Sequences Of Numbers

A large area of discrete math is studying number sequences. Some examples include:

- Natural numbers: $\{1, 2, 3, 4, 5, \ldots\}$
- Even numbers: $\{2, 4, 6, 8, 10, \ldots\}$
- Prime numbers: $\{2, 3, 5, 7, 11, 13, \ldots\}$
- Powers of 2: $\{1, 2, 4, 8, 16, 32, \ldots\}$
- Reciprocals of powers of 2: $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$

Definition: A **sequence of numbers** is a set where the order in which elements appear matters.

Fibonacci Numbers

One very famous number sequence is the **Fibonacci sequence**, named after Leonardo de Pisa (1170), a famous mathematician during the European Middle Ages.

Leonardo De Pisa was born in the Bonacci family, and Fibonacci is short for *Filius Bonacci* (son of Bonacci).

Fibonacci frequently traveled to Egypt, Syria, Greece, where he became convinced that the Indo-Arabic numbering system was superior to the Roman numeral system.

He wrote *Liber Abaci* (1202), where he introduces the Indo-Arabic system to Europe.

The Rabbit Problem

In *Liber Abaci*, Fibonacci proposes the following problem:

- Start with 2 newborn rabbits in January, one male and one female
- Each rabbit pair takes one month to mature
- Each pair produces a mixed pair (M/F) once they reach adulthood (from the second month of their life)

Q: If no rabbits die during the year, how many **pairs of rabbits** are there at the end of the year?

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Adults	0							
Babies	1							
Total	1							

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	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Adults	0	1	**1**					
Babies	1	0	**1**					
Total	1	1	**2**					

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	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Adults	0	1	1	**2**				
Babies	1	0	1	**1**				
Total	1	1	2	**3**				

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	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Adults	0	1	1	2	**3**			
Babies	1	0	1	1	**2**			
Total	1	1	2	3	**5**			

The Rabbit Problem

Continuing in this fashion, we obtain the following number of rabbits for each month (1 is January, 12 is December):

Month	1	2	3	4	5	6	7	8	9	10	11	12
# Pairs	1	1	2	3	5	8	13	21	34	55	89	144

The sequence of numbers $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ is called the **Fibonacci sequence**.

It was named in 1876 by French mathematician François Édouard Anatole Lucas.

Fibonacci Numbers

Q: How many pairs of rabbits do we have at month n?

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
1	1	2	3	5	8	13	21	34

The Fibonacci numbers can be calculated using the following:

$$F_n = (\#\text{ pairs already alive}) + (\#\text{ new baby pairs})$$

which is the same as

$$\boxed{F_n = F_{n-1} + F_{n-2}}$$

so the current Fibonacci number is the sum of the two previous values.

Fibonacci Coincidences

It turns out that the Fibonacci numbers appear in the most unexpected places. For the state of Illinois, we have

- admitted to the Union on December **3**
- the **5$^{\text{th}}$** largest state by population (2010)
- the **13$^{\text{th}}$** state alphabetically
- the **21$^{\text{st}}$** state to join the Union
- the postal abbreviation "IL" adds up to $9 + 12 = \mathbf{21}$
- the interstate *I*-**55** starts in Chicago and runs roughly along the **89$^{\text{th}}$** parallel to New Orleans

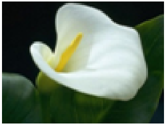
It is possible that these are pure coincidences, as we have learned when talking about the birthday problem.

Fibonacci In Nature

Fibonacci And Petals

Fibonacci numbers often appear on petals of flowers.

1



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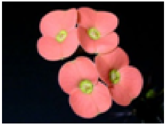

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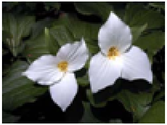

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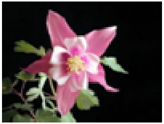

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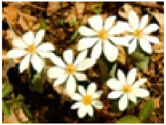

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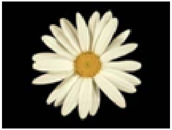
21



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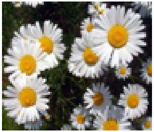


Fibonacci And Petals

The following is a table of plants and their petal count:

Plant	# Petals
Calla lily	1
Enchanter's nightshade	2
Iris, lily	3
Buttercup, columbine, delphinium, wall lettuce	5
Celandine, delphinium	8
Chamomile, cineraria, corn marigold	13
Aster, doronicum, hawkbit	21
Daisy, gailliardia, hawkweed	34

