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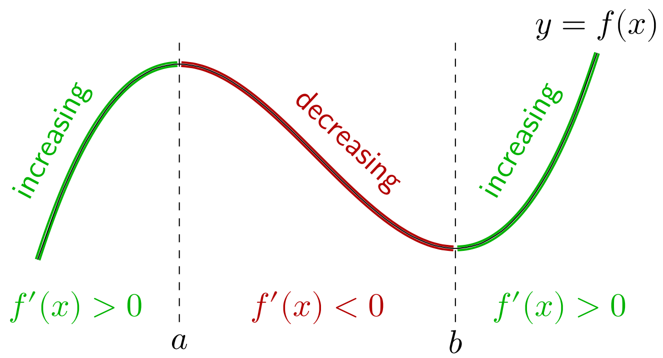
Chapter 9

Differentiation II

9.1 Increasing and Decreasing Functions

A function can be said to be *increasing* for values of x for which a tangent to the curve has a *positive gradient*, and *decreasing* where a tangent to the curve has a *negative gradient*.



A function $f(x)$ is:

- Increasing when $f'(x) > 0$.
- Decreasing when $f'(x) < 0$.

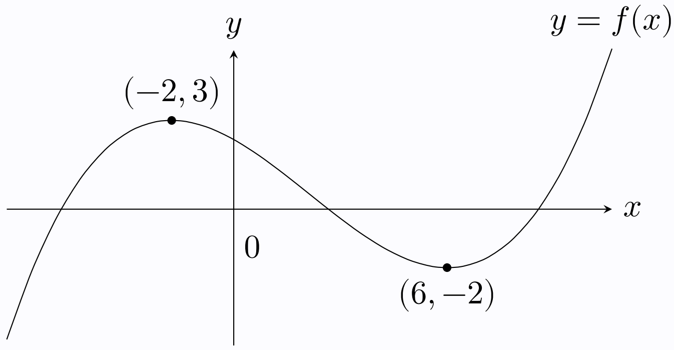
Function f above is *increasing* for $x < a,\ x > b$ and *decreasing* for $a < x < b$.

Example 9.1.1

Part of the graph of a function $f(x)$ is below. State the values of x for which $f(x)$ is:

(a) Increasing.

(b) Decreasing.



(a) The function $f(x)$ is increasing for $x < -2$ and $x > 6$.

(b) The function $f(x)$ is decreasing for $-2 < x < 6$.

Evaluating $f'(x)$ can show whether a function is increasing or decreasing at that point:

Example 9.1.2

Determine whether the function $f(x) = 5x^2 - 6\sqrt{x}^3$ is increasing or decreasing when $x = 4$.

$$\begin{aligned} f(x) &= 5x^2 - 6x^{\frac{3}{2}} && \longleftarrow \text{prepare to differentiate} \\ f'(x) &= 10x - 9x^{\frac{1}{2}} && \longleftarrow \text{differentiate} \\ &= 10x - 9\sqrt{x} && \longleftarrow \text{write in radical form} \\ f'(4) &= 10(4) - 9\sqrt{4} && \longleftarrow \text{substitute} \\ &= 22 && \longleftarrow \text{evaluate} \end{aligned}$$

Since $f'(4) > 0,$ the function f is increasing when $x = 4$.

The range of values of x for which a function is increasing or decreasing can also be determined:

Example 9.1.3

Determine the range of values of x for which the function $f(x) = 2x^2 - 8x + 11$ is increasing.

$$\text{Increasing} \implies f'(x) > 0$$

$$\begin{aligned} f'(x) &= 4x - 8 && \longleftarrow \text{differentiate} \\ 4x - 8 &> 0 && \longleftarrow \text{apply condition} \\ 4x &> 8 && \\ x &> 2 && \longleftarrow \text{solve} \end{aligned}$$

So $f'(x)$ is increasing when $x > 2$

Quadratic inequations may arise when finding ranges, which require a sketch when solving.

Example 9.1.4

Determine the range of values of x for which $y = \frac{2}{3}x^3 + 3x^2 - 8x + 7$ is decreasing.

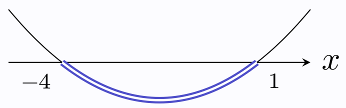
$$\text{Decreasing} \implies \frac{\mathrm{d}y}{\mathrm{d}x} < 0$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 2x^2 + 6x - 8 \\ 2x^2 + 6x - 8 &< 0 \\ x^2 + 3x - 4 &< 0 \end{aligned}$$

quadratic inequation

Consider roots:

$$\begin{aligned} x^2 + 3x - 4 &= 0 \\ (x+4)(x-1) &= 0 \\ x &= -4, x = 1 \end{aligned}$$



So $f'(x)$ is decreasing when $-4 < x < 1$

A function can be said to be **strictly increasing** if $f'(x) > 0$ across its domain.

A function can be said to be **strictly decreasing** if $f'(x) < 0$ across its domain.

Example 9.1.5

Show that $f(x) = \frac{1}{3}x^3 - 4x^2 + 21x - 10$ is strictly increasing for all values of x.

Note that $x^2 - 8x + 1$ *can be expressed as* $(x-4)^2 + 5$.

$$\begin{aligned} f'(x) &= x^2 - 8x + 21 && \longleftarrow \text{differentiate} \\ &= (x-4)^2 + 5 && \longleftarrow \text{complete the square} \end{aligned}$$

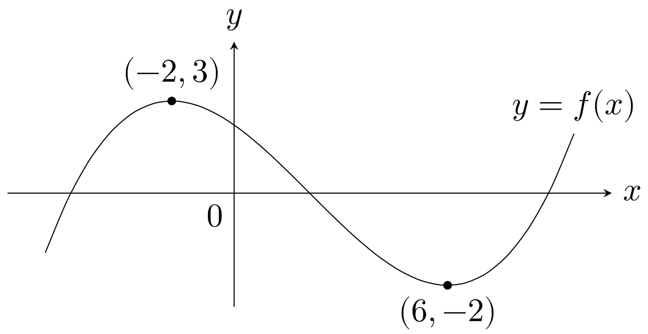
$\therefore$ minimum value of $f'(x)$ is 5 $\longleftarrow$ state minimum value of f'

Since $5 > 0$, $f'(x) > 0$ for all x, $\longleftarrow$ compare to zero

so the function f is strictly increasing. $\longleftarrow$ conclusion about f

9.2 Stationary Points

A *stationary point* can be found where a tangent to a curve has a *gradient of zero*, or:



Stationary points occur where $\frac{dy}{dx} = 0$

The graph of $y = f(x)$ has stationary points at $(-2, 3)$ and $(6, -2)$.

Example 9.2.1

Find the coordinates of the stationary points on the curve $y = x^3 - \frac{9}{2}x^2 + 6x$.

Find x-coordinate(s)

Stationary points occur where $\frac{dy}{dx} = 0$

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 9x + 6\\ 3x^2 - 9x + 6 &= 0\\ x^2 - 3x + 2 &= 0\\ (x-1)(x-2) &= 0\\ x &= 1, x = 2\end{aligned}$$

Find y-coordinates

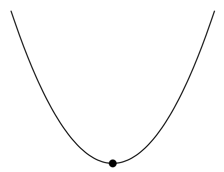
When $x = 1$, $\quad y = (1)^3 - \frac{9}{2}(1)^2 + 6(1)$
$\quad = \frac{5}{2}$

When $x = 2$, $\quad y = (2)^3 - \frac{9}{2}(2)^2 + 6(2)$
$\quad = 2$

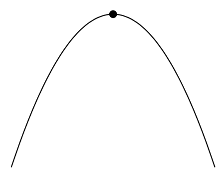
$\therefore$stationary points occur at $\left(1, \frac{5}{2}\right)$ and $(1, 2)$

Either Leibniz notation $\left(\frac{dy}{dx}\right)$ or function notation $(f'(x))$ may be used depending on the context.

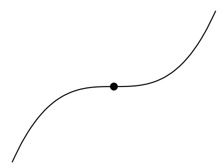
Determining the nature of a stationary point means to find out *what type* it is. There are four:



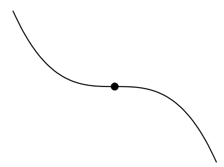
minimum turning point



maximum turning point



rising point of inflection



falling point of inflection

The **second derivative**, $f''(x)$ or $\frac{d^2y}{dx^2}$, is obtained by **differentiating the first derivative**.

For example:

$$\begin{aligned}f(x) &= 4x^3 - 3x^2 + 7x - 1\\ f'(x) &= 12x^2 - 6x + 7\\ f''(x) &= 24x - 6\end{aligned}$$

Stationary point (a, b) is a:

- **maximum turning point** if $f''(a) < 0$
- **minimum turning point** if $f''(a) > 0$

Example 9.2.2

Determine the coordinates and nature of the stationary points of $f(x) = x^3 - 12x^2 + 36x$.

Find x-coordinate(s)

SPs occur where $f'(x) = 0$

$$\begin{aligned} f'(x) &= 3x^2 - 24x + 36 \\ 3x^2 - 24x + 36 &= 0 \\ x^2 - 8x + 12 &= 0 \\ (x-2)(x-6) &= 0 \\ x = 2, x &= 6 \end{aligned}$$

Find y-coordinates

$$\begin{aligned} f(2) &= (2)(2-6)^2 \\ &= 32 \end{aligned}$$

$$\begin{aligned} f(6) &= (6)(6-6)^2 \\ &= 0 \end{aligned}$$

Determine Nature

$$f''(x) = 6x - 24$$

$$\begin{aligned} f''(2) &= 6(2) - 24 \\ &= -12 \end{aligned}$$

$f''(2) < 0 \therefore$ max TP

$$\begin{aligned} f''(6) &= 6(6) - 24 \\ &= 12 \end{aligned}$$

$f''(6) > 0 \therefore$ min TP

$\therefore$maximum turning point at $(2, 32)$ and minimum turning point at $(6, 0)$

If the second derivative is 0, a **nature table** can be used instead, where $\frac{\mathrm{d}y}{\mathrm{d}x}$ is evaluated (> 0 or < 0) for values of x either side of a stationary point, to visualise its nature. For stationary point (a, b):

x	$\longrightarrow$	a	$\longrightarrow$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$-ve$	0	$+ve$
slope	\	—	/

minimum turning point

x	$\longrightarrow$	a	$\longrightarrow$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+ve$	0	$-ve$
slope	/	—	\

maximum turning point

x	$\longrightarrow$	a	$\longrightarrow$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+ve$	0	$+ve$
slope	/	—	/

rising point of inflection

x	$\longrightarrow$	a	$\longrightarrow$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$-ve$	0	$-ve$
slope	\	—	\

falling point of inflection

Example 9.2.3

Find the coordinates and nature of the stationary point on the curve $y = 13 + 6x^2 - 12x - x^3$.

Find x-coordinate(s)

SPs occur where $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 12x - 12 - 3x^2$$

$$12x - 12x - 3x^2 = 0$$

$$4x - 4 - x^2 = 0$$

$$x^2 - 4x - 4 = 0$$

$$(x-2)(x-2) = 0$$

$$x = 2$$

Find y-coordinate

When $x = 2$,

$$y = 13 + 6(2)^2 - 12(2) - (2)^3$$
$$= 13 + 24 - 24 - 8$$
$$= 5$$

Determine Nature

$$\frac{d^2y}{dx^2} = -12 - 6x$$

When $x = 2$,

$$\frac{d^2y}{dx^2} = -12 - 6(2)$$
$$= 0$$

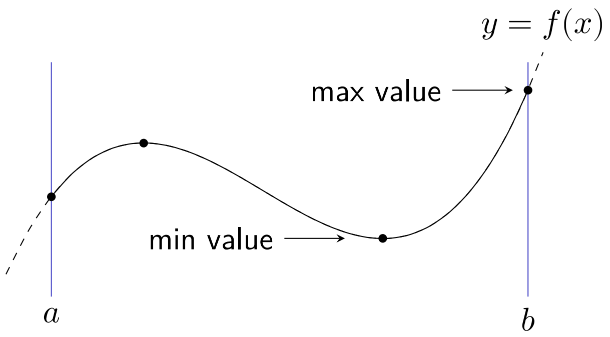
∴nature table

x	$\xrightarrow{1}$	2	$\xrightarrow{3}$
$\frac{dy}{dx}$	$-ve$	0	$-ve$
slope	╲	—	╲

∴falling point of inflection at $(2, 5)$

9.3 Closed Intervals

Given a smooth, continuous function across a *closed interval*, finding the coordinates and nature of stationary points can allow the maximum and minimum values to be determined.



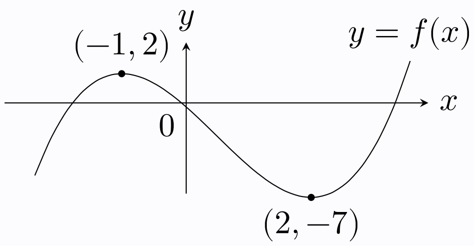
Maximum and minimum values can occur at:

- Turning points
- *Bounds* of the interval

For $f(x)$, the maximum value occurs when $x = b$ and the minimum at the minimum turning point.

Example 9.3.1

Function $f(x) = \frac{2}{3}x^3 - x^2 - 4x - \frac{1}{3}$ has two stationary points, and part of the graph of $y = f(x)$ is shown below. Find the maximum and minimum values of f on the interval $-3 \leqslant x \leqslant 3$.



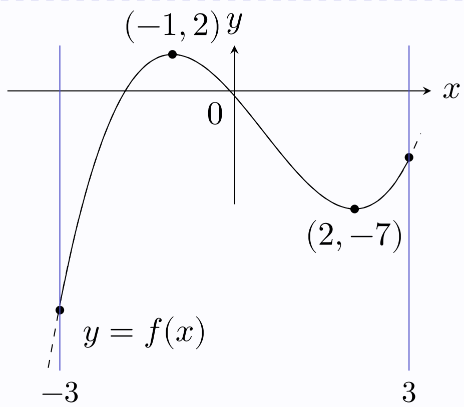
Check y-coordinates at the lower and upper bounds:

When $x = -3$, $\quad y = \frac{2}{3}(-3)^3 - (-3)^2 - 4(-3) - \frac{1}{3} = -\frac{46}{3}$

When $x = 3$, $\quad y = \frac{2}{3}(3)^3 - (3)^2 - 4(3) - \frac{1}{3} = -\frac{10}{3}$

$\therefore$ maximum value of f is 2, which occurs when $x = -1$

minimum value of f is $-\dfrac{46}{3}$, which occurs when $x = -3$



Where a function is *strictly increasing* or *strictly decreasing* in a closed interval, its maximum and minimum values can *only* be located at the bounds of the interval.

Example 9.3.2

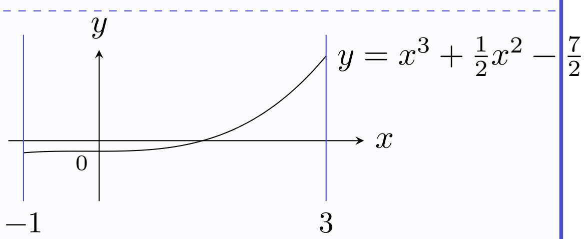
Given $y = x^3 + \frac{1}{2}x^2 - \frac{7}{2}$ is strictly increasing, find the greatest and least values of y in the interval $-1 \leqslant x \leqslant 3$.

