The Valuation of Storage

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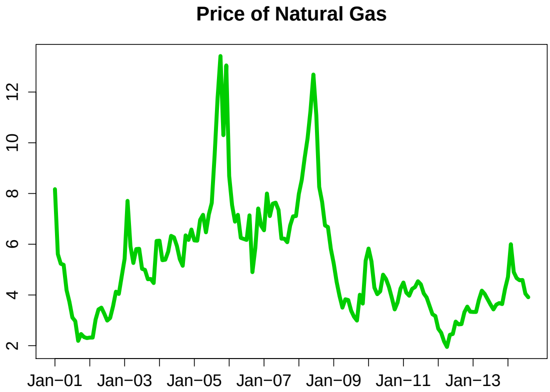
Examples of Storage

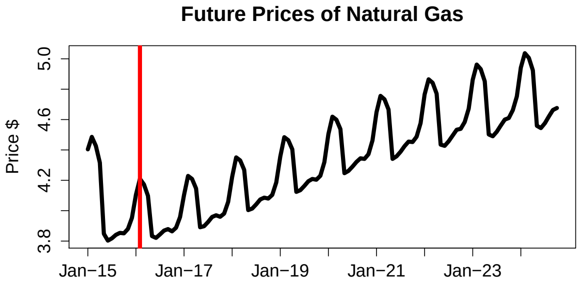
- Silos - Agricultural Commodities
- Tanks - Oil
- Caverns - Natural Gas
- Lake Reservoirs and Dams - Water $\Rightarrow$ Electricity.

Historical Prices



Data comes from U.S. Energy Information Administration.

Future Prices



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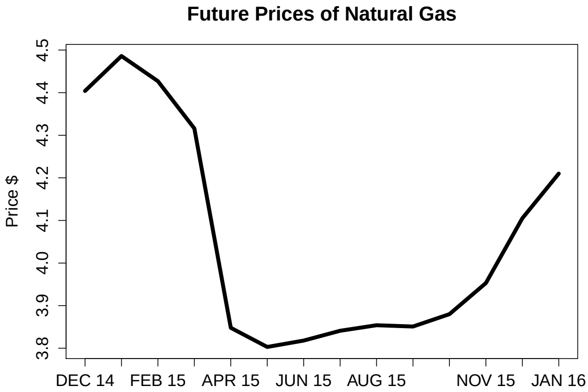

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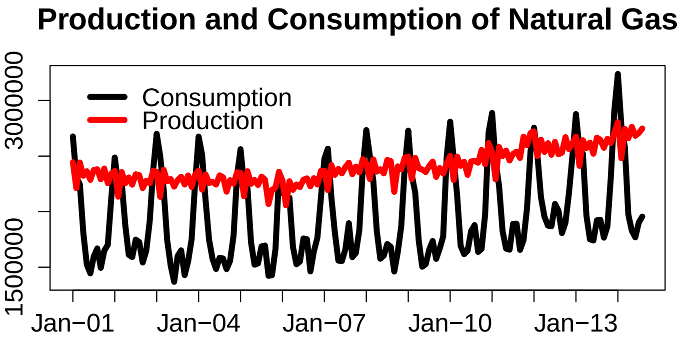
Price Dynamics

- Schwartz (1997)
- Schwartz and Smith (2000)
- Routledge, Seppi and Spatt(2000)
- Jaillet, Ronn and Tompaidis (2004)

Mismatch Between Production and Consumption



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Constraints of Storage

- Transaction costs.
- Depreciation.
- Limited delivery rate.
- Finite capacity.

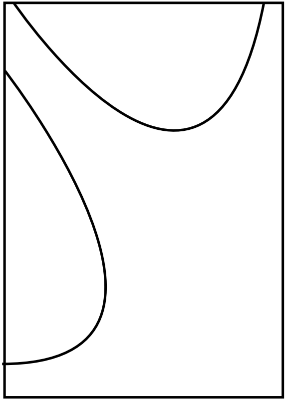
Storage Valuation

- Fackler and Livingston (2002)
- Hodges (2004)
- Chen and Forsyth (2008,2010)
- Boogert and Jong (2008)
- Thompson, Davison and Rasmussen (2009)
- Secomandi (2010)

Method

- Continuous Time Singular Control $\Rightarrow$ 2-d HJB equation.
- HJB equation (free boundary problem) is very hard to solve.
- Moving boundary method is used in 1 dimension.
  - Start with an initial guess and iteratively improve it until convergence.
  - A sequence of fixed boundary problems $\rightarrow$ free boundary problem

1 Dimension VS 2 Dimensions



Moving Boundary Method

- Muthuraman and Kumar (2006)
- Chockalingam and Muthuraman (2007,2010)
- Muthuraman and Zha (2008)
- Feng and Muthuraman (2010)

Overview of Results

- Methodology.
  - Fixed boundary problem is solved efficiently.
  - Moving boundary method is generalized to 2 dimensions.
- Value of storage.
  - The value of storage with non-trivial transaction costs and finite capacity is calculated.
  - The optimal strategy is found.

