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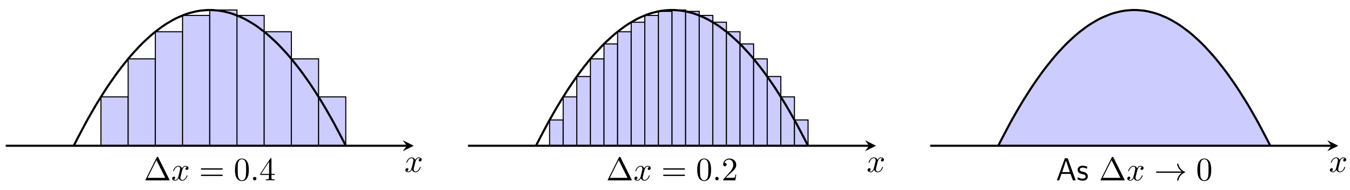
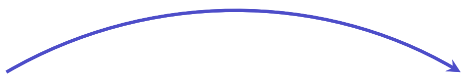
Introduction and Overview

In mathematics, *integration* can be thought of as the process of calculating some kind of *"sum"*, such as the total area under a curve, by breaking it down into sums of increasingly small parts. This technique was known about and used thousands of years ago in both Ancient Greece and other parts of the world.



In the 17th century both Gottfried Wilhelm Leibniz and Isaac Newton independently discovered the *fundamental theorem of calculus*, showing that *integration* can be performed using *antidifferentiation*, which is the 'reverse' of differentiation. Now, *integration* is used to refer to both the idea of calculating such a 'sum', and to the process of antidifferentiation.



This chapter will first introduce the use of integration to calculate an *antiderivative*, before moving on to using this concept to calculating area enclosed using curves.

Chapter Contents

Integration skills covered in this chapter will be extended upon in the later chapter of **Further Calculus**.

11.1 Indefinite Integrals

The *derivative* of the function $f(x) = x^3 - 4x^2 - 7x + 3$ is obtained by *multiplying by the power* and then *reducing the power by one* for each term.

$$\text{If } f(x) = x^3 - 4x^2 + 3$$
$$\text{Then } f'(x) = 3x^2 - 8x$$

Integration can be considered the *inverse* of differentiation, with the aim of taking the *derivative* $f'(x)$ and determining the *original function* $f(x)$. However, given only the derivative $f'(x) = 3x^2 - 8x$ it would not be possible to know any constant term the original function $f(x)$ contained. Finding the *indefinite integral* of a function requires the inclusion of a **Constant of Integration**, C:

$$\text{If } f'(x) = 3x^2 - 8x$$
$$\text{Then } f(x) = x^3 - 4x^2 + C$$

Indefinite Integrals: *"The (indefinite) integral of x^n with respect to x..."*

$$\int (x^n)\ dx = \frac{x^{n+1}}{n+1} + C$$

In other words, **increase the power by 1** then **divide by the new power.**

The *integral sign*, $\int$, should always appear accompanied by dx, for a function in x.

Note: $\int (f(x) + g(x))\ dx = \int f(x)\ dx + \int g(x\ dx)$

Example 11.1.1

Find $\int (6x^2 + 8x)\, dx$.

$$\int (6x^2 + 8x)\, dx$$
$$= \frac{6x^3}{3} + \frac{8x^2}{2} + C \qquad \longleftarrow \text{Constant of integration}$$
$$= 2x^3 + 4x^2 + C \qquad \longleftarrow \text{Simplify}$$

As with differentiation, *preparation for integration* may be needed.

Example 11.1.2

Find $\displaystyle\int \left(4\sqrt{x} - \frac{3}{x^2}\right) dx,\ , x > 0.$

$$\int \left(4\sqrt{x} - \frac{3}{x^2}\right)\, dx$$

$$= \int \left(4x^{\frac{1}{2}} - 3x^{-2}\right)\, dx \qquad \longleftarrow \text{Preparing to integrate}$$

$$= \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{-1}}{-1} + C \qquad \longleftarrow \text{Integrate: } \frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$$

$$= \frac{8}{3}x^{\frac{3}{2}} + 3x^{-1} + C \qquad \longleftarrow \text{Simplify: } 4 \div \frac{3}{2} = \frac{4}{1} \times \frac{2}{3} = \frac{8}{3}$$

The *derivative* of a *linear* term is a *constant*, so the integral of a *constant* is *linear*.

Example 11.1.3

Find $\displaystyle\int ((2x-1)(x+3))\, dx.$

$$\int ((2x-1)(x+3))\, dx$$

$$= \int (2x^2 - 5x - 3)\, dx \qquad \longleftarrow \text{Preparing to integrate}$$

$$= \frac{2x^3}{3} - \frac{5x^2}{2} - 3x + C \qquad \longleftarrow \text{Integral of } -3 \text{ is } -3x$$

Integration *with respect to variables other than* x should not use dx, and the notation used should match the variable of the function instead.

