1 Introduction

Valuing storage is a problem of significant interest in recent years, especially for energy-related commodities. Without storage, in environment with either relatively stable supply and fluctuating demand, or relatively stable demand and fluctuating supply, prices vary significantly over time. This price variation generates an incentive to shift supply from a period where it is in excess, to a period where it is in shortage. Storage derives its economic value from exploiting these predictable price fluctuations by shifting supply over time.

Overview of storage valuation literature.

In this paper, we study the problem of valuing storage specifically for the case of energy commodities. We are able to develop semi-analytical framework to price storage, allowing for mean-reverting price dynamics for the commodity, and general injection and withdrawal costs. Our framework is a generalization of the moving boundary method, described in [].

Overview of moving boundary method.

We apply our framework to study the value and optimal injection and withdrawal strategies for a calibrated example of crude oil/natural gas storage facility.

2 Model

Let S_t be the price and its evolution follows

$$dS_t = \kappa(\gamma - \log(S_t))S_t dt + \sigma S_t dZ_t. \tag{1}$$

Take $X_t = \log(S_t)$, by Ito's formula,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dZ_t, \tag{2}$$

where $\alpha = \gamma - \frac{\sigma^2}{2\kappa}$.

Denote the total amount of the commodity in the storage facility at time t as Q_t. Also, denote cumulative amounts of commodity purchased and sold up to time t as L_t and U_t, respectively. Here and after, buy and injection have the same meaning, so do sell and withdrawal. By definition, L_t and U_t are non-negative monotone cadlag processes, and

$$Q_t = Q_0 + L_t - U_t. \tag{3}$$

Implied by (3), Q_t is of bounded variation.

The decisions are when and how much shall be injected and withdrawn from the facility. In other words, every decision can be represented as a pair (L_t, U_t). An admissible control policy (L_t, U_t) must satisfy that the storage level at any time is within capabilities. That is to say, $Q_t \in [Q_{\min}, Q_{\max}]$ for all t where $Q_{\min}$ and $Q_{\max}$ are the upper and lower capability respectively. We use $\mathcal{U}$ to denote all admissible policies.

Both injection and withdrawal generate costs. As a result, injection and withdrawal won't take place at the same time. Let $\lambda(q)$ and $\mu(q)$ be the instantaneous costs of injection and withdrawal when the storage has q unit. Therefore, the costs of injection and withdrawal at time t should be $\int_{Q_{t-}}^{Q_t} \lambda(q)dq$ and $\int_{Q_t}^{Q_{t-}} \mu(q)dq$ respectfully. It is often easier to inject (withdraw) when empty (full) than full (empty). Therefore we assume that $\lambda(q)$ is increasing with respect to q while $\mu(q)$ is decreasing. To be economic meaningful, we also assume that $\lambda(q)$ and $\mu(q)$ are bounded.

The objective is to maximize discounted infinite-horizon discounted cash flows. Taking a discount factor $\beta \in (0, 1)$,

$$V(x,q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_{x,q} \left(\int_0^\infty e^{-\beta t}(e^{X_t} - M_t)dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \Lambda_t)dL_t \right) \tag{4}$$

where $X_0 = x$ and $Q_0 = q$. We use Λ_t and M_t to represent the injection and withdrawal cost at time t. By the definition of λ and μ, we have the following relations.

$$\Lambda_t = \begin{cases} \lambda(Q_t) & \text{if } L_t = L_{t-} \\ \frac{1}{Q_t - Q_{t-}} \int_{Q_{t-}}^{Q_t} \lambda(q)dq & \text{otherwise.} \end{cases} \tag{5}$$

$$M_t = \begin{cases} \mu(Q_t) & \text{if } U_t = U_{t-} \\ \frac{1}{Q_{t-} - Q_t} \int_{Q_t}^{Q_{t-}} \mu(q)dq & \text{otherwise.} \end{cases} \tag{6}$$

Because $e^{-\beta t}(e^{X_t} - M_t)$ $(e^{-\beta t}(e^{X_t} + \Lambda_t))$

In other words, the maximum value that a facility manager can obtain from a storage facility when the current spot log price is x and the amount is q is $V(x, q)$.

