# 1 Model

Let $S_t$ be the price and its evolution follows

$$dS_t = \kappa(\gamma - \log(S_t))S_t dt + \sigma S_t dZ_t. \tag{1}$$

Take $X_t = \log(S_t)$, by Ito's formula,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dZ_t, \tag{2}$$

where $\alpha = \gamma - \frac{\sigma^2}{2\kappa}$.

Denote the total amount of the commodity in the storage facility at time $t$ as $Q_t$. Also, denote cumulative amounts of commodity purchased and sold up to time $t$ as $L_t$ and $U_t$, respectively. Here and after, buy and inject have the same meaning, so do sell and withdraw.

The decisions are when and how much shall be injected and withdraw from the facility. In other words, every decision can be represented as a pair $(L_t, U_t)$. Because the change in storage is equal to the amount injected minus the amount withdrawed, we have

$$dQ_t = dL_t - dU_t. \tag{3}$$

An admissible control policy $(L_t, U_t)$ must satisfy that the storage level at any time is within capabilities. That is to say, $Q_t \in [Q_{\min}, Q_{\max}]$ for all $t$ where $Q_{\min}$ and $Q_{\max}$ are the upper and lower capability respectively. We use $\mathcal{U}$ to denote all admissible policies.

Both injection and withdrawal generate costs. As a result, injection and withdrawal won't take place at the same time. Let $\lambda(q)$ and $\mu(q)$ be the instantaneous costs of injection and withdrawal when the storage has $q$ unit. Therefore, the costs of injection and withdrawal at time $t$ should be $\int_{Q_{t-}}^{Q_t} \lambda(q)dq$ and $\int_{Q_t}^{Q_{t-}} \mu(q)dq$ respectfully. It is often easier to inject(withdraw) when empty(full) than full(empty). Therefore we assume that $\lambda(q)$ is increasing with respect to $q$ while $\mu(q)$ is decreasing. To be economic meaningful, we also assume that $\lambda(q)$ and $\mu(q)$ are bounded.

The objective is to maximize discounted infinite-horizon discounted cash flows. Taking a discount factor $\beta \in (0, 1)$,

$$V(x,q) = \max_{(L,U)\in\mathcal{U}} \mathbb{E}_{x,q} \left( \int_0^\infty e^{-\beta t}(e^{X_t} - M_t)dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \Lambda_t)dL_t \right) \tag{4}$$

where $X_0 = x$ and $Q_0 = q$. We use $\Lambda_t$ and $M_t$ to represent the injection and withdrawal cost at time $t$. By the definition of $\lambda$ and $\mu$, we have the following relations.

$$\Lambda_t = \begin{cases} \lambda(Q_t) & \text{if } L_t = L_{t-} \\ \frac{1}{Q_t - Q_{t-}} \int_{Q_{t-}}^{Q_t} \lambda(q) & \text{otherwise.} \end{cases} \tag{5}$$

$$M_t = \begin{cases} \mu(Q_t) & \text{if } U_t = U_{t-} \\ \frac{1}{Q_{t-} - Q_t} \int_{Q_t}^{Q_{t-}} \mu(q)dq & \text{otherwise.} \end{cases} \tag{6}$$

In other words, the maximum value that a facility manager can obtain from a storage facility when the current spot log price is $x$ and the amount is $q$ is $V(x, q)$.

## 2 No Transaction Costs

No transaction cost means that $\lambda$ (buying cost) $\equiv 0$ and $\mu$ (selling cost) $\equiv 0$. Since the profit (cost) of selling (buying) one unit at the same price level is the same, the optimal strategy should be bang-bang. Whenever the drift of $X_t$ exceeds interest rate, we should buy. Vice versa. Let this critical point be $X^*$, and then

$$\kappa(\alpha - X^*) = r \Rightarrow X^* = \alpha - \frac{r}{\kappa}$$

The optimal strategy is as following

- If $X_t > X^*$, sell to empty.
- If $X_t \leq X^*$, buy to full.

