \frac{x}{y} \right), \quad \frac{d}{dx}(x^2 \sqrt{y}), \quad \text{and} \quad \frac{d}{dx}[y \cos(xy)].$$

# Integration

The definite integral

- the area problem
- Riemann sums
- definition

## Fundamental Theorem of Calculus:

I: Suppose  $f$  is continuous on  $[a, b]$ . Then the function given by  $F(x) := \int_a^x f(t) dt$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , with derivative

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

II: Suppose that  $F(x)$  is continuous on the interval  $[a, b]$  and that  $F'(x) = f(x)$  for all  $a < x < b$ . Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Remarks:

- Part I says there is always a formal antiderivative on  $(a, b)$  to continuous  $f$ . A vertical shift of one antiderivative results in another antiderivative (so, if one exists, infinitely many do). But if an antiderivative is to pass through a particular point (an *initial value problem*), there is often just one satisfying this additional criterion.
- Part II indicates the definite integral is equal to the total change in any (and all) antiderivatives.

The *average value of  $f$  over  $[a, b]$*  is defined to be

$$\frac{1}{b-a} \int_a^b f(x) dx,$$

when this integral exists.

Integration by substitution:

- Counterpart to the chain rule (Q: What rules for integration correspond to the other differentiation rules?)
- Examples:

1.  $\int e^{3x} dx$

$$2. \int_0^5 \frac{dx}{2x+1}$$

$$3. \int_0^{\sqrt{\pi}/2} 2x \cos(x^2) dx$$

$$4. \int \frac{\ln x}{x} dx$$

$$5. \int \frac{dx}{1+(x-3)^2}$$

$$6. \int \frac{dx}{x\sqrt{4x^2-1}}$$

$$7. \int \cos(3x) \sin(3x) dx$$

$$8. \int \frac{\arctan(2x)}{1+4x^2} dx$$

$$9. \int \tan^m x \sec^2 x dx$$

$$10. \int \tan x dx \quad (\text{worth extra practice})$$

$$11. \int \sec x dx \quad (\text{worth extra practice})$$

MATH 162: Calculus II  
Framework for Tues., Jan. 30  
Integration by Parts

Integration by parts formula:

$$\int u \, dv = uv - \int v \, du$$

Remarks

- Counterpart to product rule for differentiation
- Applicable for definite integrals as well:

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du.$$

- Technique used when integrands have the form:
  - $p(x)e^{ax}$ ,  $p(x)e^{ax}$ , where  $p(x)$  is a polynomial,  $a$  a constant  
Let  $u$  be the polynomial part; number of iterations equals degree of  $p$
  - $p(x)\cos(bx)$ ,  $p(x)\sin(bx)$ , where  $p(x)$  is a polynomial,  $b$  a constant  
Let  $u$  be the polynomial part; number of iterations equals degree of  $p$
  - $e^{ax}\cos(bx)$ ,  $e^{ax}\sin(bx)$ , where  $a, b$  are constants  
No rule for which part  $u$  equals; requires 2 iterations and algebra

Some integrals (less obviously) done by parts:

- $\int \ln x \, dx$  (see p. 450)
- $\int \arccos x \, dx$  (and other inverse fns; see p. 454)
- $\int \sec^3 x \, dx$  (see p. 459)

# MATH 162: Calculus II

## Framework for Wed., Jan. 31

### Trigonometric Integrals

Types we handle and relevant trig. identities:

<u>integral type</u>	<u>trig. identity</u>
$\int \sin^m x \cos^n x \, dx, \quad m \text{ or } n \text{ odd}$	$\sin^2 \theta + \cos^2 \theta = 1$
$\int \sin^m x \cos^n x \, dx, \quad m, n \text{ even}$	$\sin^2 \theta = \frac{1}{2}[1 - \cos(2\theta)]$ and $\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)]$
$\int \sqrt{1 + \cos(mx)} \, dx$	$2 \cos^2 \theta = 1 + \cos(2\theta)$
$\int \tan^m x \sec^n x \, dx, \quad m, n > 1$	$1 + \tan^2 \theta = \sec^2 \theta$
$\int \cot^m x \csc^n x \, dx, \quad m, n > 1$	$1 + \cot^2 \theta = \csc^2 \theta$
$\int \sin(mx) \sin(nx) \, dx, \quad m \neq n$	$\sin(m\theta) \sin(n\theta) = \frac{1}{2}[\cos((m-n)\theta) - \cos((m+n)\theta)]$
$\int \sin(mx) \cos(nx) \, dx, \quad m \neq n$	$\sin(m\theta) \cos(n\theta) = \frac{1}{2}[\sin((m-n)\theta) + \sin((m+n)\theta)]$
$\int \cos(mx) \cos(nx) \, dx, \quad m \neq n$	$\cos(m\theta) \cos(n\theta) = \frac{1}{2}[\cos((m-n)\theta) + \cos((m+n)\theta)]$

But, frequently the above identities only serve to reduce things to a special case you must be able to handle some other way. Some of these special cases:

$$\int \tan x \, dx, \text{ handled via substitution}$$

$$\int \sec^3 x \, dx, \text{ handled by parts (see p. 459)}$$

$$\int \tan^m x \sec^2 x \, dx, \text{ handled via substitution}$$

$$\int \sec x \, dx, \text{ handled via substitution, after multiplying through by } \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$\int \tan x \sec^n x \, dx, \text{ handled via substitution}$$

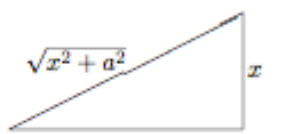
# MATH 162: Calculus II

## Framework for Fri., Feb. 2

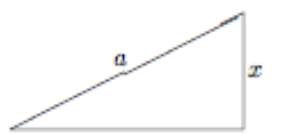
### Trigonometric Substitutions

Trigonometric substitution

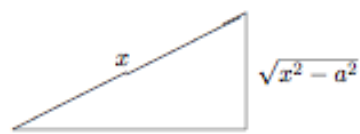
- a technique used in integration
- useful most often when integrands involve  $(a^2 + x^2)^m$ ,  $(a^2 - x^2)^m$  or  $(x^2 - a^2)^m$ , where  $a$  and  $m$  are constants.  $m$  is often, but not exclusively, equal to  $1/2$ .  
Note: To get one of these forms, often completion of a square is required.
- The usual substitutions



$$\begin{aligned}
 x &= a \tan \theta \\
 dx &= a \sec^2 \theta d\theta \\
 \sqrt{x^2 + a^2} &= a \sec \theta \\
 -\frac{\pi}{2} < \theta < \frac{\pi}{2}
 \end{aligned}$$



$$\begin{aligned}
 x &= a \sin \theta \\
 dx &= a \cos \theta d\theta \\
 \sqrt{a^2 - x^2} &= a \cos \theta \\
 -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}
 \end{aligned}$$



$$\begin{aligned}
 x &= a \sec \theta \\
 dx &= a \sec \theta \tan \theta d\theta \\
 \sqrt{x^2 - a^2} &= a |\tan \theta| \\
 0 \leq \theta < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < \theta \leq \pi
 \end{aligned}$$

Some examples:

$$\int \frac{dx}{x^2 \sqrt{9 - x^2}}$$

$$\int \frac{dx}{(x^2 + 1)^{3/2}}$$

$$\int \frac{x}{\sqrt{x^2 - 3}} dx$$

MATH 162: Calculus II  
Framework for Mon., Feb. 5  
Integrals using Partial Fraction Expansion

**Definition:** A *rational function* is a function that is the ratio of polynomials.

Examples:  $\frac{2x}{x^2 + 6}$        $\frac{x^2 - 1}{(x^2 + 3x + 1)(x - 2)^2}$        $\frac{1}{\sqrt{x + 7}}$

**Definition:** A quadratic (2nd-degree) polynomial function with real coefficients is said to be *irreducible* (over the reals) if it has no real roots.

A quadratic polynomial is reducible if and only if it may be written as the product of linear (1st-degree polynomial) factors with real coefficients

Examples:

$$x^2 + 4x + 3$$

$$x^2 + 4x + 5$$

## Partial fraction expansion

- Reverses process of “combining rational fns. into one”
  - Input: a rational fn.    Output: simpler rational fns. that sum to input fn.
  - degree of numerator in input fn. must be less than or equal to degree of denominator (You may have to use long division to make this so.)
  - denominator of input fn. must be factored completely (i.e., into linear and quadratic polynomials)
- Why a “technique of integration”?
- leaves you with integrals that you must be able to evaluate by other means. Some examples:

$$\int \frac{5}{2(x - 5)} dx \qquad \int \frac{2}{(3x + 1)^3} dx \qquad \int \frac{x + 1}{(x^2 + 4)^2} dx$$

MATH 162: Calculus II  
Framework for Wed., Feb. 7  
Numerical Integration

## Numerical approximations to definite integrals

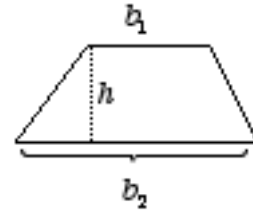
- Riemann (rectangle) sums already give us approximations

Main types: left-hand, right-hand and midpoint rules

- Question: Why rectangles?

– trapezoids

- \* Area of a trapezoid with bases  $b_1$ ,  $b_2$ , height  $h$

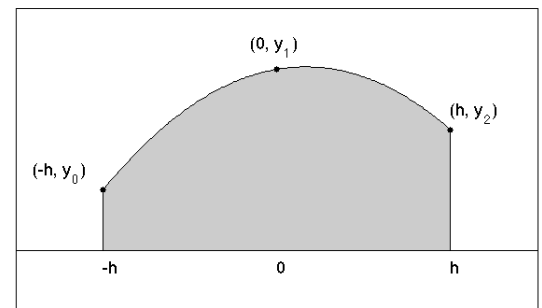


- \* Approximation to  $\int_a^b f(x) dx$  using  $n$  steps all of width  $\Delta x = (b - a)/n$   
(Trapezoid Rule)

- \* Remarkable fact: Trapezoid rule does not improve over midpoint rule.

– parabolic arcs

- \*  $\int_{-h}^h g(x) dx$ , when  $g(x) = Ax^2 + Bx + C$   
is chosen to pass through  $(-h, y_0)$ ,  $(0, y_1)$   
and  $(h, y_2)$



- \* Approximation to  $\int_a^b f(x) dx$  using  $n$  (even) steps all of width  $\Delta x$  (Simpson's Rule)



- Error bounds

- No such thing available for a general integrand  $f$
- Formulas (available when  $f$  is sufficiently differentiable)

- \* **Trapezoid Rule.** Suppose  $f''$  is continuous throughout  $[a, b]$ , and  $|f''(x)| \leq M$  for all  $x \in [a, b]$ . Then the error  $E_T$  in using the Trapezoid rule with  $n$  steps to approximate  $\int_a^b f(x) dx$  satisfies

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

- \* **Simpson's Rule.** Suppose  $f^{(4)}$  is continuous throughout  $[a, b]$ , and  $|f^{(4)}(x)| \leq M$  for all  $x \in [a, b]$ . Then the error  $E_S$  in using Simpson's rule with  $n$  steps to approximate  $\int_a^b f(x) dx$  satisfies

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

- Use

- \* For a given  $n$ , gives an upper bound on your error
- \* If a desired upper bound on error is sought, may be used to determine *a priori* how many steps to use

# MATH 162: Calculus II

## Framework for Fri., Feb. 9

### Improper Integrals

Thus far in MATH 161/162, definite integrals  $\int_a^b f(x) dx$  have:

- been over regions of integration which were finite in length (i.e.,  $a \neq -\infty$  and  $b \neq \infty$ )
- involved integrands  $f$  which are finite throughout the region of integration

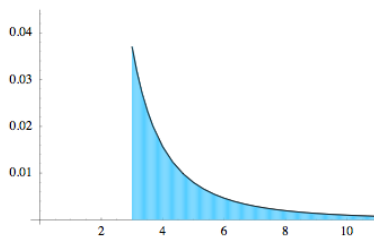
Q: How would we make sense of definite (*improper*, as they are called) integrals that violate one or both of these assumptions?

A: As limits (or sums of limits), when they exist, of definite integrals.

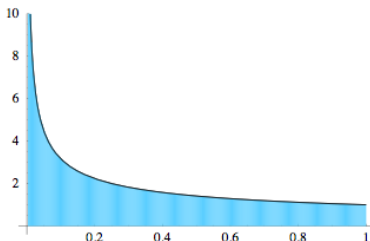
- When all of the limits involved exist, the integral is said to *converge*.
- When even one of the limits involved does not exist, the integral is said to *diverge*.

Examples:

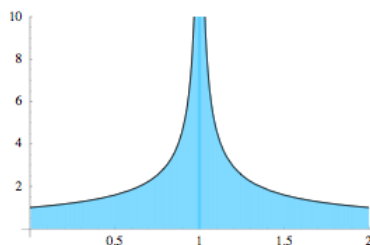
$$\int_3^{\infty} \frac{dx}{x^3}$$



$$\int_0^1 \frac{dx}{\sqrt{x}}$$



$$\int_0^2 \frac{dx}{(x-1)^{2/3}}$$



$$\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$$

Evaluating them:

$$\int_{-\infty}^0 e^x dx$$

$$\int_0^1 \ln x dx$$

$$\int_1^{\infty} \frac{dx}{x^p}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Even when an improper integral cannot be evaluated exactly, one might be able to determine if it converges or not. One of several possible theorems which address this issue:

**Theorem (Direct Comparison Test):** Suppose  $f, g$  satisfy  $0 \leq f(x) \leq g(x)$  for all  $x \geq a$ . Then

(i)  $\int_a^{\infty} f(x) dx$  converges if  $\int_a^{\infty} g(x) dx$  converges.

(ii)  $\int_a^{\infty} g(x) dx$  diverges if  $\int_a^{\infty} f(x) dx$  diverges.

# MATH 162: Calculus II

## Framework for Tues., Feb. 13

### Introduction to Series

**Example:** Application of the direct comparison test

Suppose  $g(x) = x^{-p}$ , with  $p > 0$  (so  $g$  has the general shape of the blue curve for  $x > 0$ ), and  $f$  is the step function pictured in black.

- By the direct comparison test, if  $p > 1$  then

$$\begin{aligned} \int_1^\infty f(x) dx &= 2^{-p} + 3^{-p} + 4^{-p} + \cdots + n^{-p} + \cdots \\ &= \sum_{n=2}^\infty n^{-p} \end{aligned}$$

converges. And, since it is the case that

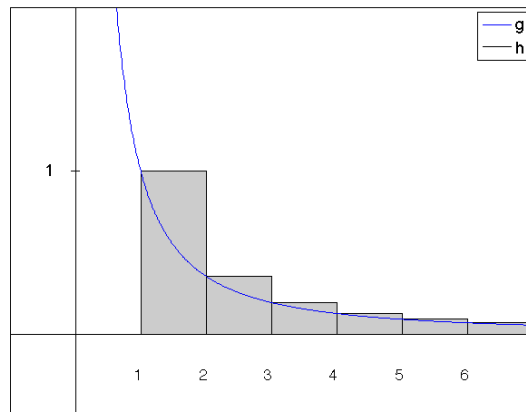
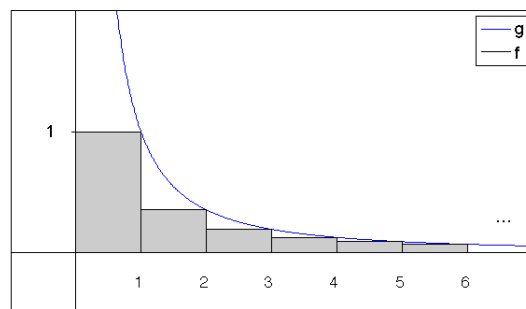
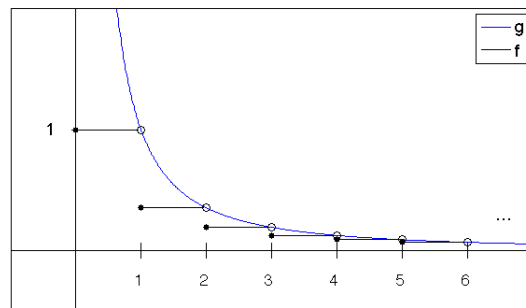
$$\sum_{n=1}^\infty n^{-p} = 1 + \sum_{n=2}^\infty n^{-p},$$

$\sum_{n=1}^\infty n^{-p}$  converges as well when  $p > 1$ .

- To conclude  $\sum_{n=1}^\infty n^{-p}$  diverges for  $p \leq 1$ , we must deal with a function like  $f$  that stays above  $g$ . The function  $h(x) = f(x-1)$  will do (see at right). The improper integral

$$\begin{aligned} \int_1^\infty h(x) dx &= 1^{-p} + 2^{-p} + 3^{-p} + \cdots \\ &= \sum_{n=1}^\infty n^{-p} \end{aligned}$$

diverges since  $\int_1^\infty x^{-p} dx$  diverges for  $p \leq 1$ .

