The heat equation

We study the following 1D heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{\pi^2}\dfrac{\partial^2 u}{\partial x^2} = 0 \text{ on } x \in (0;1); t \geq 0 \\ u(t=0,x) = \sin(\pi x), \quad \forall x \in (0;1) \\ u(t,x=0) = u(t,x=1) = F, \quad \forall t > 0 \end{cases}$$

1. Recall the matrix A corresponding to the part $\dfrac{\partial^2 u}{\partial x^2}$, using finite differences, as seen in the 1D Laplace practical sessions.

In order to write numerical schemes for the heat equation, we will start with space derivatives obtained using matrix A. We recall the explicit and implicit 2nd order schemes:

- explicit: the space derivative is expressed at time n:

$$\frac{-u_{j+1}^n + 2u_j^n - u_{j-1}^n}{\delta x^2}$$

- implicit: the space derivative is expressed at time $n+1$:

$$\frac{-u_{j+1}^{n+1} + 2u_j^{n+1} - u_{j-1}^{n+1}}{\delta x^2}$$

1 – Implementation and stability conditions

2. [PYTHON] Implement the following numerical schemes:
   (a) Euler in time, explicit second order in space;
   (b) Leap-frog in time, explicit second order in space;
   (c) Euler in time, implicit second order in space.
3. [PYTHON] Numerically check the theoretical stability conditions for these schemes.

2 – Order and diffusivity

Order

4. Compute the exact solution for $F = 0$.
5. [PYTHON] Compute the errors for all schemes and numerically infer their order.

Diffusivity For this paragraph, we consider a different boundary condition:

$$u(t, x = 0) = u(t, x = 1) = 0, \quad \forall t > 0$$

as well as a different initial condition:

$$u_0(x) = 1 - 8|x - \frac{1}{2}|, \text{ if } x \in [\frac{3}{8}; \frac{5}{8}], \quad u_0(x) = 0 \text{ elsewhere}$$

6. [PYTHON] Implement the scheme: Euler in time, 4th order explicit in space.
7. [PYTHON] Compare the time evolution of the solution using this scheme and the same with only 2nd order in space. Comment.

3 – Other schemes in 1D

Implement the following schemes and check numerically their stability and convergence:

8. Dufort-Frankel:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\delta t} + \nu \frac{-u_{j-1}^n + u_j^{n+1} + u_j^{n-1} + u_{j+1}^n}{\delta x^2} = 0$$

9. Gear:

$$\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\delta t} + \nu \frac{-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}}{\delta x^2} = 0$$

4 – 2D heat equation

And now? If you still have time, implement the following 2D scheme:

$$\frac{u_{j,k}^{n+1} - u_{j,k}^n}{\delta t} - \nu \frac{u_{j+1,k}^n - 2u_{j,k}^n + u_{j-1,k}^n}{\delta x^2} - \nu \frac{u_{j,k+1}^n - 2u_{j,k}^n + u_{j,k-1}^n}{\delta y^2} = 0$$

and check that:

10. it is stable in norm L^∞;
11. it satisfies the maximum principle.

5 – Homework for next time

For next time you should:

- carry on with this session within the acceptable personal time limit;
- fill the self-assessment sheet on the moodle page.

This should take you around 3 hours of personal work, not exceeding 5 hours.