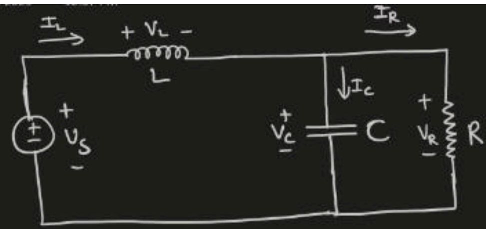
1) a)



$u(t) = V_S(t)$

$y(t) = V_C(t)$

$x_1(t) = I_L(t)$

$x_2(t) = V_C(t)$

$$\Longrightarrow \; x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$$

$\Downarrow$

$\dot{x}(t) = f(x(t), u(t))$

$y(t) = h(x(t), u(t))$

$\Longrightarrow$

$I_L(t) = I_C(t) + I_R(t)$

$I_C(t) = C\dot{V}_C(t)$

$I_R(t) = \frac{1}{R} V_R(t) = \frac{1}{R} V_C(t)$

$I_L(t) = C\dot{V}_C(t) + \frac{1}{R} V_C(t)$

$x_1(t) = C\dot{x}_2(t) + \frac{1}{R} x_2(t)$

$\dot{x}_2(t) = \frac{1}{C} x_1(t) - \frac{1}{RC} x_2(t)$

$\dot{I}_L(t) = \frac{1}{L} V_L(t) = \frac{1}{L}\left(V_S(t) - V_C(t)\right)$

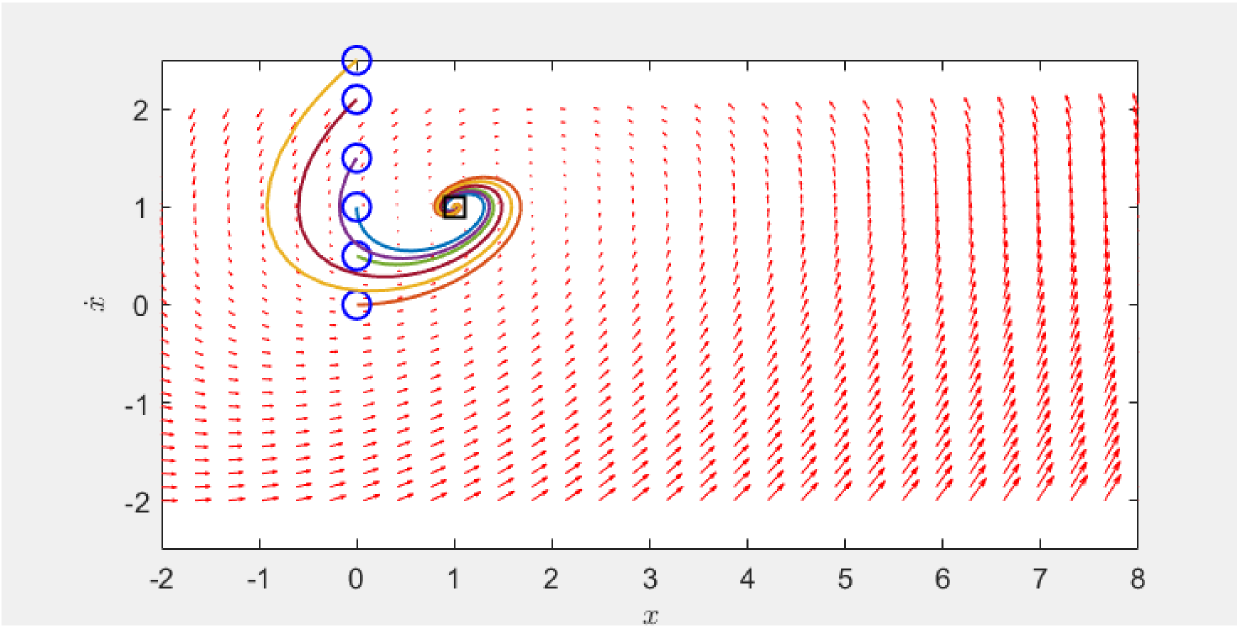
$\dot{x}_1(t) = \frac{1}{L}\left(u(t) - x_2(t)\right)$