3 No Transaction Costs

No transaction cost means that λ (buying cost) $\equiv 0$ and μ (selling cost) $\equiv 0$. Since the profit (cost) of selling (buying) one unit at the same price level is the same, the optimal strategy should be bang-bang. Whenever the drift of

X_t exceeds interest rate, we should buy. Vice versa. Let this critical point be X^*, and then

$$\kappa(\alpha - X^*) = r \Rightarrow X^* = \alpha - \frac{r}{\kappa}$$

The optimal strategy is as following

- If $X_t > X^*$, sell to empty.
- If $X_t \leq X^*$, buy to full.

4 Constant Transaction Costs

Let the constant buying (selling) costs be λ (μ). Since the cost (profit) of buying (selling) one unit at the same price level is the same, the optimal strategy should still be bang-bang. Introduce the following two functions of log price x,

$$\begin{aligned} V(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(e^{X_\tau} - \mu)\} \\ J(x) &= \sup_{\nu \in \mathcal{T}} \mathbb{E}_x\{e^{-r\nu}(V(x^\nu) - e^{X_\nu} - \lambda)\}. \end{aligned}$$

$V(x)$ is the maximum value that can be gained from selling 1 unit commodity when log price is x. $J(x)$ is the maximal profit of buying and then selling 1 unit commodity when log price is x. Tim Leung et al.(2014) use concavity to prove that the optimal policy for $V(x)$ is an upper threshold policy. In other words, if log price exceeds the threshold d^*, then sell. Using similar idea, they also prove the optimal policy for $J(x)$ is an upper-lower threshold policy. That is to say, one will buy if and only if log price is within two thresholds, namely [a^*,b^*]. Based on the existence and conditions of a^* and b^*, there are two cases.

1. $J(x) = 0$. In this case, it is not optimal enter the market at all.
2. $-\infty < a^* < b^* < d^* < \infty$. This means that commodity is bought when log price is within $[a^*, b^*]$ and will be sold when log price achieves d^*.

5 Non-Constant Transaction Costs

The stochastic control problem with non-constant transaction costs can be transformed into a Hamilton Jacobi Bellman (HJB) equation via dynamic programming principle and Ito's formula. The HJB equation characterize the value function $V(x, q)$ (**??**).

We will use V_x and V_q separately to denote the partial differential of V with respect to x and q. V_{xx} represents $\frac{\partial^2 V}{(\partial x)^2}$. The value function $V(x,q)$ is expected to satisfy the HJB equation

$$\max\left[\mathcal{L}V(x,q), \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0. \tag{7}$$

where

$$\begin{aligned}
\mathcal{L}V(x,q) &= \frac{1}{2}\sigma^2 V_{xx}(x,q) + k(\alpha - x)V_x(x,q) - \beta V(x,q), \\
\mathcal{S}V(x,q) &= -V_q(x,q) + e^x - \mu(q), \\
\mathcal{B}V(x,q) &= V_q(x,q) - e^x - \lambda(q).
\end{aligned} \tag{8}$$

Because the three terms in equation (7) are the profits utilizing holding, selling, and buying policy respectively, they are called holding, selling, and buying profit.

The region where the holding profit is the highest is called the optimal holding region and denoted as H^*. Similarly, the optimal selling (buying) region is defined and denoted as S^* (B^*). Mathematically speaking,

$$\begin{aligned}
\mathcal{L}V(x,q) &= \max\left[\mathcal{L}V(x,q), \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0 \quad (x,q) \in H^* \\
\mathcal{S}V(x,q) &= \max\left[\mathcal{L}V(x,q),, \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0 \quad (x,q) \in S^* \\
\mathcal{B}V(x,q) &= \max\left[\mathcal{L}V(x,q),, \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0 \quad (x,q) \in B^*
\end{aligned}$$

