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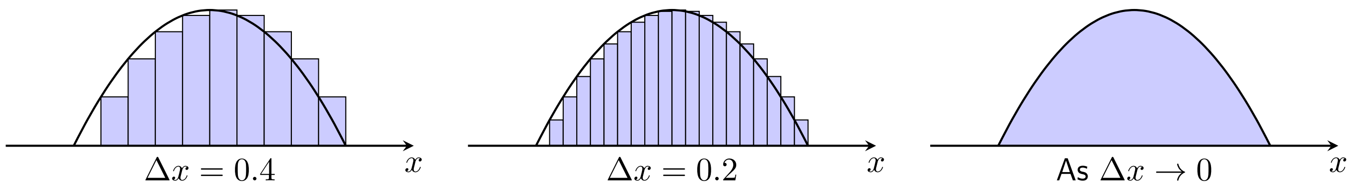
## Introduction and Overview

In mathematics, *integration* can be thought of as the process of calculating some kind of *"sum"*, such as the total area under a curve, by breaking it down into sums of increasingly small parts. This technique was known about and used thousands of years ago in both Ancient Greece and other parts of the world.




In the 17th century both Gottfried Wilhelm Leibniz and Isaac Newton independently discovered the *fundamental theorem of calculus*, showing that *integration* can be performed using *antidifferentiation*, which is the 'reverse' of differentiation. Now, *integration* is used to refer to both the idea of calculating such a 'sum', and to the process of antidifferentiation.

This chapter will first introduce the use of integration to calculate an *antiderivative*, before moving on to using this concept to calculating area enclosed using curves.

## Chapter Contents



Integration skills covered in this chapter will be extended upon in the later chapter of **Further Calculus**.

## 11.1 Indefinite Integrals

The *derivative* of the function $f(x) = x^3 - 4x^2 - 7x + 3$ is obtained by *multiplying by the power* and then *reducing the power by one* for each term.

$$\text{If } f(x) = x^3 - 4x^2 + 3$$
$$\text{Then } f'(x) = 3x^2 - 8x$$

*Integration* can be considered the *inverse* of differentiation, with the aim of taking the *derivative* $f'(x)$ and determining the *original function* $f(x)$. However, given only the derivative $f'(x) = 3x^2 - 8x$ it would not be possible to know any constant term the original function $f(x)$ contained. Finding the *indefinite integral* of a function requires the inclusion of a **Constant of Integration**, $C$:

$$\text{If } f'(x) = 3x^2 - 8x$$
$$\text{Then } f(x) = x^3 - 4x^2 + C$$

**Indefinite Integrals:** *"The (indefinite) integral of $x^n$ with respect to $x$..."*

$$\int (x^n)\, dx = \frac{x^{n+1}}{n+1} + C$$

In other words, **increase the power by 1** then **divide by the new power.**

The *integral sign*, $\int$, should always appear accompanied by $dx$, for a function in $x$.

Note: $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x\, dx)$

Example 11.1.1

Find $\int (6x^2 + 8x)\, dx$.

$$\int (6x^2 + 8x)\, dx$$
$$= \frac{6x^3}{3} + \frac{8x^2}{2} + C \qquad \longleftarrow \text{Constant of integration}$$
$$= 2x^3 + 4x^2 + C \qquad \longleftarrow \text{Simplify}$$

As with differentiation, *preparation for integration* may be needed.

**Example 11.1.2**

Find $\int \left(4\sqrt{x} - \frac{3}{x^2}\right) dx,\ , x > 0.$

$$\int \left(4\sqrt{x} - \frac{3}{x^2}\right)\, dx$$

$$= \int \left(4x^{\frac{1}{2}} - 3x^{-2}\right)\, dx \qquad \longleftarrow \text{Preparing to integrate}$$

$$= \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{-1}}{-1} + C \qquad \longleftarrow \text{Integrate: } \frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$$

$$= \frac{8}{3}x^{\frac{3}{2}} + 3x^{-1} + C \qquad \longleftarrow \text{Simplify: } 4 \div \frac{3}{2} = \frac{4}{1} \times \frac{2}{3} = \frac{8}{3}$$

The *derivative* of a *linear* term is a *constant*, so the integral of a *constant* is *linear*.

**Example 11.1.3**

Find $\int ((2x-1)(x+3))\, dx.$

$$\int ((2x-1)(x+3))\, dx$$

$$= \int \left(2x^2 - 5x - 3\right)\, dx \qquad \longleftarrow \text{Preparing to integrate}$$

$$= \frac{2x^3}{3} - \frac{5x^2}{2} - 3x + C \qquad \longleftarrow \text{Integral of } -3 \text{ is } -3x$$

Integration *with respect to variables other than* $x$ should not use $dx$, and the notation used should match the variable of the function instead.

**Example 11.1.4**

Find $\int (3t^2 - 5)\, dt.$

$$\int \left(3t^2 - 5\right)\, dt$$

$$= \frac{3t^3}{3} - 5t + C$$

$$= t^3 - 5t + C$$

## 11.2 Differential Equations

## 11.3 Definite Integrals

The indefinite integral of the function $f(x)$ can be notated as $F(x)$:

$$\int f(x)\,dx = F(x)$$

The *definite integral* of $f(x)$ from $a$ to $b$ is the difference between $F(b)$ and $F(a)$:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Together, they form a core part of the **Fundamental Theorem of Calculus**:

**Fundamental Theorem of Calculus:**

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where

$$F(x) = \int f(x)\,dx$$

Since any *constant of integration* within $F(x)$ will cancel through subtraction $(C - C)$, it is *not included* when calculating a *definite integral*.

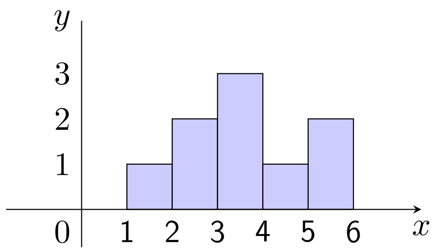
Example 11.3.1

Find $\int_1^2 (3x^2 + 6x - 2)\,dx$.

$$\begin{aligned}
&\int_1^2 (3x^2 + 6x - 2)\,dx \\
&= \left[\frac{3x^3}{3} + \frac{6x^2}{2} - 2x\right]_1^2 && \longleftarrow \text{Integrate} \\
&= \left[x^3 + 3x^2 - 2x\right]_1^2 && \longleftarrow \text{Simplify} \\
&= ((2)^3 + 3(2)^2 - 2(2)) - ((1)^3 + 5(1)^2 - 2(1)) && \longleftarrow \text{Substitute} \\
&= 16 - 4 \\
&= 12 && \longleftarrow \text{Evaluate}
\end{aligned}$$

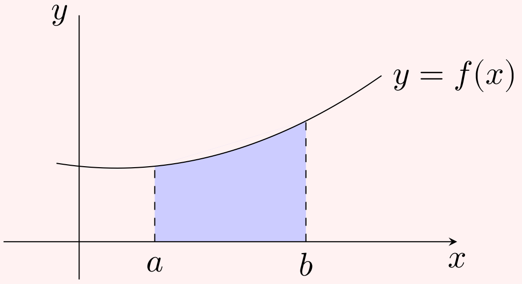
## 11.4 The Area Under a Curve

The area shaded below, with each *"bar"* having width $1$ unit, is given by the *sum* $1 + 2 + 3 + 1 + 2$.




An integral, $\int$, can be seen as *"sum"*, but for a continuous function. Given a function $f(x)$, the area enclosed between a section of the curve $y = f(x)$ and the $x$-axis can be calculated using a definite integral.

**Area Under a Curve:**

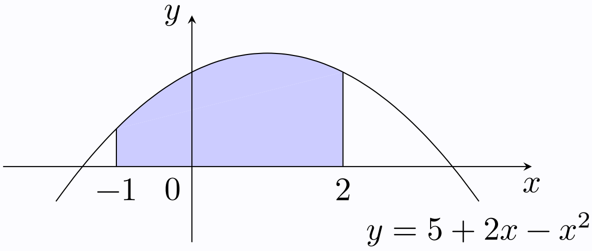



$$\text{Area} = \int_a^b f(x)\,dx = F(b) - F(a)$$

The values of $a$ and $b$ can be referred to as the *bounds* of the integration.

