The Valuation of Storage

Long Zhao
McCombs School of Business, University of Texas - Austin

Joint work with Kumar Muthuraman and Stathis Tompaidis

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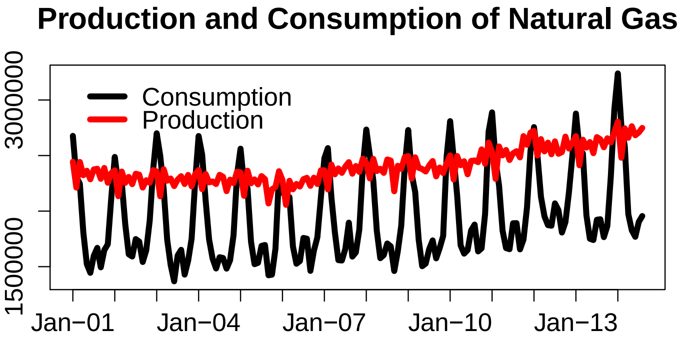
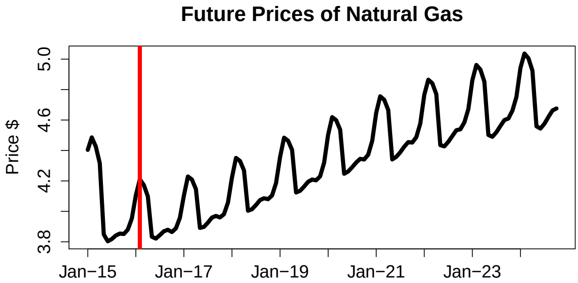
Examples of Storage

- Silos - Agricultural Commodities
- Tanks - Oil
- Caverns - Natural Gas
- Lake Reservoirs and Dams - Water $\Rightarrow$ Electricity.

Value of Storage



Prices

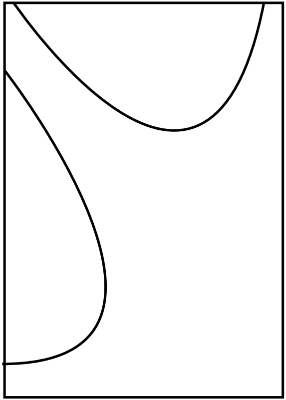


Data comes from NYMEX.

Method

- Continuous Time Singular Control $\Rightarrow$ HJB equation.
- HJB equation (free boundary problem) is very hard to solve.
- Moving boundary method is used in 1 dimension.

1 Dimension VS 2 Dimensions



Bibliography

- Commodity price.
  - Mean reversion: Gibson and Schwartz (1990); Brennan and Hughes (1991); Cortazar and Schwartz (1994); Bessembinder, Coughenour, Seguin, Smoller (1995);
  - Price Dyanmics: Schwartz (1997); Schwartz and Smith (2000);
- Valuation of storage: Fackler and Livingston (2002); Manoliu (2004); Hodges (2004); Chen and Forsyth (2008, 2010); Boogert and Jong (2008); Matt, Davison and Rasmussen (2009); Secomandi (2010, 2014); Zhao and Wijnbergen (2015)
- Moving boundary method: Muthuraman and Kumar (2006); Chockalingam and Muthuraman (2007, 2010, 2014); Muthuraman and Zha (2008); Feng and Muthuraman (2010);

Model

- One factor model

$$dS_t = \kappa(\mu - \ln S_t)S_t dt + \sigma S_t dW_t$$

- By Ito's formula, $X_t = \ln(S_t)$ is an Ornstein-Uhlenbeck process,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t.$$

where $\alpha = \gamma - \sigma^2/(2\kappa)$.

Model

- Storage level at time t is Q_t. L_t, U_t represent cumulative injections and withdrawals at time t.

$$dQ_t = dL_t - dU_t$$

- Admissible if $Q_t \in (Q_{min}, Q_{max}) \quad \forall t \geq 0$.
- Costs of injection and withdrawal, $\lambda(X_t, Q_t)$ and $\mu(X_t, Q_t)$, are continuous and bounded.

Model

- Objective: to maximize discounted infinite-horizon cash flows.

$$V(x,q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_{x,q}\Bigg(\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t^1))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t^2))dL_t\Bigg) \tag{1}$$

where $X_0 = x$ and $Q_0 = q$.