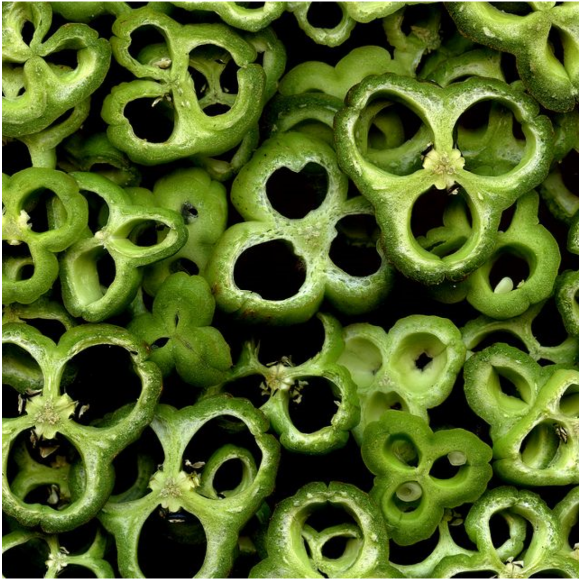
Plant Cross Sections

Fibonacci numbers also appear in other areas of a plant's anatomy, such as cross sections of fruits. Here's a picture of apple seeds seen from above:

Plant Cross Sections

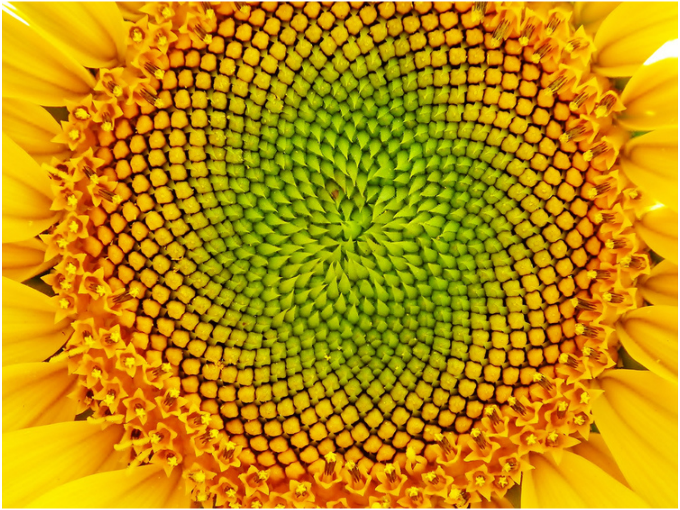
Plant Cross Sections

Plant Cross Sections



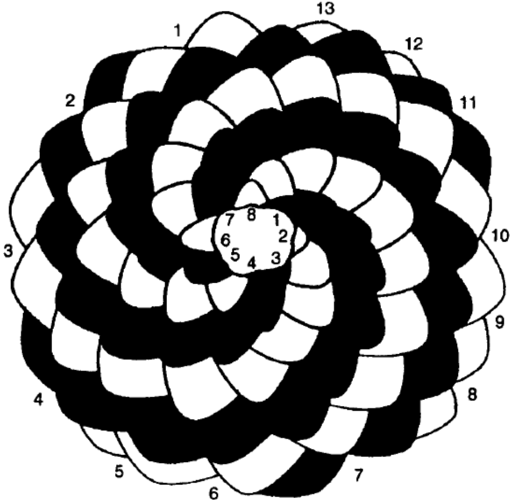
Sunflowers

Sunflowers exhibit even more remarkable patterns. They have petals (21, 34, 55) and spirals (CW, CCW)!



Fibonacci Spirals

The same spiral pattern appears in pine cones, pineapples, daisy centers, artichokes, etc...

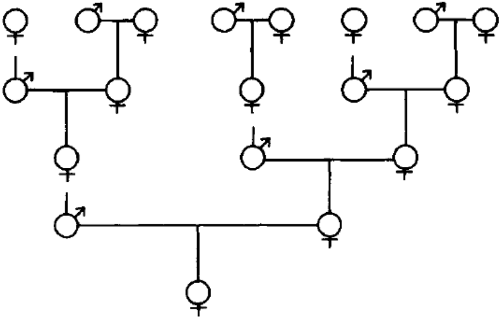


Fibonacci Spirals

Fibonacci Spirals

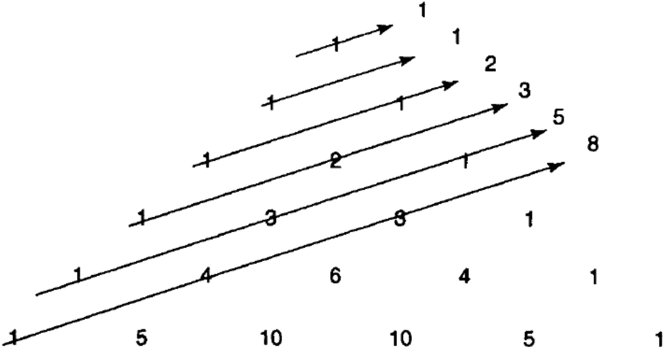
Fibonacci And Bee Ancestry

Male bees come from unfertilized eggs, so a male bee has a mother, but no father. A female bee comes from a fertilized egg, so it has both parents. Below is the ancestry tree of a female bee. Count the number of ancestors from each generation.