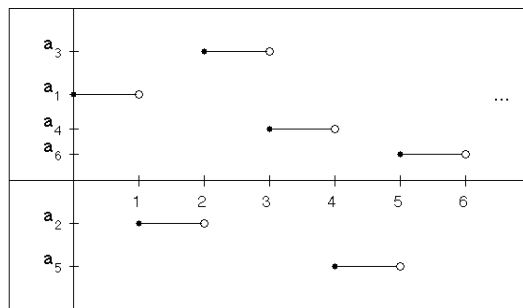


# Infinite Series

- An infinite sum of numbers:  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ .
- Can be thought of as an improper integral

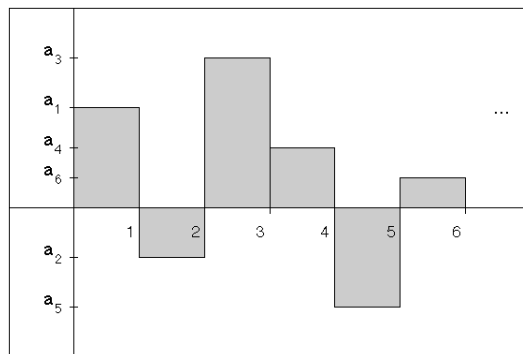
$$\text{Define } f(x) := \begin{cases} a_1, & 0 \leq x < 1, \\ a_2, & 1 \leq x < 2, \\ \vdots & \vdots \\ a_n, & n-1 \leq x < n, \\ \vdots & \vdots \end{cases}$$

(See graph of step fn. at right.)



Then our given series may be expressed as an improper integral of  $f$ :

$$\sum_{n=1}^{\infty} a_n = \int_0^{\infty} f(x) dx.$$



- Will be said to *converge* or *diverge*. As with other improper integrals, convergence requires the existence of a limit of “proper sums” (actually called *partial sums*). Define

$$\begin{aligned} s_1 &:= a_1, \\ s_2 &:= a_1 + a_2, \\ s_3 &:= a_1 + a_2 + a_3, \\ \vdots & \quad \quad \quad \vdots \\ s_n &:= a_1 + a_2 + \cdots + a_n, \\ \vdots & \quad \quad \quad \vdots \end{aligned}$$

The series  $\sum_{n=1}^{\infty} a_n$  converges precisely when the *sequence*  $s_n$  converges to some real number limit.

**Examples:**

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \quad \text{diverges.}$$

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \quad \text{converges.}$$

MATH 162: Calculus II  
Framework for Thurs., Feb. 15  
Geometric Series and Series Introduction

## Geometric Series

- Form of series under this classification

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n,$$

$a, r$  nonzero constants

- Zeno's paradox about crossing a room

- If  $L$  is length of room, then he is looking at adding up distances

$$L \cdot \left(\frac{1}{2}\right) + L \cdot \left(\frac{1}{2}\right)^2 + L \cdot \left(\frac{1}{2}\right)^3 + \cdots = \sum_{n=0}^{\infty} ar^n,$$

with  $a = L/2, r = 1/2$ .

- Evidence that (some) geometric series converge

- Partial sums  $s_n$

- Define in customary way:

$$s_1 = a, \quad s_2 = a + ar, \quad s_3 = a + ar + ar^2, \quad \text{etc.}$$

- $n$ th partial sum has nice closed-form formula:

$$s_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & \text{when } r \neq 1, \\ na, & \text{when } r = 1. \end{cases}$$

- **Main Result:** Geometric series  $\sum_{n=0}^{\infty} ar^n$  converges to  $\frac{a}{1-r}$  when  $|r| < 1$ , and diverges otherwise.

- Note the *divergence* when  $|r| = 1$ :

$$r = 1 : \sum_{n=0}^{\infty} a = a + a + \cdots + a + \cdots \quad (\text{divergent})$$

$$r = -1 : \sum_{n=0}^{\infty} a = a - a + a - a + a - a + \cdots \quad (\text{divergent})$$

Remarks concerning infinite series (general, not just geometric ones)  $\sum_{n=1}^{\infty} a_n$ :

1. Convergence relies on the partial sums  $s_n := a_1 + \cdots + a_n$  approaching a limit as  $n \rightarrow \infty$
2. Assessing the limit of partial sums directly requires a nice closed-form expression for  $s_n$ . Such an expression exists only in rare cases, such as the following examples we've already done

Geometric series:  $\sum_{n=0}^{\infty} ar^n$

“Telescoping series”:  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$

3. When no closed-form expression for  $s_n$  is available, determining if limit exists is usually more difficult.

**Example:**  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}, p \geq 0$

4. Systematic tests can help

- Some of the tests that have been developed (\* indicate ones we will study)
  - **\*nth-term test for divergence** (p. 519)
  - **integral test** (p. 525): formalization of the approach we used to determine which  $p$ -series  $\sum_{n=1}^{\infty} n^{-p}$  converge/diverge
  - **direct comparison test** (p. 529): practically a restatement of the one of the same name for improper integrals
  - **limit comparison test** (p. 530): did not do comparable result for improper integrals
  - **\*ratio test** (p. 533)
  - **root test** (p. 535)
  - **alternating series test** (p. 538): formalization of the approach we used to show  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$  converges for  $p \geq 0$
  - **\*absolute convergence test** (p. 540)
- Must be cognizant of
  - the situations in which a test may be applied
  - what conclusions may and may not be drawn from such tests

MATH 162: Calculus II  
Framework for Fri., Feb. 16  
Absolute and Conditional Convergence

## $p$ -Series Results Revisited

- Results we have shown: The series whose terms are all positive

$$\sum_{n=1}^{\infty} n^{-p} = 1 + 2^{-p} + 3^{-p} + 4^{-p} + \cdots \quad (1)$$

converges for  $p > 1$ , and diverges for  $p \leq 1$ . The series with alternating signs

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-p} = 1 - 2^{-p} + 3^{-p} - 4^{-p} + \cdots \quad (2)$$

converges for  $p > 0$ , and diverges for  $p \leq 0$ .

- The above results apply narrowly—only to series in the forms (1) and (2) respectively. Thus, nothing we have learned tells us whether

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$$

converges.

- The “borderline” case of (1), the one with  $p = 1$ ,

$$\sum_{n=1}^{\infty} n^{-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \quad (3)$$

is divergent, and has been named the *harmonic series*.

- When the convergence/divergence of a series  $\sum a_n$  is known, then the convergence/divergence of certain modified forms of that series can be known as well. In particular,
  - Any nonzero multiple of a series that converges (resp. diverges) will also converge (resp. diverge). Thus,

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \cdots = \frac{1}{3} \sum_{n=1}^{\infty} n^{-1},$$

diverges, being a multiple of the harmonic series (3).



- Suppose  $\sum a_n$  is a series whose convergence/divergence is known. Any series which has the same “tail” as that of  $\sum a_n$  will converge (resp. diverge) based on what  $\sum a_n$  does. For instance, since we know

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/2} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

converges, we can conclude

$$\sum_{n=4}^{\infty} (-1)^{n-1} n^{-1/2} = -\frac{1}{2} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

and

$$b_1 + b_2 + \cdots + b_{50} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

converge as well. Here  $b_1, \dots, b_{50}$  is any arbitrary list of 50 numbers. The important thing is not the values of these  $b_j$ 's, but that there are only finitely many (in this case, 50) of them.

## Absolute and Conditional Convergence

**Definition:** Let  $\sum a_n$  be a convergent series. If the corresponding series  $\sum |a_n|$  in which every term has been made positive diverges, then the original series  $\sum a_n$  is said to be *conditionally convergent*.

**Example:** The series  $\sum_{n=1}^{\infty} (-1)^n n^{-1} = 1 - 1/2 + 1/3 - 1/4 + \cdots$  is conditionally convergent.

**Definition:** Let  $\sum a_n$  be a given series (i.e., one for which the values of the terms  $a_j$  are known). If the corresponding series  $\sum |a_n|$  with all positive terms converges, then the original series  $\sum a_n$  is said to be *absolutely convergent*.

**Theorem (Absolute Convergence Test):** All absolutely convergent series are convergent.

**Example:** The series

$$\frac{11}{3} + \frac{11}{6} - \frac{11}{12} + \frac{11}{24} - \frac{11}{48} - \frac{11}{96} - \frac{11}{192} + \cdots$$

is absolutely convergent, since

$$\frac{11}{3} + \frac{11}{6} + \frac{11}{12} + \frac{11}{24} + \frac{11}{48} + \cdots = \sum_{n=0}^{\infty} \left(\frac{11}{3}\right) \left(\frac{1}{2}\right)^n$$

converges (being geometric, with  $r = 1/2$ ). By the absolute convergence test, the original series converges as well.

MATH 162: Calculus II  
Framework for Mon., Feb. 19  
The Ratio Test

## Several Ideas Coming Together

- **Power fns.:** ones of form  $x^{-p}$ .

- **Identification.** The base  $x$  varies, while the exponent  $(-p)$  remains fixed. These fns. grow without bound as  $x \rightarrow \infty$  when  $p < 0$ , and approach zero as  $x \rightarrow \infty$  when  $p > 0$ .
- **Improper integrals.** Only in the case  $p > 0$  might we *hope*  $\int_1^\infty x^{-p} dx$  converges. Our finding is that, in actuality,  $\int_1^\infty x^{-p} dx$  converges only when  $p > 1$ . That is, only when  $p > 1$  do such fns. approach 0 *quickly enough* so that the stated integral converges. Our corresponding finding for series:  $\sum_{n=1}^\infty n^{-p}$  converges for  $p > 1$  and diverges for  $p \leq 1$ .

- **Exponential fns.:** ones of form  $b^x$

- **Identification.** The base  $b$  remains fixed, and the exponent  $x$  varies. Such fns. grow without bound as  $x \rightarrow \infty$  when  $b > 1$ ; they approach zero as  $x \rightarrow \infty$  when  $0 < b < 1$ .
- **Improper integrals.** When  $0 < b < 1$ , the exponential  $b^x$  decays to zero faster than any power fn.  $x^p$  with  $p > 0$ . Not surprisingly, then,

$$\int_a^\infty b^x dx$$

( $a$  any real number) converges for any value of  $b$  in  $(0, 1)$ . The corresponding types of series are geometric series

$$\sum_{n=0}^\infty r^n$$

which we know converge for  $-1 < r < 1$ , and diverge for  $|r| \geq 1$ .

- **It's the tail that matters.** The “fate” of a series  $\sum a_n$  (in terms of whether or not it converges) does not depend on its first few, say 100 trillion or so, terms, but on the *tail* (the remaining infinitely many terms).

## The Ratio Test

The previous facts point to the following conjecture concerning series:

Given some series  $\sum a_n$ , if it may be determined that, eventually, the terms shrink in magnitude at least as quickly as an exponential decay function, then the series should converge.

There might be several ways to formulate this conjecture into a mathematical statement. One way, which has been proved (so it is a theorem), is:

**Theorem (Ratio test, p. 533):** Let  $\sum a_n$  be a series whose terms  $a_n > 0$  (all positive), and suppose  $\lim_{n \rightarrow \infty} a_{n+1}/a_n = \rho$  (i.e., suppose that this limit exists; we'll call it  $\rho$ ).

- (i) If  $\rho < 1$ , the series  $\sum a_n$  converges.
- (ii) If  $\rho > 1$ , the series  $\sum a_n$  diverges.

Note: If  $\rho = 1$ , this test does not allow us to draw a conclusion either way.

**Examples:**

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2 3^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

When a series  $\sum a_n$  has negative terms, we may apply the ratio test to  $\sum |a_n|$ . If the test reveals that  $\sum |a_n|$  converges, then so does  $\sum a_n$  by the absolute convergence test.

**Example:** Determine those  $x$ -values for which the series converges.

$$(a) \sum_{n=0}^{\infty} \frac{(3x-2)^n}{n 3^n}$$

$$(b) \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$(c) \sum_{n=0}^{\infty} n! x^n$$

MATH 162: Calculus II  
Framework for Tues., Feb. 20  
Introduction to Power Series

**Definition:** A function of  $x$  which takes the series form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \quad (1)$$

is called a *power series about  $x = a$* . The number  $a$  is called the *center*, and the coefficients  $c_0, c_1, c_2, \dots$  are constants.

Remarks:

- As for any function, a power series has a domain. The acceptable inputs  $x$  to a power function are those  $x$  for which the series converges.
- If it were not for the coefficients  $c_j$  (if, say, each  $c_j = 1$ ), a power series would look geometric. Indeed, the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (2)$$

is a power series about  $x = 0$ , that is known to converge to  $(1-x)^{-1}$  when  $-1 < x < 1$  and to diverge when  $|x| \geq 1$ . So, the domain of power series (2) is  $(-1, 1)$ .

More generally, for a special type of power series about  $x = a$  with coefficients  $c_j = \beta^j$  (i.e., whose coefficients are ascending powers of some fixed number  $\beta$ )

$$\sum_{n=0}^{\infty} \beta^n(x-a)^n = 1 + \beta(x-a) + \beta^2(x-a)^2 + \beta^3(x-a)^3 + \cdots, \quad (3)$$

we have that this series

converges to  $\frac{1}{1 - \beta(x-a)}$  when  $|\beta(x-a)| < 1$ , that is, for  $a - \frac{1}{|\beta|} < x < a + \frac{1}{|\beta|}$ ,

and diverges when  $|(x-a)| \geq \frac{1}{|\beta|}$ .

Thus, the domain of series (3) is  $(a - 1/|\beta|, a + 1/|\beta|)$ .

- In the most general case, where the coefficients  $c_j$  in (1) do not, in general, equal  $\beta^j$  for some number  $\beta$ , the determination of the domain usually requires

1. the use of the ratio test on the series  $\sum_{n=0}^{\infty} |c_n(x-a)^n|$ . That is, one looks at

$$\lim_{n \rightarrow \infty} \frac{|c_{n+1}||x-a|^{n+1}}{|c_n||x-a|^n} = \left( \lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} \right) |x-a|.$$

If  $\lim_{n \rightarrow \infty} |c_{n+1}|/|c_n|$  exists, and if we let

$$\rho = \left( \lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} \right) |x-a|,$$

then part (i) of the ratio test imposes constraints on what values  $x$  may take. Specifically, we generally wind up with a number  $R \geq 0$ , called the *radius of convergence*, for which the series (1)

converges if  $x$  is inside the open interval  $(a-R, a+R)$ , and

diverges if  $|x-a| > R$  (i.e., if  $x$  is outside the closed interval  $[a-R, a+R]$ ).

In those cases where  $\lim_{n \rightarrow \infty} |c_{n+1}|/|c_n| = 0$ , the value of  $R := +\infty$ , and when this happens there is no need to proceed to step 2.

2. the determination (by some other means than the ratio test) of whether the series converges when  $|x-a| = R$  (i.e., at the points  $x = a \pm R$ ).

The upshot is that the domain of a power series whose radius of convergence  $R$  is nonzero is always an interval, an interval that has  $x = a$  at its center and, in the case  $R \neq +\infty$ , may include one or both of its endpoints  $x = a \pm R$ . For this reason the domain of a power series is usually called its *interval of convergence*.

**Example:** Determine the interval of convergence for

(a)  $\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n3^n}$

(b)  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(c)  $\sum_{n=0}^{\infty} n!x^n$

(d)  $\sum_{n=0}^{\infty} \frac{x^n 3^n}{n^{3/2}}$

**Power Series Expressions for Some Fns. (building new series from known ones)**

We know that, for  $|x| < 1$ , the fn.  $f(x) = 1/(1 - x)$  may be expressed as a power series:

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Thus, we may express similar-looking fns. as power series:

$$\frac{1}{1 - 3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n, \quad |3x| < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$$

$$\frac{1}{2 - x} = \frac{1}{2} \cdot \frac{1}{1 - (x/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}, \quad \left|\frac{x}{2}\right| < 1 \Rightarrow -2 < x < 2.$$

$$\frac{x^3}{1 - x} = x^3 \cdot \frac{1}{1 - x} = x^3 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3}, \quad -1 < x < 1.$$

$$\frac{1}{1 + x} = \frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} [(-1)x]^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |-x| < 1 \Rightarrow -1 < x < 1.$$

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x^2| < 1 \Rightarrow -1 < x < 1.$$

All of the above are power series about  $x = 0$ . We show how the 2nd one (in the above list of 5) could also be written as a power series centered around  $x = 1$ :

$$\frac{1}{2 - x} = \frac{1}{1 - (x - 1)} = \sum_{n=0}^{\infty} (x - 1)^n, \quad |x - 1| < 1 \Rightarrow 0 < x < 2.$$

# MATH 162: Calculus II

## Framework for Wed., Feb. 21

### Differentiation and Integration of Power Series

While power series are allowed to have nonzero numbers as centers, for today's results we will assume all power series we discuss are centered about  $x = 0$ ; that is, are of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \quad (1)$$

We will also assume each has a positive radius of convergence  $R > 0$ , so that the series converges at least for those  $x$  satisfying  $-R < x < R$ .

