03.02 interpolation: error

1 interpolation error formula

theorem 04

assume that $P(x)$ is the (degree $n-1$ or less) interpolating polynomial fitting the n points $(x_1, y_1), \ldots, (x_n, y_n)$. interpolation error,

$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{n!} f^{(n)}(c),$$

where c lies between the smallest and largest of $\{x : x_1, \ldots, x_n\}$.

continuing example 07,

$$sin\ x - P(x) = \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} f''''(c),$$

where $0 < c < \frac{\pi}{2}$, $f''''(c) = sin\ c \in [0,1]$. ie, at worst, $|sin\ c| = |1|$ so,

$$|sin\ x - P(x)| \leq \frac{|(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})|}{24} |1|.$$

$$\Downarrow \quad \text{worst case@ } x = 1$$

$$|sin\ 1 - P(1)| \leq \frac{|(1-0)(1-\frac{\pi}{6})(1-\frac{\pi}{3})(1-\frac{\pi}{2})|}{24} |1| \approx 0.0005348.$$

computational error at $x = 1$ from example 07 (below) is less than this upper bound. ✓

x (rad)	np.sin()	N_3,mod	error
1	0.841471	0.841076	0.000394766
2	0.909297	0.910169	0.000871189
3	0.14112	0.142842	0.0017216
4	-0.756802	-0.755661	0.0011416
14	0.990607	0.992824	0.00221669
1000	0.82688	0.826294	0.000585579

consider upper bound at $x = 0.2$, near the end of the evaluation interval.

$$|sin\ 0.2 - P(0.2)| \leq \frac{|(0.2-0)(0.2-\frac{\pi}{6})(0.2-\frac{\pi}{3})(0.2-\frac{\pi}{2})|}{24} |1| \approx 0.00313,$$

about six times larger. correspondingly, the actual error is larger as well.

$$|sin\ 0.2 - P(0.2)| = |0.19867 - 0.20056| = 0.00189.$$

x (rad)	np.sin()	N_3,mod	error
0.2	0.198669	0.200546	0.00187652

example 08

find upper bound for difference at $x = 0.25, x = 0.75$ between $f(x) = e^x$ and polynomial that interpolates it at $x \in \{-1, -0.5, 0, 0.5, 1\}$.

no need to construct interpolating polynomial.

$$f(x) - P_4(x) = \frac{(x+1)(x+\frac{1}{2})x(x-\frac{1}{2})(x-1)}{5!} f^{(5)}(c),$$

$$\Downarrow \quad \text{where } -1 < c < 1,\ f^{(5)}(c) = e^c \Rightarrow |f^{(5)}| \leq e^1 \text{ on } [-1,1]$$

$$|e^x - P_4(x)| \leq \frac{(x+1)(x+\frac{1}{2})x(x-\frac{1}{2})(x-1)}{5!} e$$

$$\Downarrow \quad x = 0.25$$

$$|e^{0.25} - P_4(0.25)| \leq \frac{(0.25+1)(0.25+\frac{1}{2})(0.25)(0.25-\frac{1}{2})(0.25-1)}{120} e \approx 0.000995$$

$$\Downarrow \quad x = 0.75$$

$$|e^{0.75} - P_4(0.75)| \leq \frac{(0.75+1)(0.75+\frac{1}{2})(0.75)(0.75-\frac{1}{2})(0.75-1)}{120} e \approx 0.002323.$$

code

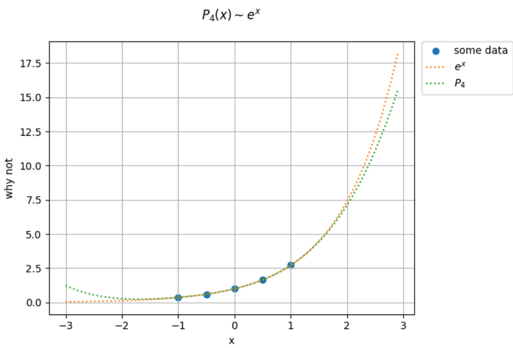
```
#https://pythonnumericalmethods.berkeley.edu/notebooks/chapter17.05-Newtons-Polynomial-Interpolation.html

def divided_diff(x, y):
  '''
  function to calculate the divided
  differences table
```

```
import matplotlib.pyplot as plt
import numpy as np

def main():
  # known
  xs = np.array([-1,-0.5,0,0.5,1])

