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Commodity Storage Valuation

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We present a general valuation framework for commodity storage facilities, for non-perishable commodities. We consider the case of a storage facility small enough so that injections and withdrawals do not influence the price of the underlying commodity. We allow for mean-reversion and seasonality in the price of the commodity, and allow for injection and withdrawal costs. To find the optimal actions for the storage owner we present an iterative numerical algorithm and prove its convergence. We illustrate our framework with numerical examples for the case of storage facilities for oil, natural gas, and water.

1. Introduction

The production, trade, and preservation or storage of commodities has been an important human activity long before financial instruments existed.[1] Commodities such as olive oil, wheat, rice, as well as iron, gold, and silver, among others, where among the first to be traded in organized markets. As markets evolved, more complicated, derivative, contracts appeared, that allowed not only the immediate exchange of commodities but also the trade of future production, and contingent payments depending on future commodity demand and price.[2] While in recent times financial contracts dominate trading in the commodity markets, where most contracts are settled before maturity and delivery is rarely taken, the physical nature of commodities presents challenges in pricing commodity-linked products unlike those faced in purely financial products such as stocks, bonds, or foreign exchange.

Storage of commodities is valuable when there is a temporal mismatch between supply and demand; for example electricity demand peaks during the daytime in weekdays; natural gas demand peaks during the winter; agricultural commodities are in supply during part of the year, etc. Storage

[1] An early documented example is given by Aristotle in *Politics 1259 a 6-23*, where he describes what today would be thought of as a European call on the spread between the price of olives and the price of olive oil. Aristotle relates how Thales of Miletus made a fortune by being correct on a forecast of a bountiful year for olive production in Ancient Greece. Based on this prediction, months ahead of olive production, Thales made a downpayment to secure the use of olive presses. When the large harvest materialized, with a large supply of olives, and a limited number of olive presses, the spread between the price of olives and the price of olive oil was particularly large, and Thales was able to rent the olive presses for a large profit.

[2] The Dojima rice exchange in Japan traded futures contracts as early as 1697.

has the benefit of avoiding immediate consumption and makes the commodity available at times when either production is low or zero, or when demand peaks. Storage facilities range from silos for agricultural commodities, to tanks used to store oil, to underground natural coverns where natural gas can be stored, to lake reservoirs and dams, where water is stored for future electricity production and possibly irrigation.

Two elements determine the value of a storage facility: the characteristics of the underlying commodity; and the characteristics of the storage facility. Commodity prices have several features that distinguish them from the prices of financial instruments. Due to either the mean-reversion of demand and the inability to adjust production, or to the adjustment in production over time to respond to demand shocks, commodity prices are mean-reverting. In addition, when either demand or supply of the commodity varies over the course of the year, commodity prices exhibit seasonal patterns. Storage characteristics include the capacity and cost of operating the storage facility, the flexibility of storing and removing amounts of the commodity, and whether the stored commodity deteriorates over time or can randomly increase.[3] The framework we develop accounts for both the features of commodity prices and the characteristics of the storage technology.

In this paper we determine the value of a storage facility, as well as the optimal injection and withdrawal policies for the case of non-perishable commodities when the size of the facility is small and its actions do not influence the price of the underlying commodity, or the production decisions of the commodity producers. We derive a closed form solution for the valuation of a very simple storage technology where injection or withdrawal can only occur once at levels known in advance. We use this solution to develop an iterative numerical algorithm, that is an efficient implementation of policy iteration, to find the optimal injection and withdrawal policies. The algorithm is based on solving, at each step, a system of linear algebraic equations and finding the roots of linear combinations of hypergeometric functions.

In addition to proving the convergence of the algorithm, we provide numerical illustrations for the value of storage facilities in calibrated examples for the case of storage facilities for natural gas which allows for seasonality in the prices of the commodity, and water reservoirs, which allow for random inflows to the storage facility.

To discuss: references ....

- Manoliu
- Secomandi

[3] An example where the amount of the commodity in storage can increase is the case of lake reservoirs for dams, where the water stored can increase by a random amount due to rainfall or melting snow.

