1)a) $\frac{e(\infty)}{r} < 0.02$

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1+L(s)}$$

$$E(s) = R(s) - Y(s)$$

$$Y(s) = U(s)P(s)$$

$$U(s) = C(s)\,E(s)$$

$$E(s) = R(s) - E(s)L(s)$$

$$(1+L(s))E(s) = R(s)$$

$$E(s) = \frac{R(s)}{1+L(s)}$$

$$r(t) = r\cdot u(t) \xrightarrow{\ s\ } R(s) = \frac{r}{s}$$

$$\lim_{s\to 0} s\frac{R(s)}{1+L(s)} = \lim_{s\to 0}\frac{\left(\frac{\cancel{s}r}{\cancel{s}}\right)}{1+L(s)} = \frac{r}{1+L(0)}$$

$$\frac{\left(\frac{\cancel{r}}{1+L(0)}\right)}{\cancel{r}} < 0.02$$

$$\frac{1}{1+L(0)} < 0.02$$

$$1+L(0) > 50$$

$$L(0) > 49$$

$$\boxed{\text{The requirement on } L(s) \text{ is } \lim_{s\to 0} L(s) > 49}$$

b) $r(t) = \sin(\omega t)$

$$\left|\frac{E(i\omega)}{R(i\omega)}\right| = \left|\frac{1}{1+L(i\omega)}\right| < 0.1$$

$$|1+L(i\omega)| > 10$$

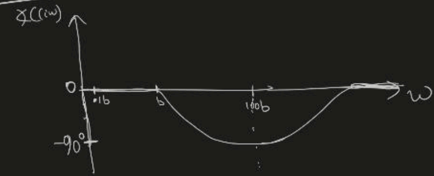
$$1+L(i\omega) > 10 \rightarrow L(i\omega) > 9$$
$$1+L(i\omega) < -10 \rightarrow L(i\omega) < -9 \quad \Rightarrow \quad |L(i\omega)| > 9$$

$\boxed{\text{The requirement on } L(s) \text{ is } |L(i\omega)| > 9.}$

c) $C(s) = K\frac{s+a}{s+b}$

Lag Compensator:

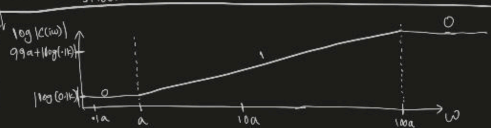
$C(s) = K\frac{s+100b}{s+b}$

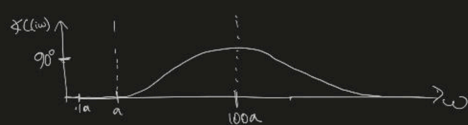


Lag compensation allows for higher magnitude $|L(i\omega)|$ at lower frequencies and the magnitude decreases as the frequency increases. There is also no phase at very low & high frequencies, which sets $\measuredangle L(i\omega)$ at 0 at these frequencies. For the closed-loop system, the magnitude $\left|\frac{L(i\omega)}{1+L(i\omega)}\right|$ will also be higher at low frequencies, and since $L(i\omega) = 100KP(i\omega)$, the sensitivity $\frac{1}{1+100KP(i\omega)}$ will be very small, which attenuates disturbance and K can be selected to meet the desired reference tracking error specifications.

Lead Compensator:

$C(s) = K\frac{s+a}{s+100a}$





Lead Compensation causes an increase in the magnitude $|L(i\omega)|$ as frequency increases. There is zero phase at very low and very high frequencies. For the closed-loop system, the magnitude $\left|\frac{L(i\omega)}{1+L(i\omega)}\right|$ increases with frequency and the sensitivity will be high at lower frequencies because of this, which will cause low disturbance attenuation at these frequencies. However, K can be adjusted to increase the phase margin at the gain crossover frequency.

d) Specs:

1) $\frac{e(\infty)}{r} < 2\%$

$\lim_{s\to 0} L(s) > 49$

$L(0) = P(0)C(0) > 49$

$\lim_{s\to 0} P(s) = 1$

$C(0) > 49$

$C(s) = K\frac{s+z}{s+p} \Rightarrow C(0) = K\frac{z}{p} > 49$

2) Tracking error $< 10\%$

$|L(i\omega)| > 9 \Rightarrow |L(i)| > 9$

$\Downarrow$

$|P(i)C(i)| > 9$

$P(0) = 1$

$$P(i) = \frac{1-e^{-0.25i}}{0.25i\,(1+i)} = \frac{1-(\cos(0.25)-i\sin(0.25))}{0.25i - 0.25} \approx \frac{0.031+0.247i}{-0.25+0.25i}$$

$$|P(i)| = \sqrt{\frac{0.031+0.247i}{-0.25+0.25i} \cdot \frac{0.031-0.247i}{-0.25-0.25i}} = \sqrt{\frac{0.000966+0.012}{0.125}} \approx 0.705$$

$$|P(i)C(i)| > 9 \Rightarrow |C(i)| > \frac{9}{0.705} \approx 12.761$$

Let $z = 10P$

$C(0) = K\frac{10P}{P} = 10K > 49 \Rightarrow K > 4.9$

Let $K = 5$

$$C(s) = 5\,\frac{s+z}{s+p}$$

$$C(i) = 5\,\frac{10p+i}{p+i}$$

$$|C(i)| = 5\sqrt{\frac{10p+i}{p+i}\cdot\frac{10p-i}{p-i}} = 5\sqrt{\frac{100p^2+1}{p^2+1}} > 12.761$$

$$\sqrt{\frac{100p^2+1}{p^2+1}} > 2.552$$

$$\frac{100p^2+1}{p^2+1} > 6.514$$

$$100p^2 + 1 > 6.514p^2 + 6.514$$

$$93.486p^2 - 5.514 > 0$$

$$p > 0.24286, \quad p < -0.24286$$

Let $p = 0.25 \implies z = 2.5$

$$C(0) = 5(10) = 50$$

$$|C(i)| = 5\sqrt{\frac{626}{1.625}} > 12.761$$

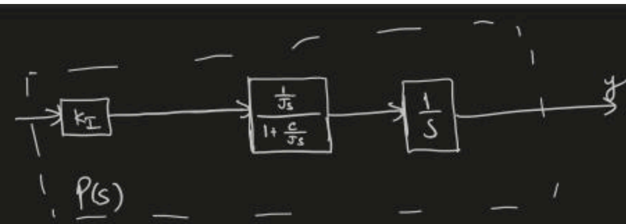
Let $C(s) = 5\,\frac{s+2.5}{s+0.25}$ be the system's lag controller

The tracking error with this controller is about 11.1%. This needs to be decreased.

I will try with a higher proportional gain value.

Making the controller $C(s) = 6\,\frac{s+2.5}{s+0.25}$ fixes this problem. The steady state error is 1.72% and reference tracking error is 9.25%. The phase margin is 32.6°.

2) a)



$$P(s) = \frac{\left(\frac{k_I}{Js}\right)\left(\frac{1}{s}\right)}{1+\frac{c}{Js}} = \frac{\frac{k_I}{Js^2}}{\left(\frac{Js+c}{Js}\right)} = \frac{k_I}{\cancel{J}s^{\cancel{2}}} \cdot \frac{\cancel{Js}}{Js+c} = \frac{k_I}{s(Js+c)} = \frac{1}{s(2s+1)}$$

$$\boxed{P(s) = \frac{1}{s(2s+1)}}$$

b) $$P(i\omega) = \frac{1}{i\omega(2i\omega+1)} = \frac{1}{-2\omega^2+\omega i} \cdot \frac{-2\omega^2-\omega i}{-2\omega^2-\omega i} = \frac{-2\omega^2-\omega i}{4\omega^4+\omega^2}$$

$$\operatorname{Re}\{P(i\omega)\} = \frac{-2\omega^2}{4\omega^4+\omega^2} \qquad \operatorname{Im}\{P(i\omega)\} = -\frac{\omega}{4\omega^4+\omega^2}$$

$\omega = 0$:

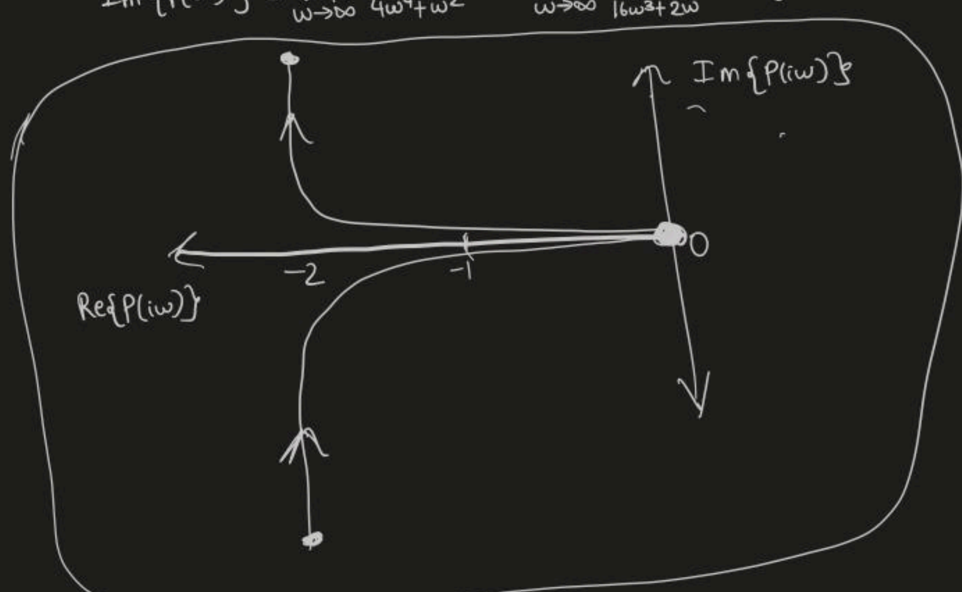
$$\operatorname{Re}\{P(0)\} = \lim_{\omega\to 0} \frac{-2\omega^2}{4\omega^4+\omega^2} = \lim_{\omega\to 0} \frac{-4\omega}{16\omega^3+2\omega} = \lim_{\omega\to 0} \frac{-4}{48\omega^2+2} = -2$$

$$\operatorname{Im}\{P(0)\} = \lim_{\omega\to 0} \frac{-\omega}{4\omega^4+\omega^2} = \lim_{\omega\to 0} \frac{-1}{16\omega^3+2\omega} = \pm\infty$$

$\omega = \infty$:

$$\operatorname{Re}\{P(\infty)\} = \lim_{\omega\to\infty} \frac{-2\omega^2}{4\omega^4+\omega^2} = \lim_{\omega\to\infty} \frac{-4\omega}{16\omega^3+2\omega} = \lim_{\omega\to\infty} \frac{-4}{48\omega^2+2} = 0$$

$$\operatorname{Im}\{P(\infty)\} = \lim_{\omega\to\infty} \frac{-\omega}{4\omega^4+\omega^2} = \lim_{\omega\to\infty} \frac{-1}{16\omega^3+2\omega} = 0$$



