1) a) When we apply $u = e^{st}$:

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)} Be^{s\tau}\,d\tau + De^{st}$$

$$= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}\,d\tau\; B + De^{st}$$

Assuming $(sI-A)$ is invertible:

$$y(t) = Ce^{At}x(0) + Ce^{At}\left((sI-A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{\tau=t}\right) B + De^{st}$$

$$= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I\right)B + De^{st}$$

$$= Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st}$$

$$= \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}B\right)}_{\text{transient response}} + \underbrace{\left(C(sI-A)^{-1}B + D\right)e^{st}}_{\text{steady-state response}}$$

b) When $s=0$:

$$y(t) = Ce^{At}\left(x(0) + A^{-1}B\right) + \left(-CA^{-1}B + D\right)$$

$$\boxed{y_{ss} = -CA^{-1}B + D}$$

c) When $u(t) = e^{st} = 1$, $s = 0$:

$$\dot{x} = Ax + B = 0$$

$$Ax + B = 0$$

$$A^{-1}[Ax] = A^{-1}(-B)$$

$$x_{ss} = -A^{-1}B$$

$$y_{ss} = Cx_{ss} + D(1) = C(-A^{-1}B) + D = -CA^{-1}B + D$$

$$\boxed{y_{ss} = -CA^{-1}B + D}$$

d) $A = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = 0$

$$G_{yu}(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \frac{1}{s(s+1)+2} \begin{bmatrix} s & -2 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2+s+2} \underset{1\times 2}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} s & -2 \\ 1 & s+1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2+s+2} \underset{1\times 2}{\begin{bmatrix} 1 & s+1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$= \frac{1}{s^2+s+2}$$

$$\boxed{G_{yu}(s) = \frac{1}{s^2+s+2}}$$

$$y_{ss} = -CA^{-1}B + D = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{2} \underset{1\times 2}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{2} \underset{1\times 2}{\begin{bmatrix} -1 & -1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \frac{1}{2}$$

$$y_{ss} = \frac{1}{2}$$

2) $m\ddot{q}(t) + c\dot{q}(t) + kq(t) = F(t)$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

$m = 1 = k \quad c = 0.2$

a) 1) $G_{yu}(s) = C(sI - A)^{-1}B + D$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s+0.2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \frac{1}{s(s+0.2)+1} \begin{bmatrix} s+0.2 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+0.2s+1} \underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} s+0.2 & 1 \\ -1 & s \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+0.2s+1} \underset{1\times 2}{\begin{bmatrix} s+0.2 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{s^2+0.2s+1}$$

$$\boxed{G_{yu}(s) = \frac{1}{s^2+0.2s+1}}$$

2) $m(s^2 q(s)) + c(sq(s)) + kq(s) = F(s)$

$s^2 q(s) + 0.2s\, q(s) + q(s) = F(s)$

$q(s)\left[s^2 + 0.2s + 1\right] = F(s)$

$$G(s) = \frac{q(s)}{F(s)} = \frac{1}{s^2+0.2s+1} \Rightarrow \boxed{G(s) = \frac{1}{s^2+0.2s+1}}$$

$\boxed{\text{The results of both methods are consistent.}}$

b) $y(t) = M\sin(\omega t + \phi)$

$M = |G(i\omega)| = |G_{yu}(i\omega)|$

$$G_{yu}(i\omega) = \frac{1}{(i\omega)^2 + 0.2i\omega + 1}$$

$$= \frac{1}{-\omega^2 + 0.2\omega i + 1}$$

$$= \frac{1}{(1-\omega^2) + 0.2\omega i}$$

$$|G_{yu}(i\omega)| = \sqrt{G_{yu}(i\omega)\, G^*_{yu}(i\omega)}$$

$$= \sqrt{\frac{1}{(1-\omega^2)+0.2\omega i} \cdot \frac{1}{(1-\omega^2) - 0.2\omega i}}$$

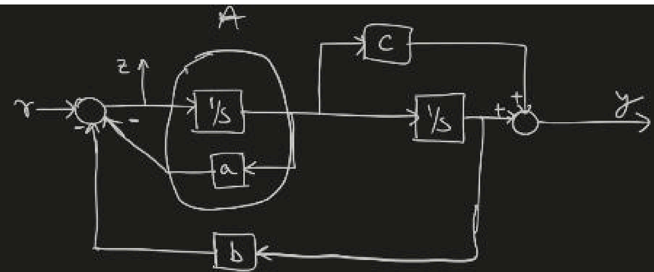
$$= \sqrt{\frac{1}{(1-\omega^2)^2 + 0.04\omega^2}} \Rightarrow \boxed{M = \sqrt{\frac{1}{(1-\omega^2)^2 + 0.04\omega^2}}}$$

$$\phi = \measuredangle G(i\omega) = -\tan^{-1}\left(\frac{Im\{G(i\omega)\}}{Re\{G(i\omega)\}}\right) = -\tan^{-1}\left(\frac{0.2\omega}{1-\omega^2}\right) \Rightarrow \boxed{\phi = -\tan^{-1}\left(\frac{0.2\omega}{1-\omega^2}\right)}$$

