01.02 rootfinding : fixed point iteration

1 fixed point of a function

definition 04

real number r is a **fixed point** of function g if $g(r) = r$.

form

1. restate problem $f(x) = 0$ into form $g(x) = x$

$$
\begin{aligned}
f(x) &= cos\, x - x = 0 \quad \Rightarrow \quad & g(x) &= cos\, x \quad & \overset{g(x)=x}{\Longrightarrow} \quad & r \approx 0.7390851332; \\
f(x) &= x^3 - x = 0 \quad \Rightarrow \quad & g(x) &= x^3 \quad & \Longrightarrow \quad & r = -1, 0, 1.
\end{aligned}
$$

2. then iterate from initial guess x_0.

algorithm

```
x[0] = initial guess
x[i+1] = g(x[i]) for i = 0,1,2,...
```

```
x[1] = g(x[0])
x[2] = g(x[1])
x[3] = g(x[2])
...
```

code, fpi

```
# algorithm, basic

def fpi(g,x,tol=1e-8,max_iter=100):
  count = 0
  gx = g(x)
  while (abs(gx-x) > tol) and (count < max_iter):
    x = gx
    gx = g(x)
    count += 1
  return x

```

```
# algorithm, expanded for lecture

def fpi_expanded(g,x,tol=1e-8,max_iter=100,worksheet=False):⋯
```

2 geometry

fpi may not converge! but if g is continuous and x_i converge to r, then r is a fixed point.

$$g(r) = g\left(\lim_{i\to\infty} x_i\right) = \lim_{i\to\infty} g(x_i) = \lim_{i\to\infty} x_{i+1} = r.$$

example 01, revisited

$x^3 + x - 1 = 0.$

code, example 01-1

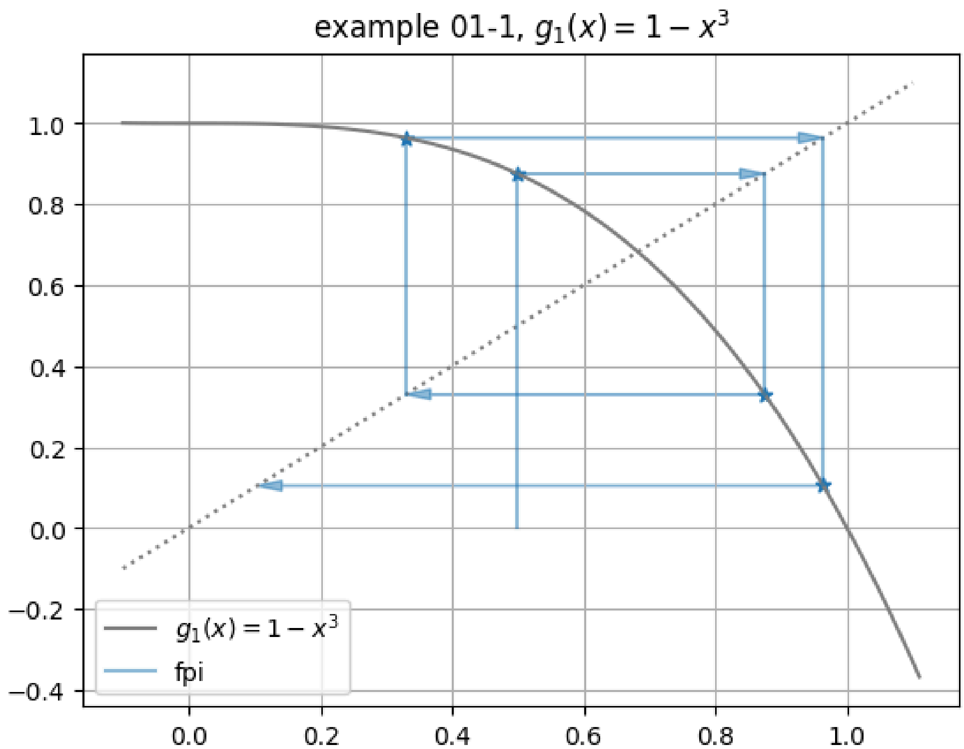
1. $x = \overbrace{1 - x^3}^{g_1(x)} \quad \Rightarrow \quad g_1(x) = x. \quad \checkmark$

```
# example 01-1, g(x) = 1 - x^3

if __name__ == "__main__":⋯
```

```
example 01-1

|   i |       x[i] |
|-----|------------|
| 000 | 0.50000000 |
| 001 | 0.87500000 |
| 002 | 0.33007812 |
| 003 | 0.96403747 |
| 004 | 0.10405419 |
| 005 | 0.99887338 |
|-----|------------|
| 008 | 0.00000012 |
| 009 | 1.00000000 |
| 010 | 0.00000000 |
| 011 | 1.00000000 |
| 012 | 0.00000000 |
```