We want to have a sequence of regions, (H^n, S^n, B^n), to approach the optimal regiones (H^*, S^*, B^*). The fixed boundary problem (H^n, S^n, B^n) is defined as a system of equations which states that holding (selling, buying) profit is 0 in H^n (S^n, B^n). Mathematically speaking,

$$\begin{aligned}
\mathcal{L}V^n(x,q) &= 0 \quad (x,q) \in H^n \\
\mathcal{S}V^n(x,q) &= 0 \quad (x,q) \in S^n \\
\mathcal{B}V^n(x,q) &= 0 \quad (x,q) \in B^n.
\end{aligned} \tag{9}$$

Here the solution to the fixed boundary problem is denoted as $V^n(x,q)$.

5.1 Algorithm

1. Find a large enough number M_0 such that the optimal selling region contains the inititial selling region $S^0 = \{(x,q) | x \geq M_0,\ Q_{\min} < q \leq Q_{\max}\}$. Set holding region $H^0 = (S^0)^c$ and buying region $B^0 = \emptyset$.

2. Keep doing the following steps until convergence.

   (a) Calculate the value function $V^n(x,q)$.

(b) If the maximal selling profit is positive, $max_x \mathcal{S}V^n(x,q) > 0$, then set the selling region $S^{n+1} = \{(x,q)|x \geq x_s^{n+1}(q)\}$ where

$$x_s^{n+1}(q) = \arg\max_x \mathcal{S}V^n(x,q)$$

(c) If the maximal buying profit is positive, $max_x \mathcal{B}V^n(x,q) > 0$, then set the buying region $B^{n+1} = \{(x,q)|x_l^{n+1}(q) \leq x \leq x_u^{n+1}(q)\}$. There are two possibilities.

i. Buying region is empty at iteration n.

$$x_l^{n+1}(q) = x_u^{n+1}(q) = \arg\max_x \mathcal{B}V^n(x,q).$$

ii. Buying region is not empty at iteration n.

$$x_u^{n+1}(q) = \arg\max_{x \geq x_u^n(q)} \mathcal{B}V^{n+1}(x,q),$$
$$x_l^{n+1}(q) = \arg\max_{x \leq x_l^n(q)} \mathcal{B}V^{n+1}(x,q).$$

(d) Set the holding region as $H^{n+1} = (S^{n+1} \cup B^{n+1})^c$.

(e) $n = n + 1$.

In order to show this algorithm works, four things need to be proved.

1. The existence of M_0.
2. V^n is monotone increasing.
3. The boundaries can keep moving.
4. The convergence function is the optimal.

5.2 The existence of M_0

Lemma 1. *For any positive storage level, $\tilde{q} > 0$, there exists a number $M_0(\tilde{q})$ such that region $\{(x,q)|x \geq M_0(\tilde{q}), q = \tilde{q}\}$ is contained by the optimal selling region S^*.*

Proof. First, in order to make money, for any buying price, there must be a higher selling price. As a result, the optimal buying region won't contain a region with infinite upper bound. Thus if lemma 1 doesn't hold, it is the optimal holding region H^* that the region $\{(x,\tilde{q})|x > M_0(\tilde{q})\}$ belongs to. In other words, for any log price x larger than the number $M_0(\tilde{q})$,

$$\begin{aligned} &\mathcal{L}V(x,\tilde{q}) = 0 \\ &\mathcal{S}V(x,\tilde{q}) \leq 0. \\ &\mathcal{B}V(x,\tilde{q}) \leq 0 \end{aligned} \tag{10}$$

Define

$$h(x) = -\mathcal{S}V(x,\tilde{q}) = V_q(x,\tilde{q}) - (e^x - \mu(\tilde{q})). \tag{11}$$

Because $-\mathcal{S}V(x,q) = \mathcal{B}V(x,q) + (\lambda(q) + \mu(q))$, together with the last two inequalities of (10), we have

$$0 \le h(x) \le \lambda(\tilde{q}) + \mu(\tilde{q}) \quad \forall x > M_0(\tilde{q}). \tag{12}$$

