1) a) $G_{yr}(s) = \dfrac{C(s)P(s)}{1 + C(s)P(s)}$

b) $C(s) = K$

$P(s) = \dfrac{1}{(s+2)(s+1)}$

$$G_{yr}(s) = \frac{\frac{K}{(s+2)(s+1)}}{1 + \frac{K}{(s+2)(s+1)}} = \frac{K}{\cancel{(s+2)(s+1)}} \cdot \frac{\cancel{(s+2)(s+1)}}{(s+2)(s+1)+K} = \frac{K}{(s+2)(s+1)+K} = \frac{b(s)}{a(s)}$$

$a(s) = (s+2)(s+1) + K = s^2 + 3s + 2 + K$

s^2	$a_2 = 1$	$a_0 = 2+K$
s^1	$a_1 = 3$	0
s^0	$b_1 = \dfrac{-\begin{vmatrix} 1 & 2+K \\ 3 & 0 \end{vmatrix}}{3} = 2+K$	0

Stability requires all first column entries to be positive

$1 > 0$ ✓

$3 > 0$ ✓

$2 + K > 0 \Rightarrow K > -2$

$\boxed{\text{Range: } K > -2}$

c) $C(s) = K \qquad P(s) = \frac{1}{(s+6)(s+3)(s-1)}$

$$G_{yr}(s) = \frac{\frac{K}{(s+6)(s+3)(s-1)}}{1 + \frac{K}{(s+6)(s+3)(s-1)}} = \frac{\frac{K}{(s+6)(s+3)(s-1)}}{\frac{(s+6)(s+3)(s-1)+K}{(s+6)(s+3)(s-1)}} = \frac{K}{\cancel{(s+6)(s+3)(s-1)}} \cdot \frac{\cancel{(s+6)(s+3)(s-1)}}{(s+6)(s+3)(s-1)+K} = \frac{K}{(s+6)(s+3)(s-1)+K} = \frac{K}{(s^2+9s+18)(s-1)+K} = \frac{K}{s^3-s^2+9s^2-9s+18s-18+K}$$

$$= \frac{K}{s^3+8s^2+9s+(K-18)} = \frac{b(s)}{a(s)}$$

$a(s) = s^3 + 8s^2 + 9s + (K-18)$

s^3	$a_3 = 1$	$a_1 = 9$
s^2	$a_2 = 8$	$a_0 = K-18$
s^1	$b_2 = -\frac{\begin{vmatrix} 1 & 9 \\ 8 & K-18 \end{vmatrix}}{8} = -\frac{(K-18)-72}{8} = \frac{90-K}{8}$	0
s^0	$c_2 = \frac{-\begin{vmatrix} 8 & K-18 \\ \frac{K-90}{8} & 0 \end{vmatrix}}{\frac{K-90}{8}} = K-18$	0

Stability requires all first column entries to be positive

$1>0$ ✓ $\quad 8>0$ ✓

$\frac{90-K}{8} > 0 \Rightarrow 90-K>0 \Rightarrow K<90$

$K-18>0 \Rightarrow K>18$

$\boxed{\text{Range: } 18<K<90}$

2) a) $G_1(s) = b\frac{s+a}{s} = \frac{b(s+a)}{s}$

$G_1(i\omega) = b\left(\frac{a+i\omega}{i\omega}\right) = b + \frac{ba}{i\omega} = b - \frac{ba}{\omega}i$

$|G_1(i\omega)| = \left|\frac{b(a+i\omega)}{i\omega}\right| = \sqrt{\frac{b(a+i\omega)}{i\omega} \cdot \frac{b(a-i\omega)}{-i\omega}} = \sqrt{\frac{b^2(a^2+\omega^2)}{\omega^2}} = \frac{b}{\omega}\sqrt{a^2+\omega^2}$

$\log|G_1(i\omega)| = \log(b) + \frac{1}{2}\log(a^2+\omega^2) - \log(\omega)$

When $\omega \ll a$:

$|G_1(i\omega)| = \frac{ab}{\omega}$

When $\omega \gg a$:

$|G_1(i\omega)| = b$

$$\log|G_1(i\omega)| = \begin{cases} \log(ab) - \log(\omega), & \omega \ll a \\ \log(b), & \omega \gg a \end{cases}$$

$\measuredangle G_1(i\omega) = \frac{180}{\pi}\tan^{-1}\left(\frac{-\frac{ba}{\omega}}{b}\right) = -\frac{180}{\pi}\tan^{-1}\left(\frac{a}{\omega}\right)$

When $\omega \ll a$:

$\measuredangle G_1(i\omega) = -\frac{180}{\pi}\tan^{-1}(\infty) = -90°$

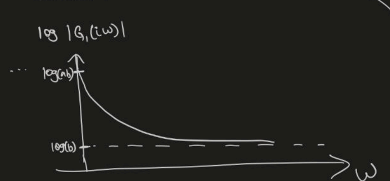
When $\omega \gg a$:

$\measuredangle G_1(i\omega) = -\frac{180}{\pi}\tan^{-1}(0) = 0°$

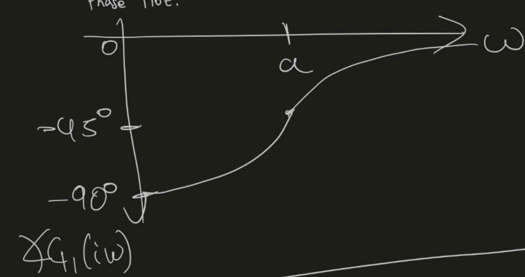
When $\omega = a$:

$\measuredangle G_1(i\omega) = -\frac{180}{\pi}\tan^{-1}(1) = -45°$

Magnitude Plot:



