1 Model

Let S_t be the price and its evolution follows

$$dS_t = \kappa(\gamma - \log(S_t))S_t dt + \sigma S_t dZ_t. \tag{1}$$

Take $X_t = \log(S_t)$, by Ito's formula,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dZ_t, \tag{2}$$

where $\alpha = \gamma - \frac{\sigma^2}{2\kappa}$.

Let $Q_t \in [Q_{\min}, Q_{\max}]$ denote the total amount of commodity in the storage facility at time t where $Q_{\min}, Q_{\max}$ are the minimum and maximum capacity. Here for simplicity, take them as 0 and 100 separately.

The decisions that a facility manager can make is when and how much shall be injected and withdraw from the facility. Denote cumulative amounts of commodity purchased and sold up to time t as L_t and U_t, respectively. Here and after, buy and inject have the same meaning, so do sell and withdraw. Then

$$dQ_t = dL_t - dU_t. \tag{3}$$

An admissible control policy is a pair (L_t, U_t) such that $Q_t \in [Q_{\min}, Q_{\max}]$ for all t. $\mathcal{U}$ is used to denote all admissible policies.

Both injection and withdraw generate costs. Let $\lambda(q)$ and $\mu(q)$ be the instantaneous costs of injection and withdrawal when the storage has q unit already and $\mu(q)$ of withdrawing. Therefore, the costs of injection and withdrawal at time t should be $\int_{L_{t-}}^{L_t} \lambda(q)dq$ and $\int_{U_{t-}}^{U_t} \mu(q)dq$ respectfully. We assume that $\lambda(q)$ and $\mu(q)$ are continuous and bounded. It is often easier to inject(withdraw) when empty(full) than full(empty). Therefore we assume that $\lambda(q)$ is increasing with respect to q while $\mu(q)$ is decreasing.

The objective is to maximize discounted infinite-horizon discounted cash flows. Taking a discount factor $\beta \in (0, 1)$,

$$V(x,q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_{x,q} \left(\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t^1))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t^2))dL_t \right) \tag{4}$$

where $X_0 = x$ and $Q_0 = q$. Q_t^1 and Q_t^2 are introduced as following to make the objective simpler,

$$Q_t^1 = \mu^{-1} \left(\frac{1}{U_t - U_{t-}} \int_{U_{t-}}^{U_t} \mu(q) dq \right) \tag{5}$$

$$Q_t^2 = \lambda^{-1} \left(\frac{1}{L_t - L_{t-}} \int_{L_{t-}}^{L_t} \lambda(q) dq \right). \tag{6}$$

Since $V(x, q)$ is the maximum value that a facility manager can obtain from a storage facility, this is the value of the facility when the current spot log price is x and the amount is q.

2 No Transaction Costs

No transaction cost means that λ (buying cost) $\equiv 0$ and μ (selling cost) $\equiv 0$. Since the profit (cost) of selling (buying) one unit at the same price level is the same, the optimal strategy should be bang-bang. Whenever the drift of X_t exceeds interest rate, we should buy. Vice versa. Let this critical point be X^*, then

$$\kappa(\alpha - X^*) = r \Rightarrow X^* = \alpha - \frac{r}{\kappa}$$

The optimal strategy is as following

- If $X_t > X^*$, sell to empty.
- If $X_t \leq X^*$, buy to full.

3 Constant Transaction Costs

Let the constant transaction cost be λ (buying cost) and μ (selling cost). Since the profit (cost) of selling (buying) one unit at the same price level is the same, the optimal strategy should be bang-bang. Introduce the following two functions of log price x,

$$\begin{aligned} V(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \{ e^{-r\tau} (e^{X_\tau} - \mu) \} \\ J(x) &= \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \{ e^{-r\nu} (V(x^\nu) - e^{X_\nu} - \lambda) \}. \end{aligned}$$

$V(x)$ is the maximum value that can be gained from 1 unit commodity when log price is x. $J(x)$ is the value of an empty storage when log price is

x. Tim Leung et al.(2014) use concavity to prove that the optimal policy for $V(x)$ is an upper threshold policy. This means that if log price exceeds the threshold sell. d^* is denoted as the threshold. Using similar idea, they also prove the optimal policy for $J(x)$ is an upper-lower threshold policy. This means that one will buy if and only if log price is inside two thresholds. a^* and b^* are denoted as the lower and upper threshold. Based on the existence and conditions of a^* and b^*, there are two cases.

