Linear 1$^{\text{st}}$-order systems of DEs

From the last set of notes, we have that a set of n differential equations in the dependent variables $x_1, x_2, \ldots, x_n$, is **linear** when its normal form is

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}, \tag{1}$$

(Note: the matrix is *square*!) or, more succinctly,

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) \tag{2}$$

This should not be surprising, as the form of a linear 1$^{\text{st}}$-order system mirrors the form of a linear 1$^{\text{st}}$-order DE (the type in Chapter 2)

$$y' = a(t)y + f(t). \tag{3}$$

Like with (3), we say (2) is **homogeneous** when $\mathbf{f}(t) = \mathbf{0}$, **nonhomogeneous** otherwise.

Initial value problems for systems

The system (2) invokes vector notation $\mathbf{x}$ as a way to discuss the multiple dependent variables (unknown functions) $x_1(t), \ldots, x_n(t)$. If we are to invoke initial conditions, it means assigning a value to each of these dependent variables at some set time t_0. We find it convenient to employ vector notation for that, too:

$$\mathbf{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} = \mathbf{x}_0.$$

We call

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \qquad \text{subject to} \qquad \mathbf{x}(t_0) = \mathbf{x}_0,$$

an **initial value problem** for the 1$^{\text{st}}$ linear system (2). Just as initial value problems for the linear DEs (3) of Chapter 2 have a theorem addressing existence, uniqueness, and interval of validity, so do IVPs for 1st-order systems. That is the content of Section 3.3. The theorem works much like Corollary 2.2 on p. 106, with the only difference being that, whereas before you had to look for an open interval containing t_0 on which *both* functions $a(t)$, $f(t)$ in (3) are continuous, in Lemma 3.1, p. 144, you must look for an open interval containing t_0 on which every one of the functions $a_{11}(t)$, $a_{12}(t), \ldots, a_{nn}(t)$ as well as $f_1(t), \ldots, f_2(t)$ are continuous.

Lemma 3.1, like Corollary 2.2, serves mainly to reassure that there is a solution—only one—and it will last (remain valid) during a certain time frame. It doesn't help you to find that solution.

Linear paradigm remains valid

We have previously encountered problems we called *linear*—namely, the problems of m algebraic equations in n unknowns of Chapter 1 $\mathbf{A}\mathbf{x} = \mathbf{b}$, and the linear DEs $y' = a(t)y + f(t)$ of Chapter 2. In both instances, the solutions break into two parts, a part that solves the homogeneous problem and contains whatever freedoms are inherent there, and a second (no-freedom) part called a particular solution. Section 3.4 assures us that this paradigm remains *in play* for 1st-order linear systems (2). That paradigm says to solve

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) \tag{4}$$

by first finding the general solution $\mathbf{x}_h(t)$ of the related homogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x},$$

then find a particular solution $\mathbf{x}_p(t)$ of the original problem (4) and put them together as

$$\mathbf{x}(t) \;=\; \mathbf{x}_h(t) + \mathbf{x}_p(t).$$

It is here that we decide to take the easy route. We only have time in this course to explain how to solve (4) when dealing with a constant matrix $\mathbf{A}$, not a matrix function $\mathbf{A}(t)$ which changes with t.

Homogeneous 1st-order linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (constant matrix A)

Past experience with the homogeneous problem $y' = ay$ (constant a) in Chapter 2 led to exponential solutions, scalar multiples of the basis solution $\phi(t) = e^{at}$. Intuition suggests we may be able to find exponential basis solutions to the homogeneous problem with contant matrix

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}. \tag{5}$$

Since solutions are *vector* functions, we don't look for anything of the form $Ce^{\lambda t}$ to work, but rather $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ is a constant vector in $\mathbb{R}^n$ (just as $\mathbf{x}(t) = \langle x_1(t), \ldots, x_n(t)\rangle$ is in $\mathbb{R}^n$). It's just a proposal, one whose derivative is $\mathbf{x}'(t) = \lambda\mathbf{v}e^{\lambda t}$. And if we insert the proposal as well as its derivative into (5), we obtain

$$\lambda\mathbf{v}e^{\lambda t} \;=\; \mathbf{A}\mathbf{v}e^{\lambda t}, \qquad \text{or, after cancelling } e^{\lambda t}, \qquad \mathbf{A}\mathbf{v} = \lambda\mathbf{v},$$

our *eigenvector equation*. That is, our proposal works precisely when $(\lambda, \mathbf{v})$ form an eigenpair of the matrix $\mathbf{A}$.

Please watch `https://drive.google.com/file/d/1ozhaO08ydYdFVXh-V64MNtB47EmAzziF/view?usp=sharing` to view some examples of solutions to homogeneous 1st-order systems.

Addendum

If you have watched the video, you will know that I made the claim there that $e^{3t}\langle -1, 1\rangle$ and $e^{6t}\langle 1, 2\rangle$ are solutions of the system

$$\mathbf{x}' = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix} \mathbf{x}.$$

What I mean by that is that you can plug either one of these into the equation (both sides) and get a true statement. Doing this for $e^{3t}\langle -1, 1\rangle$, the left side becomes

$$\mathbf{x}' \;=\; \frac{d}{dt} e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \;=\; 3e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \end{bmatrix},$$

while the right side is

$$\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} \;=\; -e^{3t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \;=\; \begin{bmatrix} -4e^{3t} \\ -2e^{3t} \end{bmatrix} + \begin{bmatrix} e^{3t} \\ 5e^{3t} \end{bmatrix} \;=\; \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \end{bmatrix},$$

the same result as the left side. In fact, sticking the entire matrix (the one I called the **fundamental matrix solution**) in for $\mathbf{x}$ produces the true statement:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{becomes} \qquad \frac{d}{dt} \begin{bmatrix} -e^{3t} & e^{6t} \\ e^{3t} & 2e^{6t} \end{bmatrix} \;=\; \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -e^{3t} & e^{6t} \\ e^{3t} & 2e^{6t} \end{bmatrix}.$$

Note that, while I leave you to check the details, the idea of differentiating a matrix, which is what the left side calls for now that we put an entire matrix in for $\mathbf{x}$, is a first. By that, we simply mean that you differentiate each component:

$$\frac{d}{dt} \begin{bmatrix} -e^{3t} & e^{6t} \\ e^{3t} & 2e^{6t} \end{bmatrix} \;=\; \begin{bmatrix} -3e^{3t} & 6e^{6t} \\ 3e^{3t} & 12e^{6t} \end{bmatrix}.$$

When you multiply the two matrices on the right, you will get this same (2-by-2 matrix) result.