Example 11.1.4

Find $\displaystyle\int (3t^2 - 5)\, dt.$

$$\int (3t^2 - 5)\, dt$$

$$= \frac{3t^3}{3} - 5t + C$$

$$= t^3 - 5t + C$$

11.2 Differential Equations

11.3 Definite Integrals

The indefinite integral of the function $f(x)$ can be notated as $F(x)$:

$$\int f(x)\,dx = F(x)$$

The *definite integral* of $f(x)$ from a to b is the difference between $F(b)$ and $F(a)$:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Together, they form a core part of the **Fundamental Theorem of Calculus**:

Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where

$$F(x) = \int f(x)\,dx$$

Since any *constant of integration* within $F(x)$ will cancel through subtraction $(C - C)$, it is *not included* when calculating a *definite integral*.

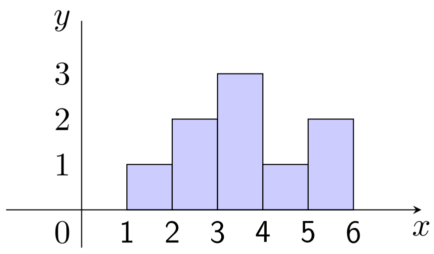
Example 11.3.1

Find $\int_1^2 (3x^2 + 6x - 2)\,dx$.

$$\begin{aligned}
&\int_1^2 (3x^2 + 6x - 2)\,dx \\
&= \left[\frac{3x^3}{3} + \frac{6x^2}{2} - 2x\right]_1^2 && \longleftarrow \text{Integrate} \\
&= \left[x^3 + 3x^2 - 2x\right]_1^2 && \longleftarrow \text{Simplify} \\
&= ((2)^3 + 3(2)^2 - 2(2)) - ((1)^3 + 5(1)^2 - 2(1)) && \longleftarrow \text{Substitute} \\
&= 16 - 4 \\
&= 12 && \longleftarrow \text{Evaluate}
\end{aligned}$$

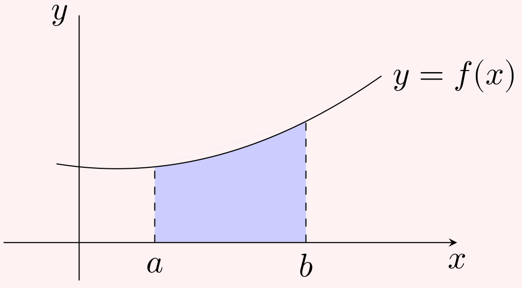
11.4 The Area Under a Curve

The area shaded below, with each *"bar"* having width 1 unit, is given by the *sum* $1+2+3+1+2$.



An integral, $\int$, can be seen as *"sum"*, but for a continuous function. Given a function $f(x)$, the area enclosed between a section of the curve $y = f(x)$ and the x-axis can be calculated using a definite integral.

Area Under a Curve:

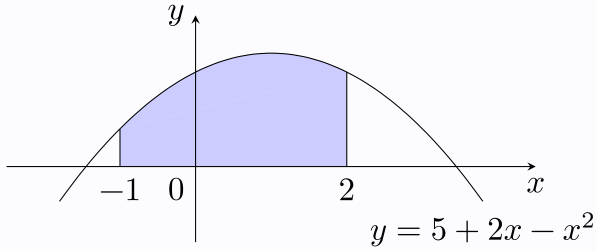


$$\text{Area} = \int_a^b f(x)\, dx = F(b) - F(a)$$

The values of a and b can be referred to as the *bounds* of the integration.

Example 11.4.1

Part of the graph of $y = 5 + 2x - x^2$ is shown. Calculate the shaded area.



$$\int_{-1}^{2} (5 + 2x - x^2)\, dx \qquad \longleftarrow \text{Area}$$

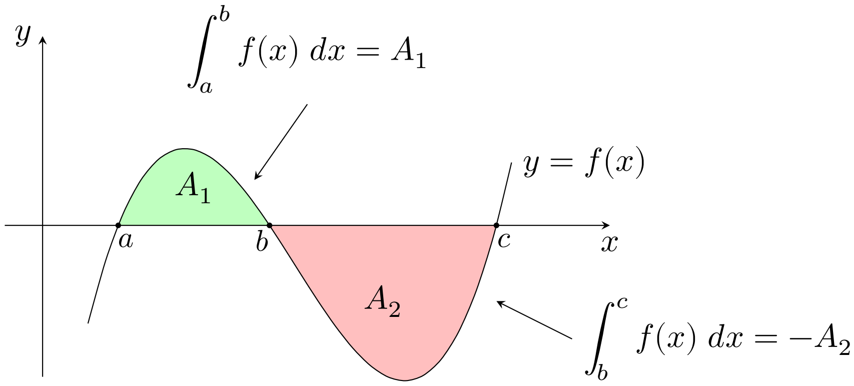
$$= \left[5x + x^2 - \frac{x^3}{3}\right]_{-1}^{2} \qquad \longleftarrow \text{Integrate}$$

$$= \left(5(2) + (2)^2 - \frac{(2)^3}{3}\right) - \left(5(-1) + (-1)^2 - \frac{(-1)^3}{3}\right) \qquad \longleftarrow \text{Substitute}$$

$$= \frac{34}{3} - \left(-\frac{11}{3}\right) \qquad \longleftarrow \text{Evaluate}$$

$= 15$ square units

Where the area between a curve and the x-axis lies *under* the x-axis, the definite integral will be negative.