Example 11.4.1

Part of the graph of $y = 5 + 2x - x^2$ is shown. Calculate the shaded area.




$$\int_{-1}^{2} (5 + 2x - x^2)\,dx \qquad \longleftarrow \text{Area}$$

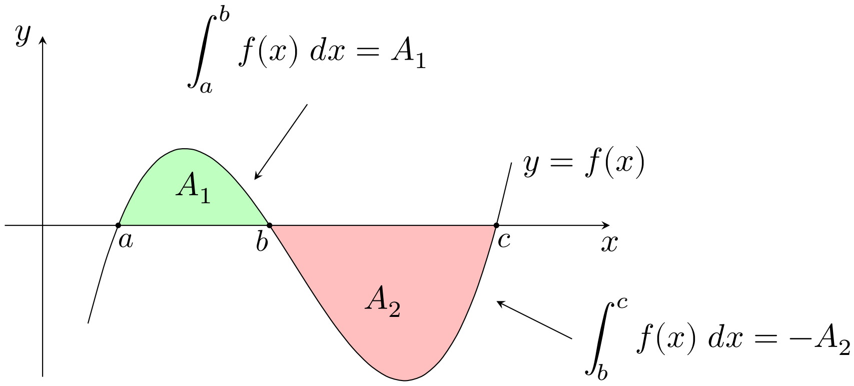
$$= \left[5x + x^2 - \frac{x^3}{3}\right]_{-1}^{2} \qquad \longleftarrow \text{Integrate}$$

$$= \left(5(2) + (2)^2 - \frac{(2)^3}{3}\right) - \left(5(-1) + (-1)^2 - \frac{(-1)^3}{3}\right) \qquad \longleftarrow \text{Substitute}$$

$$= \frac{34}{3} - \left(-\frac{11}{3}\right) \qquad \longleftarrow \text{Evaluate}$$

$= 15$ square units

Where the area between a curve and the $x$-axis lies *under* the $x$-axis, the definite integral will be negative.

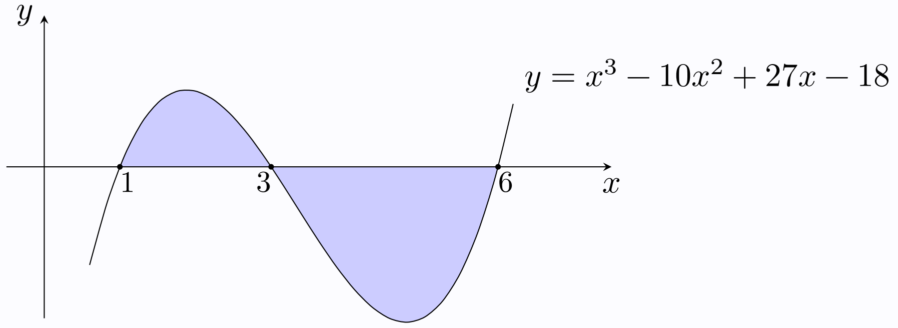



To avoid *"positve"* and *"negative"* areas cancelling each other out, such sections must be calculated as separate integrals, and their *absolute values* added.

$$\text{Area } = A_1 + A_2$$

**Example 11.4.2**

Part of the graph of $y = x^3 - 10x^2 + 27x - 18$ is shown. Calculate the shaded area.




$$\int_1^3 (x^3 - 10x^2 + 27x - 18)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x\right]_1^3$$

$$= \left(\frac{(3)^4}{4} - \frac{10(3)^3}{3} + \frac{27(3)^2}{2} - 18(3)\right)$$

$$- \left(\frac{(1)^4}{4} - \frac{10(1)^3}{3} + \frac{27(1)^2}{2} - 18(1)\right)$$

$$= -\frac{9}{4} - \left(-\frac{91}{12}\right)$$

$$= \frac{16}{3}$$

$$\int_3^6 (x^3 - 10x^2 + 27x - 18)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x\right]_3^6$$

$$= \left(\frac{(6)^4}{4} - \frac{10(6)^3}{3} + \frac{27(6)^2}{2} - 18(6)\right)$$

$$- \left(\frac{(3)^4}{4} - \frac{10(3)^3}{3} + \frac{27(3)^2}{2} - 18(3)\right)$$

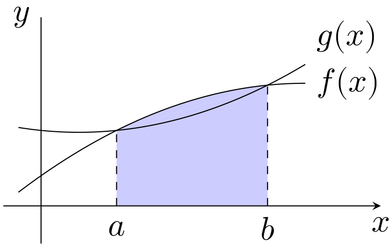
$$= -18 - \left(-\frac{9}{4}\right)$$

$$= -\frac{63}{4}$$

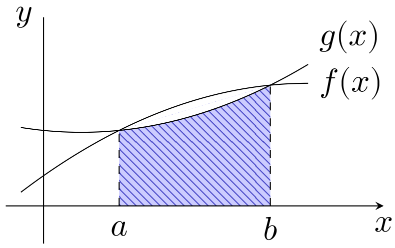
$$\therefore \text{Area} = \frac{16}{3} + \frac{63}{4} = \frac{253}{12} \text{ square units}$$

## 11.5 The Area Between Two Curves

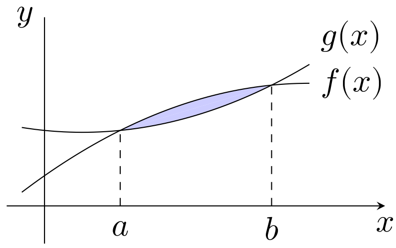
The area **between** two curves can be calculated using the subtraction of one definite integral from another:




$$\int_a^b f(x)\,dx$$



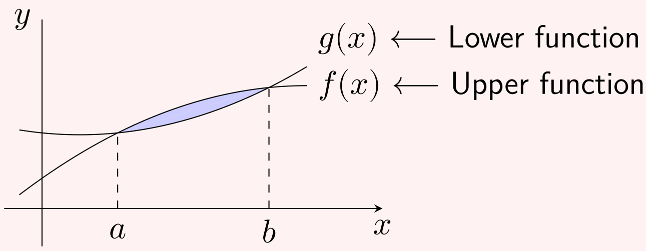

$$\int_a^b g(x)\,dx$$




$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Note that $\displaystyle\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$, leading to the following formula:

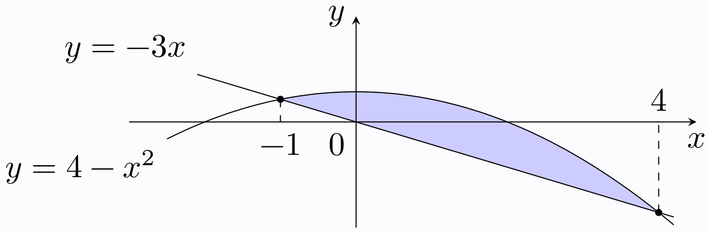
**Area Between Two Curves:**




$$\text{Area} = \int_a^b (f(x) - g(x))\,dx \quad \textbf{or} \quad \text{Area} = \int_a^b (\text{Upper} - \text{Lower})\,dx$$

Example 11.5.1

Part of the graphs of $y = -3x$ and $y = 4 - x^2$ are shown. Calculate the shaded area.




