1) $H(t)$ – hares

$G(t)$ – tigers

$$\dot{H}(t) = \overbrace{rH(t)}^{\text{growth}} - \overbrace{aG(t)}^{\text{diminishing}}, \quad H(t) \geq 0$$

$$\dot{G}(t) = \frac{\overbrace{bH(t)G(t)}^{\text{growth}}}{\underbrace{c+H(t)}_{\text{consumption}}} - \underbrace{eG(t)}_{\text{mortality}}, \quad G(t) \geq 0$$

a) At equilibrium, $\dot{H} = \dot{G} = 0$

$$\dot{H}(t) = 0 = rH(t) - aG(t) \Rightarrow rH(t) = aG(t) \Rightarrow H(t) = \frac{a}{r}G(t)$$

$$\dot{G}(t) = 0 = \frac{b\left(\frac{a}{r}G(t)\right)G(t)}{c + \frac{a}{r}G(t)} - eG(t)$$

$$\frac{\frac{ba}{r}G^2(t)}{c+\frac{a}{r}G(t)} = eG(t)$$

$$\frac{ba}{r}G^{\cancel{2}}(t) = \cancel{G(t)}\left(ce + \frac{ea}{r}G(t)\right)$$

$$\frac{ba}{r}G(t) = ce + \frac{ea}{r}G(t)$$

$$\frac{ba-ea}{r}G(t) = ce$$

$$G(t) = \frac{cer}{a(b-e)} \longrightarrow H(t) = \frac{\cancel{a}}{\cancel{r}}\left(\frac{ce\cancel{r}}{\cancel{a}(b-e)}\right) = \frac{ce}{b-e}$$

$$\boxed{(H^*, G^*) = \left(\frac{ce}{b-e}, \frac{cer}{a(b-e)}\right)}$$

b) $r=0.1 \quad e=0.1 \quad c=100 \quad b=0.2 \quad a=0.5$

$$(H^*, G^*) = \left(\frac{100(0.1)}{0.2-0.1}, \frac{100(0.1)(0.1)}{0.5(0.2-0.1)}\right) = \left(\frac{10}{0.1}, \frac{1}{0.05}\right) = (100, 20)$$

$$\boxed{(H^*, G^*) = (100, 20)}$$

$$\dot{x} = Ax \qquad x(t) = \begin{bmatrix} H(t) - H^* \\ G(t) - G^* \end{bmatrix}$$

$$\dot{H}(t) = f(H(t), G(t))$$

$$\dot{G}(t) = g(H(t), G(t))$$

$$\left.\frac{\partial f}{\partial H}\right|_{(100,20)} = r = 0.1 \qquad \left.\frac{\partial f}{\partial G}\right|_{(100,20)} = -a = -0.5$$

$$\left.\frac{\partial g}{\partial H}\right|_{(100,20)} = \frac{bG(t)[c+H(t)] - bH(t)G(t)}{(c+H(t))^2} = \frac{0.2(20)[100+100] - 0.2(100)(20)}{(100+100)^2} = \frac{800-400}{40{,}000} = \frac{\cancel{400}}{\cancel{40{,}000}} = \frac{1}{100}$$

$$\left.\frac{\partial g}{\partial G}\right|_{(100,20)} = \frac{bH(t)}{c+H(t)} - e = \frac{0.2(100)}{100+100} - 0.1 = \frac{2\cancel{0}}{20\cancel{0}} - 0.1 = 0$$

$$J(100, 20) = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix}$$

$$\boxed{\dot{x} = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix} x}$$

Equilibrium is also achieved at $\boxed{(H^*, G^*) = (0, 0)}$

$$\left.\frac{\partial f}{\partial H}\right|_{(0,0)} = 0.1 \qquad \left.\frac{\partial f}{\partial G}\right|_{(0,0)} = -0.5$$

$$\left.\frac{\partial g}{\partial H}\right|_{(0,0)} = 0 \qquad \left.\frac{\partial g}{\partial G}\right|_{(0,0)} = -0.1$$

$$J(0,0) = \begin{bmatrix} 0.1 & -0.5 \\ 0 & -0.1 \end{bmatrix}$$

The linearization at (0,0) is:

$$\boxed{\dot{x} = \begin{bmatrix} 0.1 & -0.5 \\ 0 & -0.1 \end{bmatrix} x}$$

c) $(H^*, G^*) = (100, 20)$:

$$A = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 0.1 & 0.5 \\ -0.01 & \lambda \end{vmatrix} = \lambda(\lambda - 0.1) + 0.05 = \lambda^2 - 0.1\lambda + 0.05$$

$$\lambda = \frac{0.1 \pm \sqrt{0.01 - 0.2}}{2} = 0.05 \pm \sqrt{0.19}\, i$$

The system is unstable at (100,20) because the eigenvalues of the Jacobian matrix have a positive real part.

$(H^*, G^*) = (0, 0)$:

$$A = \begin{bmatrix} 0.1 & -0.5 \\ 0 & -0.1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 0.1 & 0.5 \\ 0 & \lambda + 0.1 \end{vmatrix} = (\lambda - 0.1)(\lambda + 0.1)$$

$\lambda_1 = 0.1$, $\lambda_2 = -0.1$

The system is unstable at (0,0) because the Jacobian matrix has an eigenvalue with a positive real part.

e) $A = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix}$ $\quad (H^*, G^*) = (100, 20)$

i) $\dot{x} = Ax + u(t)$ $\quad u(t) = \begin{bmatrix} w(H(t)-100) \\ 0 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix} \begin{bmatrix} H(t)-100 \\ G(t)-20 \end{bmatrix} + \begin{bmatrix} w(H(t)-100) \\ 0 \end{bmatrix}$$

$$\dot{x}_1(t) = 0.1(H(t)-100) - 0.5(G(t)-20) + w(H(t)-100) = 0.1H(t) - \cancel{10} - 0.5G(t) + \cancel{10} + wH(t) - 100w = \underline{(0.1+w)}H(t) - 0.5G(t) - 100w = f(H(t), G(t))$$

$$\dot{x}_2(t) = 0.01(H(t)-100) = 0.01H(t) - 1 = g(H(t), G(t))$$

$$\left.\frac{\partial f}{\partial H}\right|_{(100,20)} = 0.1 + w \qquad \left.\frac{\partial f}{\partial G}\right|_{(100,20)} = -0.5$$

$$\left.\frac{\partial g}{\partial H}\right|_{(100,20)} = 0.01 \qquad \left.\frac{\partial g}{\partial G}\right|_{(100,20)} = 0$$

$$J(100,20) = \begin{bmatrix} 0.1+w & -0.5 \\ 0.01 & 0 \end{bmatrix} = \bar{A}$$

$$\boxed{\dot{x} = \begin{bmatrix} 0.1+w & -0.5 \\ 0.01 & 0 \end{bmatrix} x}$$

ii) $\dot{x} = \begin{bmatrix} -0.51 & -0.5 \\ 0.01 & 0 \end{bmatrix} x$

$$\bar{A} = \begin{bmatrix} -0.51 & -0.5 \\ 0.01 & 0 \end{bmatrix}$$

$$|\lambda I - \bar{A}| = \begin{vmatrix} \lambda + 0.51 & 0.5 \\ -0.01 & \lambda \end{vmatrix} = \lambda(\lambda + 0.51) + 0.005$$

$$= \lambda^2 + 0.51\lambda + 0.005$$

$$\lambda = \frac{-0.51 \pm \sqrt{0.51^2 - 4(0.005)}}{2} < 0$$

$-4(0.05)$

$\boxed{\text{The system will be stable at } (100,20) \text{ because both the eigenvalues of the Jacobian matrix have negative real parts.}}$

3) $e^X = I + X + \frac{1}{2}X^2 + \cdots + \frac{1}{n!}X^n + \cdots$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^n t^n + \cdots$$

$$\frac{d}{dt}(e^{At}) = A + A^2 t + \cdots + \frac{1}{(n-1)!}A^n t^{n-1} + \cdots$$

$$= A\left(I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^n t^n + \cdots\right)$$

$$= A e^{At}$$

$$\frac{d}{dt}\left(e^{At}x(0)\right) = A e^{At} x(0) \quad ✓ \quad \Longrightarrow \quad \dot{x} = Ax$$

$\therefore$ $x(t) = e^{At}x(0)$ is a general solution to the ODE $\dot{x} = Ax$ with $x(0) \in \mathbb{R}^n$.

