1 Model

Let S_t be the price and its evolution follows

$$dS_t = \kappa(\gamma - \log(S_t))S_t dt + \sigma S_t dZ_t. \tag{1}$$

Take $X_t = \log(S_t)$, by Ito's formula,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dZ_t, \tag{2}$$

where $\alpha = \gamma - \frac{\sigma^2}{2\kappa}$.

Let $Q_t \in [Q_{\min}, Q_{\max}]$ denote the total amount of commodity in the storage facility at time t where $Q_{\min}, Q_{\max}$ are the minimum and maximum capacity. Here for simplicity, take them as 0 and 100 separately.

The decisions that a facility manager can make is when and how much shall be injected and withdraw from the facility. Denote cumulative amounts of commodity purchased and sold up to time t as L_t and U_t, respectively. Here and after, buy and inject have the same meaning, so do sell and withdraw. Then

$$dQ_t = dL_t - dU_t. \tag{3}$$

An admissible control policy is a pair (L_t, U_t) such that $Q_t \in [Q_{\min}, Q_{\max}]$ for all t. $\mathcal{U}$ is used to denote all admissible policies.

Both injection and withdraw generate costs. Let $\lambda(q)$ and $\mu(q)$ be the instantaneous costs of injection and withdrawal when the storage has q unit already and $\mu(q)dq$ of withdrawing. Therefore, the costs of injection and withdrawal at time t should be $\int_{L_{t-}}^{L_t} \lambda(q)dq$ and $\int_{U_{t-}}^{U_t} \mu(q)dq$ respectfully. We assume that $\lambda(q)$ and $\mu(q)$ are continuous. It is often easier to inject when empty and to withdraw when full. Therefore we assume that $\lambda(q)$ is increasing with respect to q while $\mu(q)$ is decreasing.

The objective is to maximize discounted infinite-horizon discounted cash flows. Taking a discount factor $\beta \in (0, 1)$,

$$V(x, q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_{x,q} \left(\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t^1))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t^2))dL_t \right) \tag{4}$$

where $X_0 = x$ and $Q_0 = q$. Q_t^1 and Q_t^2 are introduced as following to make notations simpler,

$$Q_t^1 = \mu^{-1} \left(\frac{1}{U_t - U_{t-}} \int_{U_{t-}}^{U_t} \mu(q) dq \right) \tag{5}$$

$$Q_t^2 = \lambda^{-1} \left(\frac{1}{L_t - L_{t-}} \int_{L_{t-}}^{L_t} \lambda(q) dq \right). \tag{6}$$

Since $V(x, q)$ is the maximum value that a facility manager can obtain from a storage facility, this is the value of the facility when the current spot log price is x and the amount is q.

2 No Transaction Costs

No transaction cost means that λ (buying cost) $\equiv 0$ and μ (selling cost) $\equiv 0$. Since the profit (cost) of selling (buying) one unit at the same price level is the same, the optimal strategy should be bang-bang. Whenever the drift of X_t exceeds interest rate, we should buy. Vice versa. Let this critical point be X^*, then

$$\kappa(\alpha - X^*) = r \Rightarrow X^* = \alpha - \frac{r}{\kappa}$$

The optimal strategy is as following

- If $X_t > X^*$, sell to empty.
- If $X_t \leq X^*$, buy to full.

3 Constant Transaction Costs

Let the constant transaction cost be λ (buying cost) and μ (selling cost). Since the profit (cost) of selling (buying) one unit at the same price level is the same, the optimal strategy should be bang-bang. Introduce the following two functions of log price x,

$$\begin{aligned} V(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \{ e^{-r\tau} (e^{X_\tau} - \mu) \} \\ J(x) &= \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \{ e^{-r\nu} (V(x^\nu) - e^{X_\nu} - \lambda) \}. \end{aligned}$$

$V(x)$ is the maximum value that can be gained from 1 unit commodity when log price is x. $J(x)$ is the value of an empty storage when log price is x. Tim Leung et al.(2014) use concavity to prove that the optimal policy for $V(x)$ is an upper threshold policy. d^* is denoted as the threshold. Using similar idea, they also prove the optimal policy for $J(x)$ is an upper-lower threshold policy. a^* and b^* are denoted as the lower and upper threshold. Based on the values of a^* and b^*, there are two cases.

1. $a^* = -\infty$ and $b^* = \infty$ which is equivalent as $J(x) = 0$. In this case, it is not optimal enter the market at all.
2. $-\infty < a^* < b^* < d^* < \infty$. This means that commodity is bought when log price is within $[a^*, b^*]$ and will be sold when log price achieves d^*.

4 Non Constant Transaction Costs

4.1 The HJB equation

The stochastic control problem with non constant transaction costs can be transformed into a Hamilton Jacobi Bellman (HJB) equation via dynamic programming principle and Ito's formula. The HJB equation characterize the value function $V(x,q)$ (4).