$\omega_{pc} = \infty$

$$g_m = \frac{1}{|P(i\omega_{pc})|} = \frac{1}{|P(\infty)|} = \frac{1}{\sqrt{(Re\{P(\infty)\})^2 + (Im\{P(\infty)\})^2}} = \frac{1}{0} = \infty$$

$$\boxed{g_m = \infty}$$

$$|P(i\omega_{gc})| = \sqrt{(Re\{P(i\omega_{gc})\})^2 + (Im\{P(i\omega_{gc})\})^2} = \sqrt{\left(\frac{-2\omega_{gc}^2}{4\omega_{gc}^4 + \omega_{gc}^2}\right)^2 + \left(-\frac{\omega_{gc}}{4\omega_{gc}^4 + \omega_{gc}^2}\right)^2}$$

$$= \sqrt{\frac{4\omega_{gc}^4 + \omega_{gc}^2}{16\omega_{gc}^8 + 8\omega_{gc}^6 + \omega_{gc}^4}} = 1$$

$$\frac{4\omega_{gc}^4 + \omega_{gc}^2}{16\omega_{gc}^8 + 8\omega_{gc}^6 + \omega_{gc}^4} = 1$$

$$4\omega_{gc}^4 + \omega_{gc}^2 = 16\omega_{gc}^8 + 8\omega_{gc}^6 + \omega_{gc}^4$$

$$16\omega_{gc}^8 + 8\omega_{gc}^6 - 3\omega_{gc}^4 - \omega_{gc}^2 = 0$$

$$\omega_{gc}^2 (16\omega_{gc}^6 + 8\omega_{gc}^4 - 3\omega_{gc}^2 - 1) = 0$$

$$\omega_{gc} = \cancel{-0.624811}, 0, 0.624811$$

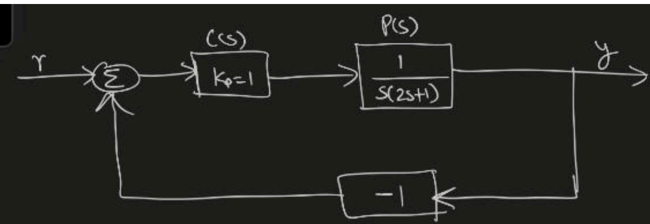
$$\omega_{gc} = 0.624811$$

$$\measuredangle P(i\omega) = \tan^{-1}\left(\frac{Im\{P(i\omega)\}}{Re\{P(i\omega)\}}\right) = \tan^{-1}\left(\frac{\cancel{-\omega}}{\cancel{4\omega^4+\omega^2}} \cdot \frac{4\omega^4 \cancel{+\omega^2}}{\cancel{-}2\omega\cancel{2}}\right) = \tan^{-1}\left(\frac{1}{2\omega}\right)$$

$$\varphi_m = \measuredangle P(i\omega_{gc}) = \tan^{-1}\left(\frac{1}{2(0.624811)}\right) + 180^\circ \approx 38.668^\circ$$

$$\boxed{\varphi_m = 38.668^\circ}$$

d)



$$G(s) = \frac{\frac{1}{s(2s+1)}}{1 + \frac{1}{s(2s+1)}} = \frac{\frac{1}{s(2s+1)}}{\frac{s(2s+1)+1}{s(2s+1)}} = \frac{1}{\cancel{s(2s+1)}} \cdot \frac{\cancel{s(2s+1)}}{s(2s+1)+1} = \frac{1}{s(2s+1)+1} = \frac{1}{2s^2+s+1}$$

$$2s^2+s+1 = 0$$

$$s = \frac{-1 \pm \sqrt{1-8}}{4} = -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$$

$$\boxed{\text{Poles (closed-loop)}: s = -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}}$$

No. of ccw encirclements around $s=-1$ $= 1$

no. of unstable open-loop poles $= 1$

$\boxed{\text{By the Nyquist stability criterion, this system is stable.}}$

f) $L_\tau(s) = L(s)e^{-\tau s}$

$$\angle e^{-j\omega\tau} = -\omega\tau$$

$$\varphi_m - \omega_{gc}\tau_{max} = 0$$

$$\varphi_m = 38.668° \approx 0.675 \text{ rad}$$

$$\omega_{gc} = 0.624811$$

$$\tau_{max} = \frac{0.675 \text{ rad}}{0.624811} \approx 1.08s.$$

$\boxed{\text{The range of stability is } 0 \leq \tau \leq 1.08s.}$

3) a)

a) $\tilde{P}(s) = P(s) + \Delta(s)$

b) $\tilde{P}(s) = P(s)(1+\delta(s))$

c) $\tilde{P}(s) = \dfrac{P(s)}{1-P(s)\Delta_{fb}(s)}$

b) Given $\delta(s) = \dfrac{\Delta(s)}{P(s)}$ $\qquad \Delta_{fb}(s) = \dfrac{\Delta(s)}{P(P+\Delta)} = \dfrac{\delta}{P(1+\delta)}$

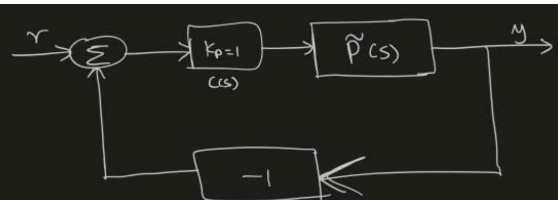
a) $\tilde{P}(s) = P(s)+\Delta(s)$

b) $\tilde{P}(s) = P(s)\left(1+\delta(s)\right) = P(s)\left(1+\dfrac{\Delta(s)}{P(s)}\right) = \cancel{P(s)}\left(\dfrac{P(s)+\Delta(s)}{\cancel{P(s)}}\right) = P(s)+\Delta(s)$

c) $\tilde{P}(s) = \dfrac{P(s)}{1-P(s)\Delta_{fb}(s)} = \dfrac{P(s)}{1-P(s)\left[\dfrac{\Delta(s)}{P(s)[P(s)+\Delta(s)]}\right]} = \dfrac{P(s)}{\left(\dfrac{P(s)[P(s)+\Delta(s)]-P(s)\Delta(s)}{P(s)[P(s)+\Delta(s)]}\right)} = \cancel{P(s)}\cdot\dfrac{\cancel{P(s)}[P(s)+\Delta(s)]}{\cancel{P(s)}\,\cancel{P(s)}} = P(s)+\Delta(s)$

All 3 expressions are equal.

c)



$$P(s) = \frac{1}{s(2s+1)} = \frac{1}{2s^2+1}$$

$$L(s) = C(s)\,P(s)$$

$$\tilde{L}(s) = C(s)\,\tilde{P}(s) = C(s)\left[P(s)+\Delta(s)\right] = C(s)P(s) + C(s)\Delta(s) = P(s)+\Delta(s)$$
$$= L(s)+\Delta(s)$$

Poles of closed-loop system are found by:

$$1+\tilde{L}(s) = 1+L(s)+\Delta(s) = 1+P(s)+\Delta(s)$$

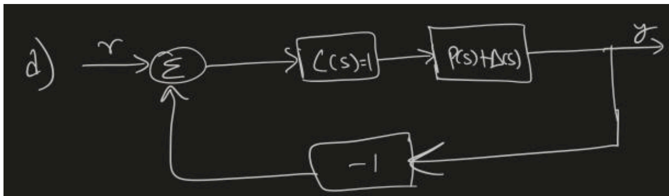
Closed-loop stability is preserved if $1+L(i\omega)+\Delta(i\omega)$ never encircles the origin

$$|\Delta(i\omega)| < |1+L(i\omega)| = |1+P(i\omega)| = \frac{1}{\left|\frac{1}{1+P(i\omega)}\right|}$$

Since $C(i\omega)=1$, $\dfrac{C}{1+L} = \dfrac{1}{1+P}$

$$\therefore\ |\Delta(i\omega)| < \frac{1}{\left|\frac{C(i\omega)}{1+L(i\omega)}\right|} = \frac{1}{\left|\frac{1}{1+P(i\omega)}\right|}$$

d)



$$\tilde{P}(s) = \frac{1}{s(2s+1)} + \frac{0.5e^{-s}}{1+s}$$

$$|\Delta(i\omega)| = \frac{0.5}{\sqrt{1+\omega^2}} \xrightarrow[\omega\to 0]{} 0.5$$

$$|1+P(i\omega)| = \sqrt{1+\frac{1}{\omega^2}} = \frac{\sqrt{1+\omega^2}}{\omega} \xrightarrow{\omega\to 0} \frac{1}{\omega} \to \infty$$

$$\frac{1}{\left|\frac{1}{1+P(i\omega)}\right|} \xrightarrow[\omega\to 0]{} 0$$

$$|\Delta(i\omega)| > \frac{1}{\left|\frac{1}{1+P(i\omega)}\right|}$$

$\boxed{\text{The proportional controller can't stabilize the true plant at small gains.}}$

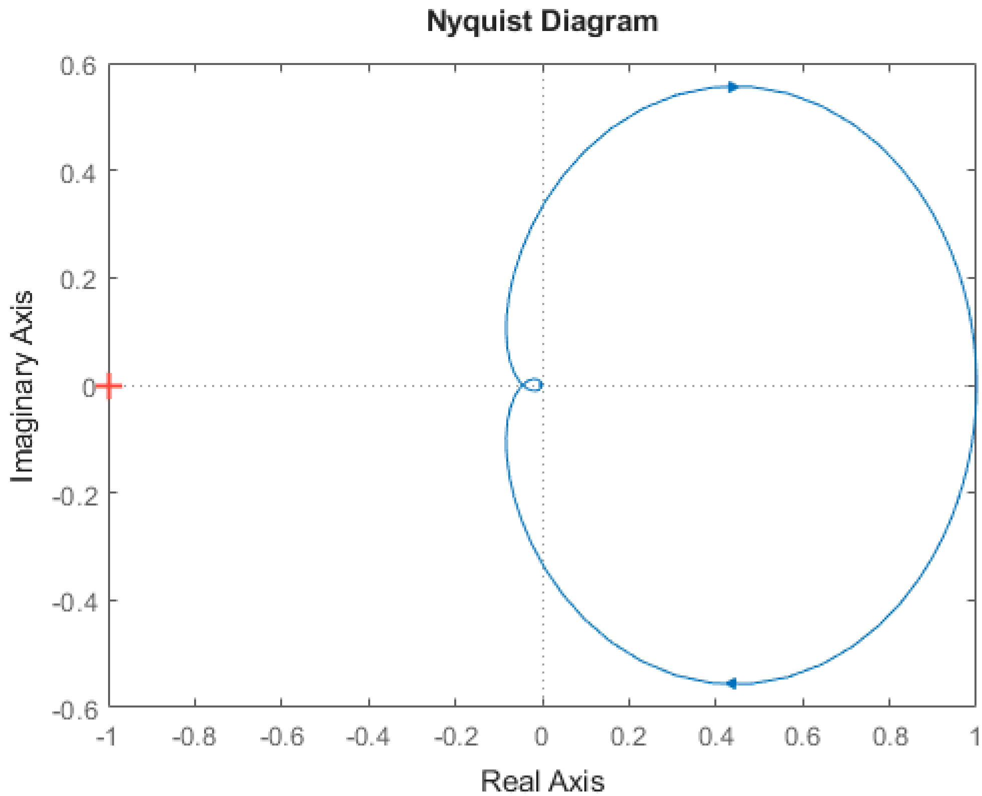
The MATLAB questions are answered in the section below.