## 3 Constant Transaction Costs

Let the constant buying (selling) costs be $\lambda$ ($\mu$). Since the cost (profit) of buying (selling) one unit at the same price level is the same, the optimal strategy should be bang-bang. Introduce the following two functions of log price $x$,

$$\begin{aligned} V(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(e^{X_\tau} - \mu)\} \\ J(x) &= \sup_{\nu \in \mathcal{T}} \mathbb{E}_x\{e^{-r\nu}(V(x^\nu) - e^{X_\nu} - \lambda)\}. \end{aligned}$$

$V(x)$ is the maximum value that can be gained from 1 unit commodity when log price is $x$. $J(x)$ is the maximal profit of buying and then selling 1 unit commodity when log price is $x$. Tim Leung et al.(2014) use concavity to prove that the optimal policy for $V(x)$ is an upper threshold policy. In other words, if log price exceeds the threshold $d^*$, then sell. Using similar idea, they also prove the optimal policy for $J(x)$ is an upper-lower threshold policy. That is to say, one will buy if and only if log price is within two thresholds, namely [$a^*$,$b^*$]. Based on the existence and conditions of $a^*$ and $b^*$, there are two cases.

1. $J(x) = 0$. In this case, it is not optimal enter the market at all.

2. $-\infty < a^* < b^* < d^* < \infty$. This means that commodity is bought when log price is within $[a^*, b^*]$ and will be sold when log price achieves $d^*$.

# 4 Non-Constant Transaction Costs

The stochastic control problem with non-constant transaction costs can be transformed into a Hamilton Jacobi Bellman (HJB) equation via dynamic programming principle and Ito's formula. The HJB equation characterize the value function $V(x, q)$ (4).

We will use $V_x$ and $V_q$ separately to denote the partial differential of $V$ with respect to $x$ and $q$. $V_{xx}$ represents $\frac{\partial^2 V}{(\partial x)^2}$. The value function $V(x, q)$ is expected to satisfy the HJB equation

$$\max\left[\mathcal{L}V(x,q), \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0. \tag{7}$$

where

$$\begin{aligned}
\mathcal{L}V(x,q) &= \frac{1}{2}\sigma^2 V_{xx}(x,q) + k(\alpha - x)V_x(x,q) - \beta V(x,q), \\
\mathcal{S}V(x,q) &= -V_q(x,q) + e^x - \mu(q), \\
\mathcal{B}V(x,q) &= V_q(x,q) - e^x - \lambda(q).
\end{aligned} \tag{8}$$

Because the three terms in equation (7) are the profits utilizing holding, selling, and buying policy respectively, they are called holding, selling, and buying profit.

The region where the holding profit is the highest is called the optimal holding region and denoted as $H^*$. Similarly, the optimal selling (buying) region is defined and denoted as $S^*$ ($B^*$). Mathematically speaking,

$$\begin{aligned}
\mathcal{L}V(x,q) &= \max\left[\mathcal{L}V(x,q), \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0 \quad (x,q) \in H^* \\
\mathcal{S}V(x,q) &= \max\left[\mathcal{L}V(x,q),, \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0 \quad (x,q) \in S^* \\
\mathcal{B}V(x,q) &= \max\left[\mathcal{L}V(x,q),, \mathcal{S}V(x,q), \mathcal{B}V(x,q)\right] = 0 \quad (x,q) \in B^*
\end{aligned}$$

We want to have a sequence of regions, $(H^n, S^n, B^n)$, to approach the optimal regiones $(H^*, S^*, B^*)$. The fixed boundary problem $(H^n, S^n, B^n)$ is defined as a system of equations which states that holding (selling, buying) profit is 0 in $H^n$ ($S^n, B^n$). Mathematically speaking,

$$\begin{aligned} \mathcal{L}V^n(x,q) = 0 \quad (x,q) \in H^n \\ \mathcal{S}V^n(x,q) = 0 \quad (x,q) \in S^n \\ \mathcal{B}V^n(x,q) = 0 \quad (x,q) \in B^n. \end{aligned} \tag{9}$$

Here the solution to the fixed boundary problem is denoted as $V^n(x,q)$.