Phase Plot:



b) $G_2(s) = \dfrac{s+a}{s+100a}$

$$G_2(i\omega) = \frac{a+i\omega}{100a+i\omega}$$

$$|G_2(i\omega)| = \sqrt{\left(\frac{a+i\omega}{100a+i\omega}\right)\left(\frac{a-i\omega}{100a-i\omega}\right)} = \sqrt{\frac{a^2+\omega^2}{10{,}000a^2+\omega^2}}$$

$$\log|G_2(i\omega)| = \frac{1}{2}\log(a^2+\omega^2) - \frac{1}{2}\log(10{,}000a^2+\omega^2)$$

When $\omega << a$:

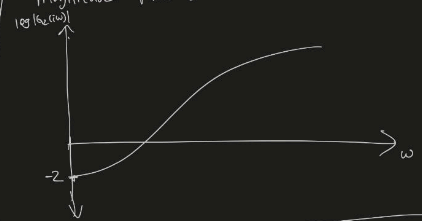
$$\log|G_2(i\omega)| = \frac{1}{2}\log(a^2) - \frac{1}{2}\log(10{,}000a^2) = \frac{1}{2}\log\left(\frac{\cancel{a^2}}{10{,}000\cancel{a^2}}\right) = \frac{1}{2}\log\left(\frac{1}{10{,}000}\right) = -2$$

When $\omega >> a$:

$$\log|G_2(i\omega)| = \frac{1}{2}\log(\omega^2) + \frac{1}{2}\log(\omega^2) = \log(\omega^2)$$

$$\log|G_2(i\omega)| = \begin{cases} -2, & \omega << a \\ \log(\omega^2), & \omega >> a \end{cases}$$

Magnitude Plot:



$$\measuredangle G_2(i\omega) = \measuredangle(a+i\omega) - \measuredangle(100a+i\omega) = \frac{180}{\pi}\tan^{-1}\left(\frac{\omega}{a}\right) - \frac{180}{\pi}\tan^{-1}\left(\frac{\omega}{100a}\right)$$

When $\omega << a$:

$$\measuredangle G_2(i\omega) = \frac{180}{\pi}\tan^{-1}(0) - \frac{180}{\pi}\tan^{-1}(0) = 0^\circ$$

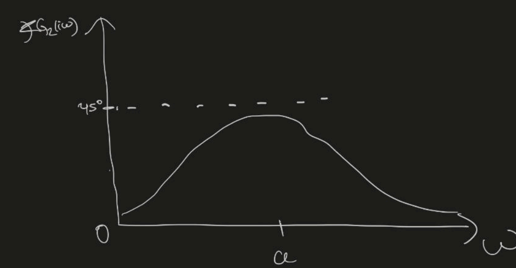
When $\omega = a$:

$$\measuredangle G_2(i\omega) = \frac{180}{\pi}\tan^{-1}(1) - \frac{180}{\pi}\tan^{-1}\left(\frac{1}{100}\right) = 45^\circ - 0.573^\circ \approx 44.427^\circ$$

When $\omega >> a$:

$$\measuredangle G_2(i\omega) = \frac{180}{\pi}\tan^{-1}(\infty) - \frac{180}{\pi}\tan^{-1}(\infty) = 90^\circ - 90^\circ = 0^\circ$$

Phase Plot:



c) $G_3(s) = \dfrac{s+100a}{s+a}$

$$G_3(i\omega) = \frac{100a+i\omega}{a+i\omega}$$

$$|G_3(i\omega)| = \sqrt{\left(\frac{100a+i\omega}{a+i\omega}\right)\left(\frac{100a-i\omega}{a-i\omega}\right)} = \sqrt{\frac{10{,}000a^2+\omega^2}{a^2+\omega^2}}$$

$$\log|G_3(i\omega)| = \frac{1}{2}\log(10{,}000a^2+\omega^2) - \frac{1}{2}\log(a^2+\omega^2)$$

When $\omega << a$:

$$\log|G_3(i\omega)| = \frac{1}{2}\log(10{,}000a^2) - \frac{1}{2}\log(a^2) = \frac{1}{2}\log(10{,}000) = 2$$

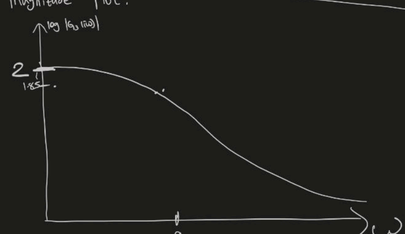
When $\omega = a$:

$$\log|G_3(i\omega)| = \frac{1}{2}\log(10{,}001a^2) - \frac{1}{2}\log(2a^2) = \frac{1}{2}\log\left(\frac{10{,}001}{2}\right) \approx 1.85$$

When $\omega >> a$:

$$\log|G_3(i\omega)| = \frac{1}{2}\log(\omega^2) - \frac{1}{2}\log(\omega^2) = 0$$

Magnitude Plot:



$$\measuredangle G_3(i\omega) = \measuredangle(100a+i\omega) - \measuredangle(a+i\omega) = \frac{180}{\pi}\tan^{-1}\left(\frac{\omega}{100a}\right) - \frac{180}{\pi}\tan^{-1}\left(\frac{\omega}{a}\right)$$

When $\omega << a$:

$$\measuredangle G_3(i\omega) = \frac{180}{\pi}\tan^{-1}(0) - \frac{180}{\pi}\tan^{-1}(0) = 0^\circ$$

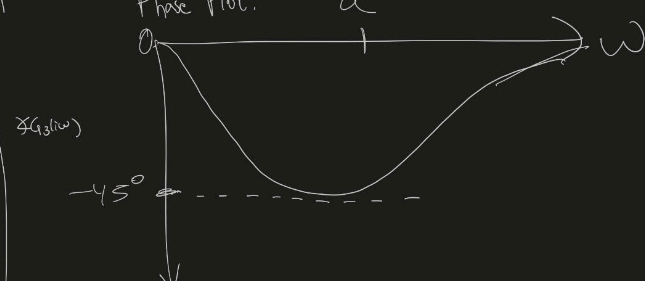
When $\omega = a$:

$$\measuredangle G_3(i\omega) = \frac{180}{\pi}\tan^{-1}\left(\frac{1}{100}\right) - \frac{180}{\pi}\tan^{-1}(1) \approx -44.427^\circ$$

