1 SEPARATION OF VARIABLES

Assume $V(x,s,q) = g(x,q)h(s)$. When $s=0$, there is no seasonal factor at all. In this case, we should have $h(0)=1$. Plug it into $\mathscr{L}V=0$ to have

$$h\left(\frac{1}{2}\sigma^2 g_{xx} + \kappa(\alpha - x)g_x - \beta g\right) + f(s)gh' = 0.$$

We can rearrange it as

$$\frac{\frac{1}{2}\sigma^2 g_{xx}(x,q) + \kappa(\alpha - x)g_x(x,q) - \beta g(x,q)}{g(x,q)} = \frac{-f(s)h'(s)}{h(s)}.$$

The left hand side is a function of x and q while the right hand side is a function of s. Therefore, the only possibility is that both are the same constant, denoted as η.

$$\frac{1}{2}\sigma^2 g_{xx}(x,q;\eta) + \kappa(\alpha - x)g_x(x,q;\eta) - (\beta+\eta)g(x,q;\eta) = 0$$
$$f(s)h'(s;\eta) + \eta h(s;\eta) = 0 \quad h(0;\eta) = 1$$

The solution to the second ODE is

$$h(s;\eta) = \exp(-\eta \int_0^s \frac{1}{f(y)} dy) \tag{1.1}$$

If we have $S(t) = \sin(t)$, then

$$f(S(t)) = S'(t) = \cos(t) = \sqrt{1-\sin(t)^2} = \sqrt{1-S(t)^2}.$$

Namely, $f(s) = \sqrt{1-s^2}$. Substitute this back into equation (1.1) to have

$$h(s;\eta) = \exp(-\eta \arcsin(s)).$$

As a result,

$$\eta = -\frac{\log(h(s;\eta))}{\arcsin(s)}.$$

On the other hand,

$$g(x,q;\eta) = aU(x,q;\eta) + bM(x,q;\eta).$$

a, b should be determined by the boundary conditions. If we already know $b=0$, then we have

$$g(x,q;\eta) = bM(x,q;\eta)$$
$$\Rightarrow g(x,q;-\frac{\log(h(s;\eta))}{\arcsin(s)}) = bM(x,q;-\frac{\log(h(s;\eta))}{\arcsin(s)})$$

2 Holding when full or empty

Notice that when empty (full), we will always hold when price is high (low). Because discretization of V_{xx} needs two boundary conditions, we need another boundary condition except the buying (selling) boundary condition. Without seasonality, by solving

$$\frac{1}{2}\sigma^2 V_{xx} + \kappa(\alpha - x)V_x - \beta V = 0$$

analytically and the using finiteness, we have

$$\frac{V(x_1,0)}{V(x_2,0)} = \frac{U\left(\kappa/\sigma^2(x_1-\alpha)^2\right)}{U\left(\kappa/\sigma^2(x_1-\alpha)^2\right)}$$

With seasonality, we don't have this boundary condition any more. Rearrange

$$\frac{1}{2}\sigma^2 V_{xx} + \kappa(\alpha - x)V_x - \beta V + f(s)V_s = 0,$$

as

$$V_x = -\frac{\frac{1}{2}\sigma^2 V_{xx} - \beta V + f(s)V_s}{\kappa(\alpha - x)}.$$

If V_{xx}, V_s is bounded, then when $|x|$ is large enough, we have V_x is approximately 0. However, if we use this formulation, the linear system turns out to have no solution at all. Therefore, we need to modify it using monotonicity.

3 Dam

Only the difference between the water levels matters. If we assume that the rain and vaporization are uniformly distributed, then none of them will affect the difference as long as none of them being full or empty. Denote the height difference as q. The injecting transaction cost $\lambda(q)$ should be

$$\lambda(q) = cq.$$

The rainfall only changes the upper bound.