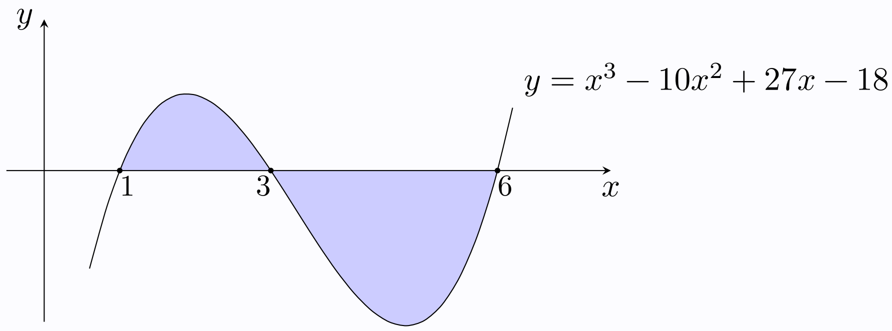


To avoid *"positve"* and *"negative"* areas cancelling each other out, such sections must be calculated as separate integrals, and their *absolute values* added.

$$\text{Area } = A_1 + A_2$$

Example 11.4.2

Part of the graph of $y = x^3 - 10x^2 + 27x - 18$ is shown. Calculate the shaded area.



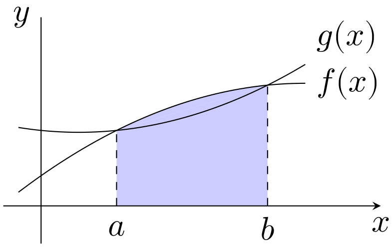
$$\int_1^3 (x^3 - 10x^2 + 27x - 18)\,dx$$
$$= \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x\right]_1^3$$
$$= \left(\frac{(3)^4}{4} - \frac{10(3)^3}{3} + \frac{27(3)^2}{2} - 18(3)\right)$$
$$- \left(\frac{(1)^4}{4} - \frac{10(1)^3}{3} + \frac{27(1)^2}{2} - 18(1)\right)$$
$$= -\frac{9}{4} - \left(-\frac{91}{12}\right)$$
$$= \frac{16}{3}$$

$$\int_3^6 (x^3 - 10x^2 + 27x - 18)\,dx$$
$$= \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x\right]_3^6$$
$$= \left(\frac{(6)^4}{4} - \frac{10(6)^3}{3} + \frac{27(6)^2}{2} - 18(6)\right)$$
$$- \left(\frac{(3)^4}{4} - \frac{10(3)^3}{3} + \frac{27(3)^2}{2} - 18(3)\right)$$
$$= -18 - \left(-\frac{9}{4}\right)$$
$$= -\frac{63}{4}$$

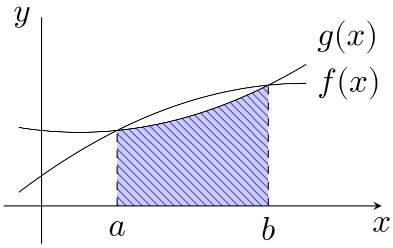
$$\therefore \text{Area} = \frac{16}{3} + \frac{63}{4} = \frac{253}{12} \text{ square units}$$

11.5 The Area Between Two Curves

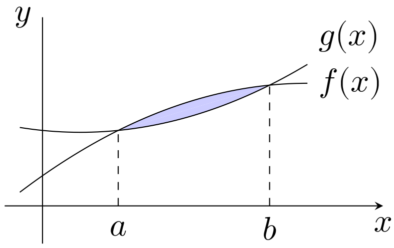
The area **between** two curves can be calculated using the subtraction of one definite integral from another:



$$\int_a^b f(x)\ dx$$



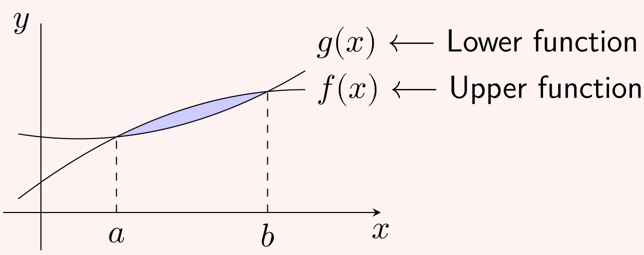
$$\int_a^b g(x)\ dx$$



$$\int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

Note that $\displaystyle\int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b (f(x) - g(x))\ dx$, leading to the following formula:

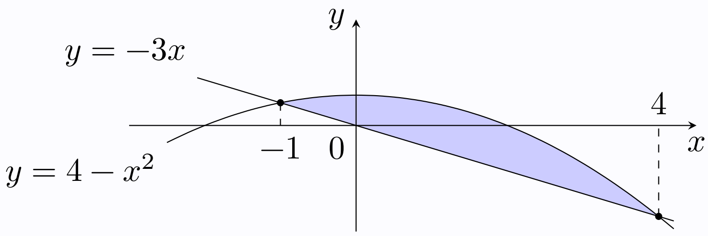
Area Between Two Curves:



$$\text{Area} = \int_a^b (f(x) - g(x))\ dx \quad \textbf{or} \quad \text{Area} = \int_a^b (\text{Upper} - \text{Lower})\ dx$$