code, example 01-2

2. $$x = (x^3)^{\frac{1}{3}} = \overbrace{(1-x)^{\frac{1}{3}}}^{g_2(x)} \quad \Rightarrow \quad g_2(x) = x. \ \checkmark$$

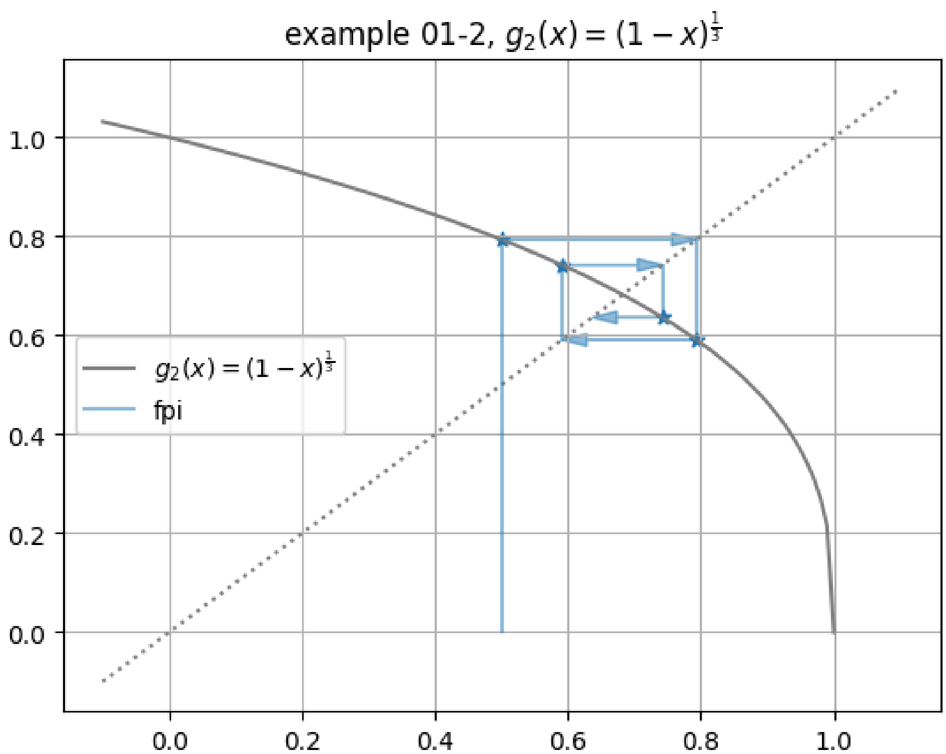
```
# example 01-2, g(x) = cbrt(1 - x)

if __name__ == "__main__":

```

```
example 01-2. x0 = mean. (endpoints diverge.)

|   i |        x[i] |
|-----|------------|
| 000 | 0.50000000 |
| 001 | 0.79370053 |
| 002 | 0.59088011 |
| 003 | 0.74236393 |
| 004 | 0.63631020 |
| 005 | 0.71380081 |
|-----|------------|
| 048 | 0.68232779 |
| 049 | 0.68232782 |
| 050 | 0.68232779 |
| 051 | 0.68232781 |
| 052 | 0.68232780 |
```



code, example 01-3

3. $g_3(x) = \frac{1+2x^3}{1+3x^2}$ bc

$$f(x) = x^3 + x - 1 = 0$$

$$x^3 + x - 1 + 2x^3 = 0 + 2x^3$$

$$(3x^2 + 1) \cdot x = +1 + 2x^3 \quad \Rightarrow \quad x = \underbrace{\frac{1+2x^3}{1+3x^2}}_{g_3(x)}. \quad \checkmark$$

