MATHEMATICS ANALYSIS AND APPROACHES INTERNAL ASSESMENT (HIGHER LEVEL)

FOURIER SERIES – MODELLING AN ECG SIGNAL

1.INTRODUCTION

After my surgery, I was told to rest for a day in the hospital without using my phone or straining my eyes for any such activity. Due to the boredom, I stared at the heart rate monitor long enough that it caught my attention. I wondered about how the movement of my heartbeat is represented in the form of lines and graphs on a screen. This made me research more about it. I was really fascinated by how interesting it was to turn a seemingly abstract biological phenomenon into a problem solely depending on mathematical concepts. As an aspiring engineering student, I wanted to take every opportunity to expand my knowledge in this concept which is greatly used in the Electrical Engineering branch for signal processing, building circuits, music identifier apps etc.

The aim of my investigation is to model an ECG signal into an infinite sum of sine and cosine functions, with the use of Fourier Series. The electrocardiogram (ECG) signal reflects the electrical activity of the heart observed from the strategic points of the human body and represented by quasi-periodic voltage signal. ("What Is an Electrocardiogram (ECG)?")

My target is to understand how heartbeats are amplified in a series of waves to a simplified function to analyze it deeply. After my research, I found out that the concept of Fourier series is used to do this, it involves a series of formulas including Fourier Coefficients. I would begin to investigate the working of a Fourier series with an example, precede to derive the formula for the Fourier coefficient. I would then begin to use my understanding to model an ECG signal wave and decompose it into a form of sine curve (called sinusoidal). Lastly, I would like to conclude with analyzing the accuracy of my results and the future implications with regards to the process.

2.FOURIER SERIES

In the early 1800s, Jean-Baptiste Joseph Fourier made the remarkable discovery that any signal in the time domain is equivalent to the sum of some (possibly infinite) number of simple sinusoidal signals, given that each component sinusoid has a certain frequency, amplitude, and phase. The series of sinusoids that together form the original time-domain signal is known as its Fourier Series (Jovanovic).

As I intend to major in Electrical/Computer Engineering, Fourier series is a powerful tool used throughout to simplify signals and then analyse them to obtain various outcomes while building circuits, however it is constrained to only periodic functions.

A periodic function $f(x)$ is a function of a real variable x that repeats itself every time x changes by a, this constant is called the period. Mathematically, we write the periodicity condition as :-

$$f(x + a) = f(x), \; a \text{ and } x \in R \quad \textit{Equation 1}$$

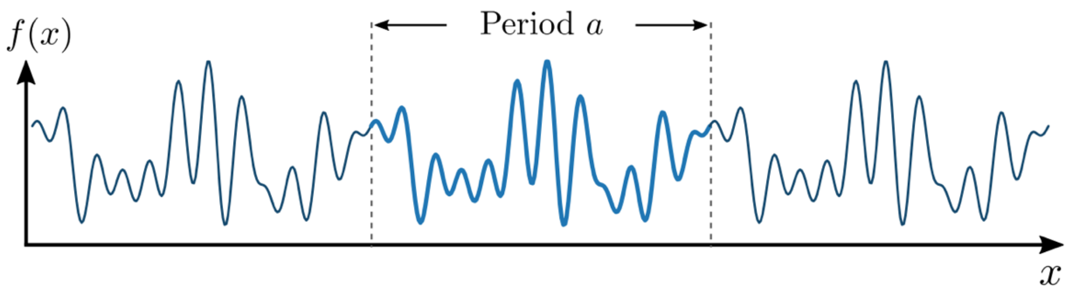

Figure 1 : An example of a periodic function (Chong)

This function is expressed as a linear combination of simpler periodic functions, consisting of sines and cosines:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \textit{Equation 2}$$

Where, L is half of the time period

n are positive integers such as 1,2,3 and so on

a_n, b_n, which are called the Fourier coefficients.

$f(x)$ can be calculated for any value of x

Before we go ahead, I would like to introduce the Euler's theorem which I will make use of in a derivation. When I was curiously reading about the theorems used in Circuit analysis under an engineering course, I came across the Euler's identity which is also called the most beautiful equation, which is why I would like to explore it in my IA. My workings are inspired from MU Prime math, as we have not learnt integration in school yet.

We can consider,

$$y = \frac{e^{ix}}{cos(x)+isin\,(x)} \textit{Equation 3}$$

We can then take the derivative of both sides in order to get i as part of the base in the numerator and attempt on converting it to a real function from complex. For RHS, we make use of the quotient rule,

$$\frac{dy}{dx} = \frac{(cosx+isinx)ie^{ix} - e^{ix}(-sinx+icosx)}{(\cos(x)+i\sin\,(x))^2}$$

We can then factorise , $e^{ix}\ outside$,

$$\frac{dy}{dx} = e^{ix}\,\frac{(icosx+i^2sinx) - (-sinx+icosx)\ dx}{(\cos(x)+i\sin\,(x))^2}$$

With the use of complex number rules we can further simplify ,

$$\frac{dy}{dx} = e^{ix}\,\frac{(icosx-sinx+sinx-icosx)dx}{(\cos(x)+i\sin\,(x))^2}$$

$$\frac{dy}{dx} = e^{ix}\,\frac{0}{(\cos(x)+i\sin\,(x))^2}$$

Therefore, the derivative of that function is 0,

Which means that the function is a constant, $y = c$

Hence, we can substitute the value of x as 0 in the function in order to find y

$$y = \frac{e^{i0}}{\cos(0) + i\sin\,(0)}$$

$$y = \frac{e^{i0}}{\cos(0) + i\sin\,(0)}$$

$$y = \frac{1}{1+0} = 1$$

If $y = 1$, then the function we assumed in the start,

$$1 = \frac{e^{ix}}{\cos(x) + i\sin\,(x)}$$

Therefore, the Euler's theorem has been proven

$$e^{ix} = cos(x) + isin\,(x) \textit{Equation 4}$$

The Fourier series can be written in two forms, the trigonometric form and the exponential form ,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{inw_0t} \text{Equation 5}$$

Where, c_n is considered to be a complex number except for c_0, and can appear in conjugate pairs also when $c_{-n=}c_n^*$

n can be any integer

$i\ is\ the\ complex\ number\ here$

$w_o\ is\ omega$

t is the time period

I will be deriving the Fourier coefficient with this formula, my findings are inspired from a website called Tutorials point India . The first step will be to multiple a particular value to both the sides to start the process of simplifying it, as we can see we have e raised to the power of inw_0t, we will multiple e^{-imw_0t} to both the sides. We multiply it to both the sides because it can then get cancelled out, hence the value stays the same.