Example 11.5.1

Part of the graphs of $y = -3x$ and $y = 4 - x^2$ are shown. Calculate the shaded area.



$$\int_{-1}^{4} (4 - x^2 - (-3x))\, dx \qquad \longleftarrow \text{Upper} - \text{Lower}$$

$$\int_{-1}^{4} (4 - x^2 + 3x)\, dx \qquad \longleftarrow \text{Simplify}$$

$$= \left[4x - \frac{x^3}{3} + \frac{3x^2}{2}\right]_{-1}^{4} \qquad \longleftarrow \text{Integrate}$$

$$= \left(4(4) - \frac{(4)^3}{3} + \frac{3(4)^2}{2}\right) - \left(4(-1) - \frac{(-1)^3}{3} + \frac{3(-1)^2}{2}\right) \qquad \longleftarrow \text{Substitute}$$

$$= \frac{56}{3} - \left(-\frac{13}{6}\right) \qquad \longleftarrow \text{Evaluate}$$

$$= \frac{125}{6} \text{ square units}$$

Review Exercise

Chapter 12 Addition Formulae

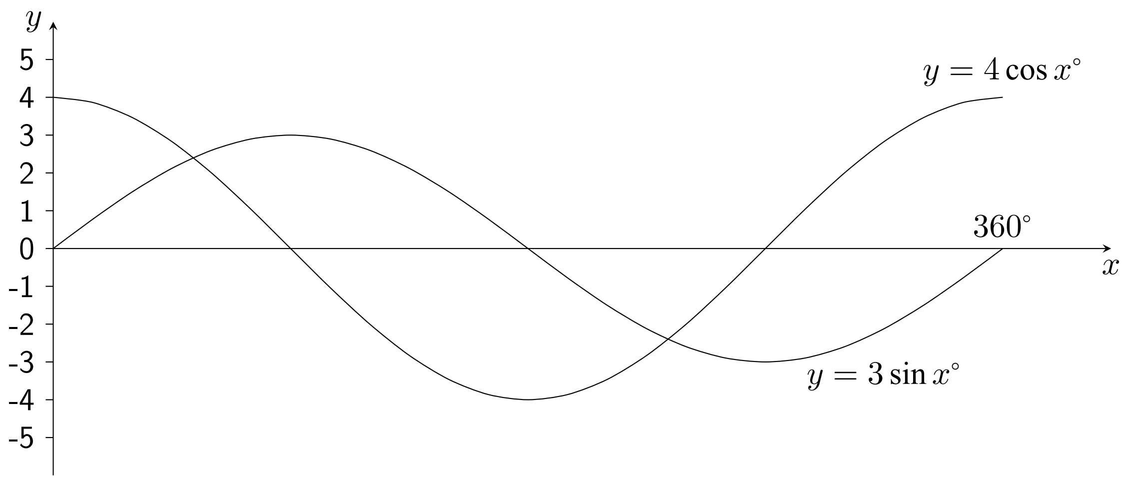
Chapter 13 The Circle

Chapter 14 The Wave Function

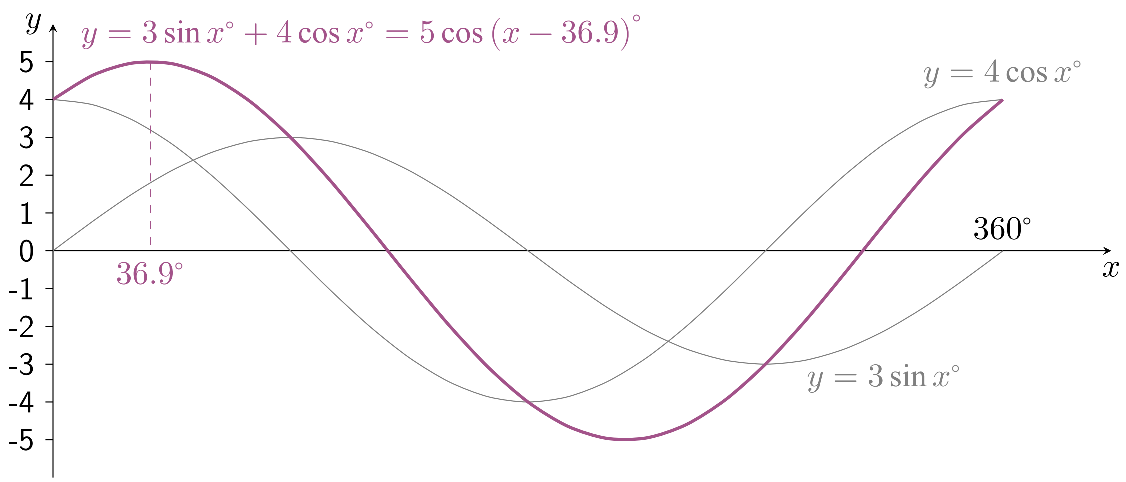
In this chapter, the sums and differences of *equal-angled* trigonometric operations will be explored.

$$\text{e.g. } 3\sin x^\circ + 4\cos x^\circ$$

The graphs of basic trigonometric functions such as $y = 3\sin x^\circ$ and $y = 4\cos x^\circ$ should be familiar:



The graph of $y = 3\sin x^\circ + 4\cos x^\circ$ is the same as that of $y = 5\cos(x - 36.9)^\circ$.