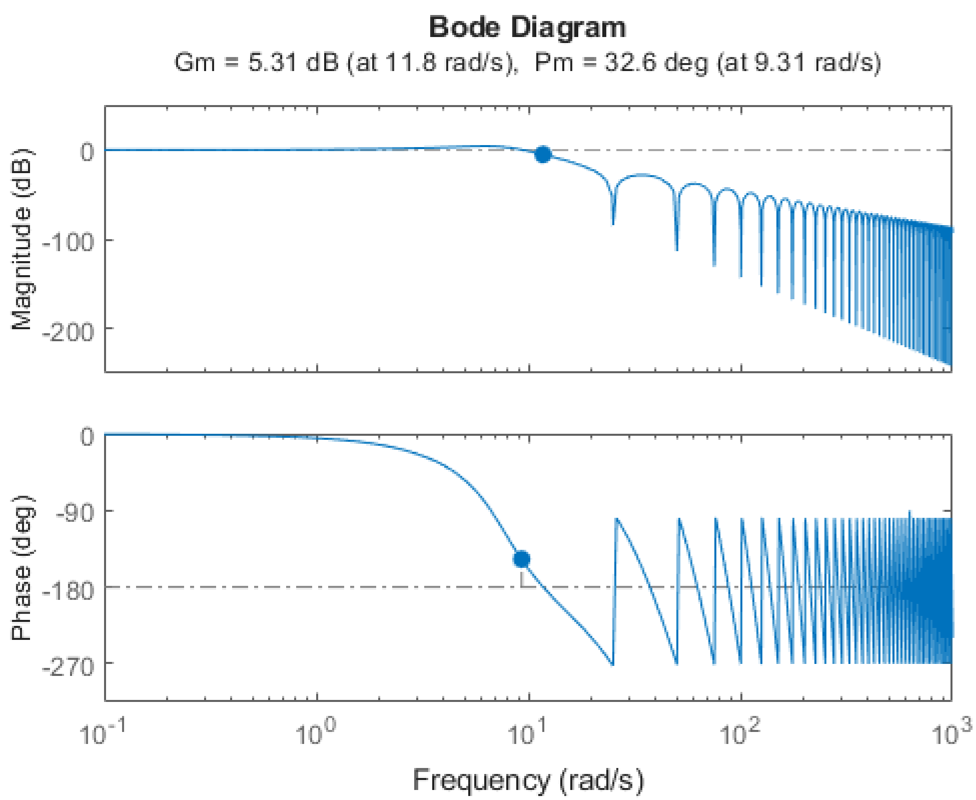
Table of Contents

1d)

```
a = 1;
tau = 0.25;
num = a;
den = [tau a*tau 0];
P = tf(num, den);
P_del = tf(num, den, 'InputDelay', tau);
P = P - P_del;
figure;
nyquist(P);
numC = [6 15];
denC = [1 0.25];
C = tf(numC, denC);
L = P*C;
G = feedback(L, 1);
figure;
margin(G);

%The Phase Margin in this plot is show to be 32.6 degrees, which is >= 30
%degrees.

[y, t] = step(G);
steady_state_error_step = abs(1 - y(end));
fprintf('Steady State Error: %f\n', steady_state_error_step);

H1 = freqresp(L, 1);
tracking_error = abs(1/(1 + H1));
fprintf('Reference Tracking Error: %f\n', tracking_error);
```

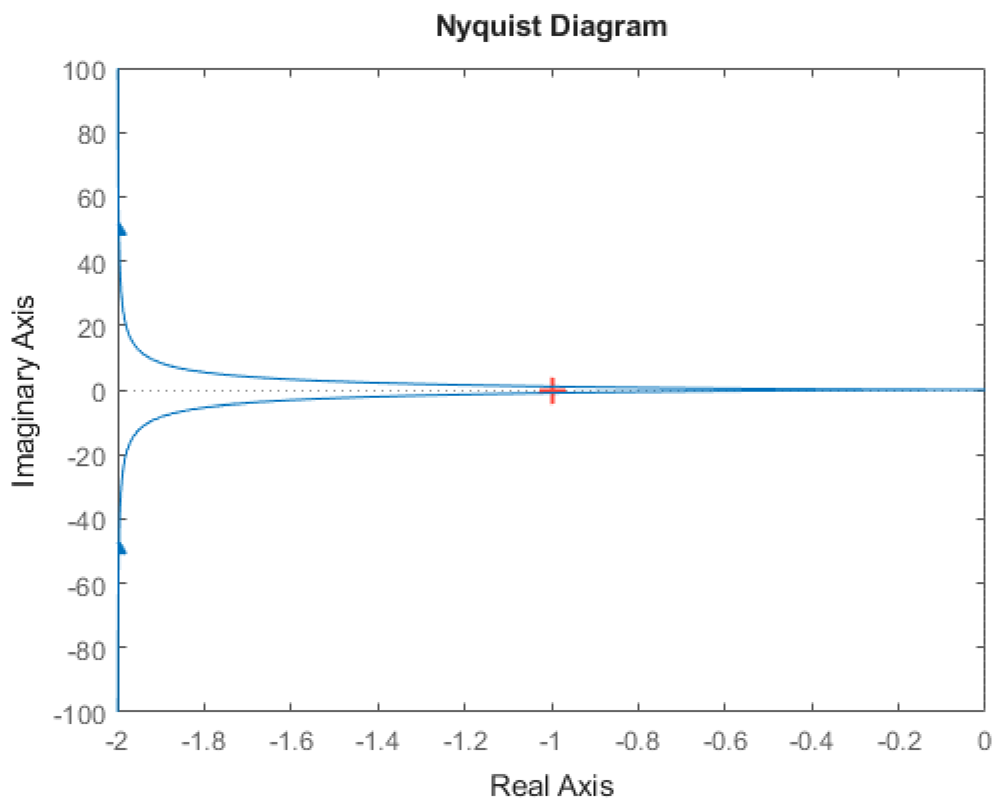
```
Steady State Error: 0.017158
Reference Tracking Error: 0.092461
```

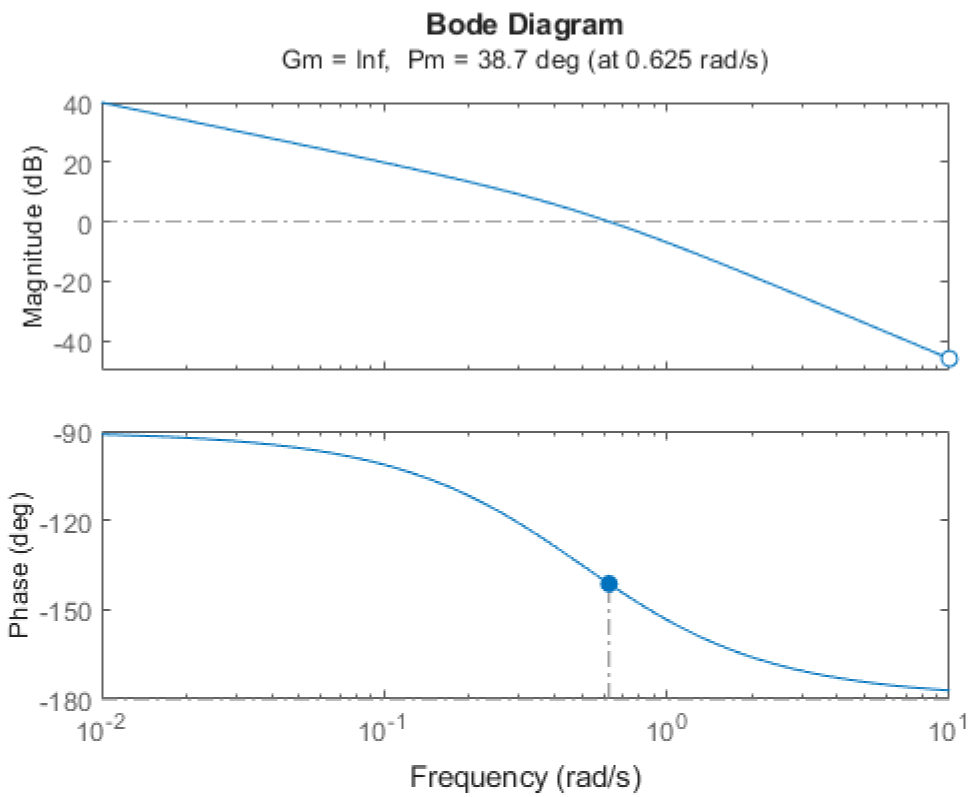

Nyquist Diagram
Imaginary Axis
Real Axis
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Bode Diagram
Gm = 5.31 dB (at 11.8 rad/s), Pm = 32.6 deg (at 9.31 rad/s)
Magnitude (dB)
0
-100
-200
Phase (deg)
0
-90
-180
-270
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

2c)

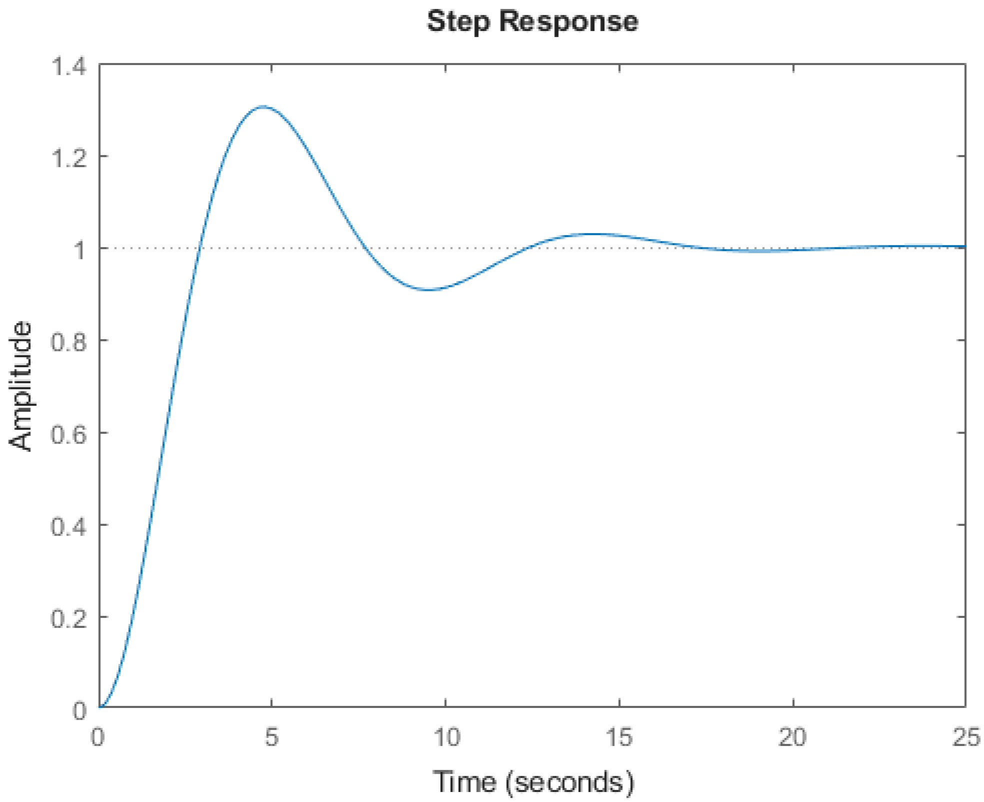
```
num = 1;
den = [2 1 0];
P = tf(num, den);
figure;
nyquist(P);
figure;
margin(P);

%These plots are consistent with my sketches in part b.
```