$$x(t) \times e^{-imw_0t} = \sum_{n=-\infty}^{\infty} c_n\, e^{inw_0t} \times e^{-imw_0t}$$

Due to this, we can now simplify the R.H.S, with the use of the exponential law that,

$$a^b \times a^c = a^{b+c},$$

$$x(t) \times e^{-imw_0t} = \sum_{n=-\infty}^{\infty} c_n\, e^{i(n-m)w_0t}$$

As we know that the main use of Fourier series is to decompose a complex function, it must make use of integration because integration is also the method of decomposing of functions to different forms. So I will integrate both the sides of this equation, the value will change however the relation will stay the same. With the limits 0 and T, where T is the time period of the respective function

$$\int_0^T x(t) \times e^{-imw_0t}dt = \int_0^T \sum_{n=-\infty}^{\infty} c_n\, e^{i(n-m)w_0t}dt$$

As performed in normal integration , I will take the constants outside the integration function.

$$\int_0^T x(t) \times e^{-imw_0t}dt = \sum_{n=-\infty}^{\infty} c_n \int_0^T e^{i(n-m)w_0t}dt$$

Whenever we see e raised to a complex number as i is present in the power , we can apply Euler's theorem to simplify, which is

$e^{ix} = \cos(x) + i \times \sin(x)$ (proved above in equation 4)

Now, continuing to derive the coefficient, the RHS can be then expanded as,

$$\int_0^T e^{i(n-m)w_0 t} dt = \int_0^T \cos(n-m)\, w_0 t dt + i \int_0^T \sin(n-m)\, w_0 t dt$$

Now, we can then substitute

$$m = n$$

We will substitute both of these into the equation and solve it, first

$$= \int_0^T \cos(n-n)\, w_0 t dt + i \int_0^T \sin(n-n)\, w_0 t dt$$

$$= \int_0^T \cos 0\, w_0 t dt + i \int_0^T \sin 0\, w_0 t dt$$

$$= \int_0^T 1\, w_0 t dt + i \int_0^T 0\, w_0 t dt$$

$$= \int_0^T 1\, w_0 t dt = T - 0$$

$$= T$$

Hence, the integral has a value only when $m = n$,

$$\int_0^T x(t) \times e^{-imw_0 t} dt = c_n T$$

Therefore,

$$c_n = \frac{1}{T} \int_0^T x(t) \times e^{-\mathrm{i}mw_0 t} dt \textit{Equation 6}$$

We can explore the second case also, when $m \neq n$, however we will get a different value which does not contribute in finding the derivation of the coefficient.

However, this is the exponential form of a Fourier coefficient which is used when complex numbers are involved, our aim is to clearly identify the a and b coefficients for the sine and cosine wave, so we will use the trigonometric form of the coefficients.

The example I will be using is of a complex wave, the square wave, also called a pulse wave, is a periodic waveform consisting of instantaneous transitions between two levels. ("Square Wave")

With the help of Khan academy's lessons I have self-taught myself and will be demonstrating my understanding of it as follows:-

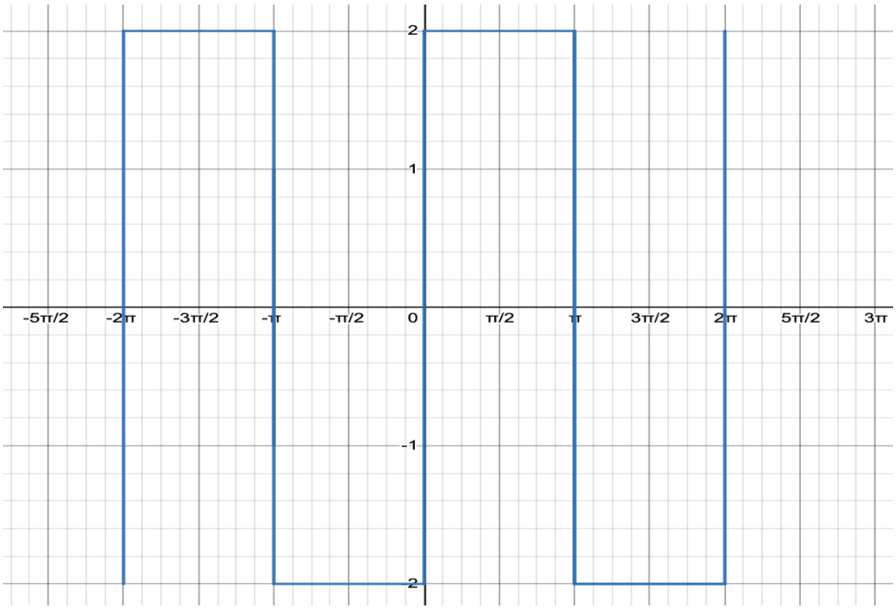

Figure 2 A square wave

In the above figure, we can see it is a periodic function as it repeats in a particular period of time which is 2π here. We can say that it takes 2π seconds per cycle.

Therefore, time period here,

$$T = 2\pi \; seconds/cycle$$

And according to the equation

$$F = \frac{1}{T}$$

$$F = \frac{1}{2\pi} \; cycles/second$$

$$y = f(T)$$

1. Starting off with the baseline constant a_0, this is an average value which helps us to determine the vertical translation/shift required from the normal position.
2. We then take $a_1 \cos(t)$ and $b_1 \sin(t)$, we take these values because the time period of our original function has a period of 2π , it will involve a sum of functions with the same time period which are the sine and cosine functions, and the coefficients a_1 *and* b_1 determine the value of sinusoidal and cosinusoidal (having the form of a cosine wave) of this function and their respective frequencies.
3. And then we continue to add more and more terms to get a more precise representation. The final equation will be

4. $f(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + a_3 \cos(3t) + b_3 \sin(3t) + \ldots$

5. With the use of given formulas, we find the coefficients ,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$$

By trying this out, we know that the integral with the limits from π *to* 2π is going to be zero, we derived that from the graph

$$=\frac{1}{2\pi}\left(\int_0^{\pi} f(t)\,dt + \int_{\pi}^{2\pi} f(t)\,dt\right)$$

With the use of the graph, we substitute 2 for the value $f(t)$ *at* π

$$= \frac{1}{2\pi}\left(\int_0^{\pi} 2\,dt + \int_{\pi}^{2\pi} 0\,dt\right)$$

$$=\frac{1}{2\pi}(2t)_0^{\pi}$$

$$= \frac{2\pi}{2\pi}$$

$$=1$$

Therefore, $a_0 = 1$

So for the next Fourier coefficients, I can focus only on the integral with limits from 0 *to* π. This eliminates one unnecessary step from our working.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)cos(nt)\,dt \text{Equation 7}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} 2\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} \cos(\text{nt})\,dt$$