$y(t) = x_2(t)$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t)$$

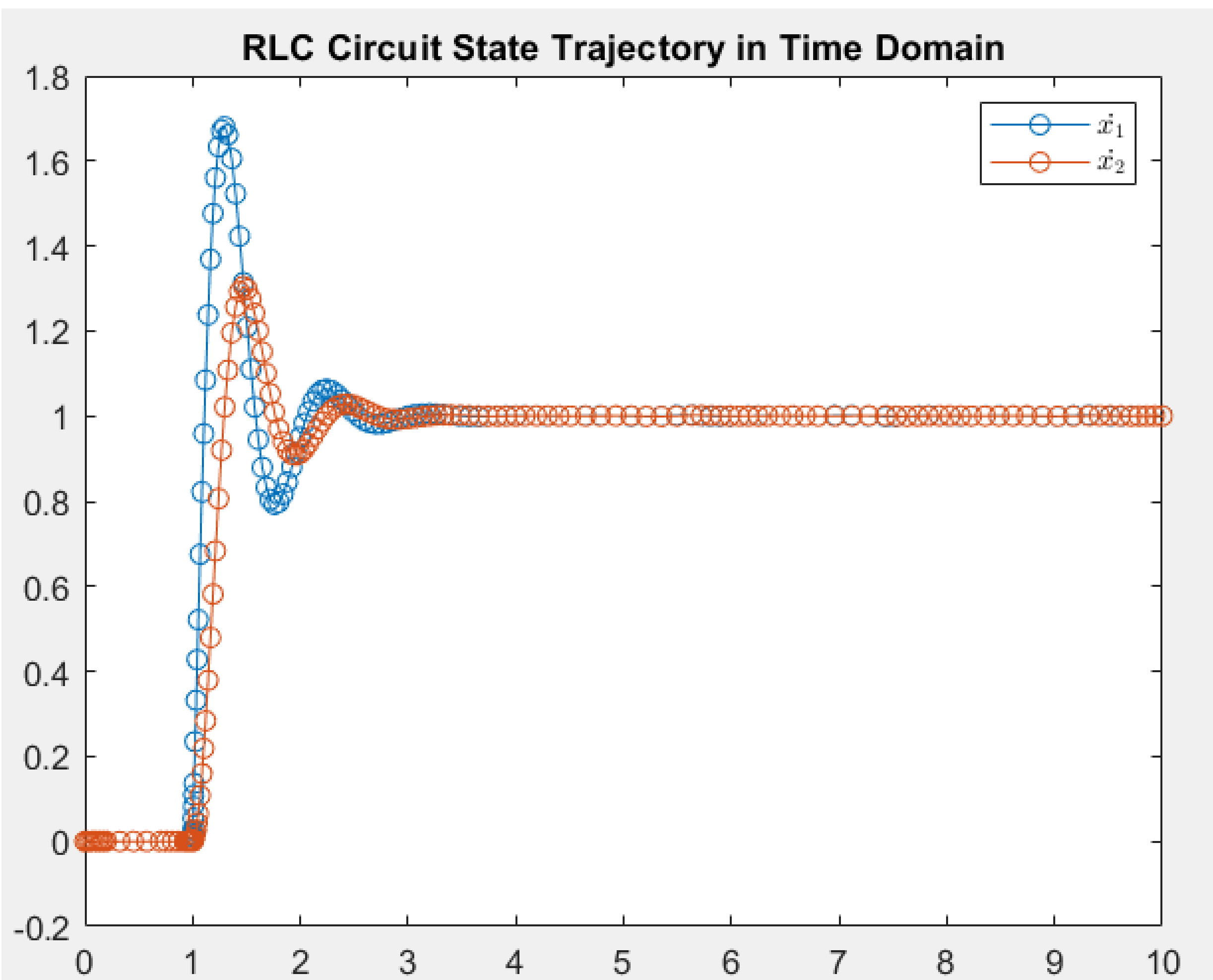
$$\boxed{\begin{aligned} y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ \dot{x}(t) &= \begin{bmatrix} 0 & -1/L \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t) \end{aligned}}$$

b)



$\dot{x}$
x

c)