- Hodges
- Zhai
- Working
- Routledge, Seppi, Spatt
- Thomson, Davison, Rasmussen
- Schwartz, Smith
- Jaillet, Ronn, Tompaidis

2. Problem Formulation

We first consider a base case of an infinite horizon problem for a non-perishable commodity whose price does not exhibit seasonality. An example of such a commodity is crude oil. Let S_t denote the price of the commodity at time t. As in the first model proposed in Schwartz (1997) we model the price evolution by

$$dS_t = \kappa(\gamma - \log S_t)S_t dt + \sigma S_t dZ_t \tag{1}$$

Here Z_t denotes a standard Brownian motion, σ denotes the volatility, κ represents the mean reversion rate and γ the mean reversion level for the logarithm of the price. Considering the evolution of the natural logarithm of the price process $X_t = \ln(S_t)$, we have the Ornstein-Uhlenbeck(OU) mean reverting process

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dZ_t. \tag{2}$$

with the mean reversion level $\alpha = \gamma - \frac{\sigma^2}{2\kappa}$.

We also denote by Q_t the total amount of the commodity in the storage facility at time t. By a "policy" we mean the strategy that a facility manager uses to decide when to buy and inject more commodity into the storage and when to withdraw and sell commodity from the facility. Injections and withdrawals of the commodity are captured by two non-decreasing right-continuous-with-left-limits (RCLL) processes L_t and U_t that represent the cumulative amounts of the commodity purchased and sold up to time t, respectively. We have for the evolution of Q_t,

$$dQ_t = dL_t - dU_t. \tag{3}$$

The cost of injecting a unit of the commodity into the storage facility when the facility contains Q units, is given by $\lambda(Q)$. Similarly, the cost of withdrawing a unit is given by $\mu(Q)$. To avoid trivial solutions we assume that $\mu(Q) \geq 0$, $\lambda(Q) \geq 0$ and $\mu(Q) + \lambda(Q) > 0$ for all Q. Hence the total cost incurred in buying dL_t units of the commodity is $e^{x_t}dL_t + \lambda(Q_t)dL_t$ and selling dU_t units of the commodity is $-e^{x_t}dU_t + \mu(Q_t)dU_t$. An admissible control policy is a pair (L_t, U_t) that keeps $Q_t \in [0, Q_{\max}]$ for all t. Here $Q_{\max}$ is the capacity of the storage facility. We denote by $\mathcal{U}$, the set of all admissible policies.

The objective is to maximize discounted infinite-horizon discounted cash flows. Taking a discount factor $\beta \in (0,1)$,

$$V(q,x) = \max_{\mathcal{U}} \; E_{q,x} \int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t))dL_t \tag{4}$$

where $X_0 = x$ and $Q_0 = q$. Since this $V(q,x)$ is the maximum value that a facility manager can obtain from a storage facility, this is the value of the facility when the current spot price is e^x and the amount in storage is q.

3. The Hamilton-Jacobi-Bellman equation

A standard way of solving for the value function V in Equation (4) is to use dynamic programming arguments in continuous time. Such arguments lead to a partial differential equation (PDE) problem that is often referred to as the Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation that characterizes the our value function $V(q,x)$ is given by

$$\max\left(\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \kappa(\alpha - x)\frac{\partial V}{\partial x} - \beta V,\ \frac{\partial V}{\partial q} - (e^x + \lambda(q)),\ -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0. \tag{5}$$

For notational convenience we will use subscripts to denote differentials.

The state space $(q,x) \in (\mathbf{R}, \mathbf{R}_+)$ can have three kinds of regions depending on which of the three terms in (5) is equal to zero.

Injection region: where $V_q - (e^x + \lambda(q)) = 0$,

Withdrawal region: where $-V_q + (e^x - \mu(q)) = 0$, and

Hold region: where $\frac{1}{2}\sigma^2 V_{xx} + \kappa(\alpha - x)V_x - \beta V = 0$.

To understand these three terms better, first consider the second expression in (5). If we were considering the possibility of injecting an infinitesimal quantity into storage, then the $\partial V/\partial q$ term captures the rate of value change and the $(e^x + \lambda(q))$ term captures the rate of control cost incurred. Hence for an optimal policy, we expect this quantity to be non-positive everywhere other than the injection region. Obviously, in the injection region we will have the term equal to zero. Similarly the third term in (5) captures the total value of withdrawing from the storage. The first term in (5) is nothing but the infinitesimal generator of the mean-reverting OU process.

