Complex numbers

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What is common in the questions?

1. *May expression* $a^2 + b^2$ *be factorized?*

2. *Why* $(-1)\times(-1) = 1$ *?*

3. *Why the equation* $x^N = 1$ *has exactly* N *roots?*

4. *Is it possible to calculate logarithm of a negative number?*

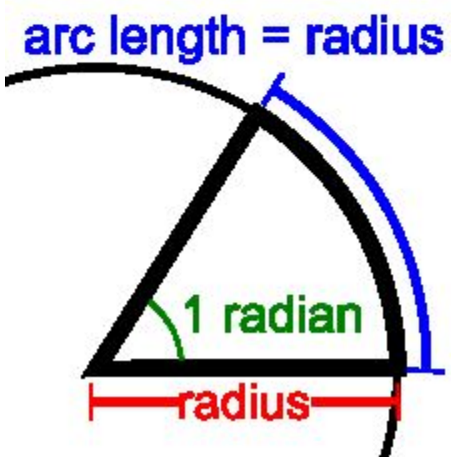
5. *Is there any formula which links together* e *and* π *?*

Degrees and radians

The textbook definition :

Radian is a unit of angular measure equal to the central angle inscribed in a circle and subtended by an arc equal in length to the radius of the circle.

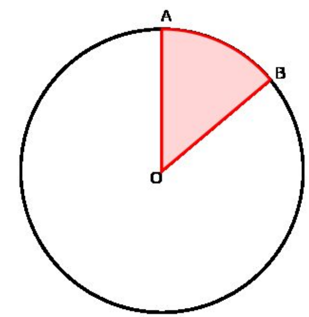
The radian measure of any angle is expressed as the ratio of the arc that the angle, with its vertex at the center of the circle, subtends to the radius of the circle.



Degrees and radians (continued)

The length of a circumference is expressed by the formula $L = 2\pi R$.

Correspondingly the length of the arc which subtends the central angle of one degree is $L_{1^\circ} = \frac{2\pi}{360^\circ} R$.



For the central angle of n degrees $L_{n^\circ} = \frac{n^\circ}{360^o} 2\pi R$.

Value $c = \frac{n^\circ}{360^o}$ *is a dimensionless coefficient,* $0 \le c \le 1$ (*since* $0 \le n^o \le 360^o$).

The $\alpha = \frac{n^\circ}{360^o} 2\pi$ *is a fraction of* 2π $(0 \le \alpha \le 2\pi)$.

It can be considered as a dimensionless measure of n° *angle and called the* ***radian measure of the angle.***

The length of an arc now can be expressed as $L = \alpha R$,

and the area of a sector as $A = \frac{1}{2} \alpha R^2$.

Degrees and radians (continued)

Correspondence between radian and degree measures of any angle.

Degrees	*Radians*
360°	2π
1°	$\dfrac{\pi}{180}$
n°	$\dfrac{n\pi}{180}$
$\dfrac{180}{\pi} \approx 57.3^\circ$	1
$\dfrac{\alpha \cdot 180}{\pi}$	α

General Solution of a Quadratic Equation

Infinite set of quadratic equations $ax^2+bx+c=0$ *with rational coefficients breaks up into three subsets in according to the sign of* $\Delta=b^2-4ac$. *Inside of each subset the general equation can be reduced to*

$$\begin{aligned} z^2 &= 1 \qquad \Delta>0 \\ z^2 &= 0 \qquad \Delta=0 \\ z^2 &= -1 \qquad \Delta<0 \end{aligned} \qquad\qquad x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}$$

The first two cases have solutions on the set of real numbers $\mathbf{R}$ *, the third case has not. So the problem is to find a number which square equals* (-1) *providing that this number does not belong to* $\mathbf{R}$ *and cannot be located on the number line because all the points on the number line from* $-\infty$ *to* $+\infty$ *are occupied by real numbers. The natural solution would be to search for the right number on another number line, let's say, orthogonal to the first one. The properties of the new number line should be different and include at least one element which square equals* (-1).

There is ready to use Cartesian coordinate system on the plane. Each point of this plane is an ordered set of two real numbers. Point $P(x,y)$ *on the plane can be considered as a radius – vector* $\vec{p}=(x,y)$ *with coordinates* x *and* y. *But Cartesian coordinate system's axes are equivalent (have the same properties) and therefore Cartesian plane cannot be used directly.*

Vectors in Cartesian Plane

In Cartesian plane the position of a point $P(x, y)$ may be represented by a radius - vector.