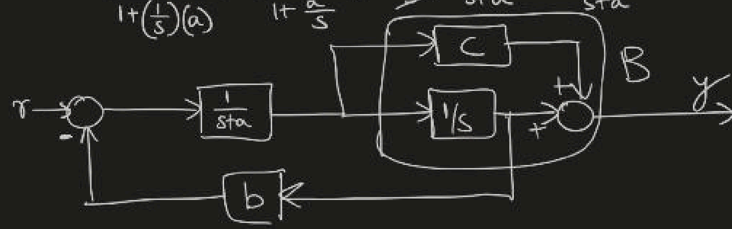
ω	0.1	0.5	1	1.5	2
M	1.01	1.322	5	0.778	0.33
ϕ(rad)	-0.02	-0.133	$-\frac{\pi}{2}$	-2.906	-3.009

These numbers are close to the approximation in the previous HW.

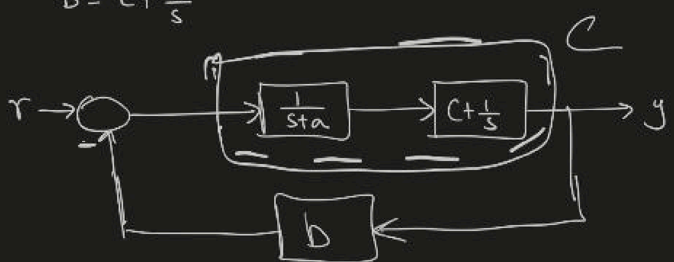
3) a)



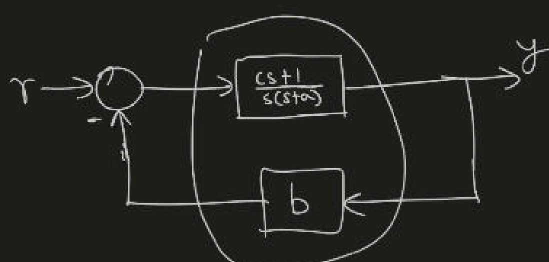
$$A = \frac{1/s}{1+\left(\frac{1}{s}\right)(a)} = \frac{1/s}{1+\frac{a}{s}} = \frac{1}{\cancel{s}} \cdot \frac{\cancel{s}}{s+a} = \frac{1}{s+a}$$



$$B = c + \frac{1}{s}$$



$$C = \frac{c+\frac{1}{s}}{s+a} = \frac{cs+1}{s(s+a)}$$



$$H_{yr}(s) = \frac{\frac{cs+1}{s(s+a)}}{1+\frac{b(cs+1)}{s(s+a)}} = \frac{\frac{cs+1}{s(s+a)}}{\frac{s(s+a)+b(cs+1)}{s(s+a)}} = \frac{cs+1}{s(s+a)+b(cs+1)}$$

$$\boxed{H_{yr}(s) = \frac{cs+1}{s^2+(a+bc)s+b}}$$

b) $A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$C = \begin{bmatrix} b_2 & b_1 \end{bmatrix}$ $\quad D = d$

$$H_{yr}(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} b_2 & b_1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d$$

$$= \begin{bmatrix} b_2 & b_1 \end{bmatrix} \begin{bmatrix} s & -1 \\ a_2 & s+a_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d$$

$$= \begin{bmatrix} b_2 & b_1 \end{bmatrix} \cdot \frac{1}{s(s+a_1) + a_2} \begin{bmatrix} s+a_1 & 1 \\ -a_2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d$$

$$= \frac{1}{s(s+a_1)+a_2} \underbrace{\begin{bmatrix} b_2 & b_1 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} s+a_1 & 1 \\ -a_2 & s \end{bmatrix}}_{2\times 2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d$$

$$= \frac{1}{s(s+a_1)+a_2} \begin{bmatrix} b_2(s+a_1) - b_1 a_2 & b_2 + b_1 s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d$$