Determining that $3\sin x^\circ + 4\cos x^\circ = 5\cos(x - 36.9)^\circ$ allows the maximum values and minimum values of $3\sin x^\circ + 4\cos x^\circ$ to be determined and the values of x which produce them, and allow equations such as $3\sin x^\circ + 4\cos x^\circ = 2$ to be solved.

Any trigonometric expression $k_1 \sin x \pm k_2 \cos x$ can be written as either $k\sin(x \pm a)$ or $k\cos(x \pm a)$.

This chapter will cover how to do this and explore the various applications of this property.

14.1 The Wave Function using $k\cos(x-a)$

Whilst any of $k\sin(x\pm a)$ or $k\cos(x\pm a)$ may be used in general, any Higher exam question is likely to specify a particular form to use. The simplest is often $k\cos(x-a)$.

To express $3\sin x^\circ + 4\cos x^\circ$ in the form $k\cos(x-a)^\circ$, two key *trigonometric identities* are needed:

$$\sin^2 x + \cos^2 x = 1 \qquad \text{and} \qquad \tan x = \frac{\sin x}{\cos x}$$

First, $k\cos(x-a)^\circ$ can be expanded using the the formula $\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$:

$$k\cos(x-a)^\circ = k\cos x^\circ\cos a^\circ + k\sin x^\circ\sin a^\circ$$

It is required that this expansion is equal to the original expression:

$$3\underline{\sin x^\circ} + 4\underline{\underline{\cos x^\circ}} = k\underline{\underline{\cos x^\circ}}\cos a^\circ + k\underline{\sin x^\circ}\sin a^\circ$$

Equating the coefficients of $\underline{\sin x^\circ}$ and $\underline{\underline{\cos x^\circ}}$ gives a set of simultaneous equations in k and a:

$$\begin{aligned} k\sin a^\circ &= 3 && \longleftarrow \text{Coefficients of } \underline{\sin x^\circ} \\ k\cos a^\circ &= 4 && \longleftarrow \text{Coefficients of } \underline{\underline{\cos x^\circ}} \end{aligned}$$

Squaring both sides of each equation and adding allows k to be determined using $\sin^2 x + \cos^2 x = 1$:

$$\begin{aligned} k^2\sin^2 a^\circ + k^2\cos^2 a^\circ &= 3^2 + 4^2 \\ k^2\left(\sin^2 x^\circ + \cos^2 x^\circ\right) &= 3^2 + 4^2 \\ k^2(1) &= 3^2 + 4^2 \\ k &= \sqrt{3^2+4^2} \\ k &= \sqrt{25} \\ k &= 5 \end{aligned}$$

Dividing $k\sin a^\circ$ by $k\cos a^\circ$ allows a to be calculated, noting $k\sin a^\circ$ and $k\cos a^\circ$ are both positive:

$$\begin{aligned} \frac{k\sin a^\circ}{k\cos a^\circ} &= \frac{3}{4} \\ \tan a^\circ &= \frac{3}{4} \\ a^\circ_{\text{acute}} &= \tan^{-1}\left(\frac{3}{4}\right) = 36.9^\circ \\ a^\circ &= 36.9^\circ \end{aligned}$$

✓S	A✓✓ a°
T	C✓

Hence $3\sin x^\circ + 4\cos x^\circ = 5\cos(x-36.9)^\circ$. Here it has been assumed that $k>0$ and $0<a<360$.

Some abbreviations of the working shown on the previous page are routinely permitted in the Higher exam. The following example demonstrates an appropriate level of detail in its solution.

Example 14.1.1

Express $3\cos x - \sin x$ in the form $k\cos(x-a)$ where $k > 0$ and $0 < a < 2\pi$.

$3\cos x - \sin x = k\cos(x-a)$

$\underline{\underline{3\cos x}} - \underline{\sin x} = \underline{\underline{k\cos x}}\cos a + \underline{k\sin x}\sin a$ $\longleftarrow$ Expand $k\cos(x-a)$

$k\sin a = -1$ $\longleftarrow$ Equate $\sin x$ coefficients

$k\cos a = 3$ $\longleftarrow$ Equate $\cos x$ coefficients

$k = \sqrt{(-1)^2 + 3^2}$ $\longleftarrow$ Calculate k

$= \sqrt{10}$

$\tan a = \dfrac{-1}{3}$ $\longleftarrow$ Find $\tan a$

$\text{acute}^\circ = \tan^{-1}\left(\dfrac{1}{3}\right) = 18.4^\circ$ $\longleftarrow$ Find acute angle first

S	A✓
✓T	C✓✓

$360^\circ - \text{acute}^\circ$

$\longleftarrow$ Negative $\sin a$: quadrants T and C

$\longleftarrow$ Positive $\cos a$: quadrants A and C

$a^\circ = 341.6^\circ$ $\longleftarrow$ Using $360^\circ - 18.4^\circ$ (Double-ticked quadrant)