From my knowledge, we know that derivative of $sin(nt)$ *is* $ncos(nt)$,the reverse for integrating will be true, in order to make it simpler we divide and multiple with n, it will not change the relation

$$= \frac{2}{\pi n}\int_0^{\pi} \text{ncos(nt)}\,dt$$

$$=\frac{2}{\pi n}\sin(\text{nt})_0^{\pi}$$

As we know from the sine curve that the value of sin *at* 0 *and* π is 0

$$= \frac{2}{\pi n}(0-0)$$

$$=0$$

Therefore, all our a coefficients are going to be equal to 0 , however as it involves sine vales and they tend to be 0 when the x values are pi and 0, we get 0. But cosine values have different values, that is why we now work out the b coefficients.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t) sin(nt)\, dt \textit{Equation 8}$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin(\mathrm{nt})\, dt$$

$$= \frac{2}{\pi}\int_0^{\pi} sin(nt)\, dt$$

Similarly, with our knowledge of the derivative of $\cos(nt)\ is\ -nsin(nt)\ is$, to make it simpler we multiply and divide with $-n$

$$= \frac{2}{-n\pi}\int_0^{\pi} -nsin(nt)\, dt$$

$$= \frac{2}{-n\pi}\cos(\mathrm{nt})_0^{\pi}$$

$$= \frac{2}{-n\pi}(\cos(\mathrm{nt}) - 1)$$

We have two different cases here depending on the value of n,

If n is even, the value will be 0 because $\cos(2\pi)\ or\ \cos(4\pi)$ and so on it will be equal to 1, resulting in $(1-1) = 0$

However, when n is odd, $\cos(3\pi)\ or\ \cos(5\pi)$ and so on, the value will be -1, so it will be -2 , hence $\frac{4}{n\pi}$

We have now got our Fourier series for this function,

Therefore we have our final equation, we can eliminate the Fourier coefficient with a, as the value will be 0

$$f(t) = 1 + \frac{4}{\pi} sin(t) + \frac{4}{3\pi} sin(3t) + \frac{4}{5\pi} sin(5t) \ldots \textit{Equation 9}$$

Graphically, shown in the diagram

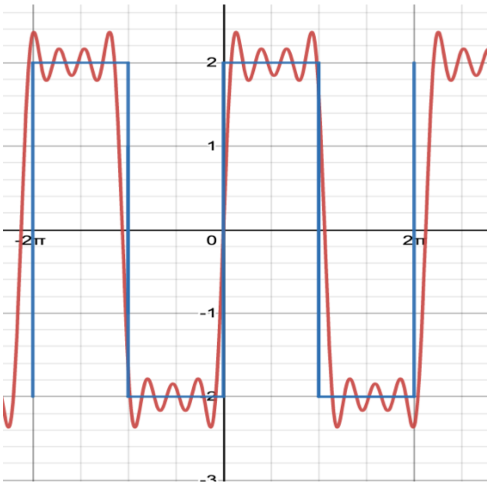

Figure 3 A square wave (Blue line) decomposed with the use of Fourier Series forming the red wave

We can see that, it does not coincide, this is due to the number of terms being plotted, if we plot more terms, it will be more accurate and precise to the original square graph. However, it is a very time consuming process and computer algorithms are used in order to plot them well.

MODELLING AN ECG SIGNAL

As stated earlier, I continued to research about ECG signals and discovered that a normal ECG signal waveform can be divided into numerous parts depending on its peaks and spikes. With the use of my understanding, I drew a diagram to help me simplify the process further in this paper.

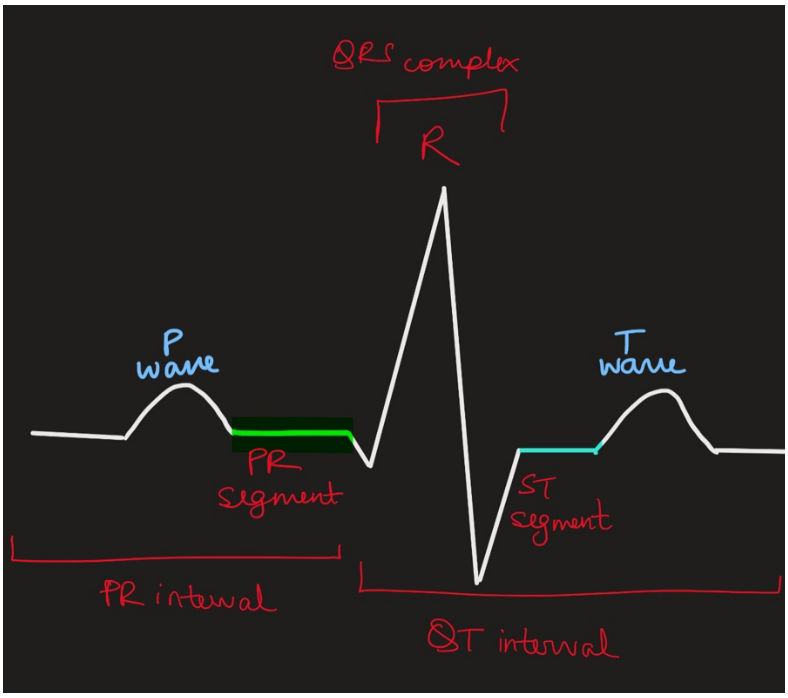

Figure 4 Rough labelled diagram of an ECG wave done by me

After this, I begin to find an ECG Signal I would like to model into a Fourier series. I came across the following (c)ECG (SubjI) in Figure 5 , this is very similar to the normal ECG signal however the peaks and spikes vary a little more, this gives room for a variety of different functions to graph. The orange dash line is a reference to the ECG, I will be neglecting that and focusing only on the black line. The x-axis represents the time in seconds and y-axis represents the amplitude in volts. I have annotated the diagram with the help of my basic understanding. We can see that it is a periodic function, as the identical waveform repeats after a particular time period which is estimated around to be

