Thou shalt not solve the problem this way

History: these two problems with correct but clumsy solutions I found on one of Internet websites.

Problem 1. **Prove identity**

$$\sin^6 a + 3\sin^2 a \cdot \cos^2 a + \cos^6 a = 1$$

The first idea which comes to one's head is to use the basic trig identity

$$\sin^2 a + \cos^2 a = 1$$

This correct but clumsy solution I found on a website.

$$\sin^2 a = 1 - \cos^2 a$$

$$\sin^6 a + 3\sin^2 a \times \cos^2 a + \cos^6 a = \left(\sin^2 a\right)^3 + 3\sin^2 a \times \cos^2 a + \cos^6 a =$$

$$\left(1 - \cos^2 a\right)^3 + 3\left(1 - \cos^2 a\right) \times \cos^2 a + \cos^6 a =$$

$$1 - 3\cos^2 a + 3\cos^4 a - \cos^6 a + 3\cos^2 a - 3\cos^4 a + \cos^6 a = 1$$

Two possible elegant solutions are

$$\sin^6 a + 3\sin^2 a \cdot \cos^2 a + \cos^6 a = \sin^6 a + 3\sin^2 a \cdot \cos^2 a\left(\sin^2 a + \cos^2 a\right) + \cos^6 a =$$

$$\sin^6 a + 3\sin^4 a \times \cos^2 a + 3\sin^2 a \cdot \cos^4 a + \cos^6 a = \left(\sin^2 a + \cos^2 a\right)^3 = 1$$

or

$$1 = \sin^2 a + \cos^2 a = \left(\sin^2 a + \cos^2 a\right)^3 = \sin^6 a + \cos^6 a + 3\sin^2 a \times \cos^2 a\left(\sin^2 a + \cos^2 a\right) =$$

$$\sin^6 a + \cos^6 a + 3\sin^2 a \cdot \cos^2 a$$

Both solutions based on the perfect cube formula

$$\left(a + b\right)^3 = a^3 + 3a^2 b + 3ab^2 + b^3 = a^3 + b^3 + 3ab\left(a + b\right)$$

Problem 2.

If ***tan α = ½*** find the value of expression $\dfrac{5cos\sim +6sin\sim}{3sin\sim -7cos\sim}$

The website solution based on the preposterous idea that for calculation of the given expression one has to know values of **sin α** and **cos α** separately. This is true if one has to calculate the numerator or denominator separately.

But to use the same idea for the ratio is an inertia of thinking.

From $$1+tan^2\sim = \frac{1}{cos^2\sim}$$ $$cos\sim = \sqrt{\frac{1}{1+tan^2\sim}} = \sqrt{\frac{1}{1+\frac{1}{4}}} = \pm\frac{2}{\sqrt{5}}$$

Correspondingly $$sin\sim = \pm\sqrt{1-cos^2\sim} = \pm\frac{1}{\sqrt{5}}.$$

Which signs one has to choose? ***sine*** and ***cosine*** are both positive if ***tan α*** belong to the first quarter or both negative if ***tan α*** belongs to the third quarter.

Therefore $$\frac{5cos\sim +6sin\sim}{3sin\sim -7cos\sim} = \frac{5\cdot\frac{2}{\sqrt{5}}+6\cdot\frac{1}{\sqrt{5}}}{3\cdot\frac{1}{\sqrt{5}}-7\cdot\frac{2}{\sqrt{5}}} = -\frac{16}{11}$$

The competent solution takes only one line. Divide numerator and denominator by ***cos α ≠ 0***.

$$\frac{5\cos a+6sina}{3sina-7cosa} = \frac{5+6tana}{3-7tana} = \frac{5+3}{3\times\frac{1}{2}-7} = -\frac{16}{11}$$

There is also an arithmetic solution based on the obvious rule: denominators of ***sine*** and ***cosine*** numerical values of the same angle α are the same. In our case numerators of sine and cosine are correspondingly equal ***1*** and ***2*** (because **tan α = ½**), and equal denominators are cancelled.

Finally, $$\frac{5cos\sim +6sin\sim}{3sin\sim -7cos\sim} = \frac{5\cdot2+6\cdot1}{3\cdot1-7\cdot2} = -\frac{16}{11}$$