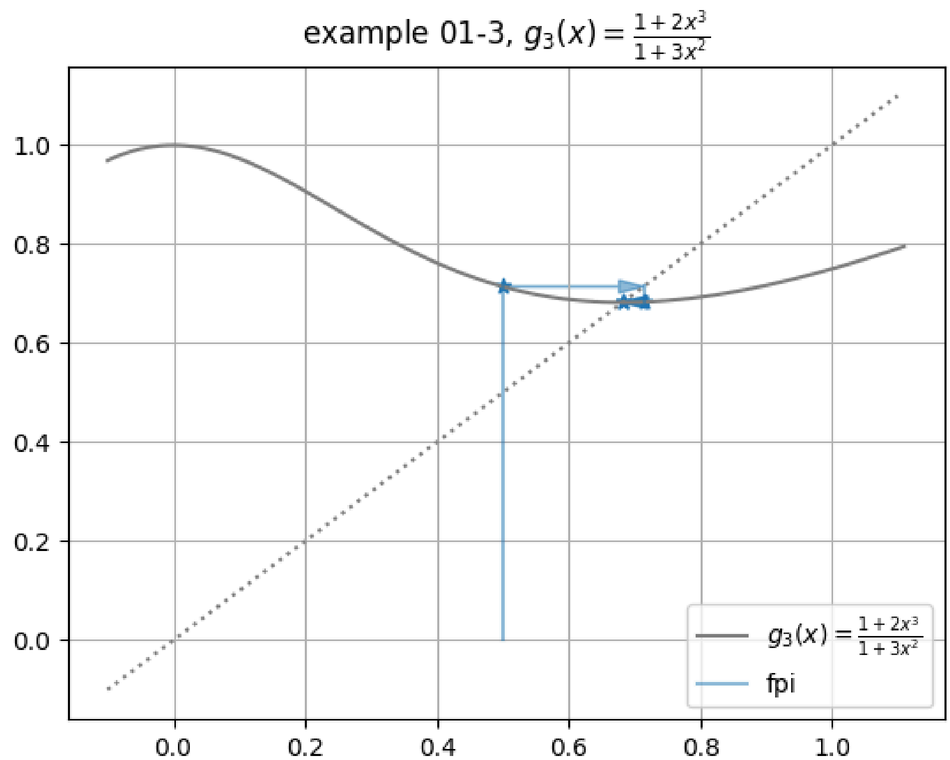
ie, add $2x^3$ to each side of $f(x)$ to reduce its order. for why "2" in "$2x^3$", consider what $g(x)$ is wrt x.

```
# example 01-3, g(x) = [1+2x^3] / [1+3x^2]]

if __name__ == "__main__":⋯
```

```
example 01-3, x0 = mean.
```

i	x[i]
000	0.50000000
001	0.71428571
002	0.68317972
003	0.68232842
004	0.68232780