On the other hand, take derivative with respect to q to both sides of first equality of (10) and we have $\mathcal{L}V_q(x,\tilde{q}) = 0$. Substitute (11) into it to have

$$\frac{1}{2}\sigma^2 h''(x) + k(\alpha - x)h'(x) - \beta h(x) = -\frac{1}{2}\sigma^2 e^x + k(x-\alpha)e^x + \beta(e^x - \mu(\tilde{q})).$$

Divide both sides by e^x,

$$\frac{1}{2}\sigma^2 e^{-x}h''(x) + ke^{-x}(\alpha - x)h'(x) - \beta e^{-x}h(x) = -\frac{1}{2}\sigma^2 + k(x-\alpha) + \beta(1 - e^{-x}\mu(\tilde{q})). \tag{13}$$

Because the buying cost, $\mu(\tilde{q})$, is bounded, the right-hand side of (13) goes to the positive infinity when x approaches the positive infinity. Meanwhile, by (12), function $h(x)$ is bounded. Combine these two, when x approaches the positive infinity, either

1. $e^{-x}h''(x) \to +\infty$ or
2. $e^{-x}(\alpha - x)h'(x) \to +\infty$.

In first case, because $e^{-x}h''(x)$ approaches the positive infinity when x approaches positive infinity, there must exist a number t_1 such that $h''(x)$ is positive for any x larger than t_1. For any t_2 larger than t_1, by Fubini's theorem, we have

$$\begin{aligned}
h(t_2) - h(t_1) &= \int_{t_1}^{t_2} h'(t)dt = \int_{t_1}^{t_2} (\int_{t_1}^{t} h''(x)dx + h'(t_1))dt \\
&= \int_{t_1}^{t_2}\int_{t_1}^{t} h''(x)dxdt + (t_2 - t_1)h'(t_1) \\
&= \int_{t_1}^{t_2}\int_{x}^{t_2} h''(x)dtdx + (t_2 - t_1)h'(t_1) \\
&= \int_{t_1}^{t_2} (t_2 - x)h''(x)dx + (t_2 - t_1)h'(t_1) \\
&\ge e^{t_1}\int_{t_1}^{t_2} (t_2 - x)e^{-x}h''(x)dx + (t_2 - t_1)h'(t_1) \\
&= e^{t_1}\int_{t_1}^{t_2} \left((t_2 - x)e^{-x}h''(x) + e^{-t_1}h'(t_1)\right)dx.
\end{aligned} \tag{14}$$

With t_1 fixed, when $t_2 \to +\infty$, the right hand side approaches positive infinity. This is a contradiction with (12).
In the second case, because $e^{-x}(\alpha - x)h'(x)$ approaches the positive infinity when x approaches positive infinity, there must exist a number $t_1 > \alpha$ such that $(\alpha - x)h'(x)$ is positive for any x larger than t_1. On the other hand, $e^x/(\alpha - x)$ is a decreasing function when x is large enough. Without losing any generality, assume it is decreasing for all x that is bigger than t_1. For any t_2 larger than t_1, we have

$$\begin{aligned} h(t_2) - h(t_1) &= \int_{t_1}^{t_2} h'(t)dt \\ &= \int_{t_1}^{t_2} \frac{e^t}{\alpha - t} \frac{(\alpha - t)h'(t)}{e^t} dt \\ &\leq \frac{e^{t_1}}{\alpha - t_1} \int_{t_1}^{t_2} \frac{(\alpha - t)h'(t)}{e^t} dt. \end{aligned} \tag{15}$$

With t_1 fixed, when $t_2 \to +\infty$, the right hand side approaches negative infinity. This is also a contradiction with (12).