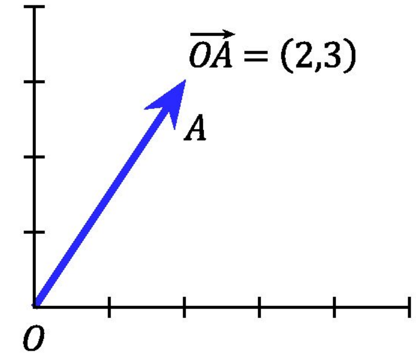
Radius - vector $\vec{p} = (x, y)$ has two properties : length and direction.

Length (or absolute value, or modulus) is defined as

$$|\vec{p}| = \sqrt{x^2 + y^2}$$

Direction (or angle φ between positive direction of X – axis and a vector taken counter - clockwise, ($0 \le \varphi < 2\pi$) is defined as

$$Arg(\vec{p}) = \arctan\left(\frac{y}{x}\right)$$



A vector in the Cartesian plane, showing the position of a point A with coordinates (2,3).

The vectors in Cartesian plane can be added in according to the rule

$$\vec{p}_1 + \vec{p}_2 = (x_1 + x_2, y_1 + y_2)$$

and multiplied by a constant in according to the rule

$$k\vec{p} = (kx, ky)$$

In Cartesian plane the commutative product of vectors with resulting vector belonging to the same plane is not defined.

Product of Vectors and Argand Plane

Introducing of the commutative product of vectors with resulting vector belonging to the same plane will disturb the equivalence of the Cartesian coordinatesystem's axes. The properties of the axes become different.

Define the product of vectors in according to the rule:

productof two plane vectors $Z_1=\left(z_{11},z_{12}\right)$ *and* $Z_2=\left(z_{21},z_{22}\right)$ *is a vector* $Z_3=\left(z_{31},z_{32}\right)$ *belonging to the same plane;*

it length $\left|Z_3\right|=\left|Z_1\right|\times\left|Z_2\right|$ *and direction* $Arg\,Z_3=Arg\,Z_1+Arg\,Z_2$.

Define the unit vector as a vector $\vec{u}=\left(u_1,u_2\right)$ *with* $\left|\vec{u}\right|=1$ *and an arbitrary direction* φ.

Then $u_1=\cos\varphi$ *and* $u_2=\sin\varphi$.

There are four special unit vectors situated on the horizontal and vertical axes:

$$\varphi=0 \qquad\qquad \varphi=\pi$$

$$\vec{1}=\left(1,0\right)=\left(\cos 0,\sin 0\right); \qquad -\vec{1}=\left(-1,0\right)=\left(\cos\pi,\sin\pi\right)$$

$$\varphi=\frac{\pi}{2} \qquad\qquad \varphi=\frac{3\pi}{2}$$

$$\vec{j}=\left(0,1\right)=\left(\cos\frac{\pi}{2},\sin\frac{\pi}{2}\right); \qquad -\vec{j}=\left(0,-1\right)=\left(\cos\frac{3\pi}{2},\sin\frac{3\pi}{2}\right)$$

Square of Unit Vector in Argand Plane

Find the squares of those unit vectors.

Properties of the horizontal axis do not change :

$$\varphi = 0 \qquad \left(\vec{1}\right)^2 = \vec{1} \times \vec{1} = (\cos 0, \sin 0) = (1,0) = \vec{1}$$

$$\varphi = \pi \qquad \left(-\vec{1}\right)^2 = \left(-\vec{1}\right) \times \left(-\vec{1}\right) = \left(\cos(\pi + \pi), \sin(\pi + \pi)\right) = (1,0) = \vec{1}$$

Properties of the vertical axis became different :

$$\varphi = \frac{\pi}{2} \qquad \left(\vec{j}\right)^2 = \vec{j} \times \vec{j} = \left(\cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right), \sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right)\right) = (-1,0) = -\vec{1}$$

$$\varphi = \frac{3\pi}{2} \qquad \left(-\vec{j}\right)^2 = \left(-\vec{j}\right) \times \left(-\vec{j}\right) = \left(\cos\left(\frac{3\pi}{2} + \frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2} + \frac{3\pi}{2}\right)\right) = (-1,0) = -\vec{1}$$

Therefore, introduction of the commutative product of vectors with resulting vector belonging to the same plane, transforms the Cartesian coordinate system into a new system in which the horizontal axis kept the properties inherited from the Cartesian coordinate system and the vertical axis acquired a new, very important property.