Model

- One factor model

$$dS_t = \kappa(\mu - \ln S_t)S_t dt + \sigma S_t dW_t$$

- By Ito's formula, $X_t = \ln(S_t)$ is an Ornstein-Uhlenbeck process,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t.$$

where $\alpha = \mu - \sigma^2/(2\kappa)$.

Model

- Storage level at time t is Q_t. L_t, U_t represent cumulative injections and withdrawals at time t.

$$dQ_t = dL_t - dU_t$$

- Admissible if $Q_t \in (Q_{min}, Q_{max}) \ \ \forall t \geq 0$.
- Costs of injection and withdrawal, $\lambda(Q_t)$ and $\mu(Q_t)$, are monotone and bounded.

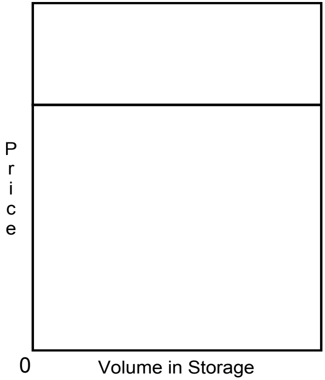
Model

- Objective: to maximize discounted infinite-horizon cash flows.

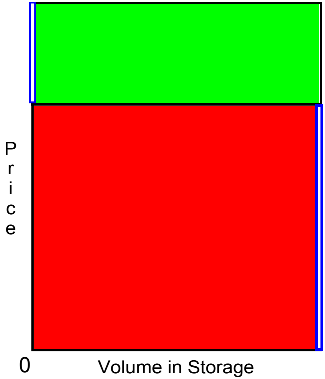
$$V(x,q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_{x,q}\left(\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t^{(1)}))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t^{(2)}))dL_t\right) \tag{1}$$

where $X_0 = x$ and $Q_0 = q$.

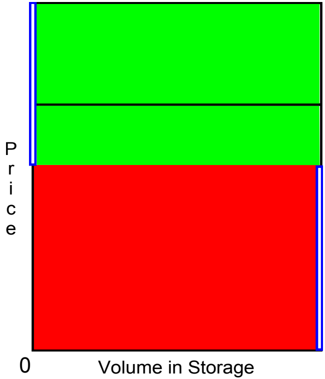
$\mu = 0,\ \lambda = 0$ and $\beta = 0$



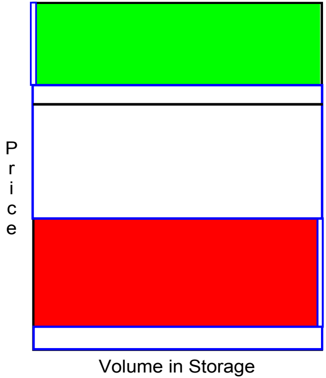
$\mu = 0, \ \lambda = 0$ and $\beta = 0$

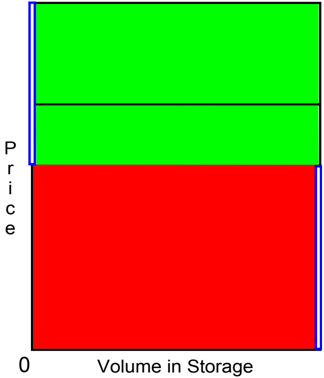


$\mu = 0, \ \lambda = 0$ **but** $\beta > 0$



$\mu =$ Constant > 0, $\lambda =$ Constant > 0 and $\beta > 0$





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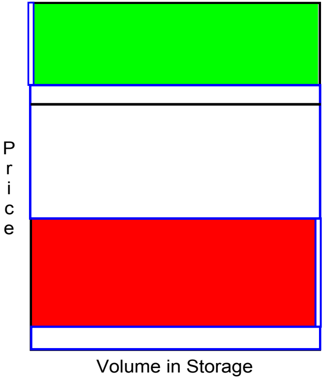

$$V(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s\{e^{-\beta\tau}(S_\tau - \mu)\}$$

$V(0+) = 0$

If $V(s) - (s + \lambda) < 0$, do not buy at Price s.