We will use V_x and V_q separately to denote the partial differential of V with respect to x and q. V_{xx} represents $\frac{\partial^2 V}{(\partial x)^2}$. The value function $V(x,q)$ is expected to satisfy the HJB equation

$$\max\left[\mathcal{L}V(x,q), V_q(x,q) - e^x - \lambda(q), -V_q(x,q) + e^x - \mu(q)\right] = 0. \tag{7}$$

where

$$\mathcal{L}V(x,q) = \frac{1}{2}\sigma^2 V_{xx}(x,q) + k(\alpha - x)V_x(x,q) - \beta V(x,q). \tag{8}$$

Because $\mathcal{L}V(x,q)$, $V_q(x,q) - e^x - \lambda(q)$ and $-V_q(x,q) + e^x - \mu(q)$ are the profits at (x,q) generated by utilizing holding, buying and selling policy respectively, they are called holding profit, buying profit and selling profit. The region that holding profit is the highest is called the optimal holding region and denoted as H^*. Similarly, optimal selling (buying) region is defined and denoted as S^* (B^*).

4.2 Algorithm

1. Find a large enough x_0 such that $S^* \supset \{(x,q)|x \geq x_0,\ Q_{\min} < q \leq Q_{\max}\}$. The selling region S^0 is set to be the latter term and holding region $H^0 = (S^0)^c$. B^0 is taken to be $\emptyset$.

2. n starts as 0. Keep doing the following steps until either regions get converged or $B^n \neq \emptyset$,

   (a) Solve the fixed boundary problem with (H^n, S^n, B^n).

   (b) Move down the selling boundary ∂S^n to the maximal selling profitable points to have S^{n+1}.

   (c) Take positive maximal buying profits points as B^{n+1}. If none, $B^{n+1} = \emptyset$.

   (d) $H^{n+1} = (S^{n+1} \cup B^{n+1})^c$.

   (e) $n = n + 1$.

3. If previous loop ends because of $B^n \neq \emptyset$, keep doing the following until convergence.

   (a) Solve the fixed boundary problem with (H^n, S^n, B^n).

   (b) If n is odd, move down the selling boundary ∂S^n to the maximal selling profitable points to have S^{n+1} and $B^{n+1} = B^n$.

   (c) If n is even, move up and down buying boundary ∂B^{n+1} to the maximal buying profitable points to have B^{n+1} and $S^{n+1} = S^n$.

   (d) $H^{n+1} = (S^{n+1} \cup B^{n+1})^c$.

   (e) $n = n + 1$.

Since there are only two activities, buy and sell, the way of making a profit must be buy low and sell high. Actually, it is rigorously proved that the optimal policy must sell when price is high enough. However, the existence of transaction costs implies that the real buying price which includes injecting cost has a lower bound $\lambda(q)$. Therefore, the minimal selling price is $\lambda(q) + \mu(q)$. In this case, we may not buy at extremely low price because it takes a long time to go back to $\lambda(q) + \mu(q)$ level which means it may be impossible to make a profit resulting from discounting. This can be seen clearly when $\lambda(q) + \mu(q)$ is larger than the mean reverting level α[1]. Here comes the difficulty of moving boundary method.

No suitable initial guess for buying region exists. A initial guess needs to generated in the moving boundary method itself.

What will the optimal strategy be without buying? It will be less aggressive because there is only one chance for selling. In other words, the selling price will be higher. On the other hand, the probability a buying strategy

[1] Is there certain constraints guarantee the existence of lower bound of buying region?

makes money depends on the selling strategy. The better selling strategy the larger probability certain buying strategy will make a profit. If a buying strategy can make money with the optimal selling strategy without buying, it must be able to make money with the optimal one with buying. Actually, the optimal selling strategy without buying is not necessary. All we need is a selling strategy that is less aggressive than the global optimal.

Here comes the method that overcome above difficulty.

1. Assume there is no buying region at all while the selling strategy is selling all in stock at a super high price.
2. Keep moving the selling boundary down to lower price until there appears a region that we can make a profit by buying.

After the buying region is generated, it is quite natural to move the selling and buying boundary alternatively because the improvement of one has a positive effect on the other. Therefore the last step is

3. Move selling and buying boundaries alternatively until converges.

The first two procedures are stage 1 while the last is stage 2. When we say move selling (or buying) boundary, it means that for each q, the price that generating the maximum selling (or buying) profit is moved to. In order to prove this method works, two things need to be done.

1. Each time the boundary moves, the value function increases. Mathematically speaking, $V^{n+1} > V^n$.
2. The moving conditions are always satisfied. This is to say, we can keep moving the boundary. Mathematically speaking, at both stages, $\left(-V_q^{n+1}(x,q) + (e^x - \mu(q))\right)_x < 0$ at $n+1$ th selling boundary. At second stages, $\left(V_q^{n+1}(x,q) - (e^x + \lambda(q))\right)_x > 0$ at $n+1$ th buying boundary.

5 Each time the boundary moves, the value function increases.

Only consider the case that selling boundary is moved, because the other part is the same.

Denote $V^k(x,q)$ as the value function after nth moving boundary procedure and S^k, B^k and H^k are the buying, selling and holding region separately.

Assume at step $n+1$, the selling boundary is moved. This implies that $S^n \subset S^{n+1}$, $B^n = B^{n+1}$ and $H^n \supset H^{n+1}$ as shown in the figure.

