Position numbers in two-, three- and n-dimensions

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Abstract

Angle-dependent rotation factors have been introduced to rotate vectors. Position numbers constructed from rotation factors have been proposed and shown to be multiplication commutative in two and higher dimensions. Based on rotation factors, complex numbers and Euler's formula have been developed in a novel way. In two-dimensions, the position numbers and complex numbers are exhibited to be equivalent. But in three and higher n-dimensions, there are no equivalent hypercomplex numbers for the position numbers.

1. Introduction

Hypercomplex numbers [5] are an extension to higher dimensions from the standard two-dimensional (2D) complex numbers. One established example of hypercomplex numbers is the quaternion number system by Hamilton [3], which is in four dimensions and non-commutative. There exist other hypercomplex systems [6]. In general, there are restrictions with respect to commutative and associative properties of hypernumbers in higher dimensions. Ferdinand Georg Frobenius [1, 2] proved that for a division algebra over the reals to be finite-dimensional and associative, it cannot be three-dimensional, and there are only three such division algebras: real numbers, complex numbers, and quaternions, which have dimension 1, 2, and 4 respectively. The consequence is that multiplication in Cartesian coordinates is valid for 1D, 2D and 4D, but not 3D. Thus far, there have been no spherical coordinate systems for hypercomplex numbers.

2. Two-dimensions

It is observed that in the convention, a complex number is defined in a Cartesian coordinate system as the addition of a real number (real-axis component) and an imaginary number (imaginary-axis component). As the extension of the 2D complex numbers, a higher dimensional hypercomplex number is also defined in a Cartesian coordinate system as the addition of components of respective axes.

By the same fashion, for two-dimensions, a new number system, which is called position number, will be defined in a polar (2D spherical) coordinate system. As the extension of the 2D position numbers, a higher dimensional position number is defined in a spherical coordinate system as well. We will then examine the similarities or differences between the new position numbers and the conventional complex / hypercomplex numbers. The position numbers in different dimensions are constructed from rotation factors that will be introduced next.

2.1. Rotation factors

Nature has its laws and properties. Mathematics is invented to discover and describe Nature. A point in a two-dimensional plane may be described by a position vector. The multiplication by a real number to the vector causes the position stretching, which represents a linear motion. On the other hand, in real life, there is another basic motion type: rotational motion. There likely also exists a number system for describing the rotation.

Real numbers can only make the position vector stretch by multiplication. By analogy, rotational numbers are introduced to make the vector rotate. Rotational numbers may also be called rotation factors in that they all have the same unit length (magnitude) and act as rotating coefficients of directional physical quantities such as position vector.

An angle-dependent rotation factor q of an arbitrary angle δ is defined as a factor for multiplying to a position vector of a point and making the vector rotate counter-clockwise by angle δ relative to the origin point of the vector with the vector's magnitude unchanged. That is

$$\mathbf{P}_\delta = \mathrm{q} \cdot \mathbf{P} = \mathbf{P} \cdot \mathrm{q} \tag{1}$$

where (see the figure) position vector $\mathbf{P}$ of point P rotates relative to origin point O by angle δ and reaches position vector $\mathbf{P}_\delta$ of point P_δ.

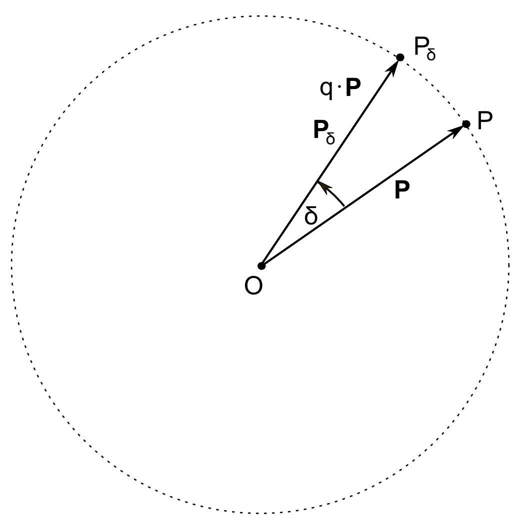

FIGURE 1. Rotation factor and target vector's direction change

Here, it is convenient to introduce the following terminologies:

Applying a rotation factor to: means multiplying a rotation factor to a position vector. Applying a rotation factor to a point means applying the factor to the point's position vector.

Target of a rotation factor: means the position vector to which a rotation factor is applied.

At this point, the existence of rotation factors such defined is a postulate. If the consequences of the postulate are consistent with existing results, it is then considered true.

Denote a rotation factor q of an arbitrary angle δ as $\mathrm{q}(\delta)$. The following basic properties of rotation factors are implied or derived from the definition (1).

Rotation factors are multiplication commutative. The order of rotation factors for multiplying to a vector will not affect the final result.

Division operation of rotation factor is allowed. The multiplication effect produced by a rotation factor may be inversed through division by the same rotation factor. This means $\mathrm{q} \cdot \frac{1}{\mathrm{q}} = 1$.

In general, a rotation factor is not a real number as the former can change a vector's direction by multiplication and the latter cannot. But the two angles 0 and π are the exceptions. The rotation factor of angle 0 produces no rotation for its target vector and becomes the real number 1. And the rotation factor of angle π produces a reverse direction for its target vector by rotating 180° and becomes the real number -1. That is, $\mathrm{q}(0) = 1$ and $\mathrm{q}(\pi) = -1$.

The rotation factor of angle $\frac{\pi}{2}$ is defined as orthogonal rotation factor in that it produces a 90° rotation for the target vector with the resultant direction being perpendicular or orthogonal to the initial direction. Notationwise, the orthogonal rotation factor is denoted as $\mathrm{q}(\frac{\pi}{2})$, which may be further succinctly denoted by a symbol i. That is

$$i = \mathrm{q}(\frac{\pi}{2}) \tag{2}$$

Applying the orthogonal rotation factor twice to a vector $\mathbf{P}$ produces a 180° rotation and reverses the vector's direction. That is, $i \cdot i \cdot \mathbf{P} = -\mathbf{P}$ or

$$i^2 = -1 \tag{3}$$

The result (3) is a property of the orthogonal rotation factor.

2.2. Fundamental equation of position vector rotation

Since a rotation factor is involved in a position vector's rotation, it is natural to study the rotation motion of the position vector.