$$(17.3\ seconds - 16.6 seconds) = 0.7\ seconds$$

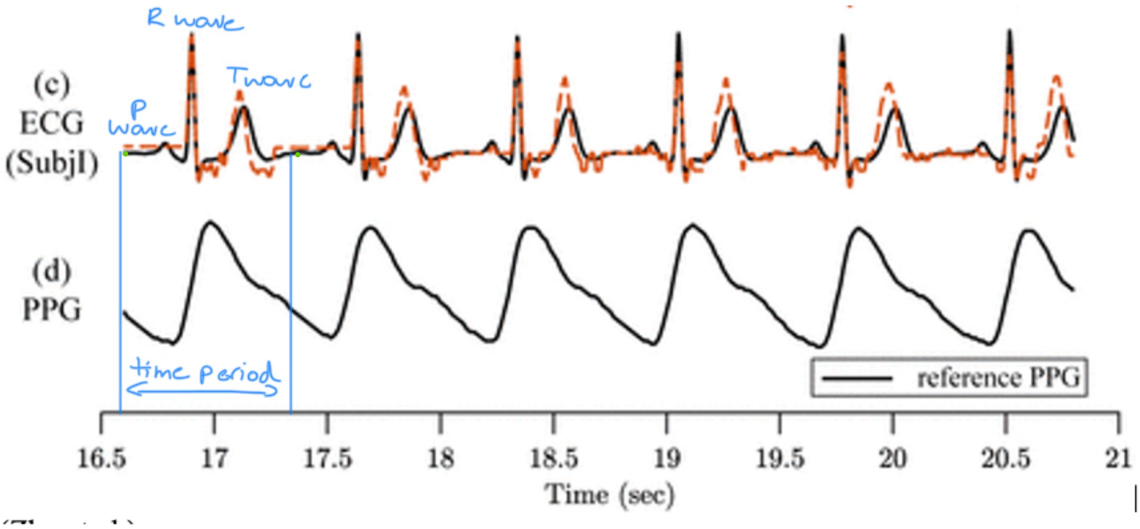

Figure 5 The ECG signal we are modelling

However, there can be human error, so I would like to verify this by inserting this image on the graphing website called Desmos.

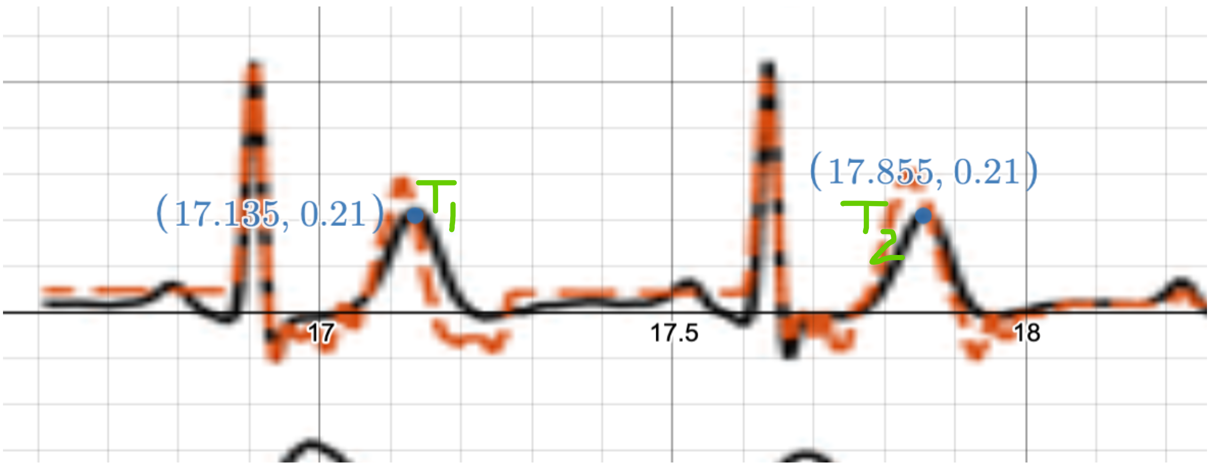

Figure 6 The labelled points in order to the find the time period

I begin to plot points on the consecutive T waves using trial and error method, naming the coordinates T1 and T2.

With the use of the respective x-coordinates, we can calculate the exact time period of this function,

$$17.855 - 17.135 = 0.72\ seconds$$

I will now label the respective points on each wave I require to form a function and find the Fourier coefficients for, I will be using the alphabets NOP(blue points), QRS(green points) and TUV(yellow points) to label them, this is to avoid any confusion, as the alphabets $a \ and \ b$ are used for Fourier coefficients. The yellow is the highlighted area is to represent the noise formed during the recording of the signal, I will be not be taking them into consideration as they will cause minimal difference to the outcome. I made this judgement following two conditions :-

Firstly I compared the distance from the x-axis, we can clearly see the peaks have a greater distance from the x-axis than any of the highlighted areas, this is how we can confirm it is the noise and not the signal.

Secondly, the mean spread of the noise is very smaller number causing the standard deviation to be a small value, if any peak is $3 \times standard\ deviations$ away , we can conclude it is the signal.

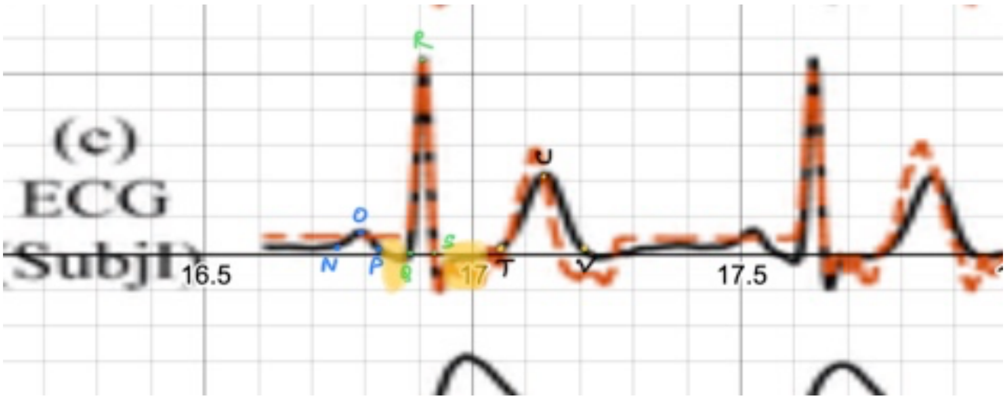

Figure 7 The three different waves/peaks I will be using the model the signal as a whole

I will now begin with the Wave NOP, the steps are as follows

a. We first find two co-ordinates on the wave NOP in order to form the function of the peak, we do this by trial and error method and making self-judgements of approximating the datapoints.

i. Point N (16.75,0.02)

ii. Point O (16.78, 0.05)

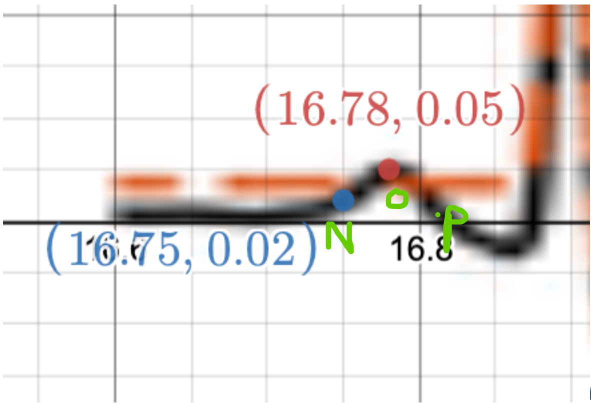

Figure 8 The co-ordinates of Points N and O

b. From my knowledge of functions, the wave NOP has a parabolic nature to it as it has an extreme point called the vertex which is point O, from point N to P it is opening downwards hinting that it will be a negative equation quadratic equation

$$y = a(x-h)^2 + k$$

Where $x = h$, the parabolic curve will be symmetric, this occurs at the maximum co-ordinate of this curve, which is point O