□

Because we use the discretization method to solve the HJB equation (7), only finite number of $M_0(\tilde{q})$ matter. The maximal of those is M_0. A proposed M_0 is large enough if it satisfies

$$(\mathcal{S}V(x,q))_x |_{x=M_0} < 0 \quad \forall q \in (Q_{\min}, Q_{\max}].$$

5.3 V^n is monotone increasing

Theorem 1. *The value function V^{n+1} with the region triplet $(H^{n+1}, S^{n+1}, B^{n+1})$ is larger than the value function V^n with the regions triplet (H^n, S^n, B^n).*

Proof. By equation (9) and the way the algorithm works, for V^n, we have

$$\begin{array}{ll} \mathcal{L}V^n(x,q) = 0 & (x,q) \in H^n \\ -V^n_q(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^n \\ -V^n_q(x,q) + e^x - \mu(q) > 0 & (x,q) \in S^{n+1}/S^n \\ V^n_q(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^n \\ V^n_q(x,q) - e^x - \lambda(q) > 0 & (x,q) \in B^{n+1}/B^n. \end{array} \tag{16}$$

For V^{n+1}, we have

$$\begin{array}{ll} \mathcal{L}V^{n+1}(x,q) = 0 & (x,q) \in H^{n+1} \\ -V^{n+1}_q(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^n \\ -V^{n+1}_q(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^{n+1}/S^n \\ V^{n+1}_q(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^n \\ V^{n+1}_q(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^{n+1}/B^n. \end{array} \tag{17}$$

The algorithm guarantees the monotonicity of H^n, S^n and B^n, that is to say, $H^n \supset H^{n+1}$, $S^n \subset S^{n+1}$ and $B^n \subset B^{n+1}$. Introduce $\Delta V^{n+1}(x,q) = V^{n+1}(x,q) - V^n(x,q)$. From (16) and (17), we can deduce

$$\begin{aligned}
\mathcal{L}\Delta V^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1} \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in S^n \\
\Delta V_q^{n+1}(x,q) &> 0 \quad (x,q) \in S^{n+1}/S^n \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in B^n \\
\Delta V_q^{n+1}(x,q) &< 0 \quad (x,q) \in B^{n+1}/B^n.
\end{aligned} \tag{18}$$

By Feynman-Kac formula, ΔV^{n+1} is the value function with regions triplet (H^n, S^n, B^n) and positive selling profit and negative buying cost [1]. As a result, for every regions triplet, the value function is positive, namely $\Delta V^{n+1} > 0$.

□

5.4 The boundaries can keep moving

From the algorithm, the boundaries can move if and only if the following theorem holds.

Lemma 2. *If $\mathcal{L}f(x) = 0$, $l \le x \le u$ and $f(l) < 0, f(u) > 0$, $f(u)$ is the maximum and $f(l)$ is the minimum.*

Proof. Assume u is not the maximum, then f must achieve the maximum at an interior point, denoted as x_M. Thus $f(x_M) > f(u) > 0$, $f'(x_M) = 0$, and $f''(x_M) \le 0$. By $\mathcal{L}f(x_M) = 0$,

$$\begin{aligned}
&\frac{1}{2}\sigma^2 f''(x_M) + \kappa(\alpha - x_M)f'(x_M) - \beta f(x_M) = 0 \\
\Rightarrow\ & 0 < \beta f(x_M) = \frac{1}{2}\sigma^2 f''(x_M) + \kappa(\alpha - x_M)f'(x_M) \le 0
\end{aligned}$$

Contradiction! We can use the similar argument to prove $f(l)$ is the minimum. □

Theorem 2. *Three inequalities below are true for all integer n and storage level q.*

$$\begin{aligned}
(\mathcal{S}V(x,q))_x \,|_{x=x_s^{n+1}(q)} &< 0 \\
(\mathcal{B}V(x,q))_x \,|_{x=x_u^{n+1}(q)} &> 0 \\
(\mathcal{B}V(x,q))_x \,|_{x=x_l^{n+1}(q)} &< 0
\end{aligned}$$

[1]It means that one can increase the volume in the storage and make a profit at the same time.

When $x_u^{n+1}(q)$ and $x_l^{n+1}(q)$ don't exist, we assume the last two inequalities are true automatically.