When $\omega >> a$:

$$\measuredangle G_3(i\omega) = \frac{180}{\pi}\tan^{-1}(\infty) - \frac{180}{\pi}\tan^{-1}(\infty) = 0^\circ$$

Phase Plot:



d) $G_4(s) = \dfrac{1}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \dfrac{1}{s}\left(\dfrac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$

$$G_4(i\omega) = \frac{1}{i\omega}\left(\frac{1}{(i\omega)^2+2\zeta\omega_n(i\omega)+\omega_n^2}\right)$$

$$|G_4(i\omega)| = \frac{1}{\omega}\sqrt{\frac{1}{(\omega_n^2-\omega^2)+2\zeta\omega_n\omega i}\cdot\frac{1}{(\omega_n-\omega)^2-2\zeta\omega_n\omega i}}$$

$$= \frac{1}{\omega}\sqrt{\frac{1}{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}}$$

$$= \frac{1}{\omega\sqrt{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}}$$

When $\omega \ll \omega_n$:

$|G_4(i\omega)| = \infty$

$\log|G_4(i\omega)| = \log(\infty) = \infty$

When $\omega = \omega_n$:

$$|G_4(i\omega)| = \frac{1}{\omega_n\sqrt{4\zeta^2\omega_n^4}} = \frac{1}{\omega_n(2\zeta\omega_n^2)} = \frac{1}{2\zeta\omega_n^3}$$

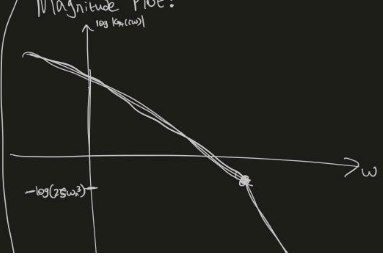
$\log|G_4(i\omega)| = -\log(2\zeta\omega_n^3)$

When $\omega \gg \omega_n$:

$$|G_4(i\omega)| = \frac{1}{\omega\sqrt{\omega^4}} = \frac{1}{\omega^3}$$

$\log|G_4(i\omega)| = -\log(\omega^3)$

Magnitude Plot:



$$\angle G_4(i\omega) = \angle\frac{1}{i\omega} + \angle\left(\frac{1}{(\omega_n^2-\omega^2)+2\zeta\omega_n\omega i}\right)$$

$$= -\angle i\omega - \angle(\omega_n^2-\omega^2)+2\zeta\omega_n\omega i$$

$$= -\frac{180}{\pi}\tan^{-1}(\infty) - \frac{180}{\pi}\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

$$= -90^\circ - \frac{180}{\pi}\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

When $\omega \ll \omega_n$:

$\angle G_4(i\omega) = -90^\circ - 0^\circ = -90^\circ$

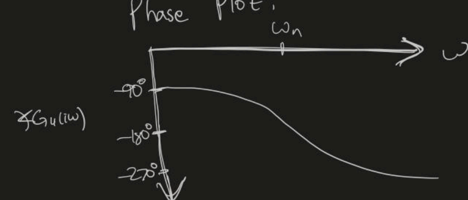
When $\omega = \omega_n$:

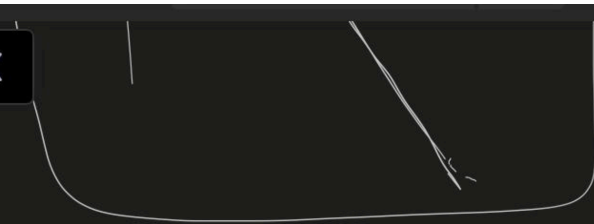
$$\angle G_4(i\omega) = -90^\circ - \frac{180}{\pi}\tan^{-1}\left(\frac{2\zeta\omega_n^2}{0}\right) = -90^\circ - \frac{180}{\pi}\tan^{-1}(\infty) = -90^\circ - 90^\circ = -180^\circ$$

When $\omega \gg \omega_n$:

$$\angle G_4(i\omega) = -90^\circ - \frac{180}{\pi}\tan^{-1}(0) = -90^\circ - 180^\circ = -270^\circ$$

Phase Plot:





e) $G_5(s) = e^{-\tau s}$

$G_5(i\omega) = e^{-\tau i\omega} = \cos(\tau\omega) - i\sin(\tau\omega)$

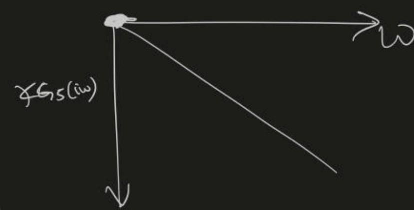
$|G_5(i\omega)| = \sqrt{e^{-\tau\omega i} \cdot e^{\tau\omega i}} = 1$

$\log|G_5(i\omega)| = \log(1) = 0$

Magnitude plot:

$$\measuredangle G_5(i\omega) = \frac{180}{\pi}\tan^{-1}\left(\frac{-\sin(\tau\omega)}{\cos(\tau\omega)}\right) = \frac{180}{\pi}\tan^{-1}(-\tan(\tau\omega)) = -\frac{180}{\pi}\tau\omega$$

Phase plot:



3) a) $L(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$

$$L(i\omega) = \frac{1}{(1+i\omega)(2+i\omega)} = \frac{1}{2+3i\omega+(i\omega)^2} = \frac{1}{(2-\omega^2)+3\omega i} \cdot \frac{(2-\omega^2)-3\omega i}{(2-\omega^2)-3\omega i} = \frac{(2-\omega^2)-3\omega i}{(2-\omega^2)^2+9\omega^2}$$