## 4.1 Algorithm

1. Find a large enough number $M_0$ such that the optimal selling region contains the inititial selling region $S^0 = \{(x,q)|x \geq M_0,\ Q_{\min} < q \leq Q_{\max}\}$. Set holding region $H^0 = (S^0)^c$ and buying region $B^0 = \emptyset$.
2. Keep doing the following steps until either convergence or buying region becomes non-empty, namely $B^n \neq \emptyset$. Here $n$ starts with 0.
   (a) Calculate the value function $V^n(x,q)$.
   (b) If the maximal buying profit is positive, $max_x \mathcal{B}V^n(x,q) > 0$, then set the buying region $B^{n+1} = \{(x,q)|x = x_p^{n+1}(q)\}$, where
   $$x_p^{n+1}(q) = \arg\max_x \mathcal{B}V^n(x,q).$$
   (c) Else, set the selling region $S^{n+1} = \{(x,q)|x \geq x_s^{n+1}(q)\}$, where
   $$x_s^{n+1}(q) = \arg\max_x \mathcal{S}V^n(x,q)$$
   (d) Set the holding region as $H^{n+1} = (S^{n+1} \cup B^{n+1})^c$.
   (e) $n = n + 1$.
3. If previous loop ends when $n = N$ because of buying region becoming non-empty, then keep doing the following until convergence. Here $n$ starts with $N$.
   (a) Calculate the value function $V^n(x,q)$.
   (b) If $n$ is odd, then the buying region remains the same, $B^{n+1} = B^n$, and the selling region is updated to $S^{n+1} = \{(x,q)|x \geq x_s^{n+1}(q)\}$.

(c) If $n$ is even, then selling region remains the same, $S^{n+1} = S^n$, and buying region is updated to $B^{n+1} = \{(x,q)|x_l^{n+1}(q) \le x \le x_u^{n+1}(q)\}$, where

$$x_u^{n+1}(q) = \arg \max_{x \ge x_u^n(q)} \mathcal{B}V^{n+1}(x,q),$$
$$x_l^{n+1}(q) = \arg \max_{x \le x_l^n(q)} \mathcal{B}V^{n+1}(x,q).$$

Here $x_u^N(q) = x_l^N(q) = x_p^N(q)$.

(d) Set the holding region as $H^{n+1} = (S^{n+1} \cup B^{n+1})^c$.

(e) $n = n+1$.

Notice that each time only one of selling and buying boundaries moves and both selling region and buying region are increasing. Here and after, the movement of a boundary has the same meaning as the change of the related region. These two facts are critical in the proof of the monotonicity of the value functions $V^n$.

In order to show this algorithm works, three things need to be proved.

1. The existence of $M_0$.
2. $V^n$ is monotone increasing.
3. The boundaries can keep moving.
4. The convergence function is the optimal.

## 4.2 The existence of $M_0$

**Lemma 1.** *For any positive storage level, $\tilde{q} > 0$, there exists a number $M_0(\tilde{q})$ such that region $\{(x,q)|x \ge M_0(\tilde{q}), q = \tilde{q}\}$ is contained by the optimal selling region $S^*$.*

*Proof.* First, in order to make money, for any buying price, there must be a higher selling price. As a result, the optimal buying region won't contain a region with infinite upper bound. Thus if lemma 1 doesn't hold, it is the optimal holding region $H^*$ that the region $\{(x,\tilde{q})|x > M_0(\tilde{q})\}$ belongs to. In other words, for any log price $x$ larger than the number $M_0(\tilde{q})$,

$$\begin{aligned} \mathcal{L}V(x,\tilde{q}) &= 0 \\ \mathcal{S}V(x,\tilde{q}) &\le 0. \\ \mathcal{B}V(x,\tilde{q}) &\le 0 \end{aligned} \tag{10}$$