In the general equation, we substitute the value of h in the function, to equate the value of y to k.

$$f(h) = k$$

So we substitute, h =16.78 and k =0.05

$$f(x) = a(x-16.78)^2 + 0.05$$

c. In order to find the value of a, this is the vertically stretch factor of the equation, we substitute the second set of co-ordinates the point N in the equation.

$$0.02 = a(16.75-16.78)^2 + 0.05$$

$$0.02 - 0.05 = a(0.03)^2$$

$$-0.03 = 0.0009a$$

$$a = -\frac{0.03}{0.0009}$$

$$a = -33.3$$

Hence the final equation is,

$$f(x) = -33.3(x-16.78)^2 + 0.05 \quad \textit{Equation 10}$$

d. We then divide this function into a piecewise function which will have three subfunctions in which two of them are 0, these represent the wave area when it is lying on the x-axis before and after the wave NOP. We will be using piecewise function throughout the IA, because we are focusing on only particular waves of the signal, it is required to separate expressions with their respective intervals of the functions' domain values.

e. To do this, we find the respective domains of the piecewise function by finding the zeroes of the quadratic equation,

I begin by simplifying it and equating it to zero,

$$0 = -33.3(x - 16.78)^2 + 0.05$$

$$\frac{-0.05}{-33.3} = (x - 16.78)^2$$

$$x = 16.78 \pm \sqrt{\frac{0.05}{33.3}}$$

$$x = 16.82 \; and \; x = 16.74$$

As stated earlier, $T = 0.72$, with the use of this, we can add and subtract half of this value from the x-coordinate of the maximum which is point O, to determine the exact starting and ending point of the function of the wave we derived.

$$x = 16.78 \pm \frac{0.72}{2}$$

$$x = 16.42 \; and \; x = 17.14$$

This particular bracket is used to represent piecewise functions

$$\begin{cases} 0 & 16.42 \le x \le 16.74 \\ -33.3(x - 16.78)^2 + 0.05 & 16.74 \le x \le 16.82 \\ 0 & 16.82 \le x \le 17.14 \end{cases}$$

As it is a periodic function, a horizontal translation is performed by a multiple of the time period, it is equally subtracted from each existing domain, it does not change the shape of the peak, only the position. This translation is performed to plot the function as close to the origin as possible, this simplifies our further calculations

$$\begin{matrix} 0 & 16.42 - 16.56 \le x \le 16.74 - 16.56 \\ -33.3(x - 0.22)^2 + 0.05 & 16.74 - 16.56 \le x \le 16.82 - 16.56 \\ 0 & 16.82 - 16.56 \le x \le 17.14 - 16.56 \end{matrix}$$

Hence the final domains after simplifying are as follows,

$$\begin{array}{cc} 0 & -0.14 \ \leq x \leq 0.18 \\ -33.3(x-0.22)^2+0.05 & 0.18 \ \leq x \leq 0.26 \\ 0 & 0.26 \ \leq x \leq 0.58 \end{array}$$

These piecewise equations are very important, I made sure to double check that I have plotted the graph only between the right set of domains, in this case which is $0.18 \ \leq x \leq 0.26$, if we plot it in the wrong domain, the value of our coefficients will also be affected as the limits of the integrals in the formulas will change, leading the error in the entire Fourier series.

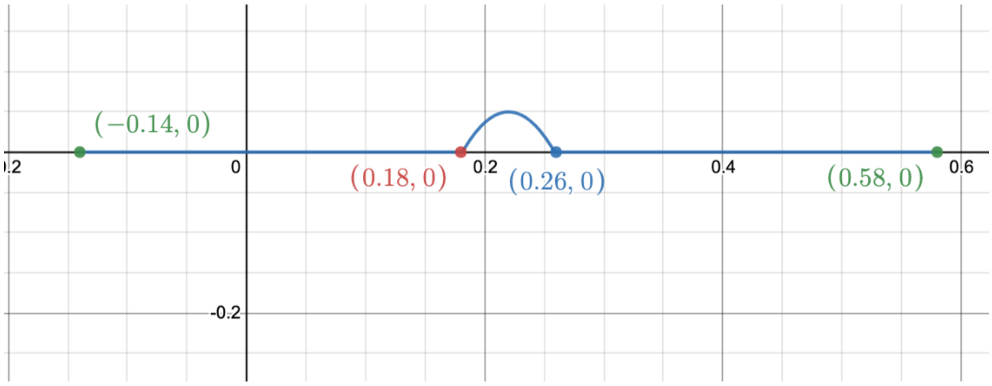

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Figure 9 The function we have derived with the calculated domains

Now similar to the example of the square wave, we begin to find the Fourier coefficients of this function with the same formulas.

The formula requires $L =$ half of time period, and the formulas have $\frac{1}{L}$, hence throughout the paper you will see,

$$\frac{1}{L} = \frac{1}{\frac{0.72}{2}} = \frac{2}{0.72}$$

The baseline constant,

$$a_0 = \frac{2}{0.72}\int_{0.18}^{0.26} (-33.3(x-0.22)^2+0.05)dx$$

I am simplifying this equation to,

$$a_0 = \frac{2}{0.72}\left(\frac{1}{20} - \frac{333\left(x-\frac{11}{50}\right)^2}{10} + C\right)$$

$$a_0 = \frac{2}{0.72}\left(\frac{x}{20} - \frac{111(50x-11)^3}{1250000} + C\right)$$

With the use of calculator then, substituting the limits in, in order to get

$$a_0 = \frac{2}{0.72} \times 0.0025792$$

$$a_0 = 0.0072$$

We perform the similar process to find rest of the coefficients, I will be showing only 5 for each of them and then the graph after plotting all of them. The formula used are:-

$$a_n = \frac{2}{0.72}\int_{0.18}^{0.26} (-33.3(x-0.22)^2 + 0.05) \times cos\ (x\frac{2n\pi}{0.72})dx \textit{Equation 11}$$

$$\textit{And } b_n = \frac{2}{0.72}\int_{0.18}^{0.26} (-33.3(x-0.22)^2 + 0.05) \times sin\ (x\frac{2n\pi}{0.72})dx \textit{Equation 12}$$

And so on, I have plotted totally 16 terms from $a_0\ to\ a_7$ and $b_1\ to\ b_8$.