When $x = -1$, $\quad y = (-2)^3 + \frac{1}{2}(-1)^2 - \frac{7}{2} = -4$

When $x = 3$, $\quad y = (3)^3 + \frac{1}{2}(3)^2 - \frac{7}{2} = 28$

$\therefore$ the maximum value of y is 28

and the minimum value of y is -4



9.4 Maximum and Minimum Values

If a smooth, continuous function has only one stationary point within its domain, and it is a *maximum turning point*, the *greatest value* of the function will be located at the turning point. The corresponding will be true for the *least* value of a function with only a *minimum* turning point.\[0.1em]

Example 9.4.1

Determine the value of x for which the function f, defined by $f(x) = x + \frac{1}{x}$, $x > 0$, has its minimum value.

Find x-coordinate(s)

Stationary points occur where $f'(x) = 0$

$$\begin{aligned} f(x) &= x + x^{-1} \\ f'(x) &= 1 - x^{-2} \\ 1 - x^{-2} &= 0 \\ 1 - \frac{1}{x^2} &= 0 \\ x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= 1 \text{ (Since } x > 0) \end{aligned}$$

Determine Nature

$$\begin{aligned} f''(x) &= 2x^{-3} \\ &= \frac{2}{x^3} \\ f''(1) &= \frac{2}{1^3} \\ &= 2 \end{aligned}$$

Since $f''(1) > 0$, min TP when $x = 1$

$\therefore$ f is minimised when $x = 1$

When using variables other than x and y (and function f), care should be taken with notation.

Example 9.4.2

Determine the maximum value of P given $P = 6t - t^3$, where $t > 0$, and the value of t for which it occurs.

Find x-coordinate(s)

Stationary points occur where $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= 6 - 3t^2 \\ 6 - 3t^2 &= 0 \\ 6 &= 3t^2 \\ 2 &= t^2 \\ \sqrt{2} &= t \end{aligned}$$

Determine Nature

$$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = -6t$$

When $t = \sqrt{2}$, $\quad \frac{\mathrm{d}^2P}{\mathrm{d}t^2} = -6\sqrt{2}$

$$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} < 0 \therefore \text{ max TP when } t = \sqrt{2}$$

Maximum Value

When $t = \sqrt{2}$, $P = 6(\sqrt{2}) - \sqrt{2}^3 = 4\sqrt{2}$

$\therefore$ max value of P is $4\sqrt{2}$, when $t = \sqrt{2}$

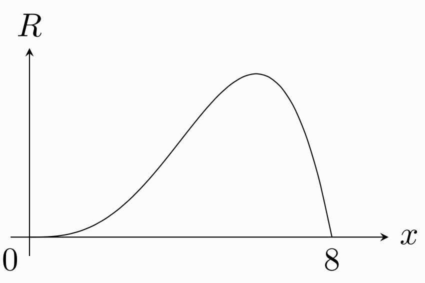
If a function has a single stationary point within an interval, which is a maximum turning point, and the *minimum* value of the function is required, then it must be located at one of the bounds, and the value of the function at each must be checked.

9.4 Optimisation

When a mathematical equation is used to model some real-world variables, finding maximum or minimum values on an interval is often desired. This is referred to as *optimisation*.

Example 9.5.1

A company has determined that a mathematical equation can model the amount of revenue in thousands of pounds, R, which may be earned by setting the price for a new product as x pounds.



The diagram illustrates the model used by the company, which is:

$$R(x) = 8x^3 - x^4 \text{ for } 0 < x \leqslant 8$$

Find the value of x which gives the maximum revenue for the product.

Stationary Points occur where $R'(x) = 0$

$$\begin{aligned} R'(x) &= 24x^2 - 4x^3 \\ 24x^2 - 4x^3 &= 0 \\ 6x^2 - x^3 &= 0 \\ x^2(6-x) &= 0 \\ x &= 6 \text{ since } x > 0 \end{aligned}$$

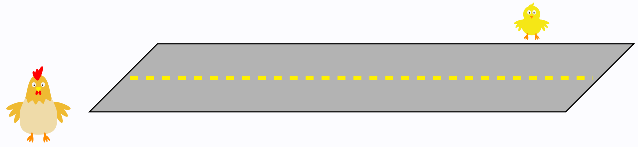
Determine Nature

$$\begin{aligned} R''(x) &= 48x - 12x^2 \\ R''(6) &= 48(6) - 12(6)^2 \\ &= -144 \end{aligned}$$

$R''(6) < 0 \therefore$ maximum revenue when $x = 6$

Example 9.5.2

A chicken is aiming to cross a road. It can fly faster than it can walk, but doing so tires it.



If it flies for x metres and walks the rest of the way then the amount of time T it will take to reach its destination is given, in seconds, by:

$$T(x) = x^2 + \frac{16}{x}$$

Determine the value of x which minimises the chicken's time to get to the other side.

Stationary Points occur where $T'(x) = 0$

$$\begin{aligned} T(x) &= x^2 + 16x^{-1} \\ T'(x) &= 2x - 16x^{-2} \end{aligned}$$

$$\begin{aligned} 2x - 16x^{-2} &= 0 \\ 2x - \frac{16}{x^2} &= 0 \\ 2x^3 - 16 &= 0 \\ x^3 - 8 &= 0 \\ x^3 &= 8 \\ x &= 2 \end{aligned}$$

$$\mathit{Determine Nature}$$

$$\begin{aligned} T''(x) &= 2 + 32x^{-3} \\ &= 2 + \frac{32}{x^3} \\ T''(2) &= 2 + \frac{32}{2^3} \\ &= 2 + 4 \\ &= 6 \end{aligned}$$

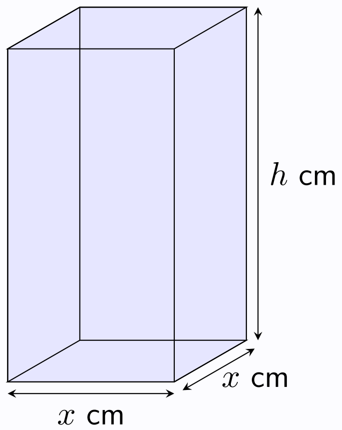
$T''(2) > 0$

$\therefore T$ is minimised when $x = 2$

It can be asked for the equation used in the mathematical model to be *shown* to be correct. Establishing the equation will require an amount of problem-solving. A common context involves the volume and surface area of a 3D object.

Example 9.5.3

A solid cuboid measures x cm by x cm by h cm. The volume of this cuboid is 125 cm^3.



(a) Show that the surface area of the cuboid can be given by:

$$A(x) = 2x^2 + \frac{500}{x}$$

(b) Find the value of x such that the surface area is minimised, and find the minimum surface area.

It is recommended to first form an equation for the surface area, A, in terms of x *and* h. By then finding the relationship between x and h, substitution allows h to be eliminated.

(a) Surface Area

$$\begin{aligned} A &= 2 \times x^2 + 4 \times xh \\ &= 2x^2 + 4xh \\ &= 2x^2 + 4x\left(\frac{125}{x^2}\right) \\ &= 2x^2 + \frac{500}{x} \text{ as required} \end{aligned}$$

Volume

$$\begin{aligned} V &= L \times B \times H \\ 125 &= x^2 h \\ \frac{125}{x^2} &= h \end{aligned}$$

It should be noted that part (b) can be answered in full even if part (a) is not completed.

(b) Stationary Points occur where $A'(x) = 0$

$$\begin{aligned} A(x) &= 2x^2 + 500x^{-1} \\ A'(x) &= 4x - 500x^{-2} \end{aligned}$$

$$\begin{aligned} 4x - 500x^{-2} &= 0 \\ 4x - \frac{500}{x^2} &= 0 \\ 4x^3 - 500 &= 0 \\ x^3 - 125 &= 0 \\ x^3 &= 125 \\ x &= 5 \end{aligned}$$

Determine Nature

$$\begin{aligned} A''(x) &= 4 + 1000x^{-3} \\ &= 4 + \frac{1000}{x^3} \\ A''(5) &= 4 + \frac{1000}{5^3} \\ &= 4 + 8 \\ &= 12 \end{aligned}$$

$A''(5) > 0$

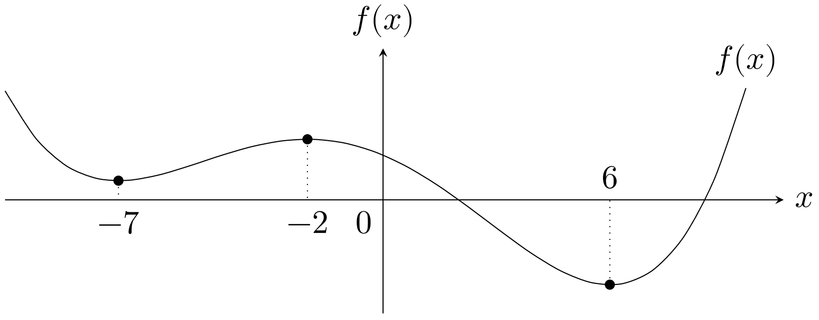
$\therefore A$ is minimised when $x = 5$

Minimum Area

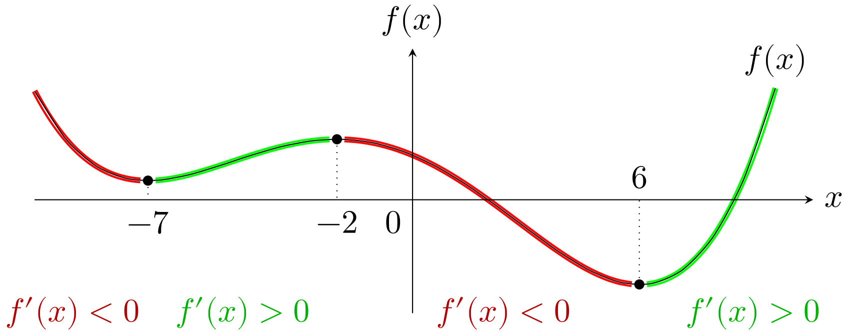
$$\begin{aligned} A(5) &= 2(5)^2 + \frac{500}{5^2} \\ &= 70\text{cm}^3 \end{aligned}$$

9.5 The Graph of the Derivative

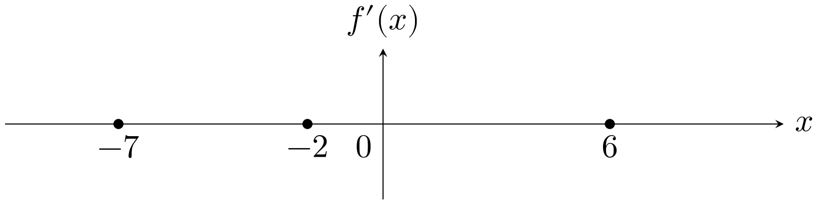
Part of the graph of $f(x)$ for function f is shown below.



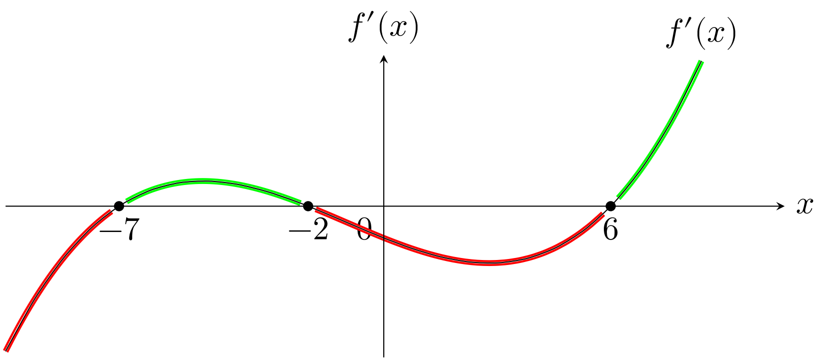
The graph of the *derived function*, $f'(x)$, plots the *rate of change* of f against values of x.\ Where $f(x)$ is *increasing*, $f'(x)$ has a positive value, $f(x)$ is *decreasing*, $f'(x)$ has a negative value



Stationary points on $f(x)$, where $f'(x) = 0$, give corresponding *roots* on the graph of $f'(x)$.

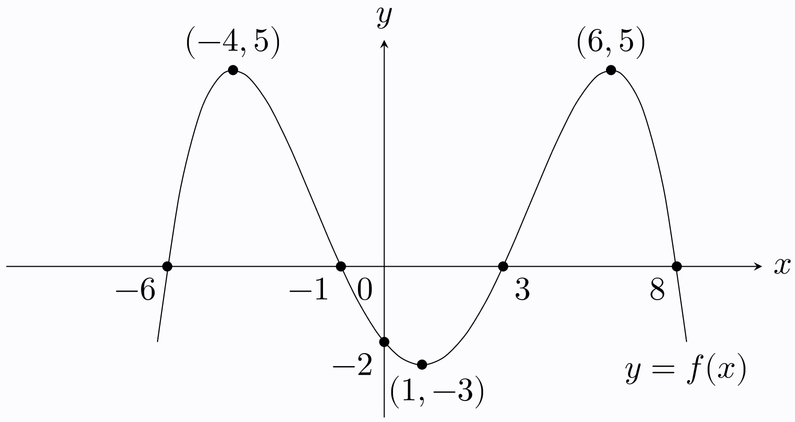


Above the x-axis when $f(x)$ is increasing, and below when $f(x)$ is decreasing, $f'(x)$ can be sketched.

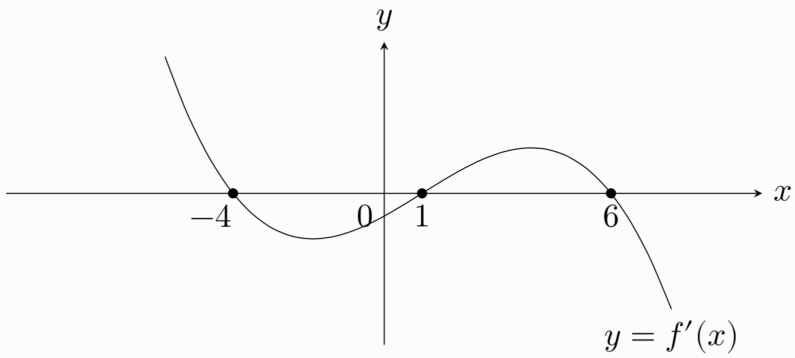


Example 9.6.1

Part of the graph of $y = f(x)$ is shown below. Sketch $y = f'(x)$.



