1 About η

The equations we have for the function g and h ($V(x,s,q) = g(x,q) \times h(s)$) are

$$\frac{1}{2}\sigma^2 g_{xx}(x,q) + \kappa(\alpha - x) g_x(x,q) - (\beta+\eta) g(x,q) = 0$$
$$f(s)h'(s) + \eta h(s) = 0 \quad h(0) = 1.$$

The solution to the second ODE is

$$h(s) = \exp(-\eta \int_0^s \frac{1}{f(y)} dy) \tag{1.1}$$

Last time, we mentioned that

$$0 = \frac{\partial V(x,s,q)}{\partial \eta} = \frac{\partial g(x,q;\eta)}{\partial \eta} h(s;\eta) + g(x,q;\eta) \frac{\partial h(s;\eta)}{\partial \eta}$$

By equation (1.1), we have

$$\frac{\partial g(x,q;\eta)}{\partial \eta} h(s;\eta) - g(x,q;\eta) h(s;\eta) \int_0^s \frac{1}{f(y)} dy = 0$$

Equation (1.1) tells us $h > 0$, therefore

$$\frac{\partial g(x,q;\eta)}{\partial \eta} - g(x,q;\eta) \int_0^s \frac{1}{f(y)} dy = 0$$
$$\Rightarrow \quad \frac{\frac{\partial g(x,q;\eta)}{\partial \eta}}{g(x,q;\eta)} = \int_0^s \frac{1}{f(y)} dy.$$

Right hand side is a function of s while the left hand is not. Contradiction!

- $\frac{\partial V(x,s,q)}{\partial \eta} = 0$ doesn't mean that the function V is not a function of η. Instead, we choose η to maximize V.
- It is also impossible to have two different ηs to have the same V. If not, assume

  $$g(x,q;\eta_1) h(s;\eta_1) = g(x,q;\eta_2) h(s;\eta_2).$$

  In this situation,

  $$\frac{g(x,q;\eta_1)}{g(x,q;\eta_2)} = \frac{h(s;\eta_1)}{h(s;\eta_2)}.$$

  The left hand side is a function of x and q while the right hand side is a function of s (because η_1 and η_2 are given constants). It is impossible to happen. As a result, it is not that

  $$V(x,s,q) = g(x,q;\eta) \times h(s;\eta)$$

  holds for all η.

2 BOUNDARY CONDITION

The boundary conditions are

$$V_q(x,s,q) = \exp(x+s) - \mu(q)$$

and

$$V_q(x,s,q) = \exp(x+s) + \lambda(q).$$

The first one is at the buying boundary while the second one is at the selling boundary. Plug g and h into previous equations to have

$$g_q(x,q) = \frac{\exp(x+s) - \mu(q)}{h(s)}$$

on the selling boundary and

$$g_q(x,q) = \frac{\exp(x+s) + \lambda(q)}{h(s)}$$

on the buying boundary. For a fixed s and any η, the problem becomes the same type as the non-seasonality problem which we can solve effectively. However, two different s_1 and s_2 are super likely to give different $g(x,q)$ which contradicts the separation of variables at the first place.
It is unlikely to satisfy the boundary condition with all different s with only one η. As a result, I propose

$$V(x,s,q) = \int_{-\infty}^{+\infty} g(x,q;\eta)\,h(s;\eta)\,t(\eta)\,d\eta.$$

I want to use the boundary condition to find an approximation of the function t, but I don't know how.

3 SEPARATION VARIABLE METHOD IN HEAT EQUATION

`http://en.wikipedia.org/wiki/Separation_of_variables`