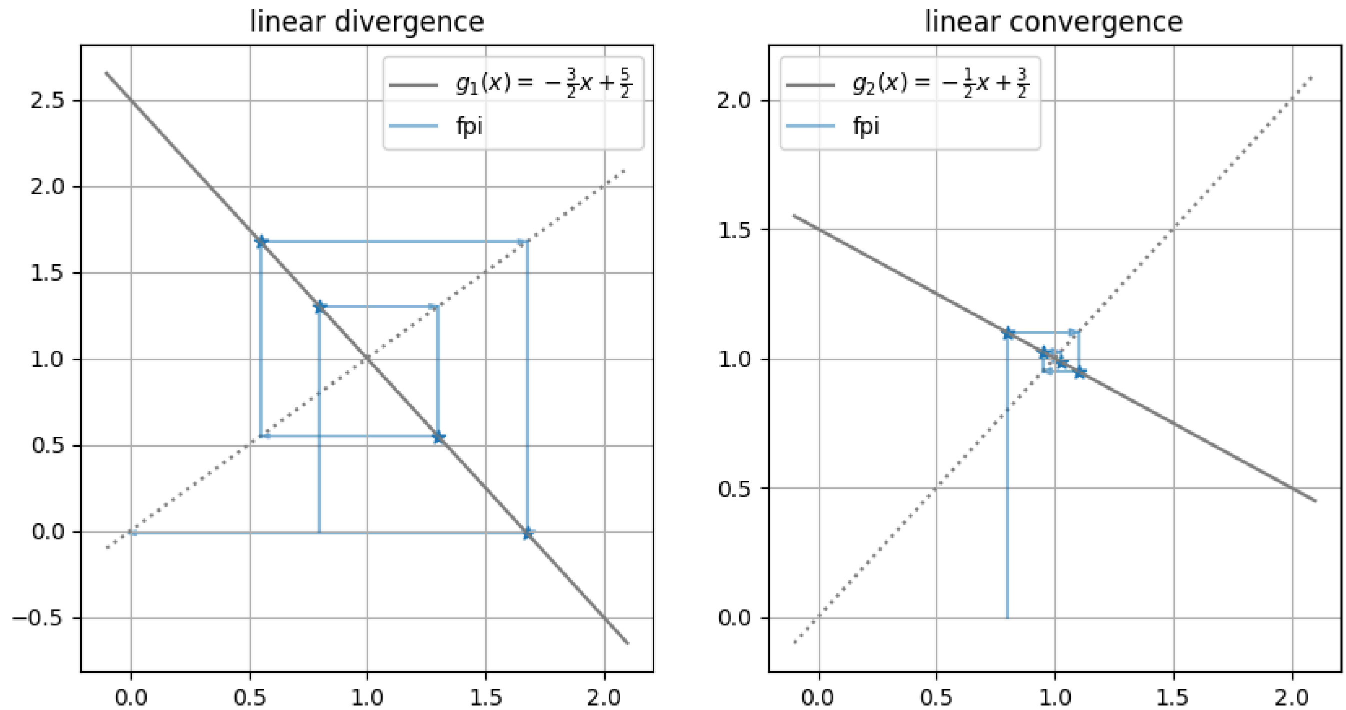
3 linear convergence

consider linear functions, $g_1(x) = -\frac{3}{2}x + \frac{5}{2}$ and $g_2(x) = -\frac{1}{2}x + \frac{3}{2}$. while both have fixed point of $r = 1$, $|g_1'(1)| = |-\frac{3}{2}| > 1$, $|g_2'(1)| = |-\frac{1}{2}| < 1$.

code, visual: convergence

```
if __name__ == "__main__":

```



$$
\begin{aligned}
g_1(x) &= -\tfrac{3}{2}(x-1)+1 \\
\Rightarrow g_1(x)-1 &= -\tfrac{3}{2}(x-1) \\
\Rightarrow x_{i+1}-1 &= -\tfrac{3}{2}(x_i-1) \\
e_i = |x_i - r| &\Rightarrow e_{i+1} = \tfrac{3}{2}e_i;
\end{aligned}
$$

$$
\begin{aligned}
g_2(x) &= -\tfrac{1}{2}(x-1)+1 \\
\Rightarrow g_2(x)-1 &= -\tfrac{1}{2}(x-1) \\
\Rightarrow x_{i+1}-1 &= -\tfrac{1}{2}(x_i-1) \\
e_i = |x_i - r| &\Rightarrow e_{i+1} = \tfrac{1}{2}e_i.
\end{aligned}
$$

definition 05

let e_i denote error at step i of an iterative method. if

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i} = S < 1,$$

the method obeys **linear convergence** with rate S.

theorem 06

assume g is continuously differentiable, $g(r) = r$ and $S = |g'(r)| < 1$. then FPI converges linearly with rate S to r for guesses x_0 sufficiently close to r.

proof

let x_i denote iterate at step i. by mean value theorem (MVT), there exists number c_i between x_i and r such that

$$x_{i+1} - r = g'(c_i)(x_i - r)$$

$$\Downarrow \quad x_{i+1} = g(x_i), r = g(r), e_i = |x_i - r|$$

$$e_{i+1} = |g'(c_i)|e_i.$$

if $S = |g'(r)| < 1$, then by continuity of g' there is small neighborhood around r for which $S < |g'(x)| < (S+1)/2 < 1$. if x_i in this neighborhood, then c_i is as well. so

$$e_{i+1} \leq \frac{S+1}{2} e_i.$$

ie, error decreases by at least $(S+1)/2$ on current and each future step. ie, as $\lim_{i \to \infty} x_i = r$,