RLC Circuit State Trajectory in Time Domain
$\dot{x_1}$
$\dot{x_2}$

2) a) $m\ddot{q}(t) + c\dot{q}(t) + kq(t) = F(t) = u(t)$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix}$$

$$\dot{q}(t) = x_2(t)$$

$$\ddot{q}(t) = \frac{u(t) - kq(t) - c\dot{q}(t)}{m}$$

$$= \frac{1}{m}u(t) - \frac{k}{m}x_1(t) - \frac{c}{m}x_2(t)$$

$$\boxed{\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)}$$

$$y(t) = q(t) = x_1(t)$$

$$\boxed{y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}$$

b) $\dot{x} = Ax + Bu$

$y = Cx$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ -\frac{1}{L} V_c(t) \end{bmatrix}$$

$$y(t) = V_c(t) = -L\,x_2(t)$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{I}_L(t) \\ -\frac{1}{L} \dot{V}_c(t) \end{bmatrix}$$

$$\dot{I}_L(t) = \frac{1}{L} V_L(t) = \frac{1}{L}\left(V_s(t) - V_c(t)\right) = \frac{1}{L} u(t) + x_2(t)$$

$$-\frac{1}{L}\dot{V}_c(t) = \frac{-1}{LC} I_c(t) = -\frac{1}{LC}\left(I_L(t) - I_R(t)\right)$$

$$= -\frac{1}{LC}\left(I_L(t) - \frac{1}{R} V_R(t)\right)$$

$$= -\frac{1}{LC}\left(I_L(t) - \frac{1}{R} V_c(t)\right)$$

$$= -\frac{1}{LC} I_L(t) - \frac{1}{RC}\left(\frac{-1}{L} V_c(t)\right)$$

$$= -\frac{1}{LC} x_1(t) - \frac{1}{RC} x_2(t)$$

$$\boxed{\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t)}$$

$$\boxed{y(t) = \begin{bmatrix} 0 & -L \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}$$

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-1}{RC} \end{bmatrix}} \quad \boxed{B = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}} \quad \boxed{C = \begin{bmatrix} 0 & -L \end{bmatrix}}$$

c) Both the spring-mass system and RLC circuit can be described by second-order differential equations written in state-space form.

Both these systems have similar structure in that the states for both systems can be defined in the form of $\dot{x}(t) = f(x(t), u(t))$ and $y(t) = h(x(t), u(t))$.

3) a) $\dot{z} = f(z, u)$

$$Z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix}$$

$$\dot{z}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos(\theta(t)) \\ V(t)\sin(\theta(t)) \\ \omega(t) \end{bmatrix}$$

$$\dot{z}_1(t) = V(t)\cos(\theta(t)) = u_1(t)\cos(z_3(t))$$

$$\dot{z}_2(t) = V(t)\sin(\theta(t)) = u_1(t)\sin(z_3(t))$$

$$\dot{z}_3(t) = \omega(t) = u_2(t)$$

$$\boxed{\dot{z}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \end{bmatrix} = \begin{bmatrix} u_1(t)\cos(z_3(t)) \\ u_1(t)\sin(z_3(t)) \\ u_2(t) \end{bmatrix}}$$

b) $\omega(t) = 0$

$$V(t) = \begin{cases} c, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$(x(0), y(0)) = (0,0)$$

$$\theta(0) = 0$$

$$\dot{\theta}(t) = \omega(t)$$

$$\theta(t) = \theta(0) + \int_0^t \omega(\tau)\, d\tau$$

$$= 0 + 0 = 0$$

$$\boxed{\theta(t) = 0}$$

$$\dot{x}(t) = V(t)\cos(\theta(t)) = V(t) = c$$

$$x(t) = x(0) + \int_0^t c\, d\tau = c\tau \Big|_0^t = ct$$

$$\boxed{x(t) = ct}$$

$$\dot{y}(t) = V(t)\sin(\theta(t)) = 0$$

$$y(t) = y(0) = 0$$

$$\boxed{y(t) = 0}$$

c) $w(t) = b$

$$v(t) = \begin{cases} c, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$x(0) = 0$