The new axes should have different names. The horizontal axis is called Real and the vertical one is called Imaginary. The plane with orthogonal Real and Imaginary axes is called Complex or Argand plane.

Product of Vectors in Coordinate Form

By means of the unit vector, any vector in the Complex plane may be expressed in the form:

$$\vec{Z} = |\vec{Z}|\vec{u} = \left(|\vec{Z}|\cos\varphi, |\vec{Z}|\sin\varphi\right)$$

Now

$$\vec{Z_1} = (a,b) = |\vec{Z_1}|\vec{u} = \left(|\vec{Z_1}|\cos\varphi_1, |\vec{Z_1}|\sin\varphi_1\right) \qquad \vec{Z_2} = (c,d) = |\vec{Z_2}|\vec{u} = \left(|\vec{Z_2}|\cos\varphi_2, |\vec{Z_2}|\sin\varphi_2\right)$$

$$\vec{Z_3} = \vec{Z_1} \times \vec{Z_2} = |\vec{Z_1}||\vec{Z_2}|\left(\cos\varphi_3, \sin\varphi_3\right)$$

where $|\vec{Z_1}| = \sqrt{a^2 + b^2}$, $|\vec{Z_2}| = \sqrt{c^2 + d^2}$ *and* $\varphi_3 = \varphi_1 + \varphi_2$.

$$\cos(\varphi_1 + \varphi_2) = \cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2 \qquad \sin(\varphi_1 + \varphi_2) = \sin\varphi_1 \cos\varphi_2 + \sin\varphi_2 \cos\varphi_1$$

Serious attention should be paid to the minus in the expression for $\cos(\varphi_1 + \varphi_2)$. *This minus is a cornerstone of the theory of complex numbers and is equivalent to the expression* $j^2 = -1$.

Omitting all the intermediate algebraic transformations, write down the final result:

$$\vec{Z_3} = \vec{Z_1} \times \vec{Z_2} = \left(ac - bd, bc + ad\right)$$

Complex Numbers

Vector $\vec{Z} = (a,b)$ *may be represented in the form of geometrical sum* $\vec{Z} = a\times\vec{1} + b\times\vec{j}$.

Traditionally vector sign is omitting and vector $\vec{Z}$ *is considered as a two – dimensional number made of two parts – real and imaginary and is called a complex number* : $Z = Re\{Z\} + jIm\{Z\}$,

where $Re\{Z\} = a$, $Im\{Z\} = b$.

Therefore $Z = a + jb$.

Properties of a complex number.

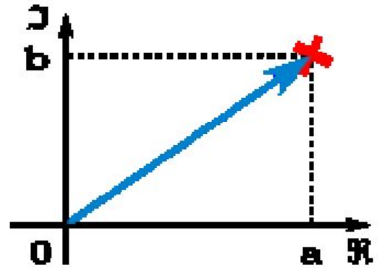
Multiplication by a constant : $kZ = ka + jkb$.

Addition of two complex numbers $Z_1 = a + jb$ *and* $Z_2 = c + jd$:

$$Z_1 + Z_2 = (a+c) + j(b+d)$$

The product of two complex numbers :

$$\vec{Z}_1 \times \vec{Z}_2 = (ac - bd) + j(bc + ad)$$



A complex number plotted as a point (red) and radius - vector (blue) on an Argand plane.

Let in complex numbers $Z_1 = a + jb$ *and* $Z_2 = c + jd$ $a = c = 0$ $b = d = 1$.

Then $Z_1 = Z_2 = j$ *and* $\vec{Z}_1 \times \vec{Z}_2 = j^2 = (0 - 1, 0 + 0) = -1$.

Conjugate Complex Numbers

The number $\bar{Z} = a - jb$ is called a complex conjugate of the number $Z = a + jb$.