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = \lim_{i \to \infty} |g'(c_i)| = |g'(r)| = S. \quad \blacksquare$$

usw

ie, the approximate error relationship

$$e_{i+1} \approx S e_i$$

holds in the limit as convergence is approached where $S = |g'(x^*)|$.

definition 07

an iterative method is **locally convergent** to r if method converges to r for initial guesses sufficiently close to r.

ie, the $g(x)$ is locally convergent to root r if there exists some neighborhood $(r-\epsilon, r+\epsilon)$ where $\epsilon > 0$ such that all initial guesses in that neighborhood converge. specifically, theorem 06 states that fpi converges locally if $|g'(r)| < 1$.

example 03

explain why $g(x) = cos\,x$ converges.

bc $g'(r) = -sin\,r \approx -sin\,0.74 \approx -0.96 \Rightarrow |g'(r)| < 0.$ ✓

example 04

find root of $cos\,x = sin\,x$ using FPI.

$\Rightarrow g(x) = x + cos\,x - sin\,x = x.$

code, example 04

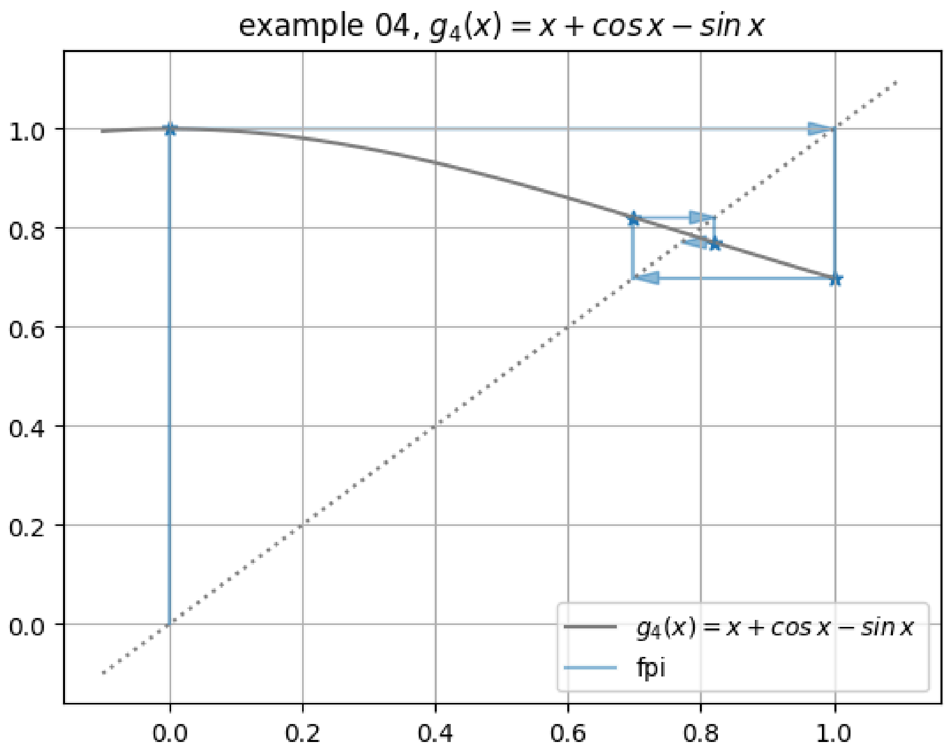
```
# example 04, g(x) = x + cosx - sinx

if __name__ == "__main__":

```

```
example 04. x0 = 0.

|   i |       x[i] |       e[i] |   "/e[i-1] |
|-----|------------|------------|------------|
| 000 | 0.00000000 | 0.78539816 |            |
| 001 | 1.00000000 | 0.21460184 | 0.27323954 |
| 002 | 0.69883132 | 0.08656684 | 0.40338351 |
| 003 | 0.82110248 | 0.03570431 | 0.41244791 |
| 004 | 0.77061968 | 0.01477848 | 0.41391311 |
| 005 | 0.79151885 | 0.00612069 | 0.41416208 |
|-----|------------|------------|------------|
| 017 | 0.78539832 | 0.00000016 | 0.41421356 |
| 018 | 0.78539810 | 0.00000006 | 0.41421356 |
| 019 | 0.78539819 | 0.00000003 | 0.41421356 |
| 020 | 0.78539815 | 0.00000001 | 0.41421356 |
| 021 | 0.78539817 | 0.00000000 | 0.41421358 |
```



