Supplemental Notes on Eigenvalues and Eigenvectors

Not everything I might want to say about eigenvalues and eigenvectors was said in class today. I showed the general process for finding **eigenpairs**—that is, eigenvalues paired with their corresponding eigenvectors. I'll give another couple of examples below.

But first, let me say that there are questions one might be asked (in homework?) which do not require so much work as today's class examples. Here are several examples:

Example 1: Given a matrix and a vector, check if it is an eigenvector

Let

$$\mathbf{A} = \begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Is $\mathbf{u}$ an eigenvector of $\mathbf{A}$?

(b) Is $\mathbf{v}$ an eigenvector of $\mathbf{A}$?

(c) If either $\mathbf{u}$ or $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, what is the eigenvalue to which it corresponds?

Answers:

(a) In order to be an eigenvector of $\mathbf{A}$, $\mathbf{u}$ must satisfy $\mathbf{Au} = \lambda\mathbf{u}$ for some scalar λ. That is, $\mathbf{Au}$ needs to be a scalar multiple of the vector $\mathbf{u}$. We have

$$\mathbf{Au} = \begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}.$$

Since $\langle 8, 5 \rangle$. is not a scalar multiple of $\langle 2, 1 \rangle$, $\langle 2, 1 \rangle$ is not an eigenvector of $\mathbf{A}$.

(b) Here,

$$\mathbf{Av} = \begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since the output vector $\langle 1, 1 \rangle$ is a scalar multiple (an exact copy, as it turns out) of the input vector, $\mathbf{v}$ is an eigenvector.

(c) The vector $\mathbf{v}$ is an eigenvector, since it satisfies $\mathbf{Av} = \lambda\mathbf{v}$ for some scalar λ. The scalar is 1, so that is the desired eigenvalue.

$\blacksquare$

Example 2: Given matrix and a number, check if that number is an eigenvalue

Let $\mathbf{A}$ be the matrix of the previous example.

(a) Is 2 an eigenvalue of $\mathbf{A}$?

(b) Is 3 an eigenvalue of $\mathbf{A}$?

Answers:

(a) If 2 is an eigenvalue, then the square matrix

$$\mathbf{A} - 2\mathbf{I} \;=\; \begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix},$$

will have a zero determinant. But

$$\begin{vmatrix} 5 & -6 \\ 4 & -5 \end{vmatrix} \;=\; (5)(-5) - (-6)(4) \;=\; -25 + 24 \;=\; -1.$$

So, 2 is not an eigenvalue.

(b) It is fine, just as in part (a), to test whether the determinant of $\mathbf{A} - 3\mathbf{I}$ is zero or not. This time, I choose instead to look at RREF for $\mathbf{A} - 3\mathbf{I}$ for, in the case of a square matrix, a nonzero determinant is equivalent to RREF being the identity matrix. We have

$$\mathbf{A} - 3\mathbf{I} \;=\; \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}.$$

As this RREF is not of full rank, we learn that 3 is an eigenvalue.

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Example 3: Find the eigenvalues of a triangular matrix

Suppose $\mathbf{A}$ is the upper-triangular matrix

$$\mathbf{A} \;=\; \begin{bmatrix} -2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

The eigenvalues are those numbers λ which satisfy $0 = \det(\mathbf{A} - \lambda\mathbf{I})$. But since $\mathbf{A}$ and $\mathbf{I}$ are both triangular, so is $\mathbf{A} - \lambda\mathbf{I}$. And it is easy to calculate the determinant of a diagonal matrix. In this case

$$\det(\mathbf{A} - \lambda\mathbf{I}) \;=\; (-2 - \lambda)(3 - \lambda)(5 - \lambda),$$

which has zeros (-2), 3, and 5, precisely the same numbers on the main diagonal of the original matrix $\mathbf{A}$. This is what always happens if $\mathbf{A}$ is lower or upper triangular.

Note, however, that the matrix

$$\mathbf{B} \;=\; \begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 2 \\ 1 & 0 & 5 \end{bmatrix},$$

is considered *neither* lower nor upper triangular. The eigenvalues are *not* 1, 3 and 5, as one might hope. When you think about the appearance of the matrix $\lambda\mathbf{I}$ being subtracted from $\mathbf{B}$, you begin to understand why it doesn't work out to the same sort of simple answer as it did for the upper triangular matrix $\mathbf{A}$ above.

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Here are several more examples of finding eigenpairs. These illustrate some situations that did not arise in class examples today, including the fact that the eigenvalues can be nonreal complex numbers.

