Developing the imaginary unit and complex number system from first principles, and extending related applications

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The complex number system is developed from first principles with the imaginary unit being completely bypassed. Vectors are widely used in physics and angle-dependent rotation factors have been introduced to rotate vectors. A fundamental equation of position vector rotation has been discovered. An orthogonal rotation factor, which turns out to be equivalent to the imaginary unit, happens to appear in the fundamental equation. Euler's formula has been derived through a novel approach. The existence of the rotation factor set has been found and the rotation factor set serves as a new useful mathematical tool for rotating vectors in physics equations. Based on the fundamental understanding gained from the development, unified vector algebra is conceived to seamlessly integrate vectors and complex numbers under one umbrella for mixed algebra manipulation and interchange, and to significantly extend complex numbers' practical applications in physics.

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I. INTRODUCTION

The discovery of the imaginary unit $i = \sqrt{-1}$ from solving the cubic equation formed the complex numbers that were eventually accepted by renaissance mathematicians[1–5] and have deep significance and profound importance to our understanding of mathematics and physics[6–8]. In 1748, Leonhard Euler obtained[9,10] Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ without using the concept of the complex plane, which was not yet introduced. The equation was called "the most remarkable formula in mathematics" by the physicist Richard Feynman[11], and lies at the heart of complex number theory[12]. Euler's identity $e^{i\pi} = -1$ as the special case for angle π, is considered to be an exemplar of mathematical beauty[12–15]. About 50 years later after the creation of Euler's formula, Caspar Wessel described complex numbers as points in the complex plane[16,17]. With the geometric significance of the complex plane for complex analysis, the complex number system has captivated more than 150 years of intensive development, finding applications in science and engineering[6,7].

Historically, $\sqrt{-1}$ has long labored under a false story of unfathomable mystery with the agonizingly prolonged, painful discovery of complex numbers[2,12]. Even today, due to basing on the same very abstract mysterious concept of $\sqrt{-1}$ or $i^2 = -1$, the enigma still exists for many learners and hinders broader active novel applications that are beyond what has been proven and taught in conventional textbooks.

II. ROTATION FACTORS

From this point on (until specifically mentioned), assume that there is no existence of the complex number system so that our mind is not influenced by the existing concepts and will focus on creating a number system from first principles.

First, from a high level perspective, one may expect the existence possibility of a new number system for describing rotation. Nature has its laws and properties. Mathematics is invented to discover and describe Nature. Real numbers represent points on a number line, and can describe stretching and displacement of a point's position by multiplication and addition. On the other hand, rotation of a point's position is also a fundamental motion type besides the stretching and displacement for translational motion. There likely also exists a number system for describing the rotation.

A point in the one-dimensional number line represented by real numbers cannot perform a rotation motion. The next higher dimension, a two-dimensional plane is needed and the position of a point in the plane may be represented by a position vector. Real numbers can only make the vector stretch by multiplication.

By analogy with real numbers, rotation numbers are introduced to make the vector rotate by multiplication. Mathematics represents abstractions of properties of the physical world. Rotation numbers may also be called rotation factors that are more in line with causality and physics convention.

A. Terminologies and conventions

Space or plane: refers to Euclidean space or plane.

Rotation factor: The term rotation factor is mainly used here. It may also be called rotation number.

Position vector: A position vector represents the position of a point. The direction of a position vector is from the coordinate origin to the point represented by the vector, and the magnitude of the vector is the distance between the point and the origin.

Position number: Almost identical to a position vector in the representation sense, a position number represents the position of a point. The direction of a position number is from the coordinate origin to the point represented by the number, and the magnitude of the number is the distance between the point and the origin.

Position vector and position number conventions: Here the terms position vector and position number are interchangeable for representing a point. For geometric representation, vector arrow may be used for a position number to explicitly indicate the direction of the position number.

Point of position vector or position number: means the point represented by the position vector or the position number, and vice versa.

Direction of point, position vector, or position number: means the direction from the origin point to the point.

Rotating (or rotation of) point, position vector, or position number: means rotating a point to another position with corresponding position vector change or position number change.

Applying a rotation factor to: means multiplying a rotation factor to a position vector or a position number. Applying a rotation factor to a point means applying the factor to the point's position vector or position number.

Target of a rotation factor: means the position vector or the position number that a rotation factor is applied to.

Position numbers versus complex numbers: Position numbers are equivalent to complex numbers. They are interchangeable.

B. Definition of rotation factors

An angle-dependent rotation factor q of an arbitrary angle δ is defined as a factor for multiplying to a position vector of a point and making the vector rotate counter-clockwise by angle δ relative to the origin point of the vector with the vector's magnitude unchanged. That is

$$\mathbf{P}_\delta = \mathrm{q} \cdot \mathbf{P} = \mathbf{P} \cdot \mathrm{q} \tag{1}$$

where (see FIG. 1) Point P represented by position vector $\mathbf{P}$ rotates relative to origin point O by angle δ and reaches Point P_δ represented by position vector $\mathbf{P}_\delta$.