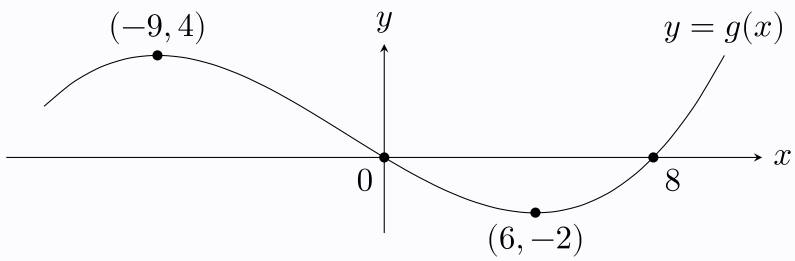
Stationary points on $y = f(x)$ are roots of $y = f'(x)$:



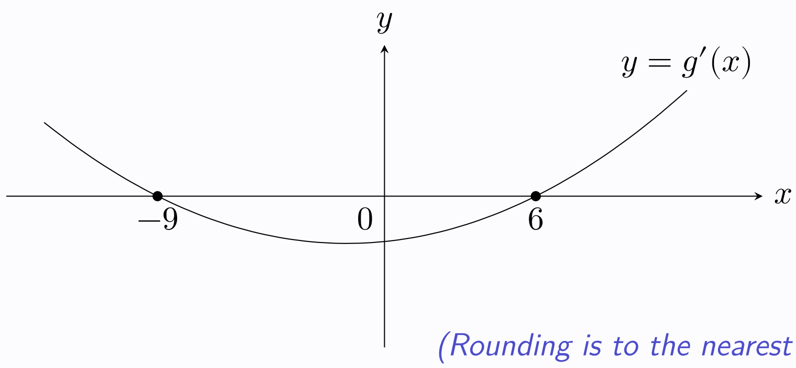
Details such as that the derivative of a *cubic* function is a *quadratic* function should be observed.

Example 9.6.2

For cubic function $g(x)$, part of the graph of $y = g(x)$ is shown below. Sketch $y = g'(x)$.



Since $g(x)$ is cubic, $g'(x)$ must be quadratic, and so the graph is parabolic:



(Rounding is to the nearest integer)

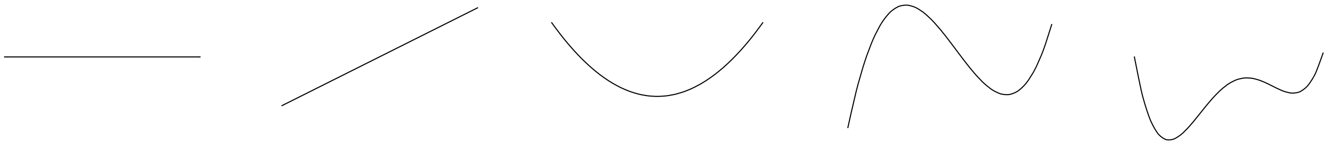
Review Exercise

Chapter 10 — **Polynomials**

10.1 Polynomials and Synthetic Division

Expressions comprised of sums of *positive, integer* powers of x are referred to as *polynomials in* x.

Constant	Linear	Quadratic	Cubic	Quartic
Degree 0	*Degree 1*	*Degree 2*	*Degree 3*	*Degree 4*
e.g. 7	e.g. $2x+4$	e.g. x^2-3x+1	e.g. $4x^3-2x$	e.g. x^4-3x^3+1



A polynomial function such as $f(x)=2x^3-7x^2+4x-1$ can be evaluated for some value of x:

$$\begin{aligned} f(3) &= 2(3)^3-7(3)^2+4(3)+1 \\ &= 54-63+12+1 \\ &= 4 \end{aligned}$$

Another way to evaluate $f(3)$ is to use synthetic division, as follows:

1) Draw a table as below and fill in the first row, then bring the first coefficient down:

3	2	-7	4	1
	$\downarrow$			
	2			

$x=3$

$\longleftarrow$ Coefficients of $2x^3-7x^2+4x+1$

2) The 6 is obtained by *multiplying* 2 by 3, and the -1 by *adding* the two values above:

3	2	-7	4	1
		6		
	2	-1		

Multiply diagonally by 3

Add columns

3) By continuing this *multiplying then adding* pattern, the value of $f(3)=4$ can be obtained:

3	2	-7	4	1
		6	-3	3
	2	-1	1	$\boxed{4}$

$\longleftarrow \therefore f(3)=4$

Example 10.1.1

Given $f(x) = x^4 - 4x^3 + 3x^2 + 9$, use synthetic division to evaluate $f(2)$.

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 3 & 0 & 9 \\ & & 2 & -4 & -2 & -4 \\ \hline & 1 & -2 & -1 & -2 & \boxed{5} \end{array} \quad \therefore f(2) = 5$$

The top line within a synthetic division box should be *ordered in decreasing powers of* x.\Any missing powers are included with a coefficient of *zero*.

10.2 Remainder Theorem

Addition, subtraction and multiplication of polynomials were all required for the National 5 course.

Since $3 \times 4 + 1 = 13$, this implies that:

$$\frac{13}{3} = 4 \textbf{ Remainder } 1$$

Since $(x-2)(x^2-3x+4)+7 = x^3-5x^2+x-1$, this implies that:

$$\frac{x^3-5x^2+x-1}{x-2} = x^2-3x+4 \textbf{ Remainder } 7$$

Note that for $f(x) = x^3-5x^2+x-1$, $f(2) = (2)^3-5(2)^2+(2)-1 = 7$.

Remainder Theorem:

If polynomial $f(x)$ is divided by a linear divisor $(x-a)$ then the remainder is $f(a)$, and $f(x) = (x-a)Q(x) + f(a)$ where quotient $Q(x)$ is a polynomial of degree one less than $f(x)$.

Example 10.2.1

Find the remainder when $3x^3 - x^2 + 5$ is divided by $(x-1)$.

$$\begin{array}{ccc|cccc}
(x-1) & & 1 & 3 & -1 & 0 & 5 \\
\rightarrow f(1) & & & & 3 & 2 & 2 \\
\hline
 & & & 3 & 2 & 2 & \boxed{7} \longleftarrow \textit{Remainder}
\end{array}$$

So the remainder is 7.

The *quotient*, $Q(x)$, can be determined from the values to the left of the remainder.\ Since $3x^3 - x^2 + 5$ is a *cubic*, the quotient $Q(x)$ is *quadratic*: $3x^2 + 2x + 2$.

$$3x^3 - x^2 + 5 = (x-1)(3x^2+2x+2) + 7$$

Example 10.2.2

Function f is defined by $f(x) = x^3 + 4x^2 - 3x - 7$. Find the remainder r when $f(x)$ is divided by $(x+1)$, and hence express $f(x)$ in the form $(x+1)Q(x) + r$ where $Q(x)$ is a quadratic.

$$
\begin{array}{r|rrr|r}
-1 & 1 & 4 & -3 & -7 \\
 & & -1 & -3 & 6 \\
\hline
\textit{Quotient}\longrightarrow & 1 & 3 & -6 & \boxed{-1} \longleftarrow \textit{Remainder}
\end{array}
$$

So the remainder is -1 and $x^3 + 4x^2 - 3x - 7 = (x+1)(x^2+3x-6) - 1$.

10.3 Factor Theorem

Since 20 can be divided by 5 to give 4 **with no remainder**, this means 5 is a **factor** of 20.

Factor Theorem:

If polynomial $f(x)$ is divided by a linear **factor** $(x-a)$ then the **remainder** $f(a)=0$, and $f(x)=(x-a)Q(x)$ where quotient $Q(x)$ is a polynomial of degree one less than $f(x)$. Or:

$$f(a)=0 \iff (x-a) \text{ is a factor of } f(x)$$

Given a linear factor, synthetic division allows a cubic expression $f(x)$ to be expressed as a the product of the *linear* factor and a *quadratic* quotient, which may potentially by further factorised:

$$\begin{aligned} f(x) &= (\textit{cubic}) \\ &= (\textit{linear})(\textit{quadratic}) && \longleftarrow \textit{Using synthetic division} \\ &= (\textit{linear})(\textit{linear})(\textit{linear}) \end{aligned}$$

Example 10.3.1

Show that $(x+4)$ is a factor of $x^3-x^2-14x+24$, and hence factorise it fully.

$$\begin{array}{r|rrr|r} -4 & 1 & -1 & -14 & 24 \\ & & -4 & 20 & -24 \\ \hline \textit{Quotient}\longrightarrow & 1 & -5 & 6 & \boxed{0} \ \longleftarrow \textit{Remainder} \end{array}$$

Remainder$=0 \therefore (x+4)$ is a factor. $\longleftarrow$ *"Show that.."*

$$\begin{aligned} &x^3-x^2-14x+24 \\ &=(x+4)(x^2-5x+6) \\ &=(x+4)(x-2)(x-3) \end{aligned}$$

For some cubic expressions, a resulting quadratic quotient may not be able to be factorised.

Example 10.3.2

Show that $(x-2)$ is a factor of $2x^3 - x^2 + x - 14$, and hence factorise it fully.

$$\begin{array}{c|cccc} 2 & 2 & -1 & 1 & -14 \\ & & 4 & 6 & 14 \\ \hline & 2 & 3 & 7 & \multicolumn{1}{|c}{0} \\ \cline{5-5} \end{array}$$

"Show that $(x-2)$ is a factor..." $\longrightarrow$ Remainder$= 0 \therefore (x-2)$ is a factor.

$$\begin{aligned} &2x^3 - x^2 + x - 14 \\ &= (x-2)(2x^2 + 3x + 7) \end{aligned}$$

Since $3^2 - 4 \times 2 \times 7 = -65$, and $-65 < 0$, the quadratic quotient cannot be factorised
Therefore the cubic cannot be further factorised.

Where a linear factor of a polynomial function is unknown, a suggested approach is to consider the factors of the constant term with the aim of obtaining a value such that the remainder is zero.

Example 10.3.3

Factorise $x^3 - x^2 - 17x = 15$.

Consider $\pm 1, \pm 3 \pm 5 \pm 15$

$f(1) = (1)^3 - (1)^2 - 17(1) - 15 = -32 \therefore$ $(x-1)$ is **not** a factor.

$f(-1) = (-1)^3 - (-1)^2 - 17(-1) - 15 = 0 \therefore$ $(x+1)$ **is** a factor.

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -17 & -15 \\ & & -1 & 2 & 15 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$\begin{aligned} & x^3 - x^2 - 17x - 15 \\ &= (x+1)(x^2 - 2x - 15) \\ &= (x+1)(x+3)(x-5) \end{aligned}$$

Repeated use of synthetic dvision may allow a *quartic* (x^4) to be expressed as a product of up to four linear factors.

$$\begin{aligned} f(x) &= (\textit{quartic}) \\ &= (\textit{linear})(\textit{cubic}) && \longleftarrow \textit{Using synthetic division} \\ &= (\textit{linear})(\textit{linear})(\textit{quadratic}) && \longleftarrow \textit{Using synthetic division} \\ &= (\textit{linear})(\textit{linear})(\textit{linear})(\textit{linear}) \end{aligned}$$

Example 10.3.4

Show that $(x-2)$ is a factor of $x^4-2x^3-7x^2+8x+12$ and hence factorise it fully.

$$\begin{array}{r|rrrr|r} 2 & 1 & -2 & -7 & 8 & 12 \\ & & 2 & 0 & -14 & -12 \\ \hline & 1 & 0 & -7 & -6 & \boxed{0} \end{array}$$

Remainder$=0 \therefore (x-2)$ is a factor.

$$x^4-2x^3-7x^2+8x+12=(x-2)(x^3-7x-6)$$

$$\begin{array}{r|rrr|r} 3 & 1 & 0 & -7 & -6 \\ & & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & \boxed{0} \end{array}$$

Try factors of 6

$$\begin{aligned} x^4-2x^3-7x^2+8x+12 &= (x-2)(x^3-7x-6) \\ &= (x-2)(x-3)(x^2+3x+2) \\ &= (x-2)(x-3)(x+1)(x+2) \end{aligned}$$

10.3 Finding Unknown Coefficients

An unknown coefficient of a polynomial expression may be calculated if a factor of the expression is known, or the remainder given a linear divisor is known. It may be preferred to use synthetic division, or to evaluate the expression directly.

Example 10.4.1

Given that $(x+2)$ is a factor of $2x^3 + kx^2 - 14x + 8$, determine the value of k.

$$\begin{array}{r|cccc}
-2 & 2 & k & -14 & 8 \\
 & & -4 & -2k+8 & 4k+12 \\
\hline
 & 2 & k-4 & -2k-6 & \boxed{4k+20}
\end{array} \longleftarrow \textit{Remainder}$$

Since $(x+2)$ is a factor, remainder$=0$.

$$\begin{aligned}
4k + 20 &= 0 \\
4k &= -20 \\
k &= -5
\end{aligned}$$

Where a polynomial has *two* unknown coefficients, *simultaneous equations* may be formed and solved.

Example 10.4.2

Find the values of a and b given that:

- $(x-2)$ is a factor of $x^3 + ax^2 + bx + 12$.
- $x^3 + ax^2 + bx + 12$ divided by $(x-1)$ has a remainder of 10.

$$\begin{array}{c|cccc} 2 & 1 & a & b & 12 \\ & & 2 & 2a+4 & 4a+2b+8 \\ \hline & 1 & a+2 & 2a+b+4 & \boxed{4a+2b+20} \end{array}$$

Since $(x-2)$ is a factor, remainder $= 0 \implies 4a + 2b + 20 = 0.$

$$\begin{array}{c|cccc} 1 & 1 & a & b & 12 \\ & & 1 & a+1 & a+b+1 \\ \hline & 1 & a+1 & a+b+1 & \boxed{a+b+13} \end{array}$$

Remainder $= 10 \implies a + b + 13 = 10.$

Solve simultaneously:

$$\left.\begin{aligned} 4a + 2b &= -20 \\ a + b &= -3 \end{aligned}\right\}_{\times 2} \qquad \begin{aligned} 4a + 2b &= -20 \\ 2a + 2b &= -6 \quad (-) \\ \hline 2a &= -14 \\ a &= -7 \end{aligned}$$

Sub $a = -7$:

$$\begin{aligned} a + b &= -3 \\ (-7) + b &= -3 \\ b &= 4 \end{aligned}$$

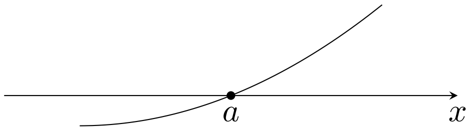
10.4 Solving Polynomial Equations

Quadratic equations such as $x^2 - 4x - 21 = 0$ are typically solved through factorising, or using the quadratic formula. There is no equivalent *general formula* for solving polynomial equations of degree 3 (*cubic*) or higher. Instead, such polynomial equations are solved through *factorising*.

10.5 Identifying Polynomial Functions

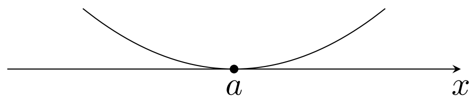
Given the form a polynomial function takes and its graph, showing any x-intercepts and at least one other coordinate, the factorised form of the polynomial may be deduced.

- An x-intercept (or *root*) when $x = a$ implies that a polynomial has a *factor* of $(x - a)$.
- A *stationary point* at an x-intercept is a *repeated root*, and its factor will be (at least) squared.



