The Valuation of Storage

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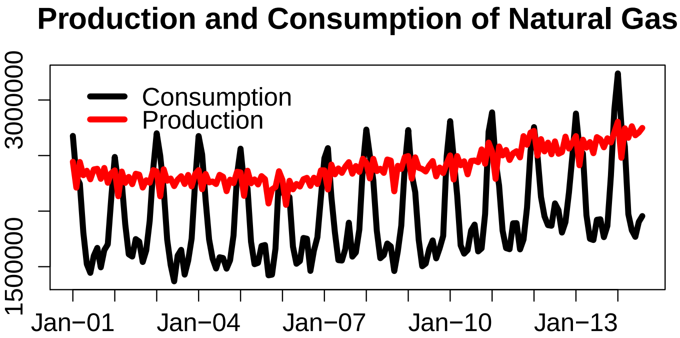
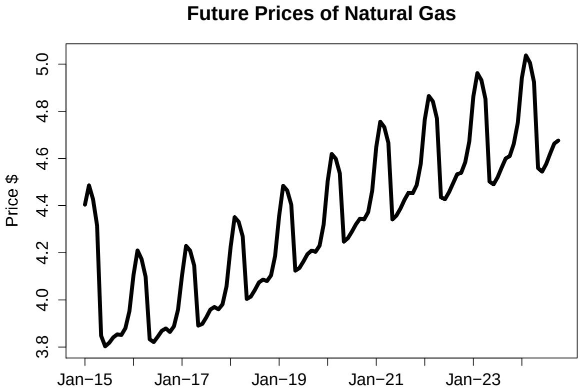
Examples of Storage

- Silos - Agricultural Commodities
- Tanks - Oil
- Caverns - Natural Gas
- Lake Reservoirs and Dams - Water $\Rightarrow$ Electricity.

Value of Storage



Prices

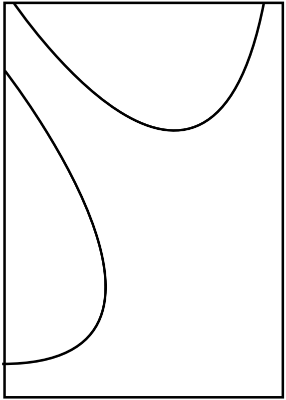


Data comes from NYMEX.

Method

- Continuous Time Singular Control $\Rightarrow$ HJB equation.
- HJB equation (free boundary problem) is very hard to solve.
- Moving boundary method is used in 1 dimension.
  - Start with an initial guess and iteratively improve it until convergence.
  - A sequence of fixed boundary problems $\rightarrow$ free boundary problem

1 Dimension VS 2 Dimensions



Overview of Results

- Fixed boundary problem is solved efficiently.
- Moving boundary method is generalized to 2 dimensions.

Bibliography

- Commodity price.
  - Mean reversion: Gibson and Schwartz (1990); Brennan and Hughes (1991); Cortazar and Schwartz (1994); Bessembinder, Coughenour, Seguin, Smoller (1995);
  - Price Dyanmics: Schwartz (1997); Schwartz and Smith (2000);
- Valuation of storage: Fackler and Livingston (2002); Manoliu (2004); Hodges (2004); Chen and Forsyth (2008,2010); Boogert and Jong (2008); Matt, Davison and Rasmussen (2009); Secomandi (2010,2014);
- Moving boundary method: Muthuraman and Kumar (2006); Chockalingam and Muthuraman (2007,2010); Muthuraman and Zha (2008); Feng and Muthuraman (2010);

Model

- One factor model

$$dS_t = \kappa(\mu - \ln S_t)S_t dt + \sigma S_t dW_t$$

- By Ito's formula, $X_t = \ln(S_t)$ is an Ornstein-Uhlenbeck process,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t.$$

where $\alpha = \gamma - \sigma^2/(2\kappa)$.

Model

- Storage level at time t is Q_t. L_t, U_t represent cumulative injections and withdrawals at time t.

$$dQ_t = dL_t - dU_t$$

- Admissible if $Q_t \in (Q_{min}, Q_{max}) \quad \forall t \geq 0$.
- Costs of injection and withdrawal, $\lambda(X_t, Q_t)$ and $\mu(X_t, Q_t)$, are continuous and bounded.

Model

- Objective: to maximize discounted infinite-horizon cash flows.

$$V(x,q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_{x,q}\left(\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t^1))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t^2))dL_t\right) \quad (1)$$

where $X_0 = x$ and $Q_0 = q$.

The Hamilton-Jacobi-Bellman Equation

- Dynamic programming arguments and Ito's formula yield the Hamilton-Jacobi-Bellman (HJB) equation.

$$\max\left(\mathcal{L}V, \frac{\partial V}{\partial q} - (e^x + \lambda(q)), -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0$$

with $\mathcal{L}V = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V$.