Define

$$h(x) = -\mathcal{S}V(x,\tilde{q}) = V_q(x,\tilde{q}) - (e^x - \mu(\tilde{q})). \tag{11}$$

Because $-\mathcal{S}V(x,q) = \mathcal{B}V(x,q) + (\lambda(q) + \mu(q))$, together with the last two inequalities of (10), we have

$$0 \leq h(x) \leq \lambda(\tilde{q}) + \mu(\tilde{q}) \quad \forall x > M_0(\tilde{q}). \tag{12}$$

On the other hand, take derivative with respect to $q$ to both sides of first equality of (10) and we have $\mathcal{L}V_q(x,\tilde{q}) = 0$. Substitute (11) into it to have

$$\frac{1}{2}\sigma^2 h''(x) + k(\alpha - x)h'(x) - \beta h(x) = -\frac{1}{2}\sigma^2 e^x + k(x-\alpha)e^x + \beta(e^x - \mu(\tilde{q})).$$

Divide both sides by $e^x$,

$$\frac{1}{2}\sigma^2 e^{-x}h''(x) + ke^{-x}(\alpha - x)h'(x) - \beta e^{-x}h(x) = -\frac{1}{2}\sigma^2 + k(x-\alpha) + \beta(1 - e^{-x}\mu(\tilde{q})). \tag{13}$$

Because the buying cost, $\mu(\tilde{q})$, is bounded, the right-hand side of (13) goes to the positive infinity when $x$ approaches the positive infinity. Meanwhile, by (12), function $h(x)$ is bounded. Combine these two, when $x$ approaches the positive infinity, either

1. $e^{-x}h''(x) \to +\infty$ or
2. $e^{-x}(\alpha - x)h'(x) \to +\infty$.

In first case, because $e^{-x}h''(x)$ approaches the positive infinity when $x$ approaches positive infinity, there must exist a number $t_1$ such that $h''(x)$ is positive for any $x$ larger than $t_1$. For any $t_2$ larger than $t_1$, by Fubini's theorem, we have

$$\begin{aligned}
h(t_2) - h(t_1) &= \int_{t_1}^{t_2} h'(t)dt = \int_{t_1}^{t_2} (\int_{t_1}^{t} h''(x)dx + h'(t_1))dt \\
&= \int_{t_1}^{t_2} \int_{t_1}^{t} h''(x)dxdt + (t_2 - t_1)h'(t_1) \\
&= \int_{t_1}^{t_2} \int_{x}^{t_2} h''(x)dtdx + (t_2 - t_1)h'(t_1) \\
&= \int_{t_1}^{t_2} (t_2 - x)h''(x)dx + (t_2 - t_1)h'(t_1) \\
&\geq e^{t_1} \int_{t_1}^{t_2} (t_2 - x)e^{-x}h''(x)dx + (t_2 - t_1)h'(t_1) \\
&= e^{t_1} \int_{t_1}^{t_2} \left((t_2 - x)e^{-x}h''(x) + e^{-t_1}h'(t_1)\right) dx.
\end{aligned} \tag{14}$$

With $t_1$ fixed, when $t_2 \to +\infty$, the right hand side approaches positive infinity. This is a contradiction with (12).

