The wave equation

We study the following 1D wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} & = 0, \qquad\qquad x \in [0, L], t > 0 \\ u(x, t = 0) & = u^0(x) \\ \dfrac{\partial u}{\partial t}(x, t = 0) & = v^0(x) \\ u(x = 0, t) & = 0 \\ u(x = L, t) & = 0 \end{cases}$$

where u^0 is the initial condition and v^0 is the initial velocity.

1 – One-dimensional case

As you did with the 1D linear transport equation, implement the following explicit-in-time centered-in-space-and-time numerical scheme:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} - c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0$$

with the two following approximations for the initial velocity:

1. [PYTHON] order 1: u^0 being given, the next iteration u^1 is computed using $u_i^1 = u_i^0 + \Delta t v^0(x_i)$, and then carry on with the numerical scheme.
2. [PYTHON] order 2: write the numerical scheme for $n = 0$, define u^{-1} using v^0 and u^1.

The tests we propose are as follows:

3. [PYTHON] *Basic testing.* Set u^0 as a Gaussian function, and $v^0 = 0$. Try various CFL conditions, try changing Δt and Δx, compare order 1 and order 2 implementations above.
4. [PYTHON] *Free end.* Change the boundary condition at $x = L$ with $\partial_x u(L, t) = 0$ (the right end of the rope is moving freely).
5. [PYTHON] *Forced end.* Change the boundary condition at $x = 0$ with $u(0, t) = g(t)$ (the left end of the rope is moving according to a prescribed movement). Try using periodical functions for $g(t)$.
6. [PYTHON] Implement the Newmark scheme:

$$\begin{cases} \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{v_j^{n+1} + v_j^n}{2} = 0 \\ \frac{v_i^{n+1} - v_i^n}{\Delta t} + A_h\left(\frac{u_j^{n+1} + u_j^n}{2}\right) = 0 \end{cases}$$

   with $A_h w = -c^2 \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$ and check the energy conservation.

2 – Advanced study

And now? If you still have time, or want a better grade, here are the propositions to increase the range of this study:

7. Implement other schemes that you have seen in the main class, or found on google. Compare their merits, illustrate their properties, comment...

8. Implement the wave equation in 2D.

3 – Homework

You should:

- carry on with this session within the acceptable personal time limit;
- fill the self-assessment sheet on the moodle page.

This should take you around 3 hours of personal work, not exceeding 5 hours.