that last table column explains the previous column. ie, the last column displays the ratio by which error e_i decreases:

$$e_i \approx 0.414\, e_{i-1}.$$

theorem 06 implies

$$S = |g'(r)| = |1 - sin\, r - cos\, r| = |1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}| = |1 - \sqrt{2}| \approx 0.414. \;\checkmark$$

example 05

find fixed points of $g(x) = 2.8\,x - x^2$.

by various non-computational shortcuts, roots $r = 0, 1.8$.

code, example 05

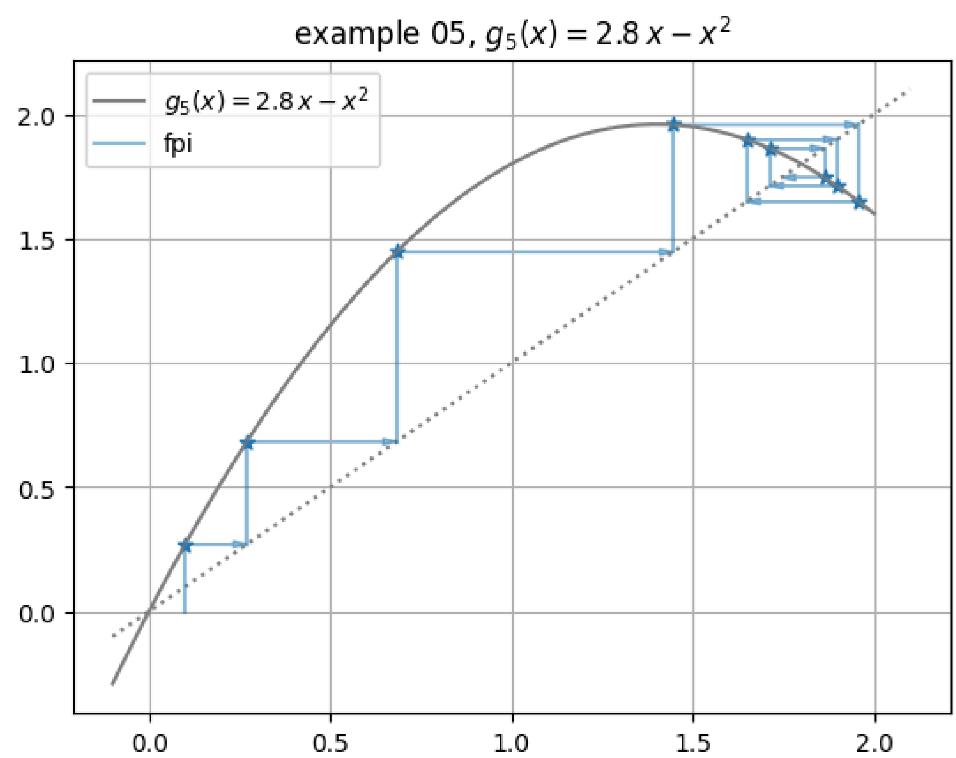
```
# example 05, g(x) = 2.8x - x^2

if __name__ == "__main__":

```

```
example 05. x0 = 0.1.

|   i |       x[i] |       e[i] |   "/e[i-1] |
|-----|------------|------------|------------|
| 000 | 0.10000000 | 1.70000000 |            |
| 001 | 0.27000000 | 1.53000000 | 0.90000000 |
| 002 | 0.68310000 | 1.11690000 | 0.73000000 |
| 003 | 1.44605439 | 0.35394561 | 0.31690000 |
| 004 | 1.95787899 | 0.15787899 | 0.44605439 |
| 005 | 1.64877103 | 0.15122897 | 0.95787899 |
|-----|------------|------------|------------|
| 077 | 1.79999999 | 0.00000001 | 0.80000002 |
| 078 | 1.80000001 | 0.00000001 | 0.79999995 |
| 079 | 1.79999999 | 0.00000001 | 0.80000003 |
| 080 | 1.80000001 | 0.00000001 | 0.79999998 |
| 081 | 1.79999999 | 0.00000001 | 0.80000010 |
```



obviously the intersection of $g_5(x)$ and $y = x$ are the roots; however, note that root $r = 1.8$ is found from $x_0 = 0.1$ despite its closer proximity to root $r = 0$. thats bc $g'(1.8) = -0.8$ vs $g'(0) = 2.8$, :D.