In the second case, because $e^{-x}(\alpha - x)h'(x)$ approaches the positive infinity when $x$ approaches positive infinity, there must exist a number $t_1 > \alpha$ such that $(\alpha - x)h'(x)$ is positive for any $x$ larger than $t_1$. On the other hand, $e^x/(\alpha - x)$ is a decreasing function when $x$ is large enough. Without losing any generality, assume it is decreasing for all $x$ that is bigger than $t_1$. For any $t_2$ larger than $t_1$, we have

$$\begin{aligned} h(t_2) - h(t_1) &= \int_{t_1}^{t_2} h'(t)dt \\ &= \int_{t_1}^{t_2} \frac{e^t}{\alpha - t} \frac{(\alpha - t)h'(t)}{e^t} dt \\ &\leq \frac{e^{t_1}}{\alpha - t_1} \int_{t_1}^{t_2} \frac{(\alpha - t)h'(t)}{e^t} dt. \end{aligned} \tag{15}$$

With $t_1$ fixed, when $t_2 \to +\infty$, the right hand side approaches negative infinity. This is also a contradiction with (12).

□

Because we use the discretization method to solve the HJB equation (7), only finite number of $M_0(\tilde{q})$ matter. The maximal of those is $M_0$. A proposed $M_0$ is large enough if it satisfies

$$(\mathcal{S}V(x,q))_x\,|_{x=M_0} < 0 \quad \forall q \in (Q_{\min}, Q_{\max}].$$

### 4.3 $V^n$ is monotone increasing

We need two lemmas to prove the monotonicity.

**Lemma 2.** *If $\mathcal{L}f = 0$, function $f$ can't achieve the positive maximal or negative minimum at an interior point.*

*Proof.* Assume that there exists $x_M$ which is the positive maximal interior point. Then $f(x_M) > 0$, $f'(x_M) = 0$ and $f''(x_M) \leq 0$. On the other hand, $\mathcal{L}f(x_M) = 0$. By the definition of $\mathcal{L}$,

$$\begin{aligned} &\frac{1}{2}\sigma^2 f''(x_M) + \kappa(\alpha - x_M)f'(x_M) - \beta f(x_M) = 0 \\ \Rightarrow &0 < \beta f(x_M) = \frac{1}{2}\sigma^2 f''(x_M) + \kappa(\alpha - x_M)f' \leq 0 \end{aligned} \tag{16}$$

Contradiction! We can use the similar argument to prove the negative minimum case. □

**Lemma 3.** *At the same log price level $\tilde{x}$, there won't exist two storage levels $q_1$ and $q_2$ such that selling is optimal at $(\tilde{x}, q_1)$ while buying is optimal at $(\tilde{x}, q_2)$.*

*Proof.* If not, without losing any generality, assume that $(\tilde{x}, q_1)$ is on the buying boundary and $(\tilde{x}, q_2)$ is on the selling boundary. Define function $f(x)$ as the buying profit function, $\mathcal{B}V(x, q_1)$. Because $(\tilde{x}, q_1)$ is on the buying boundary, $f(\tilde{x}) = 0$. By HJB equation (7), $f(x)$ achieves its maximum at point $\tilde{x}$. If function $f$ is differentiable, then $f'(\tilde{x}) = 0$ because $\tilde{x}$ is the maximal point. We need to consider two cases.

1. Holding region is above $(\tilde{x}, q_1)$.
2. Holding region is below $(\tilde{x}, q_1)$.

Because function $f(x)$ may not be second order continuous, we need to consider the left and right derivative separately. From maximal property and $f'(\tilde{x}) = 0$, we have $f''(\tilde{x}+) \leq 0$, $f''(\tilde{x}-) = 0$ for the first case and $f''(\tilde{x}+) = 0$, $f''(\tilde{x}-) \leq 0$ for the second. In the first case,

$$\mathcal{L}f(\tilde{x}+) = \frac{1}{2}\sigma^2 f''(\tilde{x}+) + k(\alpha - \tilde{x})f'(\tilde{x}) - \beta f(\tilde{x}) = \frac{1}{2}\sigma^2 f''(\tilde{x}+) \leq 0.$$