## Differentiation of Power Series about $x = 0$

**Theorem (Term-by-Term Differentiation, p. 549):** Let  $f(x)$  take the form of the power series in (1), with radius of convergence  $R > 0$ . Then the series

$$\sum_{n=1}^{\infty} n c_n x^{n-1}$$

converges for all  $x$  satisfying  $-R < x < R$ , and

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad \text{for all } x \text{ satisfying } -R < x < R.$$

Remarks:

- Since the hypotheses of this theorem now apply to  $f'(x)$ , we can continue to differentiate the series to find derivatives of  $f$  of all orders, convergent at least on the interval  $-R < x < R$ . For instance,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = (2 \cdot 1) c_2 + (3 \cdot 2) c_3 x + (4 \cdot 3) c_4 x^2 + \cdots$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) c_n x^{n-3} = (3 \cdot 2 \cdot 1) c_3 + (4 \cdot 3 \cdot 2) c_4 x + (5 \cdot 4 \cdot 3) c_5 x^2 + \cdots$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$\begin{aligned} f^{(j)}(x) &= \sum_{n=j}^{\infty} n(n-1)(n-2) \cdots (n-j+1) c_n x^{n-j} \\ &= j! c_j + [(j+1)j \cdots 2] c_{j+1} x + [(j+2)(j+1) \cdots 3] c_{j+2} x^2 + \cdots \end{aligned}$$

- The easiest place to evaluate a power series is at its center. In particular, if  $f$  has the form (1), we may evaluate  $f$  and all of its derivatives at zero to get:

$$\begin{aligned} f(0) &= c_0, \\ f'(0) &= c_1, \\ f''(0) &= (2 \cdot 1) c_2 \quad \Rightarrow \quad c_2 = \frac{1}{2} f''(0), \\ f'''(0) &= (3 \cdot 2 \cdot 1) c_3 \quad \Rightarrow \quad c_3 = \frac{1}{3!} f'''(0), \end{aligned}$$

and so on. So, we arrive at the following relationship between the coefficients  $c_j$  and the derivatives of  $f$  at the center:

**Corollary:** Let  $f(x)$  be defined by the power series (1). Then

$$c_n = \frac{f^{(n)}(0)}{n!}, \quad \text{for all } n = 0, 1, 2, \dots$$

**Example:** We know that  $f(x) = (1-x)^{-1}$  has the power series representation  $\sum_{n=0}^{\infty} x^n$  for  $x$  in the interval  $(-1, 1)$ . By the term-by-term differentiation theorem,  $f'(x) = (1-x)^{-2}$  has the power series representation

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) = \frac{d}{dx} (1 + x + x^2 + x^3 + \dots) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \dots \quad (\text{the theorem justifies this step}) \\ &= 0 + 1 + 2x + 3x^2 + 4x^3 + \dots \\ &= \sum_{n=1}^{\infty} nx^{n-1}, \end{aligned}$$

with this series representation holding at least for  $-1 < x < 1$ .

## Integration of Power Series about $x = 0$

If we can differentiate a series expression for  $f$  term-by-term in order to arrive at a series expression for  $f'$ , it may not be surprising that we may integrate a series term-by-term as well.

**Theorem (Term-by-Term Integration, p. 550):** Suppose that  $f(x)$  is defined by the power series (1) and the radius of convergence  $R > 0$ . Then

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}, \quad \text{for all } x \text{ satisfying } -R < x < R.$$



**Example:** We know that  $f(x) = (1+x)^{-1}$  has the power series representation  $\sum_{n=0}^{\infty} (-1)^n x^n$  for  $x$  in the interval  $(-1, 1)$ . Moreover,

$$\int_0^x \frac{dt}{1+t} = \left[ \ln|1+t| \right]_0^x = \ln|1+x|.$$

By the term-by-term integration theorem, for  $-1 < x < 1$  we also have

$$\begin{aligned} \int_0^x \frac{dt}{1+t} &= \int_0^x \left( \sum_{n=0}^{\infty} (-1)^n t^n \right) dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - t^5 + \cdots) dt \\ &= \int_0^x dt - \int_0^x t dt + \int_0^x t^2 dt - \int_0^x t^3 dt + \cdots \quad (\text{the theorem justifies this step}) \\ &= \sum_{n=0}^{\infty} (-1)^n \left( \int_0^x t^n dt \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left[ t^{n+1} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots . \end{aligned}$$

That is,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \ln(1+x),$$

at least for all  $x$  in the interval  $-1 < x < 1$ . In fact, though the theorem does not go so far as to guarantee convergence at the value  $x = 1$ , since the series on the left converges at  $x = 1$  (Why?), one might suspect that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

This, indeed, is the case.

**Example:** Use the power series representation for  $(1+x^2)^{-1}$  about zero to get a power series representation for  $\arctan x$ .

# MATH 162: Calculus II

## Framework for Fri., Feb. 23

### Taylor Series

In the section on power series, we seemed to be

- interested in finding power series expressions for various functions  $f$ ,
- but able to find such series only when the function  $f$ , or some order derivative/antiderivative of  $f$ , looked enough like  $(1 - x)^{-1}$  to make this feasible.

Our goal today is to find series expressions for important functions that are not so closely linked to  $(1 - x)^{-1}$ . First, a definition:

**Definition:** Suppose  $f$  is a function which has derivatives of all orders at  $x = a$ . The *Taylor series for  $f$  at  $x = a$*  is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.$$

What this definition says is that, for appropriate  $f$ , we may construct a power series about  $x = a$  employing the values of derivatives  $f^{(n)}(a)$  in the coefficients. As of yet, no assertion that this power series actually equals  $f$  has been made. (See note 2 below.)

#### Some important notes:

1. The Taylor series, like any power series, has a radius of convergence  $R$ , which may be zero.
2. Even if the radius of convergence  $R > 0$ , the function defined by the Taylor series of  $f$  might not equal  $f$  except at the single location  $x = a$ .
3. But, if  $R > 0$ , then for many “nice” functions  $f$ , the Taylor series for  $f$  equals  $f$  on its entire interval of convergence.
4. If we stop the sum at the term containing  $(x - a)^n$  (i.e., consider the partial sum of the series that includes as its last term the one with  $(x - a)$  to the  $n^{\text{th}}$  power), we get a polynomial of  $n^{\text{th}}$  degree. This polynomial is called the *Taylor polynomial of order  $n$  for  $f$  at  $x = a$* .
5. If  $a = 0$ , then the Taylor series is called the *MacLaurin series of  $f$* .
6. If  $f$  equals any power series about  $x = a$  at all, then that series must be the Taylor series.

**Some favorite Taylor series** (all of these are MacLaurin series)

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1\end{aligned}$$

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \quad -\infty < x < \infty\end{aligned}$$

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots, \quad -\infty < x < \infty\end{aligned}$$

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots, \quad -\infty < x < \infty\end{aligned}$$

$$\begin{aligned}\arctan x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \quad -1 \leq x \leq 1\end{aligned}$$

$$\begin{aligned}\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots, \quad -1 < x \leq 1\end{aligned}$$

As with expressions that were similar to  $(1-x)^{-1}$ , we may substitute into these power series to get power series expressions for other, related functions.

**Example:** The MacLaurin series converging to  $\exp(-x^2)$  is

$$\exp(-x^2) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots.$$

# MATH 162: Calculus II

## Framework for Mon., Feb. 26

### Convergence of Taylor Series

**Today's Goal:** To determine if a function equals its power series.

In a remark from the last class, it was stated that, while a certain function  $f$  may allow the construction of a Taylor series about  $x = a$  with positive radius of convergence, one may not assume this Taylor series converges to  $f$ . In our “favorite Taylor series” (see the framework for that day), however, the convergence of the MacLaurin series for  $(1 - x)^{-1}$ ,  $\arctan x$  and  $\ln(1 + x)$  to their respective functions throughout their intervals of convergence has already been established. What has yet to be established is whether the MacLaurin series for  $e^x$ ,  $\cos x$  and  $\sin x$  converge to their respective functions.

## The Remainder

Suppose  $f$  has  $(n + 1)$  derivatives throughout an interval  $I$  around  $x = a$ . Under these conditions, we can write down the  $n^{\text{th}}$ -order Taylor polynomial for  $f$  about  $x = a$ :

$$P_{n,a}(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n,$$

Here the subscript  $a$  has been added to indicate that this polynomial is about  $x = a$ . The discrepancy between the function and its Taylor polynomial is called the *remainder* term:

$$R_{n,a}(x) := f(x) - P_{n,a}(x).$$

**Theorem** (Lagrange): Suppose  $f$ ,  $P_{n,a}$  and  $R_{n,a}$  are as described above, and that  $x$  (fixed) is a number in the interval  $I$ . Then there is a number  $t$  between  $a$  and  $x$  such that

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!} (x - a)^{n+1}.$$

**Example:** We can use Lagrange's theorem to show that  $\sin x$  is equal to its MacLaurin series for every real number  $x$ . For any (fixed)  $x$ , the theorem guarantees the existence of a number  $t$  between 0 and  $x$  such that

$$|R_{n,0}(x)| = \frac{|\sin^{(n+1)}(t)|}{(n+1)!} |x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus,

$$\lim_{n \rightarrow \infty} P_{n,0}(x) = \lim_{n \rightarrow \infty} [\sin x - R_{n,0}(x)] = \sin x - \lim_{n \rightarrow \infty} R_{n,0}(x) = \sin x,$$

which says that the sequence of partial sums of the MacLaurin series for the sine function converges to sine at  $x$ . Since we did not assume anything special about the  $x$  involved in this calculation, the result holds for any real  $x$ .

A similar type of argument may be used to establish that the MacLaurin series for  $e^x$  converges to  $e^x$  for all real  $x$ , and that the MacLaurin series for  $\cos x$  converges to  $\cos x$  for all real  $x$ .

**Example** (a weird function): Let  $f$  be defined by the formula

$$f(x) := \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It can be shown that  $f^{(n)}(0) = 0$  for all integer  $n \geq 0$ . The MacLaurin series for  $f$  is thus

$$\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0,$$

the zero function (not even an infinite series, so of course it converges for all  $x$ ). A graph of  $f$  appears in Figure 8.14 on p. 558 of the text. It may not be obvious from the picture, but while  $f(0) = 0$ , for all other choices of  $x$ ,  $f(x) > 0$ . Hence, the only place its MacLaurin series equals  $f$  is at  $x = 0$ .

MATH 162: Calculus II  
Framework for Tues., Feb. 27  
Functions of Multiple Variables

**Definition:** A *function* (or *function of  $n$  variables*)  $f$  is a rule that assigns to each ordered  $n$ -tuple of real numbers  $(x_1, x_2, \dots, x_n)$  in a certain set  $D$  a real number  $f(x_1, x_2, \dots, x_n)$ . The set  $D$  is called the *domain* of the function.

**Example:** Most real-life functions are, in fact, functions of multiple variables. Here are some:

1.  $v(r, h) = \pi r^2 h$
2.  $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
3.  $g(m_1, m_2, R) = Gm_1m_2/R^2$  ( $G$  is a constant)
4.  $P(n, T, V) = nRT/V$  ( $R$  is a constant)

Of course, one can hold fixed the values of all but one of the input variables and thereby create a function of a single variable. For instance, the way the volume of a right-circular cylinder whose height is 3 varies with its radius is given by the formula

$$V(r) = 3\pi r^2, \quad r \geq 0.$$

## Graphing

### Functions of a single variable

To graph a function of a single variable requires two coordinate axes. When we write  $y = f(x)$ , it is implied that  $x$  is a possible input and the  $y$ -value is the corresponding output. We think of the domain (the set of all possible inputs) of  $f$  as consisting of some part of the real line, the graph of  $f$  (often called a “curve”) as having a point at location  $(x, f(x))$  for each  $x$  in the domain of  $f$ . Keep in mind that, given an arbitrary equation involving  $x$  and  $y$ , it is not always the case that

- (i) we *want* to make  $y$  be the dependent variable, and
- (ii) if we *do* solve for  $y$ , the result is a *function*.

**Example:**  $x^2 + y^2 = 4$

## Functions of multiple variables

We have the following analogies for functions of multiple variables:

- When nothing explicit is said about the inputs to a function of multiple variables, we take the domain to be as inclusive as possible.

**Examples:**

$$f(x, y) = \sqrt{xy}$$

$$f(x, y) = xy(x^2 + y)^{-1}$$

- The graphs of functions of  $n$  variables are  $n$ -dimensional objects drawn in a coordinate frame involving  $(n + 1)$  mutually-perpendicular coordinate axes. (Think of a curve which is the graph of  $y = f(x)$  as a 1-dimensional object weaving through 2-dimensional space.)

As a corollary: *It is not possible to produce the graph of a function of 3 or more variables.* A possible work-around: level sets.

**Definition:** Let  $f$  be a function of  $n$  variables, and  $c$  be a real number. The set of all  $n$ -tuples  $(x_1, \dots, x_n)$  for which  $f(x_1, \dots, x_n) = c$  is called the  $c$  level set of  $f$ .

**Examples:**

$$f(x, y) = y^2 - x^2$$

$$f(x, y, z) = z - x^2 - 2y^2$$

- Not every equation involving  $x$ ,  $y$  and  $z$  yields  $z$  as a single function of  $x$  and  $y$ .

**Examples:**

$$x - y^2 - z^2 = 0$$

$$x^2 + y^2 + z^2 = 4$$

- One may assume a missing variable is implied and takes on all real values.

**Example:** The meaning of  $x = 1$  in 1, 2 and 3 dimensions.

One may need more than one equation/inequality to describe certain regions of space.

**Example:**  $x^2 + (y - 1)^2 \leq 1, \quad z = -1$

MATH 162: Calculus II  
Framework for Wed., Feb. 28  
Limits and Continuity

**Today's Goal:** To understand the meaning of limits and continuity of functions of 2 and 3 variables.

## Geometry of the Domain Space

**Definition:** (Distance in  $\mathbb{R}^3$ ): Suppose that  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  are points in  $\mathbb{R}^3$ . The *distance* between these two points is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

If our two points have the same  $z$ -value (for instance, if they both lie in the  $xy$ -plane), then the distance between them is just

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We employ these definitions for distance in  $\mathbb{R}^2$ ,  $\mathbb{R}^3$  to define circles, disks, spheres and balls.

**Definition:** A *circle* in  $\mathbb{R}^2$  centered at  $(a, b)$  with radius  $r$  is the set of points  $(x, y)$  satisfying

$$(x - a)^2 + (y - b)^2 = r^2.$$

Given such a circle  $C$ , the set of all points on or inside  $C$  is a *closed disk*. The set of points inside but not on  $C$  is an *open disk*.

**Definition:** A *sphere* in  $\mathbb{R}^3$  centered at  $(a, b, c)$  with radius  $r$  is the set of points  $(x, y, z)$  satisfying

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

The inside of a sphere is called a *ball*, and can be closed or open depending on whether the (whole) sphere is included.

Open intervals in  $\mathbb{R}$  are sets of the form  $a < x < b$  (also written using interval notation  $(a, b)$ ), where neither endpoint  $a, b$  is included in the set. One observation about such sets is that, if you take any  $x \in (a, b)$ , there is a value of  $r > 0$ , perhaps quite small, for which the interval  $(x - r, x + r)$  is wholly contained inside  $(a, b)$ . We build on that idea when defining various kinds of subsets of  $\mathbb{R}^2$ .



**Definition:** Let  $R$  be a region of the  $xy$ -plane and  $(x_0, y_0)$  a point (perhaps in  $R$ , perhaps not). We call  $(x_0, y_0)$  an *interior point* of  $R$  if there is an open disk of positive radius centered at  $(x_0, y_0)$  such that every point in this disk lies inside  $R$ .

We call  $(x_0, y_0)$  a *boundary point* of  $R$  if every disk with positive radius centered at  $(x_0, y_0)$  contains both a point that is in  $R$  and a point that isn't in  $R$ .

The region  $R$  is said to be *open* if all points in  $R$  are interior points of  $R$ .

The region  $R$  is said to be *closed* if all boundary points of  $R$  are in  $R$ .

The region  $R$  is said to be *bounded* if it lies entirely inside a disk of finite radius.

### Examples:

1. An open disk is open. A closed disk is closed. For both, the boundary points are those found on the enclosing circle.
2. The upper half-plane  $R$  consisting of points  $(x, y)$  for which  $y > 0$  is an open, unbounded set. The boundary points of  $R$  are precisely those points found along the  $x$ -axis, none of which are contained in  $R$ .
3. If  $y = f(x)$  is a continuous function on the interval  $a \leq x \leq b$ , then the graph of  $f$  is a closed, bounded set made up entirely of boundary points.
4. The set of points  $(x, y)$  satisfying  $x \geq 0$ ,  $y \geq 0$  and  $y < x + 1$  is bounded, but neither open nor closed.
5. The only nonempty region of the plane which is both open and closed is the entire plane  $\mathbb{R}^2$ .

## Limits and Continuity

The idea behind the statement

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

for a function  $f$  of two variables is that you can make the value of  $f(x, y)$  as close to  $L$  as you like by focusing only on  $(x, y)$  in the domain of  $f$  which are inside an open disk centered at  $(x_0, y_0)$  of some positive radius. The official definition follows.

**Definition:** The *limit* of  $f$  as  $(x, y)$  approaches  $(x_0, y_0)$  is  $L$  if for every  $\epsilon > 0$  there is a corresponding  $\delta > 0$  such that, for all  $(x, y)$  in  $\text{dom}(f)$ ,

$$\text{if } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \epsilon.$$

Remarks:

1. All the limit laws of functions of a single variable—those stated in Section 2.2—have analogs for functions of 2 variables. For instance, the analog to Rule 5 on p. 65 goes like this:

**Theorem:** Suppose

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} g(x,y) = M.$$

(That is, suppose both these limits exist, and call them  $L$ ,  $M$  respectively.) If  $M \neq 0$ , then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}.$$

2. We recall that, for a function  $f$  of a single variable and a point  $x_0$  interior to the domain of  $f$ , the statement  $\lim_{x \rightarrow x_0} f(x) = L$  requires the values of  $f$  to approach  $L$  as the  $x$ -values approach  $x_0$  from both the left and the right. The requirement at interior points  $(x_0, y_0)$  to the domain of a function  $f$  of 2 variables is even more strict. Any path of  $(x, y)$ -values within the domain of  $f$  traversed en route to  $(x_0, y_0)$  should produce function values which approach  $L$ .