1. $J(x) = 0$. In this case, it is not optimal enter the market at all.

2. $-\infty < a^* < b^* < d^* < \infty$. This means that commodity is bought when log price is within $[a^*, b^*]$ and will be sold when log price achieves d^*.

4 Non Constant Transaction Costs

4.1 The HJB equation

The stochastic control problem with non constant transaction costs can be transformed into a Hamilton Jacobi Bellman (HJB) equation via dynamic programming principle and Ito's formula. The HJB equation characterize the value function $V(x, q)$ (4).

We will use V_x and V_q separately to denote the partial differential of V with respect to x and q. V_{xx} represents $\frac{\partial^2 V}{(\partial x)^2}$. The value function $V(x, q)$ is expected to satisfy the HJB equation

$$\max \left[\mathcal{L}V(x,q), -V_q(x,q) + e^x - \mu(q), V_q(x,q) - e^x - \lambda(q) \right] = 0. \tag{7}$$

where

$$\mathcal{L}V(x,q) = \frac{1}{2}\sigma^2 V_{xx}(x,q) + k(\alpha - x)V_x(x,q) - \beta V(x,q). \tag{8}$$

Because $\mathcal{L}V(x,q)$, $-V_q(x,q) + e^x - \mu(q)$ and $V_q(x,q) - e^x - \lambda(q)$ are the profits at (x,q) generated by utilizing holding, selling and buying policy respectively, they are called holding profit, selling profit and buying profit. The region that holding profit is the highest is called the optimal holding region and denoted as H^*. Similarly, optimal selling (buying) region is defined and denoted as S^* (B^*).

4.2 Algorithm

1. Find a large enough M_0 such that $S^* \supset S^0$ where the selling region $S^0 = \{(x,q)|x \geq M_0,\ Q_{\min} < q \leq Q_{\max}\}$. Set holding region $H^0 = (S^0)^c$ and buying region $B^0 = \emptyset$.

2. Keep doing the following steps until either convergence or $B^n \neq \emptyset$. n starts with 0.

   (a) Solve the fixed boundary problem with (H^n, S^n, B^n) and the solution is denoted as $V^n(x,q)$.

   (b) Take positive maximal buying profits points as B^{n+1}. That is to say, $B^{n+1} = \{(x,q)|x = x_M^{n+1}(q)\}$ where

   $$x_M^{n+1}(q) = \begin{cases} \arg\max_{(x,q)} BP(q) & \max_{(x,q)} BP(q) > 0 \\ \emptyset & \text{otherwise,} \end{cases}$$

   and $BP(q) = V_q^n(x,q) - e^x - \lambda(q)$.

   (c) If $B^{n+1} = \emptyset$, move down the selling boundary ∂S^n to the maximal selling profitable points. Mathematically speaking $S^{n+1} = \{(x,q)|x \geq x^{n+1}(q)\}$ where

   $$x^{n+1}(q) = \arg\max_x \left(-V_q^n(x,q) + e^x - \mu(q)\right).$$

   (d) $H^{n+1} = (S^{n+1} \cup B^{n+1})^c$.

   (e) $n = n+1$.

3. If previous loop ends when $n = N$ because of $B^N \neq \emptyset$, keep doing the following until convergence. n starts as N.

   (a) Solve the fixed boundary problem with (H^n, S^n, B^n) and the solution is denoted as $V^n(x,q)$.

   (b) If n is odd, $B^{n+1} = B^n$. Move down the selling boundary ∂S^n to the maximal selling profitable points to have S^{n+1} as previously.

   (c) If n is even, $S^{n+1} = S^n$. Move up and down buying boundary ∂B^{n+1} to the maximal buying profitable points. Mathematically speaking, $B^{n+1} = \{(x,q)|x_l^{n+1}(q) \leq x \leq x_u^{n+1}(q)\}$ where

   $$\begin{aligned} x_l^{n+1}(q) &= \arg\max_{x < x_l^n(q)} \left(V_q^n(x,q) - e^x - \lambda(q)\right) \\ x_u^{n+1}(q) &= \arg\max_{x > x_u^n(q)} \left(V_q^n(x,q) - e^x - \lambda(q)\right) \\ x_l^N(q) &= x_u^N(q) = x_M^N(q) \end{aligned}$$

(d) $H^{n+1} = (S^{n+1} \cup B^{n+1})^c$.