$$\int_{-1}^{4} (4 - x^2 - (-3x))\,dx$$ ⟵ Upper − Lower

$$\int_{-1}^{4} (4 - x^2 + 3x)\,dx$$ ⟵ Simplify

$$= \left[4x - \frac{x^3}{3} + \frac{3x^2}{2}\right]_{-1}^{4}$$ ⟵ Integrate

$$= \left(4(4) - \frac{(4)^3}{3} + \frac{3(4)^2}{2}\right) - \left(4(-1) - \frac{(-1)^3}{3} + \frac{3(-1)^2}{2}\right)$$ ⟵ Substitute

$$= \frac{56}{3} - \left(-\frac{13}{6}\right)$$ ⟵ Evaluate

$$= \frac{125}{6}$$ square units

## Review Exercise

# Chapter 12 Addition Formulae

# Chapter 13

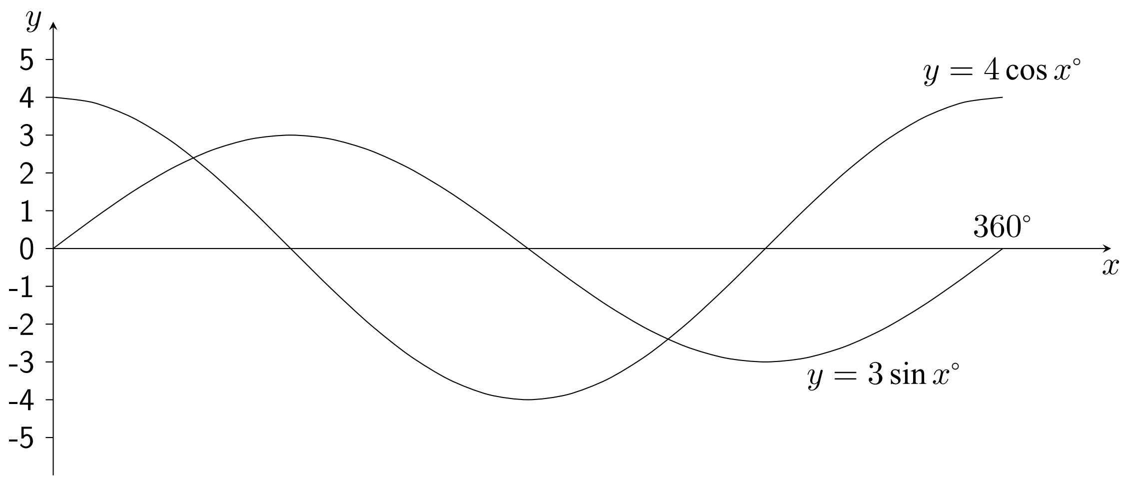
# The Circle

# Chapter 14 The Wave Function

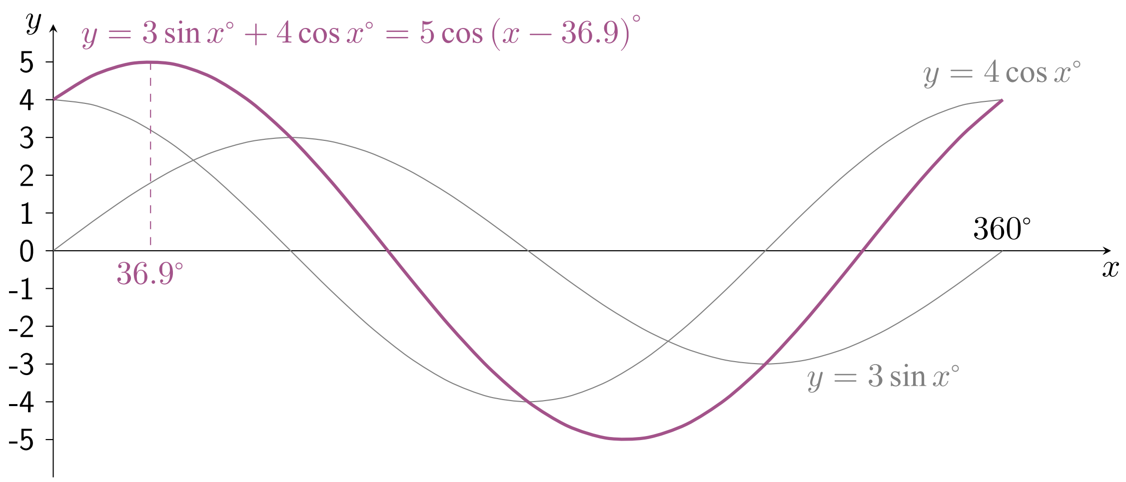
In this chapter, the sums and differences of *equal-angled* trigonometric operations will be explored.

$$\text{e.g. } 3\sin x^\circ + 4\cos x^\circ$$

The graphs of basic trigonometric functions such as $y = 3\sin x^\circ$ and $y = 4\cos x^\circ$ should be familiar:




The graph of $y = 3\sin x^\circ + 4\cos x^\circ$ is the same as that of $y = 5\cos(x - 36.9)^\circ$.




Determining that $3\sin x^\circ + 4\cos x^\circ = 5\cos(x - 36.9)^\circ$ allows the maximum values and minimum values of $3\sin x^\circ + 4\cos x^\circ$ to be determined and the values of $x$ which produce them, and allow equations such as $3\sin x^\circ + 4\cos x^\circ = 2$ to be solved.

*Any* trigonometric expression $k_1 \sin x \pm k_2 \cos x$ can be written as either $k\sin(x \pm a)$ or $k\cos(x \pm a)$.

This chapter will cover how to do this and explore the various applications of this property.

## 14.1 The Wave Function using $k\cos(x-a)$

Whilst any of $k\sin(x\pm a)$ or $k\cos(x\pm a)$ may be used in general, any Higher exam question is likely to specify a particular form to use. The simplest is often $k\cos(x-a)$.

To express $3\sin x^\circ + 4\cos x^\circ$ in the form $k\cos(x-a)^\circ$, two key *trigonometric identities* are needed:

$$\sin^2 x + \cos^2 x = 1 \qquad \text{and} \qquad \tan x = \frac{\sin x}{\cos x}$$

First, $k\cos(x-a)^\circ$ can be expanded using the the formula $\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$:

$$k\cos(x-a)^\circ = k\cos x^\circ\cos a^\circ + k\sin x^\circ\sin a^\circ$$

It is required that this expansion is equal to the original expression:

$$3\underline{\sin x^\circ} + 4\underline{\underline{\cos x^\circ}} = k\underline{\underline{\cos x^\circ}}\cos a^\circ + k\underline{\sin x^\circ}\sin a^\circ$$

Equating the coefficients of $\underline{\sin x^\circ}$ and $\underline{\underline{\cos x^\circ}}$ gives a pair of simultaneous equations in $k$ and $a$:

$$\begin{aligned} k\sin a^\circ &= 3 && \longleftarrow \text{Coefficients of } \underline{\sin x^\circ} \\ k\cos a^\circ &= 4 && \longleftarrow \text{Coefficients of } \underline{\underline{\cos x^\circ}} \end{aligned}$$

Squaring both sides of each equation and adding allows $k$ to be determined using $\sin^2 x + \cos^2 x = 1$:

$$\begin{aligned} k^2\sin^2 a^\circ + k^2\cos^2 a^\circ &= 3^2+4^2 \\ k^2\left(\sin^2 x^\circ + \cos^2 x^\circ\right) &= 3^2+4^2 \\ k^2(1) &= 3^2+4^2 \\ k &= \sqrt{3^2+4^2} \\ k &= \sqrt{25} \\ k &= 5 \end{aligned}$$

Dividing $k\sin a^\circ$ by $k\cos a^\circ$ allows $a^\circ$ to be calculated. Since the positive $\sin a^\circ$ means $a^\circ$ must lie in either the 1st quadrant (A) or the 2nd quadrant (S), and the positive $\cos a^\circ$ means $a^\circ$ must lie in the 1st quadrant (A) of the 4th quadrant (C), $a^\circ$ must lie in the 1st quadrant (A).

$$\begin{aligned} \frac{k\sin a^\circ}{k\cos a^\circ} &= \frac{3}{4} \\ \tan a^\circ &= \frac{3}{4} \\ a^\circ_{\text{acute}} &= \tan^{-1}\left(\frac{3}{4}\right) = 36.9^\circ \\ a^\circ &= 36.9^\circ \end{aligned}$$

$$\begin{array}{r|l} & a^\circ \\ ✓\text{S} & \text{A}✓✓ \\ \hline \text{T} & \text{C}✓ \end{array}$$

Hence $3\sin x^\circ + 4\cos x^\circ = 5\cos(x-36.9)^\circ$. Here it has been assumed that $k>0$ and $0<a<360$.