$$= \frac{b_2 + b_1 s}{s(s+a_1)+a_2} + d$$

$$\boxed{H_{yr}(s) = \frac{b_2 + b_1 s}{s^2 + a_1 s + a_2}}$$

$b_1 = c$ $\qquad a_1 = a + bc$

$b_2 = 1$ $\qquad a_2 = b$

$d = 0$

4) a) $m_1\ddot{q}_1 + c_1\dot{q}_1 + k_1 q_1 + k_2 q_1 - k_2 q_2 = F$

$$m_2\ddot{q}_2 + k_2 q_2 - k_2 q_1 = 0$$

Let $q_1 = e^{st}$ and $q_2 = e^{st}$

$$m_1(s^2 q_1) + c_1(s q_1) + (k_1+k_2) q_1 - k_2 q_2 = F$$

$$m_2 s^2 q_2 + k_2 q_2 - k_2 q_1 = 0$$

$$(m_1 s^2 + c_1 s + k_1 + k_2) q_1 - k_2 q_2 = F$$

$$-k_2 q_1 + (m_2 s^2 + k_2) q_2 = 0$$

$$\begin{bmatrix} F \\ 0 \end{bmatrix} = \begin{bmatrix} m_1 s^2 + c_1 s + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$D(s) = \begin{vmatrix} m_1 s^2 + c_1 s + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + k_2 \end{vmatrix}$$

$$= (m_1 s^2 + c_1 s + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2$$

$$Q_1(s) = \frac{\begin{vmatrix} F & -k_2 \\ 0 & m_2 s^2 + k_2 \end{vmatrix}}{D(s)} = \frac{F(m_2 s^2 + k_2)}{D(s)}$$

$$\boxed{G_{q_1,F}(s) = \frac{m_2 s^2 + k_2}{(m_1 s^2 + c_1 s + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}}$$

$$m_1 m_2 s^4 + \underline{m_2 c_1} s^3 + (m_1 k_2 + m_2(k_1+k_2))s^2 + k_2 c_1 s + k_2(k_1+k_2) - k_2^2$$

$$Q_2(s) = \frac{\begin{vmatrix} m_1 s^2 + c_1 s + k_1 + k_2 & F \\ -k_2 & 0 \end{vmatrix}}{D(s)} = \frac{k_2 F}{D(s)}$$

$$\boxed{G_{q_2,F}(s) = \frac{k_2}{(m_1 s^2 + c_1 s + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}}$$

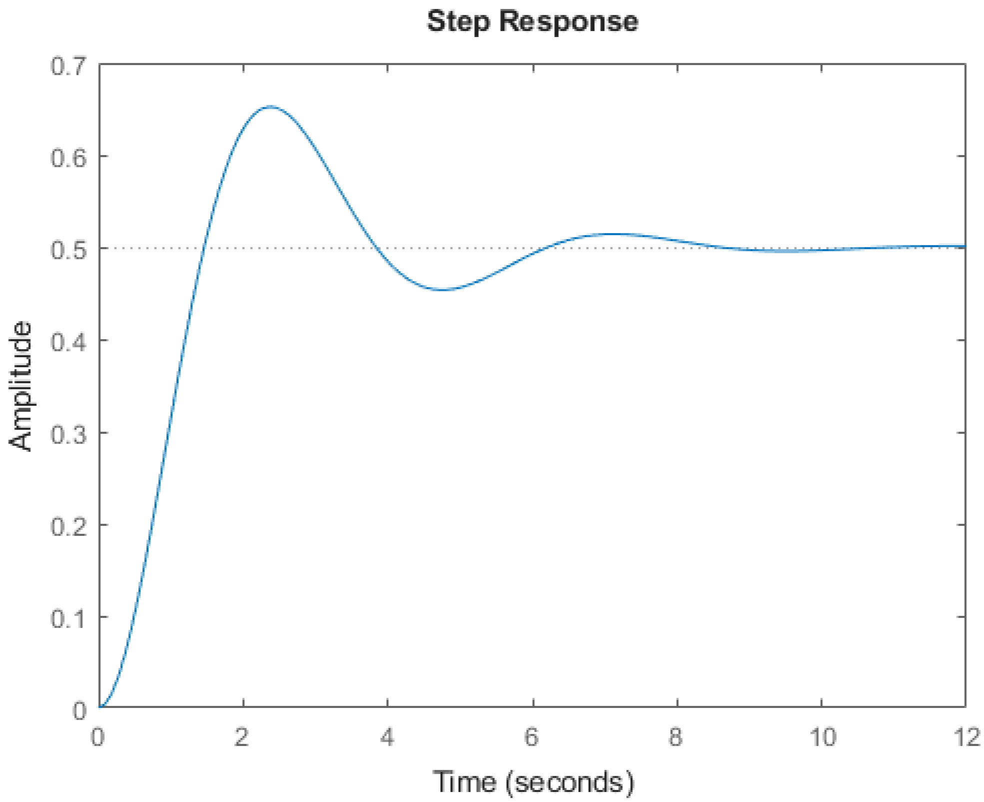
Questions 1d, 2c, 4b, and 4c are answered in the section below.