I have done this because, Fourier series is usually infinite number of terms, hence more terms, the graph will be more accurate to the original function.

Fourier Coefficient ($a_{n)}$	Value	Fourier Coefficient ($b_{n)}$	Value
a_0	0.007164	b_1	0.006663
a_1	0.002423	b_2	-0.004402
a_2	-0.005247	b_3	−0.003230
a_3	-0.005595	b_4	0.005853
a_4	0.001032	b_5	−0.00524
a_5	−0.00524	b_6	0.0023122
a_6	0.002312	b_7	0.0024953

a_7	0.002495	b_8	−0.002938

Table 1 Fourier coefficients of wave NOP

According to the domains we used which is closest to the origin, the final equation plotted is this :

$$n(x) = \frac{0.007164}{2} + -0.002423cos\frac{2\pi x}{0.72} + 0.0066\,sin\frac{2\pi x}{0.72} \ldots \ldots \ldots + 0.002495cos\frac{14\pi x}{0.72} - 0.002938\,sin\frac{16\pi x}{0.72}$$ *Equation 13*

And graphically, I have also included the graph comparing it with the original signal, we can observe that the lines are coinciding at certain points.

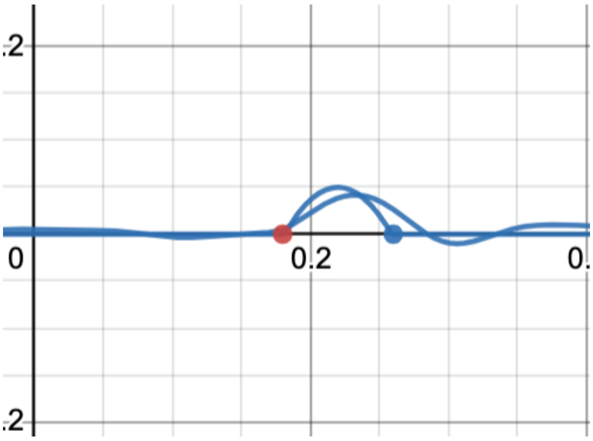

Figure 10 The plotted Fourier series against the original function between the horizontally translated domains

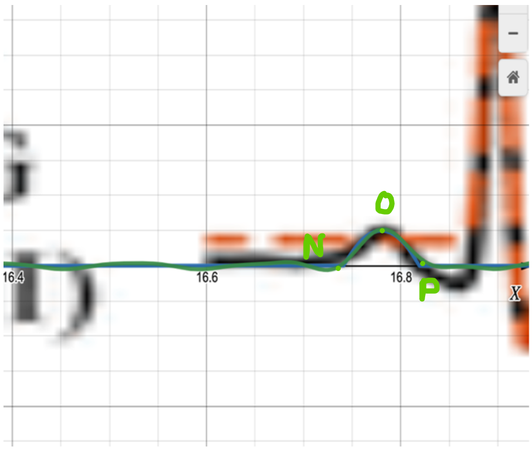

Figure 11 Comparing the Fourier series to the original signal

Now we will perform the same steps for the Wave QRS, as it a sharper peak, we require an equation which has a degree more than 2, it will be a quartic function, which is in the form of because no general symmetry is noticed, and it is steeper than a parabola, and also has a vertex point R.

$$y = a(x-h)^4 + k$$

Just like the previous equation, after performing the same steps, I can conclude that this equation will be

$$f_2(x) = -1356800(x-16.9)^4 + 0.53 \textit{Equation 14}$$

We then solve the equation to find the roots of it which are,

$x = 16.875$ and x =16.925

Like I mentioned previously in the first wave NOP, we perform the same method of horizontal translation to get it closer to the origin in order to simplify further calculations we finally get,

$$f_2(x) = \begin{cases} 0 & -0.02 \le x \le 0.315 \\ -1356800(x-0.34)^4 + 0.53 & 0.315 \le x \le 0.365 \\ 0 & 0.365 \le x \le 0.7 \end{cases}$$

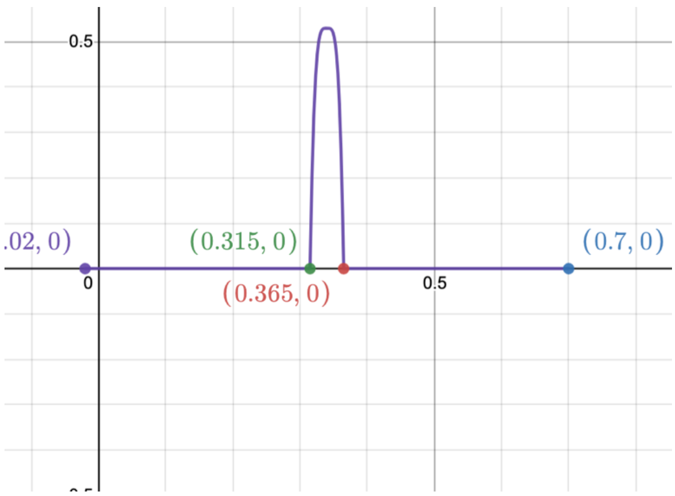

Figure 12 The function we derived of the Wave QRS with horizontally translated domains

With the same formulas like the previous wave, we find the Fourier coefficients for this displayed in the following table

The formulas used here are :-

$$a_0 = \frac{2}{0.72}\int_{0.315}^{0.365} -1356800(x-16.9)^4 + 0.53\,)dx$$

$$a_n = \frac{2}{0.72}\int_{0.315}^{0.365} (-1356800(x-16.9)^4 + 0.53) \times cos\,(x\frac{2n\pi}{0.72})dx \textit{Equation 15}$$

And,

$$b_n = \frac{2}{0.72}\int_{0.315}^{0.365} (-1356800(x-16.9)^4 + 0.53) \times sin\,(x\frac{2n\pi}{0.72})dx \textit{Equation 16}$$

Fourier Coefficient ($a_{n)}$	Value	Fourier Coefficient ($b_{n)}$	Value
a_0	0.05888	b_1	0.01016
a_1	−0.05766	b_2	−0.0196
a_2	0.05409	b_3	0.02796
a_3	−0.0484	b_4	−0.03452
a_4	0.04114	b_5	−0.03273
a_5	−0.03273	b_6	0.023825
a_6	0.02382	b_7	−0.01503
a_7	−0.01503	b_8	0.0069317