Some abbreviations of the working shown on the previous page are routinely permitted in the Higher exam. The following example demonstrates an appropriate level of detail in its solution.

**Example**

Express $3\cos x - \sin x$ in the form $k\cos(x-a)$ where $k > 0$ and $0 < a < 2\pi$.

$$3\cos x - \sin x = k\cos(x-a)$$

$$3\underline{\underline{\cos x}} - \underline{\sin x} = k\underline{\underline{\cos x}}\cos a + k\underline{\sin x}\sin a \quad \longleftarrow \text{Expand } k\cos(x-a)$$

$$k\sin a = -1 \quad \longleftarrow \text{Equate } \sin x \text{ coefficients}$$

$$k\cos a = 3 \quad \longleftarrow \text{Equate } \cos x \text{ coefficients}$$

$$k = \sqrt{(-1)^2 + 3^2} \quad \longleftarrow \text{Calculate } k$$

$$= \sqrt{10}$$

$$\tan a = \frac{-1}{3} \quad \longleftarrow \text{Find } \tan a$$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{1}{3}\right) = 18.4^\circ \quad \longleftarrow \text{Find acute angle first}$$

| S | A✓ |
|---|---|
| ✓T | C✓✓ |

$360^\circ$ − acute°

$\longleftarrow$ Negative $\sin a$: quadrants T and C

$\longleftarrow$ Positive $\cos a$: quadrants A and C

$$a^\circ = 341.6^\circ \quad \longleftarrow \text{Using } 360^\circ - 18.4^\circ \text{ (Double-ticked quadrant)}$$

$$a = 5.96 \quad \longleftarrow \text{Convert to radians: } 341.6 \times \frac{\pi}{180}$$

$$3\cos x - \sin x = \sqrt{10}\cos(x - 5.96) \quad \longleftarrow \text{State solution}$$

## Exercise 14.1

1. Express each of the following in the form $k\cos(x-a)^\circ$, where $k > 0$ and $0 < a < 360$. 🖩

   (a) $5\cos x^\circ + 12\sin x^\circ$ (b) $4\sin x^\circ + 5\cos x^\circ$ (c) $6\cos x^\circ + \sin x^\circ$

2. Express each of the following in the form $k\cos(x-a)$, where $k > 0$ and $0 < a < 2\pi$. 🖩

   (a) $8\sin x - 6\cos x$ (b) $\cos x - 3\sin x$ (c) $-2\sin x - \cos x$

3. Express each of the following in the form $k\cos(x-a)^\circ$, where $k > 0$ and $0 < a < 360$.

   (a) $\sin x^\circ + \cos x^\circ$ (b) $\cos x^\circ - \sqrt{3}\sin x^\circ$ (c) $-\sin x^\circ - \sqrt{3}\cos x^\circ$

## 14.2 Other Forms of the The Wave Function

As well as $k\cos(x-a)$, three other forms may be used, shown below including their expansions:

$$\begin{aligned} k\cos(x+a) &= k\cos x\cos a - k\sin x\sin a \\ k\sin(x+a) &= k\sin x\cos a + k\sin x\cos a \\ k\sin(x-a) &= k\sin x\cos a - k\sin x\cos a \end{aligned}$$

**Example**

Express $12\cos x^\circ - 5\sin x^\circ$ in the form $k\sin(x-a)^\circ$ where $k>0$ and $0<a<360$.

---

$$12\cos x^\circ - 5\sin x^\circ = k\sin(x-a)^\circ$$

$$12\underline{\underline{\cos x^\circ}} - 5\underline{\sin x^\circ} = k\underline{\sin x^\circ}\cos a^\circ - k\underline{\underline{\cos x^\circ}}\sin a^\circ \quad \longleftarrow \text{Expand } k\sin(x-a)^\circ$$

$$-k\sin a^\circ = 12 \implies k\sin a^\circ = -12 \quad \longleftarrow \text{Equate } \sin x^\circ \text{ coefficients}$$

$$k\cos a^\circ = -5 \quad \longleftarrow \text{Equate } \cos x^\circ \text{ coefficients}$$

$$\begin{aligned} k &= \sqrt{(-12)^2+(-5)^2} \quad \longleftarrow \text{Calculate } k \\ &= 13 \end{aligned}$$

$$\tan a^\circ = \frac{-12}{-5} = \frac{12}{5} \quad \longleftarrow \text{Find } \tan a^\circ$$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{12}{5}\right) = 67.4^\circ \quad \longleftarrow \text{Find acute angle first}$$

$$\begin{array}{r|l} ✓\text{S} & \text{A} \\ \hline ✓✓\text{T} & \text{C}✓ \end{array}$$

$180^\circ + \text{acute}^\circ$

$\longleftarrow$ Negative $\sin a^\circ$: quadrants T and C

$\longleftarrow$ Negative $\cos a^\circ$: quadrants S and T

$$a^\circ = 247.4^\circ \quad \longleftarrow \text{Using } 180^\circ + 67.4^\circ \text{ (Double-ticked)}$$

$$12\cos x^\circ - 5\sin x^\circ = 13\sin(x-247.4)^\circ \quad \longleftarrow \text{State solution}$$

**Exercise 14.2**

1. Express each of the following in the form $k\sin(x+a)^\circ$, where $k>0$ and $0<a<360$.

   (a) $3\cos x^\circ + 2\sin x^\circ$ (b) $4\sin x^\circ + 3\cos x^\circ$ (c) $7\cos x^\circ - \sin x^\circ$

2. Express each of the following in the form $k\cos(x+a)$, where $k>0$ and $0<a<2\pi$.

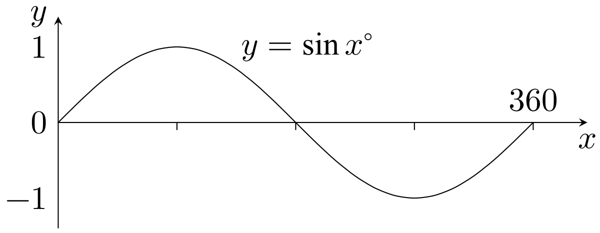
   (a) $3\sin x + 8\cos x$ (b) $-3\cos x - 4\sin x$ (c) $-6\sin x + 2\cos x$

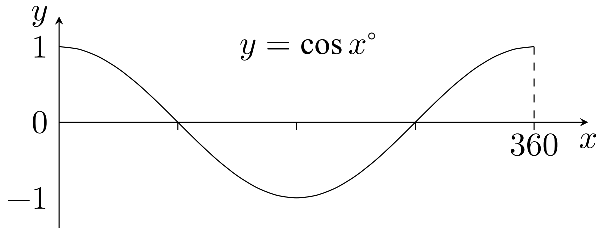
3. Express each of the following in the form $k\sin(x-a)$, where $k>0$ and $0<a<2\pi$.

   (a) $\sqrt{3}\sin x + \cos x$ (b) $\cos x - \sin x$ (c) $2\sqrt{3}\sin x - 2\cos x$

## 14.3 Maximum and Minimum Values using the Wave Function

The graphs of $y = \sin x°$ and $y = \cos x°$, including their turning points, should already be familiar:



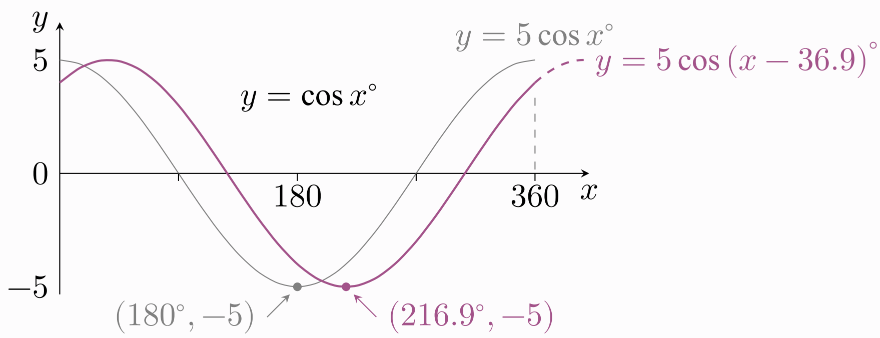




Graphs of the form $k \sin(x \pm a)$ and $k \cos(x \pm a)$ have maximum/minimum values of $\pm k$, and their *horizontal translation* is described by $\pm a$ following the rules covered in the Graph Transformations chapter.