(e) $n = n + 1$.

Notice that each time only one of selling and buying boundaries moves and both selling region and buying region are increasing. These are critical in the proof of the monotonicity V^n.

In order to show this algorithm works, 3 things need to be proved.

1. The existence of M_0.
2. V^n is monotone increasing.
3. The boundaries can be kept moving.

4.3 The existence of M_0

Lemma 1. $\forall \tilde{q} > 0$, $\exists M_0(\tilde{q})$ *s.t.* $\{(x, q) | x \geq M_0(\tilde{q}), q = \tilde{q}\} \subset S^*$

Proof. First, in order to make money, for any buying price, there must be a higher selling price. Mathematically speaking, $\forall (x_1, q) \in B^*$ $\exists x_2 > x_1$ *s.t.* $(x_2, q) \in S^*$.

If the lemma doesn't hold, combining the observation above, $\exists M_0(\tilde{q})$, $\tilde{q} > Q_{\min}$ such that $\{(x, \tilde{q}) | x > M_0(\tilde{q})\} \subset H^*$. Without losing any generality, let $M_0(\tilde{q}) > \alpha$. In other words, $\forall x > M_0(\tilde{q})$

$$
\begin{aligned}
V_q(x, \tilde{q}) &\leq e^x + \lambda(\tilde{q}) \\
\mathcal{L}V(x, \tilde{q}) &= 0 \\
V_q(x, \tilde{q}) &\geq e^x - \mu(\tilde{q}).
\end{aligned}
$$

Take derivative with respect to q to both sides of second equality,

$$
\begin{aligned}
V_q(x, \tilde{q}) &\leq e^x + \lambda(\tilde{q}) \\
\mathcal{L}V_q(x, \tilde{q}) &= 0 \\
V_q(x, \tilde{q}) &\geq e^x - \mu(\tilde{q}).
\end{aligned}
$$

Define

$$h(x) = V_q(x, \tilde{q}) - (e^x - \mu(\tilde{q})). \tag{9}$$

Therefore, $\forall x > M_0(\tilde{q})$

$$0 \leq h(x) \leq \lambda(\tilde{q}) + \mu(\tilde{q}). \tag{10}$$

Substitute (9) into $\mathcal{L}V_q(x,\tilde{q}) = 0$,

$$\frac{1}{2}\sigma^2 h''(x) + k(\alpha - x)h'(x) - \beta h(x) = -\frac{1}{2}\sigma^2 e^x + k(x-\alpha)e^x + \beta(e^x - \mu(\tilde{q})).$$

Divide both sides by e^x,

$$\frac{1}{2}\sigma^2 e^{-x}h''(x) + ke^{-x}(\alpha - x)h'(x) - \beta e^{-x}h(x) = -\frac{1}{2}\sigma^2 + k(x-\alpha) + \beta(1 - e^{-x}\mu(\tilde{q})). \tag{11}$$

Because $\mu(\tilde{q})$ is bounded, right hand side of (11) goes to $+\infty$ when $x \to +\infty$. Furthermore, by (10), $h(x)$ is bounded. Combine these two, when $x \to +\infty$, either

1. $e^{-x}h''(x) \to +\infty$.
   or

2. $e^{-x}(\alpha - x)h'(x) \to +\infty$

Let $t_2 > t_1 > \alpha$ and without losing any generality both $e^{-x}h''(x)$ and $e^{-x}(\alpha - x)h'(x)$ are positive on $[t_1, t_2]$.

