Developing two-, three- and n-dimensional complex number spherical coordinate systems from first principles

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Abstract

Angle-dependent rotation factors have been introduced to describe position vector rotation in a plane. A fundamental equation of position vector rotation has been discovered with an orthogonal rotation factor directly appearing in the equation. The solution to the fundamental equation leads to a rotation factor formula for all rotation angles. Based on rotation factors with the imaginary unit being bypassed, the two-dimensional complex number system has been developed with results showing that a new independent derivation of Euler's formula is achieved and that the orthogonal rotation factor is equivalent to the imaginary unit. The concept of rotation factors for rotating position vectors and numbers with positioning directions is extended to other dimensions, and the constructions of three and higher n-dimensional complex number spherical coordinate systems have been realized. The rotation factors are shown to be commutative in the spherical systems. The obtained spherical system's coordinate formula is succinct and consistent across dimensions with one rotation factor for each dimension, indicating that the rotation factors are natively fit for the spherical systems.

1. Introduction

The discovery of the imaginary unit $i = \sqrt{-1}$ from solving the cubic equation formed the complex numbers that were eventually accepted by renaissance mathematicians [1,2,11,12,15] and have deep significance and profound importance to our understanding of mathematics and physics [8,17,18]. In 1748, Leonhard Euler obtained [4,5] Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ without using the concept of the complex plane, which was not yet introduced. The equation was called "the most remarkable formula in mathematics" by the physicist Richard Feynman [6], and lies at the heart of complex number theory [16]. Euler's identity $e^{i\pi} = -1$ as the special case for angle π, is considered to be an exemplar of mathematical beauty [13, 16, 19, 24]. About 50 years later after the creation of Euler's formula, Caspar Wessel described complex numbers as points in the complex plane [22, 23]. With the geometric significance of the complex plane for complex analysis, the complex number system has captivated more than 150 years of intensive development, finding applications in science and engineering [17, 18].

Historically, $\sqrt{-1}$ has long labored under a false story of unfathomable mystery with the agonizingly prolonged, painful discovery of complex numbers [15, 16]. This paper presents a novel approach to develop the complex number system and derive Euler's formula from first principles with $\sqrt{-1}$ or $i^2 = -1$ being bypassed. This provides a base to further construct spherical coordinate systems for hypercomplex numbers.

Hypercomplex numbers [10] are an extension to higher dimensions from the standard two-dimensional (2D) complex numbers. One established example of hypercomplex numbers is the quaternion number system by Hamilton [9], which is in four dimensions and non-commutative. There exist other hypercomplex systems [14]. In general, there are restrictions with respect to commutative and associative properties of hypernumbers in higher dimensions. Ferdinand Georg Frobenius [3, 7] proved that for a division algebra [3] over the reals to be finite-dimensional and associative, it cannot be three-dimensional, and there are only three such division algebras: real numbers, complex numbers, and quaternions, which have dimension 1, 2, and 4 respectively. Thus far, there have been no spherical coordinate systems for hypercomplex numbers.

2. Two-dimensions

From this point on (until specifically mentioned), assume that there is no existence of the complex number system so that our mind is not influenced by the existing concepts and will focus on creating a number system from first principles.

First, from a high level perspective, one may expect the existence possibility of a new number system for describing rotation. Nature has its laws and properties. Mathematics is invented to discover and describe Nature. Real numbers represent points on a number line, and can describe stretching and displacement of a point's position by multiplication and addition. On the other hand, rotation of a point's position is also a fundamental motion type besides the stretching and displacement for translational motion. There likely also exists a number system for describing the rotation.

A point in the one-dimensional number line represented by real numbers cannot perform a rotation motion. The next higher dimension, a two-dimensional plane is needed and the position of a point in the plane may be represented by a position vector. Real numbers can only make the vector stretch by multiplication. By analogy with real numbers, rotation numbers are introduced to make the vector rotate by multiplication. Rotation numbers may also be called rotation factors.

2.1. Terminologies and conventions

Space or plane: refers to Euclidean space or plane.

Position vector: A position vector represents the position of a point. The direction of a position vector is from the coordinate origin to the point represented by the vector, and the magnitude of the vector is the distance between the point and the origin.

Position number: Almost identical to a position vector in the representation sense, a position number represents the position of a point, and has the same direction and magnitude as those of the corresponding position vector. In general, the direction of a position number is implied without being denoted by an arrow.

Position vector and position number conventions: Here the terms position vector and position number are interchangeable for representing a point. For geometric representation, vector arrow may be used for a position number to explicitly indicate the direction of the position number.

Point of position vector or position number: means the point represented by the position vector or the position number, and vice versa.

Direction of point, position vector, or position number: means the direction from the origin point to the point.

Rotating (or rotation of) point, position vector, or position number: means rotating a point to another position with corresponding position vector change or position number change.

Applying a rotation factor to: means multiplying a rotation factor to a position vector or a position number. Applying a rotation factor to a point means applying the factor to the point's position vector or position number.

Target of a rotation factor: means the position vector or the position number that a rotation factor is applied to.

Position numbers versus complex numbers: Position numbers are equivalent to complex numbers. They are interchangeable.

2.2. Definition of rotation factors

An angle-dependent rotation factor q of an arbitrary angle δ is defined as a factor for multiplying to a position vector of a point and making the vector rotate counter-clockwise by angle δ relative to the origin point of the vector with the vector's magnitude unchanged. That is

$$\mathbf{P}_\delta = \mathrm{q} \cdot \mathbf{P} = \mathbf{P} \cdot \mathrm{q} \tag{1}$$

where (see Figure 1) Point P represented by position vector $\mathbf{P}$ rotates relative to origin point O by angle δ and reaches Point P_δ represented by position vector $\mathbf{P}_\delta$.

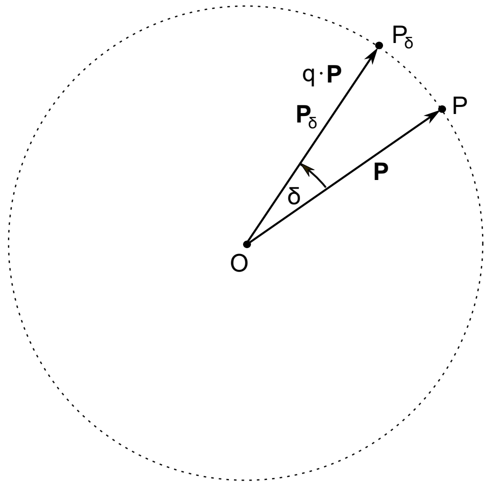

FIGURE 1. Rotation factor and target vector's direction change

By the definition, the multiplication between a rotation factor and its target vector is commutative. Left-multiplication and right-multiplication to a target vector by a rotation factor produce the same rotation result for the target vector.