Theorem 1, stated below, is often referred to as the verification theorem and establishes that a function that solves the HJB equation is the value function for the original control problem and a policy that achieves this value function is the optimal policy. The verification theorem is proved here assuming that the value function is twice differentiable. If the solution is not assumed to be twice differentiable, the proof of the verification theorem relies on the theory of viscosity solutions since one would need to interpret the non-twice differentiable function as a solution to a second order differential equation, only in the weak sense. Often (as in Shreve and Soner (1994)), the theory of viscosity solutions eventually leads to a rigorous proof of the twice differentiability of the value function.

Theorem 1 *Assume that $f(q,x)\in C^{1,2}$ satisfies the Hamilton-Jacobi-Bellman equation*

$$\max\left(\frac{1}{2}\sigma^2 f_{xx}+\kappa(\alpha-x)f_x-\beta f,\ f_q-(e^x+\lambda(q)),\ -f_q+(e^x-\mu(q))\right)=0. \tag{6}$$

Then $f\equiv V$, the optimal value function defined in (4). Moreover, a control policy that achieves this value can be constructed by identifying the corresponding term that maximizes the left hand side at each state.

Proof: All proofs are collected in the appendix.

Theorem 1 reduces our storage valuation problem to a free-boundary PDE problem, described by the HJB equation (5). We call this the free boundary problem since the boundaries that divide the three regions are not known a priori and need to be solved for at the same time as the value function, V. To help us construct a methodology to determine the boundaries, we first establish in Theorem 2 that the regions are in fact connected. Theorem 2 also shows that the boundaries of the injection and withdrawal regions can be modeled as functions of the state variable x.

Theorem 2 *Say it is optimal to inject at a state $(x,\bar{q})$. That is,*

$$V_q-(e^x+\lambda(q)) = 0, \tag{7}$$

$$\frac{1}{2}\sigma^2 V_{xx}+\kappa(\alpha-V)V_x-\beta V \le 0 \qquad \textit{and} \tag{8}$$

$$-V_q+(e^x-\mu(q)) \le 0. \tag{9}$$

at $(x,\bar{q})$. Then for any $q\in(0,\bar{q})$, we will have $V_q-(e^x+\lambda(q))=0$. Similarly, if it is optimal to withdraw at a state $(x,\bar{q})$ then it would be optimal to withdraw at any state (q,x) with $q\ge\bar{q}$.

Proof: All proofs are collected in the appendix.

Let $w(x)$ and $i(x)$ denote the boundaries of the withdrawal and injection regions. For any arbitrary pair of boundaries i^0 and w^0 we can define the value associated with the policy dictated by those boundaries, V^0, as the solution to the following set of equations

$$\frac{1}{2}\sigma^2 V_{xx}+\alpha(\kappa-x)V_x-\beta V=0 \qquad \text{for } q\in(i(x),w(x)) \tag{10}$$

with boundary conditions

$$V_q = (e^x + \lambda(q)) \qquad \text{when } q \le i(x) \tag{11}$$

$$V_q = (e^x - \mu(q)) \qquad \text{when } q \ge w(x) \tag{12}$$

We are looking for the unique pair (i, w) that satisfies the above and equation (5).

Further, we can establish the behavior of these boundaries i and w as x becomes very large or very small. Theorem 3 shows that as the price becomes very large, it is always optimal to withdraw and as price becomes very small, it is always optimal to hold.

Theorem 3 *As* $x \to \infty$, $-V_q + (e^x - \mu(q)) \to 0$ *for all* $q \in (0, Q_{\max})$. *As* $x \to -\infty$, $\frac{1}{2}\sigma^2 V_{xx} + \alpha(\kappa - x)V_x - \beta V \to 0$ *for all* $q \in (0, Q_{\max})$.

Proof: All proofs are collected in the appendix.