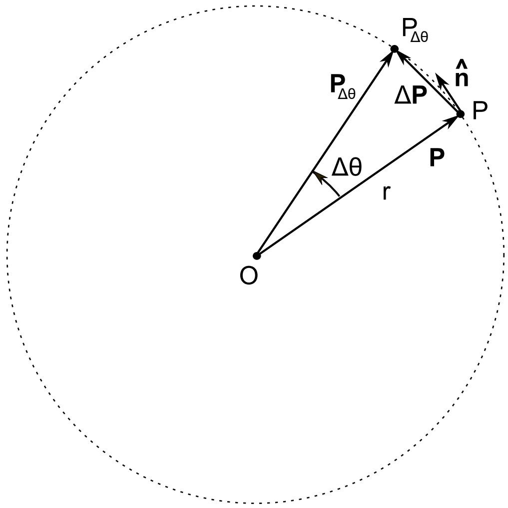

FIGURE 2. Position vector rotation relative to an origin point

In Figure 2, point P is represented by position vector $\mathbf{P}$ with magnitude r. $\hat{\mathbf{n}}$ is a unit vector perpendicular to the position vector $\mathbf{P}$. Point P rotates around origin point O by angle $\Delta\theta$ with the vector's magnitude r unchanged, and reaches point $P_{\Delta\theta}$ whose position vector is $\mathbf{P}_{\Delta\theta}$. $\mathbf{\Delta P}$ is the vector change, which is the difference between vector $\mathbf{P}_{\Delta\theta}$ and vector $\mathbf{P}$.

Since the vector's magnitude r is unchanged, point P moves along the circumference of the circle during the rotation. Let Δs be the length of the arc corresponding to the angle change $\Delta\theta$, and Δl be the length of the chord of the arc. The Δl is also the magnitude of the vector change $\mathbf{\Delta P}$. Denote $\hat{\mathbf{u}}$ as the unit vector of $\mathbf{\Delta P}$. Then

$$\mathbf{\Delta P} = \|\mathbf{\Delta P}\|\,\hat{\mathbf{u}} = \Delta l \hat{\mathbf{u}}$$

For a circle, the ratio of the arc length to the arc angle span ($\Delta\theta$) is always equal to the radius regardless of the value of the angle span. That is, $\frac{\Delta s}{\Delta\theta} = r$. We then have

$$\frac{\mathbf{\Delta P}}{\Delta\theta} = (\frac{\Delta \mathrm{l}}{\Delta s}) r \hat{\mathbf{u}}$$

As $\Delta\theta \to 0$, $(\frac{\Delta \mathrm{l}}{\Delta s}) \to 1$ and $\hat{\mathbf{u}} \to \hat{\mathbf{n}}$, the limit of the equation above for the derivative with respect to θ leads to

$$\frac{\partial\mathbf{P}}{\partial\theta} = r\hat{\mathbf{n}}$$

It is noted that multiplying the orthogonal rotation factor i to the vector $\mathbf{P}$ makes the vector perform an orthogonal rotation with the vector's magnitude r unchanged. In other words, the resultant vector has the magnitude r and the direction $\hat{\mathbf{n}}$. Thus, $i\mathbf{P} = r\hat{\mathbf{n}}$. It follows that

$$\frac{\partial\mathbf{P}}{\partial\theta} = i\mathbf{P} \tag{4}$$

The result (4) serves as a fundamental equation of position vector rotation. The orthogonal rotation factor i happens to appear in this fundamental equation.

2.3. Position numbers

Referring to Figure 3, in a plane, real number r represents point P_r in the x-axis, and has an implied direction, which is the same as that of the positive x-axis. Therefore, the real number r in the x-axis can be treated as a position vector, and rotated by a rotation factor.

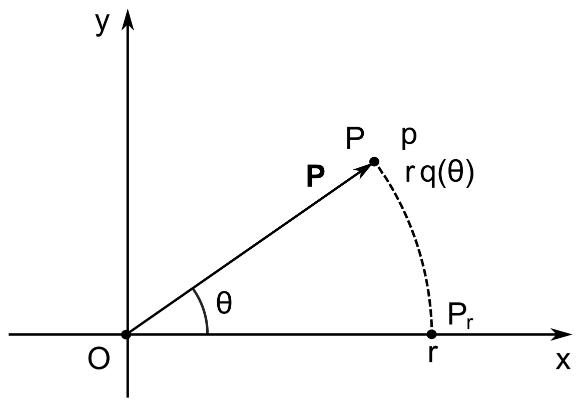

FIGURE 3. Position number

Multiplying the rotation factor $q = q(\theta)$ of angle θ to the real number r makes its implied direction rotate by angle θ with magnitude r unchanged. As a result, point P_r is rotated to point P, which is represented by an entity p. That is

$$p = rq(\theta) \tag{5}$$

The same point P is represented by position vector **P** as well.

As mentioned before, a rotation factor is a new type of number. Thus, the entity p as the product of a real number and a rotation factor is also a new type of number. Since point P is represented with a position vector, it is natural and lucid to use the counterpart term, position number as the naming for the entity p defined in (5).

Let's compare position numbers and position vectors. As illustrated in the figure, position vector **P** has an explicit direction with an arrow pointing to the direction of the vector. Position number p represents the same point P and also has the same direction (i.e., from the origin point O to the point P). For the position number, the direction is implied without the explicit direction arrow. Both have the same direction and magnitude r. Previously, a rotation factor of an arbitrary angle is applied to a position vector, which has a direction. In fact, the same rotation factor concept can also be applied to a position number. The rotation factor will generate the same rotation for the position number regardless of the target's direction being explicit or implied.

Since a position number and its corresponding position vector have the same direction and magnitude, using the same methodology and Figure 2 with position vector notation **P** replaced by position number p, gives the fundamental equation of position number rotation as

$$\frac{\partial \mathrm{p}}{\partial \theta} = i\mathrm{p} \tag{6}$$

2.4. Angle dependence formula of rotation factors

To solve the equation (6), the dependence of the position number p on the rotation angle θ is needed, and has been provided by the position number definition $\mathrm{p} = \mathrm{rq}(\theta)$. With the dependence, the equation (6) becomes

$$\frac{\partial(r\mathrm{q})}{\partial \theta} = ir\mathrm{q}$$

where $\mathrm{q} = \mathrm{q}(\theta)$.