Table 2 the Fourier coefficients of the wave QRS

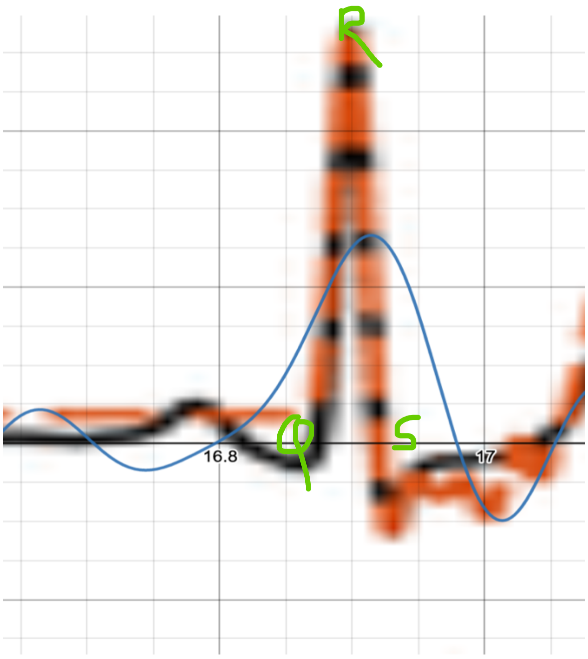

Figure 13 The plotted Fourier series against the original signal of QRS with total of 8 terms

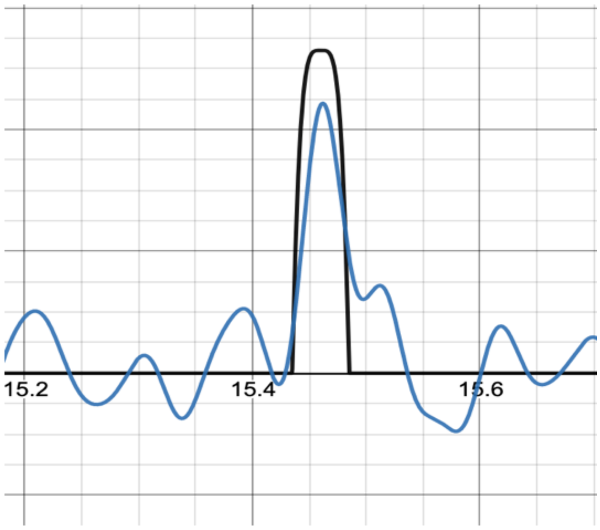

Figure 14 The plotted Fourier series with total of 16 terms

$$q(x) = \frac{0.0588}{2} - 0.0576cos\frac{2\pi x}{0.72} + 0.0101sin\frac{2\pi x}{0.72} \dots\dots -0.01503cos\frac{14\pi x}{0.72} + 0.00693sin\frac{16\pi x}{0.72}$$

Equation 17

I have inserted two graphs, the first one with 8 terms and the second one with 16 terms however, we can see there is a lot of noise we don't require, this is due to the lack of Fourier coefficients, as it is a Quartic. But we can notice that it decreases with the number of terms,

it needs more than a hundred terms to be graphed to give a more accurate graph, which coincided with our function, as the series is defined to be infinite. If we compare we can see the amplitude of the noise is decreasing with the larger number of terms, and the main wave is getting sharper in order to coincide with the original function.
Lastly, the wave TUV, we observe that the shape of the graph is like a bell curve, so for this we will be using a gaussian function to graph it . A gaussian function, which can be defined as a function for arbitrary real constants a, b and c, when c is non zero constant,((Wikipedia Contributors) they are used in statistics and probability to describe normal distributions.
I have judged this by sight but also compared common properties, a gaussian function is symmetric bell curve like shape and secondly it has inflexion points where the change of concavity takes place, as it tends to go in the upwards direction from the point V.
The general form of it is written as,

$$y = ae^{-\frac{(x-b)^2}{2c^2}} \quad \textit{Equation 18}$$

, a, b , c are constants and $c \neq 0$. (Wikipedia Contributors)
According to this, when $x = b$, it is symmetric at the maximum point of the peak of this wave TUV which is point U , as mentioned earlier I am neglecting the orange graph as it is only for reference. So I will be using the blue point's co-ordinates.

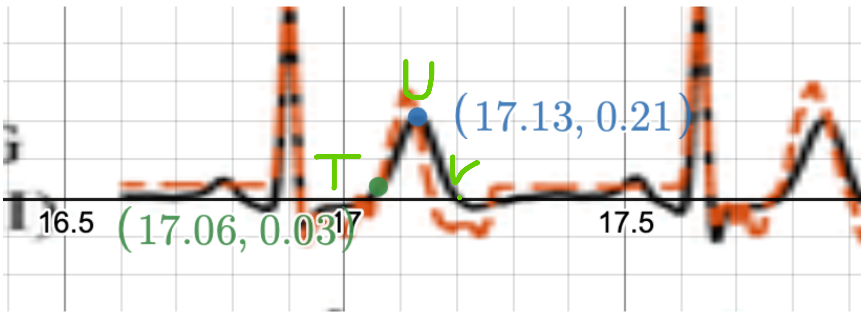

Figure 15 The points used to find the gaussian function

$$0.21 = ae^{-\frac{(17.13-17.13)^2}{2c^2}}$$

$$0.21 = ae^0$$

$$0.21 = a$$

Now that we have two of our values, we will then find the value of c by substituting the second pair of co-ordinates we found which is (17.06,0.03)
We begin by making e our subject,

$$0.03 = ae^{-\frac{(17.06-17.13)^2}{2c^2}}$$

$$0.03 = 0.21e^{-\frac{(0.07)^2}{2c^2}}$$

$$\frac{0.03}{0.21} = e^{-\frac{(0.07)^2}{2c^2}}$$

With the use of log rules,

$$ln\frac{1}{7} = -\ \frac{(0.07)^2}{2c^2}$$

$$-ln\frac{1}{7} = \frac{(0.07)^2}{2c^2}$$

$$ln7 = \frac{(0.07)^2}{2c^2}$$

Lastly, we want to make c the subject

$$c^2 = \frac{(0.07)^2}{2ln7}$$

$$c = \pm\frac{0.07}{\sqrt{2ln7}}$$

$$c = \pm 0.0355$$

Hence, the final equation is , the positive and negative sign will not make a difference, as in the general form of the equation it is squared resulting in a positive value.

$$y = 0.21e^{-\frac{(17.13-17.13)^2}{2(0.355)^2}}$$ *Equation 19*

Now, I will be repeating the same method as the previous two graphs to find the domain of this function, the starting point is 16.77 and the ending point is 17.49, then we horizontally translate the domain to the left by $23 \times T\ (0.72)$, this is by self-judgement, to get it closer to the origin simplifying calculations.

