Solutions

1. By Green's Theorem, we have

$$\begin{aligned}\oint_C 6xy\,dx + (2x^3y-1)\,dy &= \int_1^2\int_1^3 (6x^2y-6x)\,dy\,dx = \int_1^2 \Big[3x^2y^2-6xy\Big]_1^3\,dx \\ &= \int_1^2 [(27x^2-18x)-(3x^2-6x)]\,dx = \int_1^2 (24x^2-12x)\,dx \\ &= \Big[8x^3-6x^2\Big]_1^2 = 38.\end{aligned}$$

2. If we use D to denote the solid box (the surface and interior), then by the Divergence Theorem,

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\mathbf{N}\,dS &= \iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \left[\frac{\partial}{\partial x}(xy^2)+\frac{\partial}{\partial z}(-xz)\right]dV \\ &= \int_0^1\int_0^2\int_0^3 (y^2-x)\,dx\,dy\,dz = \int_0^1\int_0^2 \Big[xy^2-\frac{1}{2}x^2\Big]_0^3\,dy\,dz \\ &= \int_0^1\int_0^2 \Big(3y^2-\frac{9}{2}\Big)\,dy\,dz = \int_0^1 \Big[y^3-\frac{9}{2}y\Big]_0^2\,dz \\ &= \int_0^1 (-1)\,dz = -1.\end{aligned}$$

3. (a) The field $\mathbf{F}$ has curl

$$\begin{aligned}\text{curl}\,\mathbf{F} &= \left\langle \tfrac{\partial}{\partial y}(2yz-3)-\tfrac{\partial}{\partial z}(x+z^2),\, \tfrac{\partial}{\partial z}(y)-\tfrac{\partial}{\partial x}(2yz-3),\, \tfrac{\partial}{\partial x}(x+z^2)-\tfrac{\partial}{\partial y}(y)\right\rangle \\ &= \langle 2z-2z, 0-0, 1-1\rangle = \mathbf{0}.\end{aligned}$$

Since the components of $\mathbf{F}$ are continuously differentiable throughout 3D-space, a zero curl means $\mathbf{F}$ is conservative, and is the gradient of some potential function $f(x,y,z)$. We have

$$\begin{aligned} f_x(x,y,z) = y &\quad\Rightarrow\quad f(x,y,z) = \int y\,dx = xy+g(y,z). \\ f_y(x,y,z) = x+g_y(y,z) = x+z^2 &\quad\Rightarrow\quad f(x,y,z) = xy+\int z^2\,dy = xy+yz^2+h(z) \\ f_z(x,y,z) = 2yz+h'(z) = 2yz-3 &\quad\Rightarrow\quad f(x,y,z) = xy+yz^2-\int 3\,dz = xy+yz^2-3z+C\end{aligned}$$

The question asks for a single potential, so we may take $C=0$.

(b) Using the potential function, the line integral equals the difference of values of f at the endpoints of the curve:

$$f(0,1,1)-f(1,1,0) = (0+1-3)-(1+0-0) = -3.$$

4. The one on the right is orientable. There are two edge curves, and both must be oriented so that travel along an edge results in the surface staying on one's left.

5. Since along our surface $z=f(x,y)=8-x^2-y^2$, it follows that, if we employ the surface parametrization (with parameters x, y) $\mathbf{r}(x,y)=x\mathbf{i}+y\mathbf{j}+(1-x^2-y^2)\mathbf{k}$, for $-2\le x\le 2$, $-2\le y\le 2$, we then get

$$dS = \|\mathbf{r}_x\times\mathbf{r}_y\|\,dA = \sqrt{f_x^2+f_y^2+1}\,dA = \sqrt{1+4x^2+4y^2}\,dA\,.$$

Thus, our surface area is

$$\iint_S dS \;=\; \int_{-2}^{2}\int_{-2}^{2} \sqrt{1+4x^2+4y^2}\,dy\,dx\ .$$

6. We have

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_0^1 (t^4\mathbf{i}+t^2\mathbf{j})\cdot(\mathbf{i}+2t\mathbf{j})\,dt \;=\; \int_0^1 (t^4+2t^3)\,dt \;=\; \frac{1}{5}t^5+\frac{1}{2}t^3\Big|_0^1 \;=\; \frac{7}{10}.$$

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Coordinate changes:

$x = \rho\sin\phi\cos\theta$	$\rho^2 = x^2+y^2+z^2$
$y = \rho\sin\phi\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = \rho\cos\phi$	$\cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$
$r = \rho\sin\phi$	$\rho^2 = r^2+z^2$
$z = \rho\cos\phi$	$\cos\phi = \dfrac{z}{\sqrt{r^2+z^2}}$

Trig identities:

$$\begin{aligned}\sin^2\theta &= \tfrac{1}{2}[1-\cos(2\theta)]\\ \cos^2\theta &= \tfrac{1}{2}[1+\cos(2\theta)]\\ \sin(2\theta) &= 2\cos\theta\sin\theta\end{aligned}$$

Expansion factors:

$$\begin{aligned}dA &= r\,dr\,d\theta\\ dV &= r\,dr\,d\theta\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta\end{aligned}$$

For parametrized curve C: $\mathbf{r}(t)$, $a\le t\le b$,

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,dt$$

For parametrized surface S: $\mathbf{r}(u,v)$, (u,v) in D,

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\,\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA$$

For a 3-D vector field $\mathbf{F} = \langle P,Q,R\rangle$,

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}$$

Fundamental Theorems: Under various requirements of differentiability, smoothness, and orientation,

FT of Calculus: If $F'(x)=f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(b)-F(a)$.

FT of Line Integrals: If $\mathbf{F}=\nabla f$, and the curve C has endpoints A and B, then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = f(B)-f(A)$.

Green's Theorem 1: For $\mathbf{F}=\langle P,Q\rangle$, $\displaystyle\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx+Q\,dy$

Stokes' Theorem: $\displaystyle\iint_S \nabla\times\mathbf{F}\cdot\mathbf{N}\,dS = \oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the edge curve of S

Green's Theorem 2: For $\mathbf{F}=\langle P,Q\rangle$, $\displaystyle\iint_R\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dA = \oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \oint_C -Q\,dx+P\,dy$

Divergence Theorem: $\displaystyle\iiint_R \nabla\cdot\mathbf{F}\,dV = \oiint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$