```



2 proof of newton form and error formula

known so far: if n distinct points $x_1, \dots, x_n$ and arbitrary $y_1, \dots, y_n$, then exactly one interpolating polynomial P_{n-1} of degree $d \le n-1$ exists. lagrange gives such a polynomial. now prove that newtons divided difference does as well.

let $P(x)$ denote the (unique) polynomial that interpolates $(x_1, f(x_1)), \dots, (x_n, f(x_n))$, and denote by $f[x_1 \ \dots \ x_n]$ the degree $n-1$ coefficient of $P(x)$. thus $P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$, where $a_{n-1} = f[x_1 \ \dots \ x_n]$. also, two facts are readily apparent.

fact 01

$f[x_1 \dots x_n] = f[\sigma(x_1) \dots \sigma(x_n)]$ for any permutation σ of x_i.

proof

by uniqueness of interpolating polynomial (theorem 02).

fact 02

$P(x)$ can be written in the form

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + \dots + c_{n-1}(x - x_1)\dots(x - x_{n-1}).$$

proof

choose $c_{n-1} = a_{n-1}$. the remaining $c_{n-2}, c_{n-3}, \dots, c_0$ are defined recursively by setting c_k to be the degree k coefficient of the (degree at most k) polynomial

$$P(x) = c_{n-1}(x - x_1)\dots(x - x_{n-1}) - c_{n-2}(x - x_1)\dots(x - x_{n-2}) - \dots - c_{k+1}(x - x_1)\dots(x - x_{k+1}).$$

(this is a degree at most k polynomial due to the choice of c_{k+1}.)

theorem 05

let $P(x)$ be the interpolating polynomial of $(x_1, f(x_1)), \dots, (x_n, f(x_n))$ where the x_i are distinct. then

$$\text{(a)} \quad P(x) = f[x_1] + f[x_1 \ x_2](x - x_1) + f[x_1 \ x_2 \ x_3](x - x_1)(x - x_2) + \dots + f[x_1 \ x_2 \ \dots \ x_n](x - x_1)(x - x_2)\dots(x - x_{n-1}), \text{ and}$$

$$\text{(b)} \quad \text{for } k > 1, f[x_1 \ \dots \ x_k] = \frac{f[x_2 \ \dots \ x_k] - f[x_1 \ \dots \ x_{k-1}]}{x_k - x_1}.$$

proof

(a) prove that $c_{k-1} = f[x_1 \ \dots \ x_k]$ for $k = 1, \dots, n$.

it is already clear for $k = n$ by definition. successively substitute $x_1, \dots, x_k$ into the form of $P(x)$ in fact 02. only the first k terms are nonzero. conclude that the polynomial consisting of the first k terms of $P(x)$ is sufficient to interpolate $x_1, \dots, x_k$, and so by definition 02 and the uniqueness of interpolating polynomial, $c_{k-1} = f[x_1 \ \dots \ x_k]$. $\quad \checkmark$