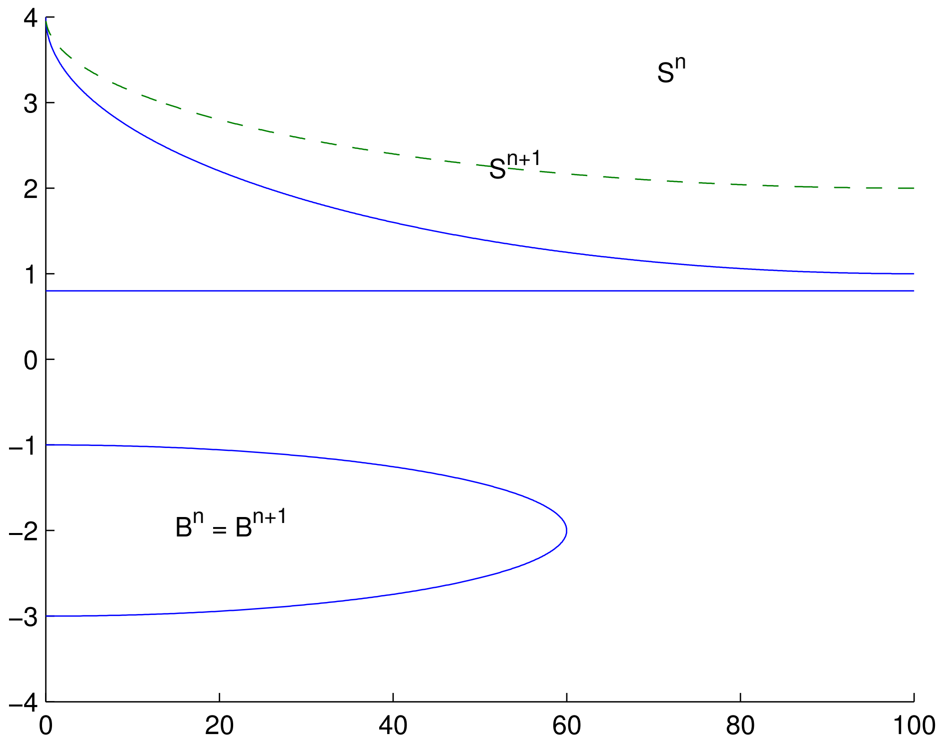

Figure 1: Regions of S^n, S^{n+1} and $B^n = B^{n+1}$

Define $\Delta V^{n+1}(x,q) = V^{n+1}(x,q) - V^n(x,q)$. According to the conditions under which the selling boundary is moved, we have

$$\begin{aligned}
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in S^n \\
\Delta V_q^{n+1}(x,q) &> 0 \quad (x,q) \in S^{n+1}/S^n \\
\mathcal{L}\Delta V^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1} \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in B^n = B^{n+1}
\end{aligned} \tag{9}$$

Since H^{n+1} doesn't have isolated vertical segment except $V_1 = \{(x,0) \in H^{n+1}\}$ and $V_2 = \{(x,100) \in H^{n+1}\}$, we can take derivative with respect to

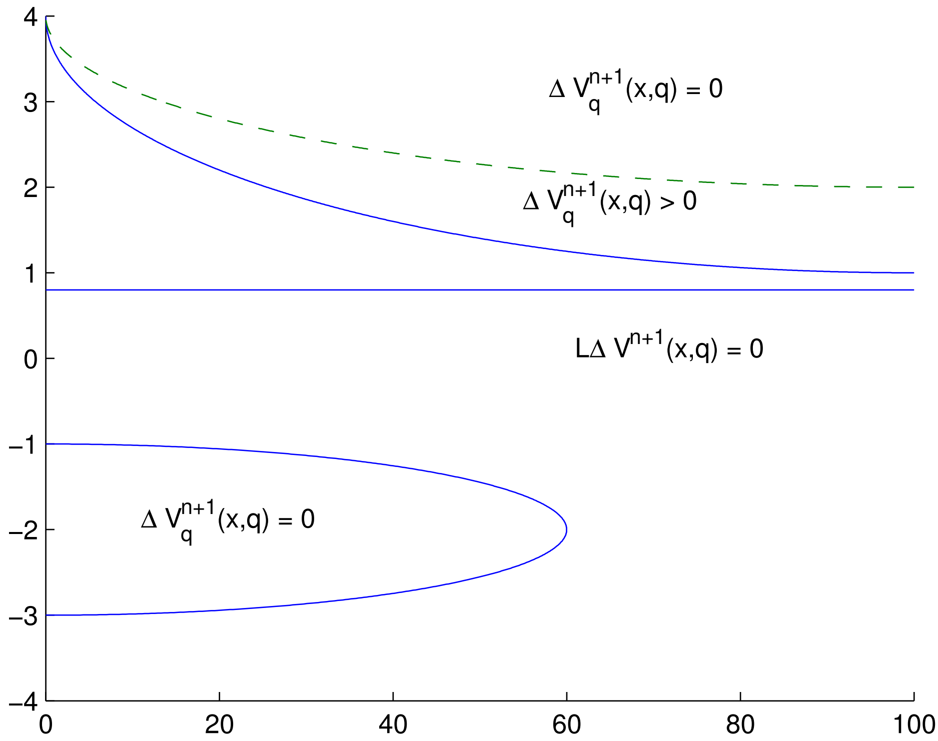

Figure 2: The equations that ΔV^{n+1} satisfies.