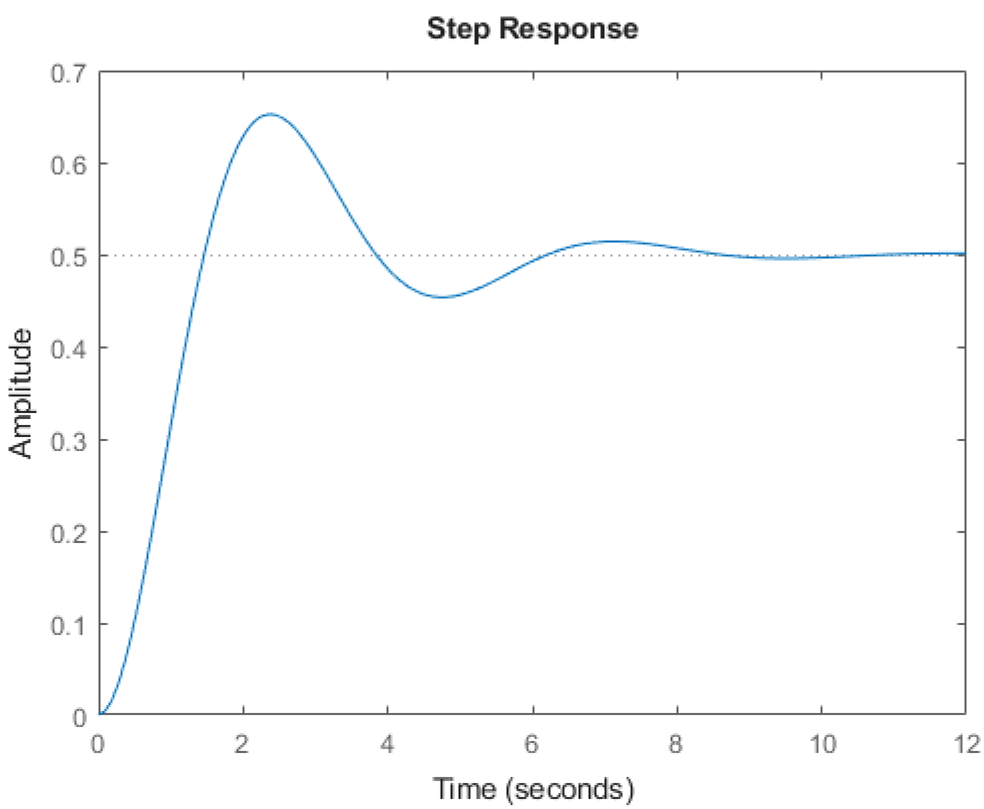
Table of Contents

1d) 1
2c) 3
4b) 3
4c) 6

1d)