At this point, the existence of rotation factors such defined is a postulate. If the consequences of the postulate are consistent with existing results, it is then considered true.

2.3. Properties of rotation factors

Denote a rotation factor q of an arbitrary angle δ as $\mathrm{q}(\delta)$. The following basic properties of rotation factors are implied or derived from the definition (1).

Rotation factors are multiplication commutative. The order of rotation factors for multiplying to a vector will not affect the final result. That is

$$\mathrm{q}(\delta_1) \cdot \mathrm{q}(\delta_2) = \mathrm{q}(\delta_2) \cdot \mathrm{q}(\delta_1)$$

Rotation factors are multiplication associative. The grouping of rotation factors for multiplying to a vector will not affect the final result. That is

$$\mathrm{q}(\delta_1) \cdot \Big(\mathrm{q}(\delta_2) \cdot \mathrm{q}(\delta_3)\Big) = \Big(\mathrm{q}(\delta_1) \cdot \mathrm{q}(\delta_2)\Big) \cdot \mathrm{q}(\delta_3)$$

Division operation of rotation factor is allowed. The multiplication effect produced by a rotation factor may be inversed through division by the same rotation factor. This means

$$\mathrm{q} \cdot \frac{1}{\mathrm{q}} = 1$$

In general, a rotation factor is not a real number as the former can change a vector's direction by multiplication and the latter cannot. But the two angles 0 and π are the exceptions. The rotation factor of angle 0 produces no rotation for its target vector and becomes the real number 1. And the rotation factor of angle π produces a reverse direction for its target vector by rotating 180° and becomes the real number -1. That is

$$\mathrm{q}(0) = 1$$

$$\mathrm{q}(\pi) = -1$$

The rotation factor of angle $\frac{\pi}{2}$ is defined as orthogonal rotation factor in that it produces a 90° rotation for the target vector with the resultant direction being perpendicular or orthogonal to the initial direction. Notationwise, the orthogonal rotation factor is denoted as $\mathrm{q}(\frac{\pi}{2})$, which may be further succinctly denoted by a symbol i. That is

$$i = \mathrm{q}(\frac{\pi}{2}) \tag{2}$$

Applying the orthogonal rotation factor twice to a vector $\mathbf{P}$ produces a 180° rotation and reverses the vector's direction. That is, $i \cdot i \cdot \mathbf{P} = -\mathbf{P}$ or

$$i^2 = -1 \tag{3}$$

The result (3) is a property of the orthogonal rotation factor and also directly shows that the orthogonal rotation factor is not a real number as the square of a real number is always positive.

2.4. Fundamental equation of position vector rotation

Since a rotation factor is involved in a position vector's rotation, it is natural to study the rotation motion of the position vector.

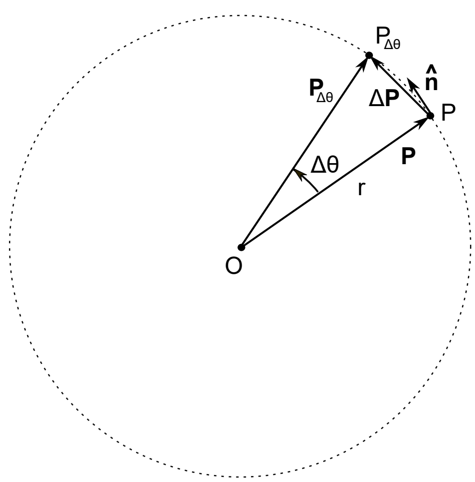

FIGURE 2. Position vector rotation relative to an origin point

In Figure 2, Point P is represented by position vector $\mathbf{P}$ with magnitude r. $\hat{\mathbf{n}}$ is a unit vector perpendicular to the position vector $\mathbf{P}$. Point P rotates around origin Point O by angle $\Delta\theta$ with the vector's magnitude r unchanged, and reaches Point $\mathrm{P}_{\Delta\theta}$ whose position vector is $\mathbf{P}_{\Delta\theta}$. $\mathbf{\Delta P}$ is the vector change, which is the difference between vector $\mathbf{P}_{\Delta\theta}$ and vector $\mathbf{P}$.

Since the vector's magnitude r is unchanged, Point P moves along the circumference of the circle during the rotation. Let Δs be the length of the arc corresponding to the angle change $\Delta\theta$, and Δl be the length of the chord of the arc. The Δl is also the magnitude of the vector change $\mathbf{\Delta P}$. Denote $\hat{\mathbf{u}}$ as the unit vector of $\mathbf{\Delta P}$. Then

$$\mathbf{\Delta P} = \|\mathbf{\Delta P}\|\, \hat{\mathbf{u}} = \Delta l \hat{\mathbf{u}} \tag{4}$$

For a circle, the ratio of the arc length to the arc angle span ($\Delta\theta$) is always equal to the radius regardless of the value of the angle span. That is, $\frac{\Delta s}{\Delta\theta} = r$ or

$$\Delta\theta = \frac{\Delta s}{r} \tag{5}$$

From (4) and (5), the ratio of the vector change to the angle change is expressed as

$$\frac{\mathbf{\Delta P}}{\Delta\theta} = (\frac{\Delta \mathrm{l}}{\Delta s}) r \hat{\mathbf{u}} \tag{6}$$

As $\Delta\theta \to 0$, $(\frac{\Delta \mathrm{l}}{\Delta s}) \to 1$ and $\hat{\mathbf{u}} \to \hat{\mathbf{n}}$, the limit of (6) for the derivative with respect to θ leads to

$$\frac{\partial \mathbf{P}}{\partial\theta} = r\hat{\mathbf{n}} \tag{7}$$

It is noted that multiplying the orthogonal rotation factor i to the vector $\mathbf{P}$ makes the vector perform an orthogonal rotation with the vector's magnitude r unchanged. In other words, the resultant vector has the magnitude r and the direction of $\hat{\mathbf{n}}$. That is

$$i\mathbf{P} = r\hat{\mathbf{n}} \tag{8}$$

This makes (7) become

$$\frac{\partial \mathbf{P}}{\partial\theta} = i\mathbf{P} \tag{9}$$

The result (9) serves as a fundamental equation of position vector rotation. The orthogonal rotation factor i happens to appear in this fundamental equation.