**Example:** The limits  $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy}{x^2 + y^2}$  and  $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^4}$  do not exist.

3. One defines continuity for functions of two variables in an identical fashion as for functions of a single variable.

**Definition:** Suppose  $(x_0, y_0)$  is in the domain of  $f$ . We say that  $f$  is *continuous at*  $(x_0, y_0)$  if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = f(x_0, y_0)$$

(i.e., if this limit exists and has the same value as  $f(x_0, y_0)$ ). We say  $f$  is *continuous* (or  $f$  is *continuous throughout its domain*) if  $f$  is continuous at each point in its domain.

**Example:** The functions  $f(x, y) := \frac{4xy}{x^2 + y^2}$  and  $g(x, y) := \frac{4xy^2}{x^2 + y^4}$  are continuous at all points  $(x, y)$  except  $(0, 0)$ .

4. The notions of *distance*, *open* and *closed balls*, *boundary point*, *interior point*, *open* and *closed sets*, *limits* and *continuity* may all be generalized to functions of  $n$  variables, where  $n$  is any integer greater than 1.

# MATH 162: Calculus II

## Framework for Thurs., Mar. 1

### Partial Derivatives

**Today's Goal:** To understand what is meant by a partial derivative.

In the **framework that introduced multivariate functions**, we indicated that one can always turn a function of  $n$  variables (like the one depicted at top right) into a function of one particular variable by holding the others constant. Suppose  $(x_0, y_0)$  is an interior point to the  $\text{dom}(f)$  (the black dot, for example). Holding  $y = y_0$  fixed and letting  $x$  take on values near  $x_0$  traces out a curve on this surface. This curve is the intersection of our surface  $z = f(x, y)$  and the plane  $y = y_0$ . (See the bottom figure.)

One might ask what the slope of this curve (the one where the plane and surface intersect) is above the point  $(x_0, y_0)$ —that is, at the location of the point  $(x_0, y_0, f(x_0, y_0))$  (the yellow dot). The answer would be found by taking a derivative of the function of  $x$  that results by holding  $y = y_0$  fixed:

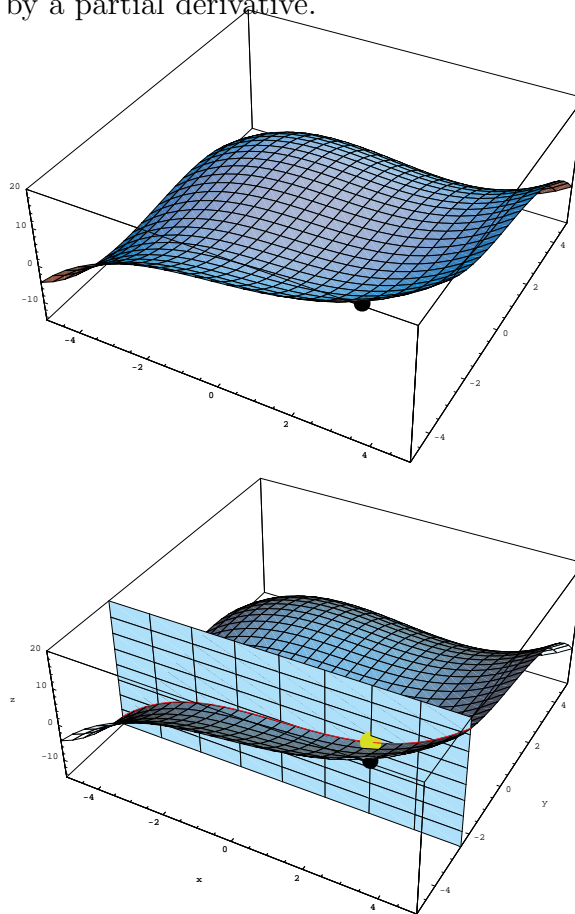
$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} := \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

called the *partial derivative of  $f$  with respect to  $x$  at  $(x_0, y_0)$* . This partial derivative has various other notations:

$$\frac{\partial f}{\partial x}(x_0, y_0), \quad f_x(x_0, y_0), \quad \left. \frac{\partial z}{\partial x} \right|_{(x_0, y_0)}.$$

Of course, the partial derivative is a function of  $x$  and  $y$  in its own right. When we think of it that way, we write

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$



We can also take partial derivatives with respect to  $y$  (or other variables, if  $f$  is a function of more than 2 variables). The resulting partial derivatives may be differentiated again:

$\frac{\partial^2 f}{\partial y^2}$ , or  $f_{yy}$  get this by differentiating  $f_y$  with respect to  $y$ .

$\frac{\partial^2 f}{\partial x \partial y}$ , or  $f_{yx}$  get this by differentiating  $f_y$  with respect to  $x$ .

$\frac{\partial^2 f}{\partial y^2 \partial x}$ , or  $f_{xyy}$  get this by differentiating  $f_x$  twice with respect to  $y$ .

The usual theorems that provide shortcuts to taking derivative may be applied, keeping in mind which variable(s) is being held constant.

**Examples:**

$$\frac{2x^2 - y}{3x - xy^2}$$

$$\exp(x/y^2)$$

$$\ln(xy^2)$$

# MATH 162: Calculus II

## Framework for Tues., Mar. 6

### Differentiability

**Today's Goal:** To understand the relationship between partial derivatives and continuity.

## The Mixed Partial Derivatives

We have learned that the partial derivative  $f_x$  at  $(x_0, y_0)$  may be interpreted geometrically as providing the slope at the point  $(x_0, y_0, f(x_0, y_0))$  along the curve that results from slicing the surface  $z = f(x, y)$  with the plane  $y = y_0$ . If one thinks of the  $x$ -axis as “facing east”, then what we are talking about is akin to standing on a patch of (possibly) hilly ground and asking what slope you would immediately experience heading eastward from your current position. Now imagine moving northward (i.e., in the direction of the positive  $y$ -axis), but still determining eastward slopes. The rate at which those eastward slopes changed as you moved northward is precisely what  $f_{xy} = \partial/\partial y(f_x)$  provides.

One might ask the following question: Suppose I mark a particular spot on this hypothetical terrain. Then I cross over the mark twice. The first time, I do so heading northward, noting the rate of change of eastward-facing slopes as I cross (that is,  $f_{xy}$ ). The second time, I do so heading eastward, noting the rate at which northward-facing slopes change as I cross (i.e.,  $f_{yx}$ ). Should these two rates of change be equal? There does not seem to be a particular reason why they should be, but experimenting with various formulas  $f(x, y)$  we find, nevertheless, that they often are.

**Example:**  $f(x, y) = \cos(x^2y)$

This phenomenon has much to do with our natural inclination to choose “nice” functions. In general,  $f_{xy}$  and  $f_{yx}$  are not equal. But, under the conditions of the following theorem, they are.

**Theorem: (The Mixed Derivative Theorem, p. 26)** If  $f(x, y)$  and its partial derivatives  $f_x$ ,  $f_y$ ,  $f_{xy}$  and  $f_{yx}$  are defined throughout an open region of the plane containing the point  $(x_0, y_0)$ , and are all continuous at  $(x_0, y_0)$ , then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

## Differentiability and Continuity

In MATH 161, we learn

- how to differentiate a function of a single variable

- at points of differentiability, the function
  - is also continuous.
  - looks (locally) like a straight line.

For functions of multiple variables, we have learned how to take partial derivatives, and what these partial derivatives represent. Unfortunately, existence of partial derivatives does not, by itself, imply continuity.

**Example:** For the function

$$f(x, y) := \begin{cases} 1, & \text{if } xy = 0, \\ 0, & \text{if } xy \neq 0, \end{cases}$$

the partial derivatives exist at  $(0, 0)$ . However,  $f$  is not continuous at  $(0, 0)$ . (The graph of this function is given on p. 725 of your text.)

We would like functions of multiple variables, like their single-variable counterparts, to be continuous whenever they are differentiable. In light of the previous example, we will require more of such a function than just “its partial derivatives exist” before we call it *differentiable*.

**Definition:** A function  $z = f(x, y)$  is said to be *differentiable at*  $(x_0, y_0)$ , a point in the domain of  $f$ , if  $f_x(x_0, y_0)$  and  $f_y(x_0, y_0)$  both exist, and  $\Delta z := f(x, y) - f(x_0, y_0)$  satisfies the equation

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y, \quad (1)$$

where

$$\Delta x := x - x_0, \quad \Delta y := y - y_0,$$

and  $\epsilon_1, \epsilon_2 \rightarrow 0$  as  $(x, y) \rightarrow (x_0, y_0)$ .

If we drop the terms in equation (1) that become more and more negligible as  $(x, y) \rightarrow (x_0, y_0)$  (the ones involving  $\epsilon_1$  and  $\epsilon_2$ ), then we obtain the approximation

$$\Delta z \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y, \quad (2)$$

or

$$f(x, y) - f(x_0, y_0) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

or

$$f(x, y) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

The right-hand side of this last version of the approximation is in the form

$$Ax + By + C.$$

Later in the course, we shall see that this is one form of the equation of a plane. Thus, the definition says that  $z = f(x, y)$  is differentiable at  $(x_0, y_0)$  if, locally speaking, the surface at the point looks like (is well-approximated by) the plane

$$f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

We know (from the example above) that existence of partial derivatives at the point  $(x_0, y_0)$  alone is not sufficient to guarantee that a function is differentiable there. However, the following theorem provides a stronger condition that guarantees it.

**Theorem:** Let  $f$  be a function of 2 variables whose partial derivatives  $f_x$  and  $f_y$  are continuous throughout an open region  $R$  of the plane. Then  $f$  is differentiable at each point of  $R$ .

Given our notion of differentiability, we may prove this analog to the theorem from MATH 161 relating differentiability and continuity.

**Theorem:** If a function  $f$  of two variables is differentiable at  $(x_0, y_0)$ , then  $f$  is continuous there.

## Differential Notation and Linear Approximation

For functions of one variable  $y = f(x)$ , we sometimes write  $dy = f'(x)dx$ . What does this mean?

- $dx$  is an independent variable (think of it like  $\Delta x$ )
- $dy$  is a dependent variable, a function of both  $x$  and  $dx$ .
- This “differential notation” is another way of writing the linear approximation to  $f$ .

Now, for the function  $z = f(x, y)$ , we may analogously write

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Compare this to equation (2).

**Example:** The volume of a right circular cylinder is given by  $v(r, h) = \pi r^2 h$ . Thus

$$dv = 2\pi r h dr + \pi r^2 dh.$$

Thus, if a cylinder of radius 2 in. and height 5 in. is deformed to a different cylinder, now of radius 1.98 in. and height 5.03 in., then the approximate change in volume is

$$2\pi(2)(5)(-0.02) + \pi(2)^2(0.03) = -0.8797$$

cubic inches. (The actual change is -0.8809 cubic inches.)

# MATH 162: Calculus II

## Framework for Thurs., Mar. 8

### Chain Rules

**Today's Goal:** To extend the chain rule to functions of multiple variables.

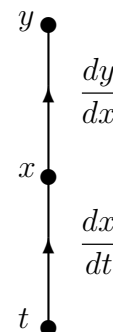
### Chain rule, single (independent) variable case

**Setting:**  $y$  is a function of  $x$ , while  $x$  is a function of  $t$ .  
More explicitly,  $y = y(x)$ , and  $x = x(t)$  (so  $y = y(x(t))$ ).

**Chain Rule:** 
$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

Note here that

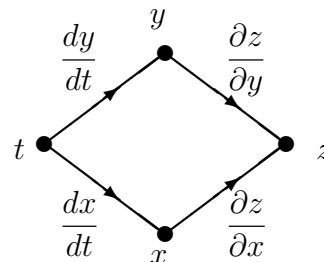
- $y$  is the (final) dependent variable.
- $t$  is the independent variable.
- $x$  is an intermediate variable.



### Many Multivariate Chain Rules

**Setting 1:**  $z = f(x, y)$ , with  $x = x(t)$ ,  $y = y(t)$

**Chain Rule:** 
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$



**Setting 2:**  $w = f(x, y, z)$ , with  $x = x(t)$ ,  $y = y(t)$ ,  $z = z(t)$

**Chain Rule:** 
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

**Setting 3:**  $z = f(x, y)$ , with  $x = x(u, v)$ ,  $y = y(u, v)$ ,

**Chain Rules:** 
$$\begin{cases} \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \end{cases}$$



## Another Look at Implicit Differentiation

Many problems from MATH 161 in which implicit differentiation was used involved equations which could be put in the form  $F(x, y) = 0$ . Assuming that this equation defines  $y$  implicitly as a function of  $x$  (an assumption that is generally true), then by the chain rule

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = F_x + F_y \frac{dy}{dx}.$$

This is the  $x$ -derivative of one side of the equation  $F(x, y) = 0$ . The  $x$ -derivative of the other side is, naturally, 0. Thus, we have

$$F_x + F_y \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

# MATH 162: Calculus II

## Framework for Fri., Mar. 9

### Vectors

**Today's Goal:** To understand vectors and be able to manipulate them.

## Vectors in 2 and 3 Dimensions

**Definition:** A *vector* is a directed line segment. If  $P$  and  $Q$  are points in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , then the directed line segment from the *initial* point  $P$  to the *terminal* point  $Q$  is denoted  $\overrightarrow{PQ}$ .

- **Vector names:** bold-faced letters (usually lower-case)  $\mathbf{v}$ , or letters with arrows  $\vec{v}$
- There are two things that distinguish a vector  $\mathbf{v}$ : its *length* and its *direction*. Thus, two directed line segments which are parallel, have the same length, and are oriented in the same direction (arrow pointing the same way) are considered to be the same (equal) even if their initial and terminal points are different.
- **Component form:** Given what was said above, any vector  $\mathbf{v}$  may be moved rigidly so as to make its initial point be the origin. Writing  $(v_1, v_2, v_3)$  for the resulting terminal point, we then say that  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ . This is called the *component form* of  $\mathbf{v}$ . The numbers  $v_1$ ,  $v_2$  and  $v_3$  are the *components* of  $\mathbf{v}$ .
- **Equality of vectors:** Two vectors are considered equal when they are equal in each component.
- If  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  then the *length* (or *magnitude*) of  $\mathbf{v}$ , denoted  $|\mathbf{v}|$ , is given by

$$|\mathbf{v}| := \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

The only vector whose length is zero is the one whose components are all zero. We call this the *zero vector*, denoting it by  $\mathbf{0}$ . The other vectors (the ones with nonzero length) are collectively referred to as *nonzero* vectors.

## Vector Operations

**Vector Addition:** If  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ , we define

$$\mathbf{u} + \mathbf{v} := \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$

**Scalar Multiplication:** If  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  and  $c$  is a real number (a *scalar*), then we define

$$c\mathbf{v} := \langle cv_1, cv_2, cv_3 \rangle.$$

Note that:

- Our definitions for vector addition and scalar multiplication are enough to give us the notion of *vector subtraction* as well, since we may think of  $\mathbf{u} - \mathbf{v}$  as

$$\mathbf{u} + (-1)\mathbf{v} = \langle u_1, u_2, u_3 \rangle + \langle -v_1, -v_2, -v_3 \rangle = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

- We make sense of an expression like  $\mathbf{v}/c$  (i.e., dividing a vector by a scalar) by thinking of it as  $(1/c)\mathbf{v}$  (i.e., the reciprocal of  $c$  multiplied by  $\mathbf{v}$ ). For any nonzero vector  $\mathbf{v}$ ,  $\mathbf{v}/|\mathbf{v}|$  is a vector whose length is 1, called the *direction* of  $\mathbf{v}$ .
- No attempt has been made to define any type of multiplication (not yet) nor division (never!) between two vectors.

## Unit Vectors

Any vector whose length is 1 is called a *unit vector*.

**Example:** For each  $\mathbf{v} \neq \mathbf{0}$ , the direction  $\frac{\mathbf{v}}{|\mathbf{v}|}$  of  $\mathbf{v}$  is a unit vector. Thus, in 2D, the vector  $\mathbf{v} = \langle -2, 5 \rangle$  has direction

$$\mathbf{d} = \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + 5^2}} = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$

We may then write  $\mathbf{v}$  as a product of its magnitude times its direction

$$\mathbf{v} = \sqrt{29}\mathbf{d}.$$

**Standard unit vectors** (the ones parallel to the coordinate axes):  $\mathbf{i} := \langle 1, 0, 0 \rangle$ ,  $\mathbf{j} := \langle 0, 1, 0 \rangle$ , and  $\mathbf{k} := \langle 0, 0, 1 \rangle$ .