(b)

by (a) the interpolating polynomial of $x_2, x_3, \dots, x_{k-1}, x_1, x_k$ is

$$\begin{aligned} P_1(x) = f[x_2] &+ f[x_2 \ x_3](x - x_2) + \dots \\ &+ f[x_2 \ x_3 \ \dots \ x_{k-1} \ x_1](x - x_2)\dots(x - x_{k-1}) \\ &+ f[x_2 \ x_3 \ \dots \ x_{k-1} \ x_1 \ x_k](x - x_2)\dots(x - x_{k-1})(x - x_1) \end{aligned}$$

and the interpolating polynomial of $x_2, x_3, \dots, x_{k-1}, x_k, x_1$ is

$$\begin{aligned} P_2(x) = f[x_2] &+ f[x_2 \ x_3](x - x_2) + \dots \\ &+ f[x_2 \ x_3 \ \dots \ x_{k-1} \ x_k](x - x_2)\dots(x - x_{k-1}) \\ &+ f[x_2 \ x_3 \ \dots \ x_{k-1} \ x_k \ x_1](x - x_2)\dots(x - x_{k-1})(x - x_k) \end{aligned}$$

by uniqueness, $P_1 = P_2$. set $P_1(x_k) = P_2(x_k)$ and canceling terms yields

$$
\begin{aligned}
&f[x_2\ \dots\ x_{k-1}\ x_1](x_k-x_2)\dots(x_k-x_{k-1})+f[x_2\ \dots\ x_{k-1}\ x_1\ x_k](x_k-x_2)\dots(x_k-x_{k-1})(x_k-x_1)\\
&=f[x_2\ \dots\ x_{k-1}\ x_k](x_k-x_2)\dots(x_k-x_{k-1})+f[x_2\ \dots\ x_{k-1}\ x_k\ x_1](x_k-x_2)\dots(x_k-x_{k-1})(x_k-x_k)
\end{aligned}
$$

$$\Downarrow$$

$$f[x_2\ \dots\ x_{k-1}\ x_1]+f[x_2\ \dots\ x_{k-1}\ x_1\ x_k](x_k-x_1)=f[x_2\ \dots\ x_k]$$

$$\Downarrow$$

$$f[x_1\ \dots\ x_k]=\frac{f[x_2\ \dots\ x_k]-f[x_1\ \dots\ x_{k-1}]}{x_k-x_{k-1}}. \quad \blacksquare$$

theorem 04 again

proof of interpolation error

now that its easier, add one more point x to n distinct points. then

$$P_n(t)=P_{n-1}(t)+f[x_1\ \dots\ x_n\ x](t-x_1)\dots(t-x_n).$$

evaluated at extra point x, $P_n(x)=f(x)$,

$$f(x)=P_{n-1}(x)+f[x_1\ \dots\ x_n\ x](x-x_1)\dots(x-x_n).$$

this is true for all x. define

$$h(t)=f(t)-P_{n-1}(t)-f[x_1\ \dots\ x_n\ x](t-x_1)\dots(t-x_n).$$

$h(x)=0$ and $0=h(x_1)=\dots=h(x_n)$ bc P_{n-1} interpolates f at these points. between each neighboring pair of the $n+1$ points $x,x_1,\dots,x_n$, there must be a new point where $h'=0$ by rolles theorem. there are n of these points. between each pair of these, there must be a new point where $h''=0$; there are $n-1$ of these. usw, there must be one point c for which $h^{(n)}(c)=0$ lies between the smallest and largest of $\{x:x_1,\dots,x_n\}$. note

$$h^{(n)}(t)=f^{(n)}(t)-n!f[x_1\ \dots\ x_n\ x],$$

bc nth derivative of polynomial $P_{n-1}(t)$ is zero. substituting c gives

$$f[x_1\ \dots\ x_n\ x]=\frac{f^{(n)}(c)}{n!}$$

$$\Downarrow$$

$$f(x)=P_{n-1}(x)+\frac{f^{(n)}(c)}{n!}(x-x_1)\dots(x-x_n). \quad \checkmark$$

runge phenomenon

polynomials can fit any set of points but they prefer some shapes over others.

example 09

interpolate $f(x)=(1+12x^2)^{-1}$ at evenly spaced points in $[-1,1]$.

code

```
import matplotlib.pyplot as plt
import numpy as np

f = lambda x: 1/(1+12*pow(x,2))

def main():⋯

if __name__ == "__main__":⋯
```