In first case, by Fubini's theorem,

$$\begin{aligned}
h(t_2) - h(t_1) &= \int_{t_1}^{t_2} h'(t)dt = \int_{t_1}^{t_2}(\int_{t_1}^{t} h''(x)dx + h'(t_1))dt \\
&= \int_{t_1}^{t_2}\int_{t_1}^{t} h''(x)dxdt + (t_2 - t_1)h'(t_1) \\
&= \int_{t_1}^{t_2}\int_{x}^{t_2} h''(x)dtdx + (t_2 - t_1)h'(t_1) \\
&= \int_{t_1}^{t_2}(t_2 - x)h''(x)dx + (t_2 - t_1)h'(t_1) \\
&\geq e^{t_1}\int_{t_1}^{t_2}(t_2 - x)e^{-x}h''(x)dx + (t_2 - t_1)h'(t_1) \\
&= e^{t_1}\int_{t_1}^{t_2}\left((t_2 - x)e^{-x}h''(x) + e^{-t_1}h'(t_1)\right)dx.
\end{aligned} \tag{12}$$

With t_1 fixed, when $t_2 \to +\infty$, the right hand side approaches positive infinity. This is a contradiction with (10).

In the second case,

$$\begin{aligned} h(t_2) - h(t_1) &= \int_{t_1}^{t_2} h'(t)dt \\ &= \int_{t_1}^{t_2} \frac{e^t}{a-t} \frac{(a-t)h'(t)}{e^t} dt \\ &\leq \frac{e^{t_1}}{a-t_1} \int_{t_1}^{t_2} \frac{(a-t)h'(t)}{e^t} dt. \end{aligned} \tag{13}$$

With t_1 fixed, when $t_2 \to +\infty$, the right hand side approaches negative infinity. This is also a contradiction with (10).

□

We have proved that for each $\tilde{q}$, there exists a suitable $M_0(\tilde{q})$. Because discretization is used when solving the HJB equation, only finite number of $M_0(\tilde{q})$ matter. The maximal of those is M_0.

If the condition below holds, a proposed M_0 is large enough. Otherwise, it is not.

$$\left(-V_q(x,q) + (e^x - \mu(q))\right)_x |_{x=M_0} < 0 \; \forall q \in (Q_{\min}, Q_{\max}]$$

This criterion guarantees M_0 can be found in practice.

4.4 V^n is monotone increasing

Two lemmas are needed.

Lemma 2. *If $\mathcal{L}f = 0$, f achieves the positive maximal and negative minimum at the boundary.*

Proof. Assume that there exists x_M which is the positive maximal interior point. Then $f(x_M) > 0$, $f'(x_M) = 0$ and $f''(x_M) \leq 0$. On the other hand, $\mathcal{L}f(x_M) = 0$. By the definition of $\mathcal{L}$,

$$\begin{aligned} &\frac{1}{2}\sigma^2 f''(x_M) + \kappa(\alpha - x_M) f'(x_M) - \beta f(x_M) = 0 \\ \Rightarrow & 0 < \beta f(x_M) = \frac{1}{2}\sigma^2 f''(x_M) + \kappa(\alpha - x_M) f' \leq 0 \end{aligned} \tag{14}$$

Contradiction! Negative minimum case can be proved using similar method. □

Lemma 3. *Denote $I_{\tilde{x}} = \{(x,q) | x = \tilde{x}, q \in [Q_{\min}, Q_{\max}]\}$. $\nexists \tilde{x}$ such that both $I_{\tilde{x}} \cap S^* \neq \emptyset$ and $I_{\tilde{x}} \cap B^* \neq \emptyset$ hold simultaneously.*

Proof. If not, there must exist $(\tilde{x}, q_1)$ on then buying boundary and $(\tilde{x}, q_2)$ on the selling boundary.

Define

$$f(x) = V_q(x, q_1) - (e^x + \lambda(q_1)). \tag{15}$$

By definition, $f(\tilde{x}) = 0$. If V_q is continuous across boundary, $f'(\tilde{x}) = 0$. By HJB equation, $f(x)$ achieves its maximum at point $\tilde{x}$. Because $f(x)$ may not be second order continuous, we need to consider two cases.