$V(0+) - (0 + \lambda) = -\lambda < 0.$

$\mu =$ Constant > 0, $\lambda =$ Constant > 0 and $\beta > 0$

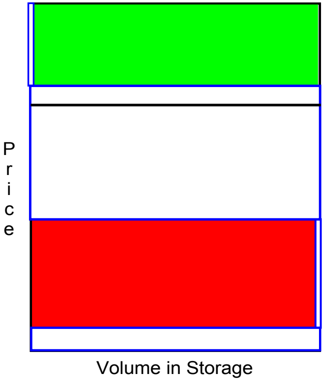

Figure : Small λ

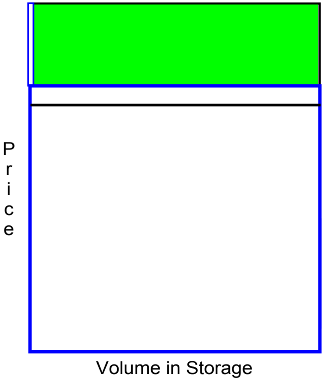

Figure : Large λ

Observation

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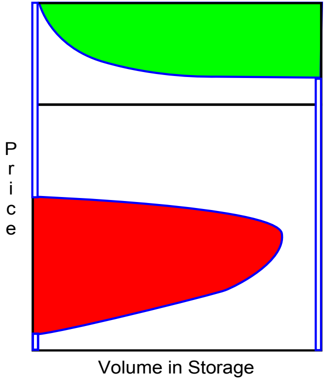
When the price is high enough, regardless of storage, selling is the optimal strategy.

$\mu(q) \downarrow,\ \lambda(q) \uparrow$ and $\beta > 0$



The Hamilton-Jacobi-Bellman Equation

- Dynamic programming arguments and Ito's formula yield the Hamilton-Jacobi-Bellman (HJB) equation.

$$\max\left(\mathcal{L}V, \frac{\partial V}{\partial q} - (e^x + \lambda(q)), -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0$$

with $\mathcal{L}V = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V$.

- A verification theorem assures us that a function that solves the HJB equation is the value function for the original control problem and a policy that achieves this value function is the optimal policy.

The Hamilton-Jacobi-Bellman Equation

- Assume V is known and the change of policy at one point (x_0, q_0) won't affect it.
  Now at (x_0, q_0), ϵ is bought at price e^{x_0}. The average buying profit is

$$\frac{[V(x_0, q_0+\epsilon) - V(x_0, q_0)] - \epsilon(e^x + \lambda(q))}{\epsilon} \xrightarrow{\epsilon\to 0} \frac{\partial V}{\partial q} - (e^x + \lambda(q))$$

- $\mathcal{L}V(x, q)$: holding profit at (x, q).
- $\frac{\partial V}{\partial q}(x, q) - (e^x + \lambda(q))$: buying profit at (x, q).
- $-\frac{\partial V}{\partial q}(x, q) + (e^x - \mu(q))$: selling profit at (x, q).
- HJB equation.

$$\max\left(\mathcal{L}V, \frac{\partial V}{\partial q} - (e^x + \lambda(q)), -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0$$

Holding, Selling and Buying Regions

- The state space $(x, q) \in \mathbb{R}^2_+$ is divided into three kinds of regions.
- Holding region: holding profit $= 0$, selling & buying profit < 0
- Selling region: selling profit $= 0$, holding & buying profit < 0
- Buying region: buying profit $= 0$, holding & selling profit < 0

Solving the Fixed Boundary Problem

- In the holding region

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V = 0$$

- Defining $y = \kappa(x-\alpha)^2/\sigma^2$, we have

$$y\frac{\partial^2 V}{\partial y^2} + (0.5 - y)\frac{\partial V}{\partial y} - \frac{\beta}{2\kappa}V = 0$$

which is the Kummer Equation. The solution is the sum of hypergeometric1F1 and the hypergeometricU functions.

$$V(x,q) = A(q)\mathrm{HyperGeoU}\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right) + B(q)\mathrm{HyperGeo1F1}\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right)$$

Boundary conditions can determine $A(q)$ and $B(q)$.

The Moving Boundary Method

Idea: Start with an initial guess and iteratively improve it until convergence.

- Challenges
    - Initial guess.
    - 2 dimensions.
        - Direction.
        - Distance.