Notice that, for  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ , it is the case that

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

# MATH 162: Calculus II

## Framework for Mon., Mar. 12

### Dot Products and Projections

**Today's Goal:** To define the dot product and learn of some of its properties and uses

## The Dot Product

**Definition:** For vectors  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ , we define the *dot product* of  $\mathbf{u}$  and  $\mathbf{v}$  to be

$$\mathbf{u} \cdot \mathbf{v} := u_1v_1 + u_2v_2 + u_3v_3.$$

Notes:

- The dot product  $\mathbf{u} \cdot \mathbf{v}$  is a scalar (number), not another vector.
- Properties
  1.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  2.  $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ .
  3.  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4.  $\mathbf{0} \cdot \mathbf{v} = 0$
  5.  $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$

**Theorem:** If  $\mathbf{u}$  and  $\mathbf{v}$  are nonzero vectors, then the angle  $\theta$  between them satisfies

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Note that, when  $\theta = \pi/2$ , the numerator on the right-hand side must be zero. This motivates the following definition.

## Orthogonality

**Definition:** Two vectors  $\mathbf{u}$  and  $\mathbf{v}$  are said to be *orthogonal* (or *perpendicular*) if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

**Example:** The zero vector  $\mathbf{0}$  is orthogonal to every other vector. In 2D, the vectors  $\langle a, b \rangle$  and  $\langle -b, a \rangle$  are orthogonal, since

$$\langle a, b \rangle \cdot \langle -b, a \rangle = (a)(-b) + (b)(a) = 0.$$

**Example:** Find an equation for the plane containing the point  $(1, 1, 2)$  and perpendicular to the vector  $\langle A, B, C \rangle$ .

## Projections

**Scalar component of  $\mathbf{u}$  in the direction of  $\mathbf{v}$ :**  $|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$

**Vector projection of  $\mathbf{u}$  onto  $\mathbf{v}$ :**

$$\text{proj}_{\mathbf{v}} \mathbf{u} := \left( \begin{array}{c} \text{scalar component of } \mathbf{u} \\ \text{in direction of } \mathbf{v} \end{array} \right) \left( \begin{array}{c} \text{direction} \\ \text{of } \mathbf{u} \end{array} \right) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v}.$$

## Work

The *work* done by a constant force  $\mathbf{F}$  acting through a displacement vector  $\mathbf{D} = \overrightarrow{PQ}$  is given by

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}| |\mathbf{D}| \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{F}$  and  $\mathbf{D}$ .

# MATH 162: Calculus II

## Framework for Tues., Mar. 13

### Cross Products

**Today's Goal:** To define the cross product and learn of some of its properties and uses

## The Cross Product

**Definition:** For nonzero, non-parallel 3D vectors  $\mathbf{u}$  and  $\mathbf{v}$ , we define the *cross product* of  $\mathbf{u}$  and  $\mathbf{v}$  to be

$$\mathbf{u} \times \mathbf{v} := (|\mathbf{u}||\mathbf{v}| \sin \theta) \mathbf{n},$$

where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ , and  $\mathbf{n}$  is a unit vector perpendicular to both  $\mathbf{u}$  and  $\mathbf{v}$ , and in the direction determined by the “right-hand rule.”

If either  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ , we define  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ . Similarly, if  $\mathbf{u}$  and  $\mathbf{v}$  are parallel, we take  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ .

Notes:

- There is no corresponding concept for 2D vectors.
- The dot product between two vectors produces a scalar. The cross product of two vectors yields another vector.
- Properties

1.  $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

2.  $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$

3.  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$

4. The cross product is not associative! This means that, in general, it is *not* the case that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \quad \text{and} \quad \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

are equal.

- The cross product  $\mathbf{u} \times \mathbf{v}$  may be computed from the following symbolic determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} := \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k},$$

where  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ .

## Applications

- $\mathbf{r} \times \mathbf{F}$  is the *torque* vector resulting from a force  $\mathbf{F}$  applied at the end of a lever arm  $\mathbf{r}$ .
- $|\mathbf{u} \times \mathbf{v}|$  (the length of the cross product  $\mathbf{u} \times \mathbf{v}$ ) is the area of a parallelogram determined by  $\mathbf{u}$  and  $\mathbf{v}$ .
- $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$  (the absolute value of the scalar  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ ) is the volume of the parallelepiped determined by  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$ .
- Finding normal vectors to planes.

**Example:** The vectors  $\mathbf{u} = \langle 1, 2, -1 \rangle$  and  $\mathbf{v} = \langle -2, 3, 1 \rangle$

- are not parallel,
- so they determine a family of parallel planes.

Find a vector that is normal to these planes. Then determine an equation for the particular one of these planes passing through the point  $(1, 1, 1)$ .

MATH 162: Calculus II  
Framework for Thurs., Mar. 15  
Vector Functions and Differential Calculus

**Today's Goal:** To understand parametrized curves and their derivatives.

In yesterday's lab, we called a set of continuous functions over a common interval  $I$

$$\begin{aligned}x &= x(t), \\y &= y(t), \quad t \in I, \\z &= z(t),\end{aligned}\tag{1}$$

a *parametrized curve*. ( $I$  may be a finite interval, like  $I = [a, b]$ , or one of infinite length.) Another name for a parametrized curve is *path*, as one may think of tracing out the location of a particle  $(x(t), y(t), z(t))$  at various  $t$ -values in  $I$ .

## Equations of lines in space

Whereas lines in the  $xy$ -plane may be characterized by a slope, lines in  $xyz$ -space are most easily characterized by a vector that is parallel to the line in question. Since there are infinitely many parallel vectors, there are infinitely many ways to describe a given line. Say we want the line passing through the point  $P = (x_0, y_0, z_0)$  parallel to the vector  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . We describe it parametrically. We might arbitrarily decide to associate the point  $P$  with the parameter value  $t = 0$ , and integer values of  $t$  correspond to integer leaps of length  $|\mathbf{v}|$ :

$$\begin{aligned}x &= x_0 + v_1t, \\y &= y_0 + v_2t, \quad -\infty < t < \infty. \\z &= z_0 + v_3t,\end{aligned}$$

**Example:** Find 3 possible parametrizations of the line through  $(2, -1, 4)$  in the direction of  $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ . Make one of these parametrizations be by arc length.

## The position vector

One might take the functions (1) and create from them a *vector function*, with  $x(t)$ ,  $y(t)$ , and  $z(t)$  as *component functions*:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Following the idea that the path (1) describes the locations of a moving particle,  $\mathbf{r}(t)$  is often called a *position vector*—that is, when drawn in *standard position* (i.e., with its initial point at the origin), the terminal point of  $\mathbf{r}(t)$  moves so as to trace out the curve.



**Example: Equation of a line in space, vector form.** For the line passing through the point  $P = (x_0, y_0, z_0)$  parallel to the vector  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ , we have the vector form

$$\mathbf{r}(t) = (x_0 + v_1t)\mathbf{i} + (y_0 + v_2t)\mathbf{j} + (z_0 + v_3t)\mathbf{k}.$$

## Limits and continuity of vector functions

While the following definition is not identical to the one given in the text, the two are logically equivalent.

**Definition:** Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  (so the component functions of  $\mathbf{r}(t)$  are  $x(t)$ ,  $y(t)$  and  $z(t)$ ). We say that

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$$

precisely when each corresponding limit of the component functions

$$\lim_{t \rightarrow t_0} x(t) = L_1, \quad \lim_{t \rightarrow t_0} y(t) = L_2, \quad \text{and} \quad \lim_{t \rightarrow t_0} z(t) = L_3$$

holds.

We say that  $\mathbf{r}(t)$  is continuous at  $t = t_0$  precisely when each of the component functions  $x(t)$ ,  $y(t)$  and  $z(t)$  are continuous at  $t = t_0$ .

**Example:** The vector function  $\mathbf{r}(t) = t/(t-1)^2\mathbf{i} + (\ln t)\mathbf{j}$  is continuous at all points  $t$  where its component functions  $x(t) = t/(t-1)^2$  and  $y(t) = \ln t$  are continuous—that is for  $t > 0$ . Thus,  $\lim_{t \rightarrow t_0} \mathbf{r}(t)$  exists whenever  $t_0 > 0$ .

## Derivatives of vector functions

**Definition:** A vector function  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  is *differentiable at  $t$*  if the limit

$$\mathbf{r}'(t) := \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

exists.

Notes:

- An equivalent definition to the one above would be that  $\mathbf{r}(t)$  is differentiable at a given  $t$ -value precisely when each of its component functions  $x(t)$ ,  $y(t)$  and  $z(t)$  are differentiable there. When this is so, we have

$$\frac{d\mathbf{r}}{dt} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

- If a vector function  $\mathbf{r}(t)$  is differentiable, then the derivative  $\mathbf{r}'(t)$  is itself another vector function, which may be differentiable as well. When this is so, we have

$$\frac{d^2\mathbf{r}}{d^2t} = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}.$$

- If the position vector function  $\mathbf{r}(t)$  is differentiable, then  $d\mathbf{r}/dt$  is the corresponding *velocity* vector function. What we call *speed* is actually the length  $|d\mathbf{r}/dt|$  of the velocity function.

If  $d\mathbf{r}/dt$  is differentiable, then we call  $d^2\mathbf{r}/dt^2$  the *acceleration* vector function.

- One check that we have defined dot and cross products between vectors in a useful fashion is whether they obey “product rules.” In fact, all of the rules for differentiation that hold for scalar functions, and are appropriate to apply to vector functions, still hold:

$$1. \quad \frac{d}{dt}\mathbf{C} = \mathbf{0} \quad (\text{constant function rule})$$

$$2. \quad \frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t) \quad (\text{constant multiple rule})$$

$$3. \quad \frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \quad (\text{product of scalar and vector fn.})$$

$$4. \quad \frac{d}{dt}\left[\frac{\mathbf{u}(t)}{f(t)}\right] = \frac{f'(t)\mathbf{u}(t) - f(t)\mathbf{u}'(t)}{[f(t)]^2} \quad (\text{quotient of vector and scalar fn.})$$

$$5. \quad \frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t) \quad (\text{sum and difference rules})$$

$$6. \quad \frac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \quad (\text{dot product rule})$$

$$7. \quad \frac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \quad (\text{cross product rule})$$

$$8. \quad \frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)) \quad (\text{chain rule})$$

MATH 162: Calculus II  
Framework for Fri., Mar. 16  
Vector Functions and Integral Calculus

**Today's Goal:** To understand how to integrate vector functions.

## Indefinite integrals

Recall: For scalar functions  $f(t)$ ,

- A function  $F$  is called an antiderivative of  $f$  on the interval  $I$  if  $F'(t) = f(t)$  at each point  $t \in I$ .
- Given any antiderivative  $F$  of  $f$  and any constant  $C$ ,  $F(t) + C$  is also an antiderivative of  $f$ .
- The indefinite integral

$$\int f(t) dt$$

stands for the set of all antiderivatives of  $f$ .

**For a vector function**  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ ,

- An antiderivative of  $\mathbf{r}(t)$  on an interval  $I$  is another vector function  $\mathbf{R}(t)$  for which  $\mathbf{R}'(t) = \mathbf{r}(t)$  at each  $t \in I$ .
- Finding an antiderivative of  $\mathbf{r}(t)$  comes down to finding antiderivatives for its component functions. That is, if  $F$ ,  $G$  and  $H$  are antiderivatives of  $f$ ,  $g$  and  $h$  on the interval  $I$ , then

$$\mathbf{R}(t) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}$$

is an antiderivative of  $\mathbf{r}(t)$ .

- Given any antiderivative  $\mathbf{R}(t)$  of  $\mathbf{r}(t)$  and any constant vector  $\mathbf{C}$ ,  $\mathbf{R}(t) + \mathbf{C}$  is also an antiderivative of  $\mathbf{r}(t)$ .
- The symbol

$$\int \mathbf{R}(t) dt$$

stands for the set of all antiderivatives of  $\mathbf{r}$ .

**Example:**  $\int \left[ \left( \frac{1}{1+t^2} \right) \mathbf{i} + (\sin t \cos t) \mathbf{j} + \left( \frac{t}{\sqrt{1+3t^2}} \right) \mathbf{k} \right] dt$

## Definite Integrals

**Definition:** Suppose that the component functions of  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  are all integrable over the interval  $[a, b]$  (true, say, if each of  $f$ ,  $g$  and  $h$  are continuous over that interval). Then we say the vector function  $\mathbf{r}(t)$  is *integrable over*  $[a, b]$ , and define its integral to be

$$\int_a^b \mathbf{r}(t) dt := \left( \int_a^b f(t) dt \right) \mathbf{i} + \left( \int_a^b g(t) dt \right) \mathbf{j} + \left( \int_a^b h(t) dt \right) \mathbf{k}.$$

Since the fundamental theorem of calculus holds for the components of  $\mathbf{r}(t)$ , it holds for  $\mathbf{r}(t)$  as well. Here we state just part II.

**Theorem:** If  $\mathbf{r}(t)$  is continuous at each point of the interval  $[a, b]$  and if  $\mathbf{R}$  is any antiderivative of  $\mathbf{r}$  on  $[a, b]$ , then

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(b) - \mathbf{R}(a).$$

### Application: Projectile motion

For projectiles near enough to sea level, we think of them as having constant acceleration  $\mathbf{a}(t) = -g\mathbf{k}$ , where  $g$  has the value  $9.8 \text{ m/s}^2$  or  $32 \text{ ft/s}^2$ . Since velocity is an antiderivative of acceleration, we may write

$$\int_0^t \mathbf{a} d\tau = -gt\mathbf{k} = \mathbf{v}(t) - \mathbf{v}(0),$$

or, abbreviating  $\mathbf{v}(0)$  by  $\mathbf{v}_0$ ,

$$\mathbf{v}(t) = \mathbf{v}_0 - gt\mathbf{k}.$$

The position  $\mathbf{r}(t)$  is an antiderivative of velocity, so

$$\int_0^t \mathbf{v}(\tau) d\tau = t\mathbf{v}_0 - \frac{1}{2}gt^2\mathbf{k} = \mathbf{r}(t) - \mathbf{r}(0),$$

or

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}_0 - \frac{1}{2}gt^2\mathbf{k},$$

where  $\mathbf{r}_0 := \mathbf{r}(0)$  is the initial position.

# MATH 162: Calculus II

## Framework for Mon., Mar. 26

### Quadric Surfaces

**Today's Goal:** To review how equations in three variables are graphed, and to identify special graphs known as quadric surfaces.

**What we already know:** A 2nd-order polynomial in  $x$  and  $y$  takes the general form

$$p(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Such a polynomial is called a *quadratic polynomial* (in  $x$  and  $y$ ).

**Some special cases**, usually treated in high school classes:

- **Case**  $B = C = 0$ ,  $A \neq 0$ :  $p(x, y) = Ax^2 + Dx + Ey + F$ 
  - The level sets of  $p$  are parabolas.
  - By symmetry of argument, the level sets are parabolas opening sideways when  $A = B = 0$  and  $C \neq 0$ .
- **Case**  $B^2 < 4AC$ :
  - Some level sets of  $p$  are ellipses.
  - When  $B = 0$  and  $A = C$ , the ellipses are actually circles.
- **Case**  $AC < 0$ : Almost all level sets of  $p$  are hyperbolas.

## Quadric Surfaces

Similar to the above, a quadratic polynomial in  $x$ ,  $y$  and  $z$  is a 2nd-order polynomial having general form

$$p(x, y, z) = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 + Gx + Hy + Iz + J. \quad (1)$$

- The graph of  $p$  would require 4 dimensions, but the level sets of  $p$  are *surfaces* in 3D.
- The solutions of the equation  $p(x, y, z) = k$  ( $k$  a fixed number) coincide with the  $k$ -level surface for the quadratic function  $p$ .

**Definition:** For a quadratic polynomial  $p$  in the form (1), the set of points  $(x, y, z)$  which satisfy the level surface equation  $p(x, y, z) = k$  is called a *quadric surface*.

- When the coefficient  $F = 0$  and at least one of  $D$ ,  $E$  or  $I$  is nonzero, the level surface equation may be manipulated algebraically to solve for  $z$  as a function of  $x$  and  $y$ . In these cases, what we learned about graphing functions of 2 variables still applies.

**Example:**  $9x^2 - y^2 - 4z = 0$  (hyperbolic paraboloid)

By symmetry of argument, the level surface equation  $p(x, y, z) = k$  can be written as a function if

- $A = 0$  and at least one of  $B$ ,  $D$  or  $G$  is nonzero, in which case  $x$  may be written as a function of  $y$  and  $z$ , or
- $C = 0$  and any one of  $B$ ,  $E$  or  $H$  is nonzero, in which case  $y$  may be written as a function of  $x$  and  $z$ .
- Even when the equation  $p(x, y, z) = k$  cannot be re-written as a function of two variables, a good way to get an idea of the graph of the level surface is to consider cross-sections:
  - slices by planes parallel to the  $xy$ -plane are the result of setting  $z = z_0$ .
  - slices by planes parallel to the  $xz$ -plane are the result of setting  $y = y_0$ .
  - slices by planes parallel to the  $yz$ -plane are the result of setting  $x = x_0$ .

**Examples:**

$$9x^2 + y^2 - 4z^2 = 1 \quad (\text{an hyperboloid in one sheet})$$

$$9x^2 + y^2 - 4z^2 = 0 \quad (\text{an elliptic cone})$$

$$9x^2 - y^2 - 4z^2 = 1 \quad (\text{an hyperboloid in two sheets})$$

$$9x^2 + y^2 + 4z^2 = 1 \quad (\text{an ellipsoid})$$

# MATH 162: Calculus II

## Framework for Tues., Mar. 27

### More about Lines and Planes in Space

**Today's Goal:** To review how lines and planes in space are represented, and use these notions to derive some useful formulas and algorithms involving points, lines and planes.