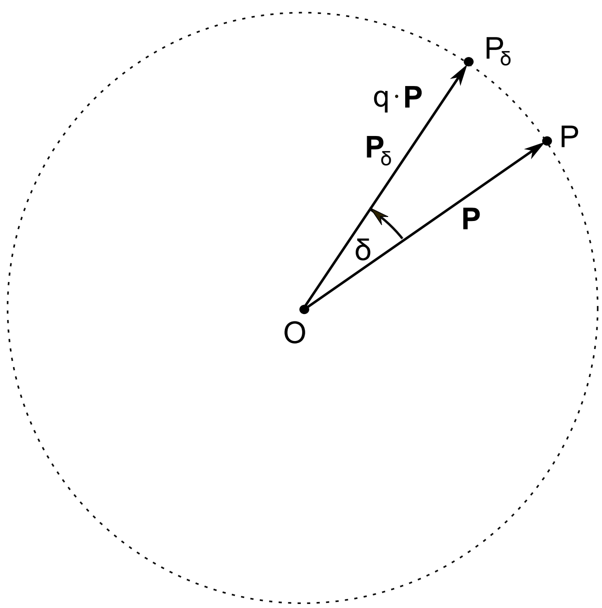

FIG. 1: Rotation factor and target vector's direction change

By the definition, the multiplication between a rotation factor and its target vector is commutative. Left-multiplication and right-multiplication to a target vector by a rotation factor produce the same rotation result for the target vector.

At this point, the existence of rotation factors such defined is a postulate. If the consequences of the postulate are consistent with existing results, it is then considered true.

C. Properties of rotation factors

Denote a rotation factor q of an arbitrary angle δ as $\mathrm{q}(\delta)$. The following basic properties of rotation factors are implied or derived from the definition (1).

Property 1. Being multiplication commutative

The order of rotation factors for multiplying to a vector will not affect the final result. That is

$$\mathrm{q}(\delta_1) \cdot \mathrm{q}(\delta_2) = \mathrm{q}(\delta_2) \cdot \mathrm{q}(\delta_1)$$

Property 2. Being multiplication associative

The grouping of rotation factors for multiplying to a vector will not affect the final result. That is

$$\mathrm{q}(\delta_1) \cdot \Big(\mathrm{q}(\delta_2) \cdot \mathrm{q}(\delta_3)\Big) = \Big(\mathrm{q}(\delta_1) \cdot \mathrm{q}(\delta_2)\Big) \cdot \mathrm{q}(\delta_3)$$

Property 3. Division operation of rotation factor being allowed

The multiplication effect produced by a rotation factor may be inversed through division by the same rotation factor. This means

$$\mathrm{q} \cdot \frac{1}{\mathrm{q}} = 1$$

Property 4. Multiplications to a vector by multiple rotation factors being equivalent to one rotation factor with the sum of the individual angles

If the total number of the multiple rotation factors is n, this property is expressed by

$$\mathrm{q}(\delta_1) \cdot \mathrm{q}(\delta_2) \cdot ... \cdot \mathrm{q}(\delta_n) = \mathrm{q}(\delta_1 + \delta_2 + ... + \delta_n)$$

Property 5. Becoming real numbers 1 and -1 respectively for angles 0 and π

In general, a rotation factor is not a real number as the former can change a vector's direction by multiplication and the latter cannot. But the two angles 0 and π are the exceptions. The rotation factor of angle 0 produces no rotation for its target vector and becomes the real number 1. And the rotation factor of angle π produces a reverse direction for its target vector by rotating 180° and becomes the real number -1. That is

$$\mathrm{q}(0) = 1$$

$$\mathrm{q}(\pi) = -1$$

III. ORTHOGONAL ROTATION FACTOR

The rotation factor of angle $\frac{\pi}{2}$ is defined as orthogonal rotation factor in that it produces a 90° rotation for the target vector with the resultant direction being perpendicular or orthogonal to the initial direction. Notationwise, the orthogonal rotation factor is denoted as $\mathrm{q}(\frac{\pi}{2})$, which may be further succinctly denoted by a symbol i. That is