Product of a complex number and its conjugate is a real number:

$$Z \times \bar{Z} = (a + jb)(a - jb) = a^2 + b^2$$

Therefore $$|Z| = \sqrt{Z \times \bar{Z}}\,.$$

Division of two complex numbers:

Now division of two complex numbers $Z_1 = a + jb$ and $Z_2 = c + jd$ may be represented in the form:

$$\frac{Z_1}{Z_2} = \frac{Z_1}{Z_2} \times \frac{\bar{Z}_2}{\bar{Z}_2} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}$$

Inverse number $Z^{-1} = \dfrac{1}{Z}$ for $Z = a + jb$ is:

$$\frac{1}{Z} = \frac{1}{Z} \times \frac{\bar{Z}}{\bar{Z}} = \frac{a}{a^2 + b^2} - j\frac{b}{a^2 + b^2}$$

Ordered Numbers

Real numbers are *ordered*.

For any two numbers $a \in \mathbf{R}$ *and* $b \in \mathbf{R}$ *one of the following three statements takes place:*

$$a > b$$

$$a = b$$

$$a < b$$

Complex numbers are *not ordered*.

Numbers $Z_1 = a + jb \in \mathbf{C}$ *and* $Z_2 = c + jd \in \mathbf{C}$ *are called equal only if*

$Re(Z_1) = Re(Z_2)$ *and* $Im(Z_1) = Im(Z_2)$

Statements $Z_1 > Z_2$ *and* $Z_1 < Z_2$ *do not have sense.*

This is the price paid for extension from the set of Real numbers $\mathbf{R}$ *to the set of Complex numbers* $\mathbf{C}$.

Trigonometric Form of Complex Numbers

Let $Z = a + jb$ *and* $\rho = |Z| = \sqrt{a^2 + b^2}$.

Then formally $Z = Z \times \dfrac{|Z|}{|Z|} = |Z|\left(\dfrac{a + jb}{|Z|}\right) = |Z|\left(\dfrac{a}{\sqrt{a^2 + b^2}} + j\dfrac{b}{\sqrt{a^2 + b^2}}\right)$.

Since $\varphi = \arctan\dfrac{b}{a}$

then $\dfrac{b}{\sqrt{a^2 + b^2}} = \sin\varphi$, $\dfrac{a}{\sqrt{a^2 + b^2}} = \cos\varphi$

and complex number in ***trigonometric form*** *is* $Z = \rho(\cos\varphi + j\sin\varphi)$.

Correspondingly $\bar{Z} = \rho(\cos\varphi - j\sin\varphi)$ *and* $\dfrac{1}{Z} = \dfrac{1}{\rho}(\cos\varphi - j\sin\varphi)$.

De Moivre Formula

Multiply $Z = \cos\varphi + j\sin\varphi$ *by itself* n *times* (*rise* Z *into power* n).

Formally

$$Z^n = \left(\cos\varphi + j\sin\varphi\right)^n$$

But in accordance with the rule of multiplication for vectors in Argand plane

$$Z^n = \left(\cos n\varphi + j\sin n\varphi\right)$$

Therefore

$$\left(\cos\varphi + j\sin\varphi\right)^n = \left(\cos n\varphi + j\sin n\varphi\right)$$

for any natural n.

This result is called ***De Moivre Formula*** .

Exponential Form of a Complex Number

In according to Macloren expansion

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!} + \frac{f^{(2)}(0)}{2!} + \frac{f^{(3)}(0)}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Substitute formally $j\varphi$ for x to the expansion series of e^x.

Then

$$e^{j\varphi} = \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots\right) + j\left(\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots\right)$$

or

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

Euler Formula

$$\boxed{e^{j\varphi} = \cos\varphi + j\sin\varphi}$$

This very important result is called Euler formula.

Now any complex number Z may be written in exponential form :

$$Z = \rho\, e^{j\varphi}$$

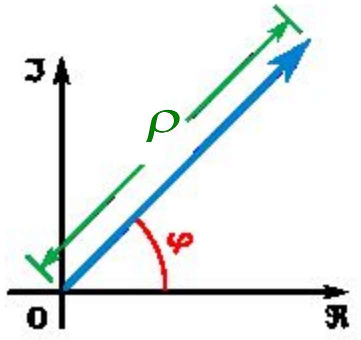
For $\varphi = \pi$

$$e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$$\boxed{e^{j\pi} + 1 = 0}$$

Polar Form of a Complex Number

Trigonometrical and exponential forms are two ways of representating a complex number in the polar coordinate system. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a fixed point and an angle from a fixed direction. The fixed point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole with the fixed direction is the polar axis. The distance from the pole is called the radius, and the angle is the polar angle, or azimuth.