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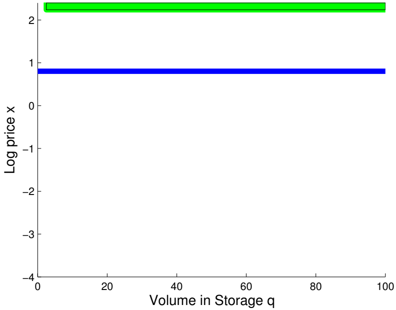

Figure : Initial Guess

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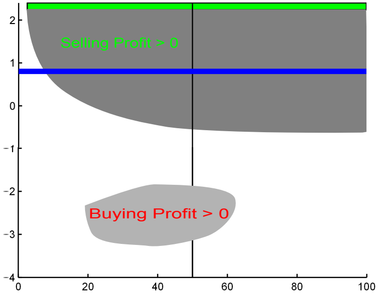

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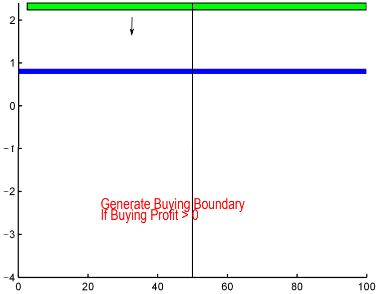


Figure : Initial Guess

Algorithm

1. Begin with selling at very high price for all $q > 0$.
2. Move selling and buying boundaries along price x alternatively until convergence.

Distance

Sell

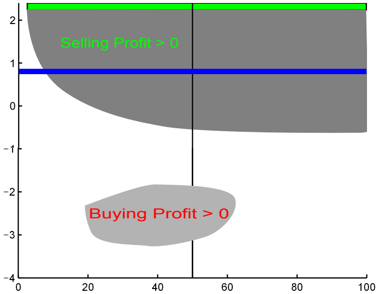
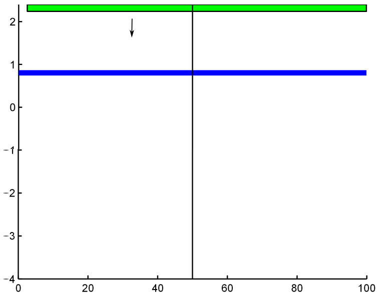


Figure : Current Policy

Distance

Sell



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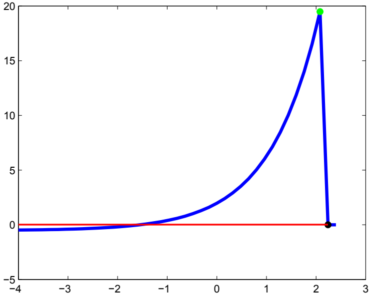
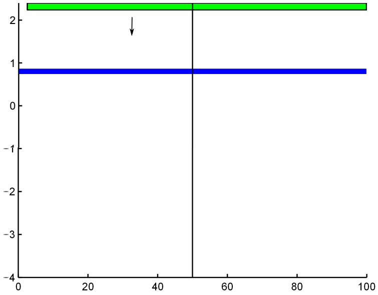


Figure : Selling Profit

Distance

Sell



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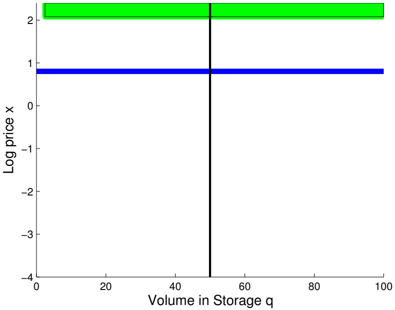
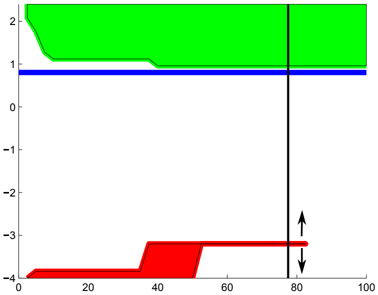


Figure : After Movement

Buy



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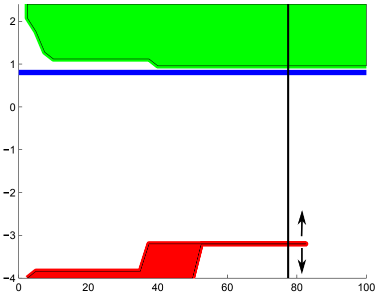

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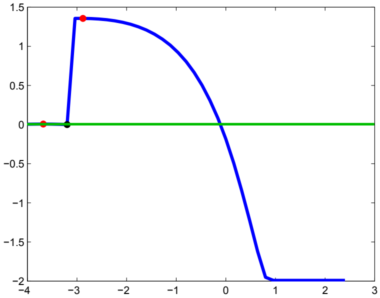
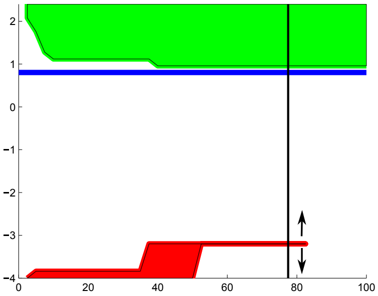