On the other hand,

$$\begin{aligned}\mathcal{L}f(\tilde{x}+) = \mathcal{L}(\mathcal{B}V(\tilde{x}+, q_1)) &= \mathcal{L}V_q(\tilde{x}+, q_1) - \mathcal{L}(e^x + \lambda(q_1))|_{x=\tilde{x}} \\ &= \left(\mathcal{L}V(\tilde{x}+, q_1)\right)_q - \mathcal{L}(e^x + \lambda(q_1))|_{x=\tilde{x}} \\ &= -\mathcal{L}(e^x + \lambda(q_1))|_{x=\tilde{x}}.\end{aligned}$$

As a result,

$$\frac{1}{2}\sigma^2 e^{\tilde{x}} + k(\alpha - \tilde{x})e^{\tilde{x}} - \beta(e^{\tilde{x}} + \lambda(q_1))) = \mathcal{L}(e^x + \lambda(q_1))|_{x=\tilde{x}} \geq 0. \tag{17}$$

In the second case, we can achieve the same equality using similar derivation above.

Define $g(x) = \mathcal{S}V(x, q_2)$. We can replicate the same analysis above to have

$$\frac{1}{2}\sigma^2 e^{\tilde{x}} + k(\alpha - \tilde{x})e^{\tilde{x}} - \beta(e^{\tilde{x}} - \mu(q_2))) \leq 0. \tag{18}$$

Because $\lambda$ and $\mu$ are positive, (17) and (18) can't hold simultaneously which is a contradiction.

□

**Theorem 1.** *The value function $V^{n+1}$ with the region triplet $(H^{n+1}, S^{n+1}, B^{n+1})$ is larger than the value function $V^n$ with the regions triplet $(H^n, S^n, B^n)$.*

*Proof.* By equation (9) and the way the algorithm works, for $V^n$, we have

$$\begin{array}{ll} \mathcal{L}V^n(x,q) = 0 & (x,q) \in H^n \\ -V_q^n(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^n \\ -V_q^n(x,q) + e^x - \mu(q) > 0 & (x,q) \in S^{n+1}/S^n \\ V_q^n(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^n \\ V_q^n(x,q) - e^x - \lambda(q) > 0 & (x,q) \in B^{n+1}/B^n. \end{array} \tag{19}$$

For $V^{n+1}$, we have

$$\begin{array}{ll} \mathcal{L}V^{n+1}(x,q) = 0 & (x,q) \in H^{n+1} \\ -V_q^{n+1}(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^n \\ -V_q^{n+1}(x,q) + e^x - \mu(q) = 0 & (x,q) \in S^{n+1}/S^n \\ V_q^{n+1}(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^n \\ V_q^{n+1}(x,q) - e^x - \lambda(q) = 0 & (x,q) \in B^{n+1}/B^n. \end{array} \tag{20}$$

Here we also use the monotonicity of $H^n$, $S^n$ and $B^n$, that is to say, $H^n \supset H^{n+1}$, $S^n \subset S^{n+1}$ and $B^n \subset B^{n+1}$. Introduce $\Delta V^{n+1}(x,q) = V^{n+1}(x,q) - V^n(x,q)$. From (19) and (20), we can deduce

$$\begin{array}{rl} \mathcal{L}\Delta V^{n+1}(x,q) = 0 & (x,q) \in H^{n+1} \\ \Delta V_q^{n+1}(x,q) = 0 & (x,q) \in S^n \\ \Delta V_q^{n+1}(x,q) > 0 & (x,q) \in S^{n+1}/S^n \\ \Delta V_q^{n+1}(x,q) = 0 & (x,q) \in B^n \\ \Delta V_q^{n+1}(x,q) < 0 & (x,q) \in B^{n+1}/B^n. \end{array} \tag{21}$$

Because only one of selling and buying region changes, we can further divide it into two cases.