Since q depends on θ only and r is considered as constant with respect to change in θ, it follows that $\frac{\partial(r\mathrm{q})}{\partial \theta} = r\frac{\partial \mathrm{q}}{\partial \theta} = r\frac{\mathrm{dq}}{d\theta}$. Thus, we have

$$\frac{\mathrm{dq}}{d\theta} = i\mathrm{q}$$

From the previous discussion, division is allowed for rotation factor including the orthogonal rotation factor i. That is, $\frac{\mathrm{dq}}{d(i\theta)} = \mathrm{q}$. Let $\phi = i\theta$. We have $\frac{\mathrm{dq}}{d\phi} = \mathrm{q}$ with the exponential function solution $\mathrm{q} = e^{\phi}$. This leads to

$$\mathrm{q}(\theta) = e^{i\theta} \tag{7}$$

The result (7) may be called rotation factor formula, which gives the angle dependence of rotation factors.

The rotation factor formula reveals the existence of rotation factor set. All rotation factors of different angles represented by the formula form a field. The set of rotation factors $\mathbb{E}$ is expressed as $\mathbb{E} = \{e^{i\theta} \mid \theta \in \mathbb{R}, -\pi < \theta \leq \pi\}$. The position number p defined in (5) is now expressed as $\mathrm{p} = re^{i\theta}$. Let $\mathbb{R}_{\geq 0}$ denote the set of zero and positive real numbers, and $\mathbb{P}$ the set of all position numbers. $\mathbb{P}$ represents all points in a plane. A member in $\mathbb{P}$ may be constructed by a member in $\mathbb{R}_{\geq 0}$ and a member in $\mathbb{E}$. The real number set and the rotation factor set complement each other and complete the representation for the whole plane. The existence of the rotation factor set with wonderful properties and characteristics is a great gift from Nature.

2.5. Developing complex numbers and Euler's formula based on rotation factors

With reference to Figure 4, in a plane, real numbers a, b, and r represent points in the x-axis.

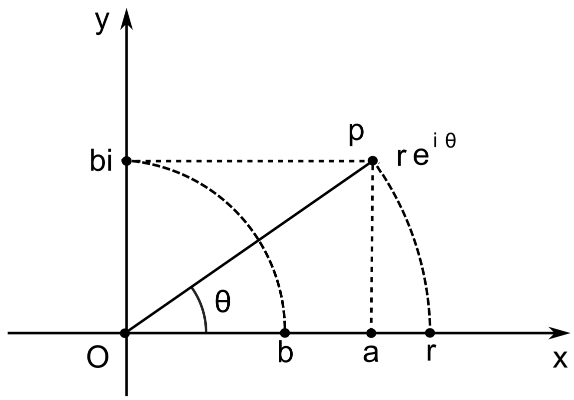

FIGURE 4. Cartesian coordinate system formed from real numbers and rotation factors

Applying a rotation factor of an arbitrary angle θ to number r makes the point rotate by angle θ to the point represented by position number p or $\mathrm{r}e^{i\theta}$. That is

$$\mathrm{p} = \mathrm{r}e^{i\theta} \tag{8}$$

The line through points a and p is parallel to the y-axis. The distance between the points O and b is the same as the distance between the points a and p. Applying the orthogonal rotation factor i to number b makes the point rotate to the point bi in the y-axis. In the Cartesian coordinate system, the position number p is expressed as

$$\mathrm{p} = a + bi \tag{9}$$

where the terms a and bi are orthogonal to each other, with the former direction pointing to the x-axis unit vector times the sign of a and the latter direction to the y-axis unit vector times the sign of b.

From the definition of sine and cosine, $a = \mathrm{r}\cos(\theta)$ and $b = \mathrm{r}\sin(\theta)$. Thus, (9) becomes $\mathrm{p} = \mathrm{r}\cos(\theta) + i\mathrm{r}\sin(\theta)$, which, together with (8), leads to Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{10}$$

The position number p expressed in Cartesian coordinates by (9) is equivalent to the complex number. The orthogonal rotation factor i here is equivalent to the imaginary unit. This shows that we can develop the complex number system including Euler's formula in a new way based on rotation factors without even using and relying on $i = \sqrt{-1}$ or $i^2 = -1$.

In two-dimensions, the position numbers are equivalent to the complex numbers. However, in other dimensions, position numbers constructed from rotation factors are not equivalent to conventional hypercomplex numbers.

2.6. Causality origin of imaginary unit

It is noted that from a first principle perspective, $\sqrt{-1}$ is not based on a thing that is already known. A first principle is a fundamental assumption that cannot be further deduced. Ancient Greek philosopher Aristotle (384 - 322 BCE) [4] expressed a first principle as "the first basis from which a thing is known." It is no wonder that many learners are unsatisfied with $\sqrt{-1}$ that originates from nowhere (the unknown).

Up to now, the orthogonal rotation factor i is still a symbol as $\mathrm{q}(\frac{\pi}{2})$ defined in (2). One may still ask what is $\mathrm{q}(\frac{\pi}{2})$? To answer this question, it is noticed that $\mathrm{q}(\frac{\pi}{2})$ is a constant that is denoted by i. And π is also a constant. From the causality viewpoint, investigating the cause of the π may provide a hint for the cause of the i.

Since the ratio of a circle's circumference C to its diameter d is a constant, this leads to the definition of π. That is

$$\pi = \frac{\mathrm{C}}{\mathrm{d}} \tag{11}$$

where C and d are variables, but their ratio π is a constant.

By the same token, from the fundamental equation of position vector rotation, the constant i may be defined as

$$i = \frac{\frac{\partial \mathbf{P}}{\partial \theta}}{\mathbf{P}} \tag{12}$$

where the derivative of position vector with respective to rotation angle $\frac{\partial \mathbf{P}}{\partial \theta}$ and the position vector $\mathbf{P}$ are equivalent to variables, but their ratio is a constant. It is the existence of the ratio being constant that causes the existence of the i and the existence of the rotation factor formula $\mathrm{q}(\theta) = e^{i\theta}$, which indicates the existence of the rotation factor set. It is important to point out that the term $\frac{\partial \mathbf{P}}{\partial \theta}$ and the term $\mathbf{P}$ have their respective directions, which are perpendicular to each other. As a result, their ratio i is a special non-real number constant and represents an orthogonal rotation.