Proof. Introduce function $f(x) = \Delta V_q^{n+1}(x,q)$ with any fixed storage level q. Differentiate (18)the first equation with respect to q, we have $\mathcal{L}f(x) = 0$, $x_u^{n+1}(q) \leq x \leq x_s^{n+1}(q)$. By Lemma (2), $f(x)$ achieves its maximum at point $x_s^{n+1}(q)$. Therefore $f'(x_s^{n+1}(q)) > 0$ or $f'(x_s^{n+1}(q)) = 0$. Because $\mathcal{L}f(x_s^{n+1}(q)) = 0$, if the latter is true, then we have $0 < \beta f(x_s^{n+1}(q)) = \frac{1}{2}\sigma^2 f''(x_s^{n+1}(q)) \leq 0$ which is a contradiction. As a result, we proved that $f'(x_s^{n+1}(q)) > 0$.

$$
\begin{aligned}
&(\Delta V_q^{n+1})_x(x_s^{n+1}(q),q) = f'(x_s^{n+1}(q)) > 0 \\
\Rightarrow &(V_q^{n+1})_x(x_s^{n+1}(q),q) - (V_q^{n})_x(x_s^{n+1}(q),q) > 0 \\
\Rightarrow &(V_q^{n+1})_x(x_s^{n+1}(q),q) - e^{x_s^{n+1}(q)} > 0 \\
\Rightarrow &\left(V_q^{n+1}(x,q) - (e^x - \mu(q))\right)_x \big|_{x=x_s^{n+1}(q)} > 0 \\
\Rightarrow &\left(-V_q^{n+1}(x,q) + (e^x - \mu(q))\right)_x \big|_{x=x_s^{n+1}(q)} < 0. \\
\Rightarrow &\left(\mathcal{S}V(x,q)\right)_x \big|_{x=x_s^{n+1}(q)} < 0
\end{aligned}
\tag{19}
$$

On the other hand, $f(x)$ achieves the minimum at $x_u^{n+1}(q)$ and $x_l^{n+1}(q)$. Using similar arguments as $x_s^{n+1}(q)$ to have

$$
\begin{aligned}
&\left(\mathcal{B}V(x,q)\right)_x \big|_{x=x_u^{n+1}(q)} > 0 \\
&\left(\mathcal{B}V(x,q)\right)_x \big|_{x=x_l^{n+1}(q)} < 0
\end{aligned}
$$

□

6 HJB equation and verification theorem

Theorem 3. *Suppose $f(x,q) \in C^{2,1}(\mathbb{R} \times \mathbb{R})$ and both f and f_x are bounded. If f satisfies*

$$
\max\left(\mathcal{L}f, \mathcal{B}f, \mathcal{S}f\right)(x,q) \leq 0, \quad (x,q) \in \mathbb{R}^2, \tag{20}
$$

we have

$$
f(x,q) \geq V(x,q),
$$

where $V(x,q)$ is defined in (4).

Proof. Because of (3) and the monotonicity of processes L_t and U_t, Q_t is of finite variation. Combined with the assumption that $f \in C^{2,1}(\mathbb{R} \times \mathbb{R})$, we are able to use Ito's formula from (Protter 2005) to have

$$\begin{aligned}
e^{-\beta t} f(X_t, Q_t) - f(x,q) = &\int_0^t e^{-\beta s} \mathcal{L} f(X_s, Q_{s-}) ds + \int_0^t e^{-\beta s} f_x(X_s, Q_{s-}) dW_s \\
&+ \int_0^t e^{-\beta s} f_q(X_s, Q_{s-}) dQ_s^c + \sum_{0 \le s \le t} \left(e^{-\beta s} f(X_s, Q_s) - e^{-\beta s} f(X_s, Q_{s-}) \right),
\end{aligned}$$

where $X_0 = x$, $Q_{0-} = q$ and $Q^c_\cdot$ is the continuous part of $Q_\cdot$. This equation is valid for arbitrary Q_t that satisfies (3). Because both f and f_x are bounded, we can take expectation to both sides and then take $t \to \infty$,

$$\begin{aligned}
f(x,q) = &- \mathbb{E} \int_0^\infty e^{-\beta t} \mathcal{L} f(X_t, Q_{t-}) dt - \mathbb{E} \int_0^\infty e^{-\beta t} f_q(X_t, Q_{t-}) dQ_t^c \\
&- \mathbb{E} \sum_{0 \le t < \infty} \left(e^{-\beta t} f(X_t, Q_t) - e^{-\beta s} f(X_t, Q_{t-}) \right).
\end{aligned}$$