Figure : Buying Profit

Buy



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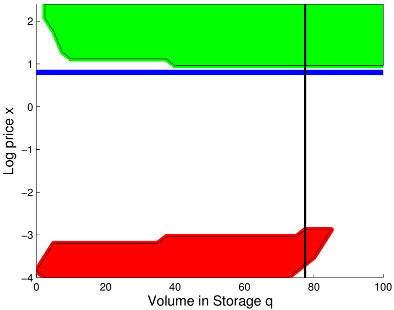
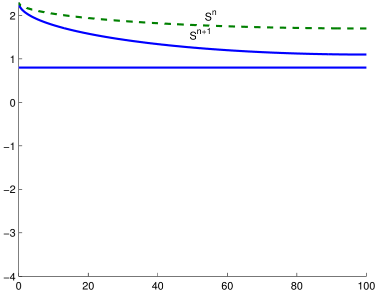


Figure : After Movement

Proof of Convergence

Theorem

Each movement improves value function. Namely $\Delta V^{n+1} = V^{n+1} - V^n \geq 0.$



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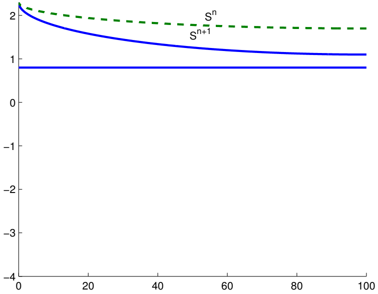

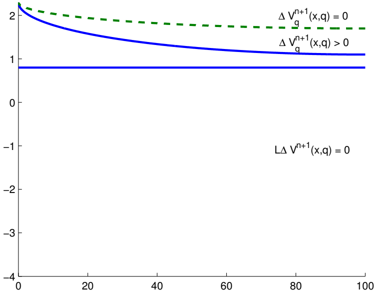

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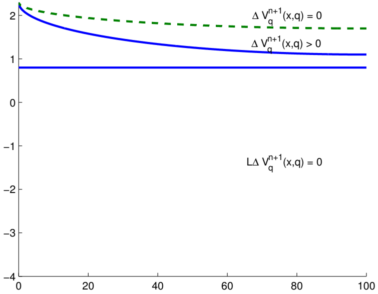

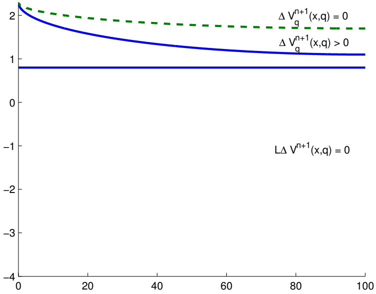

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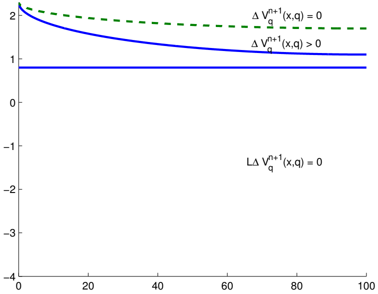

- $\mathcal{L}\Delta V^{n+1}(x,0) = 0 \ \forall x \in \mathbb{R}$.
  $\Rightarrow \Delta V^{n+1}(x,0) = 0$.
- $\mathcal{L}V = \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V$
  Maximum Principle
  $\Rightarrow \Delta V_q^{n+1}(x,q) \geq 0$

Proof of Convergence

Theorem

The boundaries can be kept moving.

Proof.

$$\Leftrightarrow \left(-V_q^{n+1}(x,q) + (e^x - \mu(q))\right)_x |_{S^{n+1}} < 0$$
$$\Leftrightarrow \left(-V_q^{n+1}(x,q) + e^x\right)_x |_{S^{n+1}} < 0$$
$$\Leftrightarrow \left(-V_q^{n+1}(x,q) + V_q^n(x,q)\right)_x |_{S^{n+1}} < 0$$
$$\Leftrightarrow \left(\Delta V_q^{n+1}(x,q)\right)_x |_{S^{n+1}} > 0$$

□

Extensions

- Seasonality and Finite time.
- Depreciation.
- Random injection and withdrawal.
- Buying and selling price follows different but related stochastic processes.