$y(0) = -\frac{c}{b}$

$\theta(0) = 0$

$\dot{\theta}(t) = w(t) = b$

$$\theta(t) = \theta(0) + \int_0^t b \, d\tau$$

$$= b\tau\Big|_0^t = bt$$

$$\boxed{\theta(t) = bt}$$

$$\dot{x}(t) = v(t)\cos(\theta(t)) = c\cos(bt)$$

$$x(t) = x(0) + \int_0^t c\cos(b\tau)\,d\tau = c\int_0^t \cos(b\tau)\,d\tau$$

$u = b\tau$

$du = b\,d\tau$

$d\tau = \frac{du}{b}$

$$\frac{c}{b}\int_0^t \cos(u)\,du = \frac{c}{b}\sin(u)\Big|_0^t = \frac{c}{b}\sin(b\tau)\Big|_0^t = \frac{c}{b}\sin(bt)$$

$$x(t) = \frac{c}{b}\sin(bt)$$

$$\dot{y}(t) = v(t)\sin(\theta(t)) = \begin{cases} c\sin(bt), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$y(t) = y(0) + \int_0^t c\sin(b\tau)\,d\tau = \frac{-c}{b} + c\int_0^t \sin(b\tau)\,d\tau$$

$u = b\tau$

$du = b\,d\tau$

$d\tau = \frac{du}{b}$

$$\frac{c}{b}\int_0^t \sin(u)\,du = \frac{-c}{b}\cos(u)\Big|_0^t = \frac{-c}{b}\cos(b\tau)\Big|_0^t = \frac{-c}{b}\cos(bt) + \frac{c}{b}$$

$$y(t) = \cancel{-\frac{c}{b}} - \frac{c}{b}\cos(bt) + \cancel{\frac{c}{b}}$$

$$\boxed{y(t) = -\frac{c}{b}\cos(bt)}$$

$$r = \sqrt{x(t)^2 + y(t)^2} = \sqrt{\left(\frac{c}{b}\sin(bt)\right)^2 + \left(\frac{-c}{b}\cos(bt)\right)^2}$$

$$= \sqrt{\frac{c^2}{b^2}\sin^2(bt) + \frac{c^2}{b^2}\cos^2(bt)}$$

$$= \sqrt{\frac{c^2}{b^2}} = \frac{c}{b}$$

$$\boxed{r = \frac{c}{b}}$$

$\boxed{\text{The radius of the position is equal to } \frac{v(t)}{\omega(t)}.}$

Question 4 is answered in the section below.

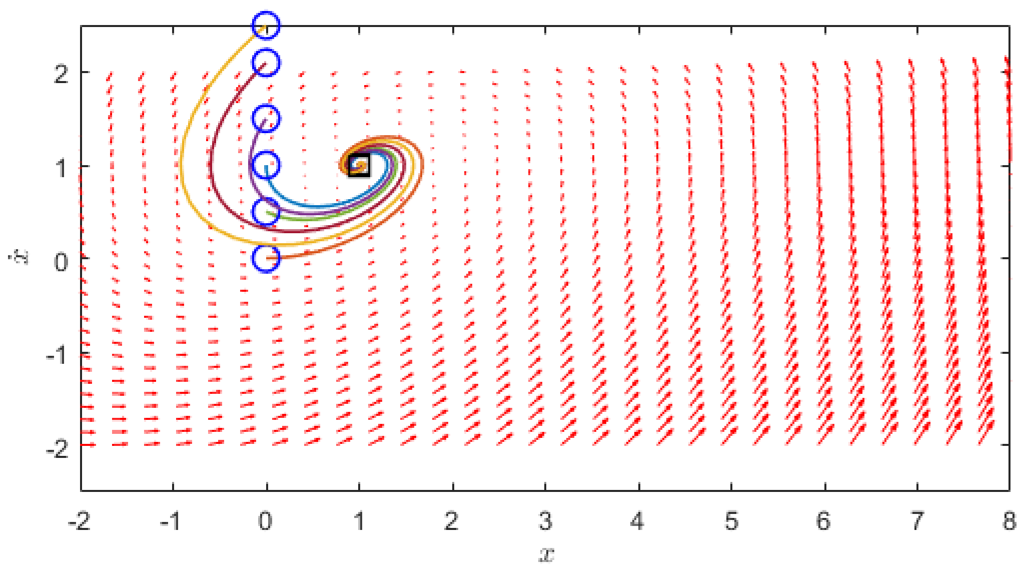
Table of Contents

1b)