From the perspective of dimensions, there is also a similarity between π and i. In (11) for π, the diameter d is a one-dimension line, and the circumference C is a two-dimension circle. The constant π is related to the ratio of a variable in two-dimensions to a variable in one-dimension. And in (12) for i, the $\mathbf{P}$ itself is in a one-dimension line. The rotation motion is represented by the derivative $\frac{\partial \mathbf{P}}{\partial \theta}$, which is in two-dimensions relative to the $\mathbf{P}$. Thus, the constant i is also related to the ratio of a variable in two-dimensions to a variable in one-dimension.

Both π and i are important mathematical constants. It is amazing to see that both share the similarities and connections through circle and rotation. Here we may feel in awe about the inherent elegance and magic of Nature exhibited through mathematics.

3. Three-dimensions

3.1. Position numbers

The construction of the 3D spherical coordinate system is shown in Figure 5. In our mind, real number 1 in x-axis is considered as a unit vector pointing to the positive x-axis direction. Applying orthogonal rotation factor i to unit vector 1 makes the vector rotate to unit vector i, which points to the positive y-axis direction. j is another orthogonal rotation factor. Applying the orthogonal rotation factor j to unit vector 1 makes the vector rotate to unit vector j, which points to the positive z-axis direction.

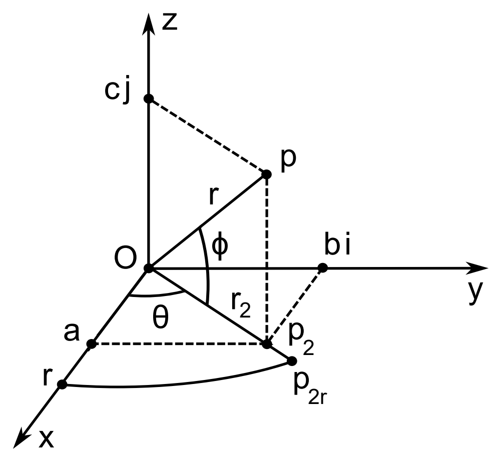

FIGURE 5. Construction of 3D coordinates by rotation factors

p_{2r} is a position number representing a point in the second dimension plane that is formed by the x-axis and y-axis. θ is the angle between the position number p_{2r} and the x-axis. r is a real number in the x-axis. The i-associated rotation factor $e^{i\theta}$ can be used to rotate the number r and make the number rotate to p_{2r}. Thus, $p_{2r} = re^{i\theta}$. Applying a j-associated rotation factor $e^{j\phi}$ to position number p_{2r} makes the point rotate by angle ϕ to position number p in the plane p_{2r}-O-z. That is, $p = p_{2r}e^{j\phi}$. With this and the expression for p_{2r}, the position number p becomes

$$p = re^{i\theta}e^{j\phi} \tag{13}$$

This formula can be used as the definition of position numbers in three-dimensions. $e^{i\theta}$ is the rotation factor for the second dimension plane, and $e^{j\phi}$ is the rotation factor for the third dimension in the j or z-axis direction. Such defined 3D position numbers are new. Unlike 2D, there are no equivalent hypercomplex numbers for the 3D position numbers.

3.2. Spherical-to-Cartesian transformation

Referring to Figure 5, the point represented by position number p_2 is in the plane x-O-y. The line through points p_2 and p is parallel to the z-axis. The position number p_2 has a magnitude of r_2. Like i and j, the unit number 1 may also be considered as a unit vector in the direction of the positive x-axis. a, b, and c are real numbers. a, bi, and cj represent, respectively, the components of position number p in the x-, y- and z-directions. In the Cartesian coordinate system, the position number p is expressed as

$$\mathrm{p} = \mathrm{a} + \mathrm{b}i + \mathrm{c}j$$

The geometry gives

$$\mathrm{a} = rcos(\theta)cos(\phi)$$

$$\mathrm{b} = rsin(\theta)cos(\phi)$$

$$\mathrm{c} = rsin(\phi)$$

It follows that

$$\mathrm{p} = r\left(cos(\theta)cos(\phi) + isin(\theta)cos(\phi) + jsin(\phi)\right) \tag{14}$$

This is the spherical-to-Cartesian coordinate transformation formula.

Likewise, from the geometry or from the above equations, the transformation from Cartesian to spherical coordinates is expressed by

$$r = \sqrt{a^2 + b^2 + c^2}$$

$$\theta = tan^{-1}(\frac{\mathrm{b}}{\mathrm{a}})$$

$$\phi = sin^{-1}(\frac{\mathrm{c}}{\mathrm{r}})$$

3.3. Commutativity of rotation factors

In the previous Figure 5, the i-associated rotation factor $e^{i\theta}$ is first applied to the number r in the x-axis, and then is followed by the j-associated rotation factor $e^{j\phi}$.

The next Figure 6 is the same as Figure 5, except that the order of the operations by the two rotation factors is reversed.

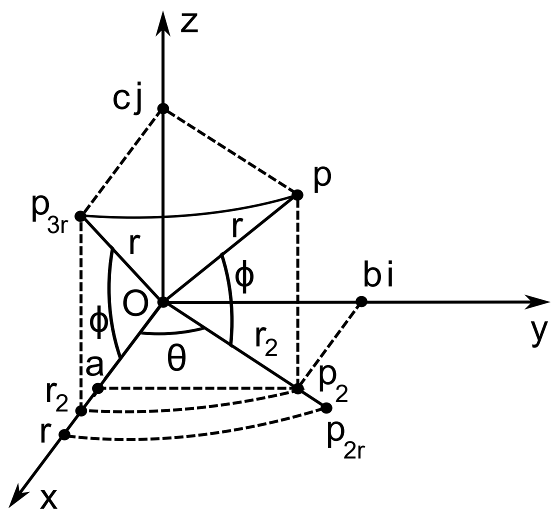

FIGURE 6. Illustration for commutativity of rotation factors

Position number p_{3r} is in the plane x-O-z. The line through p_{3r} and r_2 in the x-axis is parallel to the z-axis. Applying the rotation factor $e^{j\phi}$ to number r in the x-axis makes the point rotate by angle ϕ to position number p_{3r}. And then applying the rotation factor $e^{i\theta}$ to position number p_{3r} makes the point rotate by angle θ to position number p in the plane that is parallel to the plane x-O-y. The final position is the point p, which is the same as the point in Figure 5. That is

$$\mathrm{p} = re^{i\theta}e^{j\phi} = re^{j\phi}e^{i\theta}$$

This result represents the commutativity of the rotation factors.