The Hamilton-Jacobi-Bellman Equation

- Dynamic programming arguments and Ito's formula yield the Hamilton-Jacobi-Bellman (HJB) equation.

$$\max\left(\mathcal{L}V, \frac{\partial V}{\partial q} - (e^x + \lambda(q)), -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0$$

with $\mathcal{L}V = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V$.

- A verification theorem assures us that a function that solves the HJB equation is the value function for the original control problem and a policy that achieves this value function is the optimal policy.

The Hamilton-Jacobi-Bellman Equation

- Assume V is known and the change of policy at one point (x_0, q_0) won't affect it.
  Now at (x_0, q_0), ϵ is bought at price e^{x_0}. The average buying profit is

$$\frac{[V(x_0, q_0+\epsilon) - V(x_0, q_0)] - \epsilon(e^x + \lambda(q))}{\epsilon} \xrightarrow{\epsilon \to 0} \frac{\partial V}{\partial q} - (e^x + \lambda(q))$$

- $\mathcal{L}V(x, q)$: holding profit at (x, q).
- $\frac{\partial V}{\partial q}(x, q) - (e^x + \lambda(q))$: selling profit at (x, q).
- $-\frac{\partial V}{\partial q}(x, q) + (e^x - \mu(q))$: buying profit at (x, q).
- HJB equation.

$$\max\left(\mathcal{L}V, \frac{\partial V}{\partial q} - (e^x + \lambda(q)), -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0$$

Holding, Selling and Buying Regions

- The state space $(x, q) \in \mathbb{R}_+^2$ is divided into three kinds of regions.
- Holding region H^*: holding profit $= 0$, selling & buying profit < 0
- Selling region S^*: selling profit $= 0$, holding & buying profit < 0
- Buying region B^*: buying profit $= 0$, holding & selling profit < 0

Moving Boundary Method

- Start with an initial guess and iteratively improve it until convergence.
- A sequence of fixed boundary problems $\rightarrow$ free boundary problem

Moving Boundary Method

- Initial guess.
- How to solve fixed boundary problems efficiently.
- How to improve boundary.

Initial Guess

Theorem

When the price is high enough, regardless of storage, selling is the optimal strategy.

Solving the Fixed Boundary Problem

- In the holding region

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V = 0$$

- Defining $y = \kappa(x-\alpha)^2/\sigma^2$, we have

$$y\frac{\partial^2 V}{\partial y^2} + (0.5 - y)\frac{\partial V}{\partial y} - \frac{\beta}{2\kappa}V = 0$$

which is the Kummer Equation. The solution is the sum of hypergeometric1F1 and the hypergeometricU functions.

$$\begin{aligned} V(x,q) = {} & A(q)\mathrm{HyperGeoU}\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right) \\ & + B(q)\mathrm{HyperGeo1F1}\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right) \end{aligned}$$

Boundary conditions can determine $A(q)$ and $B(q)$.

Boundary Movement

- Direction.
- Distance.

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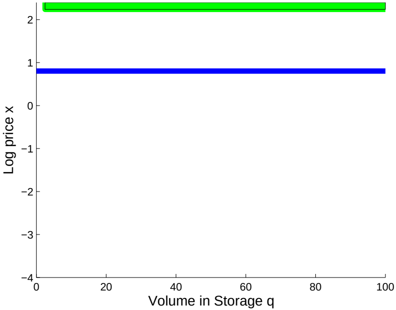


Figure : Initial Guess

Boundary Movement

- Direction.
- Distance.