With what we now know about the structure of the optimal policy, a visual representation of the regions in the state space is illustrated in Figure 3

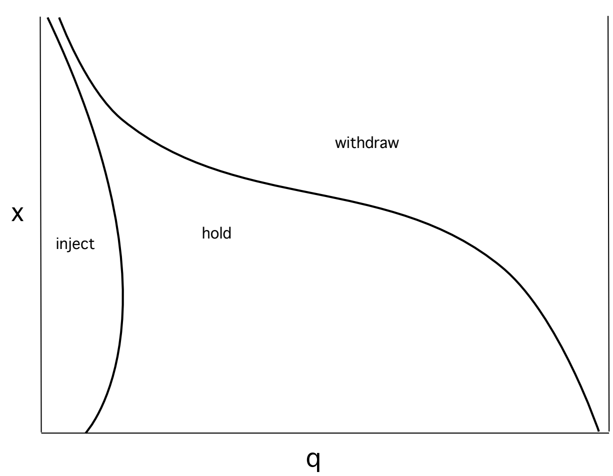

Figure 1 Control regions

4. Proposed solution method

The central idea to our solution methodology is to start with a guess policy and iteratively improve the policy until convergence. We seek to do this in a monotonic way that will assure us of convergence. To this extent we begin with an arbitrary guess for the policy (i^0, w^0), such that $i^0(x) < i(x)$

and $w^0(x) > w(x)$ for all x. Denote by V^0, the value one can obtain by using this (sub optimal) policy. Computing V^0 is easy since it is only a performance evaluation problem and can be obtained by solving the second order PDE equation (10) with boundary conditions (11) and (12).

We would like to choose i^1 and w^1 using i^0, w^0 and V^0 such that the V^1 that would be associated with the policy dictated by i^1, w^1 is such that $V^1 > V^0$, indicating policy improvement. Moreover if we can also establish that $i^1(x) < i(x)$ and $w^1(x) > w(x)$, then we can iterate this procedure to give us a monotonic sequence of boundaries. Convergence is then inevitable. We provide below boundary update equations that constructs such a sequence.

First, lets establish that if $i^0(x) < i(x)$ and $w^0(x) > w(x)$ for all x, then the derivatives of V^0 at the boundaries of the hold region will satisfy a *superset condition*. This condition is critical and is required for the initiation of the algorithm. For a arbitrarily chosen initial guess, if the superset condition fails, one simply has to arbitrarily back-out the boundaries. Since the capacity of the storage is finite this should be easy.

Proposition 1 *Let $i(x)$ and $w(x)$ be such that $i^0(x) < i(x)$ and $w^0(x) > w(x)$ for all x. Also let V be the solution to (10) with (11)-(12) as the boundary conditions. Then we have that*

$$(V_q - (e^x + \lambda(q))_q > 0 \text{ and} \tag{13}$$

$$-(V_q + (e^x - \mu(q))_q < 0 \tag{14}$$

Proof: All proofs are collected in the appendix.

As one can see from the terms in the superset conditions, they simply indicate that a policy improvement is possible by moving the boundaries towards the interior. Next we define the two boundary update equations. Define i^{n+1} by moving each point on i^n to the right (increasing q) to the point where

$$i^{n+1}(x) = \inf\left\{ q > i^n(x) \middle| V_q - (e^x + \lambda(q)) \text{ is a local maxima } \right\} \tag{15}$$

and define w^{n+1} by moving w^n to the left using

$$w^{n+1}(q) = \sup\left\{ x < w^n(q) \middle| -V_q + (e^x - \mu(q)) \text{ is a local maxima } \right\} \tag{16}$$

The following theorem establishes that the boundary update procedure improves the policy and can be iterated to convergence.

Theorem 4 *Let V^n be the solution to (10) with boundary conditions (11)-(12) within boundaries denoted by i^n and w^n. Also assume that V^n satisfies conditions (13) and (14). Let i^{n+1} and w^{n+1} be defined by the boundary update equations (15) and (16), and V^{n+1} be the solution to (10) with boundary conditions (11)-(12) within boundaries denoted by i^{n+1} and w^{n+1}. Then,*

1. *i^{n+1} and w^{n+1} exist and $i^{n+1}(x) < w^{n+1}(x)$ for all x.*
2. *$V^{n+1} > V^n$, implying policy improvement*
3. *V^{n+1} satisfies the superset conditions (13) and (14) at the boundaries i^{n+1} and w^{n+1}.*

Proof:

1. From the boundary conditions, we know that $V_q^n = (e^x - \mu(q))$ at w^n. Hence we have at w^n, $V_q - (e^x + \lambda(q)) = (e^x - \mu(q)) - (e^x + \lambda(q)) = -\mu(q) - \lambda(q) < 0$ since both $\mu(q) + \lambda(q) > 0$ for all q.