3.4. Interaction between orthogonal rotation factors i and j

The orthogonal rotation factors i and j can be considered as unit vectors. On the other hand, they are rotation factors that can rotate vectors. To specify the rotation target, we here define that the former is rotated by the latter. For example, ij means that i is rotated by j. It should be noted that we may also define the other way around. It will not change the result.

With reference to Figure 5, for ij, i is considered as a unit vector pointing to the positive y-axis direction, and is rotated by orthogonal rotation factor j with the resultant unit vector pointing to the j direction or the positive z-axis direction. That is, ij equals j. And conversely, for ji, j is considered as a unit vector pointing to the z-axis direction, and is rotated by orthogonal rotation factor i. But, the rotation generated by i is in the direction parallel to the plane x-O-y, and cannot make j rotate. That is, ji equals j. We then have this identity:

$$ij = ji = j$$

More generally, any point in the line O-p_{2r} is also rotated to the z-axis by rotation factor j. That is, $e^{i\theta}j = j$. Also, rotation factor $e^{i\theta}$ makes its target vector rotate in the direction parallel to the plane x-O-y, and cannot rotate unit vector j. Thus, $je^{i\theta} = j$. And we have the following identity:

$$e^{i\theta}j = je^{i\theta} = j \tag{15}$$

In fact, the same result can also be obtained by the transformation formula (14). In the spherical coordinate system, rotation factors are commutative. For position number $e^{i\theta}j$ or $je^{i\theta}$, the magnitude r is 1, and the angle ϕ is $\frac{\pi}{2}$. The resultant position number p from the formula is j. That is, we have the identity (15).

It is interesting to note that the transformation formula (14) can also be obtained by rotation factor multiplication algebra, as shown below. With Euler's formula for the rotation factor $e^{j\phi}$ in (13), it follows that

$$\mathrm{p} = r(e^{i\theta}cos(\phi) + e^{i\theta}jsin(\phi))$$

And with (15) and Euler's formula for $e^{i\theta}$, the above equation becomes the transformation formula (14) previously obtained by geometry.

3.5. Multiplication of position numbers

In spherical coordinate system represented by (13), let $\mathrm{p}_1 = \mathrm{r}_1 e^{i\theta_1}e^{j\phi_1}$ be one position number, and $\mathrm{p}_2 = \mathrm{r}_2 e^{i\theta_2}e^{j\phi_2}$ be another. Since rotation factors are commutative, the two position numbers are also commutative. The multiplication for the two position numbers is

$$\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{p}_2 \cdot \mathrm{p}_1 = \mathrm{r}_1\mathrm{r}_2 e^{i(\theta_1+\theta_2)}e^{j(\phi_1+\phi_2)}$$

The result is a new position number in spherical coordinates, and can be transformed by the formula (14) to Cartesian coordinates as

$$\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{r}_1\mathrm{r}_2(cos(\theta_1+\theta_2)cos(\phi_1+\phi_2) + isin(\theta_1+\theta_2)cos(\phi_1+\phi_2) + jsin(\phi_1+\phi_2)) \tag{16}$$

This formula may be used to circumvent the invalidity of the multiplication in the 3D Cartesian coordinate system. Assume that we have two position numbers, p_1 and p_2 in Cartesian coordinates as $\mathrm{p}_1 = \mathrm{a}_1 + \mathrm{b}_1 i + \mathrm{c}_1 j$, and $\mathrm{p}_2 = \mathrm{a}_2 + \mathrm{b}_2 i + \mathrm{c}_2 j$. For the multiplication of $\mathrm{p}_1 \cdot \mathrm{p}_2$, p_1 and p_2 can be first transformed into spherical coordinates as $\mathrm{p}_1 = \mathrm{r}_1 e^{i\theta_1}e^{j\phi_1}$, and $\mathrm{p}_2 = \mathrm{r}_2 e^{i\theta_2}e^{j\phi_2}$. The multiplication is then performed in the spherical system, and the result for Cartesian coordinates is given by (16).

4. N-dimensions

Before extending results obtained from a three dimensional space to a higher n-dimensional space, one may need to understand the projections from an n-dimensional space to a lower dimensional space. From here on, vectors with bold arrows denote the projections; subscript k (k=1,2,3...n) represents k-dimensional or k-th dimension; and i_k is an orthogonal rotation factor for axis x_k and k-th dimension.

4.1. Projections for n-dimensions

Three-dimensional (3D) space

The projections of position vectors in a 3D space are illustrated in the following Figure 7. Position number p_2 is in the second-dimension plane x_1-O-x_2. The projection of p_2 to the lower first-dimension is the vector corresponding to the component a_1. And the projection to the second-dimension axis x_2 is the component $a_2 i_2$. That is, $p_2 = a_1 + a_2 i_2$.

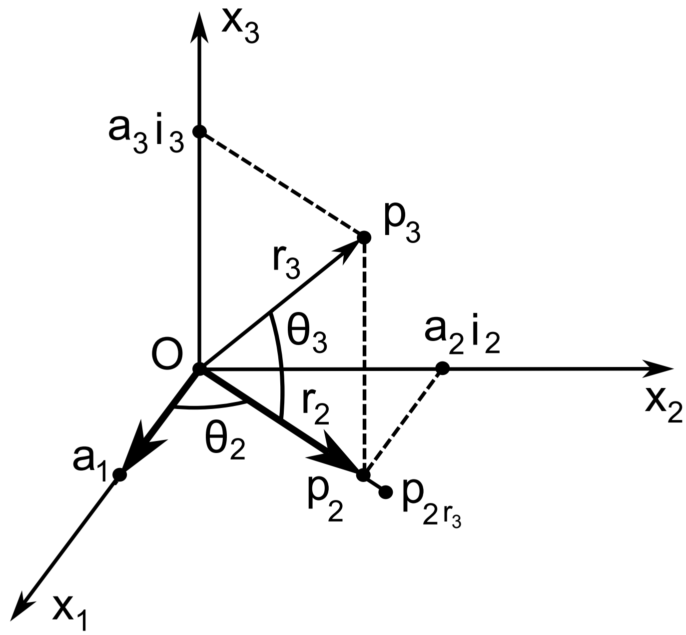

FIGURE 7. Projections of position vectors in a 3D space

Position number p_{2r_3} is in the same plane as p_2 and also in the same direction. p_{2r_3} has a magnitude of r_3. Applying an i_3-associated rotation factor $e^{i_3\theta_3}$ of angle θ_3 to p_{2r_3} makes the point rotate to position number p_3. The projection of p_3 to the lower second-dimension is p_2. And the projection to the thrid-dimension axis x_3 is the component $a_3 i_3$. The rotation and projections in the figure lead to

$$p_3 = r_3 e^{i_2\theta_2} e^{i_3\theta_3} \tag{17}$$

$$p_3 = p_2 + a_3 i_3 \tag{18}$$

$$r_2 = r_3 cos(\theta_3) \tag{19}$$

$$a_3 = r_3 sin(\theta_3) \tag{20}$$

The 3D equation set of (17)-(20) represents spherical and Cartesian coordinate systems in a 3D space.