Root when $x = a$

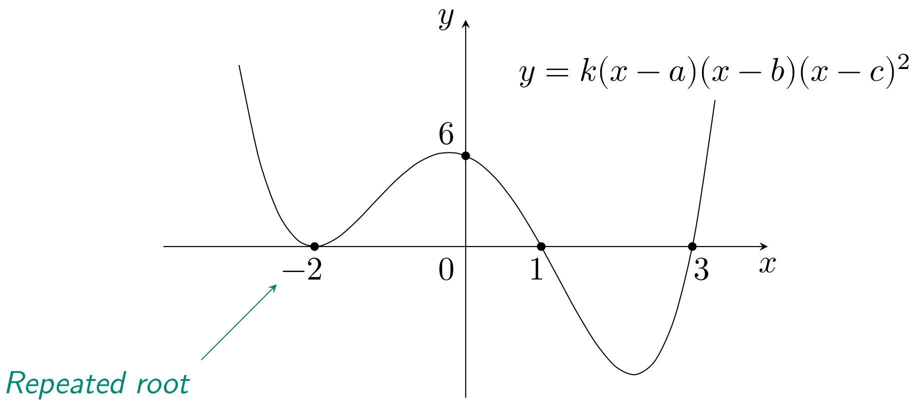
Factor of $(x - a)$



Repeated root when $x = a$

Factor of $(x - a)^n$

The diagram below shows the graph of a quartic function with equation $y = k(x - a)(x - b)(x - c)^2$.



With roots at $x = 1$ and $x = 3$, and a *repeated root* at $x = -2$, the quartic can be expressed as:

$$y = k(x - 1)(x - 3)(x + 2)^2$$

The value of k can only be found if a *non-root* coordinate is also known. By substitution:

$$\begin{aligned}
y &= k(x - 1)(x - 3)(x + 2)^2 \\
6 &= k(0 - 1)(0 - 3)(0 + 2)^2 && \textit{Substitute in } (0, 6) \\
6 &= k(-1)(-3)(2)^2 \\
6 &= 12k \\
\frac{6}{12} &= k \\
\frac{1}{2} &= k \\
y &= \frac{1}{2}(x - 1)(x - 3)(x + 2)^2 && \textit{State equation}
\end{aligned}$$

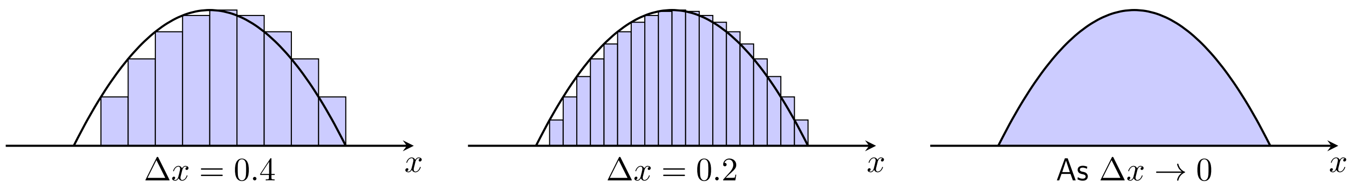
By comparing the form given and the final equation, it can be seen that: $k = \frac{1}{2}, a = 1, b = 3, c = -2$

Review Exercise

Chapter 11 Integration

Introduction and Overview

In mathematics, *integration* can be thought of as the process of calculating some kind of *"sum"*, such as the total area under a curve, by breaking it down into sums of increasingly small parts. This technique was known about and used thousands of years ago in both Ancient Greece and other parts of the world.



In the 17th century both Gottfried Wilhelm Leibniz and Isaac Newton independently discovered the *fundamental theorem of calculus*, showing that *integration* can be performed using *antidifferentiation*, which is the 'reverse' of differentiation. Now, *integration* is used to refer to both the idea of calculating such a 'sum', and to the process of antidifferentiation.

This chapter will first introduce the use of integration to calculate an *antiderivative*, before moving on to using this concept to calculating area enclosed using curves.

Chapter Contents

- 11.1 Indefinite Integrals
- 11.2 Differential Equations
- 11.3 Definite Integrals
- 11.4 The Area Under a Curve
- 11.5 The Area Between Two Curves

Integration skills covered in this chapter will be extended upon in the later chapter of **Further Calculus**.

11.1 Indefinite Integrals

The *derivative* of the function $f(x) = x^3 - 4x^2 - 7x + 3$ is obtained by *multiplying by the power* and then *reducing the power by one* for each term.

$$\text{If } f(x) = x^3 - 4x^2 + 3$$
$$\text{Then } f'(x) = 3x^2 - 8x$$

Integration can be considered the *inverse* of differentiation, with the aim of taking the *derivative* $f'(x)$ and determining the *original function* $f(x)$. However, given only the derivative $f'(x) = 3x^2 - 8x$ it would not be possible to know any constant term the original function $f(x)$ contained. Finding the *indefinite integral* of a function requires the inclusion of a **Constant of Integration**, C:

$$\text{If } f'(x) = 3x^2 - 8x$$
$$\text{Then } f(x) = x^3 - 4x^2 + C$$

Indefinite Integrals: *"The (indefinite) integral of x^n with respect to x..."*

$$\int (x^n)\, dx = \frac{x^{n+1}}{n+1} + C$$

In other words, **increase the power by 1** then **divide by the new power.**

The *integral sign*, $\int$, should always appear accompanied by dx, for a function in x.

Note: $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x\, dx)$

Example 11.1.1

Find $\int (6x^2 + 8x)\, dx$.

$$\int (6x^2 + 8x)\, dx$$
$$= \frac{6x^3}{3} + \frac{8x^2}{2} + C \qquad \longleftarrow \text{Constant of integration}$$
$$= 2x^3 + 4x^2 + C \qquad \longleftarrow \text{Simplify}$$

As with differentiation, *preparation for integration* may be needed.

Example 11.1.2

Find $\int \left(4\sqrt{x} - \frac{3}{x^2}\right) dx,\ , x > 0.$

$$\begin{aligned}
&\int \left(4\sqrt{x} - \frac{3}{x^2}\right)\, dx \\
&= \int \left(4x^{\frac{1}{2}} - 3x^{-2}\right)\, dx && \longleftarrow \text{Preparing to integrate} \\
&= \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{-1}}{-1} + C && \longleftarrow \text{Integrate: } \frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2} \\
&= \frac{8}{3}x^{\frac{3}{2}} + 3x^{-1} + C && \longleftarrow \text{Simplify: } 4 \div \frac{3}{2} = \frac{4}{1} \times \frac{2}{3} = \frac{8}{3}
\end{aligned}$$

The *derivative* of a *linear* term is a *constant*, so the integral of a *constant* is *linear*.

Example 11.1.3

Find $\int ((2x-1)(x+3))\, dx.$

$$\begin{aligned}
&\int ((2x-1)(x+3))\, dx \\
&= \int \left(2x^2 - 5x - 3\right)\, dx && \longleftarrow \text{Preparing to integrate} \\
&= \frac{2x^3}{3} - \frac{5x^2}{2} - 3x + C && \longleftarrow \text{Integral of } -3 \text{ is } -3x
\end{aligned}$$

Integration *with respect to variables other than* x should not use dx, and the notation used should match the variable of the function instead.

Example 11.1.4

Find $\int (3t^2 - 5)\, dt.$

$$\begin{aligned}
&\int \left(3t^2 - 5\right)\, dt \\
&= \frac{3t^3}{3} - 5t + C \\
&= t^3 - 5t + C
\end{aligned}$$

11.2 Differential Equations

11.3 Definite Integrals

The indefinite integral of the function $f(x)$ can be notated as $F(x)$:

$$\int f(x)\,dx = F(x)$$

The *definite integral* of $f(x)$ from a to b is the difference between $F(b)$ and $F(a)$:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Together, they form a core part of the **Fundamental Theorem of Calculus**:

Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where

$$F(x) = \int f(x)\,dx$$

Since any *constant of integration* within $F(x)$ will cancel through subtraction $(C - C)$, it is *not included* when calculating a *definite integral*.

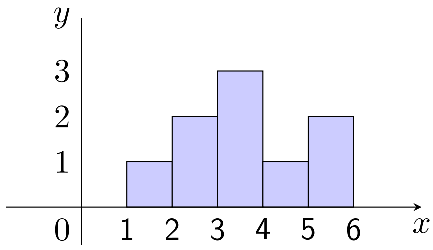
Example 11.3.1

Find $\int_1^2 (3x^2 + 6x - 2)\,dx$.

$$\begin{aligned}
&\int_1^2 (3x^2 + 6x - 2)\,dx \\
&= \left[\frac{3x^3}{3} + \frac{6x^2}{2} - 2x\right]_1^2 && \longleftarrow \text{Integrate} \\
&= \left[x^3 + 3x^2 - 2x\right]_1^2 && \longleftarrow \text{Simplify} \\
&= ((2)^3 + 3(2)^2 - 2(2)) - ((1)^3 + 5(1)^2 - 2(1)) && \longleftarrow \text{Substitute} \\
&= 16 - 4 \\
&= 12 && \longleftarrow \text{Evaluate}
\end{aligned}$$

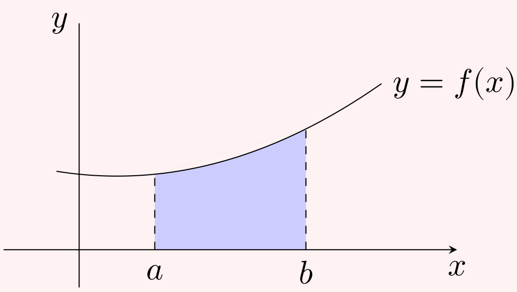
11.4 The Area Under a Curve

The area shaded below, with each *"bar"* having width 1 unit, is given by the *sum* $1 + 2 + 3 + 1 + 2$.



An integral, $\int$, can be seen as *"sum"*, but for a continuous function. Given a function $f(x)$, the area enclosed between a section of the curve $y = f(x)$ and the x-axis can be calculated using a definite integral.

Area Under a Curve:

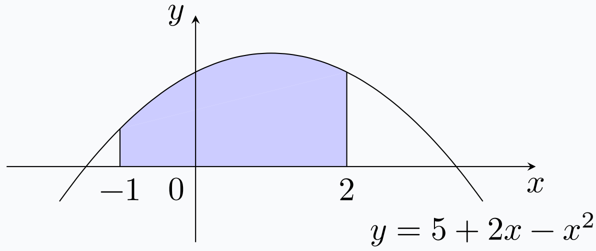


$$\text{Area} = \int_a^b f(x)\, dx = F(b) - F(a)$$

The values of a and b can be referred to as the *bounds* of the integration.

Example 11.4.1

Part of the graph of $y = 5 + 2x - x^2$ is shown. Calculate the shaded area.



$$\int_{-1}^{2} (5 + 2x - x^2)\, dx \qquad \longleftarrow \text{Area}$$

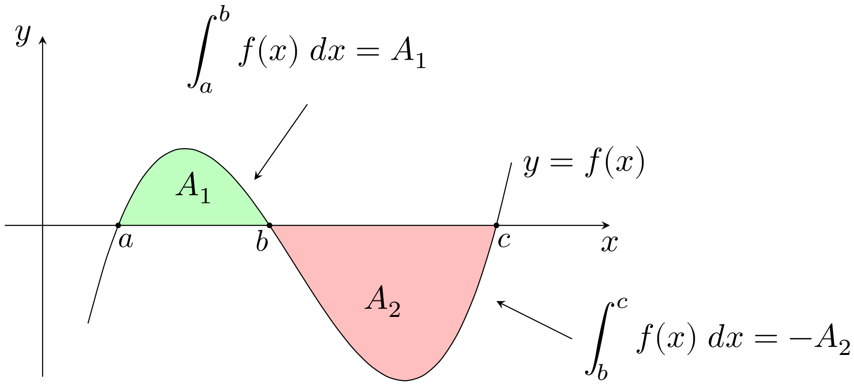
$$= \left[5x + x^2 - \frac{x^3}{3}\right]_{-1}^{2} \qquad \longleftarrow \text{Integrate}$$

$$= \left(5(2) + (2)^2 - \frac{(2)^3}{3}\right) - \left(5(-1) + (-1)^2 - \frac{(-1)^3}{3}\right) \qquad \longleftarrow \text{Substitute}$$

$$= \frac{34}{3} - \left(-\frac{11}{3}\right) \qquad \longleftarrow \text{Evaluate}$$

$= 15$ square units

Where the area between a curve and the x-axis lies *under* the x-axis, the definite integral will be negative.

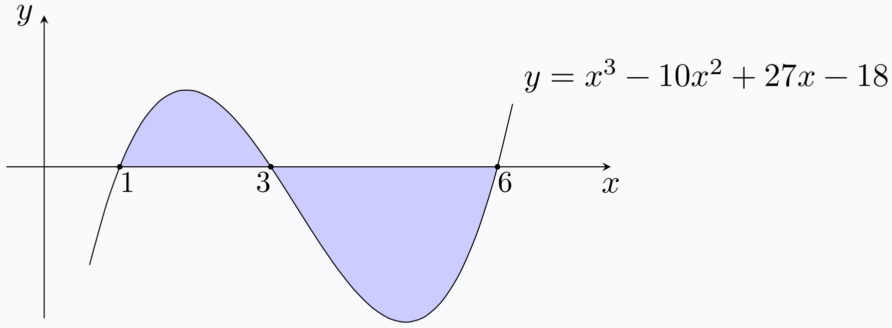


To avoid *"positve"* and *"negative"* areas cancelling each other out, such sections must be calculated as separate integrals, and their *absolute values* added.

$$\text{Area } = A_1 + A_2$$

Example 11.4.2

Part of the graph of $y = x^3 - 10x^2 + 27x - 18$ is shown. Calculate the shaded area.



$$\int_1^3 (x^3 - 10x^2 + 27x - 18)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x\right]_1^3$$

$$= \left(\frac{(3)^4}{4} - \frac{10(3)^3}{3} + \frac{27(3)^2}{2} - 18(3)\right)$$

$$- \left(\frac{(1)^4}{4} - \frac{10(1)^3}{3} + \frac{27(1)^2}{2} - 18(1)\right)$$

$$= -\frac{9}{4} - \left(-\frac{91}{12}\right)$$

$$= \frac{16}{3}$$

$$\int_3^6 (x^3 - 10x^2 + 27x - 18)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x\right]_3^6$$

$$= \left(\frac{(6)^4}{4} - \frac{10(6)^3}{3} + \frac{27(6)^2}{2} - 18(6)\right)$$

$$- \left(\frac{(3)^4}{4} - \frac{10(3)^3}{3} + \frac{27(3)^2}{2} - 18(3)\right)$$

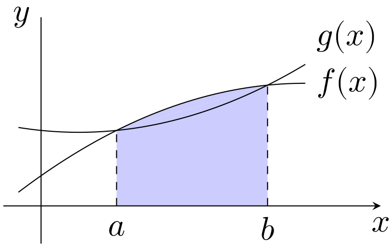
$$= -18 - \left(-\frac{9}{4}\right)$$

$$= -\frac{63}{4}$$

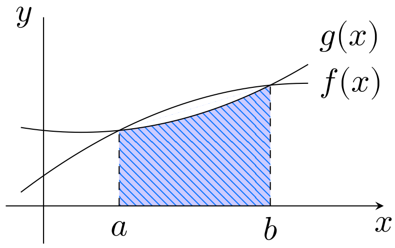
$$\therefore \text{ Area } = \frac{16}{3} + \frac{63}{4} = \frac{253}{12} \text{ square units}$$

