Exam 1

1. (10 points) Recall that acceleration is defined as the rate of change in velocity. Suppose that the acceleration of a motorboat is proportional to the difference between 50 km/h and the boat's velocity. Write a differential equation that models this situation. Then find the general solution of your equation.

2. (10 points) Use the theorem from class to discuss the existence and uniqueness of solutions to the following IVPs.

   (a) $\dfrac{dy}{dx} = \sqrt{1-y},\ y(0) = 2$

   (b) $\dfrac{dy}{dx} = \sqrt{1-x},\ y(0) = 2$

3. Solve the differential equation or IVP.

   (a) (12 points) $(3x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0$

(b) (13 points) $y' + \frac{3}{t}y = \frac{\sin t}{t^2},\ t > 0,\ y(1) = \sin 1$

(c) (20 points) $ty' + y = t\ln(t)\, y^2,\ t > 0,\ y(1) = \dfrac{1}{2}$

4. In class we discussed the logistic model for population growth. We could also add the assumption that a minimum size is necessary to maintain a healthy population. This leads to the revised model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right).$$

We consider a population with parameters $k = \frac{1}{4}$, $M = 100$, and $m = 20$.

(a) (5 points) Show that the equation with these parameters is equivalent to

$$\frac{dP}{dt} = \frac{1}{400}\left(100 - P\right)\left(P - 20\right).$$

(b) (10 points) Draw the phase portrait for this equation and describe the stability of each critical point.

(c) (5 points) Make a graph that shows the equilibrium solutions that you identified in part (b). On the same graph, sketch solution curves for the initial conditions $y(0) = 15$, $y(0) = 45$, $y(0) = 75$, and $y(0) = 115$.

(d) (5 points) For what initial values will the population go extinct?

(e) (10 points) Solve the IVP

$$\frac{dP}{dt} = \frac{1}{400}\left(100 - P\right)\left(P - 20\right),\ P(0) = 50.$$

You can leave your solution in implicit form.