Four-dimensional (4D) space

Compared with the 3D space in Figure 7, one more orthogonal axis x_4 is added in Figure 8. It is noted that the figure is viewed from the 4D perspective. Logically, if the 4D space occupies our 3D space, which is one dimension lower than the 4D, the 3D space must appear as a 2D plane from the 4D viewpoint. In the 4D space, the x_4 axis is orthogonal to the plane x_2-O-x_3 in the lower dimensions.

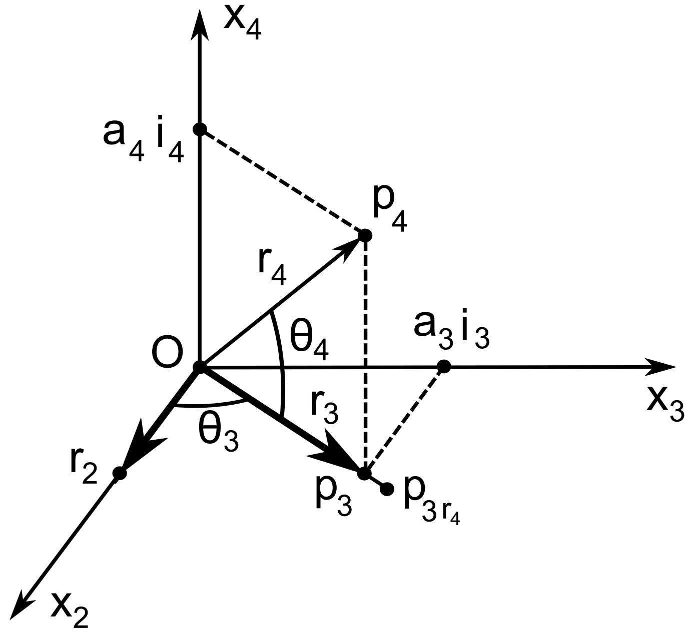

FIGURE 8. Projections of position vectors in a 4D space

Position number p_{3r_4} is in the same plane as p_3 and also in the same direction. p_{3r_4} has a magnitude of r_4. Therefore, $p_{3r_4}/p_3 = r_4/r_3$ or $p_{3r_4} = r_4 p_3/r_3 = r_4 e^{i_2\theta_2} e^{i_3\theta_3}$. Applying an i_4-associated rotation factor $e^{i_4\theta_4}$ of angle θ_4 to p_{3r_4} makes the point rotate to position number p_4. Thus, $p_4 = r_4 e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4}$. The r_4 is the magnitude of p_4. The position vector represented by p_3 in Figure 7 now becomes a projection in the current 4D space. The projection of p_4 to the lower 3D dimension is the vector corresponding to p_3. The rotation and projections lead to the corresponding 4D equation set below

$$p_4 = r_4 e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4} \tag{21}$$

$$p_4 = p_3 + a_4 i_4 \tag{22}$$

$$r_3 = r_4 cos(\theta_4) \tag{23}$$

$$a_4 = r_4 sin(\theta_4) \tag{24}$$

where p_3 is given by the 3D equation set (17)-(20).

n-dimensional (n-D) space

By the same token, Figure 9 shows the projections in an n-dimensional space. From the n-dimensional space perspective, the n-1 dimensional space appears as a plane. The $\mathrm{x_n}$ axis is orthogonal to the plane $\mathrm{x_{n-2}}$-O- $\mathrm{x_{n-1}}$ in the lower dimensions.

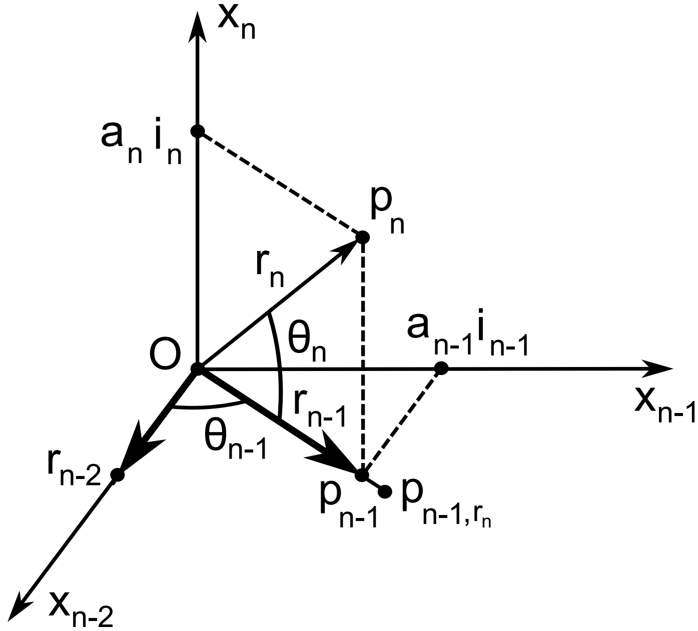

FIGURE 9. Projections of position vectors in an n-D space

Position number $\mathrm{p_{n-1,r_n}}$ is in the same plane as $\mathrm{p_{n-1}}$ and also in the same direction. $\mathrm{p_{n-1,r_n}}$ has a magnitude of $\mathrm{r_n}$. Applying an i_n-associated rotation factor $e^{i_n\theta_n}$ of angle θ_n to $\mathrm{p_{n-1,r_n}}$ makes the point rotate to $\mathrm{p_n}$. The projection of $\mathrm{p_n}$ to the lower (n-1)th dimension is $\mathrm{p_{n-1}}$. The above rotation and projections lead to

$$\mathrm{p_n} = \mathrm{r_n} e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4} ... e^{i_n\theta_n} \tag{25}$$

$$\mathrm{p_n} = \mathrm{p_{n-1}} + \mathrm{a_n i_n} \tag{26}$$

$$\mathrm{r_{n-1}} = \mathrm{r_n} cos(\theta_n) \tag{27}$$

$$\mathrm{a_n} = \mathrm{r_n} sin(\theta_n) \tag{28}$$

The n-dimensional equation set of (25)-(28) represents spherical and Cartesian coordinate systems in an n-dimensional space, with $\mathrm{p_{n-1}}$ for the lower dimension given by the corresponding n-1 dimensional equation set.