2d)

```
G = feedback(P, 1);
figure;
step(G);
```

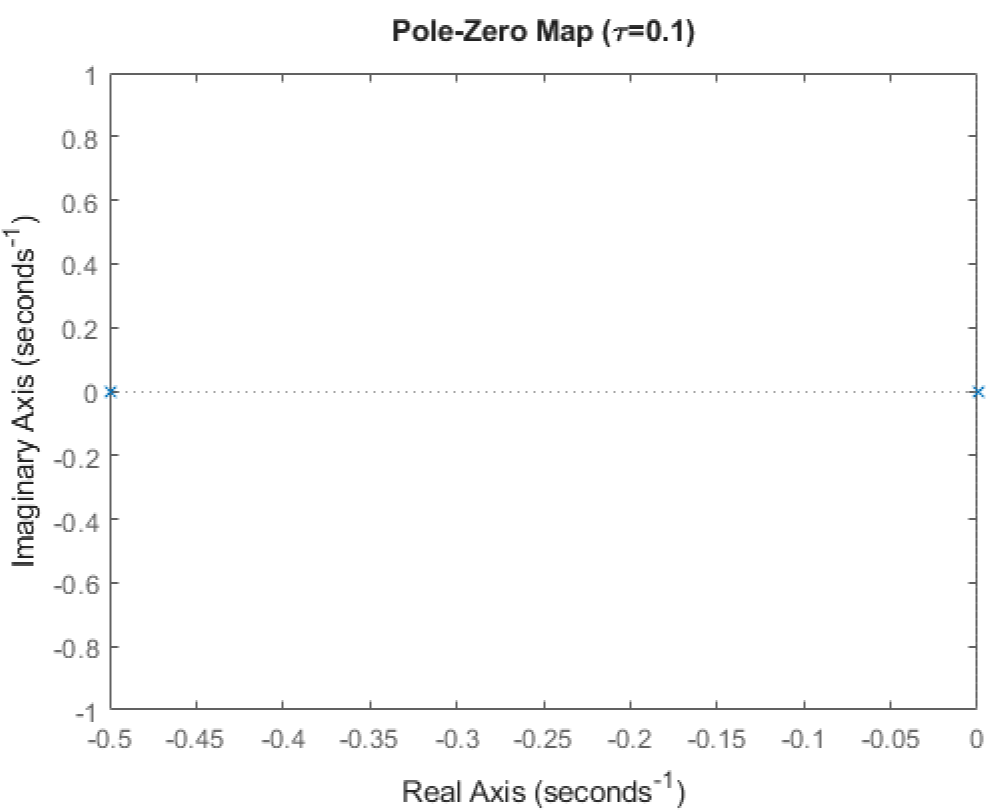


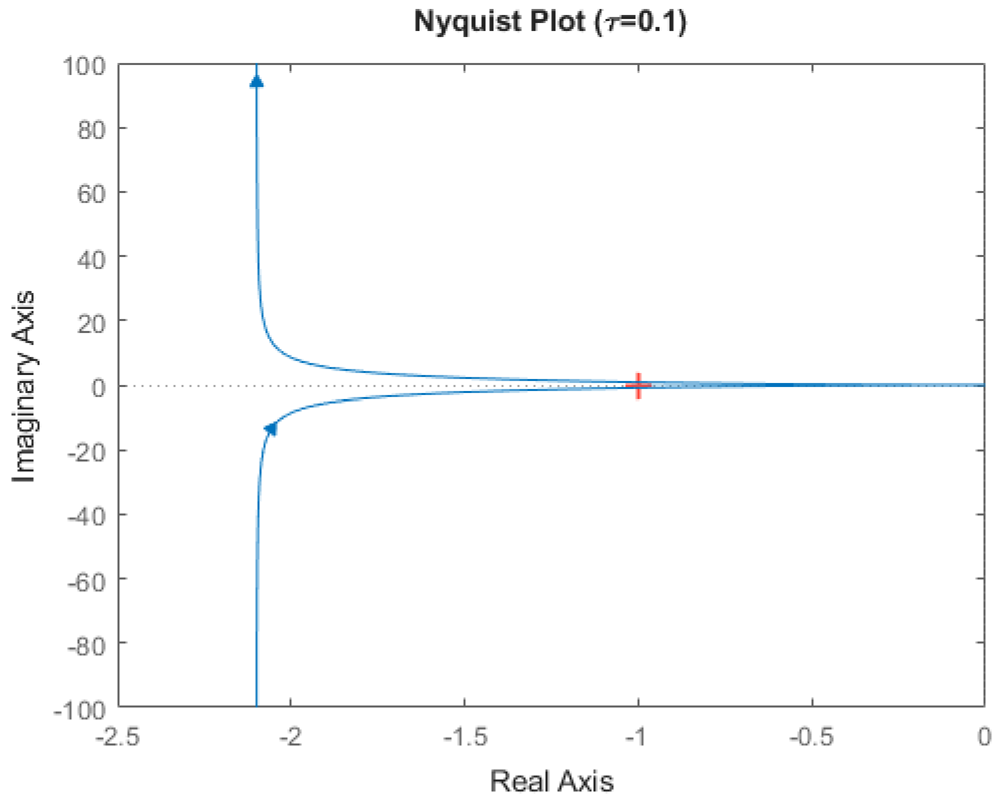
2e)

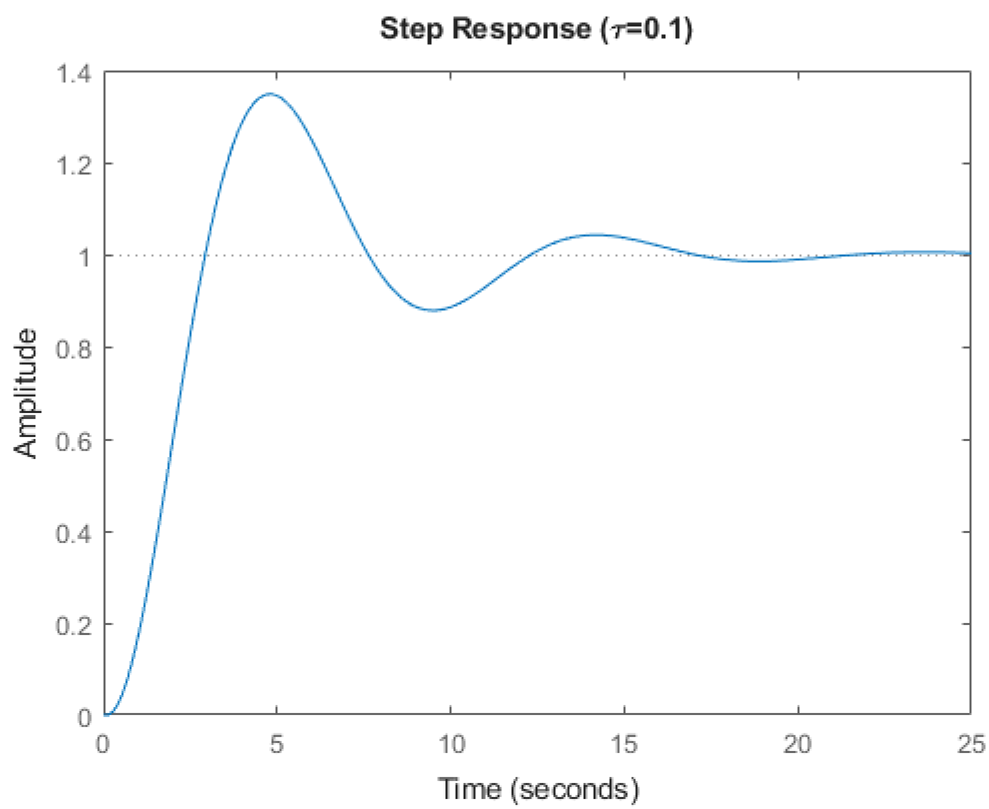
```
%tau = 0.1
tau = 0.1;
L = tf(num, den, 'InputDelay', tau);
figure;
pzmap(L);
title('Pole-Zero Map (\tau=0.1)');
figure;
nyquist(L);
title('Nyquist Plot (\tau=0.1)');
G = feedback(L, 1);
figure;
step(G);
title('Step Response (\tau=0.1)');

%no. of ccw encirclements around s=-1 = 1
%no. of unstable open-loop poles = 1

%By the Nyquist stability criterion, the closed-loop system should be
%stable, which is what is shown in the step response.

%tau = 0.5
tau = 0.5;
L = tf(num, den, 'InputDelay', tau);
```