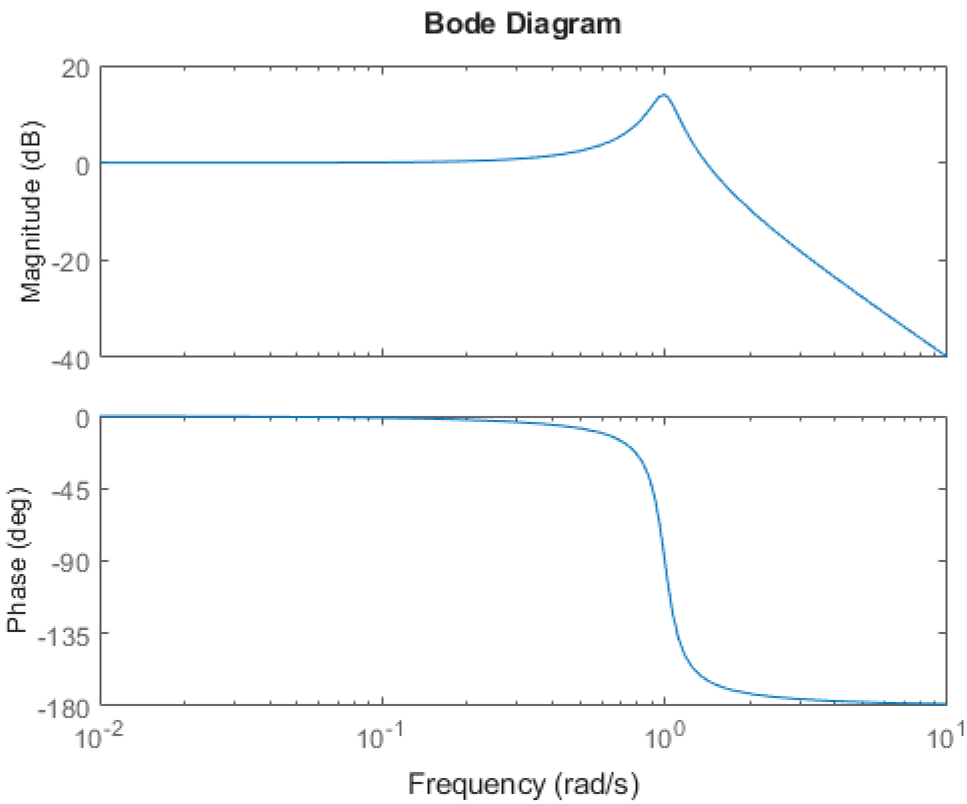
```
A = [-1 -2; 1, 0];
B = [1; 0];
C = [0 1];
D = 0;
sys = ss(A, B, C, D);
figure;
step(sys);

%Verification of Analytical Computation:
num = 1;
den = [1 1 2];
sys = tf(num, den);
figure;
step(sys);

%Since both the step responses are the same, the analytical computation of
%the transfer function is consistent with the result of the numerical
%simulation.
```


Step Response
Amplitude
Time (seconds)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
2
4
6
8
10
12


Step Response
Amplitude
Time (seconds)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
2
4
6
8
10
12

2c)

```
m = 1;
k = 1;
c = 0.2;
A = [0 1; -k/m, -c/m];
B = [0; 1/m];
C = [1 0];
D = 0;
sys = ss(A, B, C, D);
figure;
bode(sys, {0.01, 10});

%The numbers in my table, although in different units, are consistent with
%the Bode plot numbers.
```