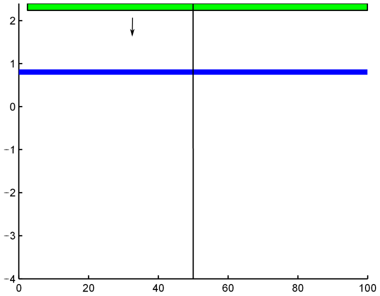


Figure : Initial Guess

Distance

Sell

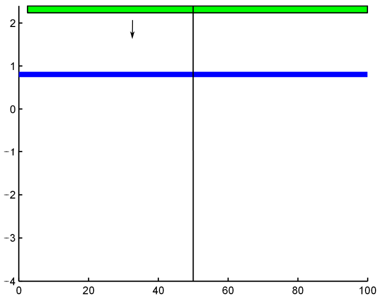


Figure : Current Policy

Distance

Sell

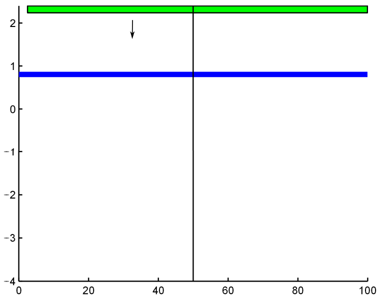


Figure : Current Policy

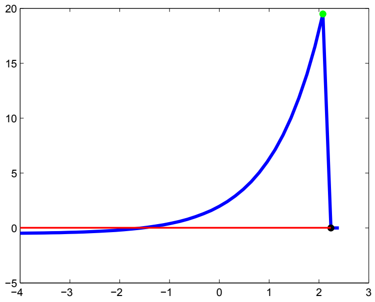


Figure : Selling Profit

Distance

Sell

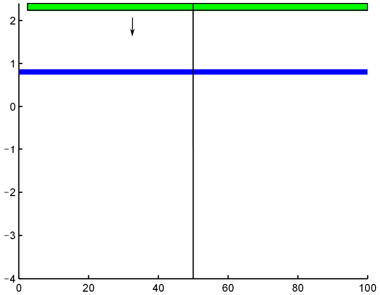


Figure : Current Policy

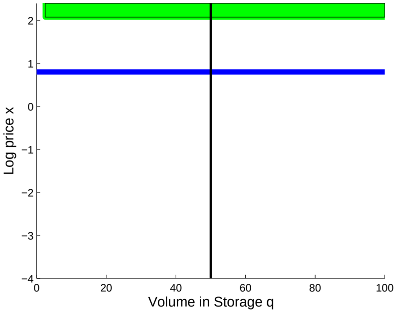


Figure : After Movement

Buy

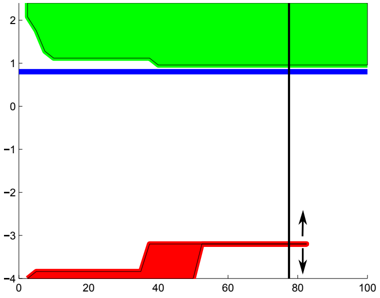


Figure : Current Policy

Buy

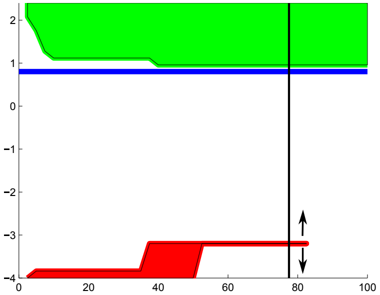


Figure : Current Policy

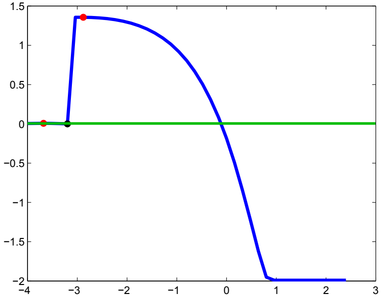


Figure : Buying Profit

Buy

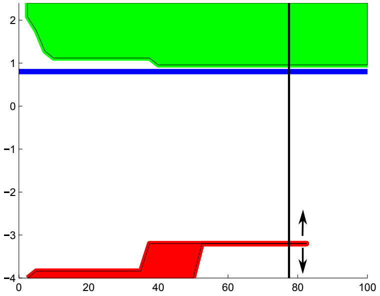


Figure : Current Policy

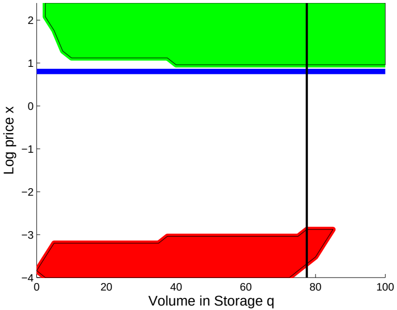


Figure : After Movement

Algorithm

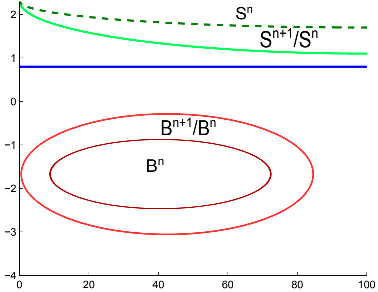
1. Begin with selling at very high price for all $q > 0$.
2. Move selling (buying) boundary along price x to local maximal selling (buying) profit points until convergence.

