Unique Boundary of Injection

Let (x_0, q_0) be the boundary point of holding and injection. What's more assume that the holding region lays in the southeast of the point while the injection region lays in the northeast.

1. $V(x,q)$ belongs to $C^{2,1}$ and V_{qxx} is also continuous everywhere including boundary.

   Proof. Because for any point (x,q) in holding region

   $$\mathcal{L}V_q(x,q) = 0$$

   holds, by the smoothness of $V(x,q)$,

   $$\mathcal{L}V_q(x_0,q_0) = 0.$$

   Similarly, for any point (x,q) in the injection region

   $$\mathcal{L}V_q(x,q) = \frac{1}{2}\sigma^2 e^x + k(\alpha - x)e^x - \beta(e^x + \lambda(q)))$$

   holds, the smoothness gives us

   $$\mathcal{L}V_q(x_0,q_0) = \frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))).$$

   Combine those two, we have

   $$\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))) = 0.$$

   If $\lambda(q)$ is strictly monotone, above equality with x_0 fixed only holds at a unique point namely there is only one boundary.

   $\square$

   Why the first assumption is wrong.

   Assume it is right, the boundary of withdrawing can be described as

   $$\frac{1}{2}\sigma^2 e^x + k(\alpha - x)e^x - \beta(e^x - \mu(q)) = 0.$$

   Reform it into the following equality

$$\beta(e^x - \mu(q)) = \frac{1}{2}\sigma^2 e^x + k(\alpha - x)e^x.$$

However, when $x \to +\infty$, the right hand side goes to $-\infty$. This means that $e^x - \mu(q) < 0$ when $x \to +\infty$. However, this is impossible since $e^x - \mu(q)$ is the profit when you sell which can not be negative.

2. $V(x,q)$ belongs to $C^{2,1}$, V_{qxx} is also continuous everywhere except boundary but V_{xx}, V_x, V are continuous with respect to q at the boundary.

Proof. Since we don't have the continuity of V_{qxx} at the boundary points, previous arguments can't hold. We only have

$$\mathcal{L}V_q(x_0-, q_0) = 0$$
$$\mathcal{L}V_q(x_0+, q_0) = \frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))).$$

On the other hand, for any point (x,q) in the holding region

$$e^x - \mu(q) < V_q(x,q) < e^x + \lambda(q).$$

What's more, at (x_0, q_0) we have $V_q(x_0, q_0) = e^{x_0} + \lambda(q_0)$ which means that function

$$f(x) = V_q(x, q_0) - (e^x + \lambda(q_0))$$

achieves its maximum at point x_0. Therefore, $f''(x_0) \leq 0$. Moreover, by the continuous conditions

$$f(x_0) = V_q(x_0, q_0) - (e^{x_0} + \lambda(q_0)) = 0$$
$$f'(x_0) = V_{qx}(x_0, q_0) - e^{x_0} = 0.$$

Thus

$$\mathcal{L}f(x_0) = \frac{1}{2}\sigma^2 f''(x_0) + k(\alpha - x_0)f'(x_0) - \beta f(x_0) = \frac{1}{2}\sigma^2 f''(x_0) \leq 0.$$

Meanwhile

$$\mathcal{L}(V_q(x_0-, q_0) - (e^{x_0} + \lambda(q_0))) = \mathcal{L}V_q(x_0-, q_0) - \mathcal{L}(e^{x_0} + \lambda(q_0))$$
$$= 0 - (\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))).$$

Combine those two

$$-(\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))) \leq 0,$$

namely

$$\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))) \geq 0. \tag{1}$$

Assume we also have another boundary point with the same level of price on the left hand side of (x_0, q_0) , i.e. (x_0, q_1) where $q_1 < q_0$ and there is no other boundary point between them. Therefore

$$\begin{aligned} V(x_0, q_0) &= V(x_0, q_1) + \int_{q_1}^{q_0} V_q(x_0, q)dq \\ &= V(x_0, q_1) + (q_1 - q_0)e^{x_0} + \int_{q_1}^{q_0} \lambda(q)dq. \end{aligned}$$