- A verification theorem assures us that a function that solves the HJB equation is the value function for the original control problem and a policy that achieves this value function is the optimal policy.

The Hamilton-Jacobi-Bellman Equation

- Assume V is known and the change of policy at one point (x_0, q_0) won't affect it.
  Now at (x_0, q_0), ϵ is bought at price e^{x_0}. The average buying profit is

$$\frac{[V(x_0, q_0 + \epsilon) - V(x_0, q_0)] - \epsilon(e^x + \lambda(q))}{\epsilon} \xrightarrow{\epsilon \to 0} \frac{\partial V}{\partial q} - (e^x + \lambda(q))$$

- $\mathcal{L}V(x, q)$: holding profit at (x, q).
- $\frac{\partial V}{\partial q}(x, q) - (e^x + \lambda(q))$: selling profit at (x, q).
- $-\frac{\partial V}{\partial q}(x, q) + (e^x - \mu(q))$: buying profit at (x, q).
- HJB equation.

$$\max\left(\mathcal{L}V, \frac{\partial V}{\partial q} - (e^x + \lambda(q)), -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0$$

Holding, Selling and Buying Regions

- The state space $(x, q) \in \mathbb{R}_+^2$ is divided into three kinds of regions.
- Holding region: holding profit $= 0$, selling & buying profit < 0
- Selling region: selling profit $= 0$, holding & buying profit < 0
- Buying region: buying profit $= 0$, holding & selling profit < 0

Solving the Fixed Boundary Problem

- In the holding region

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V = 0$$

- Defining $y = \kappa(x-\alpha)^2/\sigma^2$, we have

$$y\frac{\partial^2 V}{\partial y^2} + (0.5 - y)\frac{\partial V}{\partial y} - \frac{\beta}{2\kappa}V = 0$$

which is the Kummer Equation. The solution is the sum of hypergeometric1F1 and the hypergeometricU functions.

$$V(x,q) = A(q)\text{HyperGeoU}\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right) + B(q)\text{HyperGeo1F1}\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right)$$

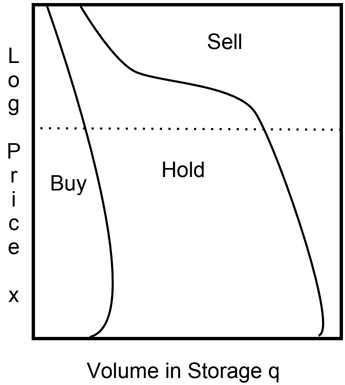
Boundary conditions can determine $A(q)$ and $B(q)$.

The Structure of Optimal Boundary

Theorem

When the price is high enough, regardless of storage, selling is the optimal strategy.

The Structure of Optimal Boundary



Theorem

Under optimal policy, there won't exist x and $q_1 \neq q_2$ such that buying happens at (x, q_1) while selling happens at (x, q_2).

The Moving Boundary Method

- Challenges
  - Initial guess.
  - 2 dimensions.
    - Direction.
    - Distance.

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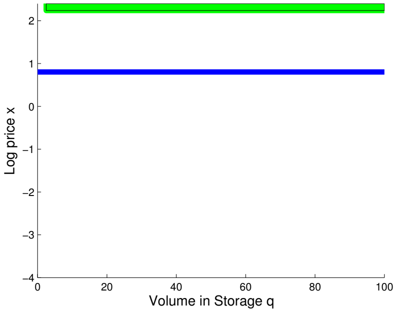

Figure : Initial Guess

The Moving Boundary Method

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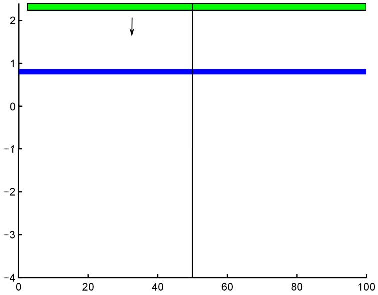

Figure : Initial Guess

Algorithm

1. Begin with selling at very high price for all $q > 0$.
2. Move selling and buying boundaries along price x alternatively until convergence.

Distance

Because there is *No Coming Back*, overshooting must be avoided.
Sell

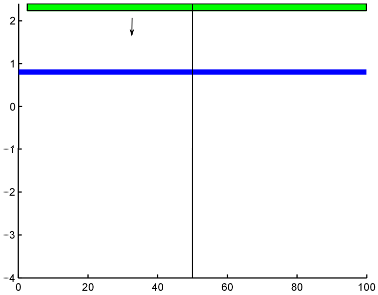

Figure : Current Policy

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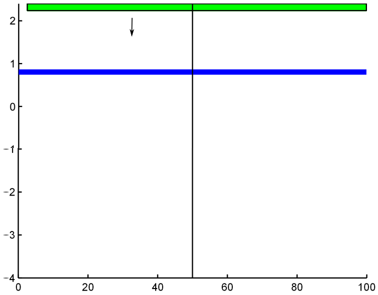