2.5. Fundamental equation of position number rotation and discovery of rotation factor set

For representing a point, a position number and its corresponding position vector have the same direction and magnitude. The direction of a position number is implicit or implied in contrast to the position vector whose direction is explicit. In the definition (1), a rotation factor of an arbitrary angle is applied to a position vector, which has a direction. In fact, the same rotation factor can also be applied to a position number. The rotation factor will generate the same rotation for the position number regardless of the target's direction being explicit or implied.

The definition (1) is here restated for position number with an implied direction. An angle-dependent rotation factor q of an arbitrary angle δ is defined as a factor for multiplying to a position number p of a point and making the position number's implied direction rotate counter-clockwise by angle δ with the position number's magnitude unchanged. That is

$$\mathrm{p}_\delta = \mathrm{q} \cdot \mathrm{p} = \mathrm{p} \cdot \mathrm{q} \tag{10}$$

where p_δ is the resultant position number after the rotation.

Since a position number and its corresponding position vector have the same direction and magnitude, using the same methodology and Figure 2 with position vector notation **P** replaced by position number p, gives the fundamental equation of position number rotation as

$$\frac{\partial \mathrm{p}}{\partial \theta} = i\mathrm{p} \tag{11}$$

To solve this equation, the dependence of the position number p on the rotation angle θ is obtained from Figure 3 based on the definition of the rotation factor.

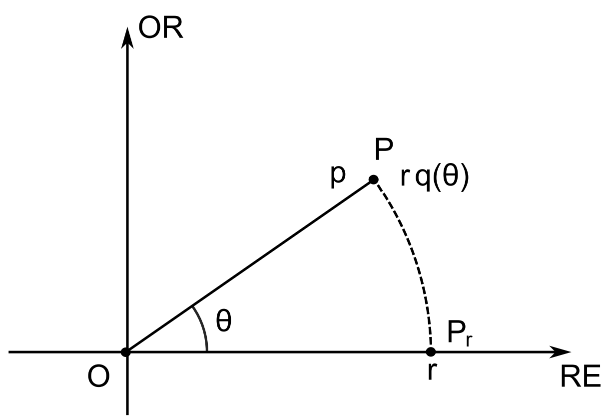

FIGURE 3. Position number represented by a real number multiplied by a rotation factor

In Figure 3, on a plane, RE denotes a real number axis and OR denotes the axis that is orthogonal to the RE axis. Point O is the origin. Point $\mathrm{P_r}$ is a point in the RE axis, and is represented by real number r. The implied direction of Point $\mathrm{P_r}$ is the positive direction of the RE axis. Multiplying the rotation factor $\mathrm{q} = \mathrm{q}(\theta)$ of angle θ to the real number r makes its implied direction rotate by angle θ with magnitude r unchanged. As a result, Point $\mathrm{P_r}$ is rotated to Point P, which is represented by position number p. That is

$$\mathrm{p} = \mathrm{rq}(\theta) \tag{12}$$

With (12), the equation (11) becomes

$$\frac{\partial (r\mathrm{q})}{\partial \theta} = ir\mathrm{q} \tag{13}$$

where $\mathrm{q} = \mathrm{q}(\theta)$.

Since q depends on θ only and r is considered as constant with respect to change in θ, it follows that $\frac{\partial(r\mathrm{q})}{\partial\theta} = r\frac{\partial\mathrm{q}}{\partial\theta} = r\frac{\mathrm{dq}}{d\theta}$ and (13) leads to

$$\frac{\mathrm{dq}}{d\theta} = i\mathrm{q} \tag{14}$$

From the previous discussion, division is allowed for rotation factor including the orthogonal rotation factor i. Then (14) becomes

$$\frac{\mathrm{dq}}{d(i\theta)} = \mathrm{q} \tag{15}$$

Let $\phi = i\theta$. It follows from (15) that

$$\frac{\mathrm{dq}}{d\phi} = \mathrm{q} \tag{16}$$

Thus, q is the exponential function. That is, $\mathrm{q} = e^{\phi}$. With $\phi = i\theta$ and $\mathrm{q} = \mathrm{q}(\theta)$, it follows that

$$\mathrm{q}(\theta) = e^{i\theta} \tag{17}$$

The result (17) is a rotation factor formula for obtaining a rotation factor of a specific angle θ. The formula shows the existence of rotation factors covering all angles and has significant implications.

2.6. Existence of rotation factor set

It should be noted that in the derivation of the rotation factor formula (17), the rotation angle θ is relative to the real number axis. But, in reality, the angle of a rotation factor in the definition (1) or (10) as well as in (17) is relative to the target's direction and independent of coordinate systems.

The rotation factors can be used to construct coordinate systems. For a polar coordinate system, from (12) and (17), the position number p is expressed as

$$\mathrm{p} = re^{i\theta} \tag{18}$$

This formula (see Figure 3) means that multiplying a rotation factor of angle θ to a real number r in a real number axis makes the real number point rotate to the point represented by the position number p. In other words, the position number is formed by multiplying a real number with a rotation factor.

All real numbers form a field [20]. The set of real numbers is denoted $\mathbb{R}$ [21]. In comparison, rotation factors are a new kind of number. The rotation factor formula (17) is an exponential function and exponential field. That is, all rotation factors of different angles represented by the formula form a field. The set of rotation factors is then given as

$$\mathbb{E} = \{e^{i\theta} \mid \theta \in \mathbb{R}, -\pi < \theta \leq \pi\} \tag{19}$$

It is noted that althrough important and special, the orthogonal rotation factor i is just one member in the set E. Let $\mathbb{P}$ denote the set of all position numbers. $\mathbb{P}$ represents all points in

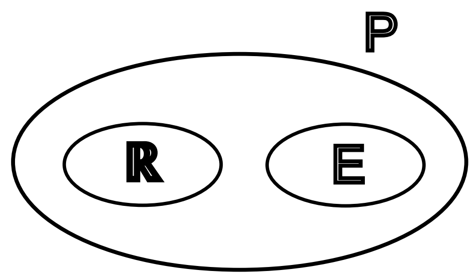

FIGURE 4. Real number set $\mathbb{R}$, rotation factor set $\mathbb{E}$ and position number set $\mathbb{P}$

a plane. A member in $\mathbb{P}$ may be constructed by a member in $\mathbb{R}$ and a member in $\mathbb{E}$. Figure 4 illustrates their relationship.

