1. (a) $\det\left(\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\right) = -1$, so this matrix has rank 2, and the columns are a basis for $\mathbb{R}^2$.

(b) $$\vec{x} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -19 \end{bmatrix}$$

(c) $\vec{b} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $[\vec{b}]_{\mathcal{B}_1} = \begin{bmatrix} 11 \\ -19 \end{bmatrix}$

(d) M is the matrix of $C_{\mathcal{B}_2} \circ id \circ C_{\mathcal{B}_1}^{-1}$

$$\Rightarrow M = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 9 \\ 25 & 14 \end{bmatrix}$$

2. The vectors $\vec{w}_1 = \langle 1, 1, 1 \rangle$, $\vec{w}_2 = \langle 1, -1, 0 \rangle$ and $\vec{w}_3 = \langle 1, 1, -2 \rangle$ are eigenvectors and mutually orthogonal already. So A is orthogonally diagonalizable. We obtain P by first turning these vectors into unit vectors:

$$\vec{u}_1 = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}}\vec{w}_1 = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$$

$$\vec{u}_2 = \frac{1}{\sqrt{1^2 + (-1)^2}}\vec{w}_2 = \frac{1}{\sqrt{2}}\langle 1, -1, 0 \rangle$$

$$\vec{u}_3 = \frac{1}{\sqrt{(-2)^2 + 1^2 + 1^2}}\vec{w}_3 = \frac{1}{\sqrt{6}}\langle 1, 1, -2 \rangle$$

So,

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

3. (a) and (g) only

4. $$\det(xI - A) = \begin{vmatrix} x-1 & -1 \\ 2 & x-3 \end{vmatrix} = (x-1)(x-3) + 2 = x^2 - 4x + 5$$

$$\Rightarrow \text{eigenvalues are roots: } \quad x = \frac{4}{2} \pm \frac{\sqrt{16 - 4(1)(5)}}{2} = \boxed{2 \pm i}$$

5. (a) $E_{-4} = \text{null}(-4I - A)$ and

$$-4I - A = \begin{bmatrix} -4 & 2 & -2 \\ 8 & -4 & 4 \\ 4 & -2 & 2 \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x_1 - \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = 0, \text{ or } x_1 = \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3$$

$x_2 = s$, $x_3 = t$ are free

eigenvectors corresponding to $\lambda = -4$ satisfy

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}s - \tfrac{1}{2}t \\ s \\ t \end{bmatrix} = \tfrac{1}{2}s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \tfrac{1}{2}t \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

So, a basis of E_{-4}: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

(b) In part (a), we learned $\lambda = -4$ has $GM = 2$, matching its algebraic multiplicity. Since the characteristic polynomial of A is degree 3, it can have only 3 roots: $\lambda = -4$ (twice) and $\lambda = 2$ (necessarily once). So, $AM = GM$ for this last eigenvalue, too. And since no eigenvalue is degenerate (i.e., with $GM < AM$), A is diagonalizable.

7. Since $A\vec{x}_1 = A\vec{x}_2$, we can subtract all to one side:

$$A\vec{x}_1 - A\vec{x}_2 = \vec{0} \quad \text{or} \quad A(\vec{x}_1 - \vec{x}_2) = \vec{0}.$$

But this, by definition, says $\vec{x}_1 - \vec{x}_2 \in \text{null}(A)$.