The argument φ and absolute value ρ locate a point on an Argand plane.

Logarithm of a Complex Number

The complex logarithm function is an "inverse" of the complex exponential function, e^{Z}.

In according to the definition of logarithm the following identity holds for any $Z \neq 0$:

$$e^{\log Z} = Z \quad or \quad e^{\log Z} = |Z| e^{j\varphi}$$

Taking formally natural logarithm of both sides

$$\log Z = \log |Z| + j\varphi$$

Note that

- logarithm of a complex number is a complex number;
- imaginary part of logZ is ambiguous because $e^{j\varphi} = e^{j(\varphi + 2\pi k)}$, where $k = 0; \pm 1; \pm 2 \ldots$

Definition of principal value

For each nonzero complex number Z, the principal value $Log\, Z$ is the logarithm whose imaginary part lies in the interval $(-\pi, \pi]$.

$$Log Z = ln|Z| + j\varphi, \quad -\pi < \varphi \leq \pi$$

Logarithm of a negative number

$Log(-a)$, where $a > 0$ is:

$$Log(-a) = Log\left(ae^{j\pi}\right) = ln\, a + j\pi$$

Root of a Complex Number

Complex number $\qquad Z = \rho e^{j\varphi}$

can be rised formally to any rational power :

$$Z^n = \rho^n \left(e^{j\varphi}\right)^n = \rho^n \left(\cos\varphi + j\sin\varphi\right)^n$$

or

$$Z^n = \rho^n e^{jn\varphi} = \rho^n \left(\cos n\varphi + j\sin n\varphi\right)$$

Therefore the De Moivre formula holds for any rational n.

Now the root of a complex number can be introdused.

$$\sqrt[n]{Z} = Z^{\frac{1}{n}} = \left(\rho e^{j\varphi}\right)^{\frac{1}{n}} = \rho^{\frac{1}{n}} e^{j\frac{\varphi}{n}} = \rho^{\frac{1}{n}}\left(\cos\frac{\varphi}{n} + j\sin\frac{\varphi}{n}\right)$$

cos and sin are periodical functions with period $T = 2\pi k$,$k = 0;1;2\ldots$

$$\sqrt[n]{Z} = \rho^{\frac{1}{n}}\left(\cos\frac{\varphi + 2\pi k}{n} + j\sin\frac{\varphi + 2\pi k}{n}\right)$$

For $k = 0;1;2\ldots n-1$ we will get n different values of $\sqrt[n]{Z}$.

The same values may by obtained for $k = -1;-2\ldots -n$.

Root of Complex Unity (de Moivre numbers)

Complex unity is a vector $\vec{1} = (1,0)$. In exponential form $\vec{1} = e^{j(2\pi + 2\pi k)}$.

An nth root of unity, where $n = 1,2,3,\cdots$, is a complex number, z, satisfying the equation $z^n = 1$.

$$1 = e^{j(2\pi + 2\pi k)}$$

Therefore

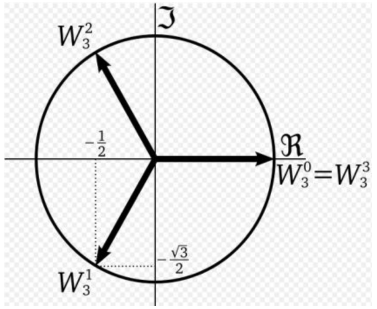
$$z^n = e^{j(2\pi + 2\pi k)}$$

$$z_k = e^{j\left(\frac{2\pi}{n} + \frac{2\pi k}{n}\right)} \qquad k = 0,1,2,3,\cdots,n-1$$

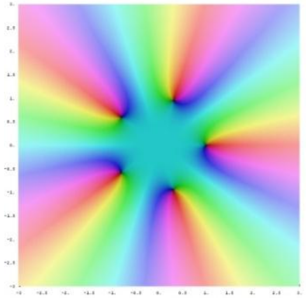
If $k = n-1$

$$\frac{2\pi}{n} + \frac{2\pi k}{n} = \frac{2\pi}{n} + \frac{2\pi(n-1)}{n} = 2\pi$$

$$\boxed{z_{n-1} = e^{j2\pi} = 1}$$



Cube root of 1



Solution of equation $z^5 - 1 = 0$.