## Lines and Planes

We have derived the following representations.

- **Lines.** The line through point  $P = (x_0, y_0, z_0)$  parallel to  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$   
**component form:** 
$$\begin{aligned} x &= x_0 + v_1t, \\ y &= y_0 + v_2t, \\ z &= z_0 + v_3t, \end{aligned} \quad -\infty < t < \infty,$$
  
**vector form:** 
$$\mathbf{r}(t) = (x_0 + v_1t)\mathbf{i} + (y_0 + v_2t)\mathbf{j} + (z_0 + v_3t)\mathbf{k}, \quad -\infty < t < \infty.$$

- **Planes.** The plane through point  $P = (x_0, y_0, z_0)$  perpendicular to  $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$$\mathbf{n} \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0, \quad \text{or} \quad ax + by + cz = d,$$

where  $d = ax_0 + by_0 + cz_0$ .

## Formulas and Algorithms for Lines and Planes

- **Distance from a point  $S$  to a line  $L$ .**

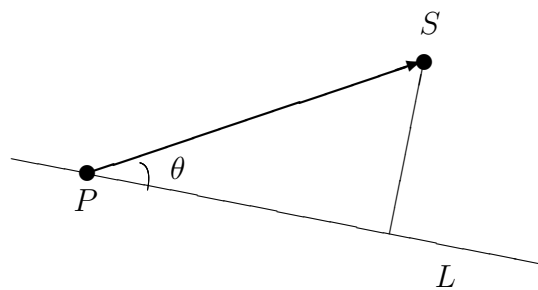
Keys to a formula:

1. Our distance is  $|\overrightarrow{PS}| \sin \theta$ , where  $P$  is any point on line  $L$ .
2. For two vectors  $\mathbf{u}$  and  $\mathbf{v}$ ,  $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$ .

From these we get

$$|\overrightarrow{PS}| \sin \theta = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|},$$

where  $\mathbf{v}$  is any vector parallel to line  $L$ .



- **Distance from a point  $S$  to a plane** containing the point  $P$  with normal vector  $\mathbf{n}$ .

Keys to a formula:

1. Our distance is  $|\overrightarrow{PS}|\cos\theta|$ , where  $\theta$  is the angle between  $\overrightarrow{PS}$  and  $\mathbf{n}$ .
2. If  $\theta$  is the angle between vectors  $\mathbf{u}$  and  $\mathbf{v}$ , then  $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$ .

Thus, we get

$$|\overrightarrow{PS}|\cos\theta| = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}.$$

- **Angle between two planes.**

**Definition:** The *angle between planes* is taken to be the angle  $\theta \in [0, \pi/2]$  between normal vectors to the planes.

By this definition, if  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are normal vectors to the two planes, then the angle between the planes is

$$\theta = \begin{cases} \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right), & \text{if } \mathbf{n}_1 \cdot \mathbf{n}_2 \geq 0, \\ \pi - \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right), & \text{if } \mathbf{n}_1 \cdot \mathbf{n}_2 < 0. \end{cases}$$

- **Line of intersection between two non-parallel planes.**

It should not be difficult to find a point on the desired line. If the two planes have equations  $a_1x + b_1y + c_1z = d_1$  and  $a_2x + b_2y + c_2z = d_2$ , then it is quite likely the line of intersection will eventually pass through a point  $P$  where the  $x$ -coordinate is zero. Assuming this is so, we may do the usually steps of solving the simultaneous equations in 2 unknowns

$$\begin{aligned} b_1y + c_1z &= d_1 \\ b_2y + c_2z &= d_2 \end{aligned}$$

for the corresponding  $y$  and  $z$  coordinates of this point. (If the solution process fails to yield corresponding  $y$  and  $z$  coordinates, one can instead look for the point  $P$  for which the  $y$  or, alternatively, the  $z$ -coordinate is zero.)

Once a point  $P$  on our line of intersection is found, we next need a vector that is parallel to our line. Such a vector would be perpendicular to normal vectors to both planes, and so could be any multiple of

$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$



MATH 162: Calculus II  
Framework for Thurs., Mar. 29–Fri. Mar. 30  
The Gradient Vector

**Today's Goal:** To learn about the gradient vector  $\vec{\nabla}f$  and its uses, where  $f$  is a function of two or three variables.

## The Gradient Vector

Suppose  $f$  is a differentiable function of two variables  $x$  and  $y$  with domain  $R$ , an open region of the  $xy$ -plane. Suppose also that

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in I,$$

(where  $I$  is some interval) is a differentiable vector function (parametrized curve) with  $(x(t), y(t))$  being a point in  $R$  for each  $t \in I$ . Then by the chain rule,

$$\begin{aligned} \frac{d}{dt}f(x(t), y(t)) &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= [f_x \mathbf{i} + f_y \mathbf{j}] \cdot [x'(t)\mathbf{i} + y'(t)\mathbf{j}] \\ &= [f_x \mathbf{i} + f_y \mathbf{j}] \cdot \frac{d\mathbf{r}}{dt}. \end{aligned} \tag{1}$$

**Definition:** For a differentiable function  $f(x_1, \dots, x_n)$  of  $n$  variables, we define the *gradient vector of  $f$*  to be

$$\vec{\nabla}f := \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle.$$

Remarks:

- Using this definition, the total derivative  $df/dt$  calculated in (1) above may be written as

$$\frac{df}{dt} = \vec{\nabla}f \cdot \mathbf{r}'.$$

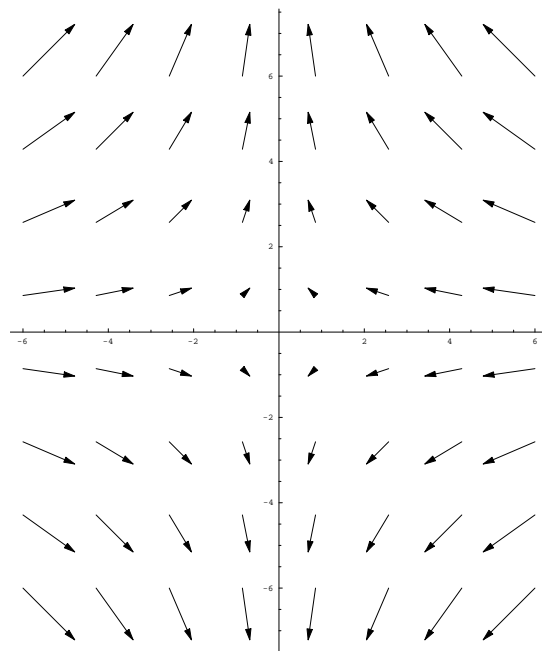
In particular, if  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ ,  $t \in (a, b)$  is a differentiable vector function, and if  $f$  is a function of 3 variables which is differentiable at the point  $(x_0, y_0, z_0)$ , where  $x_0 = x(t_0)$ ,  $y_0 = y(t_0)$ , and  $z_0 = z(t_0)$  for some  $t_0 \in (a, b)$ , then

$$\left. \frac{df}{dt} \right|_{t=t_0} = \vec{\nabla}f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0).$$

- If  $f$  is a function of 2 variables, then  $\vec{\nabla}f$  has 2 components. Thus, while the graph of such an  $f$  lives in 3D,  $\vec{\nabla}f$  should be thought of as a vector in the plane.
- If  $f$  is a function of 3 variables, then  $\vec{\nabla}f$  has 3 components, and is a vector in 3-space.

Speaking more generally, we may say that while a function  $f(x_1, \dots, x_n)$  of  $n$  variables requires  $n$  inputs to produce a single (numeric) output, the corresponding gradient  $\vec{\nabla} f$  produces from those same  $n$  inputs a *vector* with  $n$  components. Objects which assign to each  $n$ -tuple input an  $n$ -vector output are known as *vector fields*. The gradient is an example of a vector field.

**Example:** For  $f(x, y) = y^2 - x^2$ , we have  $\text{dom}(f) = \mathbb{R}^2$  and  $\vec{\nabla} f(x, y) = -2x\mathbf{i} + 2y\mathbf{j}$ . Selecting any point  $(x, y)$  in the plane, we may choose to draw  $\vec{\nabla} f(x, y)$  not as a vector in standard position, but rather one with initial point  $(x, y)$ , obtaining the picture at right.



- **Properties of the gradient operator:** If  $f, g$  are both differentiable functions of  $n$  variables on an open region  $R$ , and  $c$  is any real number (constant), then

1.  $\vec{\nabla}(cf) = c\vec{\nabla} f$  (constant multiple rule)

2.  $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$  (sum/difference rules)

3.  $\vec{\nabla}(fg) = g(\vec{\nabla} f) + f(\vec{\nabla} g)$  (product rule)

4.  $\vec{\nabla} \left( \frac{f}{g} \right) = \frac{g(\vec{\nabla} f) - f(\vec{\nabla} g)}{g^2}$  (quotient rule)

## Directional Derivatives

Let  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$  be a unit vector (i.e.,  $|\mathbf{u}| = 1$ ), and let  $\mathbf{r}(t)$  be the vector function

$$\mathbf{r}(s) = (x_0 + su_1)\mathbf{i} + (y_0 + su_2)\mathbf{j},$$

which parametrizes the line through  $P = (x_0, y_0)$  parallel to  $\mathbf{u}$  in such a way that the “speed”  $|\mathbf{dr}/dt| = |\mathbf{u}| = 1$ . We make the following definition.

**Definition:** For a function  $f$  of two variables that is differentiable at  $(x_0, y_0)$ , we define the *directional derivative of  $f$  at  $(x_0, y_0)$  in the direction  $\mathbf{u}$*  to be

$$D_{\mathbf{u}}f(x_0, y_0) := \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} = \left( \frac{df}{ds} \right)_{\mathbf{u}, P} = \vec{\nabla} f(x_0, y_0) \cdot \mathbf{u}.$$

Remarks:

- **Special cases:** the partial derivatives  $f_x, f_y$  themselves are derivatives in the directions  $\mathbf{i}, \mathbf{j}$  respectively.

$$D_{\mathbf{i}}f = \vec{\nabla}f \cdot \mathbf{i} = f_x, \quad \text{and} \quad D_{\mathbf{j}}f = \vec{\nabla}f \cdot \mathbf{j} = f_y.$$

- For  $f$  a function of 2 variables, the direction of  $\vec{\nabla}f(x, y)$  (namely  $\vec{\nabla}f(x, y)/|\vec{\nabla}f(x, y)|$ ) is the direction of maximum increase, while  $-\vec{\nabla}f(x, y)/|\vec{\nabla}f(x, y)|$  is the direction of maximum decrease.

Proof: For any unit vector  $\mathbf{u}$ ,

$$D_{\mathbf{u}}f(x, y) = \vec{\nabla}f(x, y) \cdot \mathbf{u} = |\vec{\nabla}f(x, y)||\mathbf{u}| \cos \theta = |\vec{\nabla}f(x, y)| \cos \theta,$$

where  $\theta$  is the angle between  $\vec{\nabla}f(x, y)$  and  $\mathbf{u}$ . This directional derivative is

largest when  $\theta = 0$  (i.e., when  $\mathbf{u} = \vec{\nabla}f/|\vec{\nabla}f|$ ) and smallest when  $\theta = \pi$ .

- The notion of directional derivative extends naturally to functions of 3 or more variables.

## The Gradient and Level Sets

Suppose  $f(x, y)$  is differentiable at the point  $P = (x_0, y_0)$ , and let  $k = f(x_0, y_0)$ . Then the  $k$ -level curve of  $f$  contains  $(x_0, y_0)$ . Suppose that we have a parametrization of a section of this level curve containing the point  $(x_0, y_0)$ . That is, let

- $x(t)$  and  $y(t)$  be differentiable functions of  $t$  in an open interval  $I$  containing  $t = 0$ ,
- $x_0 = x(0)$  and  $y_0 = y(0)$ , and
- $f(x(t), y(t)) = k$  for  $t \in I$  (that is, the parametrization gives at least a small part of the  $k$ -level curve of  $f$ —a part that contains the point  $P$ ).

Because we are parametrizing a level curve of  $f$ , it follows that  $df/dt = 0$  for  $t \in I$ . In particular,

$$0 = \left. \frac{df}{dt} \right|_{t=t_0} = \vec{\nabla}f(x_0, y_0) \cdot [x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j}].$$

This shows that the gradient vector at  $P$  is orthogonal to the level curve of  $f$  (or the tangent line to the level curve) through  $P$ . This is true at all points  $P$  where  $f$  is differentiable. That this result may be generalized to higher dimensions is motivation for the definition of a tangent plane.

## Tangent Planes

**Definition:** Let  $f(x, y, z)$  be differentiable at a point  $P = (x_0, y_0, z_0)$  contained in the level surface  $f(x, y, z) = k$ . We define the *tangent plane to this level surface of  $f$  at  $P$*  to be the plane containing  $P$  normal to  $\vec{\nabla} f(x_0, y_0, z_0)$ .

**Example:** Suppose  $f(x, y, z)$  is a differentiable function at the point  $P = (x_0, y_0, z_0)$  lying on the level surface  $f(x, y, z) = k$ . Derive a formula for the equation of the tangent plane to this level surface of  $f$  at  $P$ . Then use it to write the equation of the tangent plane to the quadric surface

$$f(x, y, z) = x^2 + 3y^2 + 2z^2 = 6$$

at the point  $(1, 1, 1)$ .

**Example:** Suppose  $z = f(x, y)$  is a differentiable function at the point  $P = (x_0, y_0)$ . Derive a formula for the equation of the tangent plane to the surface  $z = f(x, y)$  at the point  $P = (x_0, y_0, f(x_0, y_0))$ . Use it to get the equation of the tangent plane to  $z = 2x^2 - y^2$  at the point  $(1, 3, -7)$ .

MATH 162: Calculus II  
Framework for Wed., Apr. 4  
Unconstrained Optimization of Functions of 2 Variables

**Today's Goal:** To be able to locate and classify local extrema for functions of two variables.

**Definition:** Suppose the domain of  $f(x, y)$  includes the point  $(a, b)$ .

1.  $f(a, b)$  is called a *local maximum* (or *relative maximum*) value of  $f$  if  $f(a, b) \geq f(x, y)$  for all points from  $\text{dom}(f)$  contained in some open disk (an open disk of some positive, though perhaps quite small, radius) centered at  $(a, b)$ .
2.  $f(a, b)$  is called a *local minimum* (or *relative minimum*) value of  $f$  if  $f(a, b) \leq f(x, y)$  for all points from  $\text{dom}(f)$  contained in some open disk centered at  $(a, b)$ .

Remarks:

- As with functions of a single variable (think of the absolute value function), local extrema (maxima or minima) of functions  $f$  of two variables may occur at points where  $f$  is not differentiable.
- When an extremum occurs at an interior point  $(a, b)$  of  $\text{dom}(f)$  where  $f$  is differentiable, one would expect  $f$  to have a horizontal tangent plane there. The equation for the tangent plane to  $z = f(x, y)$  at a point  $(x_0, y_0)$  where  $f$  is differentiable is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0,$$

or

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

while the equation of a horizontal plane (one parallel to the  $xy$ -plane) is  $z = \text{constant}$ . We may, therefore, conclude:

**Theorem:** If  $f(x, y)$  has a local extremum at an interior point  $(a, b)$  of  $\text{dom}(f)$ , and if the partial derivatives of  $f$  exist there, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

This motivates the following definition.

**Definition:** Let  $f$  be a function of two variables. An interior point of  $\text{dom}(f)$  where

- (i) both  $f_x$  and  $f_y$  are zero, or
- (ii) at least one of  $f_x, f_y$  does not exist

is called a *critical point* of  $f$ .

## Classifying Critical Points

Just as with functions of one variable, not all critical points of  $f(x, y)$  correspond to a local extremum. On pp. 757–759 of the text, Figures 12.37 and 12.41 depict situations in which  $(0, 0)$  is a critical point corresponding to an extremum; Figure 12.40 depicts situations in which  $(0, 0)$  is the location of a saddle point.

**Definition:** Suppose  $f(x, y)$  is a differentiable function with critical point  $(a, b)$ . If every open disk centered at  $(a, b)$  contains both domain points  $(x, y)$  for which  $f(x, y) > f(a, b)$  and domain points  $(x, y)$  for which  $f(x, y) < f(a, b)$ , then  $f$  is said to have a *saddle point* at  $(a, b)$ .

With functions of a single variable, we have several tests (the First Derivative Test and the Second Derivative Test) for determining when a critical point corresponds to a local extremum. The following theorem provides a test for those critical points of type (i) for which  $f$  is twice continuously differentiable throughout a disk surrounding the critical point.

**Theorem:** Suppose that  $f(x, y)$  and its first and 2nd partial derivatives are continuous throughout a disk centered at  $(a, b)$ , and that  $\vec{\nabla} f(a, b) = \mathbf{0}$ . Let  $D$  be given by the following two-by-two determinant:

$$D(x, y) := \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{xy}(x, y) & f_{yy}(x, y) \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

Then

- (i)  $f$  has a local maximum at  $(a, b)$  if  $f_{xx}(a, b) < 0$  and  $D(a, b) > 0$ .
- (ii)  $f$  has a local minimum at  $(a, b)$  if  $f_{xx}(a, b) > 0$  and  $D(a, b) > 0$ .
- (iii)  $f$  has a saddle point at  $(a, b)$  if  $D(a, b) < 0$ .