Fibonacci And Pascal's Triangle

If we sum the numbers in Pascal's triangle diagonally upwards, we get the Fibonacci numbers!



The Golden Ratio

Q: How fast do Fibonacci numbers grow? We look at ratios of consecutive Fibonacci numbers:

n	F_{n+1}/F_n
1	$F_2/F_1 = 1/1 = \mathbf{1}$
2	$F_3/F_2 = 2/1 = \mathbf{2}$
3	$F_4/F_3 = 3/2 = \mathbf{1.5}$
4	$F_5/F_4 = 5/3 \approx \mathbf{1.66667}$
5	$F_6/F_5 = 8/5 = \mathbf{1.6}$
6	$F_7/F_6 = 13/8 = \mathbf{1.625}$
7	$F_8/F_7 = 21/13 \approx \mathbf{1.61538}$
8	$F_9/F_8 = 34/21 \approx \mathbf{1.61905}$
9	$F_{10}/F_9 = 55/34 \approx \mathbf{1.61765}$
10	$F_{11}/F_{10} = 89/55 \approx \mathbf{1.61818}$
11	$F_{12}/F_{11} = 144/89 \approx \mathbf{1.61810}$

It appears that the ratio values stabilize around 1.618.

The Golden Ratio

The **golden ratio** ϕ (Greek letter "phi") is the limit of ratios of consecutive Fibonacci numbers:

$$\boxed{\phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}}$$

Its value is the positive root of the equation $L^2 - L - 1 = 0$, and is equal to

$$\boxed{\phi = \frac{1+\sqrt{5}}{2} \approx 1.618033989}$$

and $1/\phi \approx 0.618033989$. In other words, ϕ satisfies

$$\boxed{\frac{1}{\phi} = \phi - 1}$$

The Golden Ratio And π

The first 10 digits of the golden ratio ϕ are

$$\phi = 1.618033988\ldots$$

and they can be permuted to get the first 10 digits of $1/\pi$:

$$\frac{1}{\pi} = 0.3183098861\ldots$$

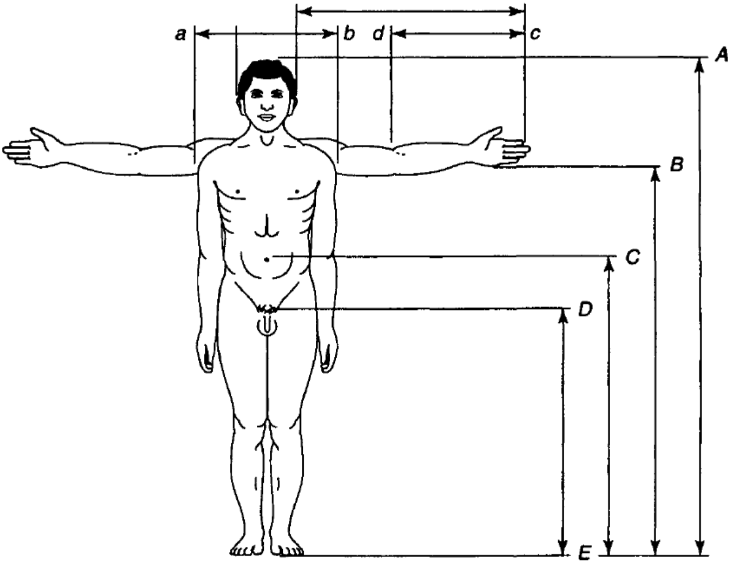
Also, the first 9 digits of $1/\phi$ are

$$\frac{1}{\phi} = 0.618033988\ldots$$

and they can be permuted to get the first 9 digits of $1/\pi$:

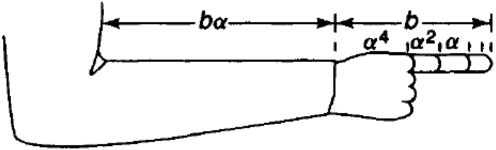
$$\frac{1}{\pi} = 0.318309886\ldots$$

The Human Body



The Human Body

Exercise: Try measuring the following distances on your arm. The ratios should be relatively close to 1.618!



The Vitruvian Man

Leonardo da Vinci drew this famous picture around 1490 while respecting the golden ratio. It is considered the most aesthetically pleasing body.