Power Series

A power series centered at c is an infinite series of the form

$$F(x) \;=\; \sum_{n=0}^{\infty} c_n(x-a)^n \;=\; a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots. \tag{1}$$

Some notes:

- What's new is that, not only does a base sequence $a_0, a_1, a_2, \ldots$ get used to build an infinite series, but the size of each term is tempered by a power of $(x-a)$. This means the series is not one fixed sum, but a different sum for every choice of x. Correspondingly, the question is no longer "Does the series converge?", but "Does it converge at this x?," or "At which choices of x does it converge?"
- The series always converges "at the center"—that is, at $x = c$.
- The phrase "centered at c" is reminiscent of Taylor polynomials. Recall that, when we begin with a sufficiently differentiable function $f(x)$ and a center c, we generate Taylor polynomials

$$T_n(x) \;=\; \sum_{j=0}^{n} \frac{f^{(j)}(c)}{j!} \;=\; f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

  Indeed, if f is differentiable at c to all orders, the extension of these Taylor polynomials is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \;=\; f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots,$$

  called the Taylor series of f centered at c. This is one way that power series arise.
- Suppose each $a_n \geqslant 0$. In that instance, if $x > c$, then the terms of the series $F(x)$ are all positive. This leads to the observations that, if $x_2 > x_1 > c$ and the series $F(x_2)$ converges, then
  - $F(x_1)$ converges, by the direct comparison test, and
  - $F(c-(x_1-c)) = a_0 - a_1(x_1-c) + a_2(x_1-c)^2 - a_3(x_1-c)^3 + \cdots$ converges, by the absolute convergence test. (Draw picture)

  Though the situation is a little more difficult to analyze when not all $a_n \geqslant 0$, even then it can be proved that one of the following situations must hold for (1):

  1. $F(x)$ converges only when $x = c$ and at no other location.
  2. There exists a positive number R such that $F(x)$ converges when $|x-c| < R$ and diverges when $|x-c| > R$. In this instance, the **interval of convergence** is one of $(c-R, c+R)$ (open at both endpoints), $[c-R, c+R)$, $(c-R, c+R]$, or $[c-R, c+R]$.

3. $F(x)$ converges for all real x. That is, the interval of convergence is $(-\infty, \infty)$.

The number R in Situation 2 is called the **radius of convergence**. It makes sense in Situation 1 to say $R = 0$, and in Situation 3 to say $R = +\infty$.

We often determine the radius of convergence using the ratio test. This differs from the use of ratio test in Section 11.5 in that consecutive terms include powers of $(x - c)$:

$$\text{rato of consecutive terms} \;=\; \left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| \;=\; \left|\frac{a_{n+1}}{a_n}\right| \cdot |x-c|.$$

It is this quantity whose limit, as $n \to \infty$, we label ρ.

Example 1:

Find the interval and radius of convergence for the given power series.

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n 3^n \sqrt{n}}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{4^n}$

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A power series can be differentiated/antidifferentiated just like a polynomial—term-by-term; its radius of convergence does not change, though the inclusion of one endpoint or the other in the interval of convergence may change. This is the content of Theorem 2 on p. 573.

Some mileage can be made out of the sum of a geometric series

$$\frac{1}{1-x} \;=\; 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{2}$$

Example 2:

1. Substitute into (2) the following "values" to see how the series changes, and the new radius of convergence.
   - $(2x)$
   - $(x - 1)$

- $(5-3x)$
- $(-x)$
- $(-x^2)$

2. Find a power series centered at 0 which equals $2x/(1-3x)$. What is this power series' radius of convergence?
3. Find a power series centered at 0 which equals $\arctan x$. What is this power series' radius of convergence?
4. What power series centered at 0 results from differentiating $1/(1-x)$?

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Example 3:

Find a power series centered at 0 associated with $\ln(5+x^4)$.

Answer: We have that $d/dx \ln(5+x^4) = \frac{4x^3}{5+x^4}$. Working with this derivative, we have

$$\begin{aligned}\frac{4x^3}{5+x^4} &= \frac{4x^3}{5}\cdot\frac{1}{1+x^4/5} = \frac{4x^3}{5}\cdot\frac{1}{1-(-x^4/5)} \qquad (1/(1-r), \text{ with } r=-x^4/5)\\ &= \frac{4x^3}{5}\sum_{n=0}^{\infty}\left(-\frac{x^4}{5}\right)^n = \frac{4x^3}{5}\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{5^n}\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{4x^{4n+3}}{5^{n+1}} = \frac{4}{5}x^3 - \frac{4}{25}x^7 + \frac{4}{125}x^{11} - \cdots .\end{aligned}$$

This series converges when

$$|r| = \left|-\frac{x^4}{5}\right| < 1 \qquad \Rightarrow \qquad |x| < \sqrt[4]{5}.$$

Now, $\ln(5+x^4)$ is an antiderivative of $4x^3/(5+x^4)$, and by Theorem 2 on p. 573 all antiderivatives of the latter (at least in the interval $(-\sqrt[4]{5}, \sqrt[4]{5})$) have the power series representation

$$\sum_{n=0}^{\infty}\int(-1)^n\frac{4x^{4n+3}}{5^{n+1}}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{4x^{4n+4}}{(4n+4)5^{n+1}} = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+4}}{(n+1)5^{n+1}}.$$

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Example 4:

Find a power series centered at 3 for $1/(1+x)$, and determine its radius of convergence.

Answer: We want to manipulate $1/(1+x)$ so that some multiple of $(x-3)$ is subtracted from

1 in the denominator.

$$\begin{aligned}\frac{1}{1+x} &= \frac{1}{4+x-3} = \frac{1}{4}\cdot\frac{1}{1-(-1)(x-3)/4} \qquad (1/(1-r) \text{ with } r=(x-3)/(-4)) \\ &= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-3}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{4^{n+1}}.\end{aligned}$$

This power series converges for

$$|r| = \left|-\frac{x-3}{4}\right| < 1 \qquad \Rightarrow \qquad |x-3| < 4,$$

that is, when the distance from x to the center at 3 does not exceed 4. Thus, the radius of convergence is 4.

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