q to the third equation.

$$
\begin{aligned}
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in S^n \\
\Delta V_q^{n+1}(x,q) &> 0 \quad (x,q) \in S^{n+1}/S^n \\
\mathcal{L}\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1}/(V_1 \cup V_2) \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in B^n = B^{n+1}
\end{aligned}
\tag{10}
$$

Before moving any further, two properties of operator $\mathcal{L}$ are needed.

1. It's impossible to have positive maximal interior point.
2. It's impossible to have negative minimum interior point.

Proof. Only the first one is proved, the second is the same.

If not, there exists x which is the positive maximal interior point. Then $f'(x) = 0$ and $f''(x) \leq 0$. On the other hand, $\mathcal{L}f(x) = 0$. By the definition of $\mathcal{L}$,

$$\begin{aligned} &\frac{1}{2}\sigma^2 f'' + \kappa(\alpha - x)f' - \beta f = 0 \\ \Rightarrow &0 < \beta f = \frac{1}{2}\sigma^2 f'' + \kappa(\alpha - x)f' \leq 0 \end{aligned} \tag{11}$$

Contradiction!

□

Here begins the proof of $\Delta V^{n+1}(x,0) \geq 0$.

Proof. Notice that with q fixed, $f(x) = \Delta V_q^{n+1}(x,q)$ is the solution of $\mathcal{L}f = 0$. Moreover, the boundary conditions are $f(x_S) > 0$, where $(x_S, q) \in S^{n+1}/S^n$ and $f(x_B) = 0$, where x_B is the largest x satisfying $(x,q) \in B^n = B^{n+1}$.

Therefore, by the two properties of $\mathcal{L}$, $f(x)$ is non-negative for all points $(x,q) \in H^{n+1}$ and it achieves its maximum at point x_S. By the fact that q is arbitrary,

$$\Delta V_q^{n+1}(x,q) \geq 0 \quad (x,q) \in H^{n+1}/(V_1 \cup V_2). \tag{12}$$

Together with (10),

$$\Delta V_q^{n+1}(x,q) \geq 0 \quad (x,q) \in \mathbb{R} \times [0,100]/(V_1 \cup V_2). \tag{13}$$

There are two cases when the selling boundary moves.

1. States of $q = 0$ is isolated.
2. States of $q = 0$ is not isolated.

In the first case, $\mathcal{L}\Delta V^{n+1}(x,0) = 0 \quad \forall x \in \mathbb{R}$. By the fact that both V^{n+1} and V^n are bounded, $\Delta V^{n+1}(x,0)$ is also bounded. Combine these two we have $\Delta V^{n+1}(x,0) = 0 \quad \forall x \in \mathbb{R}$. What's more, (13) shows that

$$\Delta V^{n+1}(x,q) = \Delta V^{n+1}(x,0) + \int_0^q \Delta V_q^{n+1}(x,\tau)d\tau \geq 0 + 0 = 0. \tag{14}$$

In the second case, we still want to show that $\mathcal{L}\Delta V^{n+1}(x,0) = 0 \quad \forall x \in \mathbb{R}$. It is trivial when $(x,0) \in H^{n+1}$. To prove the case $(x,0) \notin H^{n+1}$, another assumption is needed.

