$$f(n) := n \cdot f(n-1)$$

Solving k^{th}-degree linear recurrence relations
Come in 2 varieties

1. Homogeneous: examples include

(a) $a_n = a_{n-1} + 2a_{n-2}$

(b) $a_n = -3a_{n-5}$

2. Nonhomogeneous: examples include

(a) $a_n = 3a_{n-1} - a_{n-2} + \underline{5n - 7}$

(b) $a_n = -6a_{n-2} + \underline{2^n}$

(c) $a_n = 4a_{n-1} + \underline{2}$

presence of terms w/out a_j's make these nonhomog.

Focus on <u>homog.</u> linear recurrences w/ <u>constant coeffs.</u>

necessary for the method we discuss: Assume $a_n = r^n$

Last time: Fibonacci recurrence

$$f_n = f_{n-1} + f_{n-2}, \quad IC \quad f_0 = \underline{0},\ f_1 = \underline{1}$$

Assume $f_n = r^n$, then insert

Got

$$r^n = r^{n-1} + r^{n-2}$$

$$r^{n-2}(r^2 - r - 1) = 0$$

Solved $r^2 - r - 1 = 0$

to get roots

$$r_1 = \frac{1+\sqrt{5}}{2}, \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

Both roots generate seqs. that solve the recurrence

$$r_1 : \quad 1, \quad \frac{1+\sqrt{5}}{2}, \quad \left(\frac{1+\sqrt{5}}{2}\right)^2, \quad \left(\frac{1+\sqrt{5}}{2}\right)^3, \quad \dots \quad (r_1)^n$$

$$r_2 : \quad 1, \quad \frac{1-\sqrt{5}}{2}, \quad \left(\frac{1-\sqrt{5}}{2}\right)^2, \quad \dots \quad (r_2)^n$$

But they don't satisfy the ICs.

$$\alpha_1 r_1^n + \alpha_2 r_2^n$$

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \qquad (\star)$$

α_1, α_2 are to be determined using ICs

1st IC $\quad 0 = f_0 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha_1 + \alpha_2$

2nd IC $\quad 1 = f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$

So, we have 2 eqns. in the two unknowns α_1, α_2

$$\alpha_1 + \alpha_2 = 0 \longrightarrow \alpha_1 = -\alpha_2$$

Substitute into 2^{nd} eqn

$$\alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$-\alpha_2\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\alpha_2\left(-\cancel{\frac{1}{2}} \underline{- \frac{\sqrt{5}}{2}} + \cancel{\frac{1}{2}} \underline{- \frac{\sqrt{5}}{2}}\right) = 1$$

$$-\sqrt{5}\,\alpha_2 = 1 \implies \alpha_2 = \frac{-1}{\sqrt{5}}$$

$$\alpha_1 = \frac{1}{\sqrt{5}}$$

So, the correct mixture (rewriting (☆))

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Explicit formula for Fibonacci nos.

Ex.] $\underline{a_n = a_{n-1} + 2a_{n-2}}$, $a_0 = 1$, $a_1 = 8$

Assume $a_n = r^n$

$$r^n = r^{n-1} + 2r^{n-2}$$

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

$$r^{n-2}\left(r^2 - r - 2\right) = 0$$

char. eqn. $\quad r^2 - r - 2 = 0 \quad$ (2^{nd} degree)

$$(r-2)(r+1) = 0$$

$\Longrightarrow$ roots $\quad r_1 = -1, \quad r_2 = 2$

These roots generate sequences

$(r_1)^n$: $1, -1, 1, -1, 1, \ldots$

$\uparrow (-1)^0 \quad \uparrow (-1)^1 \quad \uparrow (-1)^2$

In a different problem w/ a ICs

$a_0 = 1$

$a_1 = 2$

$(r_2)^n$: $1, 2, 4, 8, 16, \ldots$

$\uparrow 2^0 \quad \uparrow 2^1 \quad \uparrow 2^2$

Both these seqs. satisfy the recurrence eqn., but should check whether they satisfy the ICs. Neither does.

$$\boxed{a_n = \alpha_1(-1)^n + \alpha_2(2^n) \quad (*)}$$

Choose α_1, α_2 using eqns. arising from ICs

1^{st} IC

$$1 = a_0 = \alpha_1(-1)^0 + \alpha_2(2)^0 = \alpha_1 + \alpha_2$$

2^{nd} IC

$$8 = a_1 \overset{(*)}{=} \alpha_1(-1) + \alpha_2 \cdot 2$$

2 eqs.

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ -\alpha_1 + 2\alpha_2 &= 8 \end{aligned}\right\} \text{ solve for } \alpha_1, \alpha_2$$