**Example**

Given that $3\sin x° + 4\cos x°$ can be expressed as $5\cos(x - 36.9)°$, state the minimum value of $f(x) = 3\sin x° + 4\cos x°$ and the value of $x$ at which it occurs for $0 < x < 360$.

Sketch the graph of $y = 5\cos x°$ translated $36.9°$ to the right:




Use the graph to state the answer:

Hence the minimum value of $f(x)$ is $-5$, which occurs when $x = 216.9$.

**Exercise 14.3**

1. Sketch each for $0 < x < 360$, showing the coordinates of any roots and turning points:

   (a) $7\cos(x - 10)°$ (b) $3\sin(x - 20)°$ (c) $\sqrt{5}\cos(x + 40)°$

2. Find the maximum value of each and the value(s) of $x$ for which they occur for $0 < x < 360$:

   (a) $8\sin(x - 50)°$ (b) $\sqrt{2}\cos(x + 27)°$ (c) $7\sin(x + 42.3)°$

3. Find the minimum value of each and the value(s) of $x$ for which they occur for $0 < x < 2\pi$:

   (a) $3\cos\left(x - \frac{\pi}{6}\right)$ (b) $5\sin\left(x + \frac{\pi}{3}\right) + 2$ (c) $-12\sin\left(x + \frac{\pi}{4}\right)$

4. (a) Express $6\sin x° - 7\cos x°$ in the form $k\sin(x - a)°$ where $k > 0$ and $0 < a < 360$.

   (b) Hence state the coordinates of the turning points of $12\sin x° - 14\cos x°$ for $0 < x < 360$.

## 14.4 Solving Equations using the Wave Function

Solving an equation like $3\sin x^\circ + 4\cos x^\circ = 2$ can be approached by using the wave function to rewrite it as $5\cos(x - 36.9)^\circ = 2$, before solving it in the manner covered in Chapter 6.

**Example**

It can be shown that $5\sin x^\circ - 2\cos x^\circ$ can be expressed as $\sqrt{29}\sin(x - 21.8)^\circ$.
Hence, solve the equation $5\sin x^\circ - 2\cos x^\circ = 4$ where $0 < x < 360$.

---

$$5\sin x^\circ - 2\cos x^\circ = 4$$

$$\sqrt{29}\sin(x - 21.8)^\circ = 4 \qquad \longleftarrow \text{Substitute the wave function form}$$

$$\sin(x - 21.8)^\circ = \frac{4}{\sqrt{29}} \qquad \longleftarrow \text{Rearrange to } \sin(\dots) = \dots$$

$$a = \sin^{-1}\left(\frac{4}{\sqrt{29}}\right) = 48.0^\circ \qquad \longleftarrow \text{Calculate acute angle}$$

$$\begin{array}{c|c} {\scriptstyle 180^\circ - a^\circ} & {\scriptstyle a^\circ} \\ \checkmark \text{S} & \text{A} \checkmark \\ \hline \text{T} & \text{C} \end{array} \qquad \longleftarrow \text{Positive } \sin a^\circ\text{: quadrants A and S}$$

$$x - 21.8^\circ = 48.0^\circ, 180^\circ - 48.0^\circ \qquad \longleftarrow \text{Apply ticked quadrants}$$

$$x - 21.8^\circ = 48.0^\circ, 132.0^\circ$$

$$x^\circ = 69.8^\circ, 153.8^\circ \qquad \longleftarrow \text{Add } 21.8^\circ \text{ to both sides}$$

Note that any solutions outwith the domain (often $0 < x < 360$) should have $360^\circ$ added or subtracted to bring it back within the domain, where possible.

**Exercise 14.4**

1. Solve each equation for $0 < x < 360$: 🖩

   (a) $7\sin(x - 18)^\circ = 4$ (b) $3\cos(x + 34.1)^\circ = -2$ (c) $\sqrt{5}\sin(x - 106)^\circ + 1 = 0$

2. Given $6\sin x^\circ - 8\cos x^\circ = 10\sin(x - 53.1)^\circ$, solve $6\sin x^\circ - 8\cos x^\circ = 5$ where $0 < x < 360$.

3. (a) Express $\sqrt{3}\sin x^\circ + \cos x^\circ$ in the form $k\sin(x - a)^\circ$ where $k > 0$ and $0 < a < 360$.

   (b) Hence solve the equation $\sqrt{3}\sin x^\circ + \cos x^\circ = 1$ where $0 < x < 360$.

4. Solve each equation for $0 < x < 2\pi$: 🖩

   (a) $4\sin(x + 0.31) + 2 = 1$ (b) $9\cos(x + 1.24) = 5$ (c) $2\sqrt{3}\sin(x - 0.82) = \sqrt{5}$

5. (a) Express $3\cos x + 2$ in the form $k\cos(x + a)$ where $k > 0$ and $0 < a < 2\pi$. 🖩

   (b) Hence solve the equation $2 + 6\sin x + 4\cos x = 5$ where $0 < x < 2\pi$. 🖩

## 14.5 Multiple Angles and Different Variables

The techniques covered in this chapter can be applied to trigonometric expressions beyond those containing only $\sin x$ and $\cos x$; they work for any sum or difference of *equal-angled* trigonometric operations.

e.g. $3\sin t^\circ + 4\cos t^\circ$ or $2\sin 2x - 5\cos 2x$

**Example**

Express $5\cos 2t^\circ - 3\sin 2t^\circ$ in the form $k\sin(2t+a)^\circ$ where $k > 0$ and $0 < a < 360$.

---

$$5\cos 2t^\circ - 3\sin 2t^\circ = k\sin(2t+a)^\circ$$

$$5\underline{\underline{\cos 2t^\circ}} - 3\underline{\sin 2t^\circ} = k\underline{\sin 2t^\circ}\cos a^\circ + k\underline{\underline{\cos 2t^\circ}}\sin a^\circ$$ ⟵ Expand $k\sin(2t+a)^\circ$

$$k\sin a^\circ = 5$$ ⟵ Equate $\sin 2t^\circ$ coefficients

$$k\cos a^\circ = -3$$ ⟵ Equate $\cos 2t^\circ$ coefficients

$$k = \sqrt{(5)^2 + (-3)^2}$$ ⟵ Calculate $k$

$$= \sqrt{34}$$

$$\tan a^\circ = \frac{5}{-3} = -\frac{5}{3}$$ ⟵ Find $\tan a^\circ$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{5}{3}\right) = 59.0^\circ$$ ⟵ Find acute angle first

$180^\circ - \text{acute}^\circ$

| | |
|---|---|
| ✓✓S | A✓ |
| ✓T | C |

⟵ Positive $\sin a^\circ$: quadrants A and S

⟵ Negative $\cos a^\circ$: quadrants S and T

$$a^\circ = 121.0^\circ$$ ⟵ Using $180^\circ - 59.0^\circ$ (Double-ticked)

$$5\cos 2t^\circ - 3\sin 2t^\circ = \sqrt{34}\sin(2t+121.0)^\circ$$ ⟵ State solution

### Exercise 14.5

1. Express $4\cos t^\circ - 3\sin t^\circ$ in the form $k\sin(t-a)^\circ$, where $k > 0$ and $0 < a < 360$. 🖩
2. Express $2\sin 2x - \cos 2x$ in the form $k\cos(2x-a)$ where $k > 0$ and $0 < a < 2\pi$. 🖩
3. (a) Express $12\cos t^\circ + 5\sin t^\circ$ in the form $k\sin(t+a)^\circ$, where $k > 0$ and $0 < a < 360$. 🖩
   (b) Hence state:
      i. The maximum value of the function $f(x) = 12\cos t^\circ + 5\sin t^\circ$, $0 < t < 360$.
      ii. The value(s) of $t$ for which it occurs.
4. (a) Express $\sin 2x - \sqrt{3}\cos 2x$ in the form $k\cos(2x-a)$ where $k > 0$, $0 < a < 2\pi$.
   (b) Hence solve $\sin 2x - \sqrt{3}\cos 2x - 1 = 0$, $0 < x < 2\pi$.
   (c) Sketch $y = -\sin 2x + \sqrt{3}\cos 2x - 1 = 0$ for $0 \leqslant x \leqslant 2\pi$.

