1. (a) Assuming $\lambda = 0$, the ODE is

$$v'' = 0 \quad \Rightarrow \quad v(x) = ax + b.$$

Imposing the BC at $x = 1$:

$$0 = v(1) = a + b \quad \Rightarrow \quad b = -a$$

Imposing the BC at $x = 0$:

$$0 = \beta v(0) + v'(0) = \beta b + a = \beta(-a) + a = a(1-\beta).$$

The equation

$$0 = a(1-\beta)$$

is met if

- $a = 0$. But, in that instance we would have

$$v(x) = 0 \cdot x + (-0) = 0,$$

the trivial soln. So, we rule out this possibility.

- $\boxed{\beta = 1}$

(b) Now, we know nothing about λ.

<u>Case $\lambda > 0$:</u> Write $\lambda = \omega^2$, with $\omega > 0$. Then

$$v'' - \omega^2 v = 0 \quad \Rightarrow \quad v(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

$$\text{and} \quad v'(x) = \omega c_1 e^{\omega x} - \omega c_2 e^{-\omega x}$$

The conditions (w/ $\beta = 0$) at the boundaries are

$$\left.\begin{aligned} 0 &= v'(0) = \omega c_1 - \omega c_2 \\ 0 &= v(1) = c_1 e^{\omega} + c_2 e^{-\omega} \end{aligned}\right\} \quad \text{or} \quad \begin{bmatrix} \omega & -\omega \\ e^{\omega} & e^{-\omega} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix has determinant

$$\omega e^{-\omega} + \omega e^{\omega} = \omega e^{-\omega}(1 + e^{2\omega}) > 0,$$

which means $c_1 = c_2 = 0 \quad \Rightarrow \quad v(x) = 0$ is trivial.

<u>Case $\lambda = 0$:</u> With $\beta = 0$, the possibility of a nontrivial soln. is ruled out in part (a).

<u>Case $\lambda < 0$:</u> Write $\lambda = -\omega^2$, with $\omega > 0$. Then

$$v'' + \omega^2 v = 0 \quad \Rightarrow \quad v(x) = A\cos(\omega x) + B\sin(\omega x)$$

Imposing the BC at $x=0$:

$$0 = \underbrace{\cancel{\beta v(0)}}_{=0,\ \text{since } \beta=0} + v'(0) = -\omega A \sin(0) + \omega B \cos(0) = \omega B.$$

Since $\omega > 0$, this requires $B = 0$.

So, $v(x) = A\cos(\omega x)$. But by the other boundary condition,

$$0 = v(1) = A\cos(\omega).$$

If $A=0$, then $v(x) = 0\cdot\cos(\omega x) + 0\cdot\sin(\omega x) = 0$, the trivial soln. To avoid this requires that

$$0 = \cos(\omega) \quad \Rightarrow \quad \omega = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n+1)\pi}{2}, \ldots .$$

Thus, we have nontrivial soln.

$$\cos\left(\frac{2n+1}{2}\pi x\right),$$

whenever

$$\lambda = -\left(\frac{2n+1}{2}\pi\right)^2, \qquad n = 0, 1, 2, \ldots .$$

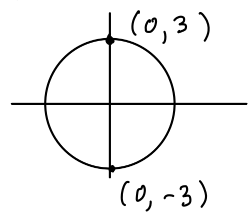
2. (a) 2^{nd}-order, linear, homogeneous

(b) 2^{nd}-order, linear, nonhomogeneous

(c) 2^{nd}-order, linear, nonhomogeneous

(d) 4^{th}-order, nonlinear

3. Since u is harmonic in Ω, the Maximum Principle says it achieves its extrema on the boundary $\partial\Omega$, where



$$u(x,y) = x^2 + y^2 - 2y = 9 - 2y$$

The extrema, then, depend on locations on $\partial\Omega$ where y is max/minimized — at the north and south poles:

$$\max_{(x,y)} u = 9 - 2(-3) = 15$$

$$\min_{(x,y)} u = 9 - 2(3) = 3.$$

4. Definition: $$\mathcal{F}u(\xi) = \int_{-\infty}^{\infty} u(x)\, e^{i\xi x}\, d\xi$$

Linearity: If u, v are functions whose Fourier transforms exist, and a, b are constants, then

$$\mathcal{F}(au + bv)(\xi) = \int_{-\infty}^{\infty} [au(x) + bv(x)]\, e^{i\xi x}\, dx$$

$$= a\int_{-\infty}^{\infty} u(x)\, e^{i\xi x}\, d\xi + b\int_{-\infty}^{\infty} v(x)\, e^{i\xi x}\, dx$$

$$= a \,.\, \hat{u}(\xi) + b \cdot v(\xi)$$

5. (a) diffusion

(b) convection

(c) source/sink

6. Let $w(x,t) = u(x,t) - xe^{-t}$, $0 \le x \le 1$, $t \ge 0$.

Then $$w_x = u_x - e^{-t} \quad \text{and} \quad w_{xx} = u_{xx}.$$

So

$$w_t = u_t + xe^{-t}$$

$$= u_{xx} + xe^{-t}$$

$$w_t = w_{xx} + xe^{-t} \quad (\text{the PDE solved by } w)$$

and
$$\begin{cases} w(0,t) = u(0,t) - 0 = 0 \\ w(1,t) = u(1,t) - e^{-t} = 0 \\ w(x,0) = u(x,0) - x = \sin\left(\frac{\pi x}{2}\right) - x. \end{cases}$$