7.26) b) Impulse invariance = aliasing

Bilinear Transformation = warping frequency axis

$\boxed{\text{Bilinear Transform}}$

d) Bilinear Transform

e) Impulse Invariance:

$$H_1(z) = \sum_{k=-\infty}^{\infty} H_{c1}\left(s - j\frac{2\pi}{T_d}k\right)$$

$$H_2(z) = \sum_{k=-\infty}^{\infty} H_{c2}\left(s - j\frac{2\pi}{T_d}k\right)$$

$$H_1(z)H_2(z) = \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} H_{c1}\left(s - j\frac{2\pi}{T_d}k\right) H_{c2}\left(s - j\frac{2\pi}{T_d}\ell\right) \neq H(z)$$

Bilinear Transformation:

$$H_1(z) = H_{c1}\left[\frac{2}{T_d}\ \frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H_2(z) = H_{c2}\left[\frac{2}{T_d}\ \frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H_1(z)H_2(z) = H_{c1}\left[\frac{2}{T_d}\ \frac{1-z^{-1}}{1+z^{-1}}\right] H_{c2}\left[\frac{2}{T_d}\ \frac{1-z^{-1}}{1+z^{-1}}\right] = H_c\left[\frac{2}{T_d}\ \frac{1-z^{-1}}{1+z^{-1}}\right] = H(z)$$

$\boxed{\text{Bilinear Transformation}}$

7.27) (a) $H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{4} \\ 0, & \pi/4 < |\omega| \le \pi \end{cases}$

$$H(e^{j\omega}) \xrightarrow{\mathcal{F}^{-1}} h[n]$$

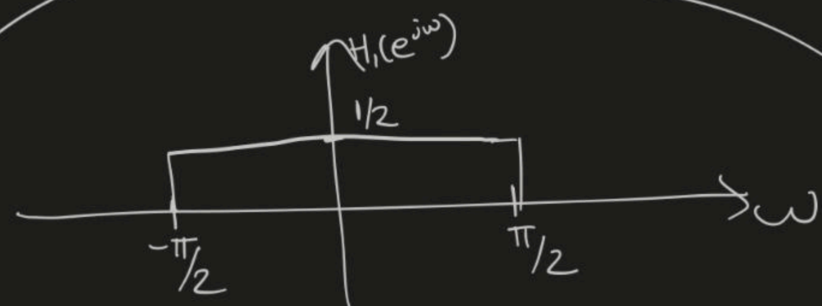
$$h[2n] \xrightarrow{\mathcal{F}} H_1(e^{j\omega})$$

$$H_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[2n]\, e^{-j\omega n}$$

$$= \sum_{\substack{n=2k,\\ k=-\infty}}^{\pi/4} h[n]\, e^{-\frac{j\omega n}{2}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(h[n] + (-1)^n h[n]\right) e^{-\frac{j\omega n}{2}}$$

$$= \frac{1}{2} H\left(e^{\frac{j\omega}{2}}\right) + \frac{1}{2} H\left(e^{j\frac{\omega+2\pi}{2}}\right)$$



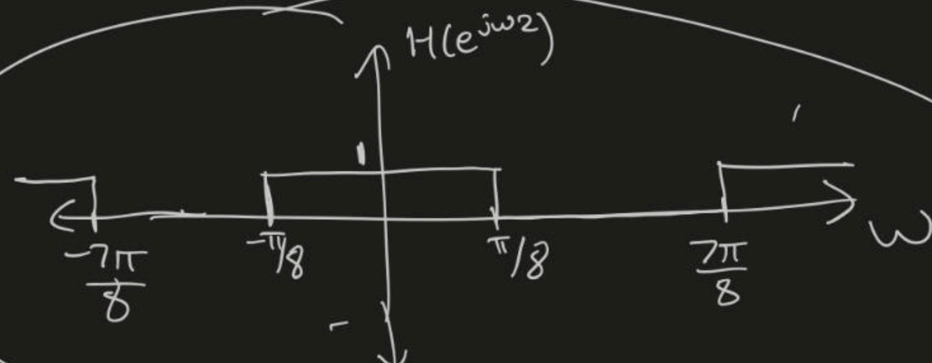
b) $h_2[n] = \begin{cases} h[n/2], & n = 0, \pm 2, \pm 4, \dots \\ 0, & \text{otherwise} \end{cases}$

$$H_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_2[n]\, e^{-j\omega n}$$

$$= \sum_{\substack{n=2k \\ k=-\infty}}^{\infty} h[n/2]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega 2n}$$