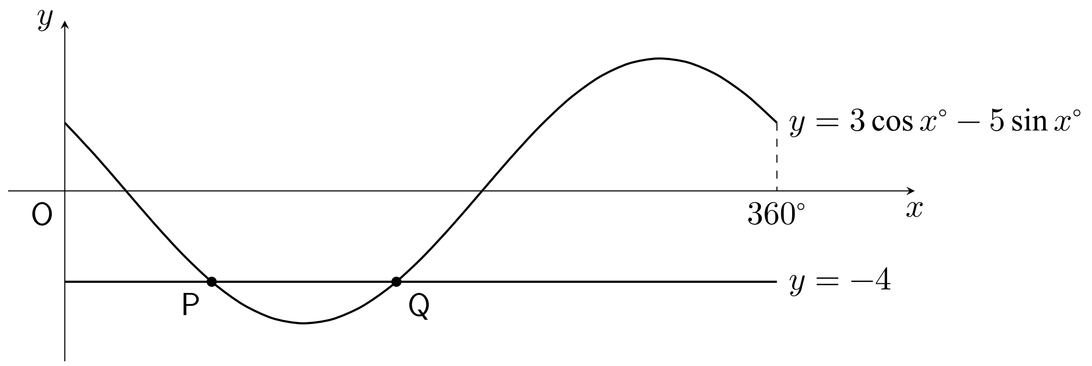
## Wave Function Review Exercise

1. Express $8\sin x^\circ + 7\cos x^\circ$ in the form $k\sin(x-a)^\circ$ where $k>0$ and $0<a<360$.

2. Express $\sqrt{5}\sin x + \cos x$ in the form $k\cos(x-a)$ where $k>0$ and $0<a<2\pi$.

3. Express $\sqrt{3}\cos t^\circ - \sin t^\circ$ in the form $k\sin(t+a)^\circ$ where $k>0$ and $0<a<360$.

4. Part of the graphs of $y = 3\cos x^\circ - 5\sin x^\circ$ and $y=-4$ are shown in the diagram below:




   Points P and Q are points of intersection.

   (a) Express $y = 3\cos x^\circ - 5\sin x^\circ$ in the form $k\cos(x+a)^\circ$ where $k>0$ and $0<a<360$.

   (b) Hence determine the coordinates of P and Q.

5. (a) Express $2\sin x^\circ - 4\cos x^\circ$ in the form $k\sin(x-a)^\circ$ where $k>0$ and $0<x<360$.

   (b) Hence sketch the graph of $y = 2\sin x^\circ - 4\cos x^\circ$ for $0<x<360$.

6. (a) Express $\cos x + \sqrt{3}\sin x$ in the form $k\cos(x-a)$ where $k>0$ and $0<a<2\pi$.

   (b) Hence sketch the graph of $y = 2\cos x + 2\sqrt{3}\sin x$ for $0<x<2\pi$.

7. (a) Express $6\cos t - 3\sin t$ in the form $k\cos(t+a)$ where $k>0$ and $0<a<2\pi$.

   (b) Hence solve $2\cos t - \sin t + 2 = 1$ where $0<t<2\pi$.

8. (a) Express $\sin 2x^\circ - \cos 2x^\circ$ in the form $k\sin(2x-a)$ where $k>0$ and $0<a<360$.

   (b) Hence solve the equation $\sin 2x^\circ = \cos 2x^\circ$ where $0<x<360$.

# Chapter 15 Logs and Exponentials

Each part of an operation of the form $a^n$ can be described using the following terminology:

"base" $\longrightarrow a^n \longleftarrow$ "power" or "exponent" or "index"

*Power functions* take the form $f(x) = x^n$, with $x$ as the *base* and a constant power, $n$.

$$\text{e.g.} \quad f(x) = x^3$$

*Exponential functions* take the form $f(x) = a^x$, with $x$ as the *exponent* and a constant base, $a > 0, a \neq 1$.

$$\text{e.g.} \quad f(x) = 3^x$$

An example of an exponential function in everyday life is that of something *appreciating* by a percentage of its value, such as an antique vase of value £4000 increasing by 20% each year. Its value after $0$ years, $1$ years, $2$ years, and so on, can be calculated as follows:

$$\begin{aligned}
4000 \times 1.2^{0} &= 4000 && \longleftarrow \text{After 0 years} \\
4000 \times 1.2^{1} &= 4800 && \longleftarrow \text{After 1 year} \\
4000 \times 1.2^{2} &= 5760 && \longleftarrow \text{After 2 years} \\
4000 \times 1.2^{3} &= 6912 && \longleftarrow \text{After 3 years} \\
4000 \times 1.2^{4} &= 8294.40 && \longleftarrow \text{After 4 years}
\end{aligned}$$

The function to describe its value $V$ after $x$ years is given by: $V(x) = 4000 \times 1.2^x$

One advantage of defining this function is the ability to calculate the value at times other that after whole years. For example, the value after three and a half years can be calculated as:

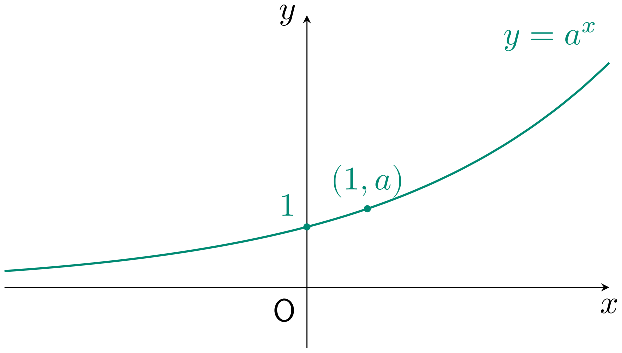
$$V(3.5) = 4000 \times 1.2^{3.5} = 7571.72$$

This chapter will introduce a range of skills required when working with exponential functions.

## 15.1 Graphs of Exponential Functions

Where $a > 1$:

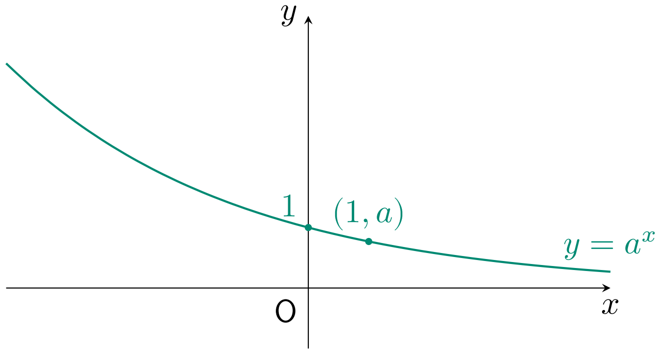
$y = a^x$ is *strictly increasing* on $x \in \mathbb{R}$:




This describes **exponential growth**.

Where $0 < a < 1$:

$y = a^x$ is *strictly decreasing* on $x \in \mathbb{R}$:




This describes **exponential decay**.

Since $a^0 = 1$, any graph of the form $y = a^x$ will pass through the point $(0, 1)$ for all $a \neq 0$.