At $\omega = 0$:

$L(i\omega) = \frac{1}{2}$

$$Re\{L(i\omega)\} = \frac{2-\omega^2}{(2-\omega^2)^2+9\omega^2} \qquad Im\{L(i\omega)\} = \frac{-3\omega}{(2-\omega^2)^2+9\omega^2}$$

At $\omega = \infty$:

$L(i\omega) = 0$

When $Re\{L(i\omega)\} = 0$:

$$\frac{2-\omega^2}{(2-\omega^2)^2+9\omega^2} = 0 \Rightarrow 2-\omega^2 = 0 \Rightarrow \omega^2 = 2 \Rightarrow \omega = \pm\sqrt{2}$$

$$\Downarrow$$

$$Im\{L(i\sqrt{2})\} = \frac{-3\sqrt{2}}{18} = \frac{-\sqrt{2}}{6}$$

$$Im\{L(-i\sqrt{2})\} = \frac{3\sqrt{2}}{18} = \frac{\sqrt{2}}{6}$$

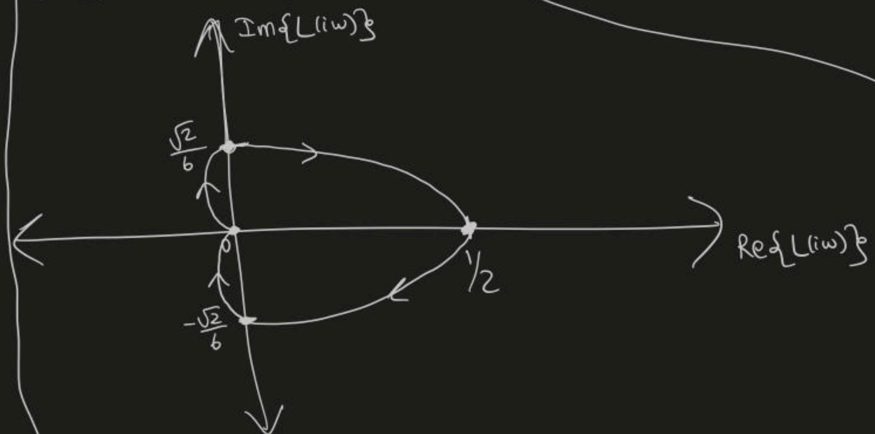
When $Im\{L(i\omega)\} = 0$:

$$\frac{-3\omega}{(2-\omega^2)^2+9\omega^2} = 0 \Rightarrow -3\omega = 0 \Rightarrow \omega = 0$$

$$\Downarrow$$

$$Re\{L(i0)\} = \frac{1}{2}$$

Nyquist Plot:



No. of encirclements around $L(i\omega) = -1 \; = 0$

No. of open loop unstable poles $= 0$

∴ The closed loop system is stable by the Nyquist Stability Criterion

b) $$\frac{L(s)}{1+L(s)} = \frac{\frac{1}{(s+1)(s+2)}}{1+\frac{1}{(s+1)(s+2)}} = \frac{\frac{1}{(s+1)(s+2)}}{\frac{(s+1)(s+2)+1}{(s+1)(s+2)}} = \frac{1}{(s+1)(s+2)+1} = \frac{1}{s^2+3s+3}$$

Poles: $$s = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm i\sqrt{3}}{2} = \frac{-3}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\boxed{\text{Poles: } s = \frac{-3}{2} \pm i\frac{\sqrt{3}}{2}}$$

The real parts of the closed loop system's poles are negative, so the system is stable.

4) a) System 2.1:

This system's magnitude response is flat until a peak at ω_0 and then has a fast decrease.

The phase response decreases from $0°$ to $-180°$.

These are the characteristics of a 2^{nd} order lower-pass filter

$$\boxed{G_1(s) = \frac{K\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}}$$

System 2.2:

This system's magnitude decreases until a flat response at ω_0 and then sharply declines

The phase response starts at $-90°$, moves up to $-45°$, and then decreases back down to $-90°$ after ω_0.

These are the characteristics of a second-order band-pass filter

$$\boxed{G_2(s) = \frac{K(s+a)}{s^2+2\zeta\omega_0 s+\omega_0^2}}$$

System 2.3:

The system's magnitude response is decreasing at a lower rate until a bump at w_0 which causes a sharp decline

The Phase response decreases from $-90°$ to $-270°$.

These are the characteristics of a system with higher order (>2) lower-pass filter

$$\boxed{G_3(s) = \frac{k w_0^2}{s(s^2+2\zeta w_0 s+w_0^2)}}$$

b) System 2.1 is 2^{nd} order lower-pass, which means there's a single loop around LHP as the phase decreases.

$\boxed{\text{System 2.1} \longrightarrow \text{N1}}$

System 2.2 is a 2^{nd} order band pass filter, which means there will be a straight line up and down the imaginary axis at $w=w_0$.

$\boxed{\text{System 2.2} \longrightarrow \text{N2}}$

System 2.3 is a 3^{rd} order lower-pass filter, which means that the curve will have multiple loops

$\boxed{\text{System 2.3} \longrightarrow \text{N3}}$

c) System 2.1:

N1 has 1 counterclockwise encirclement around -1.

System 2.1 has 0 open loop poles in RHP

The system is unstable

$\boxed{\text{System 2.1} \longrightarrow \text{S1}}$

System 2.2:

N2 has 0 counterclockwise encirclements around -1

System 2.2 has 0 open loop poles in RHP

$G_{yr,2}(0) = \frac{Ka}{w_0^2}$

$\boxed{\text{System 2.2} \longrightarrow \text{S3}}$

System 2.3:

N3 has 0 counterclockwise encirclements around -1.