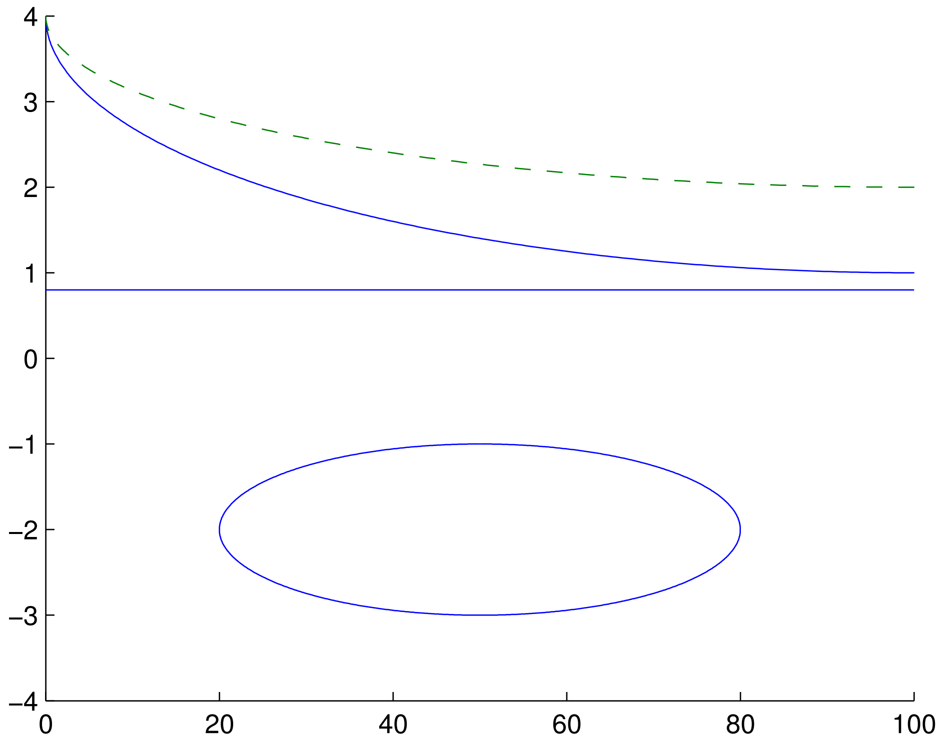

Figure 3: When states of q = 0 is isolated when selling boundary moves.

At a fixed level x, at most one of buying and selling happens which is the same as

$$\begin{aligned} (x,0) \notin H^{n+1} &\Leftrightarrow \text{Buying happens at level } x \\ &\Leftrightarrow \text{(x,100) is not a selling point} \Leftrightarrow (x,100) \in H^{n+1}. \end{aligned} \tag{15}$$

Because for the optimal policy, it is proved at most one of buying and selling happens and both buying strategy and selling strategy is less aggressive than the optimal one. This assumption is reasonable. In the case $(x,0) \notin H^{n+1}$,

$$\Delta V^{n+1}(x,0) = \Delta V^{n+1}(x,100) - \int_0^{100} \Delta V_q^{n+1}(x,q)dq. \tag{16}$$

There are only two situations for $\{(x,q)|q \in (0,100)\}$,

1. $(x,q) \in B^{n+1} \Rightarrow \Delta V_q^{n+1}(x,q) = 0$

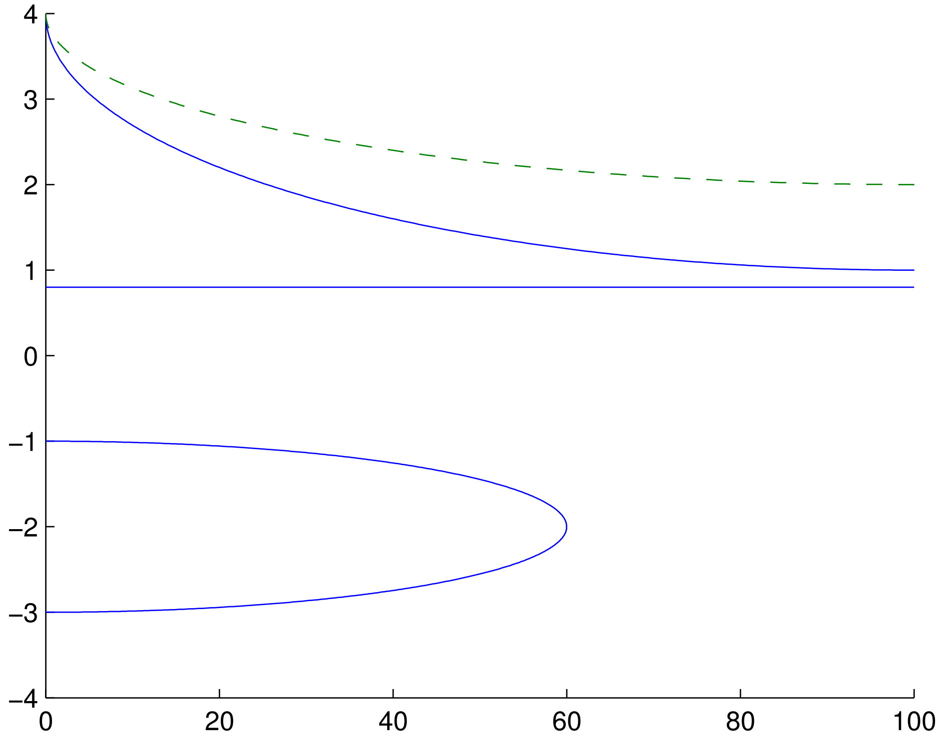

Figure 4: When states q = 0 is not isolated when selling boundary moves.

2. $(x, q) \in H^{n+1} \Rightarrow \mathcal{L}\Delta V_q^{n+1}(x, q) = 0.$

Note that $f(x) = 0$ is the solution to $\mathcal{L}f(x) = 0$. No matter which case, we always have $\mathcal{L}\Delta V_q^{n+1}(x, q) = 0$. Use operator $\mathcal{L}$ to both sides of (16),

$$\begin{aligned} \mathcal{L}\Delta V^{n+1}(x, 0) &= \mathcal{L}\Delta V^{n+1}(x, 100) - \mathcal{L}\int_0^{100} \Delta V_q^{n+1}(x, q)dq \\ &= 0 - \int_0^{100} \mathcal{L}\Delta V_q^{n+1}(x, q)dq = 0. \end{aligned} \tag{17}$$

In sum, $\mathcal{L}\Delta V^{n+1}(x, 0) = 0 \quad \forall x \in \mathbb{R}$. By previous argument, $\Delta V^{n+1}(x, q) \geq 0$.