If  $D(a, b) = 0$ , or if  $D(a, b) > 0$  and  $f_{xx}(a, b) = 0$ , then this test fails to classify the critical point  $(a, b)$ .

### Examples:

$$f(x, y) = x^3y + 12x^2 - 8y$$

$$f(x, y) = \frac{x^2y^2 - 8x + y}{xy}$$

$$f(x, y) = xy(1 - x - y)$$

MATH 162: Calculus II  
Framework for Mon., Apr. 9  
Constrained Optimization of Functions of 2 Variables

**Today's Goal:** To be able to find absolute extrema for functions of two variables on closed and bounded domains.

**Definition:** A function  $f$  of two variables is said to have

1. a *global maximum* (or *absolute maximum*) at the point  $(a, b) \in \text{dom}(f)$  if  $f(a, b) \geq f(x, y)$  for all points  $(x, y)$  in  $\text{dom}(f)$ .
2. a *global minimum* (or *absolute minimum*) at the point  $(a, b) \in \text{dom}(f)$  if  $f(a, b) \leq f(x, y)$  for all points  $(x, y)$  in  $\text{dom}(f)$ .

The value  $f(a, b)$  is correspondingly called the *global* (or *absolute*) *maximum or minimum* value of  $f$ .

Before we embark on a process of looking for absolute extrema, it would be nice to know that what we are looking for is out there to be found. The following theorem supplies such an assurance in the special case that the domain of  $f$  is closed and bounded.

**Theorem:** (*Extreme Value Theorem*) Suppose  $f(x, y)$  is a continuous function on a closed and bounded region  $R$  of the  $xy$ -plane. Then there exist points  $(a, b)$  and  $(c, d)$  in  $R$  for which

$$f(a, b) \geq f(x, y) \qquad f(c, d) \leq f(x, y)$$

for all  $(x, y)$  in  $R$ .

Remarks:

- In some simple cases, it is even possible that  $(a, b)$  and  $(c, d)$  from the theorem are the same point.
- In the theorem, we would call “continuity of  $f$  over a closed, bounded region  $R$ ” as sufficient conditions to guarantee that  $f$  reaches maximum and minimum values in  $R$ . It is possible for  $f$  to attain such extrema even if one or both of these conditions (the “continuity” or the “closed and boundedness of  $R$ ”) is not in place.

**Examples:**

1. Find the maximum value of  $f(x, y) = 49 - x^2 - y^2$  along the line  $x + 3y = 10$ .

2. Find the absolute extrema for  $f(x, y) = x^2 - y^2 - 2x + 4y$  on the region of the  $xy$ -plane bounded below by the  $x$ -axis, above by the line  $y = x + 2$ , and on the right by the line  $x = 2$ .
3. Find the absolute extrema for  $f(x, y) = x^2 + 2y^2$  on the closed disk  $x^2 + y^2 \leq 1$ .
4. Find the maximum of  $f(x, y, z) = 36 - x^2 - y^2 - z^2$  subject to the constraint  $x + 4y - z = 21$ .
5. Find the point on the graph of  $z = x^2 + y^2 + 10$  closest to the plane  $x + 2y - z = 0$ .



MATH 162: Calculus II  
Framework for Wed., Apr. 11  
Double and Iterated Integrals, Rectangular Regions

**Today's Goal:** To understand the meaning of double integrals over bounded rectangular regions  $R$  of the plane, and to be able to evaluate such integrals.

**Important Note:** In conjunction with this framework, you should look over Section 13.1 of your text.

## Riemann Sums

We assume that  $f(x, y)$  is a function of 2 variables, and that  $R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$  (i.e.,  $R$  is some bounded rectangular region of the plane whose sides are parallel to the coordinate axes). Suppose we

- divide  $R$  up into  $n$  smaller rectangles, labeling them  $R_1, R_2, \dots, R_n$ ,
- choose, from each rectangle  $R_k$ , some point  $(x_k, y_k)$ , and
- use the symbol  $\Delta A_k$  to denote the area of rectangle  $R_k$ .

Then the sum

$$\sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

is called a *Riemann sum* of  $f$  over the region  $R$ .

**Definition:** The collection of smaller rectangles  $R_1, \dots, R_n$  is called a *partition*  $P$  of  $R$ . The maximum, taken over all lengths and widths of these rectangles, is called the *norm* of the partition  $P$ , and is denoted by  $\|P\|$ .

The function  $f$  is said to be *integrable over*  $R$  if the limit

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k,$$

taken over all partitions  $P$  of  $R$ , exists. The value of this limit, called the *double integral of*  $f$  over  $R$ , is denoted by

$$\iint_R f(x, y) dA.$$

The following theorem tells us that, for certain functions, integrability is assured

**Theorem:** Suppose  $f(x, y)$  is a continuous function over the closed and bounded rectangle  $R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$  in the plane. Then  $f$  is integrable over  $R$ .

## Remarks

- The double integral of  $f$  over  $R$  is a definite integral, and has a numeric value.
- When  $f(x, y)$  is a nonnegative function,  $\iint_R f(x, y) dA$  may be interpreted as the volume under the surface  $z = f(x, y)$  over the region  $R$  in the  $xy$ -plane.
- When  $f(x, y)$  is a constant function (i.e.,  $f$  has the same value, say  $c$ , for each input point  $(x, y)$ ), then

$$\iint_R f(x, y) dA = c \cdot \text{Area}(R).$$

## Iterated Integrals and Fubini's Theorem

Except in the very special case of constant functions  $f$ , the definition of  $\iint_R f(x, y) dA$  does not, by itself, provide us with much help for evaluating a double integral. (If you are viewing this document on the web, click [here](#) for an alternate point of view, expressed by Peter A. Lindstrom of Genesee Community College.) However, the following theorem indicates that the double integral may be evaluated as an *iterated integral*.

**Theorem:** (Fubini) Suppose  $f(x, y)$  is continuous throughout the rectangle  $R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$ . Then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

**Examples:** Evaluate the double integral, with  $R$  as specified.

1.  $\iint_R (2x + x^2 y) dA$ , where  $R : -2 \leq x \leq 2, -1 \leq y \leq 1$ .

2.  $\iint_R y \sin(xy) dA$ , where  $R : 1 \leq x \leq 2, 0 \leq y \leq \pi$ .

MATH 162: Calculus II  
Framework for Fri., Apr. 13  
Double Integrals, General Regions

**Today's Goal:** To understand the meaning of double integrals over more general bounded regions  $R$  of the plane, and to be able to evaluate such integrals.

**Important Note:** In conjunction with this framework, you should look over Section 13.2 of your text.

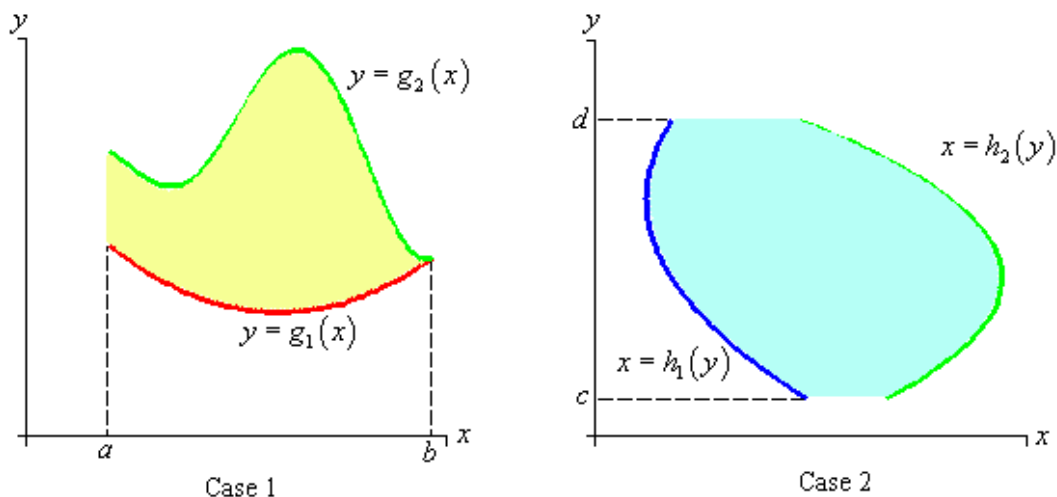
### Double Integrals as Iterated Integrals: General Treatment

**Q:** What if we seek  $\iint_R f(x, y) dA$  when  $R$  is not a rectangle whose sides are parallel to the coordinate axes?

**A1:** If  $R$  is a “nice enough” region (and, for our study, it will be), we can, once again, define  $\iint_R f(x, y) dA$  in terms of Riemann sums. The twist here is that, for any given partition of  $R$ , the rectangles will only partially fill up  $R$ .

We will not pursue this train of thought further.

**A2:** Use a more general form of Fubini's theorem. You can see the formal statement on p. 792 of your text. It deals with 2 cases (pictured):



**Case 1:** the upper and lower boundaries of  $R$  each are functions of  $x$  on a common interval; that is, a region  $R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$ . Then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

**Example:** Evaluate  $\iint_R (x + 2y) dA$  over the region  $R$  that lies between the parabolas  $y = 2x^2$  and  $y = 1 + x^2$ .

**Case 2:** the left and right boundaries of  $R$  each are functions of  $y$  on a common interval; that is, a region  $R : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$ . Then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

**Example:** Set up an integral for a function  $f(x, y)$  over the region bounded by the  $y$ -axis and the curve  $x + y^2 = 1$ .

# MATH 162: Calculus II

## Framework for Mon., Apr. 16

### Applications of Double Integrals

**Today's Goal:** To use double integrals meaningfully in solving problems.

**Important Note:** In conjunction with this framework, you should look over Section 13.3 of your text.

## Area

If  $R$  is a bounded region of the plane, then  $\iint_R dA$  gives the area of  $R$ . (This is because the volume under the curve  $z = 1$  over the region  $R$ , while it has different units, is the same as the area of  $R$ .)

## Average Value of a Function

For  $y = f(x)$  (a function of one variable), we defined the average value of  $f$  over the interval  $[a, b]$  to be

$$\frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{\text{length of interval } [a, b]} \int_a^b f(x) dx.$$

Similarly,

**Definition:** If  $f(x, y)$  is integrable over a region  $R$  of the plane, then the *average value* of  $f$  over  $R$  is defined to be

$$\frac{1}{\text{Area}(R)} \iint_R f(x, y) dA.$$

## Integral of a Density

Some functions give the amount of something per unit of measurement, as in

- $f(x)$  is the number of grams per unit length in an (idealized) 1-dimensional string.
- $f(x, y)$  is the number of grams per unit area in an (idealized) 2-dimensional plate.
- $f(x, y, z)$  is the number of molecules per unit volume of a certain gas.

Such functions are collectively known as *densities*. The examples above are 1, 2 and 3-dimensional densities respectively.

If  $f(x, y)$  gives the density (2-dimensional) of something at each point  $(x, y)$  in the region  $R$ , then  $\iint_R f(x, y) dA$  is the total of that substance found in  $R$ .

MATH 162: Calculus II  
Framework for Tues., Apr. 17  
Triple Integrals, Rectangular Coordinates

**Today's Goal:** To be able to set up and evaluate triple integrals.

**Important Note:** In conjunction with this framework, you should look over Section 13.5 of your text.

## Defining Triple Integrals

Suppose

- $D$  is a “nice” bounded region in 3-dimensional space.
- We subdivide  $D$ , creating a partition  $P$  of  $D$ , where  $P$  consists of  $n$  “boxes” wholly contained in  $D$ .
- In the  $k$ th box ( $1 \leq k \leq n$ ), we choose a point  $(x_k, y_k, z_k)$ .

We then look at sums of the form

$$\sum_{k=1}^n f(x_k, y_k, z_k) \Delta V_k,$$

where  $\Delta V_k$  denotes the volume of the  $k$ th box.

As with Riemann sums over partitions of regions of the plane, there are many functions and regions  $D$  for which the limit

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta V_k$$

exists, in which case we say that  $f$  is *integrable* over  $D$ . This limit is denoted by

$$\iiint_D f(x, y, z) dV,$$

read as the *triple integral* of  $f$  over  $D$ .

# Comparisons to Double Integrals

## 1. Evaluation.

### Double integrals:

Here our principle tool is Fubini's Theorem. We have two cases.

**Case:** 
$$\iint_R g(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} g(x, y) dy dx.$$

Here,  $a$  and  $b$  reflect the lowest and highest of a continuum of  $x$ -values encountered in the region  $R$ , while the lower and upper boundaries of  $R$  may be identified as functions of  $x$ .

**Case:** 
$$\iint_R g(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} g(x, y) dx dy.$$

Here,  $c$  and  $d$  reflect the lowest and highest of a continuum of  $y$ -values encountered in the region  $R$ , while the left and right boundaries of  $R$  may be identified as functions of  $y$ .

For most regions  $R$ , either case is applicable (sometimes one of the options requires a sum of integrals instead of a single one), meaning that the double integral may be written in either of *two orders* (i.e., either with  $y$  as the inner integral, as in  $dy dx$ , or in the order  $dx dy$ ).

### Triple integrals:

Given a bounded region  $D$  of 3-dimensional space, the triple integral  $\iiint_D f(x, y, z) dV$  may be written as an iterated integral in *six different orders*. Here are *two of the possibilities*:

- $$\iiint_D f(x, y, z) dV = \int_c^d \int_{g_1(y)}^{g_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} f(x, y, z) dx dz dy.$$

Here,  $c$  and  $d$  are the lowest and highest in a continuum of  $y$ -values encountered as one passes through the region  $D$ . For any fixed  $y \in [c, d]$ , we imagine a 2-dimensional (planar) region that results from slicing through  $D$  with a plane parallel to the  $xz$ -plane. This planar region has a starting and ending  $z$ -value, given by  $g_1(y)$  and  $g_2(y)$  respectively. At the inner-most level (the inner-most integral, which is in  $x$ ),  $y$  and  $z$  are held fixed while  $x$  is allowed to vary. The interval of possible  $x$ -values starts at  $h_1(y, z)$  and ends at  $h_2(y, z)$ .

- $$\iiint_D f(x, y, z) dV = \int_r^s \int_{g_1(z)}^{g_2(z)} \int_{h_1(x,z)}^{h_2(x,z)} f(x, y, z) dy dx dz.$$

The explanation of our region is similar to the above, but this time  $r$  and  $s$  represent lowest and highest  $z$ -values encountered in  $D$ ; for a fixed  $z$ ,  $g_1(z)$  and  $g_2(z)$  give lowest and highest  $x$ -values; for both  $x$  and  $z$  fixed,  $h_1(x, z)$  and  $h_2(x, z)$  give lowest and highest  $y$ -values.

## 2. Interpretations.

### (a) Areas, volumes and higher.

#### Double integrals:

When  $g(x, y)$  is nonnegative, the double integral  $\iint_R f(x, y) dA$  gives the volume under the surface  $z = g(x, y)$  over the region  $R$  of the  $xy$ -plane. If  $g$  changes sign in the region  $R$ , then  $\iint_R g(x, y) dA$  represents a *difference of volumes*.

A special case is when  $g(x, y) \equiv 1$ . As we have seen  $\iint_R g(x, y) dA = \iint dA$  gives the *area* of  $R$  (numerically equal to the volume under a surface over  $R$  whose height is uniformly 1).

#### Triple integrals:

When  $f(x, y, z)$  is nonnegative, we can be sure that  $\iiint_D f(x, y, z) dV$  is non-negative as well. But since the graph of  $w = f(x, y, z)$  is 4-dimensional, we would have to think of this value as a type of 4-dimensional volume (or difference of volumes, if  $f$  changes sign in  $D$ ).

When  $f(x, y, z) \equiv 1$ , then  $\iiint_D f(x, y, z) dV = \iiint dV$  gives the *volume* of the 3-dimensional region  $D$ .

### (b) Average values.

The average value of  $g(x, y)$  over a region  $R$  of the  $xy$ -plane was defined to be  $\iint_R g(x, y) dA / \iint_R dA$ . Similarly, we define the average value of  $f(x, y, z)$  over a region  $D$  of 3-dimensional space to be  $\iiint_D f(x, y, z) dV / \iiint_D dV$ .

### (c) Density integrals.

When  $g(x, y)$  gives the amount of a substance per unit area, then  $\iint_R g(x, y) dA$  tallies the amount of that substance found in a region  $R$  of the  $xy$ -plane. Similarly, when  $f(x, y, z)$  gives the amount of a substance per unit volume, then  $\iiint_D f(x, y, z) dV$  tallies the amount of that substance found in a region  $D$  of 3D space.

## Examples:

1. Evaluate  $\iiint_D z dV$  over the region enclosed by the three coordinate planes and the plane  $x + y + z = 1$ .
2. Find the average  $z$ -value in the region from problem 1.
3. Find limits of integration for  $\iiint_D \sqrt{x^2 + z^2} dy dz dx$  where  $D$  is the region bounded by the paraboloid  $y = x^2 + z^2$  and the plane  $y = 4$ .
4. Write a triple integral for  $f(x, y, z)$  over the region bounded by the ellipsoid  $9x^2 + 4y^2 + z^2 = 1$ .
5. What solid is it for which the iterated triple integral  $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dz dx dy$  gives its volume. What do other iterated triple integrals for the same expression look like?