Hence, it is $0.21\ \leq x \leq 0.93$

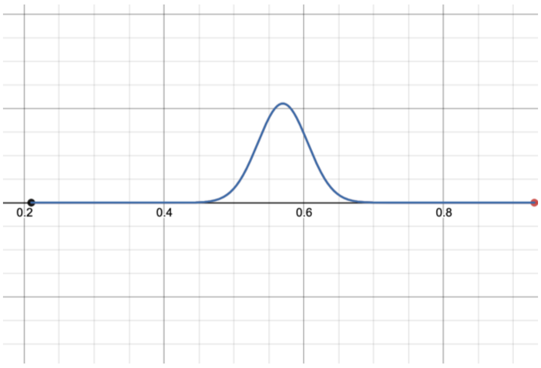

Figure 16 The derived function we found of wave TUV in the calculated domains

And now we will change the equation according to the new set of domains,

$$y = 0.21e^{-\frac{(x-0.57)^2}{2(0.0355)^2}}$$

Now we can find the Fourier coefficients,

$$a_n = \frac{2}{0.72}\int_{0.21}^{0.93} 0.21e^{-\frac{(x-0.57)^2}{2(0.0355)^2}} \times cos\left(x\frac{2n\pi}{0.72}\right) dx$$

Equation 20

$$b_n = \frac{2}{0.72}\int_{0.21}^{0.93} 0.21e^{-\frac{(x-0.57)^2}{2(0.0355)^2}} \times sin\left(x\frac{2n\pi}{0.72}\right) dx \textit{Equation 21}$$

Fourier Coefficient ($a_{n)}$	Value	Fourier Coefficient ($b_{n)}$	Value
a_0	0.0519	b_1	−0.0477
a_1	0.01279	b_2	−0.0214
a_2	−0.03709	b_3	0.023830
a_3	−0.023830	b_4	0.02086
a_4	0.0120	b_5	−0.00405

Table 3 The Fourier coefficients for the peak TUV

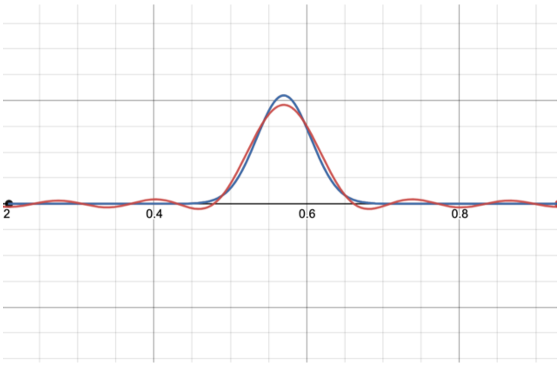

Figure 17 The plotted Fourier series against the original equation

The equation for the Fourier series is as follows,

$$t(x) = \frac{0.0519}{2} + 0.0127cos\frac{2\pi x}{0.72} - 0.0477\,sin\frac{2\pi x}{0.72} \ldots \ldots + 0.0120cos\frac{8\pi x}{0.72} - 0.00405sin\frac{10\pi x}{0.72}$$

Equation 22

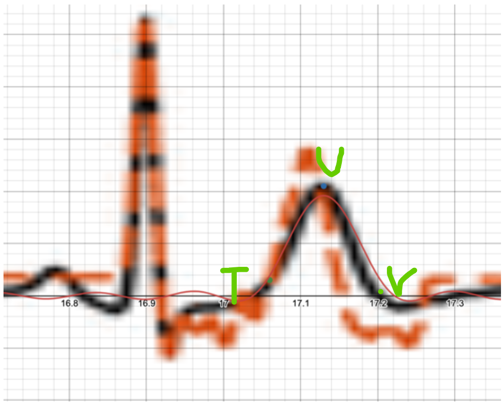

Figure 18 the plotted Fourier series against the original signal

The reason I chose to find only ten Fourier coefficients and plot them is in order to show that the accuracy and precision required varies from different types of peaks and spikes along the signal.

CONCLUSION

At the end I add all the Fourier series equations together, which are equation 13,17 and 22 together, to get the collective graph for the overall signal. When we add all three Fourier series to each other of the three different waves, we get a final graph which looks pretty close to the original signal but it doesn't coincide with it.

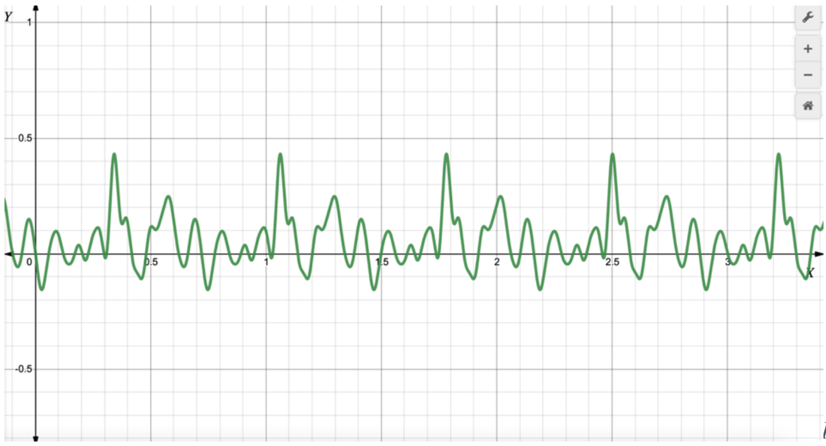

Figure 19 Collective graph of all three equations of the Fourier series of wave NOP+QRS+TUV

During the process I realized how important this could be in real life in many applications. The method is convincing, but the inaccurate results are due to human error, as graphing the original functions might have an uncertainty too which adds up during the process of forming a Fourier series. The methods and processes I went through were very detailed, however lead to imperfect outcomes due to the noise formed in the signal, leading to my ECG signal decomposed into a Fourier series which is only moderately correct as it involved certain assumptions like neglecting the reference orange line and also assuming the noise datapoints all lie on the x-axis. For a better result in the future, we must find a large number of Fourier coefficients to get closer to infinite terms. Secondly, we must use uncertainties through our calculations to display the area of errors caused.
Using these calculations, we can also include a number of proofs and graphs which show if it's a converging or a diverging function, the values of the coefficients can help us figure out if an function is even or odd. This exploration has now helped me to get an insight on engineering concepts and research more about how this works with in heart rate monitor and other real life applications such as noise cancelling headphones. Hence, I would like to conclude that decomposing a function into a Fourier series is really insightful, but to get an almost accurate and precise representation of it, we must either use a software or a computer algorithm , as heartbeat monitors are relied upon in vital areas of the medical field. In the same way, when Fourier series are used to build circuits, it must be done very precisely and accurately as electrical components are meant to be handled carefully.

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