$\square$

6 The moving conditions are always satisfied. This is to say, we can keep moving the boundary.

Proof. Still only the statement for selling boundary is proved.

1. In stage 1, the selling boundary is moved.

2. In stage 2, the buying boundary is moved.

In the first case, from previous proof, with q fixed, $f(x) = \Delta V_q^{n+1}(x,q)$ achieves its maximum at point x_S where $(x_S, q) \in S^{n+1}/S^n$. This is equivalent to $f'(x_S) > 0$.[2] Thus,

$$\begin{aligned}
&(\Delta V_q^{n+1})_x(x_S, q) = f'(x_S) > 0 \\
\Rightarrow &(V_q^{n+1})_x(x_S, q) - (V_q^n)_x(x_S, q) > 0 \\
\Rightarrow &(V_q^{n+1})_x(x_S, q) - e^{x_S} > 0 \\
\Rightarrow &\left(V_q^{n+1}(x,q) - (e^x - \mu(q)\right)_x |_{x=x_S} > 0 \\
\Rightarrow &\left(-V_q^{n+1}(x,q) + (e^x - \mu(q)\right)_x |_{x=x_S} < 0.
\end{aligned} \tag{18}$$

Noticing that the only thing used here is that x_S is the maximum. In the second case, it can be proved similarly that x_S is also the maximum. See the figure (6) for the idea.

Therefore we prove that we can continue moving the boundary.

□

7 The optimal policy will sell or buy(I need to prove that it is only sell that is possible) when the price is high enough. Namely, holding region always has a higher bound.

In this section, we would like to prove that it is impossible to hold when the price is high enough. If not, $\exists M_0 > \alpha, \tilde{q}$ such that for any $x > M_0$,

$$\begin{aligned}
V_q(x, \tilde{q}) &\le e^x + \lambda \\
\mathcal{L}V(x, \tilde{q}) &= 0 \\
V_q(x, \tilde{q}) &\ge e^x - \mu.
\end{aligned}$$

[2] If $f'(x_s) = 0$, by $\mathcal{L}f(x_s) = 0$, $0 < \beta f = \frac{1}{2}\sigma^2 f''(x) \le 0$. Contradiction!

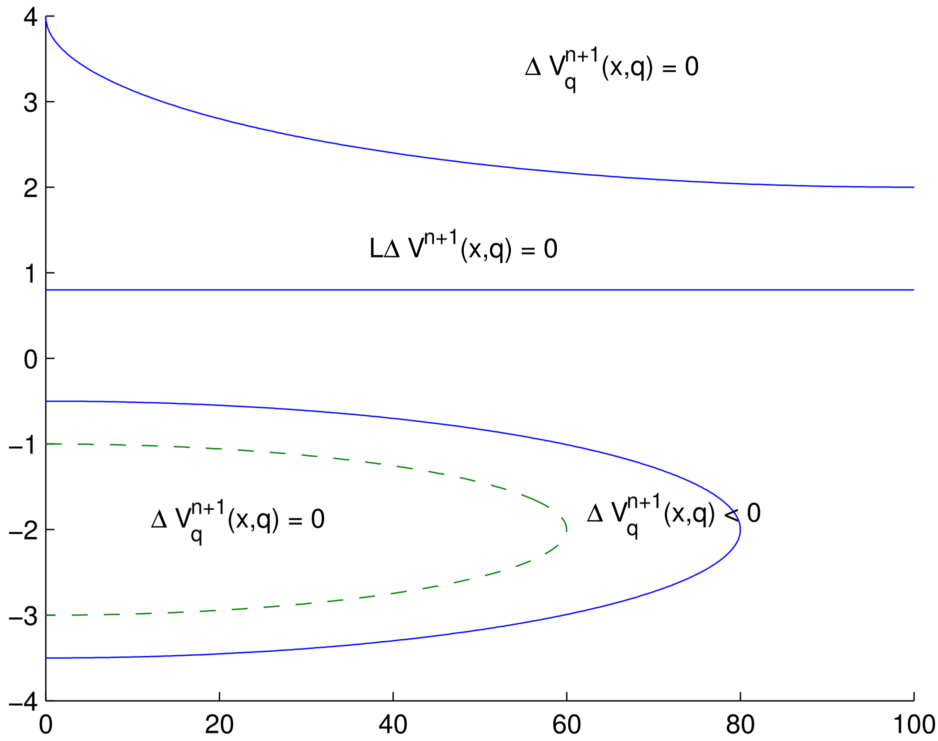

Figure 5: The equations that ΔV^{n+1} satisfies in the second stage.