MATH 162: Calculus II  
Framework for Thurs., Apr. 19  
Polar Coordinates

**Today's Goal:** To understand the use of polar coordinates for specifying locations on the plane.

**Important Note:** In conjunction with this framework, you should look over Section 9.1 of your text.

## Coordinate Systems for the Plane

1. **Rectangular coordinates** Coordinate pair  $(x, y)$  indicates how far one must travel in two perpendicular directions to arrive at point.
2. **Polar coordinates** Coordinate pair  $(r, \theta)$  indicates *signed distance* and *bearing*
  - The  $r$  value (signed distance) is written first, followed by  $\theta$ .
  - The *bearing* is an angle with the positive horizontal axis, with positive angles taken in the counterclockwise direction.
  - Any polar coordinate pair  $(r, \theta)$  with  $r = 0$  specifies the origin.
  - Each point has infinitely many specifications. For instance,

$$\dots = (-1, -\pi) = (1, 0) = (-1, \pi) = (1, 2\pi) = \dots$$

In particular, for any point other than the origin, there are infinitely-many representations  $(r, \theta)$  in polar coordinates with  $r > 0$ , and infinitely-many with  $r < 0$ .

## Relationships between Rectangular/Polar Coordinates

- **Rectangular to polar:** If a point  $(x, y)$  is specified in rectangular coordinates, it has a corresponding polar representation  $(r, \theta)$  determined by

$$r = \sqrt{x^2 + y^2}, \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}.$$

- **Polar to rectangular:** If a point  $(r, \theta)$  is specified in rectangular coordinates, it has a corresponding polar representation  $(x, y)$  determined by

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

## When Polar Coordinates Are Useful

Using the “rectangular to polar” conversion above, equations in  $x$  and  $y$  (and the curves that correspond to them) may be expressed as polar equations (equations involving polar coordinates). Often (though not always), what was a simple equation in rectangular coordinates is uglier in polar coordinates. Nevertheless, even when this is the case, integration of particular functions over particular regions is sometimes more easily carried out in polar form than in rectangular. (See examples of this in the framework for Apr. 23.)

### Examples of curves in both forms:

1. **Circles centered at the origin.**

$$\text{Rectangular form: } x^2 + y^2 = a^2 \qquad \text{Polar form: } r = \pm a$$

2. **Circles centered on coordinate axis with point of tangency at the origin.**

$$\text{Rectangular: } x^2 + (y - a)^2 = a^2 \qquad \text{Polar: } r = \pm 2a \sin \theta$$

3. **Certain ellipses.**

$$\text{Rectangular: } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Polar: } r = \frac{b}{\sqrt{1 - \epsilon^2 \cos^2 \theta}}, \quad \text{where } \epsilon = \sqrt{1 - \frac{b^2}{a^2}}$$

4. **Lines through the origin.**

$$\text{Rectangular: } y = mx \qquad \text{Polar: } \theta = c, \quad \text{where } c = \arctan m$$

5. **Horizontal and vertical lines.**

$$\text{Rectangular: } y = b \qquad \text{Polar: } r = b \csc \theta$$

## Polar Functions

For equations in rectangular coordinates, we often express  $y$  as a function of  $x$  (i.e., treat  $x$  as independent) when this is possible. Similarly, when it is possible to make  $\theta$  the independent variable (i.e., when a polar equation may be solved for  $r$ ), we tend to prefer doing so, writing  $r = f(\theta)$ . The polar forms in examples 1–3 and 5 above were expressed this way.

### Examples of families of polar curves of some interest:

Note: If you place your graphing calculator in *polar* mode, you should be able to plot any polar curve written in the form  $r = f(\theta)$ . See these and other polar curves at [this link](#).

1. **Cardioids:**  $r = a(1 \pm \cos \theta)$  or  $r = a(1 \pm \sin \theta)$
2. **Lemniscates:**  $r^2 = a^2 \cos(2\theta)$ ,  $r^2 = a^2 \sin(2\theta)$ , etc.
3. **Limaçons:**  $r = a + b \cos \theta$  or  $r = a + b \sin \theta$
4. **Rose curves:**  $r = a \cos(b\theta)$  or  $r = a \sin(b\theta)$
5. **Spirals:**  $r = a\theta$ ,  $r = e^{a\theta}$ , etc.

# MATH 162: Calculus II

## Framework for Mon., Apr. 23

### Double Integrals in Polar Coordinates

**Today's Goal:** To learn the mechanics of setting up double integrals in polar coordinates, and learn to recognize situations in double integrals that may be easier in polar form.

**Important Note:** In conjunction with this framework, you should look over Section 13.4 of your text.

### Polar Rectangles

While the specific integrand  $f(x, y)$  plays a large role in how difficult it is to *evaluate* a double integral  $\iint_R f(x, y) dA$ , it is the region  $R$  *alone* that determines what limits of integration one uses in formulating an iterated integral. When  $R$  is the rectangular region  $R : a \leq x \leq b, c \leq y \leq d$ , setting up an iterated integral is quite easy (the limits are simply the bounding  $x$  and  $y$ -values for the rectangle).

Correspondingly, if our region  $R$  is a *polar rectangle*

$$a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

then it will be easy to find limits of integration for an iterated integral in polar coordinates.

**Some examples of polar rectangles:**

- $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$
- $1 \leq r \leq 2, 0 \leq \theta \leq 2\pi$
- $1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \pi$

**Double Integrals in Polar Coordinates over a Polar Rectangle  $R :$**   
 $r_1 \leq r \leq r_2, \alpha \leq \theta \leq \beta$

- If the integrand is  $f(x, y)$  (i.e., if it is given in terms of rectangular coordinates), one must find the appropriate expression in polar coordinates by substituting  $r \cos \theta$  for  $x$ ,  $r \sin \theta$  for  $y$ .
- The  $dA$  in  $\iint_R f(x, y) dA$  becomes  $dx dy$  or  $dy dx$  when written as an iterated integral in rectangular form.

In polar form,  $dA = r dr d\theta$  because of the need for the *area expansion factor*  $r$ .

**Example:** Compute the volume under the hemisphere  $z = \sqrt{1 - x^2 - y^2}$  above the polar rectangle  $R : 0 \leq r \leq 1/2, 0 \leq \theta \leq \pi$  in the plane.

## Bounded Regions

Our regions of integration for double integrals are not always rectangles (neither in the usual sense, nor in the polar sense). Often the region  $R$  of integration for a double integral  $\iint_R f(x, y) dA$  is described as “the region bounded by the curves ...” In such instances, one step in setting up an iterated integral involves finding points where the curves intersect.

**Example:** Find the area of the region outside the circle  $r = 2$  and inside the circle  $r = 4 \sin \theta$ .

## More Examples

1. Find the volume of the region bounded by the paraboloid  $z = 10 - 3x^2 - 3y^2$  and the plane  $z = 4$ .

2. Evaluate the iterated integral  $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 dx dy$ .

MATH 162: Calculus II  
Framework for Thurs., Apr. 26–Fri., Apr. 27  
Moments, Centers of Mass

**Today's Goal:** To learn to apply the skill of setting up and evaluating triple integrals in the context of finding centers of mass.

**Important Note:** In conjunction with this framework, you should look over Section 13.6 of your text.

### Point Masses along a Line (1D)

- Assume  $n$  masses  $m_1, \dots, m_n$  sit at locations  $x_1, \dots, x_n$  on a number line.
- Call the location  $\bar{x}$  about which the total torque is zero. That is,  $\bar{x}$  is the location to place a fulcrum so that

$$\sum_{k=1}^n m_k(x_k - \bar{x}) = 0.$$

Solving for  $\bar{x}$ , we get

$$\bar{x} = \frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k} =: \frac{\text{1st moment about } x=0}{\text{total mass}} \quad (1)$$

### Continuous Mass along a Line (1D; more instructive than practical)

When

- mass is distributed throughout a continuous body (instead of being concentrated at finitely-many distinct positions),
- $\rho(x)$  gives the mass density (mass per unit length) inside the interval  $a \leq x \leq b$ ,

then the total mass is

$$\int_a^b \rho(x) dx.$$

The continuous analog to the numerator of (??) comes from the following definition:

**Definition:** If  $\rho(x)$  is the mass density (in mass per unit *length*) of a substance contained in a region  $a \leq x \leq b$  of 1D space, then the *first moment* about  $x = 0$  is given by

$$\int_a^b x \rho(x) dx.$$

Using this, the corresponding center of mass is

$$\bar{x} = \frac{\int_a^b x \rho(x) dx}{\int_a^b \rho(x) dx} \quad (2)$$

## Centers of Mass in 3D

**Definition:** If  $\rho(x, y, z)$  is the mass density (in mass per unit volume) of a substance contained in a region  $D$  of 3D space, then the *first moment* about the  $yz$ -plane ( $x = 0$ ) is given by

$$M_{yz} := \iiint_D x \rho(x, y, z) dV.$$

Similarly, the first moments about the  $xz$  and  $xy$ -planes are

$$M_{xz} := \iiint_D y \rho(x, y, z) dV \quad \text{and} \quad M_{xy} := \iiint_D z \rho(x, y, z) dV$$

respectively.

The corresponding *center of mass* will reside at a point  $(\bar{x}, \bar{y}, \bar{z})$  in space. If we denote the total mass in the region  $D$  by

$$M := \iiint_D \rho(x, y, z) dV,$$

then the **coordinates of the center of mass** are given by

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \text{and} \quad \bar{z} = \frac{M_{xy}}{M}. \quad (3)$$

Remarks:

- One can develop formulas for the position  $(\bar{x}, \bar{y})$  of the center of mass in two dimensions. In that case it is assumed  $\rho(x, y)$  gives mass density in mass per unit area, and the first moments  $M_y = \iint_R x \rho(x, y) dA$  and  $M_x = \iint_R y \rho(x, y) dA$  are *about the  $x$ -axis and  $y$ -axis* respectively.
- When there is constant mass density  $\rho(x, y, z) = \delta$  throughout the region  $D$ , then another name for the center of mass is the *centroid*. In this case,

$$\bar{x} = \frac{M_{yz}}{M} = \frac{\delta \iiint_D x dV}{\iiint_D \delta dV} = \frac{\iiint_D x dV}{\iiint_D dV} = \text{avg. } x\text{-value in } D.$$

Similar statements may be made about the other coordinates  $\bar{y}$  and  $\bar{z}$  of the centroid.

# MATH 162: Calculus II

## Framework for Mon., Apr. 30

### Integration in Cylindrical Coordinates

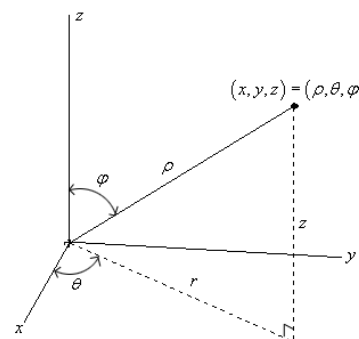
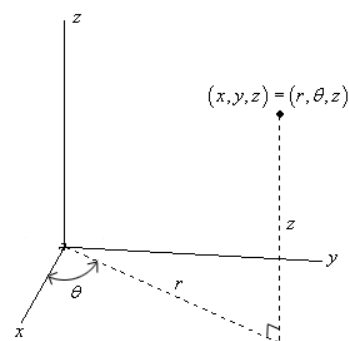
**Today's Goal:** To develop an understanding of cylindrical and spherical coordinates, and to learn to set up and evaluate triple integrals in cylindrical coordinates.

**Important Note:** In conjunction with this framework, you should look over Section 13.7 of your text.

### Coordinate Systems for 3D Space

- **Rectangular Coordinates:** Generally uses letters  $(x, y, z)$ . It tells how far to travel in directions parallel to three orthogonal coordinate axes in order to arrive at the specified point.
- **Cylindrical Coordinates** Generally uses letters  $(r, \theta, z)$ . Here  $z$  should be understood in exactly the same way as it is for rectangular coordinates, while  $r$  and  $\theta$  are polar coordinates for the shadow point in the  $xy$ -plane. (See the top figure.) Each of the three coordinates may take any value in  $\mathbb{R}$ .
- **Spherical Coordinates** Generally uses letters  $(\rho, \phi, \theta)$ . The meaning of  $\theta$  is precisely the same as with cylindrical coordinates. If a ray is drawn from the origin to the point in question, then  $\rho$  is the distance along that ray to the point, while  $\phi$  is the angle that ray makes with the  $z$ -axis. It is possible to identify all points of 3D-space using values for the three coordinates which satisfy

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$



### Conversions between Coordinates

The Pythagorean Theorem and trigonometry give

$$r = \sqrt{x^2 + y^2} = \rho \sin \phi, \quad \text{and} \quad \rho = \sqrt{x^2 + y^2 + z^2}.$$

Thus,

$$\begin{aligned} x &= r \cos \theta = \rho \sin \phi \cos \theta, \\ y &= r \sin \theta = \rho \sin \phi \sin \theta, \\ z &= \rho \cos \phi \quad (\text{by trigonometry}). \end{aligned}$$

## Triple Integrals in Cylindrical Coordinates

When computing the triple integral  $\iiint_D f(x, y, z) dV$ , one can choose any of the three coordinate systems discussed above. Cylindrical coordinates are attractive when

- the boundary of the shadow region in the plane may be expressed nicely as a combination of polar functions, and/or
- the integrand  $f(r \cos \theta, r \sin \theta, z)$  is simple.

For iterated integrals in cylindrical coordinates, the volume element is  $dV = r dz dr d\theta$ , or some permutation of this. Thus,

$$\iiint_D f(x, y, z) dV = \iiint_D f(r \cos \theta, r \sin \theta, z) r dz dr d\theta.$$

### Examples:

1. Show that the volume of a sphere with radius  $a$  is what we think it should be.
2. Find the volume of a “cored apple”—a sphere of radius  $a$  from which a cylindrical region of radius  $b$  ( $b < a$ ) has been removed.
3. Find the volume of a typical cone whose radius at the base is  $a$  and whose height is  $h$ .



MATH 162: Calculus II  
Framework for Tues., May. 1  
Integration in Spherical Coordinates

**Today's Goal:** To learn to set up and evaluate triple integrals in spherical coordinates.

**Important Note:** In conjunction with this framework, you should look over Section 13.7 of your text.

### Simple Equations in Spherical Coordinates and Their Graphs

- $\rho = \rho_0$  (a constant) corresponds to a sphere of radius  $\rho_0$ .
- $\phi = \phi_0$  corresponds to a cone with vertex at the origin and the  $z$ -axis as axis of symmetry.
- $\theta = \theta_0$  corresponds to a half-plane with  $z$ -axis as the terminal edge.

### Changing $(x, y, z)$ to $(\rho, \phi, \theta)$

Recall that we have the following relationships:

$$\begin{aligned}x &= \rho \sin \phi \cos \theta, \\y &= \rho \sin \phi \sin \theta, \\z &= \rho \cos \phi.\end{aligned}$$

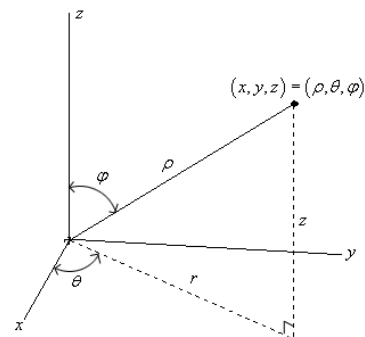
Thus, the equation (in rectangular coordinates)

$$(x - 2)^2 + y^2 + z^2 = 4$$

for a sphere of radius 2 centered at the point  $(x, y, z) = (2, 0, 0)$  may be rewritten as

$$\rho = 2 \left( \sin \phi \cos \theta + \sqrt{\sin^2 \phi \cos^2 \theta + 1} \right).$$

(Try verifying this.)



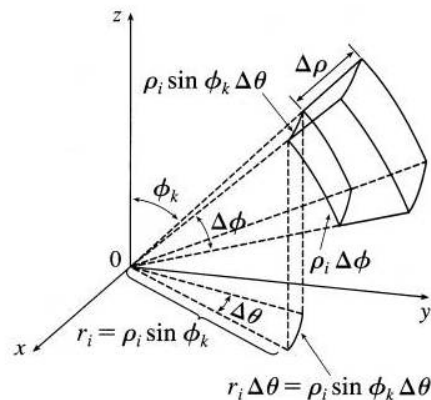
## Volume Element $dV$ in Spherical Coordinates

Pictured at right is a typical “volume element”  $\Delta V$  at a spherical point  $(\rho, \phi, \theta)$  corresponding to small changes  $\Delta\rho$ ,  $\Delta\phi$  and  $\Delta\theta$  in the spherical variables. Its sides, as can be verified using trigonometry, have approximate measures  $\Delta\rho$ ,  $(\rho\Delta\phi)$  and  $(\rho\sin\phi\Delta\theta)$ . Thus

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

As a result

$$\iiint_D f(x, y, z) \, dV = \iiint_D \rho^2 \sin \phi \, f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, d\rho \, d\phi \, d\theta,$$



### Examples:

1. Evaluate  $\iiint_D 16z \, dV$ , where  $D$  is the upper half of the sphere  $x^2 + y^2 + z^2 = 1$ .
2. Find the volume of the smaller section cut from a solid ball of radius  $a$  by the plane  $z = 1$ .