Proof of Convergence

Theorem

Each movement improves value function.

$$\mathcal{L}V^n(x,q) = 0 \qquad H^n$$
$$-V_q^n(x,q) + e^x - \mu(q) = 0 \qquad S^n$$
$$-V_q^n(x,q) + e^x - \mu(q) > 0 \qquad S^{n+1}/S^n$$
$$V_q^n(x,q) - e^x - \lambda(q) = 0 \qquad B^n$$
$$V_q^n(x,q) - e^x - \lambda(q) > 0 \qquad B^{n+1}/B^n$$

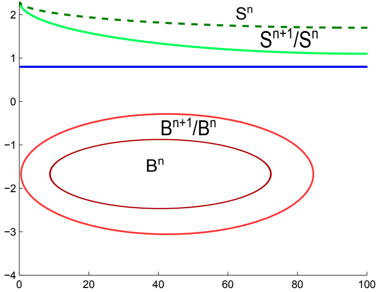


Proof of Convergence

Theorem

Each movement improves value function.

$$
\begin{array}{ll}
\mathcal{L}V^{n+1}(x,q) = 0 & H^{n+1} \\
-V_q^{n+1}(x,q) + e^x - \mu(q) = 0 & S^n \\
-V_q^{n+1}(x,q) + e^x - \mu(q) = 0 & S^{n+1}/S^n \\
V_q^{n+1}(x,q) - e^x - \lambda(q) = 0 & B^n \\
V_q^{n+1}(x,q) - e^x - \lambda(q) = 0 & B^{n+1}/B^n
\end{array}
$$

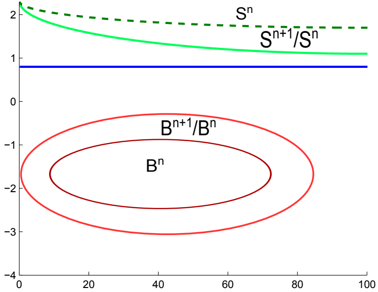


Proof of Convergence

Theorem

Each movement improves value function.

$$
\begin{aligned}
\mathcal{L}\Delta V^{n+1}(x,q) &= 0 \quad H^{n+1} \\
\Delta V_q^{n+1}(x,q) &= 0 \quad S^n \\
\Delta V_q^{n+1}(x,q) &> 0 \quad S^{n+1}/S^n \\
\Delta V_q^{n+1}(x,q) &= 0 \quad B^n \\
\Delta V_q^{n+1}(x,q) &< 0 \quad B^{n+1}/B^n
\end{aligned}
$$

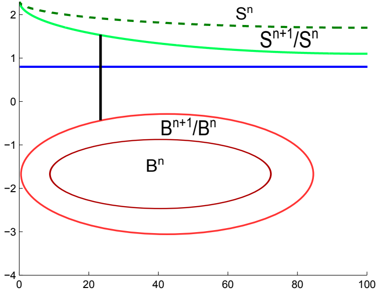


Proof of Convergence

Theorem

The boundaries can be kept moving.

$$\begin{aligned}
\mathcal{L}\Delta V^{n+1}(x,q) &= 0 \quad && H^{n+1} \\
\Delta V_q^{n+1}(x,q) &= 0 \quad && S^n \\
\Delta V_q^{n+1}(x,q) &> 0 \quad && S^{n+1}/S^n \\
\Delta V_q^{n+1}(x,q) &= 0 \quad && B^n \\
\Delta V_q^{n+1}(x,q) &< 0 \quad && B^{n+1}/B^n
\end{aligned}$$



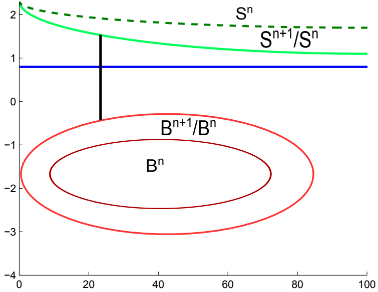
Proof of Convergence

Theorem

The boundaries can be kept moving.

At upper point

$$(\Delta V_q^{n+1}(x,q))_x \leq 0$$
$$(-V_q^n(x,q) + (e^x - \mu(q)))_x = 0$$
$$\Rightarrow (-V_q^{n+1}(x,q) + (e^x - \mu(q)))_x \leq 0$$



Extensions

- Seasonality.