11.5 The Area Between Two Curves

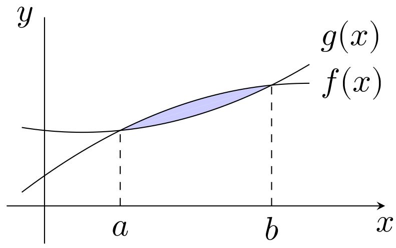
The area **between** two curves can be calculated using the subtraction of one definite integral from another:



$$\int_a^b f(x)\,dx$$



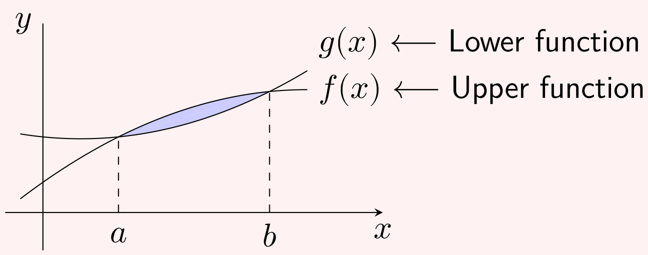
$$\int_a^b g(x)\,dx$$



$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Note that $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$, leading to the following formula:

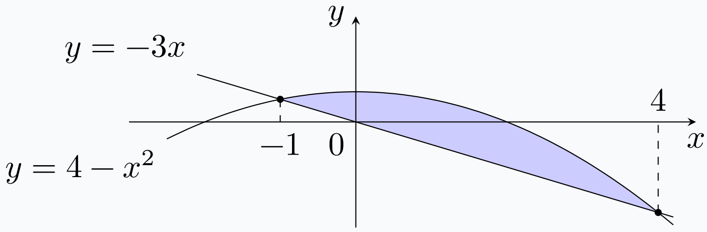
Area Between Two Curves:



$$\text{Area} = \int_a^b (f(x) - g(x))\,dx \quad \textbf{or} \quad \text{Area} = \int_a^b (\text{Upper} - \text{Lower})\,dx$$

Example 11.5.1

Part of the graphs of $y = -3x$ and $y = 4 - x^2$ are shown. Calculate the shaded area.



$$\int_{-1}^{4} (4 - x^2 - (-3x))\,dx \qquad \longleftarrow \text{Upper} - \text{Lower}$$

$$\int_{-1}^{4} (4 - x^2 + 3x)\,dx \qquad \longleftarrow \text{Simplify}$$

$$= \left[4x - \frac{x^3}{3} + \frac{3x^2}{2}\right]_{-1}^{4} \qquad \longleftarrow \text{Integrate}$$

$$= \left(4(4) - \frac{(4)^3}{3} + \frac{3(4)^2}{2}\right) - \left(4(-1) - \frac{(-1)^3}{3} + \frac{3(-1)^2}{2}\right) \qquad \longleftarrow \text{Substitute}$$

$$= \frac{56}{3} - \left(-\frac{13}{6}\right) \qquad \longleftarrow \text{Evaluate}$$

$$= \frac{125}{6} \text{ square units}$$

Review Exercise

Chapter 12 Compound Angles

12.1 The Addition Formulae

It should be noted that $\sin(A+B) \neq \sin A + \sin B$:

$$\sin(60^\circ + 30^\circ) = \sin 90^\circ = 1$$

$$\textbf{but} \quad \sin 60^\circ + \sin 30^\circ = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2}$$

Instead, the following expansions are provided in the formula sheet, known as the **addition formulae**:

Addition Formulae *(as provided on the formula sheet)*:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

They can be expanded into four formulae, covering addition and subtraction for each of sin and cos:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

They can be used to expand the usage of known *exact values* (0°,30°,45°,60°,90°,180°,270°,360°).

Example 12.1.1

Evaluate $\sin 105$.

$$\begin{aligned}
\sin 105^\circ &= \sin(60^\circ + 45^\circ) && \longleftarrow \text{Relate to known exact values} \\
&= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ && \longleftarrow \text{Expand using formula} \\
&= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} && \longleftarrow \text{Substitute exact values} \\
&= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}} && \longleftarrow \text{Simplify}
\end{aligned}$$

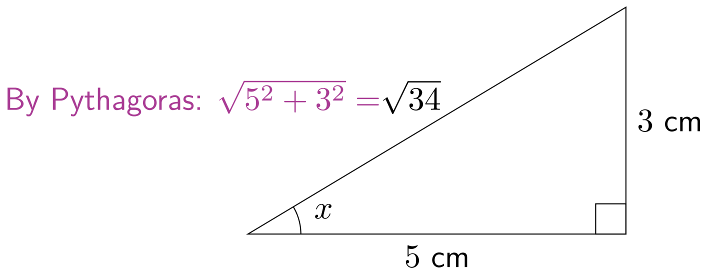
Example 12.1.2

Evaluate $\sin 40^\circ \cos 10 - \cos 40^\circ \sin 10^\circ$.

$$\begin{aligned}
\sin 40^\circ \cos 10 - \cos 40^\circ \sin 10^\circ &= \sin(40^\circ - 10^\circ) && \longleftarrow \text{Recognise expanded form of } \sin(A-B) \\
&= \sin 30^\circ \\
&= \frac{1}{2} && \longleftarrow \text{State exact value}
\end{aligned}$$

12.2 Applications to Right-Angled Triangles

Given a right-angled triangle containing an angle x, values of $\sin x$, $\cos x$ and $\tan x$ can be stated using **SOHCATOA**. For example with Pythagoras' Theorem used to find a missing side length if needed.



$$\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{3}{\sqrt{34}}$$

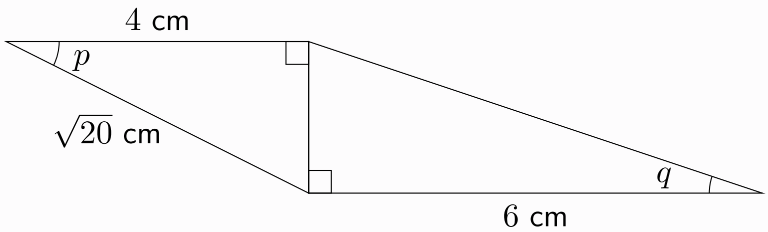
$$\cos x = \frac{\text{adj}}{\text{hyp}} = \frac{5}{\sqrt{34}}$$

$$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{3}{5}$$

These skills are often required in combination with applications of the addition formulae.

Example 12.2.1

The diagram below shows angles p and q contained within right-angled. Find $\sin(p+q)$.



Opposite side for $p = 2$ $\longleftarrow$ By Pythagoras: $\sqrt{\sqrt{20}^2 - 4^2} = \sqrt{20-16} = \sqrt{4} = 2$

Hypotenuse for $q = \sqrt{40}$ $\longleftarrow$ By Pythagoras: $\sqrt{6^2+2^2} = \sqrt{36+4} = \sqrt{40}$

Once all side length have been calculated, values of $\sin p, \cos p, \sin q, \cos q$ can be obtained.

$$\begin{aligned}
\sin(p+q) &= \sin p \cos q + \cos p \sin q && \longleftarrow \text{Expand addition formula} \\
&= \frac{2}{\sqrt{20}} \times \frac{6}{\sqrt{40}} + \frac{4}{\sqrt{20}} \times \frac{2}{\sqrt{40}} && \longleftarrow \text{Substitute using SOHCAHTOA} \\
&= \frac{12}{\sqrt{800}} + \frac{8}{\sqrt{800}} \\
&= \frac{20}{20\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} && \longleftarrow \text{Simplify (and rationalise if required)}
\end{aligned}$$

Other expansions such as $\cos(p+q)$ can be calculated similarly, and note that $\tan(p+q) = \dfrac{\sin(p+q)}{\cos(p+q)}$.

12.3 The Double Angle Formulae

Expansions for $\sin 2x$ and $\cos 2x$ in terms of $\sin x$ and $\cos x$ can be obtained from the addition formulae:

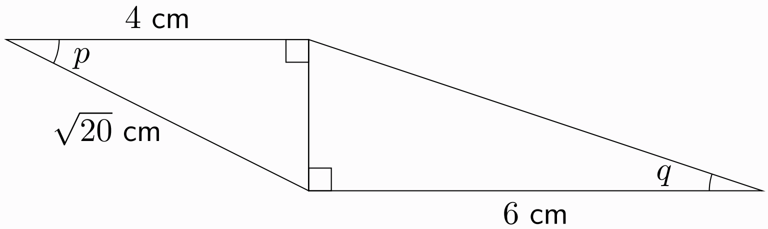
$$\begin{aligned} \sin 2x &= \sin(x+x) \\ &= \sin x \cos x + \cos x \sin x \\ &= 2 \sin x \cos x \end{aligned}$$

$$\begin{aligned} \cos 2x &= \cos(x+x) \\ &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

The other arrangements of the $\cos 2x$ expansion come from the trigonometric identify $\sin^2 x + \cos^2 x = 1$. These expansions are provided in the formula sheet. Care is needed to determine *whether* to expand a *double angle* trig term and, in the case of $\cos 2x$, *which* expansion to use.

Example 12.3.1

The diagram below shows angle p contained within a right-angled triangle. Find $\sin(2p)$.



Example 12.3.2

Given $\sin q = \dfrac{4}{11}$, where $0 < q < \dfrac{\pi}{2}$, find $\cos 2q$.

Review Exercise

Chapter 13 The Circle

13.1 The Equation of a Circle

Exercise

13.2 The General Equation of a Circle

Exercise

13.3 The Intersection of a Line and a Circle

Exercise

13.4 Finding the Equation of a Tangent to a Circle

Exercise

13.5 Points Inside and Outside Circles

Exercise

13.6 The Intersections of Circles

Exercise

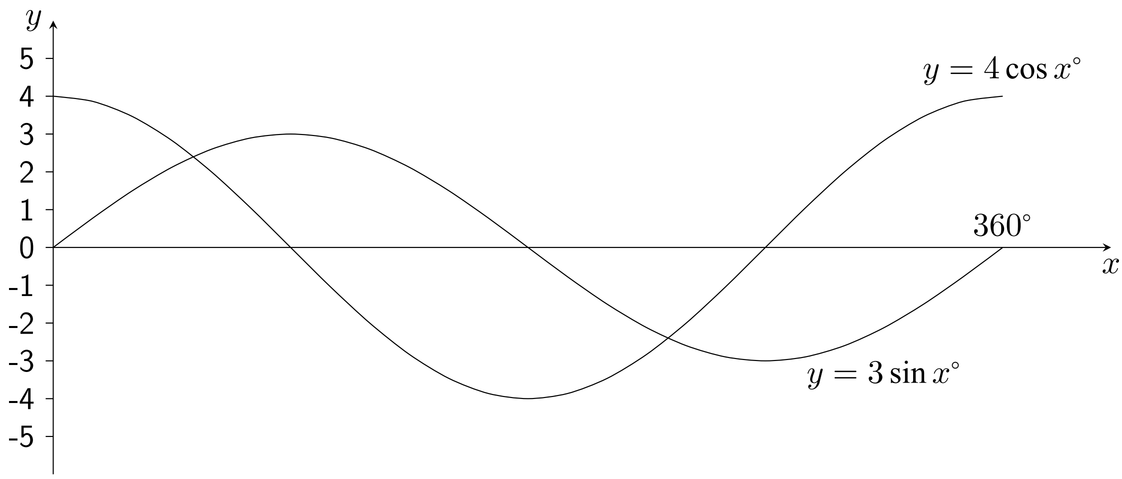
Review Exercise

Chapter 14 — The Wave Function

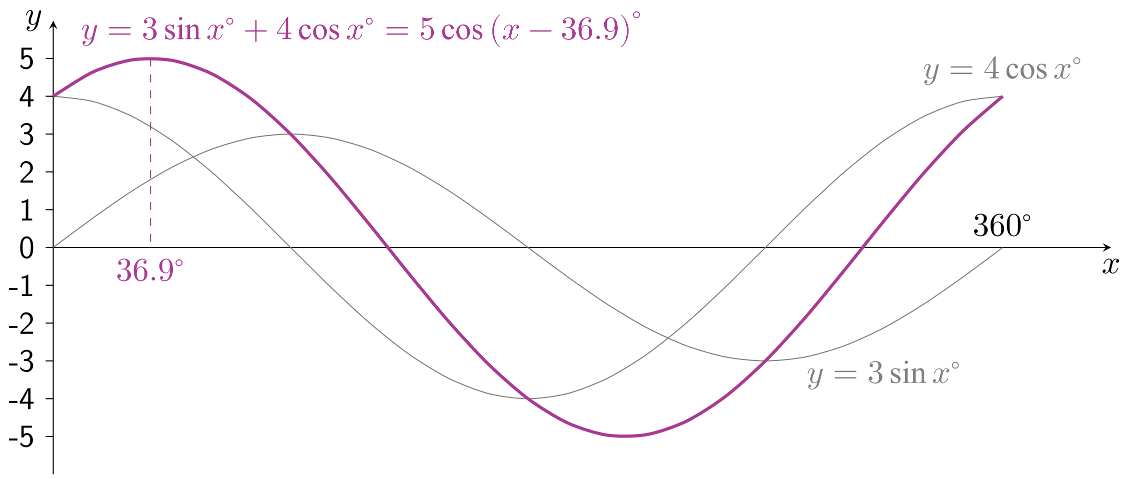
In this chapter, the sums and differences of *equal-angled* trigonometric operations will be explored.

$$\text{e.g. } 3\sin x^\circ + 4\cos x^\circ$$

The graphs of basic trigonometric functions such as $y = 3\sin x^\circ$ and $y = 4\cos x^\circ$ should be familiar:



The graph of $y = 3\sin x^\circ + 4\cos x^\circ$ is the same as that of $y = 5\cos(x - 36.9)^\circ$.



Determining that $3\sin x^\circ + 4\cos x^\circ = 5\cos(x - 36.9)^\circ$ allows the maximum values and minimum values of $3\sin x^\circ + 4\cos x^\circ$ to be determined and the values of x which produce them, and allow equations such as $3\sin x^\circ + 4\cos x^\circ = 2$ to be solved.

Any trigonometric expression $k_1 \sin x \pm k_2 \cos x$ can be written as either $k\sin(x \pm a)$ or $k\cos(x \pm a)$.

This chapter will cover how to do this and explore the various applications of this property.

14.1 The Wave Function using $k \cos (x - a)$

Whilst any of $k \sin (x \pm a)$ or $k \cos (x \pm a)$ may be used in general, any Higher exam question is likely to specify a particular form to use. The simplest is often $k \cos (x - a)$.