$$i = \mathrm{q}(\frac{\pi}{2}) \tag{2}$$

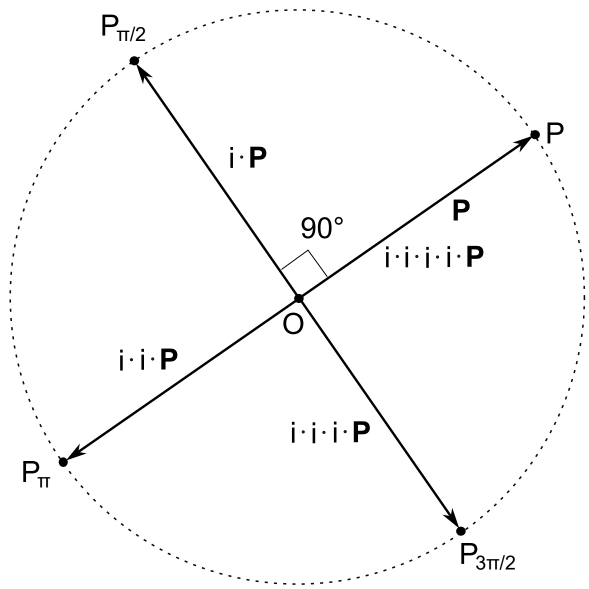

FIG. 2: Orthogonal rotation factor being successively applied to position vector

In FIG. 2, Point P relative to origin Point O forms its position vector $\mathbf{P}$. Multiplying the orthogonal rotation factor i to vector $\mathbf{P}$ makes the vector rotate by angle $\pi/2$ or 90° with the resultant vector indicated by $i \cdot \mathbf{P}$ and the resultant point position indicated by Point $\mathrm{P}_{\pi/2}$. Further, multiplying i to vector $i \cdot \mathbf{P}$ makes the vector rotate by 90° with the resultant vector indicated by $i \cdot i \cdot \mathbf{P}$ and the resultant point position indicated by Point P_{π}. It is noted that the resultant vector has a reversed direction relative to the initial position vector $\mathbf{P}$. That is, $i \cdot i \cdot \mathbf{P} = -\mathbf{P}$ or $i \cdot i = -1$ or

$$i^2 = -1 \tag{3}$$

And similarly, we have $i \cdot i \cdot i \cdot \mathbf{P} = -i \cdot \mathbf{P}$ or $i^3 = -i$; and $i \cdot i \cdot i \cdot i \cdot \mathbf{P} = \mathbf{P}$ or $i^4 = 1$.

The result (3) is an important property of the orthogonal rotation factor and also directly shows that the orthogonal rotation factor is not a real number as the square of a real number is always positive.

IV. FUNDAMENTAL EQUATION OF POSITION VECTOR ROTATION

Since a rotation factor is involved in a position vector's rotation, it is natural to study the rotation motion of the position vector.

In FIG. 3, Point P is represented by position vector $\mathbf{P}$ with magnitude r. $\hat{\mathbf{n}}$ is a unit vector perpendicular to the position vector $\mathbf{P}$. Point P rotates around origin Point O by angle $\Delta\theta$ with the vector's magnitude r unchanged, and reaches Point $\mathrm{P}_{\Delta\theta}$ whose position vector is $\mathbf{P}_{\Delta\theta}$. $\mathbf{\Delta P}$ is the vector change, which is the difference between vector $\mathbf{P}_{\Delta\theta}$ and vector $\mathbf{P}$.

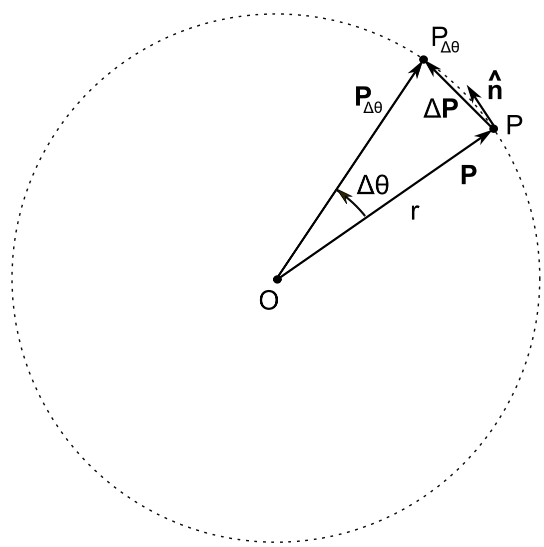

FIG. 3: Position vector rotation relative to an origin point

Since the vector's magnitude r is unchanged, Point P moves along the circumference of the circle during the rotation. Let Δs be the length of the arc corresponding to the angle change $\Delta\theta$, and Δl be the length of the chord of the arc. The Δl is also the magnitude of the vector change $\mathbf{\Delta P}$. Denote $\hat{\mathbf{u}}$ as the unit vector of $\mathbf{\Delta P}$. Then

$$\mathbf{\Delta P} = \|\mathbf{\Delta P}\|\,\hat{\mathbf{u}} = \Delta l\hat{\mathbf{u}} \tag{4}$$

For a circle, the ratio of the arc length to the arc angle span $(\Delta\theta)$ is always equal to the radius regardless of the value of the angle span. That is, $\frac{\Delta s}{\Delta\theta} = r$ or

$$\Delta\theta = \frac{\Delta s}{r} \tag{5}$$

From (4) and (5), the ratio of the vector change to the angle change is expressed as

$$\frac{\mathbf{\Delta P}}{\Delta\theta} = (\frac{\Delta \mathrm{l}}{\Delta s}) r\hat{\mathbf{u}} \tag{6}$$

As $\Delta\theta \to 0$, $(\frac{\Delta \mathrm{l}}{\Delta s}) \to 1$ and $\hat{\mathbf{u}} \to \hat{\mathbf{n}}$, the limit of (6) for the derivative with respect to θ leads to

$$\frac{\partial \mathbf{P}}{\partial\theta} = r\hat{\mathbf{n}} \tag{7}$$

It is noted that multiplying the orthogonal rotation factor i to the vector $\mathbf{P}$ makes the vector perform an orthogonal rotation with the vector's magnitude r unchanged. In other words, the resultant vector has the magnitude r and the direction of $\hat{\mathbf{n}}$. That is

$$i\mathbf{P} = r\hat{\mathbf{n}} \tag{8}$$

This makes (7) become

$$\frac{\partial \mathbf{P}}{\partial\theta} = i\mathbf{P} \tag{9}$$

The result (9) serves as a fundamental equation of position vector rotation. It governs the rule for the rotation of a point in a plane. The orthogonal rotation factor i happens to appear in this fundamental equation.