1. Holding region is above $(\tilde{x}, q_1)$.

2. Holding region is below $(\tilde{x}, q_1)$.

From maximal property and $f'(\tilde{x}) = 0$, we have $f''(\tilde{x}+) \le 0$, $f''(\tilde{x}-) = 0$ for the first and $f''(\tilde{x}+) = 0$, $f''(\tilde{x}-) \le 0$ for the second. In the first case,

$$\mathcal{L}f(\tilde{x}+) = \frac{1}{2}\sigma^2 f''(\tilde{x}+) + k(\alpha - \tilde{x})f'(\tilde{x}) - \beta f(\tilde{x}) = \frac{1}{2}\sigma^2 f''(\tilde{x}+) \le 0.$$

On the other hand,

$$\begin{aligned}\mathcal{L}f(\tilde{x}+) &= \mathcal{L}V_q(\tilde{x}+, q_1) - \mathcal{L}(e^x + \lambda(q_1))|_{x=\tilde{x}} \\ &= \left(\mathcal{L}V(\tilde{x}+, q_1)\right)_q - \mathcal{L}(e^x + \lambda(q_1))|_{x=\tilde{x}} \\ &= -\mathcal{L}(e^x + \lambda(q_1))|_{x=\tilde{x}}.\end{aligned} \tag{16}$$

Therefore,

$$\frac{1}{2}\sigma^2 e^{\tilde{x}} + k(\alpha - \tilde{x})e^{\tilde{x}} - \beta(e^{\tilde{x}} + \lambda(q_1))) \ge 0. \tag{17}$$

In the second case, the same equality can be achieved using similar arguments above.

Define

$$g(x) = -V_q(x, q_2) + (e^x - \mu(q_2)).$$

Replicate the same analysis for f to have

$$\frac{1}{2}\sigma^2 e^{\tilde{x}} + k(\alpha - \tilde{x})e^{\tilde{x}} - \beta(e^{\tilde{x}} - \mu(q_2))) \le 0. \tag{18}$$

Because λ and μ are positive, (17) and (18) can't hold simultaneously which is a contradiction.

□

Theorem 1. $V^{n+1} \ge V^n$.

Proof. By the definitions of (V^n, H^n, S^n, B^n) and the way boundary moves, we have

$$
\begin{array}{ll}
\mathcal{L}V^n(x,q) = 0 & (x,q) \in H^n \\
-V^n_q(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^n \\
-V^n_q(x,q) + e^x - \mu(q) > 0 & (x,q) \in S^{n+1}/S^n \\
V^n_q(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^n \\
V^n_q(x,q) - e^x - \lambda(q) > 0 & (x,q) \in B^{n+1}/B^n.
\end{array}
\tag{19}
$$

Here we also use the monotonicity of H^n, S^n and B^n, i.e. $H^n \supset H^{n+1}$, $S^n \subset S^{n+1}$ and $B^n \subset B^{n+1}$.

The definition of $(V^{n+1}, H^{n+1}, S^{n+1}, B^{n+1})$ gives us

$$
\begin{array}{ll}
\mathcal{L}V^{n+1}(x,q) = 0 & (x,q) \in H^{n+1} \\
-V^{n+1}_q(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^n \\
-V^{n+1}_q(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^{n+1}/S^n \\
V^{n+1}_q(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^n \\
V^{n+1}_q(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^{n+1}/B^n.
\end{array}
\tag{20}
$$

Introduce $\Delta V^{n+1}(x,q) = V^{n+1}(x,q) - V^n(x,q)$. From (19) and (20)

$$
\begin{aligned}
\mathcal{L}\Delta V^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1} \\
\Delta V^{n+1}_q(x,q) &= 0 \quad (x,q) \in S^n \\
\Delta V^{n+1}_q(x,q) &> 0 \quad (x,q) \in S^{n+1}/S^n \\
\Delta V^{n+1}_q(x,q) &= 0 \quad (x,q) \in B^n \\
\Delta V^{n+1}_q(x,q) &< 0 \quad (x,q) \in B^{n+1}/B^n.
\end{aligned}
\tag{21}
$$

Recall that in the algorithm, only one of selling and buying boundary moves. (21) can be further divided into two cases.