Define the variables

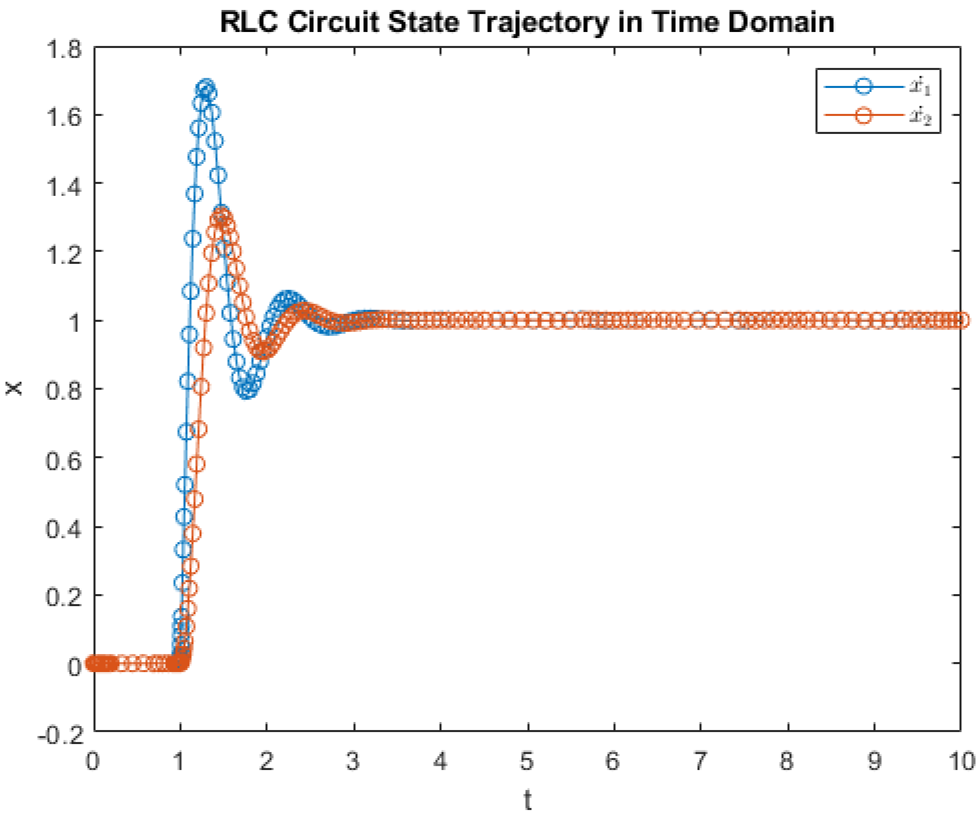
```
L = 0.1;
Vs = 1;
C = 0.2;
R = 1;

% Define the functions
f = @(t,x) [(-1/L) * x(2) + (1/L) * Vs;
            (1/C) * x(1) - (1/(R*C)) * x(2)];

% Step 2: Create a grid of, e.g., 30x30 points.
y1 = linspace(-2,8,30);
y2 = linspace(-2,2,30);

% Step 3: creates two matrices one for all the x-values on the grid, and one
for
% all the y-values on the grid.
% Note that x and y are matrices of the same
% size and shape, in this case 20 rows and 20 columns
[x,y] = meshgrid(y1,y2);

% Step 4: computing the vector field
u = zeros(size(x));
v = zeros(size(x));

% we can use a single loop over each element to compute the derivatives at
% each point (y1, y2)
t=0;   % we want the derivatives at each point at t=0, i.e. the starting time
for i = 1:numel(x)
    Yprime = f(t,[x(i); y(i)]);
    u(i) = Yprime(1);
    v(i) = Yprime(2);
end

% Step 5: we use the quiver command to plot our vector field

figure; quiver(x,y,u,v,'r');
xlabel('$x$','Interpreter','latex')
ylabel('$\dot{x}$','Interpreter','latex')
axis tight equal;
```

```
set(gcf,'Position',[150 150 600 300])

% Step 6: Plotting solutions on the vector field
% Let's plot a few solutions on the vector field.
% We will consider the solutions where y1(0)=0, and values of y2(0) = [0 0.5
1 1.5 2.1 2.5],
% in otherwords, we start the pendulum at an angle of zero, with some
angular velocity.
hold on
for y20 = [0 0.5 1 1.5 2.1 2.5]
    [ts,ys] = ode45(f,[0,50],[0;y20]);      % ode45 simulations
    plot(ys(:,1),ys(:,2),'linewidth',1.2)
    plot(ys(1,1),ys(1,2),'bo','MarkerSize',10,'LineWidth',1.2)         %
starting point
    plot(ys(end,1),ys(end,2),'ks','MarkerSize',10,'LineWidth',1.2)     %
ending point
end
ylim([-2.5,2.5]); xlim([-2,8]);
hold off
```



