# 00.03 basics : more examples

## example 01

show $x^3$ is forward stable.

ie, there exists $\eta > 0$ such that $||\Delta y|| \leq \eta \cdot ||y||$ where

$$\Delta y = y - \hat{y}, \text{ where } \begin{cases} \phi : \mathbb{R} \to \mathbb{R}, x \overset{\phi}{\mapsto} x^3 & \Rightarrow y = \phi(x) \\ \hat{\phi} : \mathbb{FP} \to \mathbb{FP}, \hat{x} \overset{\hat{\phi}}{\mapsto} \hat{x}^3 & \Rightarrow \hat{y} = \hat{\phi}(\hat{x}). \end{cases}$$

$$\hat{y} = \hat{x}^3 = (\hat{x} \otimes \hat{x}) \otimes \hat{x}$$

$$= (x(1+\delta_x) \otimes x(1+\delta_x)) \otimes (x(1+\delta_x))$$

$$= x^2(1+\delta_x)^2(1+\delta_\otimes) \otimes (x(1+\delta_x))$$

$$= x^3(1+\delta_x)^3(1+\delta_\otimes)^1(1+\delta_\otimes)^1$$

$$= x^3(1+\theta_5) \quad \text{by theorem 01}$$

$$\Downarrow$$

$$\Delta y = \hat{y} - y = x^3(1+\theta_5) - x^3 = x^3\theta_5$$

$$\Downarrow$$

$$|\Delta y| = |\hat{y} - y| \leq |x^3||\theta_5| \leq |y| \cdot \gamma_5 \quad \text{by theorem 01, where } \gamma_5 = \frac{n\mu_M}{1-n\mu_M} = \frac{5\mu_M}{1-5\mu_M}$$

$$\text{choose } \eta = \gamma_5, \text{ then } |\hat{y} - y| \leq \eta \cdot |y|. \quad \blacksquare$$

example 02

show $x^3$ is numerically stable.

ie, there exists $\eta > 0$ and $\epsilon > 0$ such that $||\Delta y|| \leq \eta \cdot ||y||$ and $|\Delta x| \leq \epsilon |x|$ where

$$\Delta y = y - \hat{y}, \text{ where } \begin{cases} \phi : \mathbb{R} \to \mathbb{R}, x \overset{\phi}{\mapsto} x^3 & \Rightarrow y = \phi(x) \\ \hat{\phi} : \mathbb{FP} \to \mathbb{FP}, \hat{x} \overset{\hat{\phi}}{\mapsto} \hat{x}^3 & \Rightarrow \hat{y} = \hat{\phi}(\hat{x}). \end{cases}$$

note: $\hat{y} = x^3(1+\theta_5), \gamma_5$ from example 01, forward stability.

$$\hat{y} + \Delta y = \phi(x + \Delta x) \quad \text{see highams schematic}$$

$$x^3(1+\theta_5) + \Delta y = (x+\Delta x)^3 = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\Rightarrow \Delta y = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3\theta_5.$$

$$\text{what } \eta, \epsilon \text{ such that } |\Delta y| \leq \eta \cdot |y|, |\Delta x| \leq \epsilon \cdot |x| \text{ ?}$$

$$|\Delta y| \leq 3|x|^2|\Delta x| + 3|x||\Delta x|^2 + |\Delta x|^3 + |x|^3\theta_5$$

$$\leq 3|x|^2\epsilon|x| + 3|x|\epsilon^2|x|^2 + \epsilon^3|x|^3 + |x|^3\gamma_5 \quad \text{by definition of } \epsilon, \gamma_5$$

$$\leq 3\epsilon|x|^3 + 3\epsilon^2|x|^3 + \epsilon^3|x|^3 + |x|^3\gamma_5 = (3\epsilon + 3\epsilon^2 + \epsilon^3 + \gamma_5)|y|$$

$$\Rightarrow |\Delta y| \leq \eta \cdot |y| \Rightarrow \eta = 3\epsilon + 3\epsilon^2 + \epsilon^3 + \gamma_5. \quad \checkmark$$

ie, take any $\epsilon > 0$ and any $\Delta x$ such that $|\Delta x| \leq \epsilon|x|$. then $\eta = 3\epsilon + 3\epsilon^2 + \epsilon^3 + \gamma_5$ and $\Delta y = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3\theta_5$ and by construction, $|\Delta y| \leq \eta \cdot |y|$ and $\hat{y} + \Delta y = \phi(x + \Delta x)$. ie, $\hat{\phi}(x) = \hat{x}^3$ is numerically stable. ■

note: thats "any" $\epsilon$ vs "best, smallest" $\epsilon, \eta$. also, the only part of the derived $\eta$ outside control is $\gamma_5$, so choose $\epsilon \ll 1$ such that $\eta$ is barely above $\gamma_5$. ie, $\eta = \gamma_5 \cdot c_0$ where $c_0 \geq 1$.

example 03

calculate error for $f_1(a,b) = (a+b)^2$.

$$\hat{a} \oplus \hat{b} = a(1+\delta_a) \oplus b(1+\delta_b)$$

$$= (a + b + a\delta_a + b\delta_b)(1+\delta_\oplus)$$

$$= (a+b)\left(1 + \frac{a\delta_a + b\delta_b}{a+b}\right)(1+\delta_\oplus)$$

$$= (a+b)(1+\delta_{a,b})(1+\delta_\oplus)$$

$$\Downarrow$$

$$\hat{y} = (\hat{a} \oplus \hat{b}) \otimes (\hat{a} \oplus \hat{b})$$

$$= [(a+b)(1+\delta_{a,b})(1+\delta_\oplus)][(a+b)(1+\delta_{a,b})(1+\delta_\oplus)](1+\delta_\otimes)$$

$$= (a+b)^2(1+\delta_{a,b})^2(1+\delta_\oplus)^2(1+\delta_\otimes)^1$$

$$= (a+b)^2(1+\theta_5) \quad \text{theorem 01}$$

$$\Downarrow$$

$$|\Delta y| = |y - \hat{y}| = |(a+b)^2 - [(a+b)^2(1+\theta_5)]|$$

$$= |(a+b)^2\theta_5| = |y| \cdot \theta_5 \leq \gamma_5 \cdot |y|.$$

example 03, continued (as homework, lol)

1. calculate error for the binomial expansion of $f_1 \to f_2(a,b) = a^2 + 2ab + b^2$.
2. discuss $f_1, f_2$ if exponent $2$ replaced with $n$ and $n \to \infty$.