V. FUNDAMENTAL EQUATION OF POSITION NUMBER ROTATION AND DISCOVERY OF ROTATION FACTOR SET

For representing a point, a position number and its corresponding position vector have the same direction and magnitude. The direction of a position number is implicit or implied in contrast to the position vector whose direction is explicit. In the definition (1), a rotation factor of an arbitrary angle is applied to a position vector, which has a direction. In fact, the same rotation factor can also be applied to a position number. The rotation factor will generate the same rotation for the position number regardless of the target's direction being explicit or implied.

The definition (1) is here restated for position number with an implied direction. An angle-dependent rotation factor q of an arbitrary angle δ is defined as a factor for multiplying to a position number p of a point and making the position number's implied direction rotate counter-clockwise by angle δ with the position number's magnitude unchanged. That is

$$\mathrm{p}_\delta = \mathrm{q} \cdot \mathrm{p} = \mathrm{p} \cdot \mathrm{q} \tag{10}$$

where p_δ is the resultant position number after the rotation.

Since a position number and its corresponding position vector have the same direction and magnitude, using the same methodology and FIG. 3 with position vector notation **P** replaced by position number p, gives the fundamental equation of position number rotation as

$$\frac{\partial \mathrm{p}}{\partial \theta} = i\mathrm{p} \tag{11}$$

To solve this equation, the dependence of the position number p on the rotation angle θ is obtained from FIG. 4 based on the definition of the rotation factor.

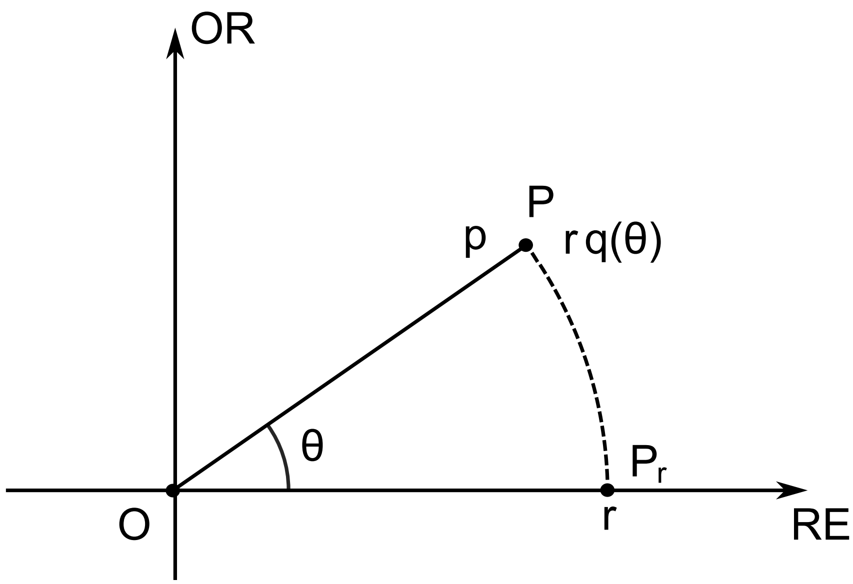

FIG. 4: Position number represented by a real number multiplied by a rotation factor

In FIG. 4, on a plane, RE denotes a real number axis and OR denotes the axis that is orthogonal to the real number axis. Point O is the origin. Point P_r is a point in the real number axis, and is represented by real number r. The implied direction of Point P_r is the positive direction of the real number axis. $\mathrm{q}(\theta)$ is the rotation factor of angle θ. Multiplying the rotation factor $\mathrm{q} = \mathrm{q}(\theta)$ to the real number r makes its implied direction rotate by angle θ with magnitude r unchanged. As a result, Point P_r is rotated to Point P, which is represented by position number p. Before the rotation, the position number is r. After the rotation, the position number is $\mathrm{rq}(\theta)$. By definition, the resultant position number, which is p, equals $\mathrm{rq}(\theta)$. That is

$$\mathrm{p} = \mathrm{rq}(\theta) \tag{12}$$

With (12), the equation (11) becomes

$$\frac{\partial(r\mathrm{q})}{\partial\theta} = ir\mathrm{q} \tag{13}$$

where $\mathrm{q} = \mathrm{q}(\theta)$.