System 2.3 has 0 open loop poles in RHP

$G_{yr,3}(0) = 1$

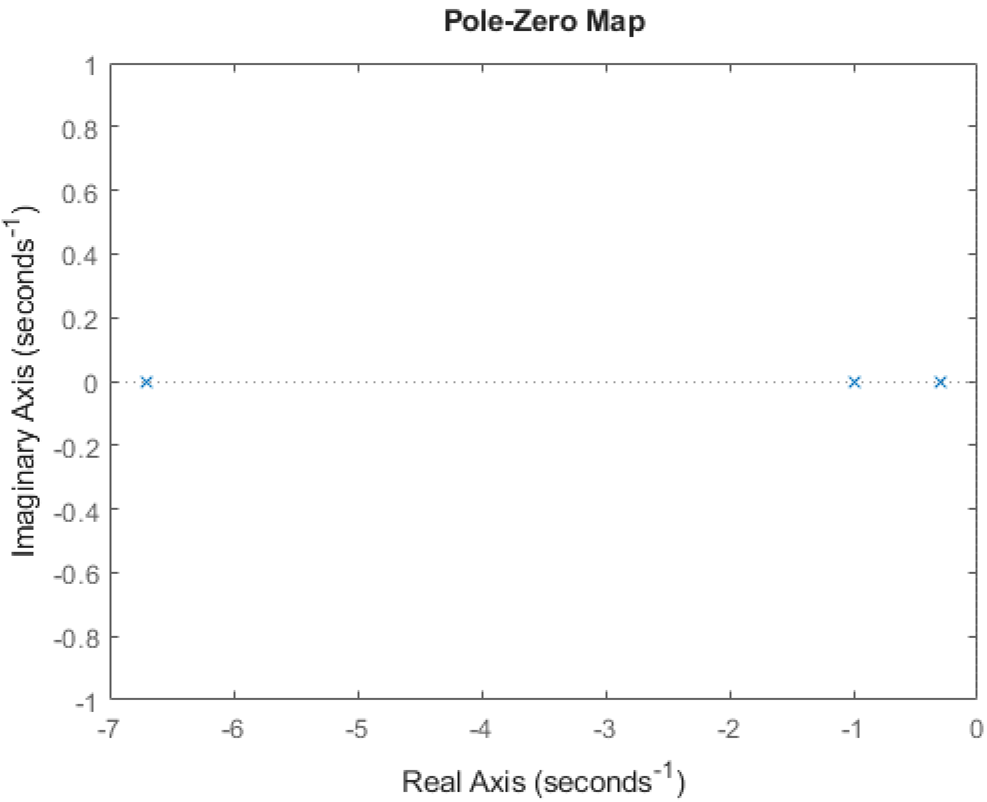
$\boxed{\text{System 2.3} \longrightarrow \text{S2}}$

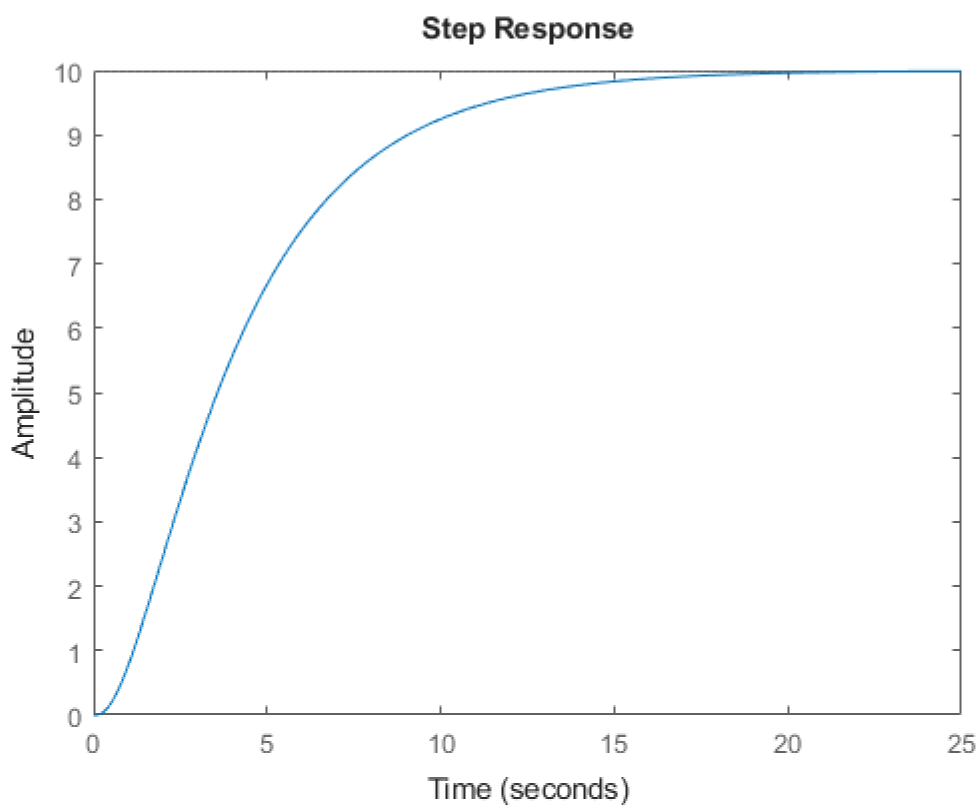
Table of Contents

1d)

```
k = 20;
C = k;
den = [1 8 9 -18];
P = tf(1, den);
Gyr = feedback(P*C, 1);
disp(Gyr);
figure;
pzmap(Gyr);

%The poles of this system are all on the left half plane, which indicates
%that the system is stable. According to my answer in part c, in order for
%a system to be stable, the range of k must be 18<k<90. Since k=20 falls
%within the range, the system is expected to be stable.

figure;
step(Gyr);
```

```
  tf with properties:

       Numerator: {[0 0 0 20]}
     Denominator: {[1 8 9 2]}
        Variable: 's'
         IODelay: 0
      InputDelay: 0
     OutputDelay: 0
       InputName: {''}
       InputUnit: {''}
      InputGroup: [1×1 struct]
      OutputName: {''}
      OutputUnit: {''}
     OutputGroup: [1×1 struct]
           Notes: [0×1 string]
        UserData: []
            Name: ''
              Ts: 0
        TimeUnit: 'seconds'
    SamplingGrid: [1×1 struct]
```