Therefore we have that $V_q - (e^x + \lambda(q)) = 0$ and $V_q - (e^x + \lambda(q)) < 0$ at i^n and w^n, respectively, with $(V_q - (e^x + \lambda(q)))_q > 0$ at i^n. These together imply that there exists at least one local maxima in (i^n, w^n). Being defined as the infimum of such local maxima, i^{n+1} exists. The exact argument will show the existence of w^{n+1} as well.

Moreover, there exists a point $\bar{i} > i^{n+1} > i^n$ such that $(V_q - (e^x + \lambda(q))) > 0$ in $(i^n, \bar{i})$. Similarly there exists a point $\bar{w} < w^{n+1} < w^n$ such that $(-V_q + (e^x - \mu(q))) > 0$ in $(\bar{w}, w^n)$. The intervals $(i^n, \bar{i})$ and $(\bar{w}, w^n)$ cannot overlap because $V_q - e^x > \lambda(q) > -\mu(q)$ and $(V_q - e^x)) < -\mu(q)$ in these intervals respectively. Implying that $i^{n+1} < w^{n+1}$.

2. Let Ω^n denote the hold region in the n^{th} iteration, that is, $\Omega^n = \{(q,x) | q \in (i^n(x), w^n(x))\}$. We know that V^n solves (10) with boundary conditions (11) and (12) at boundaries i^n and w^n, respectively. Similarly, V^{n+1} solves (10) with boundary conditions (11) and (12) at boundaries i^{n+1} and w^{n+1}, respectively.

Take $W = V^{n+1} - V^n$. Now subtracting the PDE satisfied by V^n from the PDE satisfied by V^{n+1}, we have in Ω^{n+1},

$$\frac{1}{2}\sigma^2 W_{xx} + \alpha(\kappa - x)W_x - \beta W = 0. \tag{17}$$

At i^{n+1}, $\frac{\partial V^{n+1}}{\partial q} = (e^x + \lambda(q))$ and $\frac{\partial V^n}{\partial q} > (e^x + \lambda(q))$. This gives us $\frac{\partial W}{\partial q} < 0$ at i^{n+1}. Similarly, we can argue that $\frac{\partial W}{\partial q} > 0$ at w^{n+1}

Given that the partial of W is negative at i^{n+1} and positive at w^{n+1}, for each x, there exists a point $\hat{q}(x)$ that is interior local minima along the horizontal axis. From the necessary conditions, we have that $\frac{\partial W}{\partial q} = 0$ and $\frac{\partial^2 W}{\partial q^2} \geq 0$. Substitution and rearrangement in (20) gives us $W(\hat{q}(x), x) \geq 0$ for all x.

3. To show that the superset condition holds for V^{n+1}, it would be sufficient to show that

$$W_{qq} \geq 0 \text{ at } i^{n+1}. \tag{18}$$

This is because W_{qq} can be expressed as,

$$W_{qq} = (V_q^{n+1} - (e^x + \lambda(q)))_q - (V_q^n - (e^x + \lambda(q)))_q \tag{19}$$

and we know that the second term above is zero at i^{n+1}.

Now to show that $W_{qq} \geq 0$ at i^{n+1} lets assume the opposite and show a contradiction. Say $W_{qq} < 0$. Now since $W_q < 0$ at i^{n+1} and $W_q > 0$ at w^{n+1}, we will have $W_{qq} < 0$ at i^{n+1} implying a interior local minima for W_q along each x. Hence the global minima of W_q should exist in the interior or as x goes to $+/-\infty$.

Say it is achieved in the interior at a $\hat{q}, \hat{x}$. Differentiating (20) by q we have

$$\frac{1}{2}\sigma^2 P_{xx} + \alpha(\kappa - x)P_x - \beta P = 0. \tag{20}$$

where $P = W_q$. Substituting the necessary conditions for the global/local minima of P, that is $P_x = 0$ and $P_{xx} \geq 0$, we have that $P > 0$ at $\hat{q}, \hat{x}$ a contradiction since $P = W_q < 0$ at the boundary.