Plug (3) into,

$$\begin{aligned}
f(x,q) = &-\mathbb{E} \int_0^\infty e^{-\beta t} \mathcal{L} f(X_t, Q_{t-}) dt \\
&- \mathbb{E} \int_0^\infty e^{-\beta t} f_q(X_t, Q_{t-}) dL_t^c + \mathbb{E} \int_0^\infty e^{-\beta t} f_q(X_t, Q_{t-}) dU_t^c \\
&- \mathbb{E} \sum_{0 \le t < \infty} \left(e^{-\beta t} f(X_t, Q_{t-} + \Delta L_t) - e^{-\beta t} f(X_t, Q_{t-}) \right) \\
&- \mathbb{E} \sum_{0 \le t < \infty} \left(e^{-\beta t} f(X_t, Q_{t-} - \Delta U_t) - e^{-\beta t} f(X_t, Q_{t-}) \right) \\
= &-\mathbb{E} \int_0^\infty e^{-\beta t} \mathcal{L} f(X_t, Q_{t-}) dt \\
&- \mathbb{E} \int_0^\infty e^{-\beta t} f_q(X_t, Q_{t-}) dL_t^c + \mathbb{E} \int_0^\infty e^{-\beta t} f_q(X_t, Q_{t-}) dU_t^c \\
&- \mathbb{E} \sum_{0 \le t < \infty} \left(e^{-\beta t} \int_{Q_{t-}}^{Q_{t-} + \Delta L_t} f_q(X_t, q) dq \right) \\
&+ \mathbb{E} \sum_{0 \le t < \infty} \left(e^{-\beta t} \int_{Q_{t-} - \Delta U_t}^{Q_{t-}} f_q(X_t, q) dq \right).
\end{aligned} \tag{21}$$

From (20), we have $\mathcal{L} f(x,q) \le 0$ and $e^x - \mu(q) \le f_q(x,q) \le e^x + \lambda(q)$ hold for all $(x, q) \in \mathbb{R}^2$. Substitute them into (21) to have

$$\begin{aligned}
f(x,q) \geq & -\mathbb{E}\int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_{t-}))dL_t^c + \mathbb{E}\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_{t-}))dU_t^c \\
& - \mathbb{E}\sum_{0\leq t<\infty}\left(e^{-\beta t}\int_{Q_{t-}}^{Q_{t-}+\Delta L_t}(e^x + \lambda(q))dq\right) \\
& + \mathbb{E}\sum_{0\leq t<\infty}\left(e^{-\beta t}\int_{Q_{t-}-\Delta U_t}^{Q_{t-}}(e^x - \mu(q))dq\right) \\
= & \, \mathbb{E}_{x,q}\left(\int_0^\infty e^{-\beta t}(e^{X_t} - M_t)dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \Lambda_t)dL_t\right).
\end{aligned}$$

Because this inequality is true for all admissible (L_t, U_t), by definition of $V(x, q)$, namely (4), we have

$$f(x,q) \geq V(x,q).$$

□

If $\mathbb{R}^2$ is divided into holding region, H, selling region, S, and buying region, B, we can derive a related strategy (L_t, U_t) following the rules that sell in S, buy in B, and hold in H. We call (H, S, B) a region triplet and we also use it to represent related strategy.

Theorem 4. *Let v be the solution to*

$$\begin{aligned}
\mathcal{L}v = 0 \quad & (x,q) \in H \\
\mathcal{S}v = 0 \quad & (x,q) \in S \\
\mathcal{B}v = 0 \quad & (x,q) \in B,
\end{aligned}$$

where (H, S, B) is a region triplet. If $v \in C^{2,1}(\mathbb{R} \times \mathbb{R}/\partial H)$ and both v and v_x are bounded, v is the value function w.r.t (H, S, B).