Pole-Zero Map
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-7
-6
-5
-4
-3
-2
-1
0

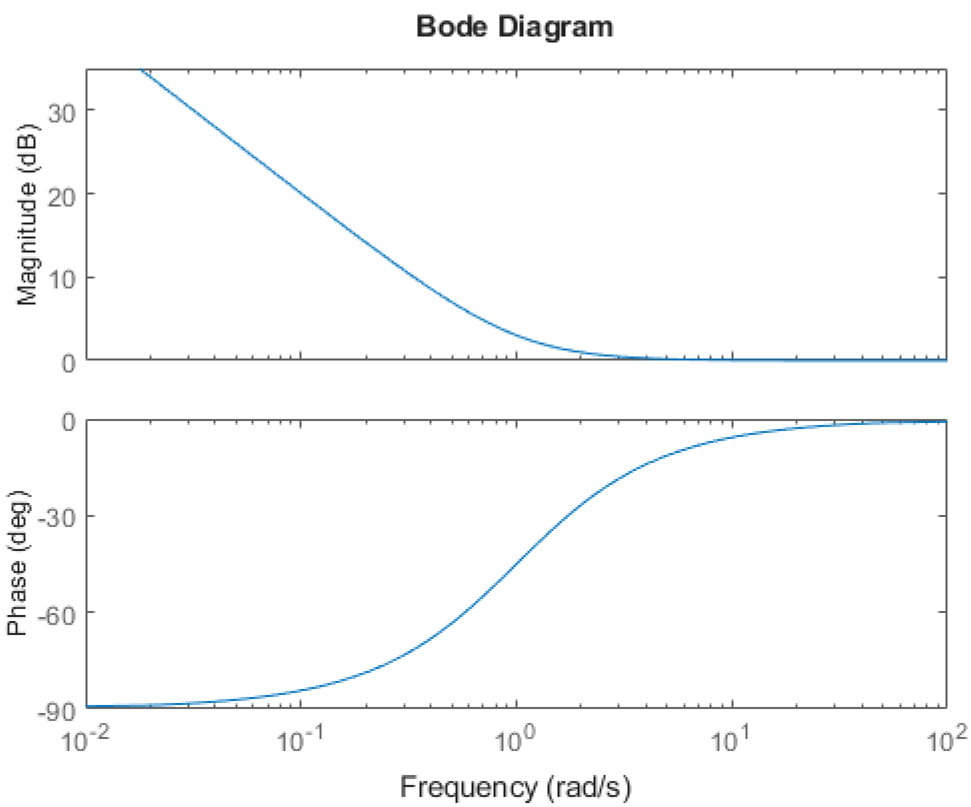

Step Response
Amplitude
Time (seconds)
10
9
8
7
6
5
4
3
2
1
0
0
5
10
15
20
25

2f)

```
a = 1;
b = 1;
omega = 1;
zeta = 0.5;
tau = 1;
```

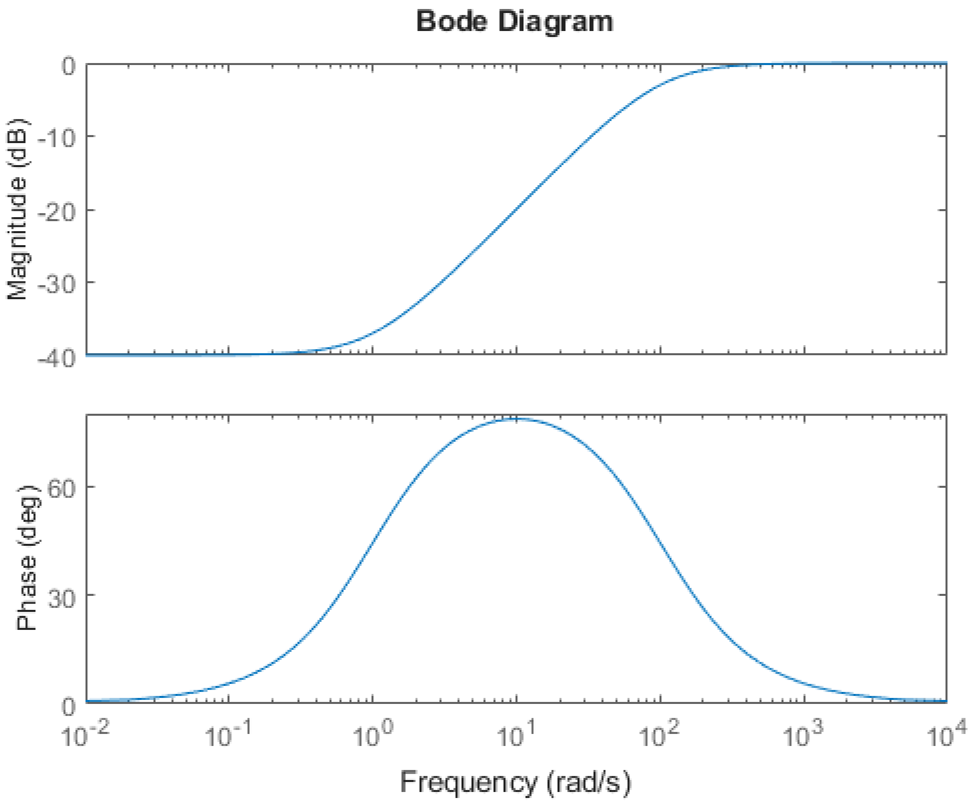
2a)

```
num = [b a*b];
den = [1 0];
sys = tf(num, den);
figure;
bode(sys);
```