Since q depends on θ only and r is considered as constant with respect to change in θ, it follows that $\frac{\partial(r\mathrm{q})}{\partial\theta} = r\frac{\partial\mathrm{q}}{\partial\theta} = r\frac{\mathrm{dq}}{d\theta}$ and (13) leads to

$$\frac{\mathrm{dq}}{d\theta} = i\mathrm{q} \tag{14}$$

From the previous discussion, division is allowed for rotation factor including the orthogonal rotation factor i. Then (14) becomes

$$\frac{\mathrm{dq}}{d(i\theta)} = \mathrm{q} \tag{15}$$

Let $\phi = i\theta$. It follows from (15) that

$$\frac{\mathrm{dq}}{d\phi} = \mathrm{q} \tag{16}$$

The equation (16) says that the derivative of q with respect to ϕ equals q itself. Thus, q is the exponential function. That is, $\mathrm{q} = e^{\phi}$. Since $\phi = i\theta$, then $\mathrm{q} = e^{i\theta}$. With $\mathrm{q} = \mathrm{q}(\theta)$, it follows that

$$\mathrm{q}(\theta) = e^{i\theta} \tag{17}$$

The result (17) is a rotation factor formula for obtaining a rotation factor of a specific angle θ. The formula shows the existence of rotation factors covering all angles and has significant implications.

VI. EXISTENCE OF ROTATION FACTOR SET

It should be noted that in the derivation of the rotation factor formula (17), the rotation angle θ is relative to the real number axis. But, in reality, the angle of a rotation factor in the definition (1) or (10) as well as in (17) is relative to the target's direction and independent of coordinate systems. This relativeness property of a rotation factor is found to be very useful.

The rotation factors can be used to construct coordinate systems. For a polar coordinate system, from (12) and (17), the position number p is expressed as

$$\mathrm{p} = re^{i\theta} \tag{18}$$

This formula (see FIG. 4) means that multiplying a rotation factor of angle θ to a real number r in a real number axis makes the real number point rotate to the point represented by the position number p. In other words, the position number is formed by multiplying a real number with a rotation factor.

All real numbers form a field[18]. The set of real numbers is denoted $\mathbb{R}$[19]. In comparison, rotation factors are a new kind of number. The rotation factor formula (17) is an exponential function and exponential field. That is, all rotation factors of different angles represented by the formula form a field. The set of rotation factors is then given as

$$\mathbb{E} = \{e^{i\theta} \mid \theta \in \mathbb{R}, 0 \leq \theta \leq 2\pi\} \tag{19}$$

It is noted that althrough important and special, the orthogonal rotation factor i is just one member in the set E.

Let $\mathbb{P}$ denote the set of all position numbers. $\mathbb{P}$ represents all points in a plane. A member in $\mathbb{P}$ may be constructed by a member in $\mathbb{R}$ and a member in $\mathbb{E}$. FIG. 5 illustrates their relationship.

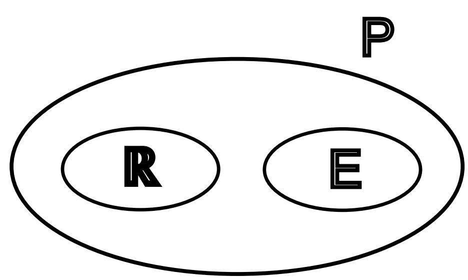

FIG. 5: Real number set $\mathbb{R}$, rotation factor set $\mathbb{E}$ and position number set $\mathbb{P}$

The position stretching and position rotation are two basic motion types in the physical world. From the perspective of describing motion of points in a plane, numbers may be classified into two types: real numbers for position stretching, and rotation factors for position rotation. Multiplying a real number to a position number or position vector stretches the target's magnitude without changing the target's direction or orientation. On the other hand, multiplying a rotation factor to a position number or position vector rotates the target without changing the target's magnitude. The existences of real number set $\mathbb{R}$ and the rotation factor set $\mathbb{E}$ complement each other and complete the representation for the two basic motion types.

VII. ORTHOGONAL ROTATION FACTOR AS A MATHEMATICAL CONSTANT

Next, orthogonal rotation factor i as a mathematical constant is discussed. In general, a mathematical constant is assigned a value. Up to now, the i is still a symbol as $\mathrm{q}(\frac{\pi}{2})$. One may still ask what is $\mathrm{q}(\frac{\pi}{2})$? It is noticed that investigating the definition of the mathematical constant π may provide a hint for the answer.

Since the ratio of a circle's circumference C to its diameter d is a constant, this leads to the definition of π. That is

$$\pi = \frac{\mathrm{C}}{\mathrm{d}} \tag{20}$$

where C and d are variables, but their ratio π is a constant.

By the same token, from (9), the constant i may be defined as

$$i = \frac{\frac{\partial \mathbf{P}}{\partial \theta}}{\mathbf{P}} \tag{21}$$

where the derivative of position vector with respective to rotation angle $\frac{\partial \mathbf{P}}{\partial \theta}$ and the position vector $\mathbf{P}$ are equivalent to variables, but their ratio is a constant. It is the existence of the ratio being constant that causes the existence of the i as well as the existence of the rotation factor set $\mathbb{E}$. It is important to point out that the i is a special non-real number constant and represents an orthogonal rotation as the direction of the derivative term $\frac{\partial \mathbf{P}}{\partial \theta}$ is perpendicular to that of the vector term $\mathbf{P}$.

VIII. CONSTRUCTING CARTESIAN COORDINATE SYSTEM WITH ROTATION FACTORS AND DERIVING EULER'S FORMULA

Now, there exists the rotation factor set $\mathbb{E}$ besides the real number set $\mathbb{R}$. Next, a Cartesian coordinate system is constructed with the two sets by applying rotation factors to real numbers.