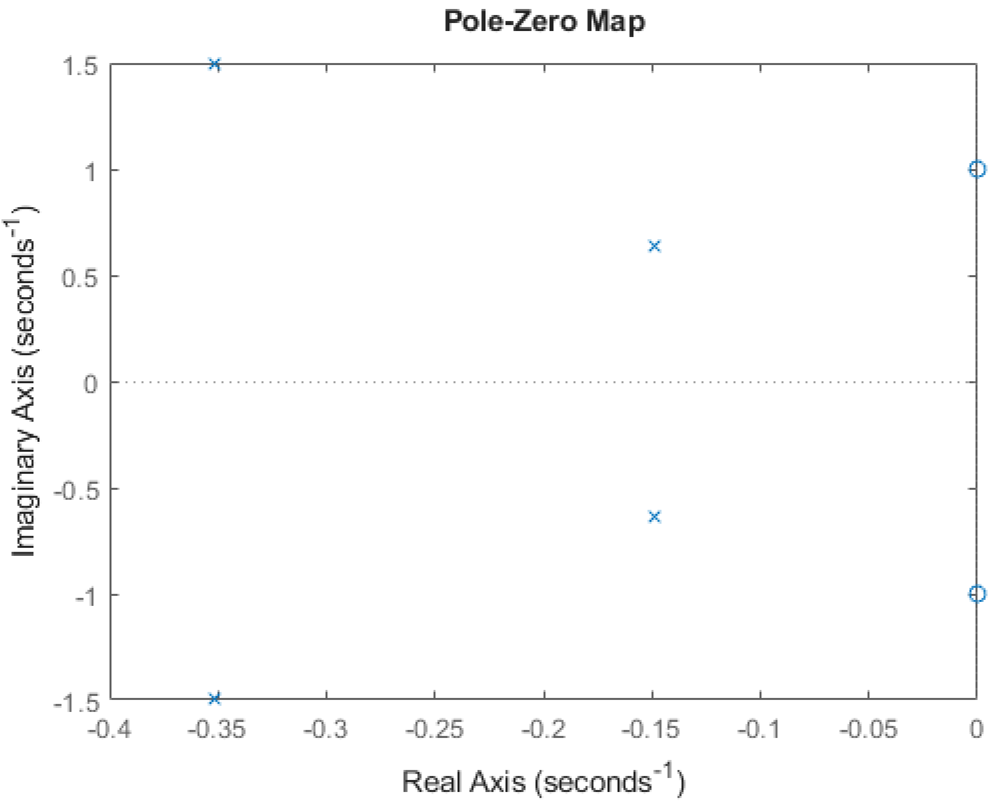


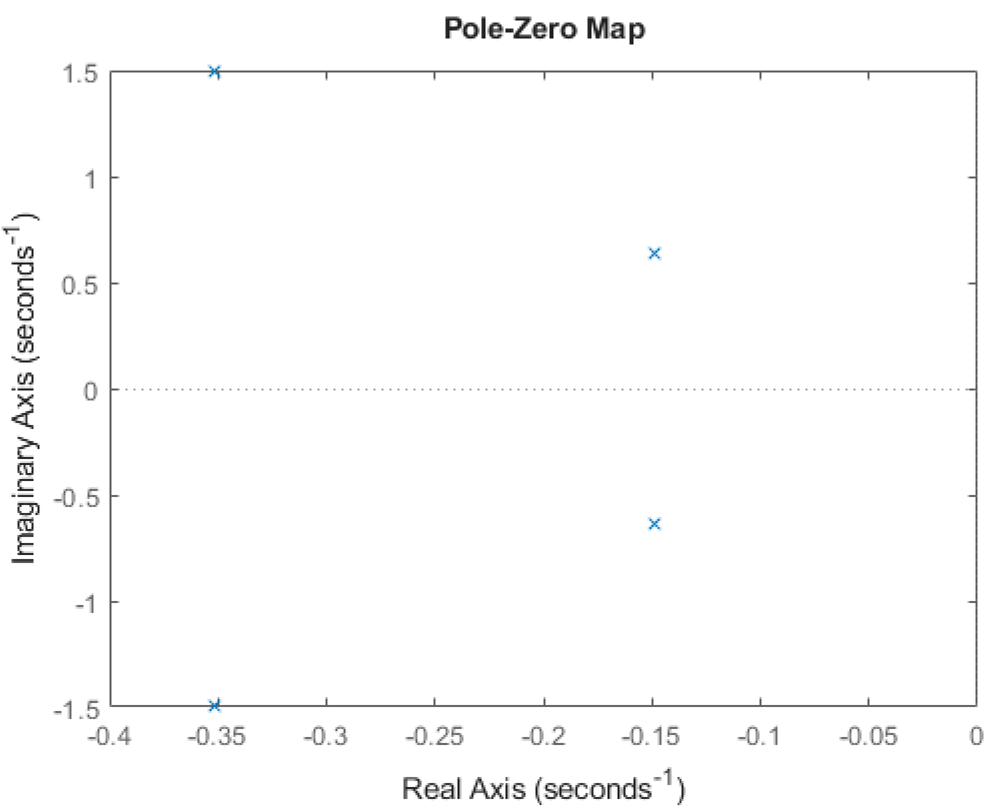
4b)

```
m1 = 1;
c1 = 1;
k1 = 1;
m2 = 1;
k2 = 1;

num1 = [m2 0 k2];
```

```
den = [m1*m2 m2*c1 ((m1*k2)+m2*(k1+k2)) k2*c1 k2*(k1+k2)-k2^2];
num2 = k2;
sys1 = tf(num1, den);
figure;
pzmap(sys1);
sys2 = tf(num2, den);
figure;
pzmap(sys2);

fprintf('Poles q_1:\n')
disp(pole(sys1));
fprintf('Zeros q_1:\n')
disp(zero(sys1));
fprintf('Poles q_2:\n')
disp(pole(sys2));
fprintf('Zeros q_1:\n')
disp(zero(sys2));

%The poles in the pole zero map correspond to the location of the points
%marked with an x while the zeros correspond to the location of the points
%marked with o. The poles of both the systems are the same as they share
%the same denominator, and they are all to the left of Real = 0.
```