Solution points represented by the black color.

Geometrically, root of unity form the vertices of an n-sided regular polygon inscribed into unit circle with one vertex on $(1,0)$.

Rotation Operator

$$Let\ z^n = 1.$$

$$z_k = e^{j\left(\frac{2\pi}{n}+\frac{2\pi k}{n}\right)} \quad k = 0, 1, \dots\ n-1 \qquad or \qquad z_k = e^{j\left(\frac{2\pi}{n}+\frac{2\pi k}{n}\right)} \quad k = -1, -2, \dots -n \quad .$$

Introduce the complex number $a = e^{j\frac{2\pi}{n}}$.

Then

$z_0 = a;$

$$z_1 = e^{j\left(\frac{2\pi}{n}+\frac{2\pi}{n}\right)} = e^{j\frac{2\pi}{n}} \cdot e^{j\frac{2\pi}{n}} = a^2; \qquad z_1 = e^{j\left(\frac{2\pi}{n}-\frac{2\pi}{n}\right)} = 1 = a^{-n};$$

$$z_2 = e^{j\left(\frac{2\pi}{n}+2\cdot\frac{2\pi}{n}\right)} = e^{j\frac{2\pi}{n}} \cdot e^{j\frac{2\pi}{n}} \cdot e^{j\frac{2\pi}{n}} = a^3; \qquad z_2 = e^{j\left(\frac{2\pi}{n}-2\cdot\frac{2\pi}{n}\right)} = a^{-1};$$

....

$$z_{n-1} = a^n = 1 \qquad\qquad z_n = a^{-(n-1)}$$

Number $a = e^{j\frac{2\pi}{n}}$ is called a rotation operator.

Rotational operator is a periodical function with period n.

$$a^{m\pm n} = e^{j\frac{2\pi}{n}(m\pm n)} = e^{j\frac{2\pi}{n}m\pm j\frac{2\pi}{n}n} = a^m$$

Roots of Unity Series

The sum of the roots of unity forms the finite geometric series $1+a+a^2+\ldots+a^{n-1}$ *with the first term equals* 1 *and the common denominator equals* a.

$$S(n)=1+a+a^2+\ldots+a^{n-1}=\frac{a^n-1}{a-1}=0, \text{ because } a^n=1.$$

Therefore

$$\boxed{1+a+a^2+\ldots+a^{n-1}=0}$$

Concider now the finite series of natural power of the roots of unity

$$S_r(n)=1+(a)^r+\left(a^2\right)^r+\ldots+\left(a^{n-1}\right)^r,\quad 1<r\le n-1$$

This series too is a geometric one with the first term equals 1 *and the common denominator equals* a^r.

$$S_r(n)=\frac{a^{nr}-1}{a-1} \text{ and } a^{nr}=\left(a^n\right)^r=1\,.$$

May we state that $S_r(n)=0$ *for any natural* $n>1$ *and* $1<r\le n-1$?

Let $n=5$ *and* $r=2$.

$$1+a^2+a^4+a^6+a^8=\frac{a^{10}-1}{a-1}=0$$

But for $n=4$ *and* $r=2$

$$1+a^2+a^4+a^6=1+a+1+a^6\ne 0$$

Root of Unity Series for Prime n

Rewrite $S_1(n)$ in the form

$$S_r(n) = 1 + \sum_{k=1}^{n-1} a^{kr}$$

Let n be a composite number, $n = p \cdot q$. Obviously $1 < p < n$, $1 < q < n$.

$k = 1, 2, \ldots p \ldots q \ldots n-1$

$r = 1, 2, \ldots p \ldots q \ldots n-1$

In other words, if n is a composite number there is always a product $kr = pq = n$ and the corresponding term in the series $a^{pq} = 1$.

Therefore, only if n is a prime number

$$\boxed{S_r(n) = 1 + (a)^r + (a^2)^r + \ldots + (a^{n-1})^r = 0}$$

for any natural $1 < r \le n-1$.