1. Only selling boundary moves, namely

$$
\begin{aligned}
\mathcal{L}\Delta V^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1} \\
\Delta V^{n+1}_q(x,q) &= 0 \quad (x,q) \in S^n \\
\Delta V^{n+1}_q(x,q) &> 0 \quad (x,q) \in S^{n+1}/S^n \\
\Delta V^{n+1}_q(x,q) &= 0 \quad (x,q) \in B^{n+1} = B^n
\end{aligned}
\tag{22}
$$

Take derivative with respect to q in the first equality above to have

$$
\begin{aligned}
\mathcal{L}\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1} \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in S^n \\
\Delta V_q^{n+1}(x,q) &> 0 \quad (x,q) \in S^{n+1}/S^n \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in B^{n+1} = B^n.
\end{aligned}
\tag{23}
$$

By Lemma 2, $\Delta V_q^{n+1} \geq 0$ holds for all points. Right now focus on the set $\{(x, Q_{\min})|x \in \mathbb{R}\}$. Only holding and buying are possible policy on this set because there is nothing to sell.

(a) Hold at $(x, Q_{\min})$. Directly from the first equality of (22), we have

$$\mathcal{L}\Delta V^{n+1}(x, Q_{\min}) = 0$$

(b) Buy at $(x, Q_{\min})$. By Lemma 3, on this log price level x, there will only be holding and buying policy. Especially, we will hold at $(x, Q_{\max})$ because there is no place to inject. Therefore,

$$\mathcal{L}\Delta V^{n+1}(x, Q_{\max}) = 0.$$

By (23),

$$\mathcal{L}\Delta V_q^{n+1}(x,q) = 0 \quad \forall (x,q) \in H^{n+1} \cup B^{n+1}.$$

As mentioned above, only holding and buying are possible. Thanks to above equality,

$$\int_{Q_{\min}}^{Q_{\max}} \mathcal{L}\Delta V_q^{n+1}(x,q)dq = 0.$$

Thus,

$$
\begin{aligned}
\mathcal{L}\Delta V^{n+1}(x, Q_{\min}) &= \mathcal{L}\left(\Delta V^{n+1}(x, Q_{\max}) - \int_{Q_{\min}}^{Q_{\max}} \Delta V_q^{n+1}(x,q)dq\right) \\
&= \mathcal{L}\Delta V^{n+1}(x, Q_{\max}) - \int_{Q_{\min}}^{Q_{\max}} \mathcal{L}\Delta V_q^{n+1}(x,q)dq = 0.
\end{aligned}
$$

We have proved that

$$\mathcal{L}\Delta V^{n+1}(x, Q_{\min}) = 0 \quad \forall x \in \mathbb{R}.$$

This implies $\Delta V^{n+1}(x, Q_{\min}) = 0$ because 0 is the only finite function that satisfy above equation. Therefore,

$$\begin{aligned}\Delta V^{n+1}(x,q) &= \Delta V^{n+1}(x,Q_{\min}) + \int_{Q_{\min}}^{q} \Delta V_q^{n+1}(x,q)dq \\ &= \int_{Q_{\min}}^{q} \Delta V_q^{n+1}(x,q)dq \geq 0\end{aligned}$$

The last inequality holds because of $\Delta V_q^{n+1} \geq 0$.

2. Only buying boundary moves, namely

$$\begin{aligned}\mathcal{L}\Delta V^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1} \\ \Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in S^{n+1} = S^n \\ \Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in B^n \\ \Delta V_q^{n+1}(x,q) &< 0 \quad (x,q) \in B^{n+1}/B^n.\end{aligned} \tag{24}$$

Using similar method to have $\Delta V_q^{n+1} \leq 0$ and $\Delta V^{n+1}(x, Q_{\max}) = 0$. Then

$$\begin{aligned}\Delta V^{n+1}(x,q) &= \Delta V^{n+1}(x,Q_{\max}) - \int_{q}^{Q_{\max}} \Delta V_q^{n+1}(x,q)dq \\ &= -\int_{Q_{\min}}^{q} \Delta V_q^{n+1}(x,q)dq \geq 0\end{aligned}$$

□

4.5 The boundaries can keep moving

From the algorithm, the boundaries can move if and only if the following Theorem holds.

Theorem 2. *Three inequalities below are true for all n and $q \in [Q_{\min}, Q_{\max}]$.*

$$\begin{aligned}\left(-V_q^{n+1}(x,q) + e^x - \mu(q)\right)_x |_{x=x^{n+1}(q)} &\leq 0 \\ \left(V_q^{n+1}(x,q) - e^x - \lambda(q)\right)_x |_{x=x_u^{n+1}(q)} &\geq 0 \\ \left(V_q^{n+1}(x,q) - e^x - \lambda(q)\right)_x |_{x=x_l^{n+1}(q)} &\leq 0\end{aligned}$$

Proof. Still only the statement for selling boundary is proved.