1c)

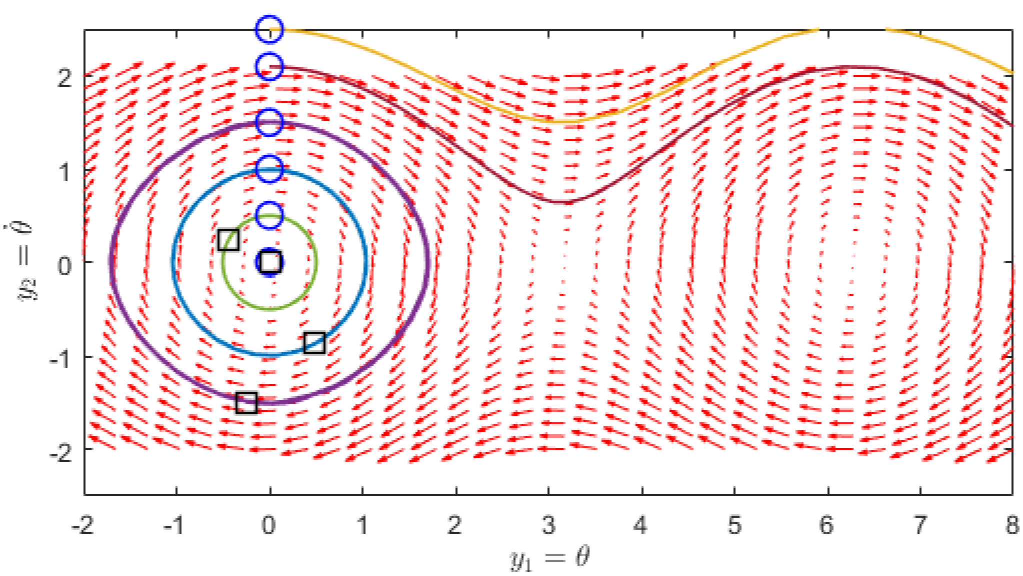
```
fc = @(t,x) [( (t < 1)*0 + (t >= 1)*1 - x(2) )/L;
             (1/C) * x(1) - (1/(R*C)) * x(2)];

figure;
ode45(fc, [0 10], [0 0]);
xlabel('t');
ylabel('x');
title('RLC Circuit State Trajectory in Time Domain');
legend('$\dot{x_1}$', '$\dot{x_2}$','Interpreter','latex');
```



4a)

```
f4a = @(t,x)[x(2); -sin(x(1))];
u4a = zeros(size(x));
v4a = zeros(size(x));

% we can use a single loop over each element to compute the derivatives at
% each point (y1, y2)
t = 0;
for i = 1:numel(x)
    Yprime = f4a(t,[x(i); y(i)]);
    u4a(i) = Yprime(1);
    v4a(i) = Yprime(2);
end

% Step 5: we use the quiver command to plot our vector field

figure; quiver(x,y,u4a,v4a,'r');
xlabel('$y_1=\theta$','Interpreter','latex')
ylabel('$y_2=\dot{\theta}$','Interpreter','latex')
axis tight equal;
set(gcf,'Position',[150 150 600 300])


% Step 6: Plotting solutions on the vector field
```

