MATH 162: Calculus II
Framework for Thurs., Mar. 15
Vector Functions and Differential Calculus

Today's Goal: To understand parametrized curves and their derivatives.

In yesterday's lab, we called a set of continuous functions over a common interval I

$$\begin{array}{ll} x = x(t), & \\ y = y(t), & t \in I, \\ z = z(t), & \end{array} \tag{1}$$

a *parametrized curve*. (I may be a finite interval, like $I = [a, b]$, or one of infinite length.) Another name for a parametrized curve is *path*, as one may think of tracing out the location of a particle $(x(t), y(t), z(t))$ at various t-values in I.

Equations of lines in space

Whereas lines in the xy-plane may be characterized by a slope, lines in xyz-space are most easily characterized by a vector that is parallel to the line in question. Since there are infinitely many parallel vectors, there are infinitely many ways to describe a given line. Say we want the line passing through the point $P = (x_0, y_0, z_0)$ parallel to the vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. We describe it parametrically. We might arbitrarily decide to associate the point P with the parameter value $t = 0$, and integer values of t correspond to integer leaps of length $|\mathbf{v}|$:

$$\begin{array}{ll} x = x_0 + v_1 t, & \\ y = y_0 + v_2 t, & -\infty < t < \infty. \\ z = z_0 + v_3 t, & \end{array}$$

Example: Find 3 possible parametrizations of the line through $(2, -1, 4)$ in the direction of $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Make one of these parametrizations be by arc length.

The position vector

One might take the functions (1) and create from them a *vector function*, with $x(t)$, $y(t)$, and $z(t)$ as *component functions*:

$$\mathbf{r}(t) \ = \ x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Following the idea that the path (1) describes the locations of a moving particle, $\mathbf{r}(t)$ is often called a *position vector*—that is, when drawn in *standard position* (i.e., with its initial point at the origin), the terminal point of $\mathbf{r}(t)$ moves so as to trace out the curve.

Example: **Equation of a line in space, vector form**. For the line passing through the point $P = (x_0, y_0, z_0)$ parallel to the vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, we have the vector form

$$\mathbf{r}(t) = (x_0 + v_1 t)\mathbf{i} + (y_0 + v_2 t)\mathbf{j} + (z_0 + v_3 t)\mathbf{k}.$$

Limits and continuity of vector functions

While the following definition is not identical to the one given in the text, the two are logically equivalent.

Definition: Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ (so the component functions of $\mathbf{r}(t)$ are $x(t)$, $y(t)$ and $z(t)$). We say that

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$$

precisely when each corresponding limit of the component functions

$$\lim_{t\to t_0} x(t) = L_1, \qquad \lim_{t\to t_0} y(t) = L_2, \qquad \text{and} \qquad \lim_{t\to t_0} z(t) = L_3$$

holds.

We say that $\mathbf{r}(t)$ is continuous at $t = t_0$ precisely when each of the component functions $x(t)$, $y(t)$ and $z(t)$ are continuous at $t = t_0$.

Example: The vector function $\mathbf{r}(t) = t/(t-1)^2\mathbf{i} + (\ln t)\mathbf{j}$ is continuous at all points t where its component functions $x(t) = t/(t-1)^2$ and $y(t) = \ln t$ are continuous—that is for $t > 0$. Thus, $\lim_{t\to t_0} \mathbf{r}(t)$ exists whenever $t_0 > 0$.

Derivatives of vector functions

Definition: A vector function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is *differentiable at* t if the limit

$$\mathbf{r}'(t) := \lim_{\Delta t\to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

exists.

Notes:

- An equivalent definition to the one above would be that $\mathbf{r}(t)$ is differentiable at a given t-value precisely when each of its component functions $x(t)$, $y(t)$ and $z(t)$ are differentiable there. When this is so, we have

$$\frac{d\mathbf{r}}{dt} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

- If a vector function $\mathbf{r}(t)$ is differentiable, then the derivative $\mathbf{r}'(t)$ is itself another vector function, which may be differentiable as well. When this is so, we have

$$\frac{d^2\mathbf{r}}{d^2t} = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}.$$

- If the position vector function $\mathbf{r}(t)$ is differentiable, then $d\mathbf{r}/dt$ is the corresponding *velocity* vector function. What we call *speed* is actually the length $|d\mathbf{r}/dt|$ of the velocity function.

  If $d\mathbf{r}/dt$ is differentiable, then we call $d^2\mathbf{r}/dt^2$ the *acceleration* vector function.

- One check that we have defined dot and cross products between vectors in a useful fashion is whether they obey "product rules." In fact, all of the rules for differentiation that hold for scalar functions, and are appropriate to apply to vector functions, still hold:

  1. $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$ (constant function rule)

  2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$ (constant multiple rule)

  3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ (product of scalar and vector fn.)

  4. $\dfrac{d}{dt}\left[\dfrac{\mathbf{u}(t)}{f(t)}\right] = \dfrac{f'(t)\mathbf{u}(t) - f(t)\mathbf{u}'(t)}{[f(t)]^2}$ (quotient of vector and scalar fn.)

  5. $\dfrac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$ (sum and difference rules)

  6. $\dfrac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ (dot product rule)

  7. $\dfrac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ (cross product rule)

  8. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (chain rule)