The position stretching and position rotation are two basic motion types in the physical world. From the perspective of describing motion of points in a plane, numbers may be classified into two types: real numbers for position stretching, and rotation factors for position rotation. Multiplying a real number to a position number or position vector stretches the target's magnitude without changing the target's direction or orientation. On the other hand, multiplying a rotation factor to a position number or position vector rotates the target without changing the target's magnitude. The existences of real number set $\mathbb{R}$ and the rotation factor set $\mathbb{E}$ complement each other and complete the representation for the two basic motion types.

2.7. Constructing Cartesian coordinate system with rotation factors and deriving Euler's formula

In Figure 5, real numbers a, b, and r represent points in the RE axis, and have the direction of the positive RE axis.

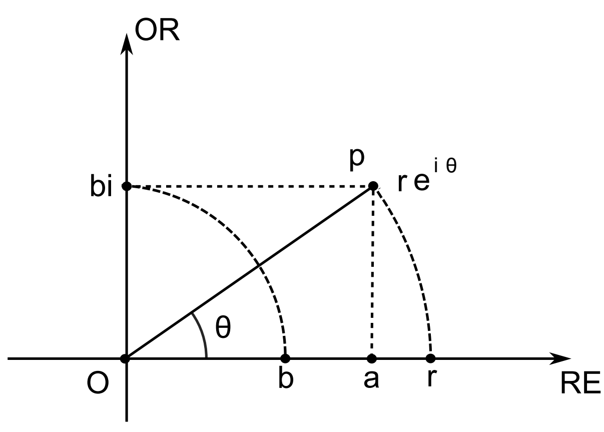

FIGURE 5. Cartesian coordinate system formed from real numbers and rotation factors

Applying a rotation factor of an arbitrary angle θ to number r makes the point rotate by angle θ to the point represented by position number p or $\mathrm{r}e^{i\theta}$. That is

$$\mathrm{p} = \mathrm{r}e^{i\theta} \tag{20}$$

The line through points a and p is parallel to the OR axis. The distance between the points O and b is the same as the distance between the points a and p. Applying the orthogonal rotation factor i to number b makes the point rotate to the point bi in the OR axis. In the Cartesian coordinate system, the position number p is expressed as

$$\mathrm{p} = \mathrm{a} + \mathrm{b}i \tag{21}$$

From the definition of sine and cosine, $\mathrm{a} = \mathrm{r}\cos(\theta)$ and $\mathrm{b} = \mathrm{r}\sin(\theta)$. Thus, (21) becomes $\mathrm{p} = \mathrm{r}\cos(\theta) + i\mathrm{r}\sin(\theta)$, which, together with (20), leads to

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{22}$$

The result (22) is Euler's formula. The position number p in (21) is equivalent to the definition of a complex number. All complex numbers in the two-dimensional plane form the complex number set $\mathbb{C}$. The orthogonal rotation factor i here is equivalent to the imaginary unit. This shows that we can develop the complex number system including Euler's formula in a new way from first principles without even using and basing on $i = \sqrt{-1}$ or $i^2 = -1$.

3. Three-dimensions

3.1. Spherical coordinate system

In Figure 6, $\mathrm{OR_i}$ is an axis that is orthogonal to the RE axis.

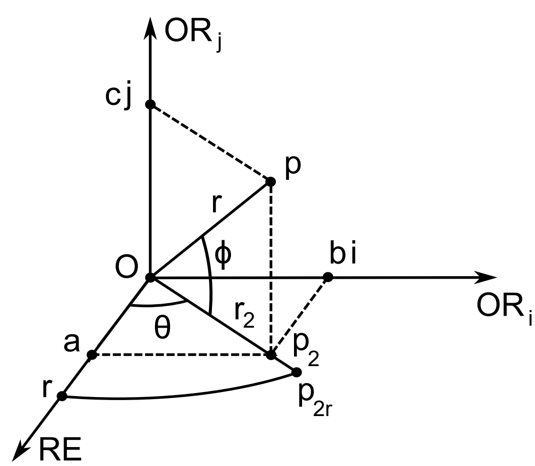

FIGURE 6. Construction of three-dimensional coordinates by rotation factors

Applying orthogonal rotation factor i to any point in the RE axis makes the point rotate and reach the $\mathrm{OR_i}$ axis. The i may also be considered as a unit vector pointing to the positive direction of the OR_i axis. OR_j is an axis that is orthogonal to the plane RE-O-OR_i. j is another orthogonal rotation factor. Similarly, applying j to any point in the RE axis makes the point rotate and reach the OR_j axis, and the j may also be considered as a unit vector pointing to the positive direction of the OR_j axis.

Real number r represents a point in the RE axis. Applying an i-associated rotation factor $e^{i\theta}$ to number r makes the point rotate by angle θ to position number p_{2r} in the plane RE-O-OR_i. Thus,

$$p_{2r} = re^{i\theta} \tag{23}$$

Applying a j-associated rotation factor $e^{j\phi}$ to position number p_{2r} makes the point rotate by angle ϕ to position number p in the plane p_{2r}-O-OR_j. That is, $p = p_{2r}e^{j\phi}$. With this and (23), the position number p becomes

$$p = re^{i\theta}e^{j\phi} \tag{24}$$

This is the formula of position numbers in the three-dimensional spherical coordinate system.

3.2. Spherical-to-Cartesian transformation

In Figure 6, the point represented by position number p_2 is in the plane RE-O-OR_i. The line through points p_2 and p is parallel to the OR_j axis. The position number p_2 has a magnitude of r_2. Like i and j, the unit number 1 may also be considered as a unit vector in the direction of the positive RE axis. a, b, and c are real numbers. a, bi, and cj represent, respectively, the components of position number p in the RE, OR_i and OR_j axis. In the Cartesian coordinate system, the position number p is expressed as

$$p = a + bi + cj \tag{25}$$

The geometry in the figure gives

$$a = rcos(\theta)cos(\phi) \tag{26}$$

$$b = rsin(\theta)cos(\phi) \tag{27}$$

$$c = rsin(\phi) \tag{28}$$

From (25) through (28), it follows that

$$p = r(cos(\theta)cos(\phi) + isin(\theta)cos(\phi) + jsin(\phi)) \tag{29}$$

This is the spherical-to-Cartesian coordinate transformation formula.

3.3. Commutativity of rotation factors

In Figure 6, the i-associated rotation factor $e^{i\theta}$ is first applied to the number r in the RE axis, and then is followed by the j-associated rotation factor $e^{j\phi}$.