Using similar arguments, one can show that the superset condition holds for V^{n+1} at the w^{n+1} boundary as well.

5. Solving the fixed boundary problem

Theorem 4 essentially converts a free boundary problem into a sequence of fixed boundary problems. These fixed boundary problems can easily be solved using standard PDE methods. However, in this case we derive a closed for expression to denote the solution of the fixed boundary problem. The boundary conditions will have to be numerically resolved to compute the constants of integration.

The problem at hand is to solve the PDE

$$\frac{1}{2}\sigma^2 V_{xx} + \alpha(\kappa - x)V_x - \beta V = 0, \tag{21}$$

in the hold region. Suppressing the iteration indices in this section, the boundary conditions are give at the boundaries $i(x)$ and $w(x)$. Lets define $y = \kappa/\sigma^2(x-\alpha)^2$, we have

$$y\frac{\partial^2 V}{\partial y^2} + (\frac{1}{2} - y)\frac{\partial V}{\partial y} - \frac{\beta}{2\kappa}V = 0 \tag{22}$$

that is, the Kummer Equation. The solution for which is known to be the sum of hypergeometric1F1 and the hypergeometricU functions. Hence we have for $V(q,x)$

$$V(q,x) = A(q)\text{HyperGeoU}\Big(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\Big) + B(q)\text{HyperGeo1F1}\Big(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\Big) \tag{23}$$

The boundary conditions provide the equations to evaluate $A(q)$ and $B(q)$. Need to add info on how we evaluate the hypergeometric functions?

6. Computational illustration

For numerical illustration, we consider the valuation of an oil storage facility that has a maximum capacity of 1000 units. The costs of injection $\lambda(q)$ and withdrawal $\mu(q)$ are taken as

$$\lambda(q) = \frac{1000}{1000-q} - 1 \text{ and} \tag{24}$$

$$\mu(q) = \frac{1000}{q} - 1. \tag{25}$$

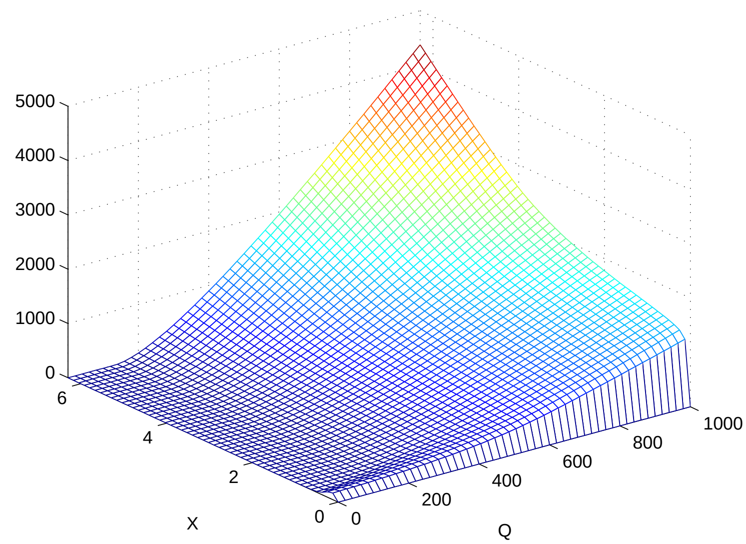

Figure 2 **Boundaries**

The price process is calibrated from oil futures data (Schwartz (1997)) from 1/2/1985 to 2/17/1995. The calibrated parameters are $\sigma = 0.334, \alpha = 0.0301$ and $\kappa = 3.15$. The discount rate β is taken to be 5% per year. The value function obtained after the numerical convergence of the procedure is shown in figure 2. The free boundaries that identify the holding, withdraw and injection regions, are shown in figure 3.