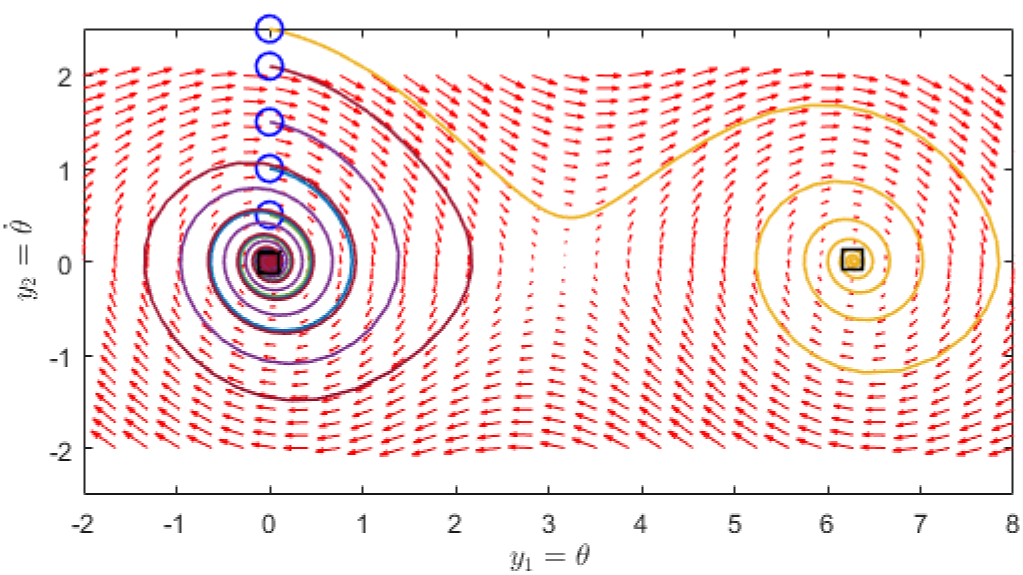
```
% Let's plot a few solutions on the vector field.
% We will consider the solutions where y1(0)=0, and values of y2(0) = [0 0.5
1 1.5 2.1 2.5],
% in otherwords, we start the pendulum at an angle of zero, with some
angular velocity.
hold on
for y20 = [0 0.5 1 1.5 2.1 2.5]
    [ts,ys] = ode45(f4a,[0,50],[0;y20]);     % ode45 simulations
    plot(ys(:,1),ys(:,2),'linewidth',1.2)
    plot(ys(1,1),ys(1,2),'bo','MarkerSize',10,'LineWidth',1.2)        %
starting point
    plot(ys(end,1),ys(end,2),'ks','MarkerSize',10,'LineWidth',1.2)    %
ending point
end
ylim([-2.5,2.5]); xlim([-2,8]);
hold off

%Figure Description:
% This figure displays a center around an equilibrium point when the
% initial y points are 0, 0.5, 1, and 1.5. When the initial points are 2.1
% and 2.5, then the shape of the plot is a curve that doesn't approach the
% equilibrium point at any point in time.
```



4b)