Take derivative with respect to q to both sides of second equality,

$$\begin{aligned} V_q(x,\tilde{q}) &\leq e^x + \lambda \\ \mathcal{L}V_q(x,\tilde{q}) &= 0 \\ V_q(x,\tilde{q}) &\geq e^x - \mu. \end{aligned}$$

Define

$$h(x) = V_q(x,\tilde{q}) - (e^x - \mu). \tag{19}$$

Therefore, $\forall x > M_0$

$$0 \leq h(x) \leq \lambda + \mu. \tag{20}$$

Substitute (19) into $\mathcal{L}V_q(x,\tilde{q}) = 0$,

$$\frac{1}{2}\sigma^2 h''(x) + k(\alpha - x)h'(x) - \beta h = -\frac{1}{2}\sigma^2 e^x + k(x-\alpha)e^x + \beta(e^x - \mu). \tag{21}$$

First, we would like to prove that $h'(x)$ must change sign on $[M_0, \infty)$. If not, assume $h'(x) \geq 0 \; \forall x > M_0$, then by (20) and (21)

$$\frac{1}{2}\sigma^2 h''(x) \geq -\frac{1}{2}\sigma^2 e^x + k(x-\alpha)e^x + \beta(e^x - \mu). \tag{22}$$

However, by Fubini's theorem, $\forall N > M_0$

$$\begin{aligned} h(N) - h(M_0) &= \int_{M_0}^{N} h'(t)dt = \int_{M_0}^{N} (\int_{M_0}^{t} h''(x)dx + h'(M_0))dt \\ &= \int_{M_0}^{N}\int_{M_0}^{t} h''(x)dxdt + (N - M_0)h'(M_0) \\ &= \int_{M_0}^{N}\int_{x}^{N} h''(x)dtdx + (N - M_0)h'(M_0) \\ &= \int_{M_0}^{N} (N-x)h''(x)dx + (N - M_0)h'(M_0). \end{aligned} \tag{23}$$

Substitute (22) and $h'(M_0) \geq 0$ into (23)

$$\begin{aligned} h(N) - h(M_0) &= \int_{M_0}^{N} (N-x)h''(x)dx + (N - M_0)h'(M_0) \\ &\geq \int_{M_0}^{N} (N-x)(-e^x + \frac{2k}{\sigma^2}(x-\alpha)e^x + \frac{2\beta}{\sigma^2}(e^x - \mu))dx \end{aligned}$$

which shows that $h(N) - h(M_0) \to +\infty$ when $N \to +\infty$. However via (20),

$$-(\lambda + \mu) \leq h(N) - h(M_0) \leq (\lambda + \mu) \tag{24}$$

holds when $\forall N > M_0$. Contradiction!
Assume that $h'(x) < 0 \; \forall x > M_0$. Define

$$g(x) = -\frac{1}{2}\sigma^2 e^x + k(x-\alpha)e^x + \beta(e^x - \mu) - k(x - \alpha).$$

Then we have

$$g'(x) = -\frac{1}{2}\sigma^2 e^x + k(x - \alpha + 1)e^x + \beta e^x - k.$$

Noticing that both $g(x) \to +\infty$ and $g'(x) \to +\infty$ when $x \to +\infty$, there exists a $M_1 > M_0$ such that $g(M_1) > 0$ and $g(x)$ is increasing on $[M_1, +\infty)$. On the other hand, combining (24) and

$$h(N) - h(M_1) = \int_{M_1}^{N} h'(t)dt,$$

there must exist a $M_2 > M_1$ which satisfies that $h'(M_2) > -1$. By (21)

$$\frac{1}{2}\sigma^2 h''(M_2) \geq -\frac{1}{2}\sigma^2 e^{M_2} + k(M_2 - \alpha)e^{M_2} + \beta(e^{M_2} - \mu) - k(M_2 - \alpha) = g(M_2) > 0$$

namely

$$h''(M_2) > 0.$$

Let $\tilde{x} = \inf\{x > M_2 | h'(x) = -1\}$. If $\tilde{x} < +\infty$ then $h''(\tilde{x}) \leq 0$. However, (21) at point $\tilde{x}$ shows us

$$\begin{aligned}
\frac{1}{2}\sigma^2 h''(\tilde{x}) &= -\frac{1}{2}\sigma^2 e^{\tilde{x}} + k(\tilde{x}-\alpha)e^{\tilde{x}} + \beta(e^{\tilde{x}} - \mu) + k(\tilde{x}-\alpha)h'(\tilde{x}) + \beta h(\tilde{x}) \\
&\geq -\frac{1}{2}\sigma^2 e^{\tilde{x}} + k(\tilde{x}-\alpha)e^{\tilde{x}} + \beta(e^{\tilde{x}} - \mu) - k(\tilde{x}-\alpha) = g(\tilde{x}).
\end{aligned}$$

Since $g(x)$ is increasing on $[M_1, +\infty)$, $g(\tilde{x}) \geq g(M_1) > 0$.

$$\frac{1}{2}\sigma^2 h''(\tilde{x}) = g(\tilde{x}) > 0.$$

Contradiction! Therefore $h'(x)$ must change signs infinite times since we can replace M_0 with a sequence whose limit is $+\infty$. Let M_3 be the point that $h'(x)$ changes from positive to negative. M_3 can be arbitrarily large and $h'(M_3) = 0$, $h''(M_3) \leq 0$. However, at point M_3 there is no chance that (21) holds when M_3 is large enough. Contradiction! We prove that we must inject or withdraw at some prices. Section 1 shows that injection is impossible here, so we must withdraw at some prices.