To express $3 \sin x^\circ + 4 \cos x^\circ$ in the form $k \cos (x - a)^\circ$, two key *trigonometric identities* are needed:

$$\sin^2 x + \cos^2 x = 1 \qquad \text{and} \qquad \tan x = \frac{\sin x}{\cos x}$$

First, $k \cos (x - a)^\circ$ can be expanded using the the formula $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$:

$$k \cos (x - a)^\circ = k \cos x^\circ \cos a^\circ + k \sin x^\circ \sin a^\circ$$

It is required that this expansion is equal to the original expression:

$$3\underline{\sin x^\circ} + 4\underline{\underline{\cos x^\circ}} = k\underline{\underline{\cos x^\circ}} \cos a^\circ + k\underline{\sin x^\circ} \sin a^\circ$$

Equating the coefficients of $\underline{\sin x^\circ}$ and $\underline{\underline{\cos x^\circ}}$ gives a set of simultaneous equations in k and a:

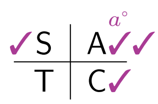
$$k \sin a^\circ = 3 \qquad \longleftarrow \text{Coefficients of } \underline{\sin x^\circ}$$
$$k \cos a^\circ = 4 \qquad \longleftarrow \text{Coefficients of } \underline{\underline{\cos x^\circ}}$$

Squaring both sides of each equation and adding allows k to be determined using $\sin^2 x + \cos^2 x = 1$:

$$\begin{aligned} k^2 \sin^2 a^\circ + k^2 \cos^2 a^\circ &= 3^2 + 4^2 \\ k^2 \left(\sin^2 x^\circ + \cos^2 x^\circ\right) &= 3^2 + 4^2 \\ k^2(1) &= 3^2 + 4^2 \\ k &= \sqrt{3^2 + 4^2} \\ k &= \sqrt{25} \\ k &= 5 \end{aligned}$$

Dividing $k \sin a^\circ$ by $k \cos a^\circ$ allows a to be calculated, noting $k \sin a^\circ$ and $k \cos a^\circ$ are both positive:

$$\begin{aligned} \frac{k \sin a^\circ}{k \cos a^\circ} &= \frac{3}{4} \\ \tan a^\circ &= \frac{3}{4} \\ a^\circ_{\text{acute}} &= \tan^{-1}\left(\frac{3}{4}\right) = 36.9^\circ \\ a^\circ &= 36.9^\circ \end{aligned}$$



Hence $3 \sin x^\circ + 4 \cos x^\circ = 5 \cos (x - 36.9)^\circ$. Here it has been assumed that $k > 0$ and $0 < a < 360$.

Some abbreviations of the working shown on the previous page are routinely permitted in the Higher exam. The following example demonstrates an appropriate level of detail in its solution.

Example

Express $3\cos x - \sin x$ in the form $k\cos(x-a)$ where $k>0$ and $0<a<2\pi$.

$3\cos x - \sin x = k\cos(x-a)$

$3\underline{\underline{\cos x}} - \underline{\sin x} = k\underline{\underline{\cos x}}\cos a + k\underline{\sin x}\sin a$ $\longleftarrow$ Expand $k\cos(x-a)$

$k\sin a = -1$ $\longleftarrow$ Equate $\sin x$ coefficients

$k\cos a = 3$ $\longleftarrow$ Equate $\cos x$ coefficients

$$k = \sqrt{(-1)^2+3^2}$$
$$= \sqrt{10}$$
$\longleftarrow$ Calculate k

$$\tan a = \frac{-1}{3}$$
$\longleftarrow$ Find $\tan a$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{1}{3}\right) = 18.4^\circ$$
$\longleftarrow$ Find acute angle first

S	A✓
✓T	C✓✓

$360^\circ - \text{acute}^\circ$

$\longleftarrow$ Negative $\sin a$: quadrants T and C

$\longleftarrow$ Positive $\cos a$: quadrants A and C

$a^\circ = 341.6^\circ$ $\longleftarrow$ Using $360^\circ - 18.4^\circ$ (Double-ticked quadrant)

$a = 5.96$ $\longleftarrow$ Convert to radians: $341.6 \times \dfrac{\pi}{180}$

$3\cos x - \sin x = \sqrt{10}\cos(x-5.96)$ $\longleftarrow$ State solution

Exercise 14.1

1. Express each of the following in the form $k\cos(x-a)^\circ$, where $k>0$ and $0<a<360$.

(a) $5\cos x^\circ + 12\sin x^\circ$ (b) $4\sin x^\circ + 5\cos x^\circ$ (c) $6\cos x^\circ + \sin x^\circ$

2. Express each of the following in the form $k\cos(x-a)$, where $k>0$ and $0<a<2\pi$.

(a) $8\sin x - 6\cos x$ (b) $\cos x - 3\sin x$ (c) $-2\sin x - \cos x$

3. Express each of the following in the form $k\cos(x-a)^\circ$, where $k>0$ and $0<a<360$.

(a) $\sin x^\circ + \cos x^\circ$ (b) $\cos x^\circ - \sqrt{3}\sin x^\circ$ (c) $-\sin x^\circ - \sqrt{3}\cos x^\circ$

14.2 Other Forms of the The Wave Function

As well as $k\cos(x-a)$, three other forms may be used, shown below including their expansions:

$$\begin{aligned} k\cos(x+a) &= k\cos x\cos a - k\sin x\sin a \\ k\sin(x+a) &= k\sin x\cos a + k\sin x\cos a \\ k\sin(x-a) &= k\sin x\cos a - k\sin x\cos a \end{aligned}$$

Example

Express $12\cos x^\circ - 5\sin x^\circ$ in the form $k\sin(x-a)^\circ$ where $k>0$ and $0<a<360$.

$$12\cos x^\circ - 5\sin x^\circ = k\sin(x-a)^\circ$$

$$12\underline{\underline{\cos x^\circ}} - 5\underline{\sin x^\circ} = k\underline{\sin x^\circ}\cos a^\circ - k\underline{\underline{\cos x^\circ}}\sin a^\circ \qquad \longleftarrow \text{Expand } k\sin(x-a)^\circ$$

$$-k\sin a^\circ = 12 \implies k\sin a^\circ = -12 \qquad \longleftarrow \text{Equate } \sin x^\circ \text{ coefficients}$$

$$k\cos a^\circ = -5 \qquad \longleftarrow \text{Equate } \cos x^\circ \text{ coefficients}$$

$$\begin{aligned} k &= \sqrt{(-12)^2+(-5)^2} \qquad \longleftarrow \text{Calculate } k \\ &= 13 \end{aligned}$$

$$\tan a^\circ = \frac{-12}{-5} = \frac{12}{5} \qquad \longleftarrow \text{Find } \tan a^\circ$$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{12}{5}\right) = 67.4^\circ \qquad \longleftarrow \text{Find acute angle first}$$

✓S	A
✓✓T	C✓

$180^\circ + \text{acute}^\circ$

$\longleftarrow$ Negative $\sin a^\circ$: quadrants T and C

$\longleftarrow$ Negative $\cos a^\circ$: quadrants S and T

$$a^\circ = 247.4^\circ \qquad \longleftarrow \text{Using } 180^\circ + 67.4^\circ \text{ (Double-ticked)}$$

$$12\cos x^\circ - 5\sin x^\circ = 13\sin(x-247.4)^\circ \qquad \longleftarrow \text{State solution}$$

Exercise 14.2

1. Express each of the following in the form $k\sin(x+a)^\circ$, where $k>0$ and $0<a<360$. 🖩

   (a) $3\cos x^\circ + 2\sin x^\circ$ (b) $4\sin x^\circ + 3\cos x^\circ$ (c) $7\cos x^\circ - \sin x^\circ$

2. Express each of the following in the form $k\cos(x+a)$, where $k>0$ and $0<a<2\pi$. 🖩

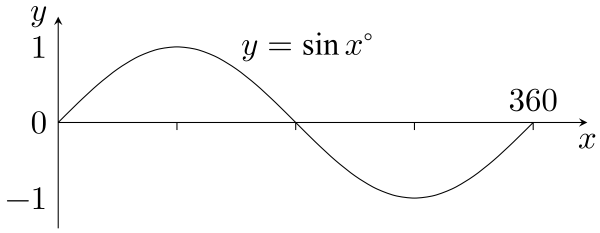
   (a) $3\sin x + 8\cos x$ (b) $-3\cos x - 4\sin x$ (c) $-6\sin x + 2\cos x$

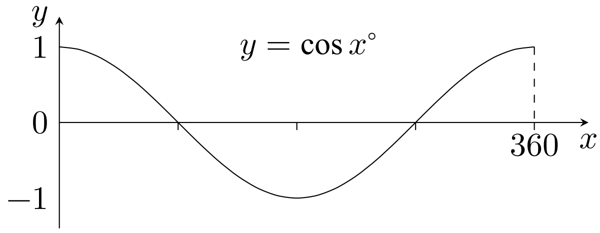
3. Express each of the following in the form $k\sin(x-a)$, where $k>0$ and $0<a<2\pi$.

   (a) $\sqrt{3}\sin x + \cos x$ (b) $\cos x - \sin x$ (c) $2\sqrt{3}\sin x - 2\cos x$

14.3 Maximum and Minimum Values using the Wave Function

The graphs of $y = \sin x^\circ$ and $y = \cos x^\circ$, including their turning points, should already be familiar:



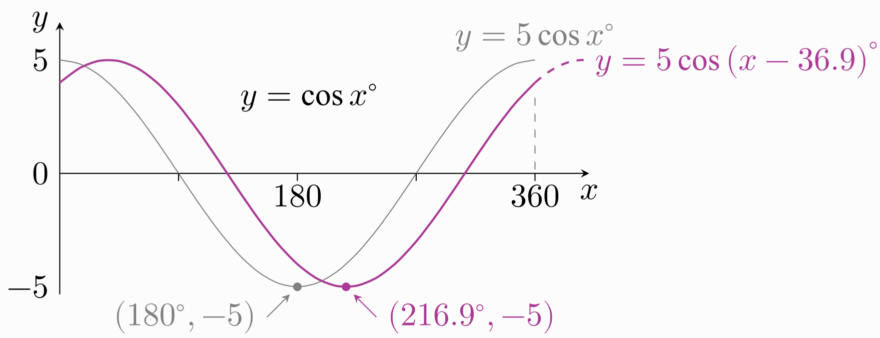


Graphs of the form $k\sin(x \pm a)$ and $k\cos(x \pm a)$ have maximum/minimum values of $\pm k$, and their *horizontal translation* is described by $\pm a$ following the rules covered in the Graph Transformations chapter.

Example

Given that $3\sin x^\circ + 4\cos x^\circ$ can be expressed as $5\cos(x - 36.9)^\circ$, state the minimum value of $f(x) = 3\sin x^\circ + 4\cos x^\circ$ and the value of x at which it occurs for $0 < x < 360$.

Sketch the graph of $y = 5\cos x^\circ$ translated 36.9° to the right:



Use the graph to state the answer:

Hence the minimum value of $f(x)$ is -5, which occurs when $x = 216.9$.

Exercise 14.3

1. Sketch each for $0 < x < 360$, showing the coordinates of any roots and turning points:

   (a) $7\cos(x - 10)^\circ$ (b) $3\sin(x - 20)^\circ$ (c) $\sqrt{5}\cos(x + 40)^\circ$

2. Find the maximum value of each and the value(s) of x for which they occur for $0 < x < 360$:

   (a) $8\sin(x - 50)^\circ$ (b) $\sqrt{2}\cos(x + 27)^\circ$ (c) $7\sin(x + 42.3)^\circ$

3. Find the minimum value of each and the value(s) of x for which they occur for $0 < x < 2\pi$:

   (a) $3\cos\left(x - \frac{\pi}{6}\right)$ (b) $5\sin\left(x + \frac{\pi}{3}\right) + 2$ (c) $-12\sin\left(x + \frac{\pi}{4}\right)$

4. (a) Express $6\sin x^\circ - 7\cos x^\circ$ in the form $k\sin(x - a)^\circ$ where $k > 0$ and $0 < a < 360$.

   (b) Hence state the coordinates of the turning points of $12\sin x^\circ - 14\cos x^\circ$ for $0 < x < 360$.

14.4 Solving Equations using the Wave Function

Solving an equation like $3\sin x^\circ + 4\cos x^\circ = 2$ can be approached by using the wave function to rewrite it as $5\cos(x - 36.9)^\circ = 2$, before solving it in the manner covered in Chapter 6.

Example

It can be shown that $5\sin x^\circ - 2\cos x^\circ$ can be expressed as $\sqrt{29}\sin(x - 21.8)^\circ$.
Hence, solve the equation $5\sin x^\circ - 2\cos x^\circ = 4$ where $0 < x < 360$.

$$5\sin x^\circ - 2\cos x^\circ = 4$$

$$\sqrt{29}\sin(x - 21.8)^\circ = 4 \qquad \longleftarrow \text{Substitute the wave function form}$$

$$\sin(x - 21.8)^\circ = \frac{4}{\sqrt{29}} \qquad \longleftarrow \text{Rearrange to } \sin(\dots) = \dots$$

$$a = \sin^{-1}\left(\frac{4}{\sqrt{29}}\right) = 48.0^\circ \qquad \longleftarrow \text{Calculate acute angle}$$

$$\begin{array}{c|c} {}^{180^\circ - a^\circ}\checkmark\text{S} & \text{A}\checkmark^{a^\circ} \\ \hline \text{T} & \text{C} \end{array} \qquad \longleftarrow \text{Positive } \sin a^\circ\text{: quadrants A and S}$$

$$x - 21.8^\circ = 48.0^\circ, 180^\circ - 48.0^\circ \qquad \longleftarrow \text{Apply ticked quadrants}$$

$$x - 21.8^\circ = 48.0^\circ, 132.0^\circ$$

$$x^\circ = 69.8^\circ, 153.8^\circ \qquad \longleftarrow \text{Add } 21.8^\circ \text{ to both sides}$$

Note that any solutions outwith the domain (often $0 < x < 360$) should have 360° added or subtracted to bring it back within the domain, where possible.

Exercise 14.4

1. Solve each equation for $0 < x < 360$:

   (a) $7\sin(x - 18)^\circ = 4$ (b) $3\cos(x + 34.1)^\circ = -2$ (c) $\sqrt{5}\sin(x - 106)^\circ + 1 = 0$

2. Given $6\sin x^\circ - 8\cos x^\circ = 10\sin(x - 53.1)^\circ$, solve $6\sin x^\circ - 8\cos x^\circ = 5$ where $0 < x < 360$.

3. (a) Express $\sqrt{3}\sin x^\circ + \cos x^\circ$ in the form $k\sin(x - a)^\circ$ where $k > 0$ and $0 < a < 360$.

   (b) Hence solve the equation $\sqrt{3}\sin x^\circ + \cos x^\circ = 1$ where $0 < x < 360$.

4. Solve each equation for $0 < x < 2\pi$:

   (a) $4\sin(x + 0.31) + 2 = 1$ (b) $9\cos(x + 1.24) = 5$ (c) $2\sqrt{3}\sin(x - 0.82) = \sqrt{5}$

5. (a) Express $3\cos x + 2$ in the form $k\cos(x + a)$ where $k > 0$ and $0 < a < 2\pi$.

   (b) Hence solve the equation $2 + 6\sin x + 4\cos x = 5$ where $0 < x < 2\pi$.

14.5 Multiple Angles and Different Variables

The techniques covered in this chapter can be applied to trigonometric expressions beyond those containing only $\sin x$ and $\cos x$; they work for any sum or difference of *equal-angled* trigonometric operations.

e.g. $3\sin t^\circ + 4\cos t^\circ$ or $2\sin 2x - 5\cos 2x$

Example

Express $5\cos 2t^\circ - 3\sin 2t^\circ$ in the form $k\sin(2t+a)^\circ$ where $k>0$ and $0<a<360$.