Questions 1d, 1f, and 2 are answered in the sections below.

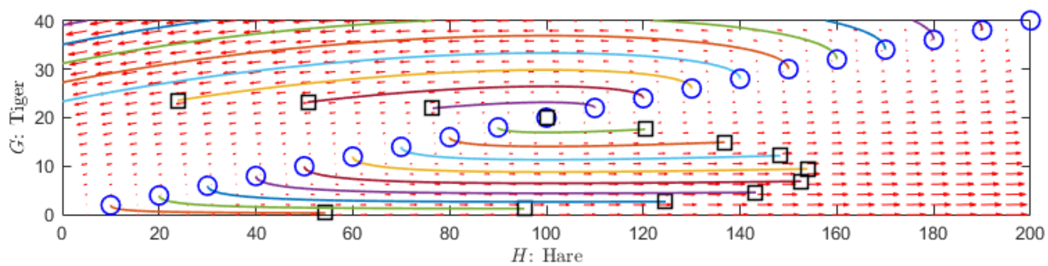
Table of Contents

1d

```
ECE171A: HW3 Problem 1 - sample code
Writen by Yang Zheng
Spring, 2025
```

```
close all;

% parameters
r = 0.1;
e = 0.1;
c = 100;
b = 0.2;
a = 0.5;
w = -0.61;


% dynamics
f = @(t,x) [r*x(1) - a*x(2); b*x(1)*x(2)/(c+x(1))-e*x(2)];

H = linspace(0,200,40);
G = linspace(0,40,20);

[x,y] = meshgrid(H,G);
u = zeros(size(x));
v = zeros(size(x));

% we can use a single loop over each element to compute the derivatives at
% each point (y1, y2)
t=0;   % we want the derivatives at each point at t=0, i.e. the starting time
for i = 1:numel(x)
    Yprime = f(t,[x(i); y(i)]);
    u(i) = Yprime(1);
    v(i) = Yprime(2);
end

% We use the quiver command to plot our vector field
figure; quiver(x,y,u,v,'r');
xlabel('$H$: Hare','Interpreter','latex')
ylabel('$G$: Tiger','Interpreter','latex')
```

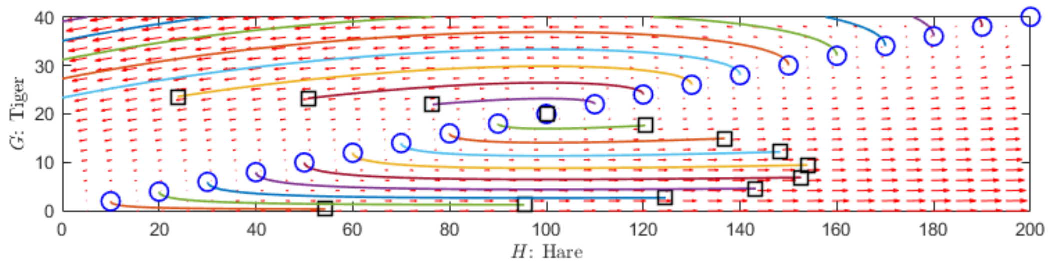
```
axis tight equal;

% Plot some trajectories
hold on

H0 = 10:10:200;

for i=1:length(H0)
    [ts,ys] = ode45(f,[0,25],[H0(i);0.2*H0(i)]);      % ode45 simulations
    plot(ys(:,1),ys(:,2),'linewidth',1.2)
    plot(ys(1,1),ys(1,2),'bo','MarkerSize',10,'LineWidth',1.2)         %
starting point
    plot(ys(end,1),ys(end,2),'ks','MarkerSize',10,'LineWidth',1.2)     %
ending point
end

ylim([0,40]); xlim([0,200]);
hold off
set(gcf,'Position',[150 150 900 250])
```