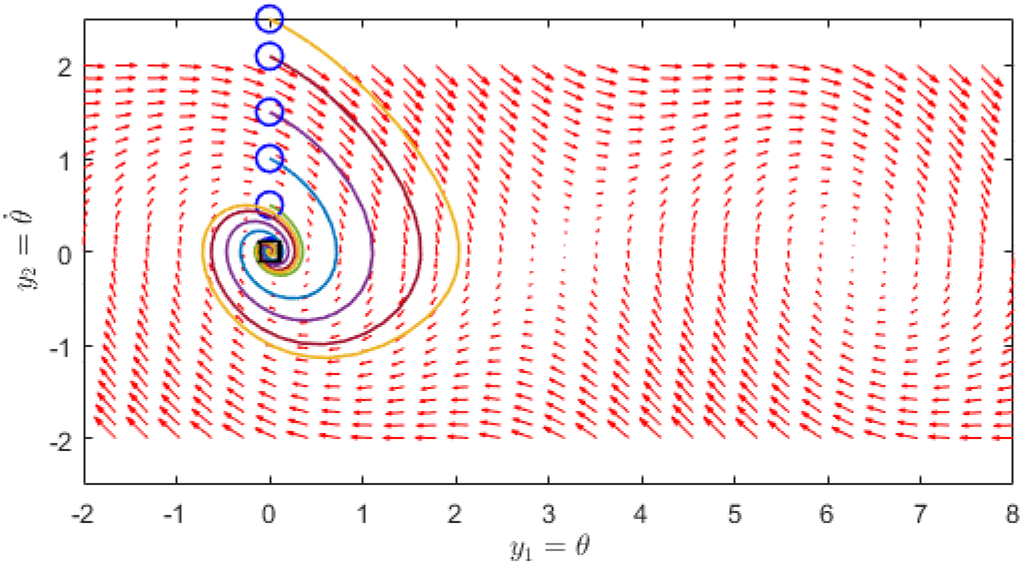
```
f4b = @(t,x)[x(2); (-0.2*x(2))-sin(x(1))];
u4b = zeros(size(x));
v4b = zeros(size(x));

% we can use a single loop over each element to compute the derivatives at
% each point (y1, y2)
t = 0;
for i = 1:numel(x)
```

```
    Yprime = f4b(t,[x(i); y(i)]);
    u4b(i) = Yprime(1);
    v4b(i) = Yprime(2);
end

% Step 5: we use the quiver command to plot our vector field

figure; quiver(x,y,u4b,v4b,'r');
xlabel('$y_1=\theta$','Interpreter','latex')
ylabel('$y_2=\dot{\theta}$','Interpreter','latex')
axis tight equal;
set(gcf,'Position',[150 150 600 300])

% Step 6: Plotting solutions on the vector field
% Let's plot a few solutions on the vector field.
% We will consider the solutions where y1(0)=0, and values of y2(0) = [0 0.5
1 1.5 2.1 2.5],
% in otherwords, we start the pendulum at an angle of zero, with some
angular velocity.
hold on
for y20 = [0 0.5 1 1.5 2.1 2.5]
    [ts,ys] = ode45(f4b,[0,50],[0;y20]);      % ode45 simulations
    plot(ys(:,1),ys(:,2),'linewidth',1.2)
    plot(ys(1,1),ys(1,2),'bo','MarkerSize',10,'LineWidth',1.2)         %
starting point
    plot(ys(end,1),ys(end,2),'ks','MarkerSize',10,'LineWidth',1.2)     %
ending point
end
ylim([-2.5,2.5]); xlim([-2,8]);
hold off

%Figure Description:
% This figure displays two sinks. This indicates that both the equilibrium
% points in this system are asymptotically stable.
```



4c)

```
f4c = @(t,x)[x(2); (-0.5*x(2))-sin(x(1))];
u4c = zeros(size(x));
v4c = zeros(size(x));

% we can use a single loop over each element to compute the derivatives at
% each point (y1, y2)
t = 0;
for i = 1:numel(x)
    Yprime = f4c(t,[x(i); y(i)]);
    u4c(i) = Yprime(1);
    v4c(i) = Yprime(2);
end

% Step 5: we use the quiver command to plot our vector field

figure; quiver(x,y,u4c,v4c,'r');
xlabel('$y_1=\theta$','Interpreter','latex')
ylabel('$y_2=\dot{\theta}$','Interpreter','latex')
axis tight equal;
set(gcf,'Position',[150 150 600 300])

% Step 6: Plotting solutions on the vector field
% Let's plot a few solutions on the vector field.
% We will consider the solutions where y1(0)=0, and values of y2(0) = [0 0.5
1 1.5 2.1 2.5],
% in otherwords, we start the pendulum at an angle of zero, with some
angular velocity.
hold on
for y20 = [0 0.5 1 1.5 2.1 2.5]
    [ts,ys] = ode45(f4c,[0,50],[0;y20]);     % ode45 simulations
    plot(ys(:,1),ys(:,2),'linewidth',1.2)
    plot(ys(1,1),ys(1,2),'bo','MarkerSize',10,'LineWidth',1.2)        %
starting point
    plot(ys(end,1),ys(end,2),'ks','MarkerSize',10,'LineWidth',1.2)    %
ending point
end
ylim([-2.5,2.5]); xlim([-2,8]);
hold off

%Figure Description:
% This figure displays a sink. This indicates that the equilibrium point is
% asymptotically stable.
```



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