Harmonic Phasor

Harmonic phasor is an uniformly rotating vector in the complex plane.

$$P = Ae^{j(\omega t+\varphi)}$$

In other words it is a complex number represented in exponential form with an argument depending on time.

If $\omega > 0$ *then the vector rotates counterclockwise.*

In electrical engineering phasor is a representation of a sine wave with amplitude A, *phase* φ *and angular frequency* ω *not depending on time.*

$$Re\{P\} = A\cos(\omega t + \varphi) \qquad \operatorname{Im}\{P\} = A\sin(\omega t + \varphi)$$

If frequency is known phasor may be written in angle notation

$$A\angle\varphi$$

Any periodical signal $S(t)$ *with period* T *may be expressed as an infinite sum of phasors (Fourier Series).*

$$S(t) = \sum_{0}^{\infty} a_k e^{j(k\omega_0 t+\varphi)}$$

where $\omega_0 = \dfrac{2\pi}{T}$ *is a fundamental frequency and* $a_k > 0$.

Phasor Arithmetic

1. *Addition*

The sum of two phasors with the same frequencies produces another phasor.

If $P_1 = A_1 e^{j(\omega t+\varphi_1)}$ *and* $P_2 = A_2 e^{j(\omega t+\varphi_2)}$

then

$$P_3 = P_1 + P_2 = A_3 e^{j(wt+\varphi_3)}$$

where

$$A_3^2 = \left(A_1 cos\,\varphi_1 + A_2 cos\,\varphi_2\right)^2 + \left(A_1 sin\,\varphi_2 + A_2 sin\,\varphi_2\right)^2$$

$$\varphi_3 = arctan\frac{A_1 sin\,\varphi_1 + A_2 sin\,\varphi_2}{A_1 cos\,\varphi_1 + A_2 cos\,\varphi_2}$$

In angle notation

$$A_3 \angle\varphi_3 = A_1 \angle\varphi_1 + A_2 \angle\varphi_2$$

2. *Multiplication by a complex constant Z*

Let $P = Ae^{j(\omega t+\varphi)}$ *and* $Z = Ce^{j\theta}$.

Then $Z \cdot P = ACe^{j(\omega t+\varphi+\theta)}$.

For example, if P *is a phasor current and* Z *is impedance then* $Z \cdot P$ *is a phasor voltage.*

Phasor Calculus *(differentiation)*

The time derivative or integral of a phasor produces another phasor.

1. *Time derivative of a phasor*

$$\frac{d}{dt}P(t)=\frac{d}{dt}Ae^{j(\omega t+\varphi)}=j\omega Ae^{j(\omega t+\varphi)}=j\omega P(t)$$

$$j=e^{j\frac{\pi}{2}}$$

Therefore

$$\frac{d}{dt}P(t)=\omega Ae^{j\left(\omega t+\varphi+\frac{\pi}{2}\right)}$$

Both forms of the result are significant.

The first one shows that $P(t)$ is an eigenfunction of the operator of differentiation.

The second one shows that the result of differentiation may be considered as a new phasor lagging in phase by $\frac{\pi}{2}$.

Phasor Calculus *(integration)*

2. *Integral of a phasor*

$$\int P(t)\,dt = \int Ae^{j(\omega t+\varphi)}\,dt = \frac{1}{j\omega}Ae^{j(\omega t+\varphi)} = \frac{1}{j\omega}P(t)$$

$$\frac{1}{j} = e^{-j\frac{\pi}{2}}$$

Therefore

$$\int P(t)\,dt = \frac{1}{\omega}Ae^{j\left(\omega t+\varphi-\frac{\pi}{2}\right)}$$

Both forms of the result again have the same significance.

The first one shows that $P(t)$ is an eigenfunction of the operator of integration.

The second one shows that the result of integration may be considered as a new phasor, now leading in phase by $\frac{\pi}{2}$.

Conclusion

Complex numbers are extention and generalization of the plane trigonometry.
Any complex number with the phase angle φ *may be constructed as a two-dimesional vector muliplyed by a real coefficient :*

$$Z = c\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

with the specified rules of addition and multiplication.
Therefore, the properties of a complex number are defined by combined behaviour of sine and cosine functions.