1. In stage 1, the selling boundary is moved.

2. In stage 2, the buying boundary is moved.

In the first case, from previous proof, with q fixed, $f(x) = \Delta V_q^{n+1}(x,q)$ achieves its maximum at point x_S where $(x_S, q) \in S^{n+1}/S^n$. This is equivalent to $f'(x_S) > 0$.[1] Thus,

$$
\begin{aligned}
&(\Delta V_q^{n+1})_x(x_S,q) = f'(x_S) > 0 \\
\Rightarrow &(V_q^{n+1})_x(x_S,q) - (V_q^n)_x(x_S,q) > 0 \\
\Rightarrow &(V_q^{n+1})_x(x_S,q) - e^{x_S} > 0 \\
\Rightarrow &\left(V_q^{n+1}(x,q) - (e^x - \mu(q)\right)_x |_{x=x_S} > 0 \\
\Rightarrow &\left(-V_q^{n+1}(x,q) + (e^x - \mu(q)\right)_x |_{x=x_S} < 0.
\end{aligned}
\tag{25}
$$

Noticing that the only thing used here is that x_S is the maximum. In the second case, it can be proved similarly that x_S is also the maximum. See the figure (4.5) for the idea.

Therefore we prove that we can continue moving the boundary.

□

5 The moving conditions are always satisfied. This is to say, we can keep moving the boundary.

Proof. □

[1]If $f'(x_s) = 0$, by $\mathcal{L}f(x_s) = 0$, $0 < \beta f = \frac{1}{2}\sigma^2 f''(x) \leq 0$. Contradiction!

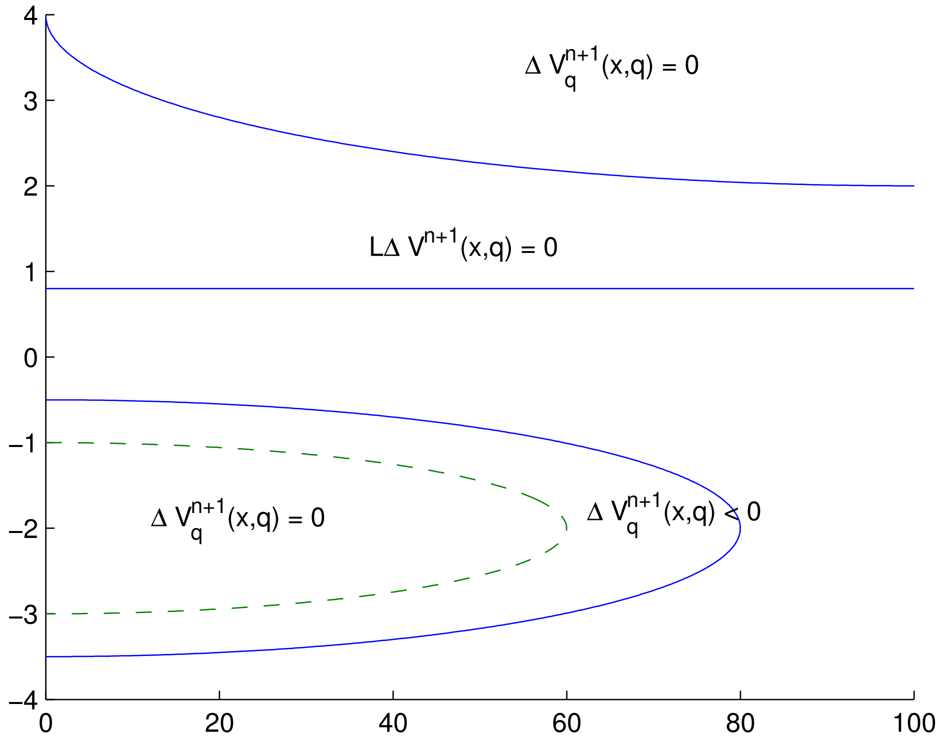

Figure 1: The equations that ΔV^{n+1} satisfies in the second stage.