Part 6.3: Linear Combinations

You by now may have noticed the formulas to the right on your formula sheet. Let's look at them one at a time to see what they mean and an example for each one. They are designed for when we want to add two distributions together or for when we multiply a distribution or add things happening to a distribution. It is very important that for these sorts of distributions that you work out the variance before working out the standard deviation, as it is very easy to make mistakes which mean that you get the wrong answer if you try and do it all in one step.

E[aX + b] = aE[X] + b

Var[aX + b] = a^2Var[X]

E[aX + bY] = aE[X] + bE[Y]

Var[aX + bY] = a^2Var[X] + b^2Var[Y]

If X, Y are independent

E[aX + b] = aE[X] + b

This formula means if I have 'a' times X plus 'b' the expected value, or the mean will be a times the expected value of X plus b.

Var[aX + b] = a^2Var[X]

This formula is the matching one for variance (standard deviation squared) and means if I have 'a' times X plus 'b' the variance will be a squared times the expected value of X.

Note, we do not add 'b' on to the variance because the b does not affect how spread out the data is.

Example

A salesman earns commission of $10 for every TV he sells plus a fixed rate of $30 a day. If he sells on average 4.3 TVs per day with a standard deviation of 0.3 TVs, calculate the mean and standard deviation for how much he will earn each day.

Answer

Let's start by working out the mean, or E(aX + b). $\mathrm{E}[aX+b] = a\mathrm{E}[X] + b = 10 \times 4.3 + 30 = \$73.00$

Now let's work out the variance. The Var(X) = $sd^2 = 0.3^2 = 0.09$. $\mathrm{Var}[aX + b] = a^2\mathrm{Var}[X] = 10^2 \times 0.09 = 9$

So the standard deviation = $\sqrt{9} = \$3.00$

E[aX + bY] = aE[X] + bE[Y]

This formula means that if I have 'a' times X plus 'b' times Y then the expected total is 'a' times the expected number of X plus 'b' times the expected number of Y.

Var[aX + bY] = a^2Var[X] + b^2Var[Y]

This is the matching formula for variance and means the variance of the sum is 'a' squared times the variance of X plus 'b' squared times the variance of Y.

Note: these last two formulas assume that X and Y are independent.

Example

In prescription glasses the weight of the lens is 9 grams with a standard deviation of 2 grams. The frames have an average weight of 6 grams with a standard deviation of 1 gram. If the material for the glass costs $3.20 per gram and the material for the frame costs $2.50 per gram, what is the mean and standard deviation for the cost of the materials in the glasses?

Answer

Lets start with the mean. $\mathrm{E}[aX + bY] = a\mathrm{E}[X] + b\mathrm{E}[Y] = \$3.20 \times 9 + \$2.50 \times 6 = \$43.80$.

Then the variance = $\mathrm{Var}[aX + bY] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y] = 3.20^2 \times 2^2 + 2.50^2 \times 1^2 = 47.21$.

(the 2^2 and 1^2 are because the variance = sd^2)

This means the standard deviation of the total weight = $\sqrt{47.21} = \$6.87$ (3sf)

Note: when you have '3 lots of' you do not square the number of them, as you are just adding it on a number of times.

Exercise 6.3

1. The length of a worm is on average 7.3 cm with a standard deviation of 0.6 cm.
    a. If 10 worms were laid end to end how long would you expect the chain to be, and what would the standard deviation of this length be?
    b. In reality laying 10 worms end to end is quite tricky, so you need a 1cm gap between them (9cm total). With this gap what would the mean and standard deviation be for total length of the line?
2. The average time it takes for a relay runner to run 400 m is 55 seconds with a standard deviation of 3 seconds, how long would you expect it to take for a team of 4 runners to complete a 4 x 400 m relay, and what would the standard deviation of this time be, if there is a delay at the start of 5 seconds as the first runner gets up to speed?
3. A pair of kiwi birds is being transported. The average weight of a female kiwi is 2.91 kg with a standard deviation of 0.404. The average weight of a male is 2.26 kg with a standard deviation of 0.274. What is the mean and standard deviation for the total cost of transporting one male and one female if the different requirements mean that it costs $125 per kilo to transport males and $135 per kilo to transport females?
4. The weight of an average adult in NZ is 81.3 kg with a standard deviation of 6.2 kg.
    a. If you have 10 NZ adults how much are they likely to weigh in total, and what would the standard deviation be of this total weight?
    b. If 5 adults were in a lift with 4 suitcases with an average weight of 15 kg and a standard deviation of 2 kg, what is the total weight of the passengers and their suitcases?