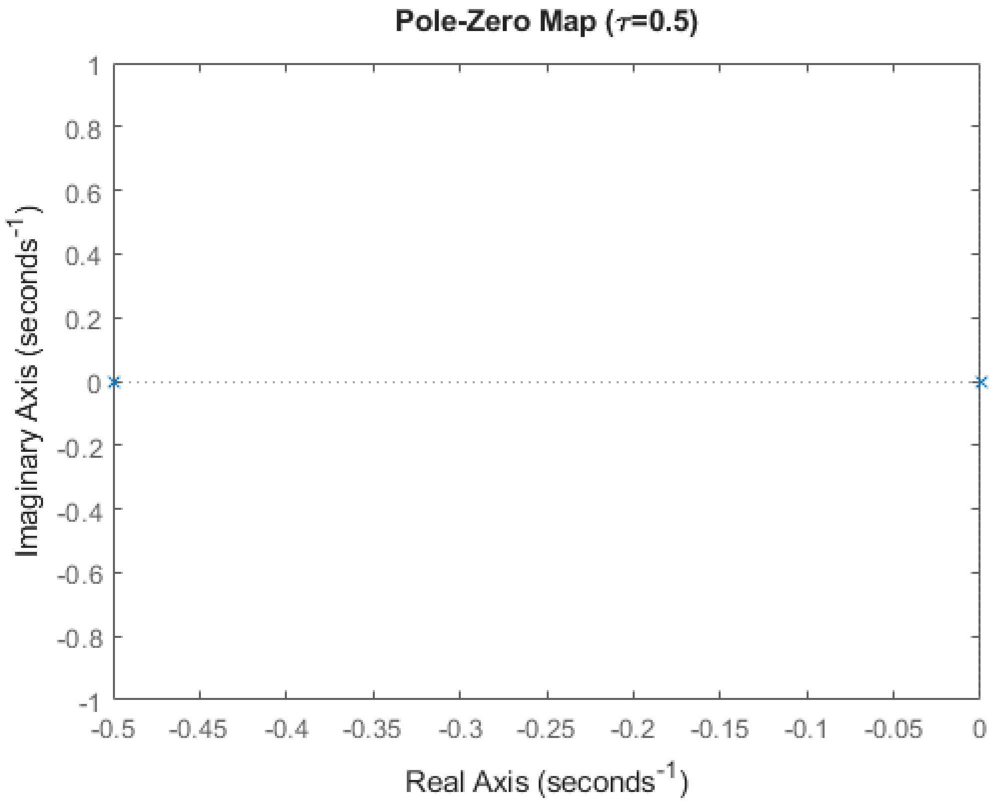
```
figure;
pzmap(L);
title('Pole-Zero Map (\tau=0.5)');
figure;
nyquist(L);
title('Nyquist Plot (\tau=0.5)');
G = feedback(L, 1);
figure;
step(G);
title('Step Response (\tau=0.5)');

%no. of ccw encirclements around s=-1 = 1
%no. of unstable open-loop poles = 1

%By the Nyquist stability criterion, the closed-loop system should be
%stable, which is what is shown in the step response.

%tau = 1
tau = 1;
L = tf(num, den, 'InputDelay', tau);
figure;
pzmap(L);
title('Pole-Zero Map (\tau=1)');
figure;
nyquist(L);
title('Nyquist Plot (\tau=1)');
G = feedback(L, 1);
figure;
step(G);
title('Step Response (\tau=1)');

%no. of ccw encirclements around s=-1 = 1
%no. of unstable open-loop poles = 1

%By the Nyquist stability criterion, the closed-loop system should be
%stable, which is what is shown in the step response as it approaches the
%steady state.


%tau = 1.5
tau = 1.5;
L = tf(num, den, 'InputDelay', tau);
figure;
pzmap(L);
title('Pole-Zero Map (\tau=1.5)');
figure;
nyquist(L);
title('Nyquist Plot (\tau=1.5)');
G = feedback(L, 1);
figure;
step(G);
title('Step Response (\tau=1.5)');

%no. of ccw encirclements around s=-1 = 0
```

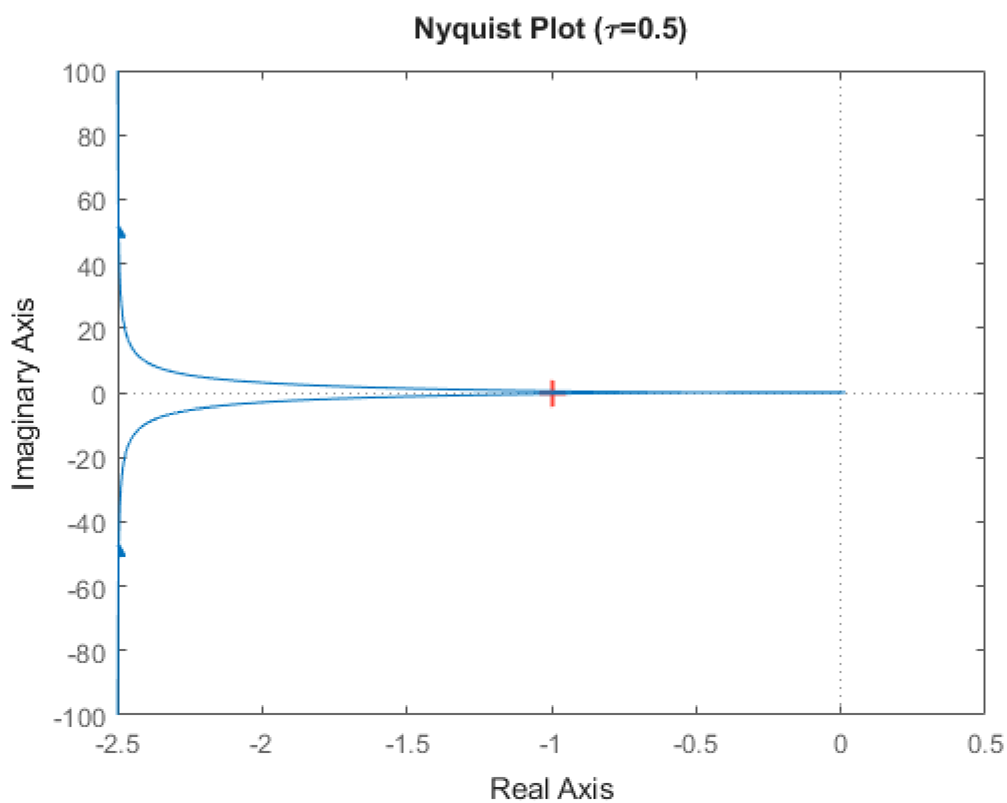
```
%no. of unstable open-loop poles = 1

%By the Nyquist stability criterion, the closed-loop system will not be
%stable, which is shown in the step response by its inability to reach
%the steady state value.

%tau = 2
tau = 2;
L = tf(num, den, 'InputDelay', tau);
figure;
pzmap(L);
title('Pole-Zero Map (\tau=2)');
figure;
nyquist(L);
title('Nyquist Plot (\tau=2)');
G = feedback(L, 1);
figure;
step(G);
title('Step Response (\tau=2)');

%no. of ccw encirclements around s=-1 = 0
%no. of unstable open-loop poles = 1

%By the Nyquist stability criterion, the closed-loop system will not be
%stable, which is shown in the step response by its inability to reach
%the steady state value.
```

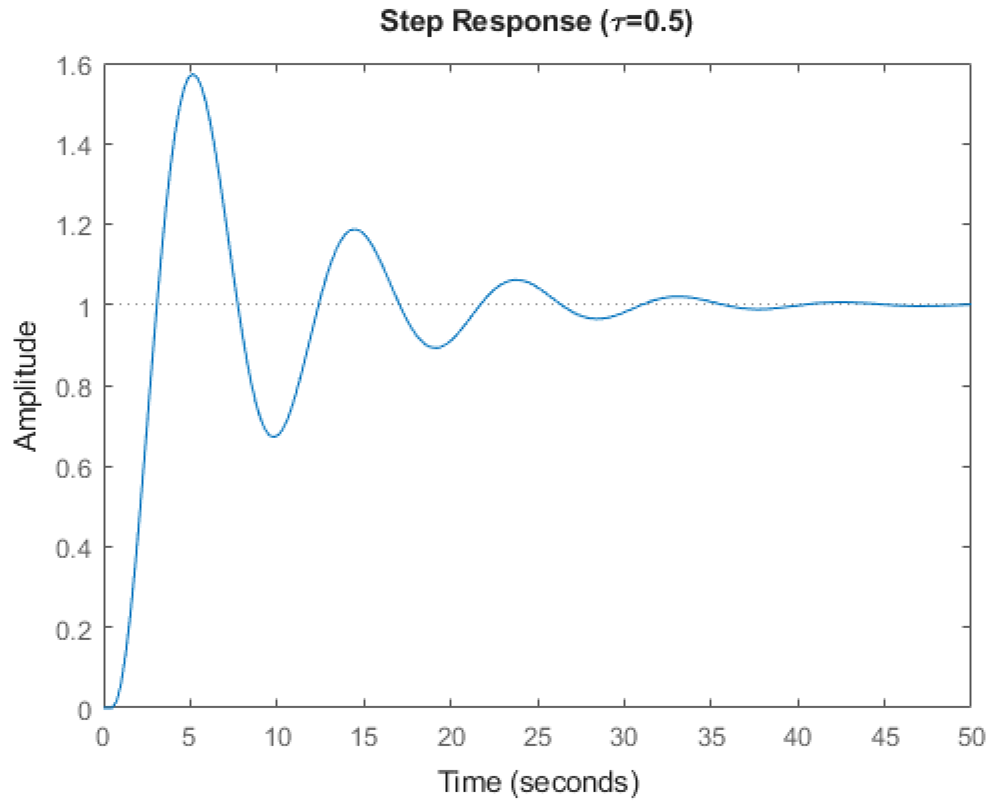


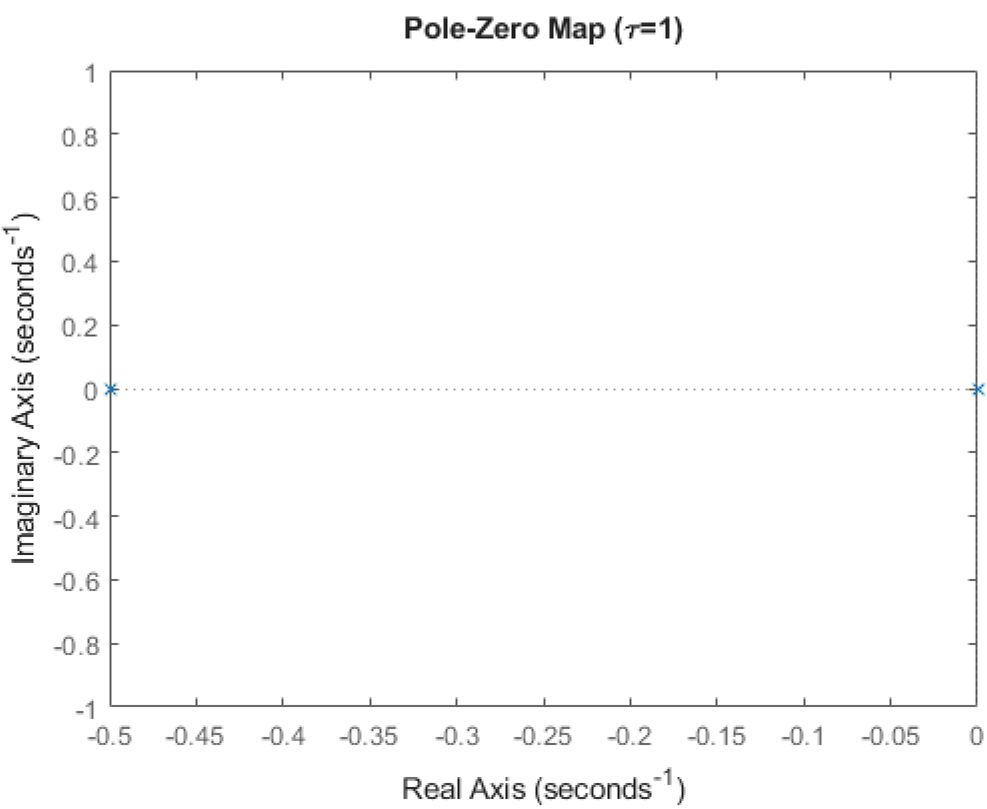

Nyquist Plot (τ=0.1)
Imaginary Axis
Real Axis
100
80
60
40
20
0
-20
-40
-60
-80
-100
-2.5
-2
-1.5
-1
-0.5
0