$a = 5.96$ $\longleftarrow$ Convert to radians: $341.6 \times \dfrac{\pi}{180}$

$3\cos x - \sin x = \sqrt{10}\cos(x - 5.96)$ $\longleftarrow$ State solution

Exercise 14.1

1. Express each of the following in the form $k\cos(x-a)^\circ$, where $k > 0$ and $0 < a < 360$.

   (a) $5\cos x^\circ + 12\sin x^\circ$ (b) $4\sin x^\circ + 5\cos x^\circ$ (c) $6\cos x^\circ + \sin x^\circ$

2. Express each of the following in the form $k\cos(x-a)$, where $k > 0$ and $0 < a < 2\pi$.

   (a) $8\sin x - 6\cos x$ (b) $\cos x - 3\sin x$ (c) $-2\sin x - \cos x$

3. Express each of the following in the form $k\cos(x-a)^\circ$, where $k > 0$ and $0 < a < 360$.

   (a) $\sin x^\circ + \sqrt{3}\cos x^\circ$ (b) $\cos x^\circ - \sqrt{3}\sin x^\circ$ (c) $-3\sin x^\circ - \sqrt{3}\cos x^\circ$

14.2 Other Forms of the The Wave Function

As well as $k\cos(x-a)$, three other forms may be used, shown below including their expansions:

$$\begin{aligned} k\cos(x+a) &= k\cos x\cos a - k\sin x\sin a \\ k\sin(x+a) &= k\sin x\cos a + k\sin x\cos a \\ k\sin(x-a) &= k\sin x\cos a - k\sin x\cos a \end{aligned}$$

Example 14.2.1

Express $12\cos x^\circ - 5\sin x^\circ$ in the form $k\sin(x-a)^\circ$ where $k>0$ and $0<a<360$.

$$12\cos x^\circ - 5\sin x^\circ = k\sin(x-a)^\circ$$

$$12\underline{\underline{\cos x^\circ}} - 5\underline{\sin x^\circ} = k\underline{\sin x^\circ}\cos a^\circ - k\underline{\underline{\cos x^\circ}}\sin a^\circ \quad \longleftarrow \text{Expand } k\sin(x-a)^\circ$$

$$-k\sin a^\circ = 12 \implies k\sin a^\circ = -12 \quad \longleftarrow \text{Equate } \sin x^\circ \text{ coefficients}$$

$$k\cos a^\circ = -5 \quad \longleftarrow \text{Equate } \cos x^\circ \text{ coefficients}$$

$$\begin{aligned} k &= \sqrt{(-12)^2 + (-5)^2} \quad \longleftarrow \text{Calculate } k \\ &= 13 \end{aligned}$$

$$\tan a^\circ = \frac{-12}{-5} = \frac{12}{5} \quad \longleftarrow \text{Find } \tan a^\circ$$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{12}{5}\right) = 67.4^\circ \quad \longleftarrow \text{Find acute angle first}$$

✓S	A
✓✓T	C✓

180° + acute$^\circ$

$\longleftarrow$ Negative $\sin a^\circ$: quadrants T and C

$\longleftarrow$ Negative $\cos a^\circ$: quadrants S and T

$$a^\circ = 247.4^\circ \quad \longleftarrow \text{Using } 180^\circ + 67.4^\circ \text{ (Double-ticked)}$$

$$12\cos x^\circ - 5\sin x^\circ = 13\sin(x-247.4)^\circ \quad \longleftarrow \text{State solution}$$

Exercise 14.2

1. Express each of the following in the form $k\sin(x+a)^\circ$, where $k>0$ and $0<a<360$. 🖩

   (a) $3\cos x^\circ + 2\sin x^\circ$ (b) $4\sin x^\circ + 3\cos x^\circ$ (c) $7\cos x^\circ - \sin x^\circ$

2. Express each of the following in the form $k\cos(x+a)$, where $k>0$ and $0<a<2\pi$. 🖩

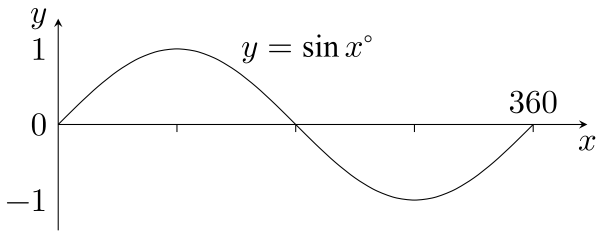
   (a) $3\sin x + 8\cos x$ (b) $-3\cos x - 4\sin x$ (c) $-6\sin x + 2\cos x$

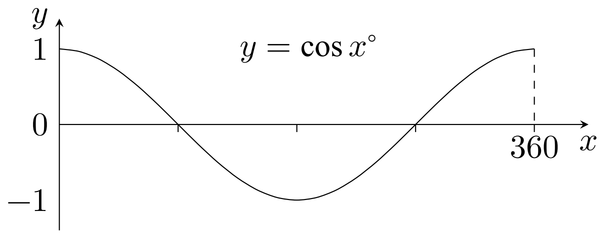
3. Express each of the following in the form $k\sin(x-a)$, where $k>0$ and $0<a<2\pi$.