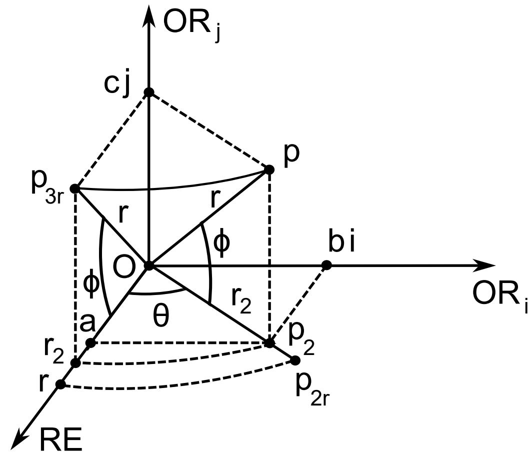

FIGURE 7. Illustration for commutativity of rotation factors

Figure 7 is the same as Figure 6, except that the order of the operations by the two rotation factors is reversed. Position number $\mathrm{p_{3r}}$ is in the plane RE-O-OR$_\mathrm{j}$. The line through $\mathrm{p_{3r}}$ and $\mathrm{r_2}$ in the RE axis is parallel to the OR$_\mathrm{j}$ axis. Applying the rotation factor $e^{j\phi}$ to number r in the RE axis makes the point rotate by angle ϕ to position number $\mathrm{p_{3r}}$. And then applying the rotation factor $e^{i\theta}$ to position number $\mathrm{p_{3r}}$ makes the point rotate by angle θ to position number p in the plane that is parallel to the plane RE-O-OR$_\mathrm{i}$. The final position is the point p, which is the same as the point in Figure 6. That is

$$\mathrm{p} = re^{i\theta}e^{j\phi} = re^{j\phi}e^{i\theta} \tag{30}$$

The result (30) represents the commutativity of the rotation factors.

3.4. Interaction between orthogonal rotation factors i and j

The orthogonal rotation factors i and j can be considered as unit vectors. On the other hand, they are rotation factors that can rotate vectors. To specify the rotation target, here, we define that the former is rotated by the latter. For example, ij means that i is rotated by j. It should be noted that we may also define the other way around. It will not change the result.

With reference to Figure 6, for ij, i is considered as a unit vector pointing to the positive OR$_\mathrm{i}$ axis direction, and is rotated by orthogonal rotation factor j with the resultant unit vector pointing to the j direction or the positive OR$_\mathrm{j}$ axis direction. That is, ij equals j. And conversely, for ji, j is considered as a unit vector pointing to the OR$_\mathrm{j}$ axis direction, and is rotated by orthogonal rotation factor i. But, the rotation generated by i is in the direction parallel to the plane RE-O-OR$_\mathrm{i}$, and cannot make j rotate. That is, ji equals j. We have

$$ij = ji = j \tag{31}$$

More generally, any point in the line O-p_{2r} is also rotated to the OR_j axis by rotation factor j. That is, $e^{i\theta}j = j$. Also, rotation factor $e^{i\theta}$ makes its target vector rotate in the direction parallel to the plane RE-O-OR_i, and cannot rotate unit vector j. Thus, $je^{i\theta} = j$. And we have

$$e^{i\theta}j = je^{i\theta} = j \tag{32}$$

In fact, the same result can also be obtained by the transformation formula. In spherical coordinate system, rotation factors are commutative. For position number $e^{i\theta}j$ or $je^{i\theta}$, the magnitude r is 1, and the angle ϕ is $\frac{\pi}{2}$. The resultant position number p from the transformation formula (29) is j. That is, we have the identity (32).

3.5. Obtaining spherical-to-Cartesian transformation by rotation factor multiplication algebra

With Euler's formula for the rotation factor $e^{j\phi}$ in (24), it follows that

$$\mathrm{p} = r(e^{i\theta}cos(\phi) + e^{i\theta}jsin(\phi))$$

And with $e^{i\theta}j = j$ in (32) and Euler's formula for $e^{i\theta}$, the above equation becomes the transformation formula (29) previously obtained by geometry.

3.6. Multiplication of position numbers

In spherical coordinate system represented by (24), let $\mathrm{p}_1 = \mathrm{r}_1 e^{i\theta_1} e^{j\phi_1}$ be one position number, and $\mathrm{p}_2 = \mathrm{r}_2 e^{i\theta_2} e^{j\phi_2}$ be another. Since rotation factors are commutative, the two position numbers are also commutative. The multiplication for the two position numbers is

$$\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{p}_2 \cdot \mathrm{p}_1 = \mathrm{r}_1\mathrm{r}_2 e^{i(\theta_1+\theta_2)} e^{j(\phi_1+\phi_2)} \tag{33}$$

The result is a new position number in spherical coordinates, and can be transformed by the formula (29) to Cartesian coordinates as

$$\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{r}_1\mathrm{r}_2(cos(\theta_1+\theta_2)cos(\phi_1+\phi_2) + isin(\theta_1+\theta_2)cos(\phi_1+\phi_2) + jsin(\phi_1+\phi_2)) \tag{34}$$

4. n-dimensions: projections

Before extending results obtained from a three dimensional space to a higher n-dimensional space, one may need to understand the projections from an n-dimensional space to a lower dimensional space. From here on, vectors with bold arrows denote the projections; subscript k (k=1,2,3...n) represents k-dimensional or k-th dimension; and i_k is an orthogonal rotation factor for axis OR_k and k-th dimension.

4.1. Three-dimensional (3D) space

Figure 8 illustrates the projections of position vectors in a 3D space.

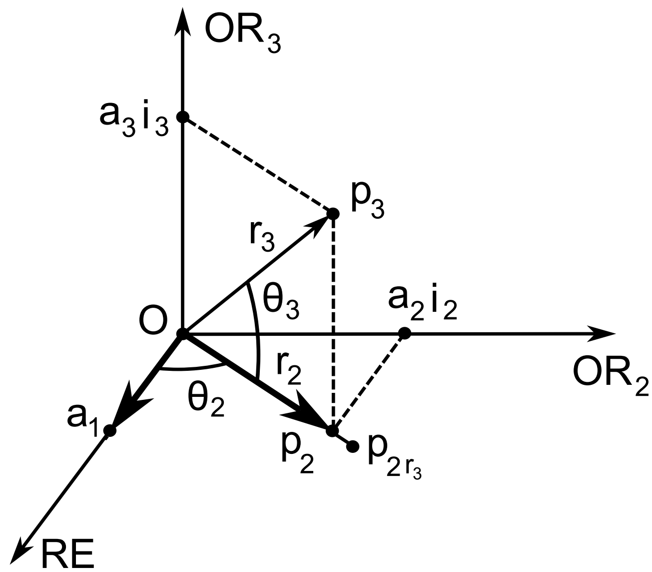

FIGURE 8. Projections of position vectors in a 3D space

Position number p_2 is in the second-dimension plane RE-O-OR_2. The projection of p_2 to the lower first-dimension is the vector corresponding to the component a_1. And the projection to the second-dimension axis OR_2 is the component $a_2 i_2$.