In FIG. 6, RE denotes a real number axis. OR denotes the axis orthogonal to RE. Point O is the origin. The point represented by a real number r in the RE axis has the direction of the positive RE axis. Applying a rotation factor of an arbitrary angle θ to number r makes the point rotate by angle θ to the point represented by position number p or $\mathrm{r}e^{i\cdot\theta}$. That is

$$\mathrm{p} = \mathrm{r}e^{i\cdot\theta} \tag{22}$$

The point represented by number a in the RE axis, and the point represented by p, form a line that is parallel to the OR axis. The distance between the origin and the point represented

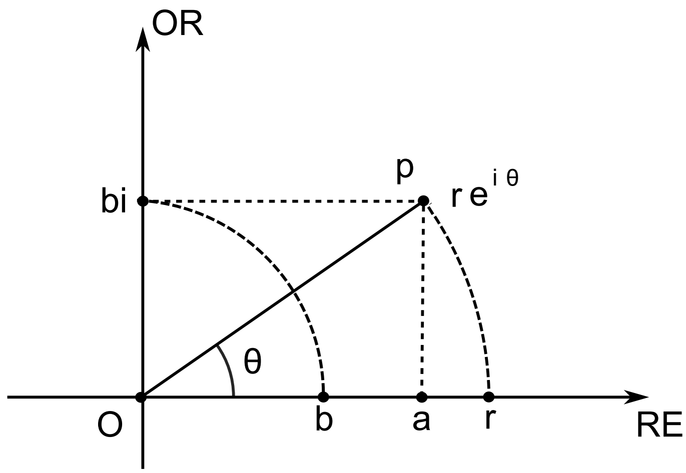

FIG. 6: Cartesian coordinate system formed from real numbers and rotation factors

by number b is the same as the distance between the point represented by p and the point represented by a. Applying the orthogonal rotation factor i to number b makes the point rotate to the point bi in the OR axis. By the rotation factor concept, bi has a direction that is perpendicular to the RE axis and points upward. In the Cartesian coordinate system, the position number p is expressed as

$$\mathrm{p} = \mathrm{a} + \mathrm{b}i \tag{23}$$

In FIG. 6, from the definition of sine and cosine, $\mathrm{a} = \mathrm{r}\cos(\theta)$ and $\mathrm{b} = \mathrm{r}\sin(\theta)$. Thus, (23) becomes $\mathrm{p} = \mathrm{r}\cos(\theta) + i\mathrm{r}\sin(\theta)$, which, together with (22), leads to

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{24}$$

The result (24) is Euler's formula. The new interpretation of the formula is that multiplying rotation factor $e^{i\theta}$ to the unit number 1 makes the number rotate to a position with the component in the real axis direction being $\cos(\theta)$ and the component in the direction orthogonal to the real axis being $i\sin(\theta)$.

The position number p in (23) is equivalent to the definition of a complex number. All complex numbers in the 2D plane form the complex number set $\mathbb{C}$. The orthogonal rotation factor i here is equivalent to the imaginary unit. This shows that we can develop the complex number system in a new way from first principles without even talking about and basing on $i = \sqrt{-1}$ or $i^2 = -1$.

IX. EXTENDING THE APPLICATIONS OF ROTATION FACTORS AND COMPLEX NUMBERS

Rotation factors cause rotations for vectors and complex numbers, and can play active roles in rotation generation and encourage creativity for novel applications. On the other hand, complex numbers have useful properties that deserve broader applications. Next, examples of extending the applications are discussed.

A. Rotating generic vectors in a plane

In addition to rotating position vectors relative to the coordinate origin, rotation factors can also rotate other generic vectors in the plane, as illustrated in Figure 7.

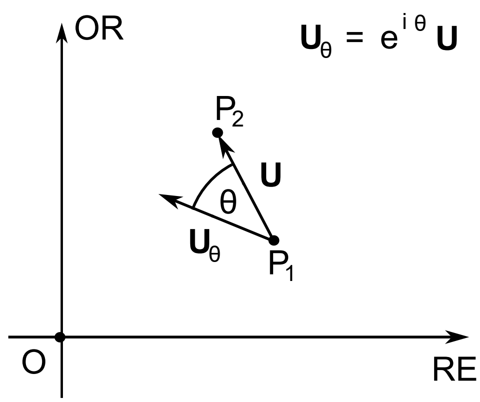

FIG. 7: Generic vector rotated by a rotation factor of angle θ

In the figure, vector $\mathbf{U}$ starts from Point P_1 and ends at Point P_2. With respect to the rotation for vector $\mathbf{U}$ by a rotation factor, it is important to note that by the rotation factor definition (1), the rotation is now relative to the vector starting point P_1, not the origin O. Thus, applying the rotation factor of angle θ to the vector $\mathbf{U}$ makes the vector rotate counter-clockwise by angle θ relative to the starting point P_1, and the resultant vector is $\mathbf{U}_\theta$ with the same magnitude. That is, $\mathbf{U}_\theta = e^{i\theta}\mathbf{U}$.