   (a) $\sqrt{3}\sin x + \cos x$ (b) $\sqrt{3}\cos x - \sin x$ (c) $2\sqrt{3}\sin x - 6\cos x$

14.3 Maximum and Minimum Values using the Wave Function

The graphs of $y = \sin x°$ and $y = \cos x°$, including their turning points, should already be familiar:



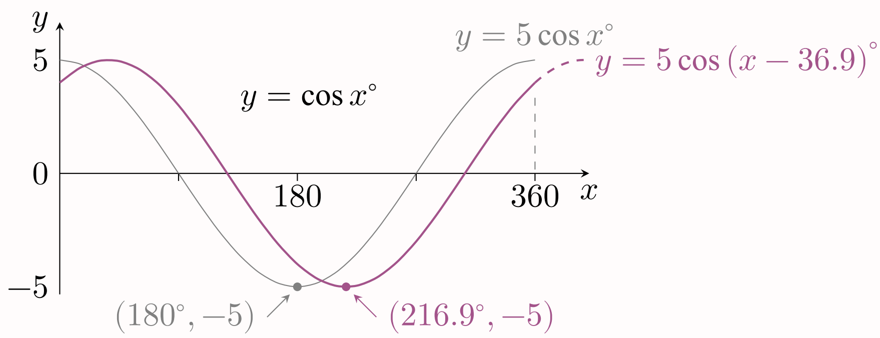


Graphs of the form $k\sin(x \pm a)$ and $k\cos(x \pm a)$ have maximum/minimum values of $\pm k$, and their *horizontal translation* is described by $\pm a$ following the rules covered in the Graph Transformations chapter.

Example 14.3.1

Given that $3\sin x° + 4\cos x°$ can be expressed as $5\cos(x - 36.9)°$, state the minimum value of $f(x) = 3\sin x° + 4\cos x°$ and the value of x at which it occurs for $0 < x < 360$.

Sketch the graph of $y = 5\cos x°$ translated $36.9°$ to the right:



Use the graph to state the answer:

Hence the minimum value of $f(x)$ is -5, which occurs when $x = 216.9$.

Exercise 14.3

1. Sketch each for $0 < x < 360$, showing the coordinates of any roots and turning points:

   (a) $7\cos(x - 10)°$ (b) $3\sin(x - 20)°$ (c) $\sqrt{5}\cos(x + 40)°$

2. Find the maximum value of each and the value(s) of x for which they occur for $0 < x < 360$:

   (a) $8\sin(x - 50)°$ (b) $\sqrt{2}\cos(x + 27)°$ (c) $7\sin(x + 42.3)°$

3. Find the minimum value of each and the value(s) of x for which they occur for $0 < x < 2\pi$:

   (a) $3\cos\left(x - \frac{\pi}{6}\right)$ (b) $5\sin\left(x + \frac{\pi}{3}\right) + 2$ (c) $-12\sin\left(x + \frac{\pi}{4}\right)$

4. (a) Express $6\sin x° - 7\cos x°$ in the form $k\sin(x - a)°$ where $k > 0$ and $0 < a < 360$.

   (b) Hence state the coordinates of the turning points of $12\sin x° - 14\cos x°$ for $0 < x < 360$.

14.4 Solving Equations using the Wave Function

Solving an equation like $3\sin x^\circ + 4\cos x^\circ = 2$ can be approached by using the wave function to rewrite it as $5\cos(x - 36.9)^\circ = 2$, before solving it in the manner covered in Chapter 6.

Example 14.4.1

It can be shown that $5\sin x^\circ - 2\cos x^\circ$ can be expressed as $\sqrt{29}\sin(x - 21.8)^\circ$.
Hence, solve the equation $5\sin x^\circ - 2\cos x^\circ = 4$ where $0 < x < 360$.

$$5\sin x^\circ - 2\cos x^\circ = 4$$

$$\sqrt{29}\sin(x - 21.8)^\circ = 4 \qquad \longleftarrow \text{Substitute the wave function form}$$

$$\sin(x - 21.8)^\circ = \frac{4}{\sqrt{29}} \qquad \longleftarrow \text{Rearrange to } \sin(\dots) = \dots$$

$$a = \sin^{-1}\left(\frac{4}{\sqrt{29}}\right) = 48.0^\circ \qquad \longleftarrow \text{Calculate acute angle}$$

$$\begin{array}{r|l} {\scriptstyle 180^\circ - a^\circ} \ \checkmark\text{S} & \text{A}\checkmark \ {\scriptstyle a^\circ} \\ \hline \text{T} & \text{C} \end{array} \qquad \longleftarrow \text{Positive } \sin a^\circ\text{: quadrants A and S}$$

$$x - 21.8^\circ = 48.0^\circ, 180^\circ - 48.0^\circ \qquad \longleftarrow \text{Apply ticked quadrants}$$

$$x - 21.8^\circ = 48.0^\circ, 132.0^\circ$$

$$x^\circ = 69.8^\circ, 153.8^\circ \qquad \longleftarrow \text{Add } 21.8^\circ \text{ to both sides}$$

Note that any solutions outwith the domain (often $0 < x < 360$) should have 360° added or subtracted to bring it back within the domain, where possible.