7. Water storage for power generation and seasonality in natural gas prices

The storage of water in dams for the purpose of electricity generation, is a closely related problem. The price of electricity can be represented by a mean-reverting process. It would hence be beneficial to withdraw water to generate electricity during periods of high prices/demand and retain water during periods of low prices/demand. Even injection of water is possible during periods of low

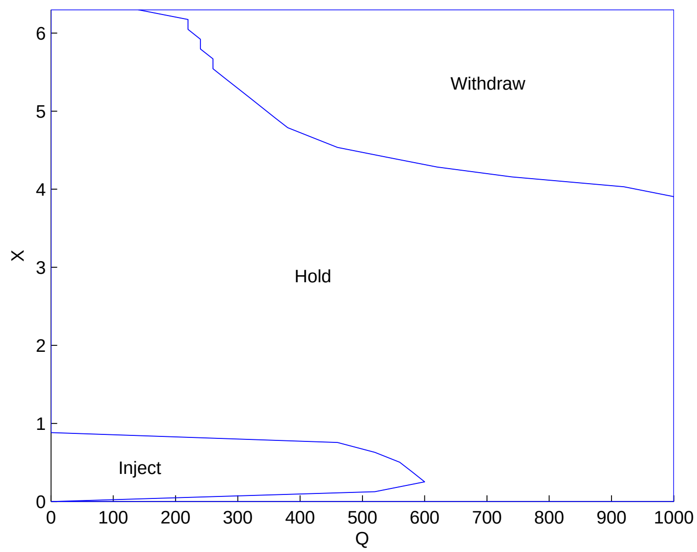

Figure 3 **Boundaries**

electricity prices since one could potentially pump water from the down stream into the dam for later use. However, the critical difference with other storage is the inflow of water due to rainfalls or the melting of ice. The occurrence of rainfall and the amount of rain fall are both random.

In the paper, we will model the occurrence of rainfall by a Poisson arrival process and allow a general distribution for the amount of rain fall. This leads to a Partial Integro Differential Equation (PIDE) that needs to be solved to obtain the optimal policy and the value function. A modified version of the moving boundary procedure will be described to compute solutions for the PIDE as well.

In the case of natural gas, seasonality in prices during the year are extremely significant due to the limited amounts of storage, decay and the primary use of natural gas in the United States for winter heating. In the paper we will account for seasonality by modeling the price process as $X_t = f_t D_t$. Here D_t the deseasonalized price process is modeled as an Ornstein-Uhlenbeck mean reverting process with f_t denoting a normalized seasonality coefficient. The state space will be three dimensional, with time of the year representing the third state variable. Though theoretical guarantees are extremely difficult to provide in three dimensions, we will provide a modified boundary update procedure that will be numerically illustrated.

References

Schwartz, E. 1997. The stochastic behavior of commodity prices: Implications for valuation and hedging. *Journal of Finance* **52** 923–973.

Shreve, S. E., H. M. Soner. 1994. Optimal investment and consumption with transaction costs. *Annals of Applied Probability* **4**(3) 609–692.

8. Appendix

Proof of Theorem 1 Say $f(q,x) \in C^{1,2}$ satisfies the Hamilton-Jacobi-Bellman equation 6

$$\max\left(\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} + \kappa(\alpha - x)\frac{\partial f}{\partial x} - \beta f,\ \frac{\partial f}{\partial q} - (e^x + \lambda(q)),\ -\frac{\partial f}{\partial q} + (e^x - \mu(q))\right) = 0 \tag{26}$$

Then $f \equiv V$, the optimal value function defined in (4). Moreover a control policy that achieves this value can be constructed by identifying the corresponding term that maximizes the LHS at each state.

Proof of Theorem 2 Say it is optimal to inject at a state $(x, \bar{q})$. That is,

$$V_q - (e^x + \lambda(q)) = 0, \tag{27}$$

$$\frac{1}{2}\sigma^2 V_{xx} + \kappa(\alpha - V)\frac{\partial f}{\partial x} - \beta V \leq 0 \qquad \text{and} \tag{28}$$

$$-V_q + (e^x - \mu(q)) \leq 0. \tag{29}$$

at $(x, \bar{q})$. Then for any $q \in (0, \bar{q})$, we will have $V_q - (e^x + \lambda(q)) = 0$. Similarly, if it is optimal to withdraw at a state $(x, \bar{q})$ then it would be optimal to withdraw at any state (q, x) with $q \geq \bar{q}$.

Proof of Theorem 3 As $x \to \infty$, $-V_q + (e^x - \mu(q)) \to 0$ for all $q \in (0, Q_{\max})$. As $x \to -\infty$, $\frac{1}{2}\sigma^2 V_{xx} + \alpha(\kappa - x)V_x - \beta V \to 0$ for all $q \in (0, Q_{\max})$.