B. Unified vector algebra

Vectors and complex numbers are similar to each other. Both have directions and magnitudes. To immediately broaden complex numbers' applications, it is helpful to directly link and integrate complex numbers with vectors that are already widely used in physics. Thus, unified vector algebra is conceived to put vectors and complex numbers under one umbrella for mixed algebra manipulation.

The unified vector algebra concept is formed based on the understanding gained from the development from first principles, which helps to view vectors and complex numbers from a new perspective. On one hand, rotation factors are complex numbers and can rotate vectors. In other words, complex numbers can interact with vectors. On the other hand, complex numbers can be considered as vectors with a basis $< 1, i >$ for the vector space (plane).

With referring to Fig. 8, Point P has coordinates (x, y). The point is represented by position vector $\mathbf{P}$ and by complex number p.

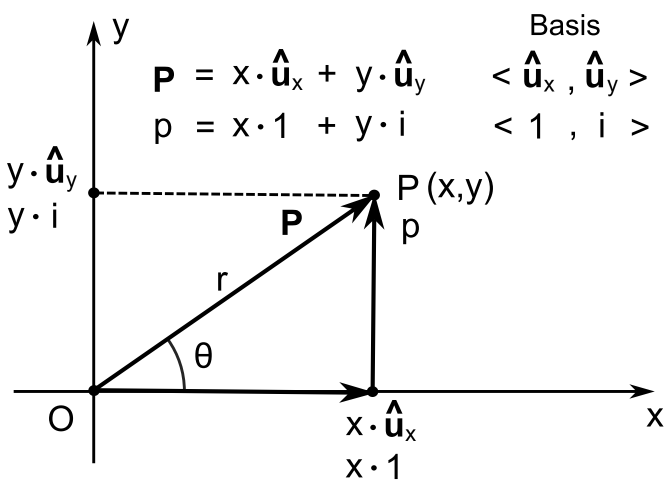

FIG. 8: Interchangeability between vectors and complex numbers in unified vector algebra

For complex numbers with basis $< 1, i >$, the 1 is considered as the unit vector in x-direction and the i as the unit vector in y-direction. For vectors, the basis is $< \hat{\mathbf{u}}_\mathrm{x}, \hat{\mathbf{u}}_\mathrm{y} >$ with $\hat{\mathbf{u}}_\mathrm{x}$ and $\hat{\mathbf{u}}_\mathrm{y}$ being unit vectors in x- an y-direction respectively. From the viewpoint of vectors, the unit vectors 1 and $\hat{\mathbf{u}}_\mathrm{x}$ are equivalent as they point to the same direction; and the same with the unit vectors i and $\hat{\mathbf{u}}_\mathrm{y}$.

In the unified vector algebra, a vector and a complex number are interchangeable with each other if they have the same direction and magnitude. For example, $\mathbf{P}$ and p are interchangeable. Further, under the condition that respective terms multiplied by the unit vectors are real numbers only, the unit vectors $\hat{\mathbf{u}}_\mathrm{x}$ and 1 are interchangeable, and the same with the unit vectors $\hat{\mathbf{u}}_\mathrm{y}$ and i. And further, rotation factors are introduced to rotate vectors as they do to complex numbers. The unified algebra includes addition and multiplication operations. For simplicity, but without losing broad usage, the multiplication for vector is limited to a vector multiplied by rotation factors and real numbers.

The example for the unified vector algebra in (25) with reference to Fig. 8 shows that vector terms and complex number terms are mixed in expressions. Vectors are converted to complex numbers for manipulation, and the result is converted back to a vector.

$$\begin{aligned} 5e^{-i\theta}\mathbf{P} + 3\mathrm{p} + i\mathbf{P} - 2re^{i\theta} &= 5r\hat{\mathbf{u}}_{\mathrm{x}} + 3\mathrm{p} + i\mathrm{p} - 2\mathrm{p} \\ &= 5r + \mathrm{p} + i(x+iy) \\ &= (5r + x - y) + (x+y)i \\ &= (5r + x - y)\hat{\mathbf{u}}_{\mathrm{x}} + (x+y)\hat{\mathbf{u}}_{\mathrm{y}} \end{aligned} \tag{25}$$

C. Example application in Newtonian mechanics

For an object's motion in a plane, Newton's second law of motion is given by

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{P}}{dt^2} = m\frac{d^2\mathrm{p}}{dt^2} \tag{26}$$

where $\mathbf{F}$ is the net force, m the object's mass, and $\mathbf{a}$ the acceleration, which is the second derivative of position vector $\mathbf{P}$ with respect to time t. The unified vector algebra concept can also be applied to the derivative, and position vector $\mathbf{P}$ is replaced with the corresponding position complex number p.