Position number p_{2r_3} is in the same plane as p_2 and also in the same direction. p_{2r_3} has a magnitude of r_3. Applying an i_3-associated rotation factor $e^{i_3\theta_3}$ of angle θ_3 to p_{2r_3} makes the point rotate to position number p_3. The projection of p_3 to the lower second-dimension is p_2. And the projection to the thrid-dimension axis OR_3 is the component $a_3 i_3$. The rotation and projections in the figure lead to

$$p_3 = r_3 e^{i_2\theta_2} e^{i_3\theta_3} \tag{35}$$

$$p_3 = p_2 + a_3 i_3 \tag{36}$$

$$r_2 = r_3 cos(\theta_3) \tag{37}$$

$$a_3 = r_3 sin(\theta_3) \tag{38}$$

The 3D equation set of (35)-(38) represent spherical and Cartesian coordinate systems in a 3D space.

4.2. Four-dimensional (4D) space

Compared with the 3D space in Figure 8, one more orthogonal axis OR_4 is added in Figure 9. It is noted that the figure is viewed from the 4D perspective. Logically, if the 4D space occupies our 3D space, which is one dimension lower than the 4D, the 3D space must appear as a 2D plane from the 4D viewpoint. In the 4D space, the OR_4 axis is orthogonal to the plane OR_2-O-OR_3 in the lower dimensions.

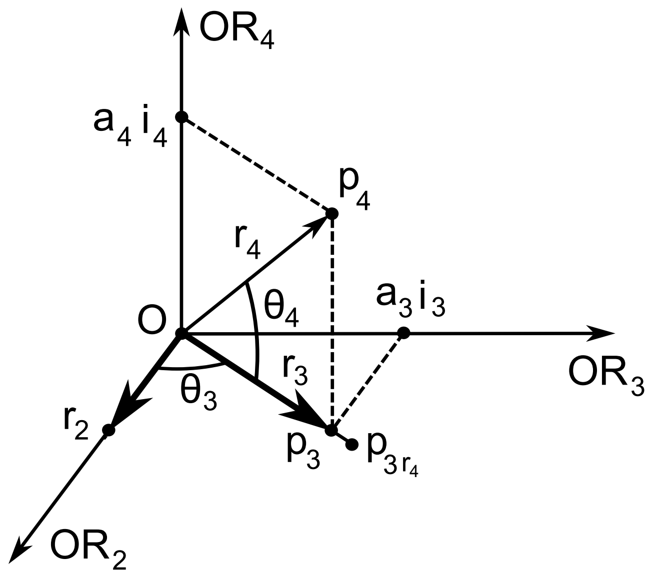

FIGURE 9. Projections of position vectors in a 4D space

Position number p_{3r_4} is in the same plane as p_3 and also in the same direction. p_{3r_4} has a magnitude of r_4. Applying an i_4-associated rotation factor $e^{i_4\theta_4}$ of angle θ_4 to p_{3r_4} makes the point rotate to position number p_4. Thus, $p_4 = r_4 e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4}$. The r_4 is the magnitude of p_4. The position vector represented by p_3 in Figure 8 now becomes a projection in the current 4D space. The projection of p_4 to the lower 3D dimension is the vector corresponding to p_3. The rotation and projections lead to the corresponding 4D equation set below

$$p_4 = r_4 e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4} \tag{39}$$

$$p_4 = p_3 + a_4 i_4 \tag{40}$$

$$r_3 = r_4 cos(\theta_4) \tag{41}$$

$$a_4 = r_4 sin(\theta_4) \tag{42}$$

where p_3 is given by the 3D equation set (35)-(38).

4.3. n-dimensional space

By the same token, Figure 10 shows the projections in an n-dimensional space. From the n-dimensional space perspective, the n-1 dimensional space appears as a plane. The OR_n axis is orthogonal to the plane OR_{n-2}-O- OR_{n-1} in the lower dimensions.

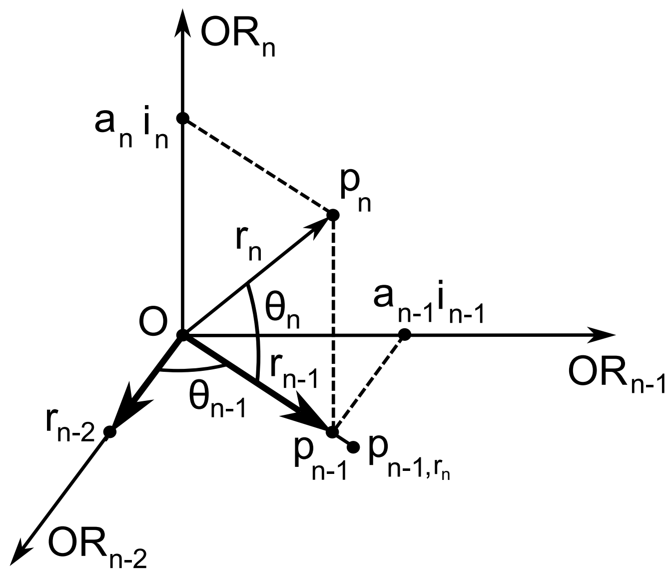

FIGURE 10. Projections of position vectors in an n-dimensional space

Position number $\mathrm{p_{n-1,r_n}}$ is in the same plane as $\mathrm{p_{n-1}}$ and also in the same direction. $\mathrm{p_{n-1,r_n}}$ has a magnitude of $\mathrm{r_n}$. Applying an i_n-associated rotation factor $e^{i_n\theta_n}$ of angle θ_n to $\mathrm{p_{n-1,r_n}}$ makes the point rotate to $\mathrm{p_n}$. The projection of $\mathrm{p_n}$ to the lower (n-1)th dimension is $\mathrm{p_{n-1}}$. The above rotation and projections lead to

$$\mathrm{p_n} = \mathrm{r_n} e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4} ... e^{i_n\theta_n} \tag{43}$$

$$\mathrm{p_n} = \mathrm{p_{n-1}} + \mathrm{a_n i_n} \tag{44}$$

$$\mathrm{r_{n-1}} = \mathrm{r_n} cos(\theta_n) \tag{45}$$

$$\mathrm{a_n} = \mathrm{r_n} sin(\theta_n) \tag{46}$$

The n-dimensional equation set of (43)-(46) represent spherical and Cartesian coordinate systems in an n-dimensional space, with $\mathrm{p_{n-1}}$ for the lower dimension given by the corresponding n-1 equation set.