1. Only selling boundary moves, namely

$$\begin{array}{rl} \mathcal{L}\Delta V^{n+1}(x,q) = 0 & (x,q) \in H^{n+1} \\ \Delta V_q^{n+1}(x,q) = 0 & (x,q) \in S^n \\ \Delta V_q^{n+1}(x,q) > 0 & (x,q) \in S^{n+1}/S^n \\ \Delta V_q^{n+1}(x,q) = 0 & (x,q) \in B^{n+1} = B^n \end{array} \tag{22}$$

2. Only buying boundary moves, namely

$$\begin{array}{rl} \mathcal{L}\Delta V^{n+1}(x,q) = 0 & (x,q) \in H^{n+1} \\ \Delta V_q^{n+1}(x,q) = 0 & (x,q) \in S^{n+1} = S^n \\ \Delta V_q^{n+1}(x,q) = 0 & (x,q) \in B^n \\ \Delta V_q^{n+1}(x,q) < 0 & (x,q) \in B^{n+1}/B^n. \end{array} \tag{23}$$

In the first case, take derivative with respect to $q$ in the first equality of equation (22) to have

$$\begin{aligned}
\mathcal{L}\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in H^{n+1} \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in S^{n} \\
\Delta V_q^{n+1}(x,q) &> 0 \quad (x,q) \in S^{n+1}/S^{n} \\
\Delta V_q^{n+1}(x,q) &= 0 \quad (x,q) \in B^{n+1} = B^{n}.
\end{aligned} \tag{24}$$

By Lemma 2, $\Delta V_q^{n+1} \geq 0$ holds for all points. Right now focus on the set $\{(x, Q_{\min})|x \in \mathbb{R}\}$. Only holding and buying are possible policy on this set because there is nothing to sell.

(a) Hold at $(x, Q_{\min})$. Directly from the first equality of (22), we have

$$\mathcal{L}\Delta V^{n+1}(x, Q_{\min}) = 0$$

(b) Buy at $(x, Q_{\min})$. By Lemma 3, at this log price level $x$, there won't be a selling policy because we already buy at $(x, Q_{\min})$. Especially, we will hold at $(x, Q_{\max})$ because there is no place to inject. Therefore, $\mathcal{L}\Delta V^{n+1}(x, Q_{\max}) = 0$. Meanwhile, by (24),

$$\mathcal{L}\Delta V_q^{n+1}(x,q) = 0 \quad \forall (x,q) \in H^{n+1} \cup B^{n+1}.$$

Since buying and holding are the only policies at $x$, together with above equality,

$$\int_{Q_{\min}}^{Q_{\max}} \mathcal{L}\Delta V_q^{n+1}(x,q)dq = 0.$$

Thus,

$$\begin{aligned}
\mathcal{L}\Delta V^{n+1}(x, Q_{\min}) &= \mathcal{L}\left(\Delta V^{n+1}(x, Q_{\max}) - \int_{Q_{\min}}^{Q_{\max}} \Delta V_q^{n+1}(x,q)dq\right) \\
&= \mathcal{L}\Delta V^{n+1}(x, Q_{\max}) - \int_{Q_{\min}}^{Q_{\max}} \mathcal{L}\Delta V_q^{n+1}(x,q)dq = 0.
\end{aligned}$$

We have proved that

$$\mathcal{L}\Delta V^{n+1}(x, Q_{\min}) = 0 \quad \forall x \in \mathbb{R}.$$

This implies $\Delta V^{n+1}(x, Q_{\min}) = 0$ because 0 is the only finite function that satisfy the above equation. Therefore,

$$\begin{aligned}
\Delta V^{n+1}(x,q) &= \Delta V^{n+1}(x, Q_{\min}) + \int_{Q_{\min}}^{q} \Delta V_q^{n+1}(x,q)dq \\
&= \int_{Q_{\min}}^{q} \Delta V_q^{n+1}(x,q)dq \geq 0
\end{aligned}$$