Step Response (τ=0.1)
Amplitude
Time (seconds)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25

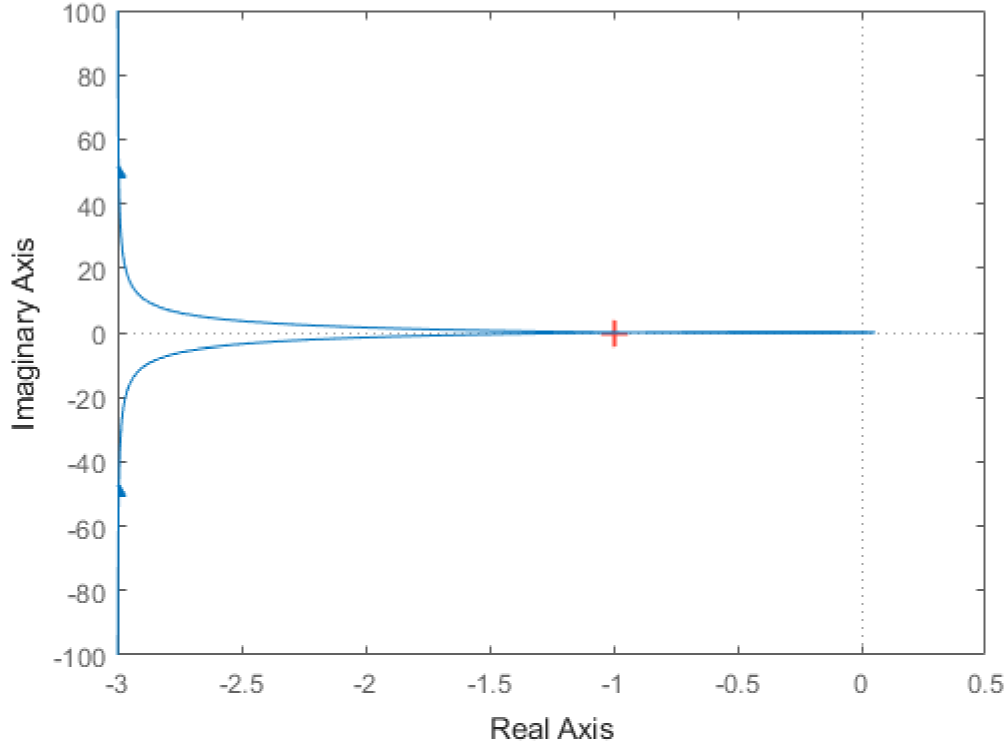

Pole-Zero Map (τ=0.5)
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
-0.45
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0


Nyquist Plot (τ=0.5)
Imaginary Axis
Real Axis
100
80
60
40
20
0
-20
-40
-60
-80
-100
-2.5
-2
-1.5
-1
-0.5
0
0.5

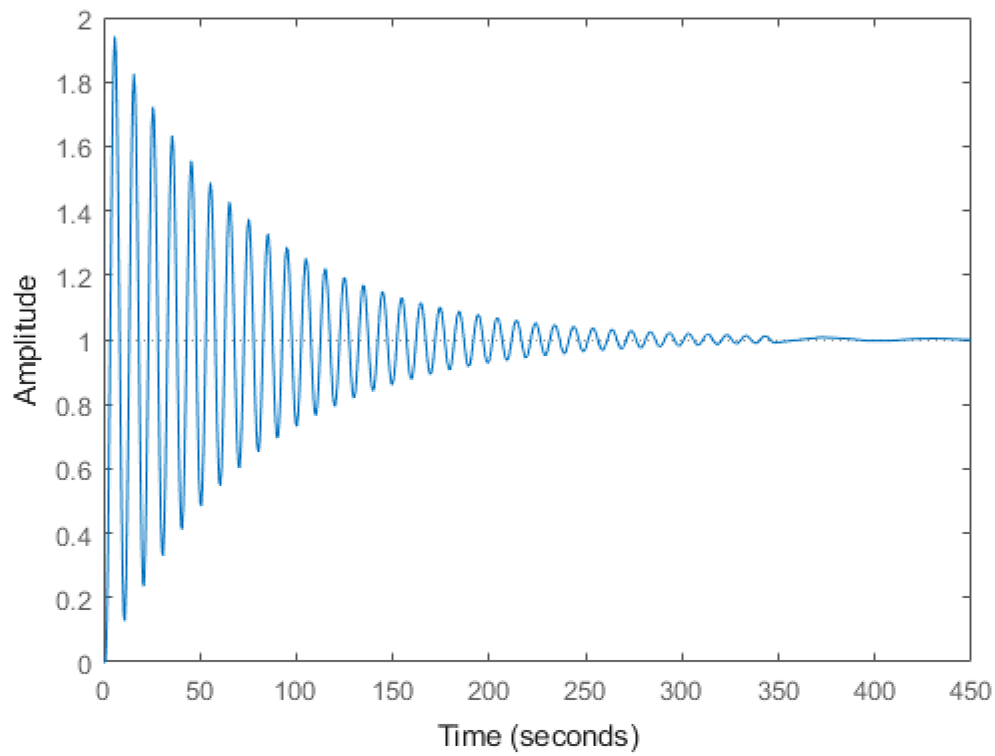

Step Response (τ=0.5)
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0
5
10
15
20
25
30
35
40
45
50

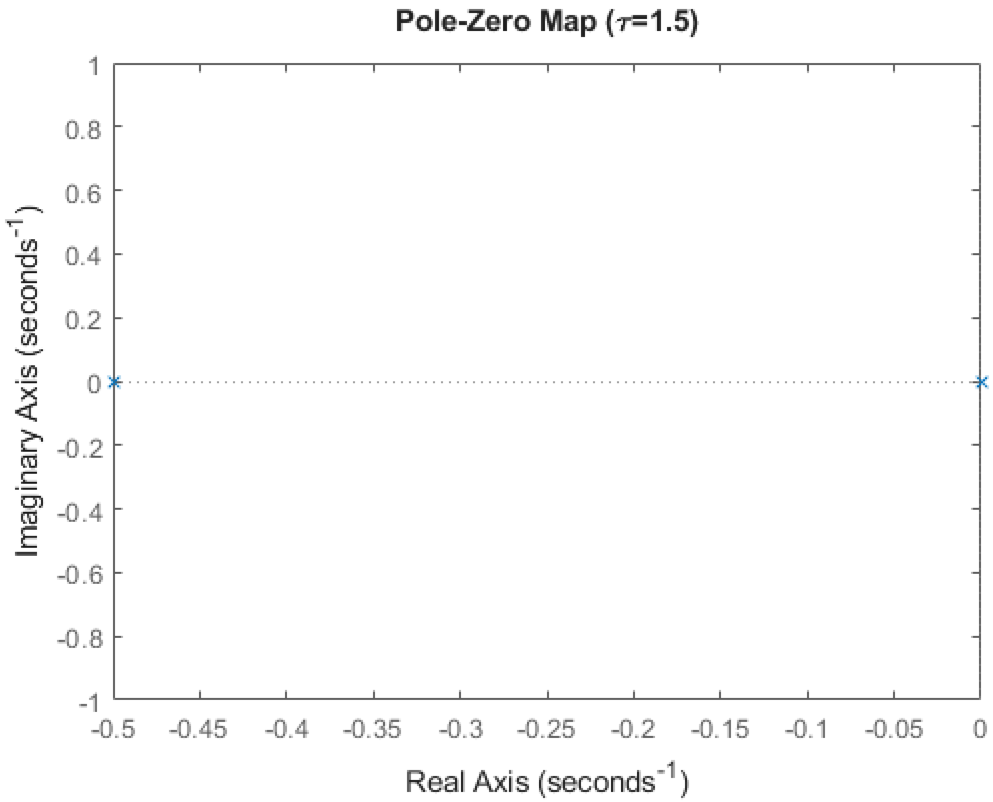

Pole-Zero Map (τ=1)
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
-0.45
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0

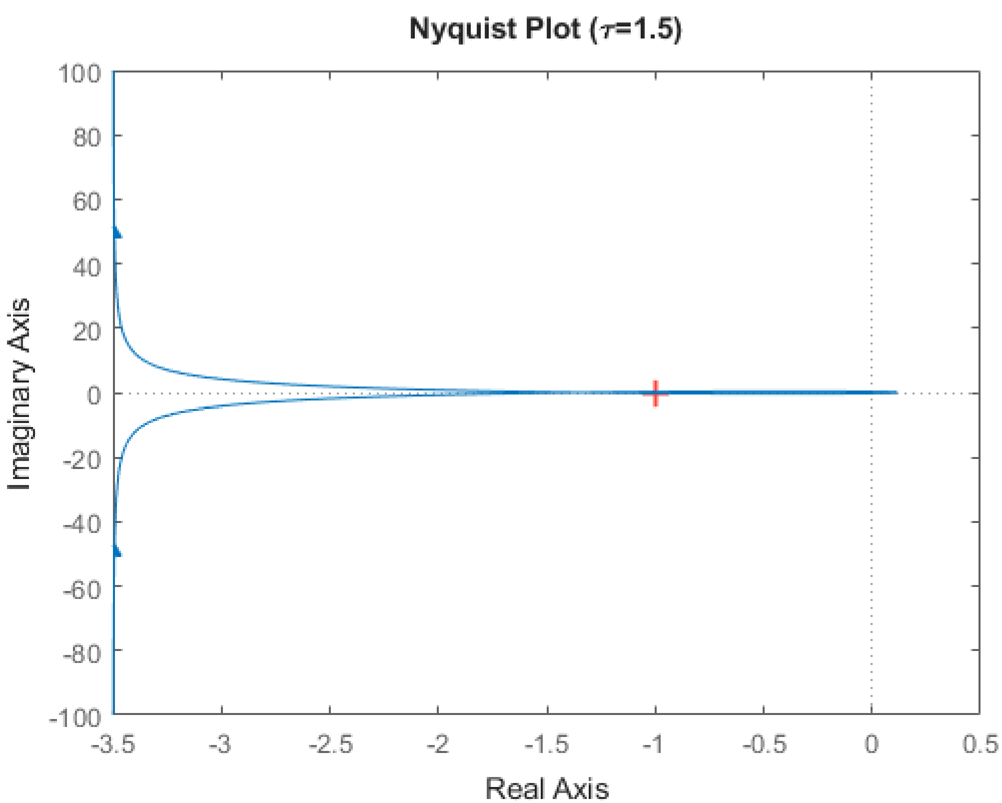
Nyquist Plot (τ=1)