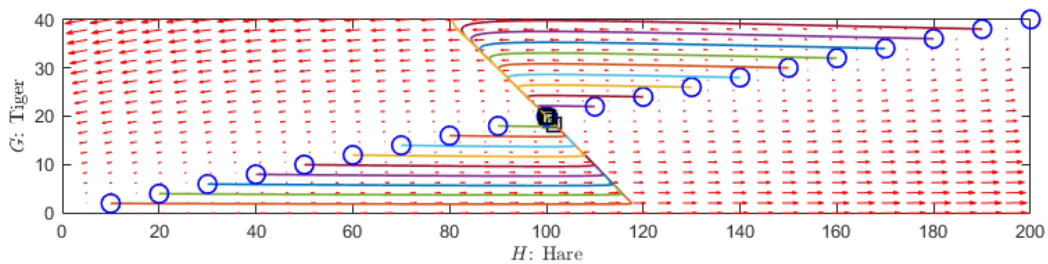


1f

```
% dynamics
ff = @(t,x) [r*x(1) - a*x(2) + w*(x(1)-100); b*x(1)*x(2)/(c+x(1))-e*x(2)];

% We use the quiver command to plot our vector field
figure; quiver(x,y,u,v,'r');
xlabel('$H$: Hare','Interpreter','latex')
ylabel('$G$: Tiger','Interpreter','latex')
axis tight equal;

% Plot some trajectories
hold on

for i=1:length(H0)
    [ts,ys] = ode45(ff,[0,500],[H0(i);0.2*H0(i)]);      % ode45 simulations
    plot(ys(:,1),ys(:,2),'linewidth',1.2)
    plot(ys(1,1),ys(1,2),'bo','MarkerSize',10,'LineWidth',1.2)         %
starting point
    plot(ys(end,1),ys(end,2),'ks','MarkerSize',10,'LineWidth',1.2)     %
ending point
```

```
end

ylim([0,40]); xlim([0,200]);
hold off
set(gcf,'Position',[150 150 900 250])

% Comparison:
% The phase portrait in part (d) shows trajectories from each initial point
% moving away from the equilibrium point. The phase portrait looks like a
% source. This happens because the system is unstable at the equilibrium
% point without any Jacobian linearization. The phase portrait in part (f)
% is much closer to a sink with the trajectories from the initial points
% spinning more directly near to the equilibrium point. This happens
% because the linearized system is stable at the equilibrium point.
```