5. n-dimensions: generalization of coordinate systems

5.1. Spherical coordinate system

For the generalization of the coordinate system construction, in the 2D plane, the orthogonal axis $\mathrm{OR_2}$ is orthogonal to the RE real number axis. The associated rotation factor for the $\mathrm{OR_2}$ axis is $e^{i_2\theta_2}$. Then, in the 3D space, the orthogonal axis $\mathrm{OR_3}$ is introduced to be orthogonal to the plane RE-O-OR_2. The associated rotation factor for the OR_3 axis is $e^{i_3\theta_3}$. Similarly, in the n-dimensional space, the orthogonal axis OR_n is introduced to be orthogonal to the plane $\mathrm{OR}_{\mathrm{n}-2}$-O-$\mathrm{OR}_{\mathrm{n}-1}$. The associated rotation factor for the OR_n axis is $e^{i_n\theta_n}$. With (43), the position number in an n-dimensional spherical coordinate system is given by

$$\mathrm{p_n} = \mathrm{r_n} \prod_{j=2}^{n} e^{i_j\theta_j} \tag{47}$$

where $\mathrm{r_n}$ is the magnitude of the position number and $e^{i_j\theta_j}$ is the j-th dimension associated rotation factor.

5.2. Spherical-to-Cartesian transformation

From equation set (36)-(38) for 3D and (40)-(42) for 4D, it follows that

$$\mathrm{p_3} = \mathrm{r_3}(cos(\theta_2)cos(\theta_3) + i_2sin(\theta_2)cos(\theta_3) + i_3sin(\theta_3))$$

$$\mathrm{p_4} = \mathrm{r_4}(cos(\theta_2)cos(\theta_3)cos(\theta_4) + i_2sin(\theta_2)cos(\theta_3)cos(\theta_4) + i_3sin(\theta_3)cos(\theta_4) + i_4sin(\theta_4))$$

Along with (44)-(46) for downward iterations, the generalization of n-dimensional position number transformation from spherical coordinates to Cartesian coordinates is given by

$$\mathrm{p_n} = \sum_{j=1}^{n} a_j i_j \tag{48}$$

where $a_j = r_n E_j$ and

$$E_j = sin(\theta_j) \prod_{k=j+1}^{n} cos(\theta_k)$$

with $i_1 = 1$, $sin(\theta_1) = 1$, and $\prod_{k=j+1}^{n} cos(\theta_k) = 1$ if $j + 1 > n$.

5.3. Cartesian-to-spherical transformation

Also from the equation sets (36)-(38), (40)-(42) and (44)-(46), the generalization of n-dimensional position number transformation from Cartesian coordinates to spherical coordinates is given by

$$\mathrm{r_n} = \sqrt{\sum_{j=1}^{n} a_j^2} \tag{49}$$

for the magnitude and

$$(\theta_2, \theta_3, \theta_4, ..., \theta_\mathrm{n}) \tag{50}$$

for (n-1)-tuple of angles with

$$\theta_{\mathrm{j}} = sin^{-1}(\frac{a_j}{\mathrm{r_j}})$$

where θ_{j} is the rotation angle for the j-th dimension with the corresponding magnitude

$$\mathrm{r_j} = \sqrt{\sum_{k=1}^{j} a_k^2} \tag{51}$$

6. n-dimensions: generalization of rotation factor multiplication algebra

The multiplication algebra involves the interactions of rotation factors and orthogonal rotation factors across different dimensions.

6.1. 3D space

For the 3D space, with notations in Figure 8, the identity (32) becomes

$$e^{i_2\theta_2} i_3 = i_3 e^{i_2\theta_2} = i_3 \tag{52}$$

And for $\theta_2 = \frac{\pi}{2}$, the above identity becomes

$$i_2 i_3 = i_3 i_2 = i_3 \tag{53}$$

This indicates that the orthogonal rotation factor i_3 is commutative with the orthogonal rotation factor i_2 and the rotation factor $e^{i_2\theta_2}$ in the lower dimension, and the multiplication result is always equal to the i_3 itself.

6.2. 4D space

In Figure 9 for the 4D space, any point in the line O-$\mathrm{p_{3r_4}}$ is rotated to the $\mathrm{OR_4}$ axis by orthogonal rotation factor i_4. Thus, $e^{i_3\theta_3} i_4 = i_4$. On the other hand, applying the rotation factor $e^{i_3\theta_3}$ to a point represented by i_4 in the $\mathrm{OR_4}$ axis will not rotate the point. Thus, $i_4 e^{i_3\theta_3} = i_4$.

Further, from the 4D perspective, the plane RE-O-$\mathrm{OR_2}$ in Figure 8 for the 3D now appears one-dimensional in Figure 9. Logically, in Figure 9, the point represented by the i_2-associated rotation factor $e^{i_2\theta_2}$ is in the $\mathrm{OR_2}$ axis. Applying the i_4 orthogonal rotation factor to the point makes it rotate to the $\mathrm{OR_4}$ axis. Thus, $e^{i_2\theta_2} i_4 = i_4$. On the other hand, applying the rotation factor $e^{i_2\theta_2}$ to a point represented by i_4 in the $\mathrm{OR_4}$ axis will not rotate the point. Thus, $i_4 e^{i_2\theta_2} = i_4$.

From the above, it follows that

$$e^{i_3\theta_3} i_4 = i_4 e^{i_3\theta_3} = e^{i_2\theta_2} i_4 = i_4 e^{i_2\theta_2} = i_4 \tag{54}$$

With angle $\frac{\pi}{2}$ for rotation factors, the above identity leads to

$$i_3 i_4 = i_4 i_3 = i_2 i_4 = i_4 i_2 = i_4 \tag{55}$$

This indicates that the orthogonal rotation factor i_4 is commutative with the orthogonal rotation factors and rotation factors i_2, $e^{i_2\theta_2}$, i_3, and $e^{i_3\theta_3}$ in the lower dimensions, and the multiplication result is always equal to the i_4 itself.