The last inequality holds because of $\Delta V_q^{n+1} \geq 0$.
In the second case, we can use very similar method to have $\Delta V_q^{n+1} \leq 0$ and $\Delta V^{n+1}(x, Q_{\max}) = 0$. Then

$$\begin{aligned}\Delta V^{n+1}(x,q) &= \Delta V^{n+1}(x, Q_{\max}) - \int_q^{Q_{\max}} \Delta V_q^{n+1}(x,q)dq \\ &= -\int_{Q_{\min}}^{q} \Delta V_q^{n+1}(x,q)dq \geq 0\end{aligned}$$

□

### 4.4 The boundaries can keep moving

From the algorithm, the boundaries can move if and only if the following theorem holds.

**Theorem 2.** *Three inequalities below are true for all integer n and storage level q.*

$$\begin{aligned}(\mathcal{S}V(x,q))_x\,|_{x=x_s^{n+1}(q)} &< 0 \\ (\mathcal{B}V(x,q))_x\,|_{x=x_u^{n+1}(q)} &> 0 \\ (\mathcal{B}V(x,q))_x\,|_{x=x_l^{n+1}(q)} &< 0\end{aligned}$$

*When $x_u^{n+1}(q)$ and $x_l^{n+1}(q)$ don't exist, we assume the last two inequalities are true automatically.*

*Proof.* Introduce function $f(x) = \Delta V_q^{n+1}(x,q)$ with any fixed storage level $q$. As mentioned before, there are two cases namely only the selling boundary moves and only the buying. In the first case, from previous proof, $f(x)$ are non-negative at all points and achieves its maximum at point $x_s^{n+1}(q)$. Therefore $f'(x_s^{n+1}(q)) > 0$ or $f'(x_s^{n+1}(q)) = 0$. Because $\mathcal{L}f(x_s^{n+1}(q)) = 0$, if the latter is true, then we have $0 < \beta f(x_s^{n+1}(q)) = \frac{1}{2}\sigma^2 f''(x_s^{n+1}(q)) \leq 0$ which is a contradiction. As a result, we proved that $f'(x_s^{n+1}(q)) > 0$.

$$\begin{aligned}&(\Delta V_q^{n+1})_x(x_s^{n+1}(q), q) = f'(x_s^{n+1}(q)) > 0 \\ \Rightarrow &(V_q^{n+1})_x(x_s^{n+1}(q), q) - (V_q^{n})_x(x_s^{n+1}(q), q) > 0 \\ \Rightarrow &(V_q^{n+1})_x(x_s^{n+1}(q), q) - e^{x_s^{n+1}(q)} > 0 \\ \Rightarrow &\left(V_q^{n+1}(x,q) - (e^x - \mu(q))\right)_x |_{x=x_s^{n+1}(q)} > 0 \\ \Rightarrow &\left(-V_q^{n+1}(x,q) + (e^x - \mu(q))\right)_x |_{x=x_s^{n+1}(q)} < 0. \\ \Rightarrow &(\mathcal{S}V(x,q))_x\,|_{x=x_s^{n+1}(q)} < 0\end{aligned} \tag{25}$$

Because the buying boundary remains, $f(x_u^{n+1}(q)) = f(x_l^{n+1}(q)) = 0$. Non-negativity and previous fact implies that $f(x)$ achieves the minimum at $x_u^{n+1}(q)$ and $x_l^{n+1}(q)$. Using similar arguments as $x_s^{n+1}(q)$ to have

$$
\begin{aligned}
(\mathcal{B}V(x,q))_x \,|_{x=x_u^{n+1}(q)} &> 0 \\
(\mathcal{B}V(x,q))_x \,|_{x=x_l^{n+1}(q)} &< 0
\end{aligned}
$$

In the second case that only the buying moves, $f(x)$ still achieves the maximum at $x_s^{n+1}(q)$ and the minimum at $x_u^{n+1}(q)$ and $x_l^{n+1}(q)$. Thus we still have those three strict inequalities hold. $\square$