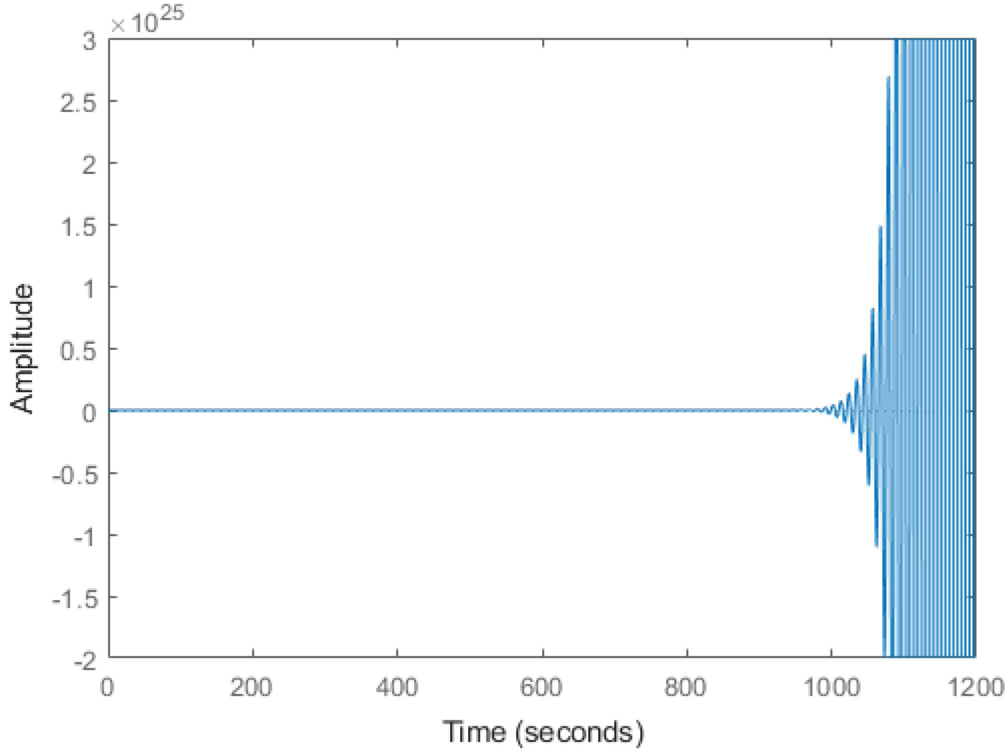
Step Response (τ=1)



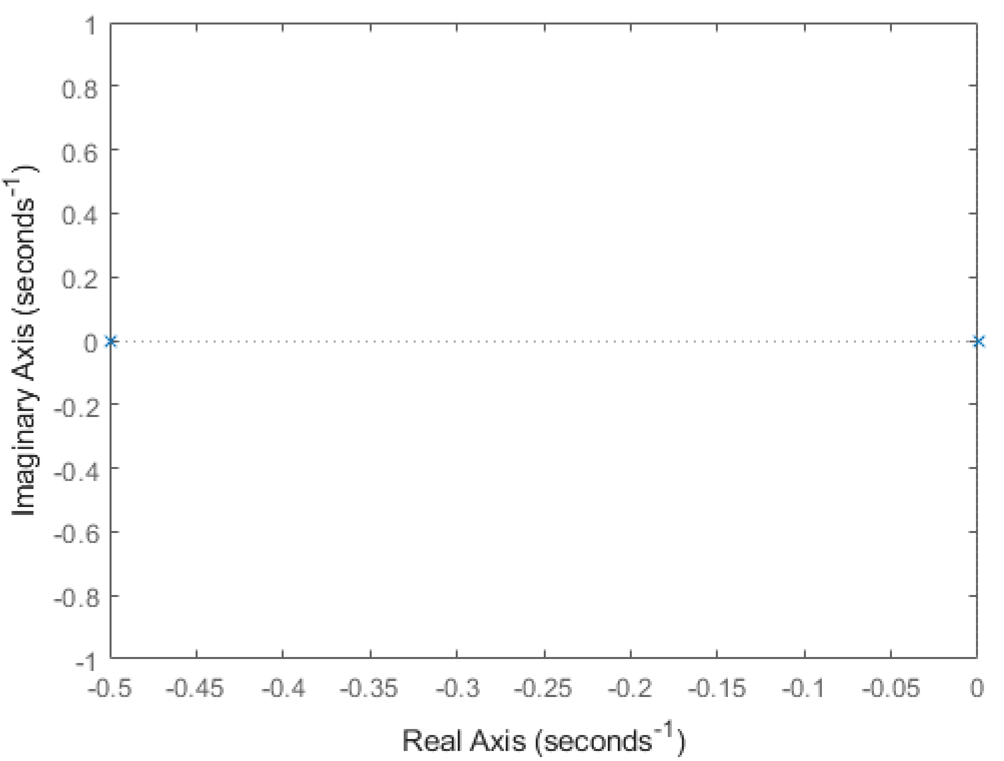

Pole-Zero Map (τ=1.5)
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})

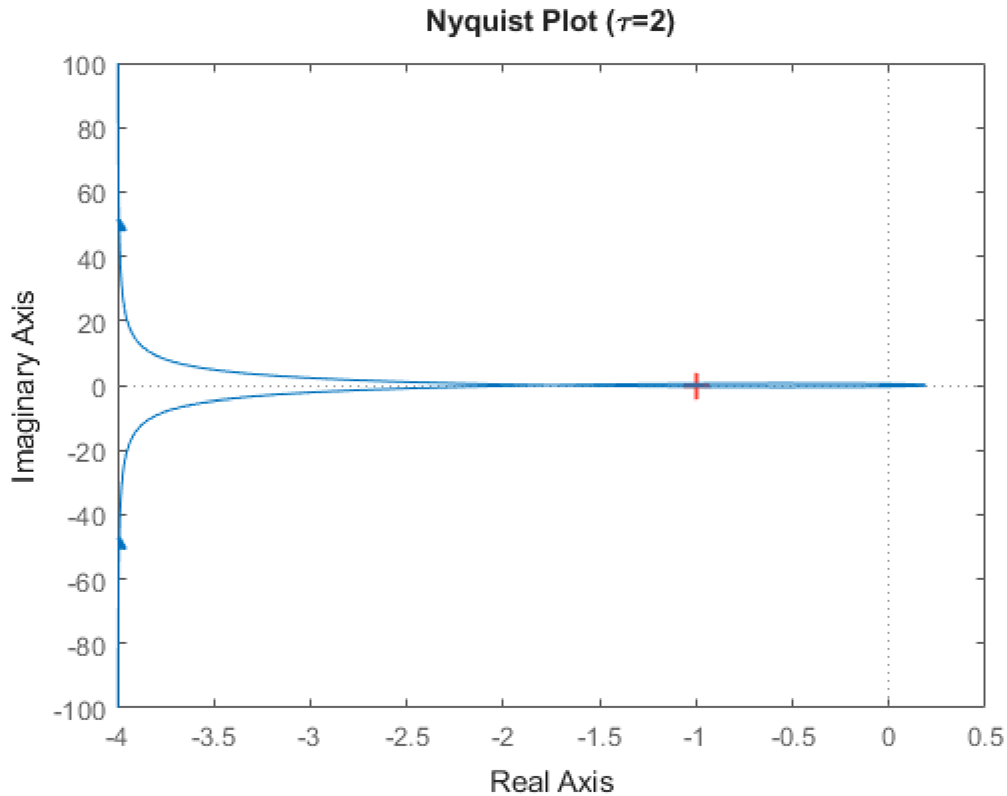

Nyquist Plot (τ=1.5)
Imaginary Axis
Real Axis

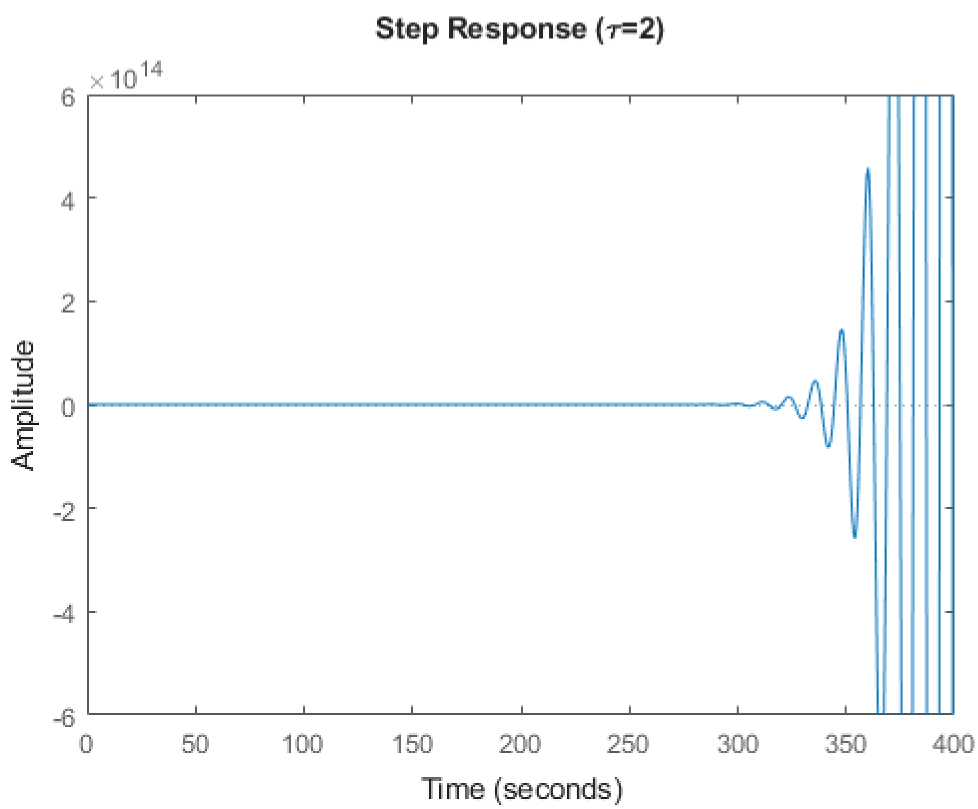
Step Response (τ=1.5)



Pole-Zero Map (τ=2)



