Answers to "Further exercises"

1. (a) $x' = 3x + 2y, \quad y' = 2x$

   (b) One eigenvalue is $\lambda = 4$ with basis eigenvector $\langle 1, 1/2\rangle$. The other eigenvalue is $\lambda = -1$ with basis eigenvector $\langle 1, -2\rangle$. A fundamental matrix has the corresponding solutions as columns

   $$\mathbf{\Phi}(t) \;=\; \begin{bmatrix} e^{4t} & e^{-t} \\ (1/2)e^{4t} & -2e^{-t} \end{bmatrix}.$$

   Since each eigenvalue is simple, there is no doubt this is a fundamental matrix.

   (c) We solve $\mathbf{\Phi}(0)\mathbf{c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or

   $$\begin{bmatrix} 1 & 1 \\ 1/2 & -2 \end{bmatrix}\mathbf{c} \;=\; \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{to get} \qquad \mathbf{c} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

   Thus, the solution is of the IVP is

   $$\mathbf{x}(t) \;=\; \mathbf{\Phi}(t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} \;=\; 2\begin{bmatrix} e^{4t} \\ (1/2)e^{4t} \end{bmatrix} + 0\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} \;=\; \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix} \;=\; e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

   (d) From (c), the solution is an exponentially-increasing scalar multiple of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   (e) We solve $\mathbf{\Phi}(0)\mathbf{c} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, or

   $$\begin{bmatrix} 1 & 1 \\ 1/2 & -2 \end{bmatrix}\mathbf{c} \;=\; \begin{bmatrix} 2 \\ -4 \end{bmatrix} \qquad \text{to get} \qquad \mathbf{c} \;=\; \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

   Thus, the solution is of the IVP is

   $$\mathbf{x}(t) \;=\; 0\begin{bmatrix} e^{4t} \\ (1/2)e^{4t} \end{bmatrix} + 2\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} \;=\; \begin{bmatrix} 2e^{-t} \\ -4e^{-t} \end{bmatrix} \;=\; 2e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

   Thus, the solution is an exponentially-decaying scalar multiple of the vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

   (f) We solve $\mathbf{\Phi}(0)\mathbf{c} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, or

   $$\begin{bmatrix} 1 & 1 \\ 1/2 & -2 \end{bmatrix}\mathbf{c} \;=\; \begin{bmatrix} 3 \\ -3 \end{bmatrix} \qquad \text{to get} \qquad \mathbf{c} \;=\; \begin{bmatrix} 6/5 \\ 9/5 \end{bmatrix}.$$

   Thus, the solution is of the IVP is

   $$\mathbf{x}(t) \;=\; \frac{6}{5}\begin{bmatrix} e^{4t} \\ (1/2)e^{4t} \end{bmatrix} + \frac{9}{5}\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} \;=\; \frac{3}{5}e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{9}{5}e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

   So, as time progresses, the component of the solution in the direction $\langle 1, -2\rangle$ becomes negligible while the component in the direction $\langle 2, 1\rangle$ dominates.

2. (a) One eigenvalue is $\lambda = 5$ with basis eigenvector $\langle 1, 1/2\rangle$. The other eigenvalue is $\lambda = 2$ with basis eigenvector $\langle 1, -1\rangle$. A fundamental matrix has the corresponding solutions as columns

$$\mathbf{\Phi}(t) \;=\; \begin{bmatrix} e^{5t} & e^{2t} \\ (1/2)e^{5t} & -e^{2t} \end{bmatrix}.$$

Since each eigenvalue is simple, there is no doubt this is a fundamental matrix.

(b) We solve $\mathbf{\Phi}(0)\mathbf{c} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, or

$$\begin{bmatrix} 1 & 1 \\ 1/2 & -1 \end{bmatrix}\mathbf{c} \;=\; \begin{bmatrix} -2 \\ -1 \end{bmatrix} \qquad \text{to get} \qquad \mathbf{c} \;=\; \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Thus, the solution is of the IVP is

$$\mathbf{x}(t) \;=\; \mathbf{\Phi}(t)\begin{bmatrix} -2 \\ 0 \end{bmatrix} \;=\; -2\begin{bmatrix} e^{5t} \\ (1/2)e^{5t} \end{bmatrix} + 0\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} \;=\; \begin{bmatrix} -2e^{5t} \\ -e^{5t} \end{bmatrix} \;=\; e^{5t}\begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

(c) From (c), the solution is an exponentially-increasing scalar multiple of the vector $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

(d) We solve $\mathbf{\Phi}(0)\mathbf{c} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, or

$$\begin{bmatrix} 1 & 1 \\ 1/2 & -1 \end{bmatrix}\mathbf{c} \;=\; \begin{bmatrix} 3 \\ -3 \end{bmatrix} \qquad \text{to get} \qquad \mathbf{c} \;=\; \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Thus, the solution is of the IVP is

$$\mathbf{x}(t) \;=\; 0\begin{bmatrix} e^{5t} \\ (1/2)e^{5t} \end{bmatrix} + 3\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} \;=\; \begin{bmatrix} 3e^{2t} \\ -3e^{2t} \end{bmatrix} \;=\; 3e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus, the solution is an exponentially-growing scalar multiple of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(e) We solve $\mathbf{\Phi}(0)\mathbf{c} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, or

$$\begin{bmatrix} 1 & 1 \\ 1/2 & -1 \end{bmatrix}\mathbf{c} \;=\; \begin{bmatrix} 2 \\ -4 \end{bmatrix} \qquad \text{to get} \qquad \mathbf{c} \;=\; \begin{bmatrix} -4/3 \\ 10/3 \end{bmatrix}.$$

Thus, the solution is of the IVP is

$$\mathbf{x}(t) \;=\; -\frac{4}{3}\begin{bmatrix} e^{5t} \\ (1/2)e^{5t} \end{bmatrix} + \frac{10}{3}\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} \;=\; -\frac{2}{3}e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{10}{3}e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So, as time progresses, the component of the solution in the direction $\langle 2, 1\rangle$ becomes increasingly dominant, and bends the trajectory in the direction of $\langle 2, 1\rangle$.

3. (a) The matrix has simple eigenvalue 2 with corresponding basis eigenvector $\langle 1,1,1\rangle$. It also has the repeated eigenvalue (-1) which has geometric multiplicity 2, having two basis eigenvectors $\langle 1,0,-1\rangle$ and $\langle 0,1,-1\rangle$. Thus

$$\phi(t) = \begin{bmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{bmatrix}.$$

(b) We can check the determinant of $\mathbf{\Phi}(0)$ and see that it is nonzero.