MATH 162: Calculus II
Framework for Mon., Mar. 12
Dot Products and Projections

Today's Goal: To define the dot product and learn of some of its properties and uses

The Dot Product

Definition: For vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, we define the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ to be

$$\mathbf{u} \cdot \mathbf{v} \ := \ u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Notes:

- The dot product $\mathbf{u} \cdot \mathbf{v}$ is a scalar (number), not another vector.
- Properties
  1. $\mathbf{u} \cdot \mathbf{v} \ = \ \mathbf{v} \cdot \mathbf{u}$
  2. $c(\mathbf{u} \cdot \mathbf{v}) \ = \ (c\mathbf{u}) \cdot \mathbf{v} \ = \ \mathbf{u} \cdot (c\mathbf{v})$.
  3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \ = \ \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4. $\mathbf{0} \cdot \mathbf{v} \ = \ 0$
  5. $\mathbf{v} \cdot \mathbf{v} \ = \ |\mathbf{v}|^2$

Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the angle θ between them satisfies

$$\cos\theta \ = \ \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Note that, when $\theta = \pi/2$, the numerator on the right-hand side must be zero. This motivates the following definition.

Orthogonality

Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be *orthogonal* (or *perpendicular*) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example: The zero vector $\mathbf{0}$ is orthogonal to every other vector. In 2D, the vectors $\langle a, b \rangle$ and $\langle -b, a \rangle$ are orthogonal, since

$$\langle a, b \rangle \cdot \langle -b, a \rangle \ = \ (a)(-b) + (b)(a) \ = \ 0.$$

Example: Find an equation for the plane containing the point $(1, 1, 2)$ and perpendicular to the vector $\langle A, B, C\rangle$.

Projections

Scalar component of u in the direction of v: $|\mathbf{u}|\cos\theta \ = \ \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$.

Vector projection of u onto v:

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} \ := \ \left(\begin{array}{c}\text{scalar component of } \mathbf{u}\\ \text{in direction of } \mathbf{v}\end{array}\right)\left(\begin{array}{c}\text{direction}\\ \text{of } \mathbf{u}\end{array}\right) \ = \ \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) \ = \ \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}.$$

Work

The *work* done by a constant force $\mathbf{F}$ acting through a displacement vector $\mathbf{D} = \overrightarrow{PQ}$ is given by

$$W \ = \ \mathbf{F}\cdot\mathbf{D} \ = \ |\mathbf{F}||\mathbf{D}|\cos\theta,$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{D}$.