4.2. Generalization for n-D position numbers

Position numbers

For the generalization of the coordinate system construction, first in the 2D plane, the 2nd dimension axis $\mathrm{x_2}$ is orthogonal to the 1st dimension axis $\mathrm{x_1}$. The associated rotation factor for the $\mathrm{x_2}$ axis is $e^{i_2\theta_2}$. Then, in the 3D space, the 3rd dimension axis $\mathrm{x_3}$ is introduced to be orthogonal to the plane $\mathrm{x_1}$-O-$\mathrm{x_2}$. The associated rotation factor for the $\mathrm{x_3}$ axis is $e^{i_3\theta_3}$. Similarly, in the n-dimensional space, the n-th dimension axis $\mathrm{x_n}$ is introduced to be orthogonal to the plane $\mathrm{x_{n-2}}$-O-$\mathrm{x_{n-1}}$. The associated rotation factor for the $\mathrm{x_n}$ axis is $e^{i_n\theta_n}$. With (25), the position number in an n-dimensional spherical coordinate system is given by

$$\mathrm{p_n} = \mathrm{r_n} \prod_{j=2}^{n} e^{i_j\theta_j} \tag{29}$$

where $\mathrm{r_n}$ is the magnitude of the position number and $e^{i_j\theta_j}$ is the j-th dimension associated rotation factor. This formula can be used as the definition of position numbers in n-dimensions.

Spherical-to-Cartesian transformation

From equation set (18)-(20) for 3D and (22)-(24) for 4D, it follows that

$$\mathrm{p_3} = \mathrm{r_3}(cos(\theta_2)cos(\theta_3) + i_2sin(\theta_2)cos(\theta_3) + i_3sin(\theta_3))$$

and

$$\begin{aligned}\mathrm{p_4} = \mathrm{r_4}(&cos(\theta_2)cos(\theta_3)cos(\theta_4)\\ &+ i_2sin(\theta_2)cos(\theta_3)cos(\theta_4)\\ &+ i_3sin(\theta_3)cos(\theta_4) + i_4sin(\theta_4))\end{aligned}$$

Along with (26)-(28) for downward iterations, the generalization of n-dimensional position number transformation from spherical coordinates to Cartesian coordinates is given by

$$\mathrm{p_n} = \sum_{j=1}^{n} a_j i_j \tag{30}$$

where $a_j = r_n E_j$ and

$$E_j = sin(\theta_j) \prod_{k=j+1}^{n} cos(\theta_k)$$

with $i_1 = 1$, $sin(\theta_1) = 1$, and $\prod_{k=j+1}^{n} cos(\theta_k) = 1$ if $j + 1 > n$.

Cartesian-to-spherical transformation

Also from the equation sets (18)-(20), (22)-(24) and (26)-(28), the generalization of n-dimensional position number transformation from Cartesian coordinates to spherical coordinates is given by

$$\mathrm{r_n} = \sqrt{\sum_{j=1}^{n} a_j^2} \tag{31}$$

for the magnitude and

$$(\theta_2, \theta_3, \theta_4, ..., \theta_\mathrm{n}) \tag{32}$$

for (n-1)-tuple of angles with

$$\theta_\mathrm{j} = sin^{-1}(\frac{a_j}{\mathrm{r_j}}) \tag{33}$$

where θ_j is the rotation angle for the j-th dimension with the corresponding magnitude

$$\mathrm{r_j} = \sqrt{\sum_{k=1}^{j} a_k^2} \tag{34}$$

4.3. Generalization of rotation factor algebra for n-dimensions

The rotation factor multiplication algebra involves the interactions of rotation factors and orthogonal rotation factors across different dimensions.

3D space

For the 3D space, with notations in Figure 7, the identity (15) becomes

$$e^{i_2\theta_2} i_3 = i_3 e^{i_2\theta_2} = i_3 \tag{35}$$

And for $\theta_2 = \frac{\pi}{2}$, the above identity becomes

$$i_2 i_3 = i_3 i_2 = i_3 \tag{36}$$

This indicates that the orthogonal rotation factor i_3 is commutative with the orthogonal rotation factor i_2 and the rotation factor $e^{i_2\theta_2}$ in the lower dimension, and the multiplication result is always equal to the i_3 itself.

4D space

In Figure 8 for the 4D space, any point in the line O-p_{3r_4} is rotated to the x_4 axis by orthogonal rotation factor i_4. Thus, $e^{i_3\theta_3}i_4 = i_4$. On the other hand, applying the rotation factor $e^{i_3\theta_3}$ to a point represented by i_4 in the x_4 axis will not rotate the point. Thus, $i_4e^{i_3\theta_3} = i_4$.

Further, from the 4D perspective, the plane x_1-O-x_2 in Figure 7 for the 3D now appears one-dimensional in Figure 8. Logically, in Figure 8, the point represented by the i_2-associated rotation factor $e^{i_2\theta_2}$ is in the x_2 axis. Applying the i_4 orthogonal rotation factor to the point makes it rotate to the x_4 axis. Thus, $e^{i_2\theta_2}i_4 = i_4$. On the other hand, applying the rotation factor $e^{i_2\theta_2}$ to a point represented by i_4 in the x_4 axis will not rotate the point. Thus, $i_4e^{i_2\theta_2} = i_4$.