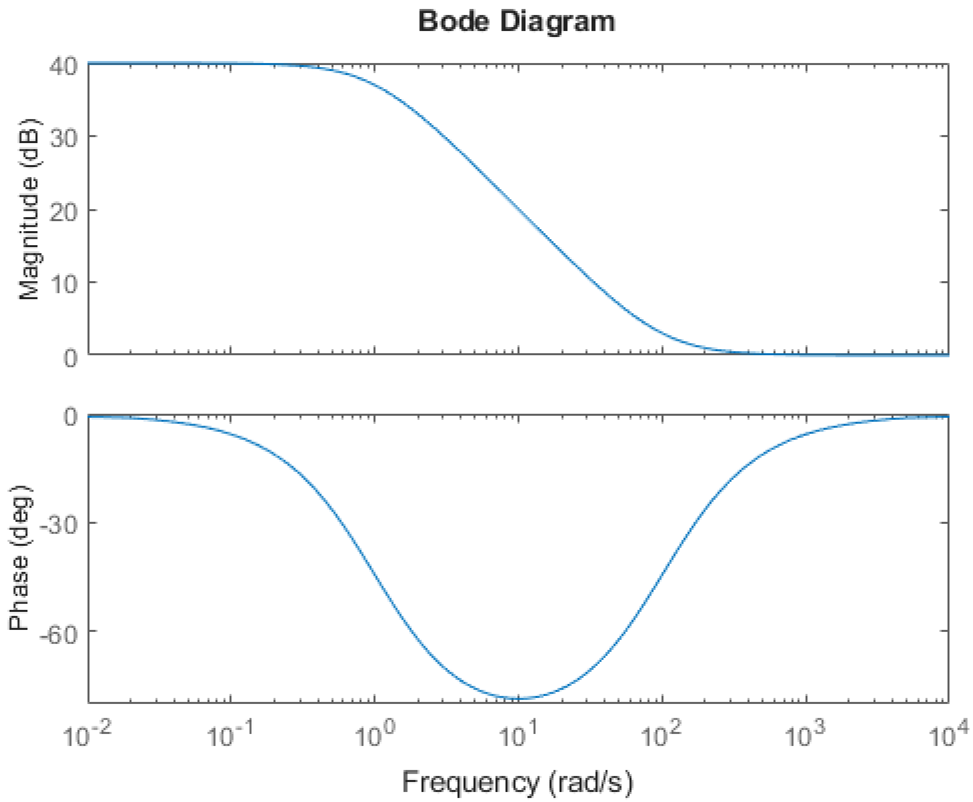
2b)

```
num = [1 a];
den = [1 100*a];
sys = tf(num, den);
figure;
bode(sys);
```



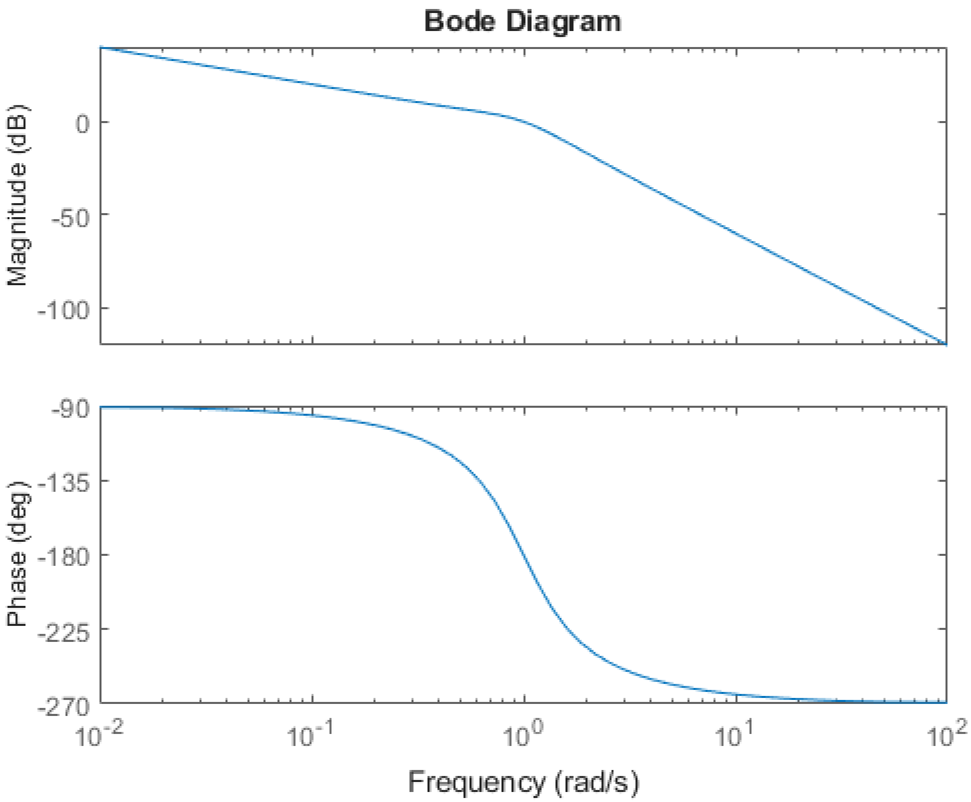
2c)

```
num = [1 100*a];
den = [1 a];
sys = tf(num, den);
figure;
bode(sys);
```