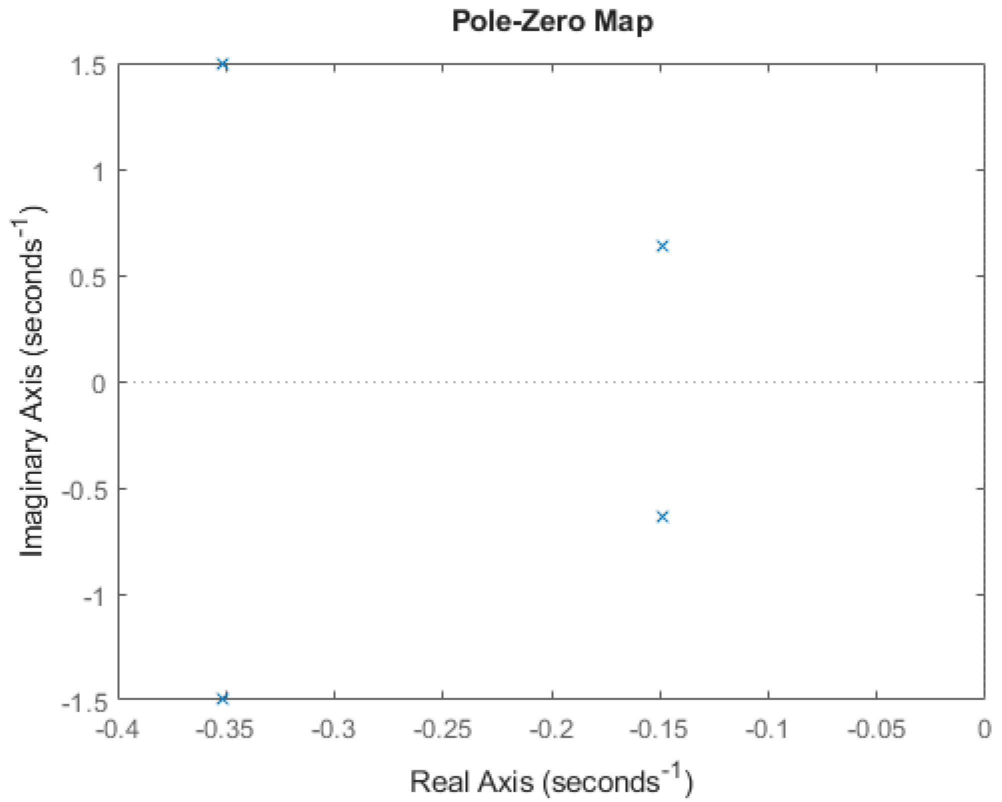
```
Poles q_1:
  -0.3516 + 1.4985i
  -0.3516 - 1.4985i
  -0.1484 + 0.6325i
  -0.1484 - 0.6325i

Zeros q_1:
   0.0000 + 1.0000i
   0.0000 - 1.0000i

Poles q_2:
  -0.3516 + 1.4985i
  -0.3516 - 1.4985i
  -0.1484 + 0.6325i
  -0.1484 - 0.6325i

Zeros q_1:
```

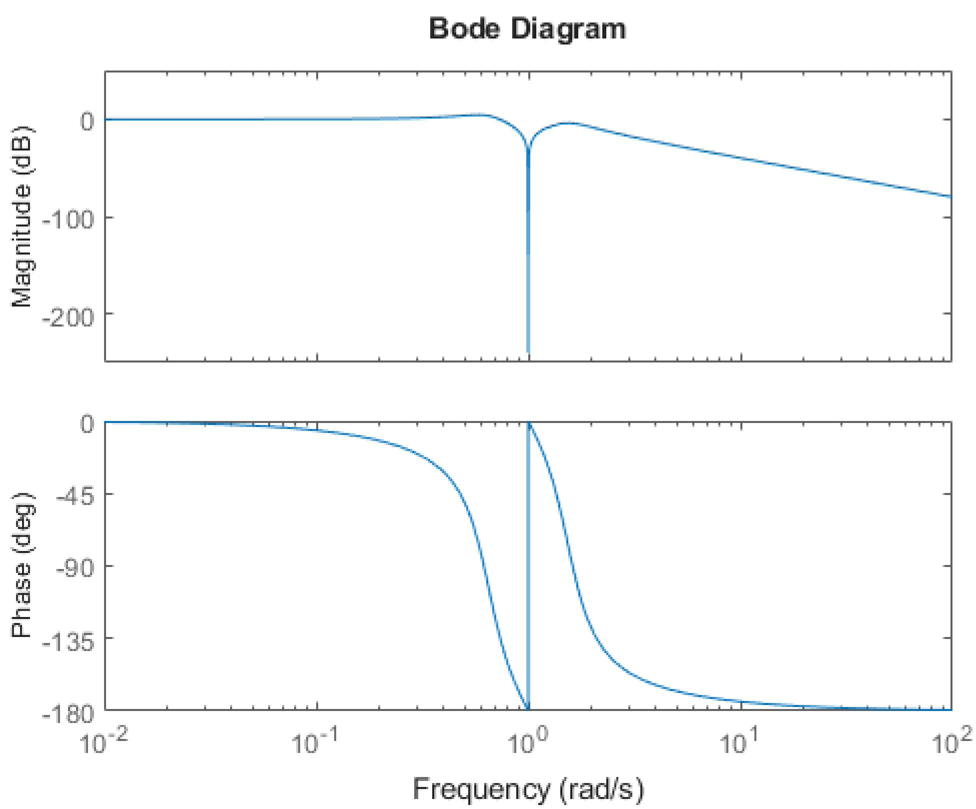

Pole-Zero Map
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
1.5
1
0.5
0
-0.5
-1
-1.5
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0