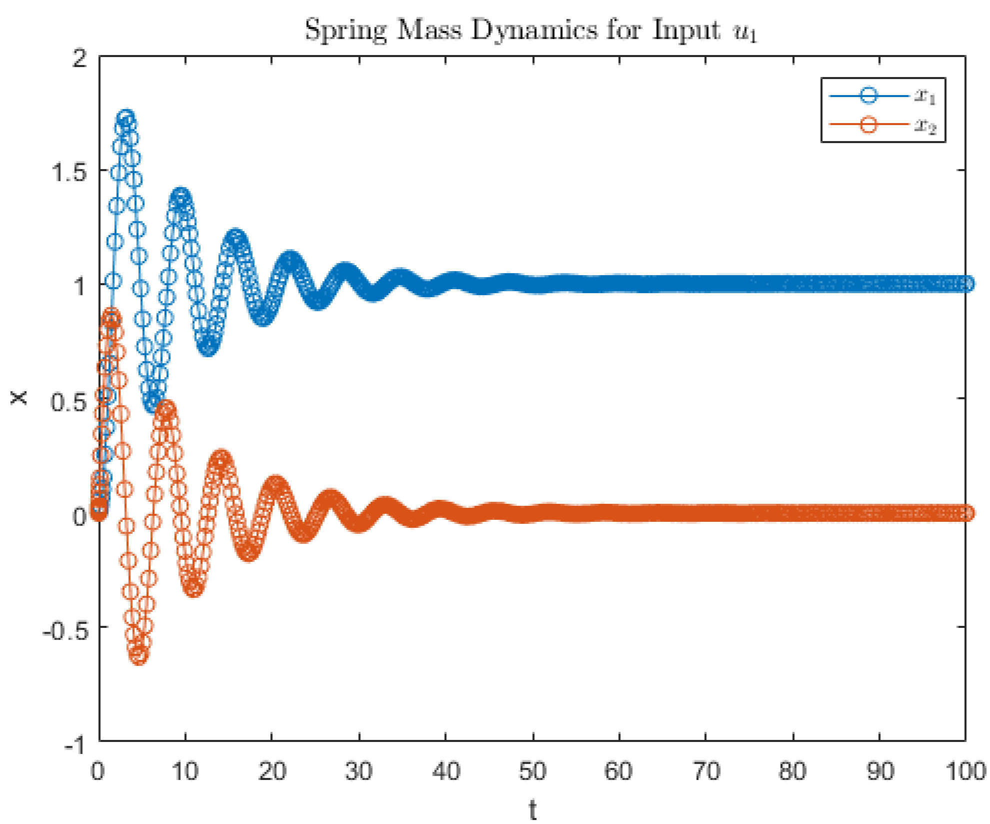


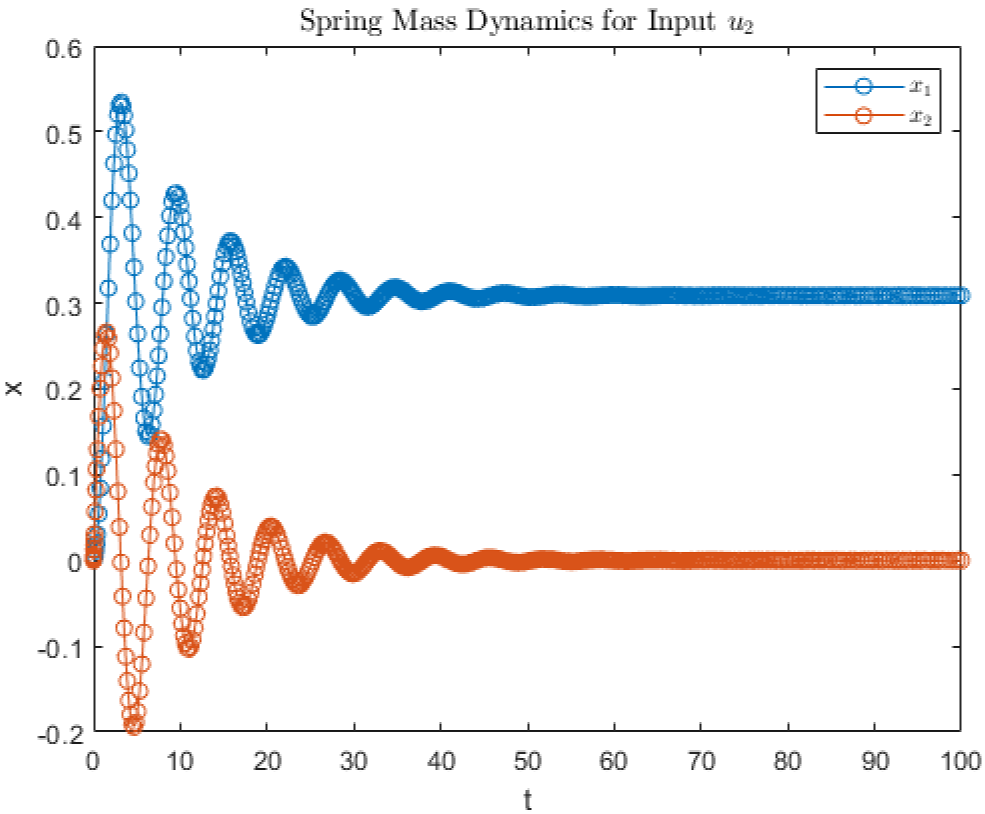


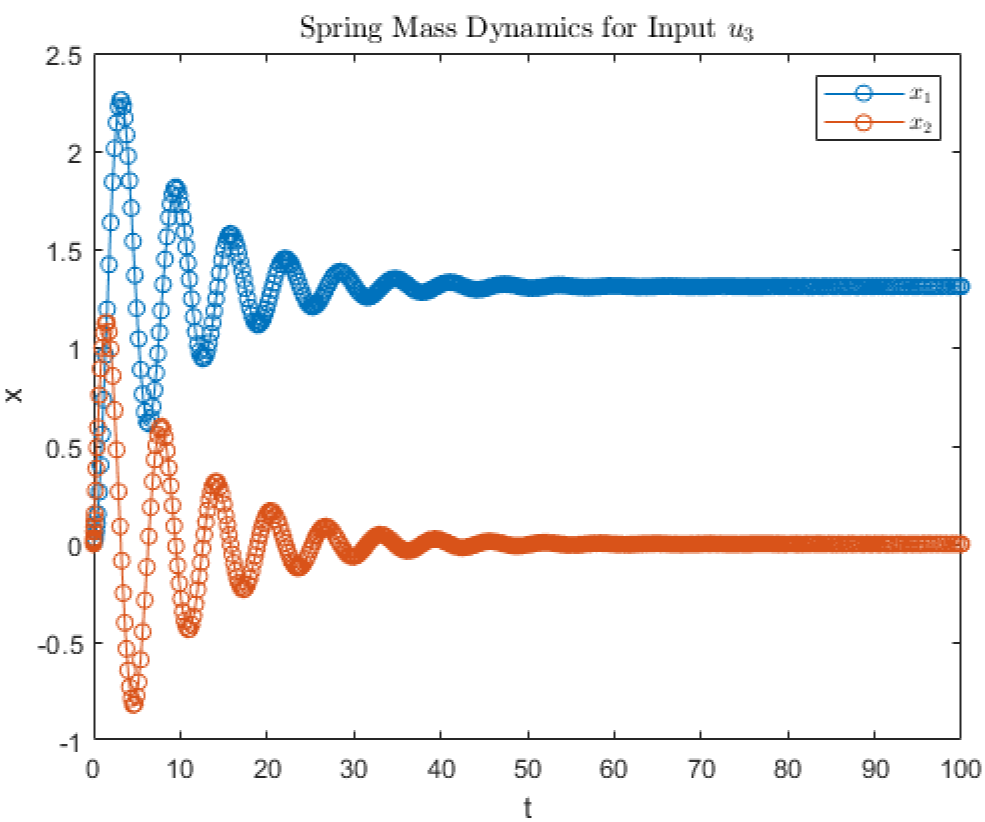
2a)

```
f2a_u1 = @(t,x) [x(2); sin((2*pi)/4)-(0.2*x(2))-x(1)];
figure;
ode45(f2a_u1,[0,100],[0;0]);
xlabel('t');
ylabel('x');
legend('$x_1$', '$x_2$', 'Interpreter', 'Latex');
title('Spring Mass Dynamics for Input $u_1$', 'Interpreter', 'Latex');


f2a_u2 = @(t,x) [x(2); sin((2*pi)/20)-(0.2*x(2))-x(1)];
figure;
ode45(f2a_u2, [0,100], [0;0]);
xlabel('t');
```

```
ylabel('x');
legend('$x_1$', '$x_2$', 'Interpreter', 'Latex');
title('Spring Mass Dynamics for Input $u_2$', 'Interpreter', 'Latex');

f2a_u3 = @(t,x) [x(2); sin((2*pi)/4)+sin((2*pi)/20)-(0.2*x(2))-x(1)];
figure;
ode45(f2a_u3, [0,100], [0;0]);
xlabel('t');
ylabel('x');
legend('$x_1$', '$x_2$', 'Interpreter', 'Latex');
title('Spring Mass Dynamics for Input $u_3$', 'Interpreter', 'Latex');
```




Spring Mass Dynamics for Input u_2
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
x
0
10
20
30
40
50
60
70
80
90
100
t
x_1
x_2


Spring Mass Dynamics for Input u_3
2.5
2
1.5
1
0.5
0
-0.5
-1
x
0
10
20
30
40
50
60
70
80
90
100
t
x_1
x_2

2b)

```
%Observations:
%The points on the plot for the outputs of the 3rd input u3 are the sum of
%the points on the plot for the outputs of the 1st input u1 and the second
%input u2. This happens because the system is linear, which means that
%if an input is the sum of inputs, then the output will be the sum of the
%outputs of each individual input.
```

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