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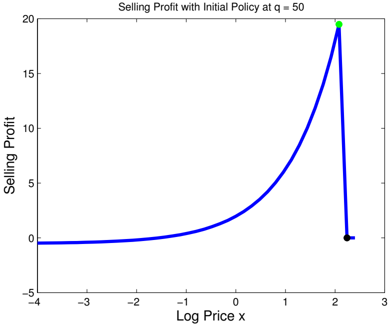

Figure : Selling Profit

Distance

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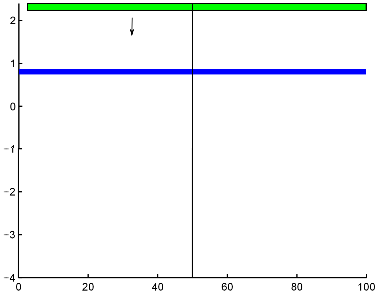
Sell



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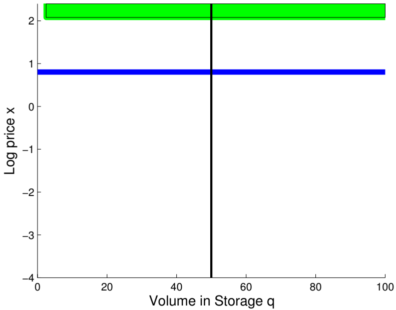

Figure : After Movement

Buy

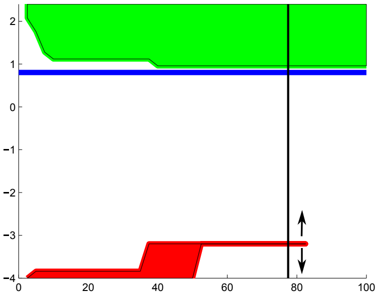

Figure : Current Policy

Buy

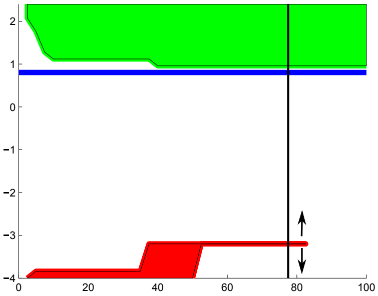

Figure : Current Policy

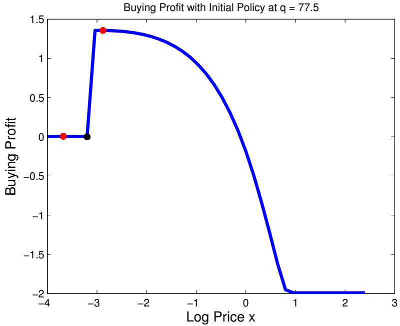

Figure : Buying Profit

Buy

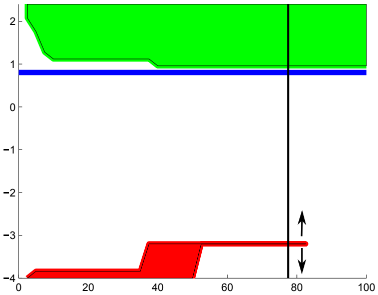

Figure : Current Policy

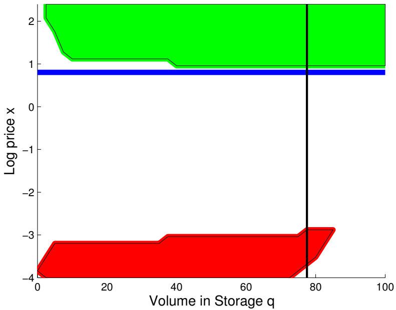

Figure : After Movement

Proof of Convergence

Theorem

Each movement improves value function.

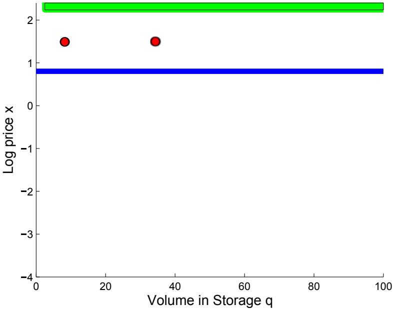

Figure : Initial Guess

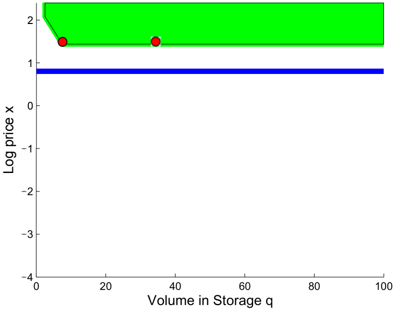

Figure : Initial Guess

Proof of Convergence

Theorem

The boundaries can be kept moving.

Extensions

- Seasonality and Finite time.
- Storage cost.
- Random injection and withdrawal.