6.3. n-dimensional space

Similarly, in Figure 10 for the n-dimensional space, any point in the line $\mathrm{O\text{-}p_{n-1,r_n}}$ is rotated to the $\mathrm{OR_n}$ axis by orthogonal rotation factor i_n. Thus, $e^{i_{n-1}\theta_{n-1}} i_n = i_n$. On the other hand, applying the rotation factor $e^{i_{n-1}\theta_{n-1}}$ to a point represented by i_n in the $\mathrm{OR_n}$ axis will not rotate the point. Thus, $i_n e^{i_{n-1}\theta_{n-1}} = i_n$.

Further, from the n-dimensional perspective, the point represented by the i_{n-2}-associated rotation factor $e^{i_{n-2}\theta_{n-2}}$ is in the $\mathrm{OR_{n-2}}$ axis. Applying the i_n orthogonal rotation factor to the point makes it rotate to the $\mathrm{OR_n}$ axis. Thus, $e^{i_{n-2}\theta_{n-2}} i_n = i_n$. On the other hand, applying the rotation factor $e^{i_{n-2}\theta_{n-2}}$ to a point represented by i_n in the $\mathrm{OR_n}$ axis will not rotate the point. Thus, $i_n e^{i_{n-2}\theta_{n-2}} = i_n$.

And still further, by the same token, in the n-dimensional coordinate system, all axes are mutually orthogonal to each other. For a j-th dimension with j lower than n-2, applying the i_n orthogonal rotation factor to a point represented by $e^{i_j\theta_j}$ in the j-th dimension makes the point rotate to the $\mathrm{OR_n}$ axis. Thus, $e^{i_j\theta_j} i_n = i_n$. On the other hand, applying the rotation factor $e^{i_j\theta_j}$ to a point represented by i_n in the $\mathrm{OR_n}$ axis will not rotate the point. Thus, $i_n e^{i_j\theta_j} = i_n$.

From the above and with the generalization, it follows that

$$e^{i_j\theta_j} i_n = i_n e^{i_j\theta_j} = i_n \tag{56}$$

and

$$i_j i_n = i_n i_j = i_n \tag{57}$$

where $2 \leq j < n$.

It is noted that the above results can also be obtained by the spherical-to-Cartesian transformation in (48). Take (56) for example. The position number $e^{i_j\theta_j} i_n = e^{i_j\theta_j} e^{i_n\frac{\pi}{2}}$ means that in (48), $r_n = 1$, $\theta_j = \theta_j$, $\theta_n = \frac{\pi}{2}$ and all other angles (θ_k with $2 \leq k \leq n-1$ excluding $k = j$) are 0. Thus, all E_j in (48) become 0 except for E_n, which is 1. That is, $\mathrm{p_n}$ $= i_n$, which is consistent with (56).

6.4. Obtaining n-dimensional spherical-to-Cartesian transformation by rotation factor multiplication algebra

Denote

$$Q_j = \prod_{k=2}^{j} e^{i_k \theta_k} \tag{58}$$

Formula (47) becomes

$$\frac{\mathrm{p_n}}{\mathrm{r_n}} = Q_n \tag{59}$$

The result in (56) means

$$Q_{j-1} i_j = i_j \tag{60}$$

With (60), it follows that

$$Q_n = Q_{n-1} e^{i_n \theta_n} = Q_{n-1}(cos(\theta_n) + i_n sin(\theta_n)) = Q_{n-1} cos(\theta_n) + i_n sin(\theta_n) \tag{61}$$

With (61), it follows that

$$Q_{n-1} = Q_{n-2} cos(\theta_{n-1}) + i_{n-1} sin(\theta_{n-1}) \tag{62}$$

and

$$Q_{n-2} = Q_{n-3} cos(\theta_{n-2}) + i_{n-2} sin(\theta_{n-2}) \tag{63}$$

Q_n may be obtained by continuing the iteration process and combining the iteration results. Inserting the obtained Q_n into (59) leads to Formula (48).

7. Concluding remarks

Angle-dependent rotation factors have been introduced to rotate vectors and the equivalents such as complex numbers. A fundamental equation of position vector rotation has been discovered. An orthogonal rotation factor happens to directly appear in the fundamental equation whose solution leads to the rotation factor formula and the finding for the existence of the rotation factor set. A new approach has been provided to derive Euler's formula by constructing Cartesian and polar (2D spherical) coordinate systems based on rotation factors. It is shown that the orthogonal rotation factor is equivalent to the imaginary unit.

In mathematics, there may be more than one method to solve a problem, derive an equation or develop a system. Each new method may provide a different perspective and enrich the understanding of the subject matter. This turns out to also happen to the complex number system. It is found that there does exist a more thorough way of developing the system rather than just relying on the imaginary unit. Developing the complex number system from first principles is achieved with $\sqrt{-1}$ or $i^2 = -1$ being entirely bypassed.

With the understandings gained from the 2D development, the concept of rotation factors for rotating position vectors and numbers with positioning directions is extended to other dimensions, and the constructions of three and higher n-dimensional complex number (position number) spherical coordinate systems are realized. The obtained spherical system's coordinate formula is succinct and consistent across dimensions with one rotation factor for each dimension, indicating that the rotation factors are natively suited for the spherical systems.

Methodology for projections of position vectors in four-dimensional and higher n-dimensional spaces is presented. A projection chain is established where a position vector that makes a projection to a position vector in a lower dimension becomes the projection of a position vector in a higher dimension. The projection patterns in the chain are consistent throughout with the similar mathematical formulas from n-th dimension, down to 4th, 3rd, and 2nd. The projections are helpful in understanding the geometric representation of position vectors in four-dimensional and higher n-dimensional spaces, and provide the qualitative insights and the quantitative results.

Generalizations of spherical-to-Cartesian and Cartesian-to-spherical transformations for n-dimensional complex number spherical coordinate systems are provided. In 3D spherical system, rotation factors are graphically shown to be commutative, and multiplication of position numbers are illustrated. The results can be extended to other dimensions. In higher n-dimensional spherical systems, the commutativity is shown by the information contained in the spherical-to-Cartesian transformation formula, and by the interactions between dimension-associated rotation factors and axis-associated unit vectors of orthogonal rotation factors.

One method for obtaining the multiplication and interactions between orthogonal rotation factors and rotation factors in the same and across dimensions is by the geometric representation where rotation factors are applied to position vectors and the resultant rotations are examined. Another method is by the spherical-to-Cartesian transformation formula, which also contains the interaction information. The rotation factor multiplication algebra method may also be used to obtain the spherical-to-Cartesian transformation. The three methods produce results that are consistent with each other.

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