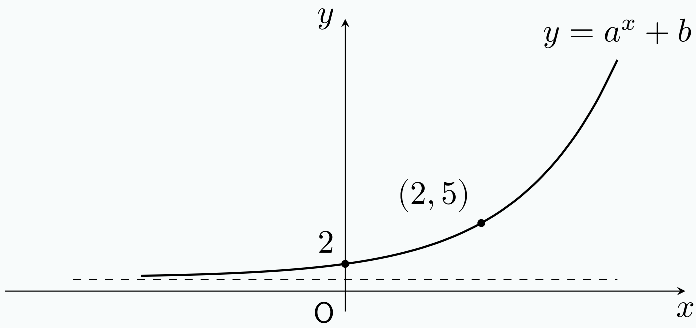
Since $a^1 = a$, any graph of the form $y = a^x$ will pass through the point $(1, a)$ for all $a$.

The graph of $y = a^x$ has the $x$-axis as a *horizontal asymptote* - a line that it approaches but never meets.

Determining the equation of the graph of an exponential function can typically be achieved using substitution or consideration of graph transformations, along with knowledge of points $(0, 1)$ and $(1, a)$.

**Example**

The graph of $y = a^x + b$ is below. Find the values of $a$ and $b$, and state the range of $f(x) = a^x + b$.




$2 = a^0 + b$ $\longleftarrow$ Substitute $(0, 2)$

$2 = 1 + b$

$1 = b$ $\longleftarrow$ Solve to obtain $b$

$5 = a^2 + 1$ $\longleftarrow$ Substitute $(2, 5)$ and $b = 1$

$4 = a^2$

$2 = a$ $\longleftarrow$ Solve to obtain $a$

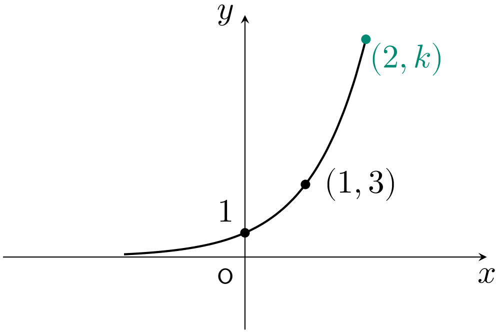
$y = 2^x + 1$ $\longleftarrow$ State equation

$f(x) \geqslant 1$ $\longleftarrow$ State range

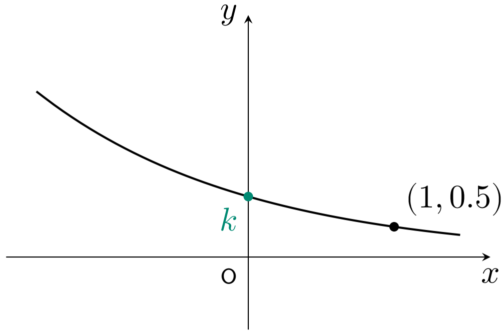
## Exercise 15.1

1. Find the equation of each exponential graph using the form given, then determine the value of $k$.

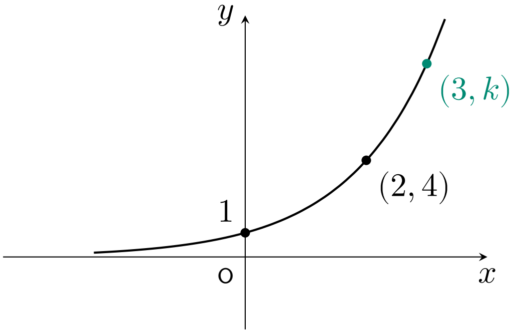
(a) $y = a^x$



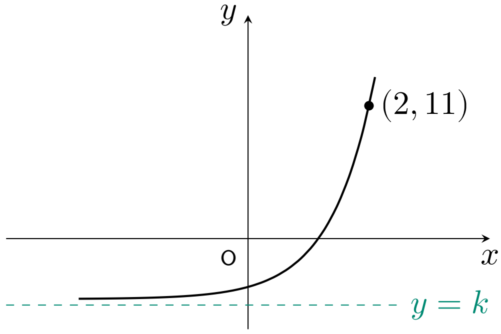

(b) $y = a^x$



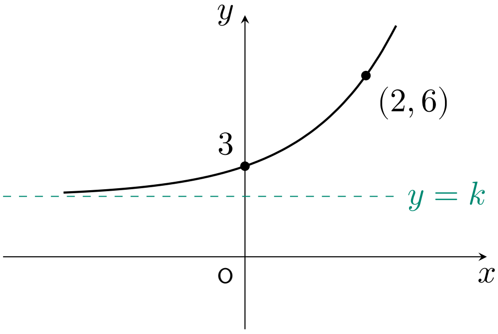

(c) $y = a^x$



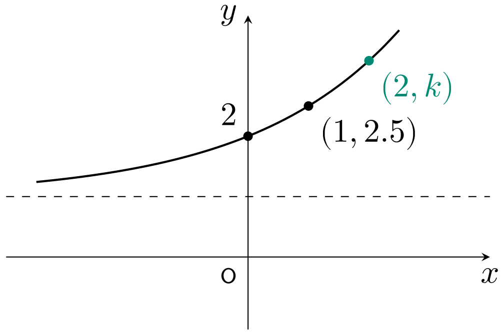

(d) $y = 4^x + b$



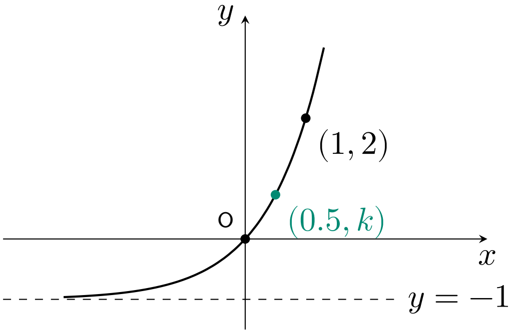

(e) $y = a^x + 2$



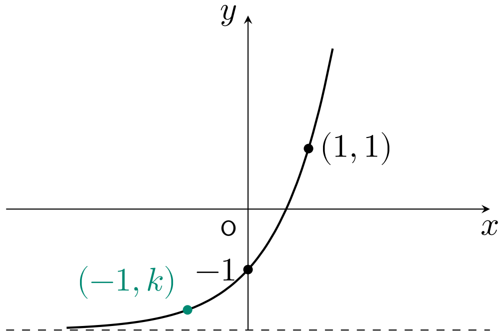

(f) $y = a^x + b$




(g) $y = a^x + b$




(h) $y = a^x + b$




2. Sketch the graph of each of the following, and state the range of $y = f(x)$:

(a) $y = 3^x$ (b) $y = 3^x + 1$ (c) $y = 3^x - 2$ (d) $y = 3^{x+2}$

(e) $y = 0.5^x$ (f) $y = 0.5^x - 1$ (g) $y = 0.5^{x-3}$ (h) $y = 0.5^{1-x}$

## 15.2 Evaluating Logarithms

The function $f(x) = \log_a x$, where $a > 0$ and $x > 0$, is defined as the inverse function to $f(x) = a^x$.

$$\begin{array}{lll} \text{If} & x = a^y & \\ \text{then} & \log_a x = y & (\text{where } a > 0, x > 0) \end{array}$$

A numerical example will help build an understanding of how to interpret a logarithm:

$$\begin{array}{ll} \text{Since} & 8 = 2^3 \\ \text{then} & \log_2 8 = 3 \end{array}$$

Hence $\log_2 8$ ("log to the base 2 of 8") can be read as: "2 raised to *which power* gives a value of 8?"

**Example**

Evaluate $\log_3 81$.

$\log_3 81 = 4 \longleftarrow$ Since $3^4 = 81$

**Example**

Evaluate $\log_5 1$.

$\log_5 1 = 0 \longleftarrow$ Since $5^0 = 1$

**Example**

Evaluate $\log_7 49$.

$\log_7 49 = 2 \longleftarrow$ Since $7^2 = 49$

**Example**

Evaluate $\log_3 \frac{1}{9}$.

