Recurrence relations

Homogeneous or not?

term introducing nonhomogeneity

1) $a_n = 2a_{n-1} + n - 3a_{n-2}$

2) $a_n = 2a_{n-1} + 3$

3) $a_n = a_{n-1} + a_{n-4}$

Linear or not

nonconstant coeff

1) $a_n = \underline{(3n)}\,\underline{a_{n-1}} + \underline{a_{n-4}}$ (linear)

2) $a_n = \underbrace{(a_{n-1})(a_{n-2})}_{\text{interaction between dep. var. values}} + 7a_{n-3}$ (nonlinear)

3) $a_n = \dfrac{1}{a_{n-1}}$ $\quad (a_{n-1})^{-1}$ — or any power would make it nonlinear

Ex) Atypical - special in being
- 1^{st}-order
- having a constant for nonhomog. term

$\underline{a_n = 2a_{n-1} + 3,}$ $\quad a_0 = 1$

Also true

$\rightarrow$ $a_{n-1} = 2a_{n-2} + 3$

$\underline{a_{n-2}} = 2a_{n-3} + 3$

and so on

Starting with

$$a_n = 2a_{n-1} + 3 = 2(2a_{n-2} + 3) + 3$$

insert

$$= 2^2 a_{n-2} + 3 + 3(2)$$

$$= 2^2(2a_{n-3} + 3) + 3 + 3(2)$$

$$= 2^3 a_{n-3} + 3 + 3(2) + 3(2^2)$$

$$= \underline{2^4} a_{n-4} + 3[1 + 2 + 2^2 + 2^3]$$

= after n steps

$$= \underline{2^n} a_0 + 3\underbrace{[1 + 2 + 2^2 + \cdots + 2^{n-1}]}_{\text{sum of terms of a geometric sequence w/ } a_0 = 3,\ r = 2}$$

From earlier in semester

$$a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^{n-1} = a_0 \cdot \frac{1 - r^n}{1 - r}$$

Thus

$$a_n = 2^n \cdot \overset{=1}{a_0} + 3 \cdot \frac{1 - 2^n}{\underset{=-1}{1 - 2}} \quad \text{explicit soln.}$$

$$= 2^n - 3(1 - 2^n)$$

$$= 2^n - 3 + 3(2^n)$$

$$= 4(2^n) - 3.$$

Ex.) Needs more general approach

$$a_n = \underbrace{a_{n-1} + 2a_{n-2}}_{\text{2nd degree recurrence}} + 2n, \qquad a_0 = 1, \ a_1 = -1$$

Characterize this as

- 2nd degree
- nonhomog.
- constant coeff

Attack method:

① Solve the related homog. recurrence ignoring ICs

② Find an appropriate form for a specific/particular soln. of the nonhomog. problem

③ Put solns. from ① and ② together

④ Choose weights α_1, α_2 (...) to fit ICs.

Applied to the above

① Solve

$$a_n = a_{n-1} + 2a_{n-2}$$

using our prior method: assume $a_n = r^n$.

$$r^n = r^{n-1} + 2r^{n-1}$$

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

$$r^{n-2}\left(r^2 - r - 2\right) = 0$$

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0 \implies r = -1, 2$$

Both $(-1)^n$, $n = 0, 1, 2, \ldots$

and 2^n, $n = 0, 1, 2, \ldots$

satisfy the homog. recurrence, but so does a lin. comb

$$\alpha_1 (-1)^n + \alpha_2 \cdot (2^n)$$

(2) Back to nonhomog. problem

$$a_n = a_{n-1} - 2a_{n-2} + \underbrace{2n}_{\text{a 1st degree poly. in } n}$$

$an + b$

propose a specific soln. to this recurrence may take the form of a 1st-degree poly. in n.

insert into nonhomog. problem

$$\tilde{a}_n = mn + b$$

(here m, b are numbers to be found later)

$$\tilde{a}_{n-1} = m(n-1) + b$$

$$\tilde{a}_{n-2} = m(n-2) + b$$

$$\underbrace{mn+b}_{\text{role of } a_n} = \underbrace{\left[m(n-1)+b\right]}_{\text{role of } a_{n-1}} + 2\underbrace{\left[m(n-2)+b\right]}_{\text{role of } a_{n-2}} + 2n$$

$$\underline{mn} + \cancel{b} = \underline{mn} - m + \cancel{b} + \underline{2mn} - 4m + 2b + 2n$$

$$-2mn + 5m - 2b = 2n + 0$$

both sides are 1st degree polys. in n

RHS: $\underline{2}\, n + \underline{0}$

LHS: $\underline{-2m}\, n + \underline{5m-2b}$

Need to be same function

$\Updownarrow$

equate coeffs

Now can determine m, b

$$\begin{cases} 2 = -2m \\ 0 = 5m - 2b \end{cases}$$

$$m = -1$$

$$b = -5/2$$

Our proposal worked w/ these m, b

$$\tilde{a}_n = -n - \frac{5}{2}$$

(3) Put together formulas

$$a_n = \underbrace{\alpha_1 (-1)^n + \alpha_2 (2^n)}_{\text{formula from (1)}} \underbrace{- n - \frac{5}{2}}_{\text{formula from (2)}} \qquad (*)$$

(4) Finally, to satisfy ICs $a_0 = 1$, $a_1 = -1$:

$$1 = a_0 = \alpha_1 (-1)^0 + \alpha_2 (2)^0 - 0 - \frac{5}{2}$$

or $\quad 1 = \alpha_1 + \alpha_2 - \frac{5}{2} \Rightarrow \alpha_1 + \alpha_2 = \frac{7}{2}$

$$-1 = a_1 = \alpha_1 (-1)^1 + \alpha_2 (2)^1 - 1 - \frac{5}{2}$$

or $\quad -1 = -\alpha_1 + 2\alpha_2 - \frac{7}{2} \Rightarrow -\alpha_1 + 2\alpha_2 = \frac{5}{2}$

Our two equations in the unknowns α_1, α_2 are

$$\begin{aligned} \alpha_1 + \alpha_2 &= \frac{7}{2} \\ -\alpha_1 + 2\alpha_2 &= 5/2 \\ \hline 3\alpha_2 &= 6 \end{aligned} \Rightarrow \alpha_2 = 2, \quad \alpha_1 = \frac{3}{2}$$

Thus we have our solution to the original recurrence relation + ICs:

$$\boxed{a_n = \frac{3}{2}(-1)^n + 2(2)^n - n - \frac{5}{2}}$$