Exercise 14.4

1. Solve each equation for $0 < x < 360$:

   (a) $7\sin(x - 18)^\circ = 4$ (b) $3\cos(x + 34.1)^\circ = -2$ (c) $\sqrt{5}\sin(x - 106)^\circ + 1 = 0$

2. Given $6\sin x^\circ - 8\cos x^\circ = 10\sin(x - 53.1)^\circ$, solve $6\sin x^\circ - 8\cos x^\circ = 5$ where $0 < x < 360$.

3. (a) Express $\sqrt{3}\sin x^\circ + \cos x^\circ$ in the form $k\sin(x - a)^\circ$ where $k > 0$ and $0 < a < 360$.

   (b) Hence solve the equation $\sqrt{3}\sin x^\circ + \cos x^\circ = 1$ where $0 < x < 360$.

4. Solve each equation for $0 < x < 2\pi$:

   (a) $4\sin(x + 0.31) + 2 = 1$ (b) $9\cos(x + 1.24) = 5$ (c) $2\sqrt{3}\sin(x - 0.82) = \sqrt{5}$

5. (a) Express $3\cos x + 2$ in the form $k\cos(x + a)$ where $k > 0$ and $0 < a < 2\pi$.

   (b) Hence solve the equation $2 + 6\sin x + 4\cos x = 5$ where $0 < x < 2\pi$.

14.5 Multiple Angles and Different Variables

The techniques covered in this chapter can be applied to trigonometric expressions beyond those containing only $\sin x$ and $\cos x$; they work for any sum or difference of *equal-angled* trigonometric operations.

$$\text{e.g.} \qquad 3\sin t^\circ + 4\cos t^\circ \qquad \text{or} \qquad 2\sin 2x - 5\cos 2x$$

Example 14.5.1

Express $5\cos 2t^\circ - 3\sin 2t^\circ$ in the form $k\sin(2t+a)^\circ$ where $k>0$ and $0<a<360$.

$$5\cos 2t^\circ - 3\sin 2t^\circ = k\sin(2t+a)^\circ$$

$$5\underline{\underline{\cos 2t^\circ}} - 3\underline{\sin 2t^\circ} = k\underline{\sin 2t^\circ}\cos a^\circ + k\underline{\underline{\cos 2t^\circ}}\sin a^\circ \qquad \longleftarrow \text{Expand } k\sin(2t+a)^\circ$$

$$k\sin a^\circ = 5 \qquad \longleftarrow \text{Equate } \sin 2t^\circ \text{ coefficients}$$

$$k\cos a^\circ = -3 \qquad \longleftarrow \text{Equate } \cos 2t^\circ \text{ coefficients}$$

$$k = \sqrt{(5)^2+(-3)^2} \qquad \longleftarrow \text{Calculate } k$$

$$= \sqrt{34}$$

$$\tan a^\circ = \frac{5}{-3} = -\frac{5}{3} \qquad \longleftarrow \text{Find } \tan a^\circ$$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{5}{3}\right) = 59.0^\circ \qquad \longleftarrow \text{Find acute angle first}$$

$180^\circ - \text{acute}^\circ$

✓✓S	A✓
✓T	C

$\longleftarrow$ Positive $\sin a^\circ$: quadrants A and S

$\longleftarrow$ Negative $\cos a^\circ$: quadrants S and T

$$a^\circ = 121.0^\circ \qquad \longleftarrow \text{Using } 180^\circ - 59.0^\circ \text{ (Double-ticked)}$$

$$5\cos 2t^\circ - 3\sin 2t^\circ = \sqrt{34}\sin(x+59.0)^\circ \qquad \longleftarrow \text{State solution}$$

Exercise 14.5

1. Express $4\cos t^\circ - 3\sin t^\circ$ in the form $k\sin(t-a)^\circ$, where $k>0$ and $0<a<360$.
2. Express $2\sin 2x - \cos 2x$ in the form $k\cos(2x-a)$ where $k>0$ and $0<a<2\pi$.
3. (a) Express $12\cos t^\circ + 5\sin t^\circ$ in the form $k\sin(t+a)^\circ$, where $k>0$ and $0<a<360$.
   (b) Hence state:
      i. The maximum value of the function $f(x) = 12\cos t^\circ + 5\sin t^\circ$, $0<t<360$.
      ii. The value(s) of t for which it occurs.
4. (a) Express $\sin 2x - \sqrt{3}\cos 2x$ in the form $k\cos(2x-a)$ where $k>0$, $0<a<2\pi$.
   (b) Hence solve $\sin 2x - \sqrt{3}\cos 2x - 1 = 0$, $0<x<2\pi$.
   (c) Sketch $y = -\sin 2x + \sqrt{3}\cos 2x - 1 = 0$ for $0 \leqslant x \leqslant 2\pi$.

Wave Function Review Exercise

1.

Chapter 15 Logs and Exponentials

Chapter 16 Further Calculus

Answers