1 HJB Equation

Denote the price process as $P(t)$; the log seasonal factor as $S(t)$; the log deseasonal price process as $X(t)$; the storage process as $Q(t)$. In other words,

$$P(t) = \exp(X(t) + S(t)).$$

What's more, we know that $X(t)$ follows an OU process as following

$$dX(t) = \kappa(\alpha - X(t))dt + \sigma dW_t, \tag{1.1}$$

where W_t is a Brownian motion.
The value function, V, should be a function of the deseasonal price x, the seasonal factor s, and the storage level q. That is to say, $V(x,s,q)$ is the value function. In the holding region, $Q(t)$ remains. As a result, in the holding region,

$$\begin{aligned} de^{-\beta t}V(X_t,S_t,Q_t) =& e^{-\beta t}\Big(V_x dX_t + \frac{1}{2}V_{xx}(dX_t)^2 + V_s dS_t + \frac{1}{2}V_{ss}(dS_t)^2 \\ &+ V_{xs}dX_t dS_t - \beta V dt\Big). \end{aligned} \tag{1.2}$$

If $S(t)$ is deterministic and differentiable, we can assume it satisfies

$$dS(t) = f(S(t))dt.$$

Equation (1.2) turns out to be

$$de^{-\beta t}V(X_t,S_t,Q_t) = e^{-\beta t}\Big(V_x dX_t + \frac{1}{2}V_{xx}(dX_t)^2 + f(S(t))V_s dt - \beta V\Big).$$

Plug equation (1.1) into previous equaiton,

$$de^{-\beta t}V(X_t,S_t,Q_t) = e^{-\beta t}\left(\kappa(\alpha - X(t))V_x + \frac{1}{2}\sigma^2 V_{xx} + f(S(t))V_s - \beta V\right)dt + \sigma e^{-\beta t}V_x dW_t.$$

Define operator $\mathscr{L}$ as

$$\mathscr{L}V(x,s,q) = \frac{1}{2}\sigma^2 V_{xx} + \kappa(\alpha - x)V_x - \beta V + f(s)V_s.$$

As a conclusion, the HJB equation can be written as

$$\max\{\mathscr{L}V, -V_q + e^x - \mu(q), V_q - e^x - \lambda(q)\} = 0.$$

2 SOLVE $\mathscr{L}V = 0$.

Assume $V(x,s,q) = g(x,q)h(s)$. When $s = 0$, there is no seasonal factor at all. In this case, we should have $h(0) = 1$. Plug it into $\mathscr{L}V = 0$ to have

$$h\left(\frac{1}{2}\sigma^2 g_{xx} + \kappa(\alpha - x)g_x - \beta g\right) + f(s)gh' = 0.$$

We can rearrange it as

$$\frac{\frac{1}{2}\sigma^2 g_{xx}(x,q) + \kappa(\alpha - x)g_x(x,q) - \beta g(x,q)}{g(x,q)} = \frac{-f(s)h'(s)}{h(s)}.$$

The left hand side is a function of x and q while the right hand side is a function of s. Therefore, the only possibility is that both are the same constant, denoted as η.

$$\begin{aligned}
&\frac{1}{2}\sigma^2 g_{xx}(x,q) + \kappa(\alpha - x)g_x(x,q) - (\beta + \eta)g(x,q) = 0 \\
&f(s)h'(s) + \eta h(s) = 0 \quad h(0) = 1
\end{aligned}$$

The solution to the second ODE is

$$h(s) = \exp(-\eta \int_0^s \frac{1}{f(y)} dy) \tag{2.1}$$

If we have $S(t) = \sin(t)$, then

$$f(S(t)) = S'(t) = \cos(t) = \sqrt{1 - \sin(t)^2} = \sqrt{1 - S(t)^2}.$$

Namely, $f(s) = \sqrt{1 - s^2}$. Substitute this back into equation (2.1) to have

$$h(s) = \exp(-\eta \arcsin(s))$$

One boundary condition is missing! Assume we will hold on the line $x = x_0,\ q = q_0$. Without another boundary condition, it is impossible to calculate η. It is not like on the line $x = x_0,\ s = s_0$. Because it is impossible to buy (sell) when the facility is full (empty), the number of the boundary conditions are the same as the number of unknown variables.

3 A QUESTION ABOUT POLICY & VALUE ITERATION

In the case without seasonality, we construct a Markov chain from the holding profit,

$$\frac{1}{2}\sigma^2 V_{xx} + \kappa(\alpha - x)V_x - \beta V.$$

Firstly, we find an approximation of above expression. For example, in the case $x \geq \alpha$,

$$\begin{aligned}
&\frac{1}{2}\sigma^2 \frac{V(x+dx) - 2V(x) + V(x-dx)}{dx^2} + \kappa(\alpha - x)\frac{V(x) - V(x-dx)}{dx} - \beta V(x) \\
=&\frac{1}{2}\frac{\sigma}{dx^2}V(x+dx) + \left(\frac{1}{2}\frac{\sigma^2}{dx^2} + \frac{\kappa(x-\alpha)}{dx}\right)V(x-dx) - \left(\frac{\sigma^2}{dx^2} + \frac{\kappa(x-\alpha)}{dx} + \beta\right)V(x)
\end{aligned}$$

Next, from this approximation, we set the transition probability of the Markov chain from x to $x+dx$ as

$$p_+ = \frac{\frac{1}{2}\frac{\sigma}{dx^2}}{\frac{\sigma^2}{dx^2} + \frac{\kappa(x-\alpha)}{dx} + \beta},$$

and from x to $x - dx$ as

$$p_- = \frac{\frac{1}{2}\frac{\sigma^2}{dx^2} + \frac{\kappa(x-\alpha)}{dx}}{\frac{\sigma^2}{dx^2} + \frac{\kappa(x-\alpha)}{dx} + \beta}.$$

In this case, the sum of p_+ and p_- is

$$p_+ + p_- = \frac{\frac{\sigma^2}{dx^2} + \frac{\kappa(x-\alpha)}{dx}}{\frac{\sigma^2}{dx^2} + \frac{\kappa(x-\alpha)}{dx} + \beta},$$

which is not 1. Is it OK? How to understand it?

4 A QUESTION ABOUT THE STORAGE CONTRACT

- Start Date: 19th December 2012
- End Date: 18th December 2013
- Initial Inventory Level: 0
- Final Inventory Level: 0

- Capacity: 29.3 GWh
- Max Injection/Withdrawal: 1.465 GWh per day
- Underlying Gas Price: NBP (National Balancing Point) pence/therm

Is it a contract that enables one to use the storage facility?