$\log_3 \frac{1}{9} = -2 \longleftarrow$ Since $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

### Exercise 15.2

1. State the value each of the following logarithms:

(a) $\log_6 36$ (b) $\log_5 125$ (c) $\log_4 16$ (d) $\log_2 16$ (e) $\log_9 81$

(f) $\log_3 81$ (g) $\log_2 32$ (h) $\log_{10} 100$ (i) $\log_8 64$ (j) $\log_4 64$

(k) $\log_7 7$ (l) $\log_3 9$ (m) $\log_8 1$ (n) $\log_5 25$ (o) $\log_5 5$

2. By considering negative and fractional indices as well as integers, evaluate each:

(a) $\log_2 32$ (b) $\log_4 4$ (c) $\log_8 1$ (d) $\log_9 3$ (e) $\log_{49} 7$

(f) $\log_9 9$ (g) $\log_9 1$ (h) $\log_2 2$ (i) $\log_2 1$ (j) $\log_2 \frac{1}{2}$

(k) $\log_3 \frac{1}{9}$ (l) $\log_{25} 5$ (m) $\log_2 \frac{1}{8}$ (n) $\log_4 \frac{1}{16}$ (o) $\log_8 4$

3. Calculate the value of each to three significant figures:

(a) $\log_2 31$ (b) $\log_7 50$ (c) $\log_{10} 99$ (d) $\log_5 120$ (e) $\log_2 0.9$

## 15.3 Laws of Logarithms

There are two key identities that can be obtained from the definition of a logarithm. For all $a > 0$:

$$\boxed{\log_a 1 = 0} \quad \text{and} \quad \boxed{\log_a a = 1}$$

There also three *"log laws"* that can be applied for logarithms of the *same base*. For all $a, x, y > 0$:

The **product law**: $\log_a (xy) = \log_a x + \log_a y$

The **quotient law**: $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

The **power law**: $\log_a (x^y) = y \log_a x$

**Example**

Evaluate $\log_6 8 + 2\log_6 3 - \log_6 2$.

$$\begin{aligned}
&\log_6 8 + 2\log_6 3 - \log_6 2 && \longleftarrow \text{Consider order of operations} \\
=&\log_6 8 + \log_6 (3^2) - \log_6 2 && \longleftarrow \text{Apply power law} \\
=&\log_6 8 + \log_6 9 - \log_6 2 \\
=&\log_6 (8 \times 9) - \log_6 2 && \longleftarrow \text{Apply product law} \\
=&\log_6 72 - \log_6 2 \\
=&\log_6 \left(\frac{72}{2}\right) && \longleftarrow \text{Apply product law} \\
=&\log_6 36 \\
=&2 && \longleftarrow \text{Evaluate}
\end{aligned}$$

### Exercise 15.3

1. Evaluate each of the following:

(a) $\log_6 9 + \log_6 4$ (b) $\log_{12} 2 + \log_{12} 6$ (c) $\log_2 3 + \log_2 \frac{1}{3}$

(d) $\log_3 54 - \log_3 2$ (e) $\log_5 10 - \log_5 2$ (f) $\log_5 100 - \log_5 4$

(g) $2\log_4 6 - \log_4 9$ (h) $\log_{12} 4 + 2\log_{12} 6$ (i) $3\log_6 2 + \log_6 9 - \log_6 2$

(j) $2\log_6 3 + \log_6 \frac{2}{3}$ (k) $\log_5 4 + 2\log_5 10$ (l) $2\log_2 6 - \log_2 12 + \log_2 \frac{1}{6}$

2. Simplify each, giving an answer in the form $\log_p q$ where $p$ and $q$ are positive integers:

(a) $\log_7 3 + \log_7 6 - \log_7 9$ (b) $4\log_6 2 - \log_6 2$ (c) $2\log_8 9 - 3\log_2 3$

3. Simplify each, giving an answer in the form $\log_a p$ where $p$ is a positive integer:

(a) $\log_a 3 + 2\log_a 4$ (b) $3\log_a 2 - \log_a 4$ (c) $\log_a 20 - \log_a 4 + \log_a \frac{1}{15}$

## 15.4 Solving Log Equations

To solve a *log equation*, it is typically desirable to have each side within a logarithm of the *same base*:

| | | |
|---|---|---|
| If | $\log_a x = \log_a y$ | |
| then | $x = y$ | (where $a > 0, x > 0, y > 0$) |

"Cancelling the logs", mathematically, is raising each side to an exponential base $a$, using $a^{\log_a b} = b$.

**Example**

Solve the equation $\log_3 x + \log_3 5 = \log_3 20$.

$$\begin{aligned} \log_3 x + \log_3 5 &= \log_3 20 \\ \log_3 5x &= \log_3 20 && \longleftarrow \text{Apply product law} \\ 5x &= 20 && \longleftarrow \text{"Cancel" } \log_3 \text{ from both sides} \\ x &= 4 && \longleftarrow \text{Solve} \end{aligned}$$

To express any term as a logarithm, multiply it by $\log_a a$ (which equals 1) and then apply the power law:

**Example**

Solve the equation $\log_2 40 - \log_2 x = 3$.

$$\begin{aligned} \log_2 40 - \log_2 x &= 3 \log_2 2 && \longleftarrow \text{Multiply the constant by } \log_2 2 \\ \log_2 \left(\frac{40}{x}\right) &= \log_2 2^3 && \longleftarrow \text{Apply quotient and power laws} \\ \frac{40}{x} &= 8 && \longleftarrow \text{"Cancel" } \log_2 \text{ from both sides} \\ 40 &= 8x && \longleftarrow \text{Multiply both sides by } x \\ 5 &= x && \longleftarrow \text{Solve} \end{aligned}$$

### Exercise 15.4

1. Solve:

   (a) $\log + \log = \log$ (b) $\log + \log = \log$ (c) $\log + \log = \log$

   (d) $\log + \log = \log$ (e) $\log + \log = \log$ (f) $\log + \log = \log$

2. Solve each:

   (a) $\log + \log = \log$ (b) $\log + \log = \log$ (c) $\log + \log = \log$

   (d) $\log + \log = \log$ (e) $\log + \log = \log$ (f) $\log + \log = \log$

Since $\log_a x$ is only defined for $a > 0$ and $x > 0$, any solutions which contradict this must be discarded.

**Example**

Solve the equation $\log_3 x + \log_3 (x-2) = \log_3 8$.

$$\begin{aligned}
\log_3 x + \log_3 (x-2) &= \log_3 8 && \longleftarrow \text{Note } x > 0 \text{ and } x-2>0 \text{ therefore } x>2 \\
\log_3 (x(x-2)) &= \log_3 8 && \longleftarrow \text{Apply product law} \\
x^2 - 2x &= 8 && \longleftarrow \text{"Cancel" } \log_3 \text{ and expand} \\
x^2 - 2x - 8 &= 0 && \longleftarrow \text{Equate quadratic equation to zero} \\
(x+2)(x-4) &= 0 && \longleftarrow \text{Factorise} \\
x+2=0, x-4 &= 0 && \\
\cancel{x=-2}, x &= 4 && \longleftarrow \text{Discard } x=-2 \text{ since } x>2
\end{aligned}$$

3.

## 15.5 Graphs of Logarithmic Functions

## Exercise 15.5

## 15.6 Exponential Growth and Decay

## Exercise 15.6

## 15.7 Experimental Data of the form $y = ab^x$

## Exercise 15.6

## 15.8 Experimental Data of the form $y = ax^b$

## Exercise 15.7

## Logs and Exponentials Review Exercise

# Chapter 16 Further Calculus

# Answers

# Challenge Problems

The following problems **do not** represent the kind of question expected to feature in a Higher Mathematics exam, either in the way they are presented or the level of difficulty. Instead, they aim to encourage a flexible approach towards problem-solving and an understanding that the skills covered in the course have applications beyond those featured in any typical exam. *Some questions may be solveable without using the skills covered in this chapter, and some questions may be unreleated to this chapter.*