By the same token, the velocity $\mathbf{v}$ can be expressed as

$$\mathbf{v} = \frac{d\mathbf{P}}{dt} = \frac{d\mathrm{p}}{dt} \tag{27}$$

Take circular motion for example. With referring to Fig. 8, $\mathrm{p} = re^{i\theta}$ and r is constant, the above equation becomes

$$\mathbf{v} = ire^{i\theta}\frac{d\theta}{dt} = i\mathrm{p}\omega = i\mathbf{P}\omega = r\omega\hat{\mathbf{n}} \tag{28}$$

where $\omega = \frac{d\theta}{dt}$ is angular velocity. The position number p is replaced back with position vector $\mathbf{P}$ after the manipulation. The term $i\mathbf{P}$ has a magnitude of r and a direction denoted by unit vector $\hat{\mathbf{n}}$ that is perpenticular to $\mathbf{P}$.

D. Three dimensional complex number spherical coordinate system

Based on the set of rotation factors, the constructions of three and higher n-dimensional complex number spherical coordinate systems are realized in a separate paper[20]. Here we only present the three dimensional (3D) result to illustrate the extended application of the rotation factors.

In Figure 9, RE denotes a real number axis. OR_i is an axis that has the direction of orthogonal rotation factor i and is orthogonal to the RE axis. OR_j axis has the direction of another orthogonal rotation factor j and is orthogonal to the plane formed by the RE axis and the OR_i axis.

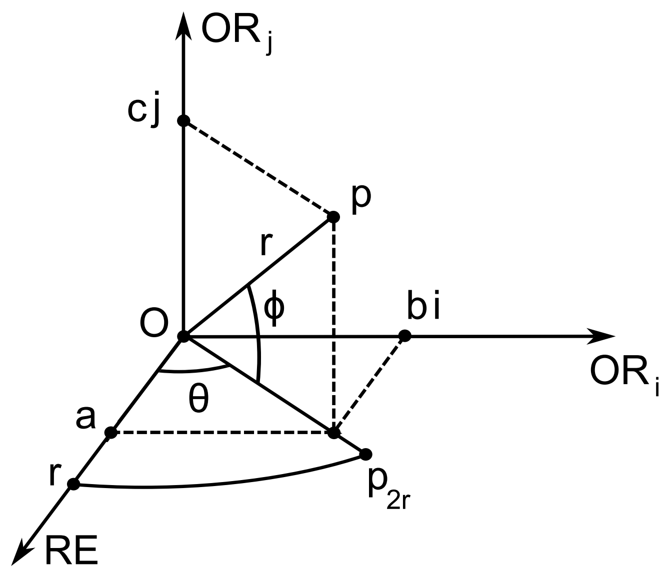

FIG. 9: Construction of three-dimensional spherical and Cartesian coordinate systems by rotation factors

Number r is a real number in the RE axis. Applying an i-associated rotation factor $e^{i\theta}$ to number r makes the point rotate to position number p_{2r}. Applying a j-associated rotation factor $e^{j\phi}$ to position number p_{2r} makes the point rotate to position number p. Thus, we have

$$\mathrm{p} = re^{i\theta}e^{j\phi} \tag{29}$$

Equation (29) gives the formula for three-dimensional position numbers in the spherical coordinate system. The formula is surprisingly succinct, showing that the rotation factors is the “native language” for the system.

By projection, the formula of the transformation from spherical coordinates to Cartesian coordinates is obtained as

$$\mathrm{p} = r(cos(\theta)cos(\phi) + isin(\theta)cos(\phi) + jsin(\phi)) \tag{30}$$

Alternatively, this transformation formula can also be obtained by rotation factor multiplication algebra with the two factors in Equation (29) being decomposed through Euler's formula.

The result here provides the base for further exploring and applying the unified vector algebra concept to 3D applications of complex numbers in physics.

X. CONCLUSION REMARKS

In mathematics and physics, there may be more than one method to solve a problem, derive an equation or develop a system. Each new method may provide a different perspective and enrich the understanding of the subject matter. This turns out to also happen to the complex number system as shown by the present paper.

Developing the complex number system from first principles is achieved with $\sqrt{-1}$ or $i^2 = -1$ being entirely bypassed. A more elementary and thorough way of developing the system rather than just relying on the imaginary unit is provided. The development shows that the complex numbers are not some arbitrary creation we have made, but are actually inherently part of mathematics and logic.

Angle-dependent rotation factors have been introduced to rotate vectors and the equivalents such as complex numbers. A fundamental equation of position vector rotation has been discovered. An orthogonal rotation factor happens to directly appear in the fundamental equation. It is found that the imaginary unit is equivalent to the orthogonal rotation factor and directly connects to the physical reality through rotation. This realness demystifies the imaginary perception, and can increase the trust and encourage complex numbers' wider practical applications.

The existence of the rotation factor set has been found and the formula for the set has been obtained. The Euler's formula has been derived through a new approach of constructing Cartesian coordinate system with rotation factors. The rotation factor set covers all angles for rotation with the imaginary unit being one of its members. The rotation factor set and real number set complement each other, with the former for vector rotation and the latter for vector stretching.

The more basic understanding of rotation factors and complex numbers further helps to extend the applications. The knowledge that rotation factors can rotate vectors provides a new mathematical means for vector rotation algebra. And further, the unified vector algebra is conceived to put vectors and complex numbers under one umbrella for mixed algebra manipulation and interchange so as to seamlessly broaden complex numbers' applications through vectors that are already familiar and widely used in physics.

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