8 At optimal, buy and sell won't occur at the same price level.

Assume $V_{xxq}(x, q)$ **exists in the holding region and** V_{xq} **is continuous with respect to** q **at the buying and selling boundaries.**

Proof. Assume (x_0, q_0) is at the buying boundary and holding region is above it.

$$\begin{aligned}
V_q(x_0, q_0) &= e^{x_0} + \lambda(q_0) \\
V_{qx}(x_0, q_0) &= (e^x + \lambda(q_0))_x|_{x=x_0} = e^{x_0} \\
\mathcal{L}V_q(x_0+, q_0) &= 0
\end{aligned} \tag{25}$$

Define

$$f(x) = V_q(x, q_0) - (e^x + \lambda(q_0)).$$

We have

$$
\begin{aligned}
f(x_0) &= V_q(x_0,q_0) - (e^{x_0} + \lambda(q_0)) = 0 \\
f'(x_0) &= V_{qx}(x_0,q_0) - e^{x_0} = 0
\end{aligned} \tag{26}
$$

and

$$
\mathcal{L}f(x_0+) = \mathcal{L}V_q(x_0+,q_0) - \mathcal{L}(e^x + \lambda(q_0))|_{x=x_0} = -\mathcal{L}(e^x + \lambda(q_0))|_{x=x_0}. \tag{27}
$$

On the other hand, for any point (x,q) in the holding region

$$
e^x - \mu(q) < V_q(x,q) < e^x + \lambda(q).
$$

and at (x_0,q_0) we have $V_q(x_0,q_0) = e^{x_0} + \lambda(q_0)$. These mean that $f(x)$ achieves its maximum at point x_0. Combined with (26), $f''(x_0-) \le 0$. Thus

$$
\mathcal{L}f(x_0+) = \frac{1}{2}\sigma^2 f''(x_0+) + k(\alpha - x_0)f'(x_0) - \beta f(x_0) = \frac{1}{2}\sigma^2 f''(x_0) \le 0.
$$

Then (27) implies that

$$
\begin{aligned}
-\mathcal{L}(e^{x_0} + \lambda(q_0)) &\le 0 \\
-(\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))) &\le 0.
\end{aligned}
$$

namely

$$
\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))) \ge 0. \tag{28}
$$

Noticing that above inequality holds no matter the holding boundary is below or above (x_0,q_0).

Similarly, **if** (x_0,q_1) **is at the selling boundary,** we have

$$
\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} - \mu(q_1))) \le 0. \tag{29}
$$

Since

$$
\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} - \mu(q_1))) > \frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0)))
$$

holds for any $q_1, q_2 \in [Q_{\min}, Q_{\max}]$, it is impossible for (28) and (29) hold at the same time. This is the same as saying that buy and sell won't occur at the same x_0.

□

9 When futures are redundant?

Let $F_{t,T}$ denote the price at time t of future whose maturity is T. Q, P are risk-neutral and historical measure respectively. If the prices are consistent with underlying asset price $exp(X_t)$,

$$\exp(-\beta t)F_{t,T} = E_Q(\exp(-\beta T)\exp(X_T)|\mathcal{F}_t)$$

The discounted profit of selling one unit commodity via this future is

$$\begin{aligned}
&\exp(-\beta t)F_{t,T} - \exp(-\beta T)\int_Q^{Q+1}\mu(q)dq \\
=&E_Q(\exp(-\beta T)\exp(X_T)|\mathcal{F}_t) - \exp(-\beta T)\int_Q^{Q+1}\mu(q)dq \\
=&E_Q(\exp(-\beta T)\exp(X_T) - \exp(-\beta T)\int_Q^{Q+1}\mu(q)dq|\mathcal{F}_t). \\
=&E_Q(e^{-\beta T}(e^{X_T} - \mu(Q_T^1)|\mathcal{F}_t).
\end{aligned}$$

where $\mu(Q_T^1) = \int_Q^{Q+1}\mu(q)dq$.

Similarly total discounted cost of one unit commodity bought via this future is

$$\begin{aligned}
&\exp(-\beta t)F_{t,T} + \exp(-\beta T)\int_Q^{Q+1}\lambda(q)dq \\
=&E_Q(\exp(-\beta T)\exp(X_T)|\mathcal{F}_t) + \exp(-\beta T)\int_Q^{Q+1}\lambda(q)dq \\
=&E_Q(\exp(-\beta T)\exp(X_T) + \exp(-\beta T)\int_Q^{Q+1}\lambda(q)dq|\mathcal{F}_t) \\
=&E_Q(e^{-\beta T}(e^{X_T} + \lambda(Q_T^2)|\mathcal{F}_t).
\end{aligned}$$

where $\lambda(Q_T^2) = \int_Q^{Q+1}\lambda(q)dq)$.

Recall the objective function without futures.

$$V(x,q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_P\left(\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t^1))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t^2))dL_t\right) \tag{30}$$

Notice, this expectation is under historical measure. If $P = Q$, the future is redundant. If $P \neq Q$, it is not.