Use operator $\mathcal{L}$ to both sides,

$$\mathcal{L}V(x_0, q_0) = \mathcal{L}V(x_0, q_1) + \mathcal{L}((q_1 - q_0)e^{x_0} + \int_{q_1}^{q_0} \lambda(q)dq)$$

By the continuity of V_{xx}, V_x, V with respect to q on the boundary, we have

$$\mathcal{L}V(x_0, q_0) = \mathcal{L}V(x_0, q_1) = 0.$$

So

$$\mathcal{L}((q_1 - q_0)e^{x_0} + \int_{q_1}^{q_0} \lambda(q)dq) = 0,$$

which gives us

$$\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \frac{1}{q_0 - q_1}\int_{q_1}^{q_0} \lambda(q)dq) = 0.$$

This can't happen if $\lambda(q)$ is strictly increasing on $[q_1, q_0]$, because by (1)

$$\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta e^{x_0} \geq \beta\lambda(q_0) > \frac{1}{q_0 - q_1}\int_{q_1}^{q_0} \lambda(q)dq.$$

Assume that there is another boundary point (x_0, q_2) at the right hand side of (x_0, q_0), namely $q_2 > q_0$ and the region between them are holding region. By the previous result, there can't be a boundary point on the right hand side of (x_0, q_2). Let (x_0, q_3) where $q_3 > q_2$, then (x_0, q_3) must belong to injection region.

Similarly

$$
\begin{aligned}
\mathcal{L}V(x_0, q_2) &= 0 \\
\mathcal{L}V(x_0, q_3) &< 0 \\
V(x_0, q_3) &= V(x_0, q_2) + (q_3 - q_2)e^{x_0} + \int_{q_2}^{q_3} \lambda(q)dq.
\end{aligned}
$$

Use operator $\mathcal{L}$ to both sides of the last equality and use the top two

$$
0 > \mathcal{L}V(x_0, q_3) = 0 + (q_3 - q_2)(\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \frac{1}{q_3 - q_2}\int_{q_2}^{q_3} \lambda(q)dq)).
$$

This shows that

$$
\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \frac{1}{q_3 - q_2}\int_{q_2}^{q_3} \lambda(q)dq) < 0.
$$

Let $q_3 \to q_2$,

$$
\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_2)) \leq 0.
$$

On the other hand, (x_0, q_2) is the boundary point, thus

$$
\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_2)) \geq 0.
$$

Combine those two,

$$
\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_2)) = 0.
$$

□

3. $V(x, q)$ belongs to $C^{2,1}$, V_{qxx} is also continuous everywhere except boundary and we don't have the continuity with respect to q.

Then the only thing we have is

$$\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))) \geq 0. \tag{2}$$

Unique Boundary of Withdraw

For the withdraw region, the same arguments can be still used to obtain similar result. The weakest conditions will give us.

$$\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} - \mu(q_0))) \leq 0.$$

Noticing that

$$\frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} - \mu(q_0))) \geq \frac{1}{2}\sigma^2 e^{x_0} + k(\alpha - x_0)e^{x_0} - \beta(e^{x_0} + \lambda(q_0))),$$

it is impossible to inject and withdraw at the same price level no matter what the value q is.

Thinking

Personally speaking, I think the assumption in the first case is too strong to be true while the third is too weak to derive sufficient properties. The second one is my favorite and I get the feeling that it may be true because adding another dimension (here is volume) sometimes improve the continuity.

We want to use solve the following linear equations without taking the inverse of the matrix

$$Ax = b \tag{3}$$

Let $B = (I - A)$, then

$$x = Bx + b.$$

We can do iteration using this the method generating from above equality.

$$x_{n+1} = Bx_n + b.$$

If x_n converges, then it must converge to the solution of $Ax = b$.