Example 4:

Find the eigenpairs (i.e., the eigenvalues, with corresponding basis eigenvectors) for the matrix

$$\mathbf{A} = \begin{bmatrix} 28 & 100 \\ -9 & -32 \end{bmatrix}.$$

Answer: First, we solve

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 28-\lambda & 100 \\ -9 & -32-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - 896 - (-900) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2.$$

Thus, $\lambda = -2$ is a repeated eigenvalue (i.e., one of algebraic multiplicity 2). We solve for the corresponding eigenvectors by finding the vectors in Null$(\mathbf{A} + 2\mathbf{I})$:

$$\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 30 & 100 \\ -9 & -30 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 10/3 \\ 0 & 0 \end{bmatrix}.$$

Since RREF has just one free column, the dimension of the nullspace is 1 (despite that the eigenvalue is repeated). The top row indicates x_2 is free and $x_1 = (-10/3)x_2$, so that eigenvectors corresponding to $\lambda = -2$ take the form

$$\left\langle -\frac{10}{3}x_2, x_2 \right\rangle = \frac{x_2}{3}\langle -10, 3\rangle.$$

Thus, the eigenvectors corresponding to $\lambda = -2$ have the single basis vector $\langle -10, 3\rangle$.

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Example 5:

Find the eigenvalues for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Answer: First, we solve

$$0 \;=\; \det(\mathbf{A} - \lambda\mathbf{I}) \;=\; \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} \;=\; \lambda^2 - 5\lambda + 4 - (6) \;=\; \lambda^2 - 5\lambda - 2.$$

This quadratic polynomial does not easily factor, so we use the quadratic formula to determine the zeros (eigenvalues)

$$\lambda \;=\; \frac{5}{2} \pm \frac{1}{2}\sqrt{25 - (4)(-2)} \;=\; \frac{5}{2} \pm \frac{\sqrt{33}}{2}.$$

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Example 6:

Find the eigenpairs (i.e., the eigenvalues, with corresponding basis eigenvectors) for the matrix

$$\mathbf{A} \;=\; \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}.$$

Answer: First, we solve

$$0 \;=\; \det(\mathbf{A} - \lambda\mathbf{I}) \;=\; \begin{vmatrix} 1-\lambda & 4 \\ -4 & 1-\lambda \end{vmatrix} \;=\; \lambda^2 - 2\lambda + 1 - (-16) \;=\; \lambda^2 - 2\lambda + 17.$$

Again, using the quadratic formula, we obtain eigenvalues

$$\lambda \;=\; \frac{2}{2} \pm \frac{\sqrt{4 - (4)(17)}}{2} \;=\; 1 \pm \frac{\sqrt{-64}}{2} \;=\; 1 \pm 4\sqrt{-1} \;=\; 1 \pm 4i,$$

where i represents $\sqrt{-1}$. We have two eigenvalues which, being nonreal, come in complex conjugate pairs. Taking the first eigenvalue as $\lambda = 1 - 4i$, we find vectors in $\mathrm{Null}(\mathbf{A} - (1-4i)\mathbf{I})$:

$$\mathbf{A} - (1-4i)\mathbf{I} \;=\; \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix}.$$

Here, if we are confident our work is correctly done to this point, we may employ our knowledge that $\mathbf{A} - (1-4i)\mathbf{I}$ cannot be of full column rank to conclude echelon form has a final zero row. Thus, an echelon form of $\mathbf{A} - (1-4i)\mathbf{I}$ is

$$\begin{bmatrix} 4i & 4 \\ 0 & 0 \end{bmatrix},$$

which tells us x_2 is free, and that

$$4ix_1 + 4x_2 = 0 \quad\Rightarrow\quad ix_1 = -x_2 \quad\Rightarrow\quad (-i)ix_1 = -(-i)x_2 \quad\Rightarrow\quad x_1 = ix_2,$$

and eigenvectors corresponding to $\lambda = 1 - 4i$ take the form

$$\langle ix_2, x_2\rangle \;=\; x_2\,\langle i, 1\rangle.$$

with single basis vector $\langle i, 1\rangle$.

We do not need to work out similar details for eigenvectors corresponding to the other eigenvalue $\lambda = 1 + 4i$, because of this fact: When the original matrix $\mathbf{A}$ has real-number entries, its nonreal eigenvalues always come in complex-conjugate pairs, and so do its eigenvectors. In particular, if we split apart the eigenvector

$$\langle i, 1\rangle \;=\; \langle 0, 1\rangle + i\,\langle 1, 0\rangle$$

that corresponds to $\lambda = 1 - 4i$, then we are assured

$$\langle 0, 1\rangle - i\,\langle 1, 0\rangle \;=\; \langle -i, 1\rangle$$

is a basis eigenvector that corresponds to $\lambda = 4 + i$.

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Linear Independence of Functions

The material here is important both now (for one of the homework problems, Exercise 1.6.8, on the next hand-checked assignment), and later, when we are solving differential equations.

Review of linear independence

The definition of linear independence is this: a collection $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of objects from the same "space" is **linearly independent** if the only linear combination

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

which produces the *zero* object in the space is when all the weights $c_1, \ldots, c_k$ are zero.