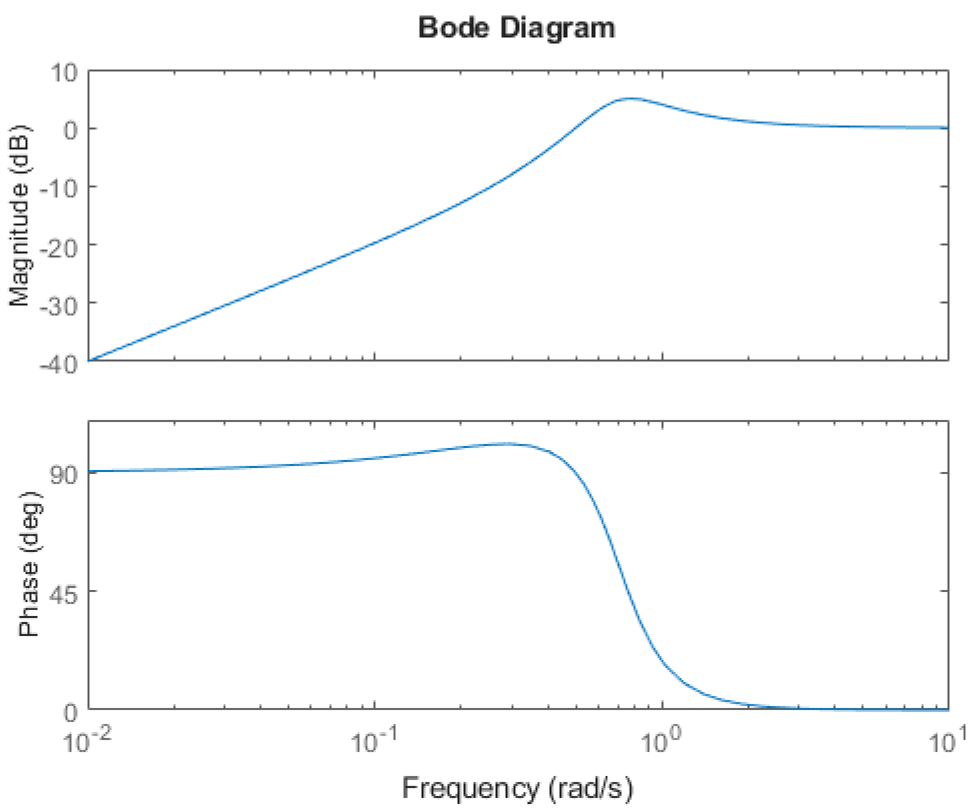

Nyquist Plot (τ=2)
Imaginary Axis
Real Axis
100
80
60
40
20
0
-20
-40
-60
-80
-100
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5


Step Response (τ=2)
×10^14
Amplitude
Time (seconds)
6
4
2
0
-2
-4
-6
0
50
100
150
200
250
300
350
400

3c)

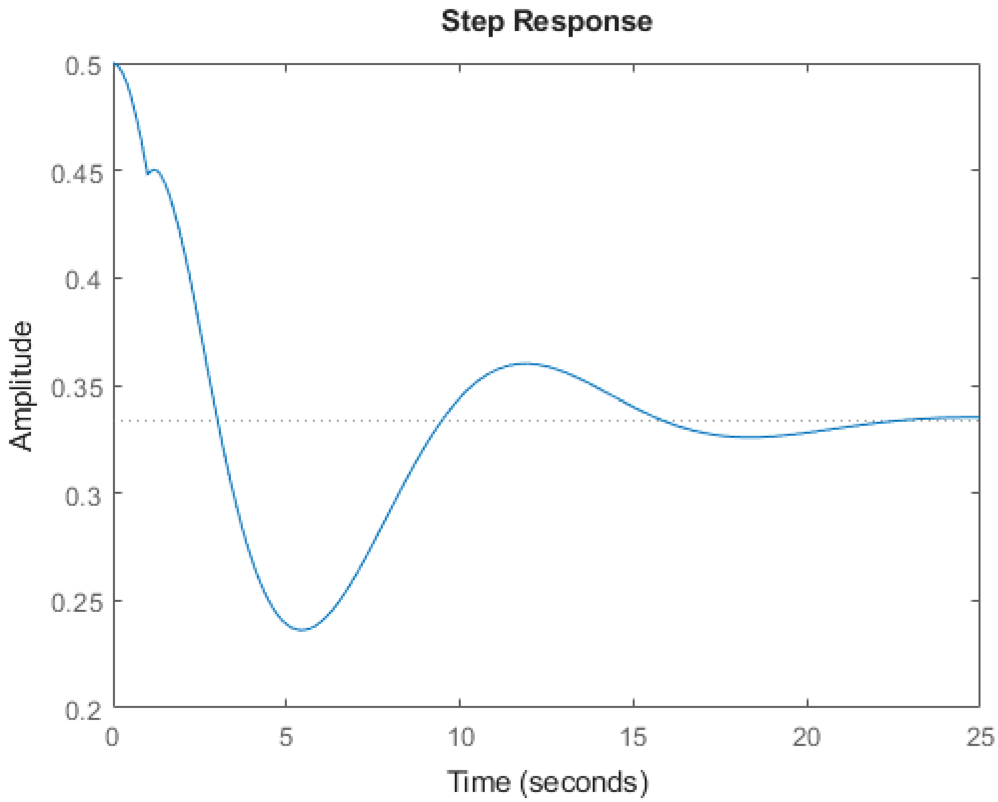
```
num = 1;
den = [2 1 0];
P = tf(num, den);
P = feedback(1, P);
[mag,~,w] = bode(P);
mag = squeeze(mag);
[peak_mag, idx] = max(mag);
fprintf('Peak Magnitude: %f\n', peak_mag);
figure;
bode(P);
```

Peak Magnitude: 1.783038



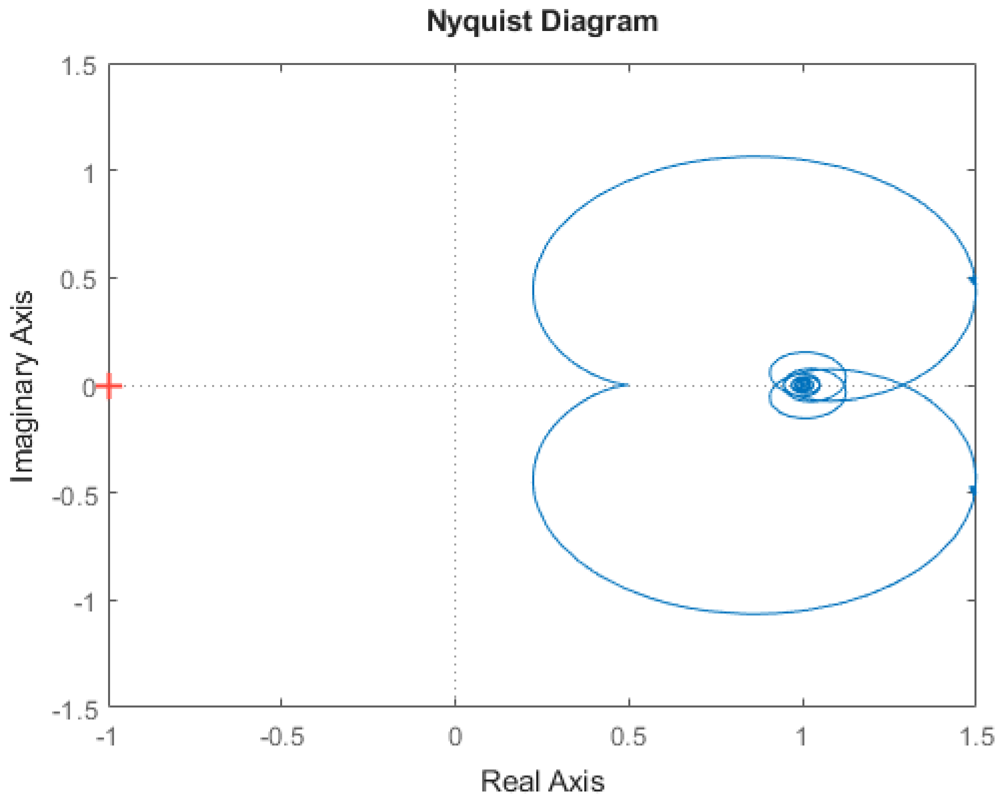
3d)

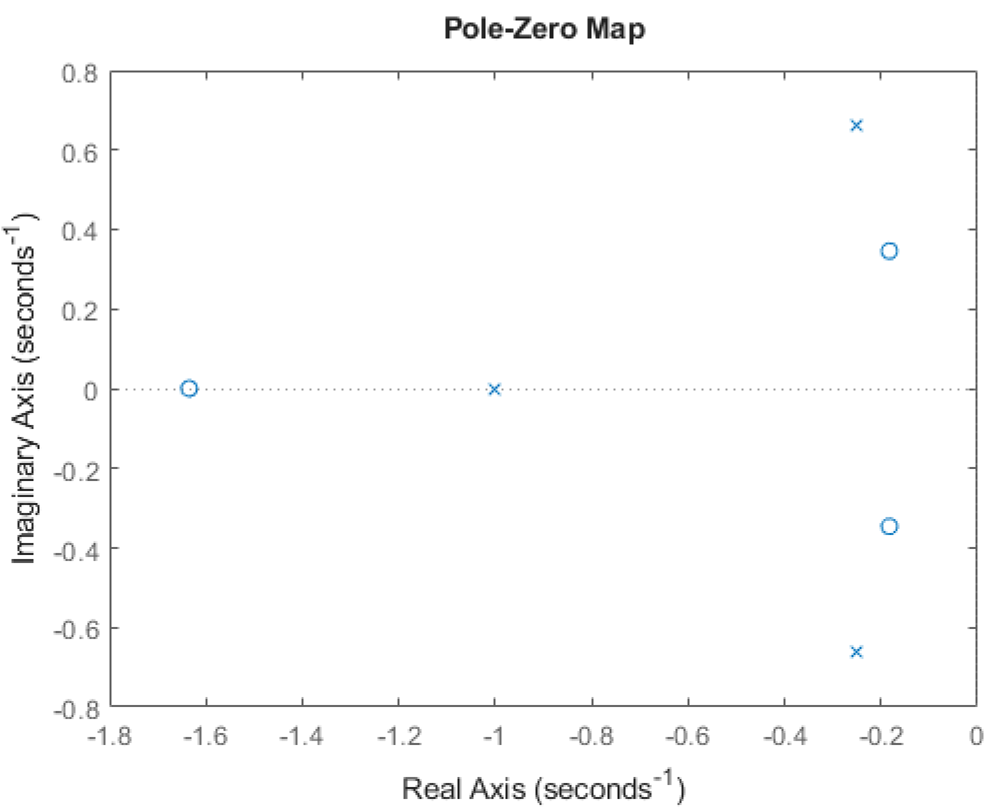
```
P_del = P + tf(0.5, [1 1], 'InputDelay', 1);
G_del = feedback(P_del, 1);
figure;
step(G_del);

%The step response validates that the proportional controller can't
%stabilize the true plant at smaller gains there is some underdamping that
%occurs at an earlier time corresponding to the earlier gains.
```



3e)

```
figure;
nyquist(P_del);
figure;
pzmap(P_del);

%no. of ccw encirclements around s=-1 = 0
%no. of unstable open-loop poles = 0

%By the Nyquist stability criterion, the closed-loop system will be
%stable. This result is consistent with the step response in part d.
```


Nyquist Diagram
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5


Pole-Zero Map
Imaginary Axis (seconds-1)
Real Axis (seconds-1)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0

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