$$= H(e^{j\omega 2})$$

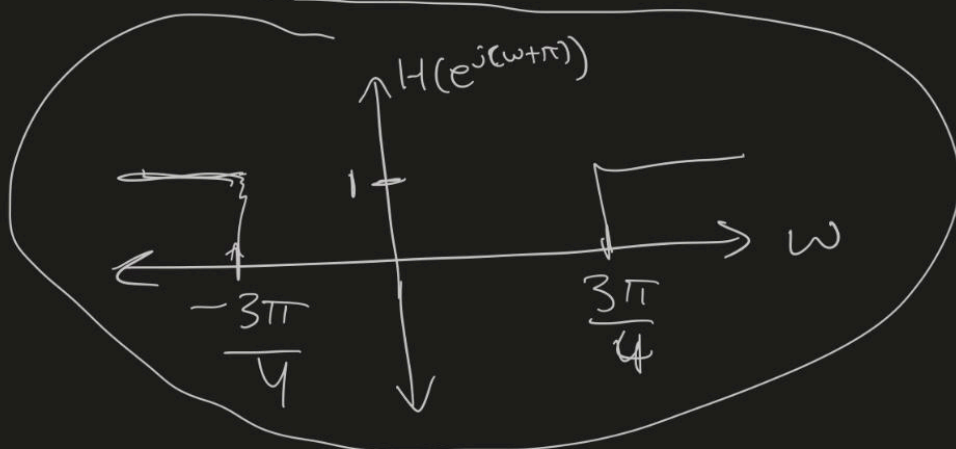


c) $h_3[n] = (-1)^n h[n]$

$$H_3(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (-1)^n h[n] e^{-j\omega n}$$

$$= \sum_{n \text{ even}} h[n] e^{-j\omega n} - \sum_{n \text{ odd}} h[n] e^{-j\omega n}$$

$$= H\left(e^{j(\omega+\pi)}\right)$$



3) Specs:

$0.98 < H(e^{j\omega}) < 1.02 \quad , \quad 0 \le |\omega| \le 0.63\pi$

$-0.15 < H(e^{j\omega}) < 0.15 \quad , \quad 0.65\pi \le |\omega| \le \pi$

Analog Filter Specs:

$0.98 < H(j\Omega) < 1.02 \quad , \quad 0 \le \Omega \le 2\tan\left(\frac{0.63\pi}{2}\right)$

$-0.15 < H(j\Omega) < 0.15 \quad , \quad 2\tan\left(\frac{0.65\pi}{2}\right) \le \Omega \le \infty$

Design Constraints:

$T_d = 1$

$|H_c(j2\tan(0.315\pi))| > 0.98$

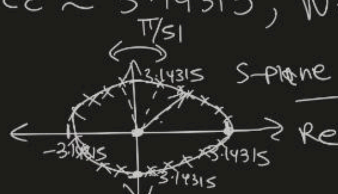
$|H_c(j2\tan(0.325\pi))| < 0.15$

For Butterworth filter, $|H(j\Omega)|^2 = \dfrac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$ and due to its monotonic response:

$$1+\left(\frac{2\tan(0.315\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.98}\right)^2$$

$$1+\left(\frac{2\tan(0.325\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.15}\right)^2$$

$$\Omega_c \simeq 3.14315 \,,\ N = 51$$



→ There will be 102 poles around the circle the S-plane

$$H_c(s) = \frac{(\Omega_c)^N}{\prod_{k=1}^{N}(s-s_k)}$$ → Rest of steps are done using MATLAB code below

4) $\omega_{pd} = \frac{2}{T_d} \tan\left(\frac{\omega_p}{2}\right)$

$\omega_{sd} = \frac{2}{T_d} \tan\left(\frac{\omega_s}{2}\right)$

$H_1(s)$ will have $T_d = T_1$ designed to pre-warp ω_p \& ω_s

$H_2(s)$ will have $T_d = T_2$ with same pre-warped frequencies

$H_2(s) = H_1\left(s\frac{T_1}{T_2}\right)$

Starting from $H_1(s) \longrightarrow H_1(z) = H_1\left(\frac{2}{T_1}\ \frac{1-z^{-1}}{1+z^{-1}}\right)$

$$H_2(z) = H_2\left(\frac{2}{T_2}\ \frac{1-z^{-1}}{1+z^{-1}}\right) = H_1\left(\frac{2}{T_2}\ \frac{1-z^{-1}}{1+z^{-1}}\ \frac{T_1}{T_2}\right)$$

$$= H_1\left(\frac{2T_1}{T_2^2}\ \frac{1-z^{-1}}{1+z^{-1}}\right)$$

In the appropriate bilinear transformation:

$$\frac{2}{T_1}\ \cancel{\frac{1-z^{-1}}{1+z^{-1}}} = \frac{2T_1}{T_2^2}\ \cancel{\frac{1-z^{-1}}{1+z^{-1}}}$$

$$\frac{\cancel{2}}{T_1} = \frac{\cancel{2}T_1}{T_2^2} \longrightarrow \frac{1}{T_1} = \frac{T_1}{T_2^2} \longrightarrow T_1^2 = T_2^2$$

Since $T_1, T_2 \geq 0$:

$T_1 = T_2$

Given a bilinear transformation where $T_d = T_1 = T_2$, both $H_1(s)$ and $H_2(s)$ will result in the same digital filter $H(z)$.