To date the objects involved have been vectors, taken from $\mathbb{R}^n$ for some n. Examples include collections such as $S = \{(1,1,0), (1,0,1)\}$, which is linearly independent since the only weights c_1, c_2 which make the following equation true are $c_1 = c_2 = 0$:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, the collection $S = \{(1,-2,2), (3,4,-1), (1,18,-12)\}$ is linearly dependent, because

$$\underline{\quad 5 \quad} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + \underline{\quad -2 \quad} \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} + \underline{\quad 1 \quad} \begin{bmatrix} 1 \\ 18 \\ -12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Of course, the task of trying to fill in blanks with scalars until either

- you find a set of scalars, not all of which are zero, that works, or
- you decide the only set of scalars that works is the one with all zeros

is not the best way of deciding which of the scenarios, *linear independence* or *linear dependence* holds. Gaussian elimination on the augmented matrix $[\mathbf{A} \mid \mathbf{0}]$ tells you vectors $\mathbf{x}$ of weights which can be used to produce $\mathbf{0}$ from the columns of $\mathbf{A}$. And, if $\mathbf{A}$ is *square* and you only want to know if the columns are linearly independent or not, $\det(\mathbf{A})$ tells you all you need to know: $S = \{(1,-2,2), (3,4,-1), (1,18,-12)\}$ is linearly dependent since

$$\begin{vmatrix} 1 & 3 & 1 \\ -2 & 4 & 18 \\ 2 & -1 & -12 \end{vmatrix} = 0.$$

Test for linear independence of k vectors from $\mathbb{R}^k$. Form the k-by-k (square) matrix $\mathbf{A}$ with the vectors to be tested as the columns of $\mathbf{A}$. If $|\mathbf{A}| \neq 0$, then the vectors are linearly independent; if $|\mathbf{A}| = 0$, then the vectors are linearly dependent.

Linear independence of functions

Functions are like vectors. The main difference is the number of coordinates. A vector from $\mathbb{R}^3$ has 3 coordinates. A function defined on the domain $a < x < b$ has a "coordinate" (a y-value) at each of the infinitely many x in the domain. The zero in this "vector space of functions defined on (a, b)" is the function $g(x) = 0$ which has the value 0 for each x.

When we have a collection of functions defined on a common domain, we can ask if that collection is linearly independent over that domain. It is, again, a question about weights in a linear combination to produce zero (the zero function). For example, $\{1, \sin^2 x, \cos^2 x\}$ is a linearly dependent collection on domain $(-\infty, \infty)$ since

$$\underline{\ -1\ }(1) + \underline{\ 1\ }(\sin^2 x) + \underline{\ 1\ }(\cos^2 x) = 0, \qquad \text{for every } -\infty < x < \infty.$$

That is, a collection of weights exists that produces the zero function without all the weights being zero. But, as before, hunting for scalars to weight the various functions can feel like looking for the proverbial needle in the haystack.

If you liked the simplicity of the process for testing independence of k vectors from $\mathbb{R}^k$ (see the box above), then here is an alternative for functions which mirrors it. It rests on the idea that, if the weights $(c_1, c_2, \ldots, c_k)$ in the linear combination of sufficiently-differentiable functions in the set $S = \{f_1, f_2, \ldots, f_k\}$ produce the zero function

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0, \qquad \text{for all } a < x < b,$$

then the same weights applied to linear combinations of derivatives, 2^{nd} derivatives, etc. also produce the zero function:

$$\begin{aligned}
c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_k f_k'(x) &= 0 \\
c_1 f_1''(x) + c_2 f_2''(x) + \cdots + c_k f_k''(x) &= 0 \\
&\ \ \vdots \\
c_1 f_1^{(j)}(x) + c_2 f_2^{(j)}(x) + \cdots + c_k f_k^{(j)}(x) &= 0 \\
&\ \ \vdots
\end{aligned}$$

If we include only those equations up to the $(k-1)^{\text{st}}$ derivative, we get a system of k equations in k unknowns, which we can arrange as the vector equation

$$c_1 \begin{bmatrix} f_1(x) \\ f_1'(x) \\ \vdots \\ f_1^{(k-1)}(x) \end{bmatrix} + c_2 \begin{bmatrix} f_2(x) \\ f_2'(x) \\ \vdots \\ f_2^{(k-1)}(x) \end{bmatrix} + \cdots + c_k \begin{bmatrix} f_k(x) \\ f_k'(x) \\ \vdots \\ f_k^{(k-1)}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Test for linear independence of sufficiently-differentiable functions on domain (a, b). If the determinant

$$W(f_1, f_2, \ldots, f_k)(t) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

is zero for all $a < x < b$, then the set of functions $S = \{f_1, f_2, \ldots, f_k\}$ is **linearly dependent** on (a, b). On the other hand, if there is *any* x in (a, b) at which this determinant $W(f_1, \ldots, f_k)(x) \neq 0$, then S is **linearly independent** on (a, b).

The determinant above is called the **Wronskian** of the functions $\{f_1, \ldots, f_k\}$.

Example 7:

Above we showed that $\{1, \sin^2 x, \cos^2 x\}$ is linearly *dependent* on $\mathbb{R}$. What about $\{1, \sin x, \cos x\}$? The Wronskian is

$$\begin{aligned} W(1, \sin, \cos)(x) &= \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} \\ &= -\cos^2 x - \sin^2 x \;=\; -1, \end{aligned}$$

for all real x. These functions are linearly independent on $(-\infty, \infty)$.

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For more examples, follow this link https://youtu.be/zw9rkAD3BEI?t=181.