example 06: the babylonians

calculate $\sqrt{2}$ using FPI.

$$x_{i+1} = \frac{x_i + \frac{2}{x+i}}{2}.$$

code, example 06

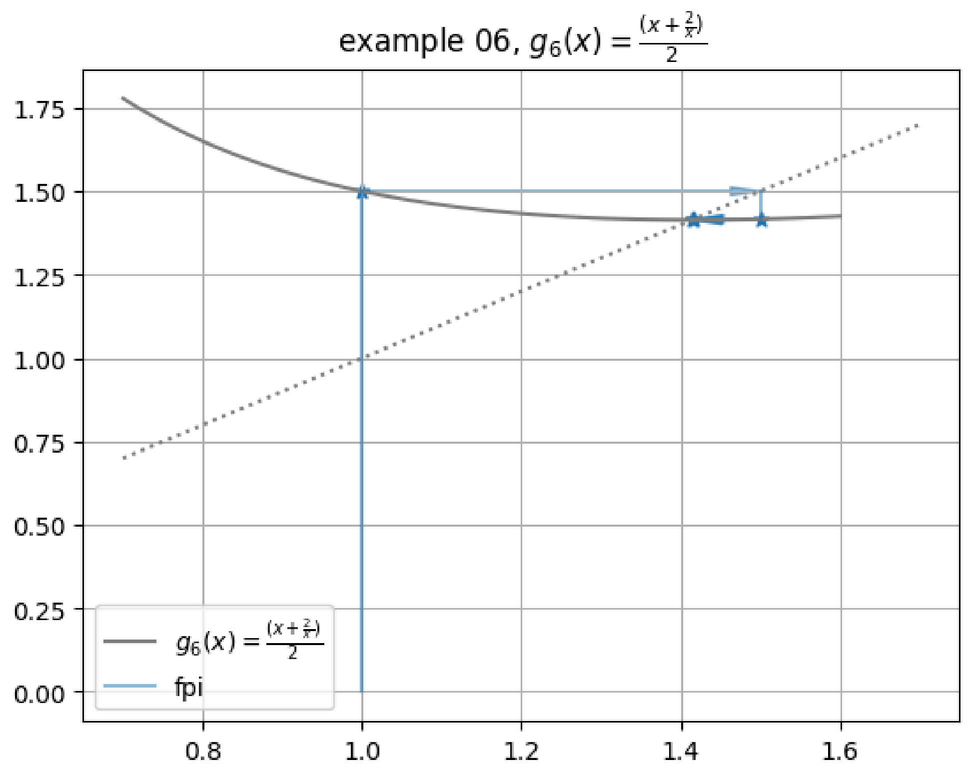
```
# example 06, g(x) = [x + 2/x]/2

if __name__ == "__main__":

```

```
example 06. x0 = 1.

|   i |       x[i] |       e[i] |   "/e[i-1] |
|-----|------------|------------|------------|
| 000 | 1.00000000 | 0.41421356 |            |
| 001 | 1.50000000 | 0.08578644 | 0.20710678 |
| 002 | 1.41666667 | 0.00245310 | 0.02859548 |
| 003 | 1.41421569 | 0.00000212 | 0.00086580 |
| 004 | 1.41421356 | 0.00000000 | 0.00000075 |
```



4 stopping criteria

bisection is predictable and guaranteed to converge; FPI might converge locally, linearly and quickly. or it might not, lol. instead of estimating steps required for a given error, specify a stopping criteria for FPI. eg, for tolerance TOL,

$$\Delta x = |x_{i+1} - x_i| < TOL \sim \epsilon$$

$$\frac{|x_{i+1} - x_i|}{|x_{i+1}|} < TOL, \quad r \text{ not near } 0$$

$$\frac{|x_{i+1} - x_i|}{\max(|x_{i+1}|, \theta)} < TOL, \quad \theta > 0 \text{ and } r \text{ near } 0.$$