$5\cos 2t^\circ - 3\sin 2t^\circ = k\sin(2t+a)^\circ$

$5\underline{\underline{\cos 2t^\circ}} - 3\underline{\sin 2t^\circ} = k\underline{\sin 2t^\circ}\cos a^\circ + k\underline{\underline{\cos 2t^\circ}}\sin a^\circ$ ⟵ Expand $k\sin(2t+a)^\circ$

$k\sin a^\circ = 5$ ⟵ Equate $\sin 2t^\circ$ coefficients

$k\cos a^\circ = -3$ ⟵ Equate $\cos 2t^\circ$ coefficients

$$k = \sqrt{(5)^2+(-3)^2} = \sqrt{34}$$ ⟵ Calculate k

$$\tan a^\circ = \frac{5}{-3} = -\frac{5}{3}$$ ⟵ Find $\tan a^\circ$

$$\text{acute}^\circ = \tan^{-1}\left(\frac{5}{3}\right) = 59.0^\circ$$ ⟵ Find acute angle first

$180^\circ - \text{acute}^\circ$

✓✓S	A✓
✓T	C

⟵ Positive $\sin a^\circ$: quadrants A and S

⟵ Negative $\cos a^\circ$: quadrants S and T

$a^\circ = 121.0^\circ$ ⟵ Using $180^\circ - 59.0^\circ$ (Double-ticked)

$5\cos 2t^\circ - 3\sin 2t^\circ = \sqrt{34}\sin(2t+121.0)^\circ$ ⟵ State solution

Exercise 14.5

1. Express $4\cos t^\circ - 3\sin t^\circ$ in the form $k\sin(t-a)^\circ$, where $k>0$ and $0<a<360$.
2. Express $2\sin 2x - \cos 2x$ in the form $k\cos(2x-a)$ where $k>0$ and $0<a<2\pi$.
3. (a) Express $12\cos t^\circ + 5\sin t^\circ$ in the form $k\sin(t+a)^\circ$, where $k>0$ and $0<a<360$.
   (b) Hence state:
      i. The maximum value of the function $f(x) = 12\cos t^\circ + 5\sin t^\circ$, $0<t<360$.
      ii. The value(s) of t for which it occurs.
4. (a) Express $\sin 2x - \sqrt{3}\cos 2x$ in the form $k\cos(2x-a)$ where $k>0$, $0<a<2\pi$.
   (b) Hence solve $\sin 2x - \sqrt{3}\cos 2x - 1 = 0$, $0<x<2\pi$.
   (c) Sketch $y = -\sin 2x + \sqrt{3}\cos 2x - 1 = 0$ for $0 \leqslant x \leqslant 2\pi$.

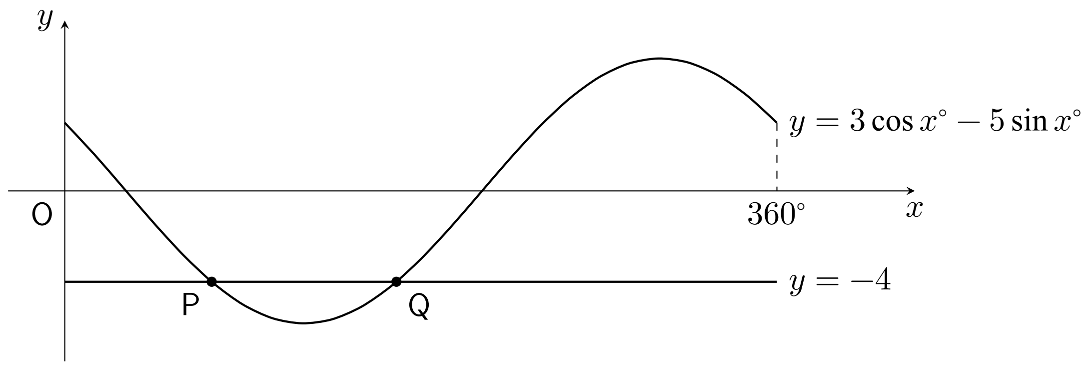
Wave Function Review Exercise

1. Express $8\sin x^\circ + 7\cos x^\circ$ in the form $k\sin(x-a)^\circ$ where $k>0$ and $0<a<360$.

2. Express $\sqrt{5}\sin x + \cos x$ in the form $k\cos(x-a)$ where $k>0$ and $0<a<2\pi$.

3. Express $\sqrt{3}\cos t^\circ - \sin t^\circ$ in the form $k\sin(t+a)^\circ$ where $k>0$ and $0<a<360$.

4. Part of the graphs of $y = 3\cos x^\circ - 5\sin x^\circ$ and $y=-4$ are shown in the diagram below:



   Points P and Q are points of intersection.

   (a) Express $y = 3\cos x^\circ - 5\sin x^\circ$ in the form $k\cos(x+a)^\circ$ where $k>0$ and $0<a<360$.

   (b) Hence determine the coordinates of P and Q.

5. (a) Express $2\sin x^\circ - 4\cos x^\circ$ in the form $k\sin(x-a)^\circ$ where $k>0$ and $0<x<360$.

   (b) Hence sketch the graph of $y = 2\sin x^\circ - 4\cos x^\circ$ for $0<x<360$.

6. (a) Express $\cos x + \sqrt{3}\sin x$ in the form $k\cos(x-a)$ where $k>0$ and $0<a<2\pi$.

   (b) Hence sketch the graph of $y = 2\cos x + 2\sqrt{3}\sin x$ for $0<x<2\pi$.

7. (a) Express $6\cos t - 3\sin t$ in the form $k\cos(t+a)$ where $k>0$ and $0<a<2\pi$.

   (b) Hence solve $2\cos t - \sin t + 2 = 1$ where $0<t<2\pi$.

8. (a) Express $\sin 2x^\circ - \cos 2x^\circ$ in the form $k\sin(2x-a)$ where $k>0$ and $0<a<360$.

   (b) Hence solve the equation $\sin 2x^\circ = \cos 2x^\circ$ where $0<x<360$.

Chapter 15 — Logs and Exponentials

Each part of an operation of the form a^n can be described using the following terminology:

"power" or "exponent" or "index"

"base" $\longrightarrow a^n$

Power functions take the form $f(x) = x^n$, with x as the *base* and a constant power, n.

$$\text{e.g.} \quad f(x) = x^3$$

Exponential functions take the form $f(x) = a^x$, with x as the *exponent* and a constant base, $a > 0, a \neq 1$.

$$\text{e.g.} \quad f(x) = 3^x$$

An example of an exponential function in everyday life is that of something *appreciating* by a percentage of its value, such as an antique vase of value £4000 increasing by 20% each year. Its value after 0 years, 1 years, 2 years, and so on, can be calculated as follows:

$$\begin{aligned}
4000 \times 1.2^0 &= 4000 && \longleftarrow \text{After 0 years} \\
4000 \times 1.2^1 &= 4800 && \longleftarrow \text{After 1 year} \\
4000 \times 1.2^2 &= 5760 && \longleftarrow \text{After 2 years} \\
4000 \times 1.2^3 &= 6912 && \longleftarrow \text{After 3 years} \\
4000 \times 1.2^4 &= 8294.40 && \longleftarrow \text{After 4 years}
\end{aligned}$$

The function to describe its value V after x years is given by: $V(x) = 4000 \times 1.2^x$

One advantage of defining this function is the ability to calculate the value at times other that after whole years. For example, the value after three and a half years can be calculated as:

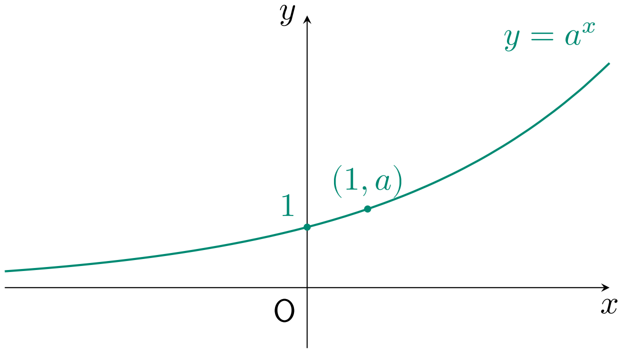
$$V(3.5) = 4000 \times 1.2^{3.5} = 7571.72$$

This chapter will introduce a range of skills required when working with exponential functions.

15.1 Graphs of Exponential Functions

Where $a > 1$:

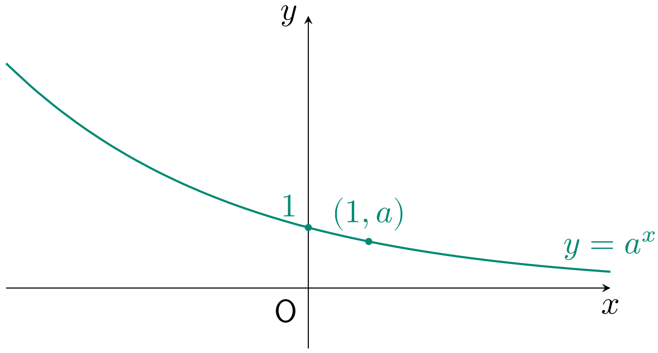
$y = a^x$ is *strictly increasing* on $x \in \mathbb{R}$:



This describes **exponential growth**.

Where $0 < a < 1$:

$y = a^x$ is *strictly decreasing* on $x \in \mathbb{R}$:



This describes **exponential decay**.

Since $a^0 = 1$, any graph of the form $y = a^x$ will pass through the point $(0, 1)$ for all $a \neq 0$.

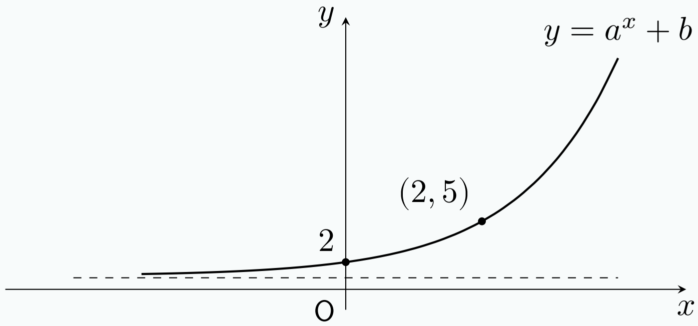
Since $a^1 = a$, any graph of the form $y = a^x$ will pass through the point $(1, a)$ for all a.

The graph of $y = a^x$ has the x-axis as a *horizontal asymptote* - a line that it approaches but never meets.

Determining the equation of the graph of an exponential function can typically be achieved using substitution or consideration of graph transformations, along with knowledge of points $(0, 1)$ and $(1, a)$.

Example

The graph of $y = a^x + b$ is below. Find the values of a and b, and state the range of $f(x) = a^x + b$.

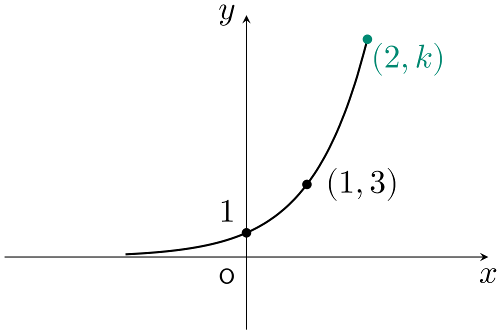


$$
\begin{aligned}
2 &= a^0 + b && \longleftarrow \text{Substitute } (0, 2) \\
2 &= 1 + b && \\
1 &= b && \longleftarrow \text{Solve to obtain } b \\
5 &= a^2 + 1 && \longleftarrow \text{Substitute } (2, 5) \text{ and } b = 1 \\
4 &= a^2 && \\
2 &= a && \longleftarrow \text{Solve to obtain } a \\
y &= 2^x + 1 && \longleftarrow \text{State equation} \\
f(x) &\geqslant 1 && \longleftarrow \text{State range}
\end{aligned}
$$

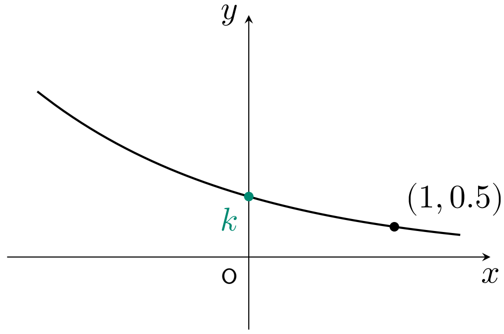
Exercise 15.1

1. Find the equation of each exponential graph using the form given, then determine the value of k.

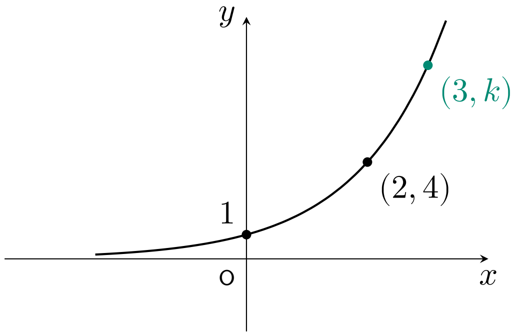
(a) $y = a^x$



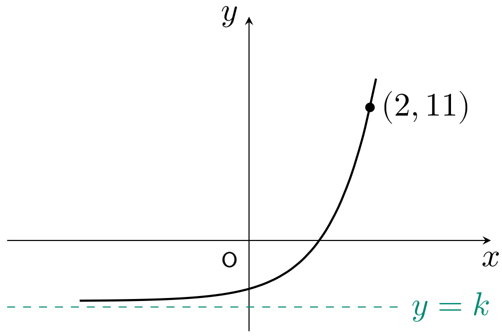
(b) $y = a^x$



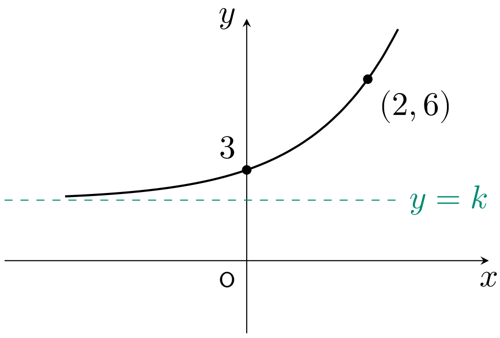
(c) $y = a^x$



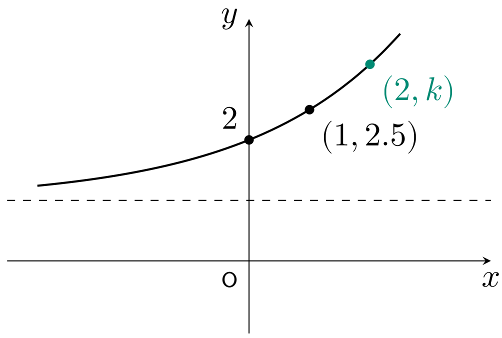
(d) $y = 4^x + b$



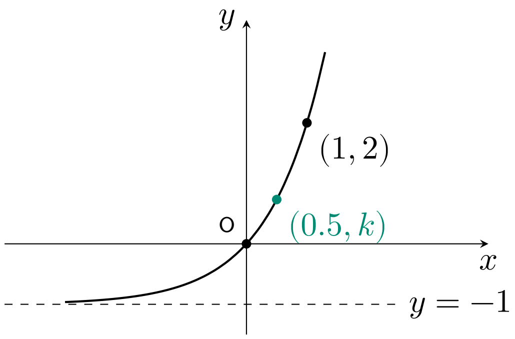
(e) $y = a^x + 2$



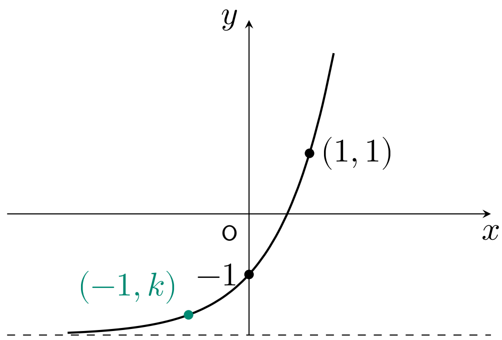
(f) $y = a^x + b$



(g) $y = a^x + b$



(h) $y = a^x + b$



2. Sketch the graph of each of the following, and state the range of $y = f(x)$:

(a) $y = 3^x$ (b) $y = 3^x + 1$ (c) $y = 3^x - 2$ (d) $y = 3^{x+2}$

(e) $y = 0.5^x$ (f) $y = 0.5^x - 1$ (g) $y = 0.5^{x-3}$ (h) $y = 0.5^{1-x}$

15.2 Evaluating Logarithms

The function $f(x) = \log_a x$, where $a > 0$ and $x > 0$, is defined as the inverse function to $f(x) = a^x$.

$$\begin{array}{lll} \text{If} & x = a^y & \\ \text{then} & \log_a x = y & (\text{where } a > 0, x > 0) \end{array}$$

A numerical example will help build an understanding of how to interpret a logarithm:

$$\begin{array}{ll} \text{Since} & 8 = 2^3 \\ \text{then} & \log_2 8 = 3 \end{array}$$

Hence $\log_2 8$ ("log to the base 2 of 8") can be read as: "2 raised to *which power* gives a value of 8?"