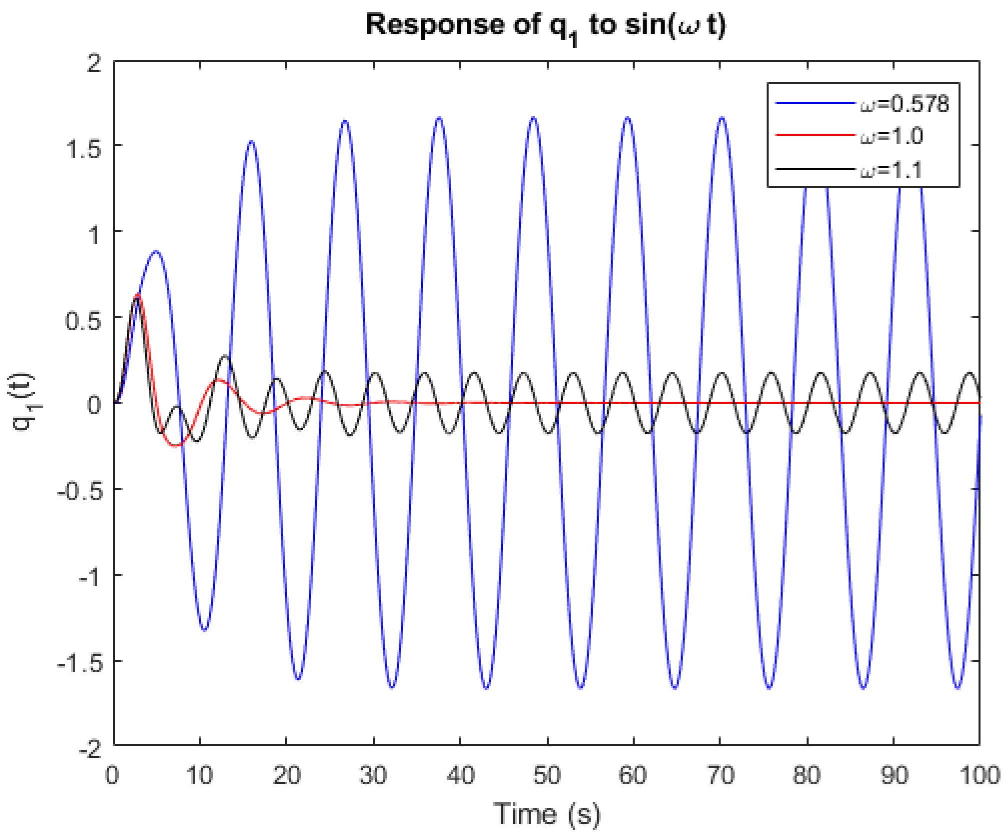

Pole-Zero Map
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
1.5
1
0.5
0
-0.5
-1
-1.5
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0

4c)

```
figure;
bode(sys1);

T  = 100;
dt = 0.01;
t  = 0:dt:T;
w = [0.578, 1.0, 1.1];
u1 = sin(w(1)*t);
u2 = sin(w(2)*t);
u3 = sin(w(3)*t);
x0 = [0 0 0 0];
y1 = lsim(sys1, u1, t, x0);
y2 = lsim(sys1, u2, t, x0);
y3 = lsim(sys1, u3, t, x0);
figure;
plot(t,y1,'b', t,y2,'r', t,y3,'k')
legend('\omega=0.578','\omega=1.0','\omega=1.1')
xlabel('Time (s)')
ylabel('q_1(t)')
title('Response of q_1 to sin(\omega t)')

%These responses are consistent with the Bode plots. The steady state
%amplitudes line up with the Bode magnitude curve. The numerator of this
%system is essentially s^2+1, which means that the system has zeros at
%s=+/-i\omega with \omega=1. At \omega=1, a sinsuoidal input will not
%produce a long term output in this system and will go down to 0. In a
%physical system, this can be seen as a tuned damper canceling out
%the motion of a primary mass at the natural frequency.
```


Pole-Zero Map
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
1.5
1
0.5
0
-0.5
-1
-1.5
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
0
-100
-200
0
-45
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}



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