Proof. Let (L_t, U_t) be the respective strategy w.r.t (H, S, B) and we have

$$\begin{aligned}
&(X_t, Q_{t-}) \in H \quad \forall t > 0 \\
&(X_t, Q_{t-}) \in \partial B \Leftrightarrow dL_t^c \neq 0 \\
&(X_t, Q_{t-}) \in \partial S \Leftrightarrow dU_t^c \neq 0 \\
&(X_t, Q_{t-}) \in B^o \Leftrightarrow \Delta L_t \neq 0 \\
&(X_t, Q_{t-}) \in S^o \Leftrightarrow \Delta U_t \neq 0
\end{aligned}$$

By (21), we have

$$\begin{aligned}
v(x,q) = & -\mathbb{E}\int_0^\infty e^{-\beta t}(e^x + \lambda(q))dL_t^c + \mathbb{E}\int_0^\infty e^{-\beta t}(e^x - \mu(q))dU_t^c \\
& - \mathbb{E}\sum_{0\leq t<\infty} e^{-\beta t}\int_{Q_{t-}}^{Q_{t-}+\Delta L_t}(e^x + \lambda(q))dq + \mathbb{E}\sum_{0\leq t<\infty} e^{-\beta t}\int_{Q_{t-}}^{Q_{t-}+\Delta L_t}(e^x + \lambda(q))dq \\
= & \, \mathbb{E}\int_0^\infty e^{-\beta t}(e^x - M(q))dU_t - \mathbb{E}\int_0^\infty e^{-\beta t}(e^x + \Lambda(q))dL_t = V_{(L_t,U_t)}(x,q)
\end{aligned}$$

□

Theorem 5. *Suppose* $f \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ *and both* f *and* f_x *are bounded. If* f *satisfies that* $\mathcal{L}f \le \epsilon$, $\max(\mathcal{B}f, \mathcal{S}f) \le 0$, *then* $f(x,q) \ge V(x,q) - \frac{\epsilon}{\beta}$

Proof. Because $f \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ and both f and f_x are bounded, (21) is valid. Plug $\mathcal{L}f \le \epsilon$, $\max(\mathcal{B}f, \mathcal{S}f) \le 0$ into (21) and we have

$$\begin{aligned} f(x,q) &\ge -\mathbb{E}\int_0^\infty e^{-\beta t}\epsilon dt + \mathbb{E}\left(\int_0^\infty e^{-\beta t}(e^{X_t} - M_t)dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \Lambda_t)dL_t\right) \\ &= V_{(L,U)}(x,q) - \frac{\epsilon}{\beta}. \end{aligned}$$

Because this inequality holds for any admissible strategy (L,U), we have $f(x,q) \ge V(x,q) - \frac{\epsilon}{\beta}$.

□

Theorem 6. *Suppose* $f \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ *and both* f *and* f_x *are bounded. If* f *satisfies that* $\max(\mathcal{L}f, \mathcal{B}f, \mathcal{S}f) \le \epsilon$, *then ???.*

Proof. Because $f \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ and both f and f_x are bounded, (21) is valid. Plug $\mathcal{L}f \le \epsilon$, $\max(\mathcal{B}f, \mathcal{S}f) \le 0$ into (21) and we have

$$\begin{aligned} f(x,q) &\ge -\mathbb{E}\int_0^\infty e^{-\beta t}\epsilon dt + \mathbb{E}\left(\int_0^\infty e^{-\beta t}(e^{X_t} - M_t - \epsilon)dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \Lambda_t + \epsilon)dL_t\right) \\ &= V_{(L,U)}(x,q) - \frac{\epsilon}{\beta} - \epsilon\mathbb{E}\int_0^\infty e^{-\beta t}(dU_t + dL_t). \end{aligned}$$

□