Example

Evaluate $\log_3 81$.

$\log_3 81 = 4 \longleftarrow$ Since $3^4 = 81$

Example

Evaluate $\log_5 1$.

$\log_5 1 = 0 \longleftarrow$ Since $5^0 = 1$

Example

Evaluate $\log_7 49$.

$\log_7 49 = 2 \longleftarrow$ Since $7^2 = 49$

Example

Evaluate $\log_3 \frac{1}{9}$.

$\log_3 \frac{1}{9} = -2 \longleftarrow$ Since $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Exercise 15.2

1. State the value each of the following logarithms:

(a) $\log_6 36$ (b) $\log_5 125$ (c) $\log_4 16$ (d) $\log_2 16$ (e) $\log_9 81$

(f) $\log_3 81$ (g) $\log_2 32$ (h) $\log_{10} 100$ (i) $\log_8 64$ (j) $\log_4 64$

(k) $\log_7 7$ (l) $\log_3 9$ (m) $\log_8 1$ (n) $\log_5 25$ (o) $\log_5 5$

(p) $\log_2 32$ (q) $\log_4 4$ (r) $\log_2 1$ (s) $\log_6 216$ (t) $\log_2 128$

2. By considering negative and fractional indices, evaluate each:

(a) $\log_9 3$ (b) $\log_{64} 8$ (c) $\log_{25} 5$ (d) $\log_1 0010$ (e) $\log_8 2$

(f) $\log_{27} 3$ (g) $\log_4 2$ (h) $\log_{16} 2$ (i) $\log_{81} 3$ (j) $\log_{11} 121$

(k) $\log_5 \frac{1}{5}$ (l) $\log_3 \frac{1}{3}$ (m) $\log_6 \frac{1}{36}$ (n) $\log_2 \frac{1}{8}$ (o) $\log_9 \frac{1}{3}$

3. Calculate the value of each to three significant figures:

(a) $\log_2 31$ (b) $\log_7 50$ (c) $\log_{10} 99$ (d) $\log_5 120$ (e) $\log_2 0.9$

15.3 Laws of Logarithms

There are two key identities that can be obtained from the definition of a logarithm. For all $a > 0$:

$$\boxed{\log_a 1 = 0} \quad \text{and} \quad \boxed{\log_a a = 1}$$

There also three *"log laws"* that can be applied for logarithms of the *same base*. For all $a, x, y > 0$:

The **product law**:

$$\log_a (xy) = \log_a x + \log_a y$$

The **quotient law**:

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

The **power law**:

$$\log_a (x^y) = y \log_a x$$

Example

Evaluate $\log_6 8 + 2\log_6 3 - \log_6 2$.

$$\begin{aligned}
&\log_6 8 + 2\log_6 3 - \log_6 2 && \longleftarrow \text{Consider order of operations} \\
=&\log_6 8 + \log_6 (3^2) - \log_6 2 && \longleftarrow \text{Apply power law} \\
=&\log_6 8 + \log_6 9 - \log_6 2 && \\
=&\log_6 (8 \times 9) - \log_6 2 && \longleftarrow \text{Apply product law} \\
=&\log_6 72 - \log_6 2 && \\
=&\log_6 \left(\frac{72}{2}\right) && \longleftarrow \text{Apply product law} \\
=&\log_6 36 && \\
=&2 && \longleftarrow \text{Evaluate}
\end{aligned}$$

Exercise 15.3

1. Evaluate each of the following:

(a) $\log_6 9 + \log_6 4$ (b) $\log_{12} 2 + \log_{12} 6$ (c) $\log_2 3 + \log_2 \frac{1}{3}$

(d) $\log_3 54 - \log_3 2$ (e) $\log_5 10 - \log_5 2$ (f) $\log_5 100 - \log_5 4$

(g) $2\log_4 6 - \log_4 9$ (h) $\log_{12} 4 + 2\log_{12} 6$ (i) $3\log_6 2 + \log_6 9 - \log_6 2$

(j) $2\log_6 3 + \log_6 \frac{2}{3}$ (k) $\log_5 4 + 2\log_5 10$ (l) $2\log_2 6 - \log_2 12 + \log_2 \frac{1}{6}$

2. Simplify each, giving an answer in the form $\log_p q$ where p and q are positive integers:

(a) $\log_7 3 + \log_7 6 - \log_7 9$ (b) $\frac{1}{2}\log_6 9 - \log_6 2$ (c) $2\log_8 9 - 3\log_2 3$

3. Simplify each, giving an answer in the form $\log_a p$ where p is a positive integer:

(a) $\log_a 3 + 2\log_a 4$ (b) $\frac{1}{3}\log_a 8 - \log_a 4$ (c) $\log_a 20 - \log_a 4 + \log_a \frac{1}{15}$

15.4 Solving Log Equations

To solve a *log equation*, it is typically desirable to have each side within a logarithm of the *same base*:

If	$\log_a x = \log_a y$	
then	$x = y$	(where $a > 0, x > 0, y > 0$)

"Cancelling the logs", mathematically, is raising each side to an exponential base a, using $a^{\log_a b} = b$.

Example

Solve the equation $\log_3 x + \log_3 5 = \log_3 20$.

$$\begin{aligned} \log_3 x + \log_3 5 &= \log_3 20 \\ \log_3 5x &= \log_3 20 && \longleftarrow \text{Apply product law} \\ 5x &= 20 && \longleftarrow \text{“Cancel” } \log_3 \text{ from both sides} \\ x &= 4 && \longleftarrow \text{Solve} \end{aligned}$$

To express any term as a logarithm, multiply it by $\log_a a$ (which equals 1) and then apply the power law:

Example

Solve the equation $\log_3 x - \log_3 2 = 2$.

$$\begin{aligned} \log_3 x - \log_3 2 &= 2\log_3 3 && \longleftarrow \text{Multiply the constant by } \log_3 3 \\ \log_3\left(\frac{x}{2}\right) &= \log_3 3^2 && \longleftarrow \text{Apply quotient and power laws} \\ \frac{x}{2} &= 9 && \longleftarrow \text{“Cancel” } \log_3 \text{ from both sides} \\ x &= 18 && \longleftarrow \text{Multiply both sides by 2 to solve} \end{aligned}$$

Exercise 15.4

1. Solve:

   (a) $\log_5 3 + \log_5 x = \log_5 18$

   (b) $\log_7 2x + \log_7 4 = \log_7 56$

   (c) $\log_6 5 + \log_5 (x - 1) = \log_6 30$

   (d) $\log_a x - 2\log_a 3 = \log_a 18$

   (e) $\log_8 2x - \log_8 3 = \log_8 1$

   (f) $\log_3 8 - \log_3 x = \log_3 2$

2. Solve each:

   (a) $\log_6 x + \log_6 12 = 2$

   (b) $\log_4 x + \log_4 8 = 2$

   (c) $\log_2 10 + \log_2 x = 4$

   (d) $\log_5 x - \log_5 2 = 1$

   (e) $\log_3 x - 2\log_3 2 = 2$

   (f) $\log_4 2 + 3\log_4 x = 2$

   (g) $\log_2 5 + 3 = \log_2 (x + 2)$

   (h) $\log_3 (2x) - 1 = \log_3 8$

   (i) $2 - \log_6 (x - 3) = 2\log_6 3$

15.5 Log Equations and Extraneous Solutions

Some algebraic manipulations used to solve an equation may produce solutions do not, in fact, satisfy the original equation. These **extraneous solutions** can arise when solving log equations.

Since $\log_a x$ is only defined for $a > 0$ and $x > 0$, any solutions which contradict this must be discarded.

Example

Solve the equation $\log_3 x + \log_3 (x-2) = \log_3 8$.

$$\begin{aligned}
\log_3 x + \log_3 (x-2) &= \log_3 8 && \longleftarrow \text{Note } x>0 \text{ and } (x-2)>0 \text{ therefore } x>2 \\
\log_3 (x(x-2)) &= \log_3 8 && \longleftarrow \text{Apply product law} \\
x^2 - 2x &= 8 && \longleftarrow \text{“Cancel” } \log_3 \text{ and expand} \\
x^2 - 2x - 8 &= 0 && \longleftarrow \text{Equate quadratic equation to zero} \\
(x+2)(x-4) &= 0 && \longleftarrow \text{Factorise} \\
x+2 = 0, x-4 &= 0 \\
\cancel{x=-2}, x &= 4 && \longleftarrow \text{Discard } x=-2 \text{ since } x>2
\end{aligned}$$

All solutions for x must be checked carefully to verify that they satisfy the *original* log equation.

Exercise 15.5

1. Solve for x::

(a) $\log_a x + \log_a (x+1) = \log_a 6$

(b) $\log_4 x + \log_4 (x-3) = \log_4 10$

(c) $\log_6 (x-1) + \log_6 x = 1$

(d) $\log_3 (x+3) - \log_3 x = 2$

(e) $\log_5 (x+2) + \log_5 (x-3) = \log_5 6$

(f) $\log_a 3 = \log_a (x+5) + \log_a 2x$

2. Solve each:

(a) $\log_5 x^2 + \log_5 4 = \log_5 36$

(b) $\log_6 2 + \log_6 x^2 = \log_6 50$

(c) $\log_a 7 + \log_a x^2 = \log_a 14$

(d) $2\log_a x + \log_a 3 = \log_a 48$

(e) $\log_9 4 + 2\log_9 x = 2\log_9 8$

(f) $2\log_7 3x + \log_7 2 = \log_7 18$

3. Solve each for a:

(a) $\log_a 3 + \log_a 2 = 1$

(b) $\log_a 18 - \log_a 2 = 1$

(c) $\log_a 4 + \log_a 2 = 3$

(d) $\log_a 12 - \log_a 4 = \frac{1}{2}$

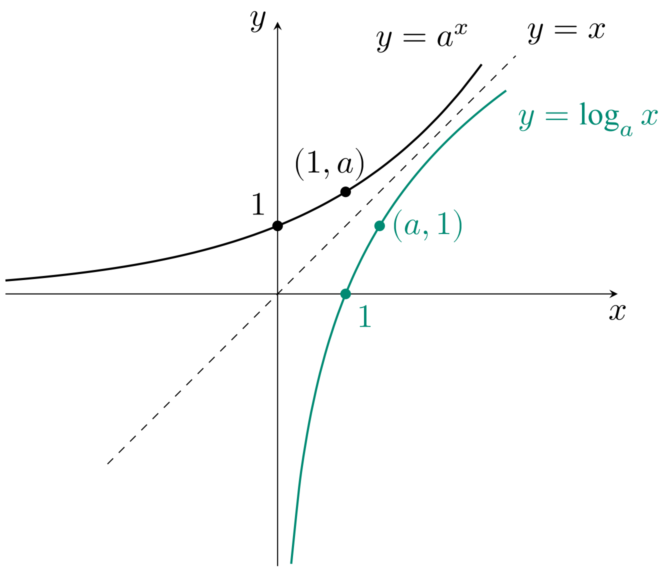
(e) $\log_a 16 + \log_a 4 = 2$

(f) $\log_a 50 - \log_a 2 = 2$

15.6 Graphs of Logarithmic Functions

Given a function f, defined on a suitable domain such that an inverse function exists, the graph of its inverse function $y = f^{-1}(x)$ can be obtained by *reflecting the graph of* $y = f(x)$ *in the line* $y = x$.

Since the inverse to the exponential function $f(x) = a^x$ is the **logarithmic function** $f^{-1}(x) = \log_a x$, knowledge of exponential graphs allows log graphs to be sketched:



The graph of $y = a^x$ passes through...

$(0, 1)$ and $(1, a)$

...and has a **horizontal** asymptote of $x = 0$.

The graph of $y = \log_a x$ passes through...

$(1, 0)$ and $(a, 1)$.

..and has a **vertical** asymptote of $y = 0$.

In general, if $y = f(x)$ passes through (x, y) then $y = f^{-1}(x)$ passes through (y, x).

Knowledge of the graph of $y = \log_a x$ may be combined with and understanding of graph transformations.

Example

Sketch the graph of $y = \log_3 (x - 1) + 2$.

Consider $y = \log_3 (x - 1) + 2$ as a $f(x - 1) + 3$ transformation of $y = log_3 x$:

Exercise 15.6

15.7 Exponential Growth and Decay

Exercise 15.7

15.8 Experimental Data of the form $y = ab^x$

Exercise 15.8

15.9 Experimental Data of the form $y = ax^b$

Exercise 15.9

Logs and Exponentials Review Exercise

Chapter 16 — Further Calculus

16.1 Differentiating Trig Functions

Exercise

16.2 Integrating Trig Functions

Exercise

16.3 The Chain Rule for Differentiation

Exercise

16.4 Integrating $(ax + b)^n$

Exercise

Review Exercise

Challenge Problems

The following problems **do not** represent the kind of question expected to feature in a Higher Mathematics exam, either in the way they are presented or the level of difficulty. Instead, they aim to encourage a flexible approach towards problem-solving and an understanding that the skills covered in the course have applications beyond those featured in any typical exam. *Some questions may be solvable without using the skills covered in this chapter, and some questions may be unrelated to this chapter.*

Warm-Up Problems

Chapter 1 Problems

Chapter 2 Problems

Chapter 3 Problems

Chapter 4 Problems

Chapter 5 Problems

Chapter 6 Problems

Chapter 7 Problems

Chapter 8 Problems

Chapter 9 Problems

Chapter 10 Problems

Chapter 11 Problems

Chapter 12 Problems

Chapter 13 Problems

Chapter 14 Problems

Chapter 15 Problems

Chapter 16 Problems

End of Course Problems

Chapter 1 Answers

Chapter 2 Answers

Chapter 3 Answers

Chapter 4 Answers

Chapter 5 Answers

Chapter 6 Answers

Chapter 7 Answers

Chapter 8 Answers

Chapter 9 Answers

Chapter 10 Answers

Chapter 11 Answers

Chapter 12 Answers

Chapter 13 Answers

Chapter 14 Answers

Chapter 15 Answers

Chapter 16 Answers