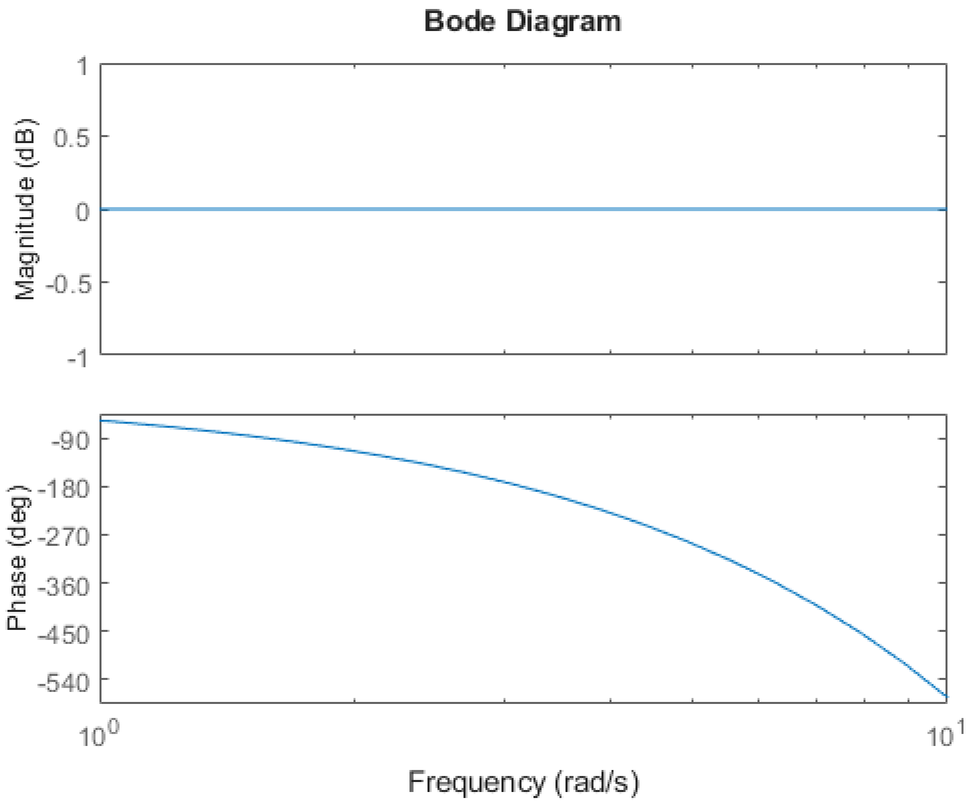
2d)

```
num = 1;
den = [1 2*zeta*omega omega^2 0];
sys = tf(num, den);
figure;
bode(sys);
```



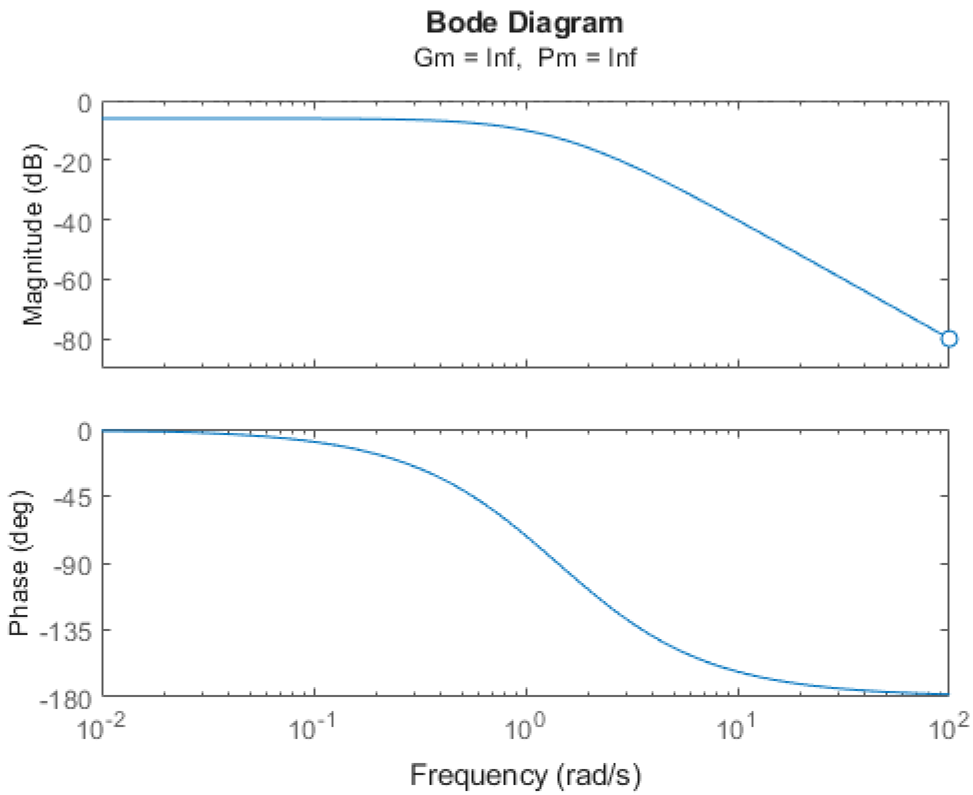
2e)

```
sys = tf(1, 1, 'InputDelay', tau);
figure;
bode(sys);

% These plots are consistent with my sketches.
```



3c)

```
num = 1;
den = [1 3 2];
sys = tf(num, den);
figure;
margin(sys);

%The gain margin is the maximum gain increase or decrease of a system that
%doesn't compromise stability. This system is already stable because it
%has poles with negative real parts, so the gain margin will be infinity.
%This same transfer function block was part of the open loop system in
%problem 1b except that it was multiplied by a constant k. The range of the
%constant k was determined to be k>-2 for stability, and this aligns with
%the infinite gain margin of L(s) because it indicates that we can increase
%k from -2 to any value without losing stability.
```



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