From the above, it follows that

$$e^{i_3\theta_3}i_4 = i_4e^{i_3\theta_3} = e^{i_2\theta_2}i_4 = i_4e^{i_2\theta_2} = i_4 \tag{37}$$

With angle $\frac{\pi}{2}$ for rotation factors, the above identity leads to

$$i_3i_4 = i_4i_3 = i_2i_4 = i_4i_2 = i_4 \tag{38}$$

This indicates that the orthogonal rotation factor i_4 is commutative with the orthogonal rotation factors (i_2, i_3) and rotation factors ($e^{i_2\theta_2}$, $e^{i_3\theta_3}$) in the lower dimensions, and the multiplication result is always equal to the i_4 itself.

n-dimensional space

Similarly, in Figure 9 for the n-dimensional space, any point in the line O-p_{n-1,r_n} is rotated to the x_n axis by orthogonal rotation factor i_n. Thus, $e^{i_{n-1}\theta_{n-1}}i_n = i_n$. On the other hand, applying the rotation factor $e^{i_{n-1}\theta_{n-1}}$ to a point represented by i_n in the x_n axis will not rotate the point. Thus, $i_ne^{i_{n-1}\theta_{n-1}} = i_n$.

Further, from the n-dimensional perspective, the point represented by the i_{n-2}-associated rotation factor $e^{i_{n-2}\theta_{n-2}}$ is in the x_{n-2} axis. Applying the i_n orthogonal rotation factor to the point makes it rotate to the x_n axis. Thus, $e^{i_{n-2}\theta_{n-2}}i_n = i_n$. On the other hand, applying the rotation factor $e^{i_{n-2}\theta_{n-2}}$ to a point represented by i_n in the x_n axis will not rotate the point. Thus, $i_ne^{i_{n-2}\theta_{n-2}} = i_n$.

And still further, by the same token, in the n-dimensional coordinate system, all axes are mutually orthogonal to each other. For a j-th dimension with j lower than n-2, applying the i_n orthogonal rotation factor to a point represented by $e^{i_j\theta_j}$ in the j-th dimension makes the point rotate to the x_n axis. Thus, $e^{i_j\theta_j}i_n = i_n$. On the other hand, applying the rotation factor $e^{i_j\theta_j}$ to a point represented by i_n in the x_n axis will not rotate the point. Thus, $i_ne^{i_j\theta_j} = i_n$.

From the above and with the generalization, it follows that

$$e^{i_j\theta_j} i_n = i_n e^{i_j\theta_j} = i_n \tag{39}$$

and

$$i_j i_n = i_n i_j = i_n \tag{40}$$

where $2 \leq j < n$.

It is noted that the above results can also be obtained by the spherical-to-Cartesian transformation in (30). Take (39) for example. The position number $e^{i_j\theta_j} i_n = e^{i_j\theta_j} e^{i_n\frac{\pi}{2}}$ means that in (30), $r_n = 1$, $\theta_j = \theta_j$, $\theta_n = \frac{\pi}{2}$ and all other angles (θ_k with $2 \leq k \leq n-1$ excluding $k = j$) are 0. Thus, all E_j in (30) become 0 except for E_n, which is 1. That is, $\mathrm{p_n} = i_n$, which is consistent with (39).

Obtaining n-dimensional spherical-to-Cartesian transformation by rotation factor algebra

Denote

$$Q_j = \prod_{k=2}^{j} e^{i_k\theta_k} \tag{41}$$

Formula (29) becomes

$$\frac{\mathrm{p_n}}{\mathrm{r_n}} = Q_n \tag{42}$$

The result in (39) means

$$Q_{j-1} i_j = i_j \tag{43}$$

With (43), it follows that

$$\begin{aligned} Q_n &= Q_{n-1} e^{i_n\theta_n} \\ &= Q_{n-1}(cos(\theta_n) + i_n sin(\theta_n)) \\ &= Q_{n-1} cos(\theta_n) + i_n sin(\theta_n) \end{aligned} \tag{44}$$

With (44), it follows that

$$Q_{n-1} = Q_{n-2} cos(\theta_{n-1}) + i_{n-1} sin(\theta_{n-1}) \tag{45}$$

and

$$Q_{n-2} = Q_{n-3} cos(\theta_{n-2}) + i_{n-2} sin(\theta_{n-2}) \tag{46}$$

Q_n may be obtained by continuing the iteration process and combining the iteration results. Inserting the obtained Q_n into (42) leads to Formula (30).

5. Conclusion

Rotation factors have been introduced to rotate vectors. A fundamental equation of position vector rotation has been discovered. An orthogonal rotation factor happens to directly appear in the fundamental equation whose solution leads to the angle-dependent rotation factor formula. Based on rotation factors, complex numbers and Euler's formula have been developed with the imaginary unit revealed to be equivalent to the orthogonal rotation factor.

The definition of position numbers in a specific dimensions is different than that of complex numbers or hypercomplex numbers. The former is defined by multiplication of rotation factors and real numbers while the latter is formed by addition of Cartesian-axis components, which are constructed with real numbers for a basis $\{1, i_2, ..., i_n\}$. In two-dimensions, position numbers are shown to be equivalent to complex numbers. However, in other higher dimensions, there are no equivalent hypercomplex numbers for position numbers. In a particular dimensions higher than 2D, the interaction between elements in the basis for position numbers is different than that for hypercomplex numbers. Compared with hypercomplex numbers, position numbers may be considered as new number system from the perspective of definition, properties and characteristics.

In Table 1, different number systems in 2D and higher dimensions are compared.

Number system	2D			3D			4D			8D			n-D		
	MC	MS	CM	MC	MS	CM	MC	MS	CM	MC	MS	CM	MC	MS	CM
Complex number	✓	✓	✓												
Quaternion							✓								
Octonion										✓					
Position number	✓	✓	✓		✓	✓		✓	✓		✓	✓		✓	✓

MC: Multiplication (Cartesian); MS: Multiplication (Spherical); CM: Multiplication Commutative

TABLE 1. Comparison of number systems

In two-dimensions, position numbers and complex numbers are equivalent, having the properties of multiplication commutative (CM) for multiplication in Cartesian coordinates (MC) and multiplication in spherical coordinates (MS). In 3D, multiplication in Cartesian coordinates is not valid, but multiplication in spherical coordinates can still be performed with commutativity. This is also applicable to any other higher dimensions for position numbers. The conventional rules of multiplication in Cartesian coordinates for hypercomplex numbers such as quaternions (4D) and octonions (8D) are not valid for position numbers in dimensions higher than 2D. However, to achieve the corresponding multiplication result, one may first convert position numbers in Cartesian to spherical coordinates and perform the multiplication in spherical coordinates and transform the result back to Cartesian. The formula for rotation-factor constructed position numbers is succinct and consistent across dimensions with one rotation factor for each dimension, indicating that the rotation factors are natively suited for the spherical systems.

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