# MAT 508 Programming Project

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**Abstract**

Let $S$ be a real skew-symmetric matrix. In this project, we compare Matlab's `expm` function with Diego Avalos's formula for $e^S$, where $S$ is of size 5, by randomly generating matrices $S$ such that the entries of $S$ are entries with $|S_{ij}| \leq 5$. The goal of this paper is to demonstrate cases where `expm` fails, but Avalos's closed formula gives an appropriate value for $e^S$. To accomplish this, we test `expm(10^k*S)` for integer values of $k$ ($0 \leq k \leq 20$), and observe that the Matlab output goes to zero, `Inf`, or `NaN`, whereas Avalos's formula gives a more valid value or $e^S$.

## Introduction

In Diego Avalos's Master's thesis *Exponentials of Real Skew-Symmetric Matrices in Terms of Their Eigenvalues*, he computes formulas for the exponential of any real-valued skew-symmetric matrix up to size 9 in terms of their eigenvalues and powers of the matrix. Particularly, in this paper we will study the effectiveness of Avalos's formulas against Matlab's `expm` function. We will look at real skew-symmetric matrices $S = -S^T$ of size five with four different purely imaginary eigenvalues $\pm\theta_1 i$, $\pm\theta_2 i$ ($\theta_2 > \theta_1 > 0$), and zero. We will rely on the Matlab function `charpoly` to compute the coefficients of the characteristic polynomial of $S$, which we denote by $p(X) = \det(XI_5 - S)$, because computing these coefficients does not cost anything, as they are simply sums and products of the entries of $S$. For instance, since $p(X)$ has the form

$$p(X) = X^5 + aX^3 + bX$$

then we can use the quadratic formula to obtain

$$\theta_1 = \sqrt{\frac{a - \sqrt{a^2 - 4b}}{2}}$$

and

$$\theta_2 = \sqrt{\frac{a + \sqrt{a^2 - 4b}}{2}}.$$

Observe that $\theta_2 > \theta_1 > 0$ because $a^2 > 4b$ is ensured by the fact that $S$ has five different eigenvalues. Hence, to get $e^S$ we can use Avalos's formula for the Case 3.5.2, namely

$$\begin{aligned} e^S =& I_5 + \frac{\theta_2^3 \sin\theta_1 - \theta_1^3 \sin\theta_2}{\theta_1\theta_2(\theta_2^2 - \theta_1^2)} S + \frac{\theta_2^4(1-\cos\theta_1) - \theta_1^4(1-\cos\theta_2)}{\theta_1^2\theta_2^2(\theta_2^2 - \theta_1^2)} S^2 \\ &+ \frac{\theta_2 \sin\theta_1 - \theta_1 \sin\theta_2}{\theta_1\theta_2(\theta_2^2 - \theta_1^2)} S^3 + \frac{\theta_2^2(1-\cos\theta_1) - \theta_1^2(1-\cos\theta_2)}{\theta_1^2\theta_2^2(\theta_2^2 - \theta_1^2)} S^4. \end{aligned} \tag{*}$$

Using the identities $\theta_1\theta_2 = \sqrt{b}$ and $\theta_2^2 - \theta_1^2 = \sqrt{a^2-4b}$ we get the more computationally friendly formula

$$
\begin{aligned}
e^S =& I_5 + \frac{\theta_2^3 \sin\theta_1 - \theta_1^3 \sin\theta_2}{\sqrt{a^2b-4b^2}} S + \frac{\theta_2^4(1-\cos\theta_1) - \theta_1^4(1-\cos\theta_2)}{b\sqrt{a^2-4b}} S^2 \\
&+ \frac{\theta_2 \sin\theta_1 - \theta_1 \sin\theta_2}{\sqrt{a^2b-4b^2}} S^3 + \frac{\theta_2^2(1-\cos\theta_1) - \theta_1^2(1-\cos\theta_2)}{b\sqrt{a^2-4b}} S^4.
\end{aligned} \tag{1}
$$

In this paper, we will randomly generate real skew-symmetric matrices of size five $S$ on Matlab whose entries $S_{ij}$ are constrained to integer values such that $|S_{ij}| \leq 5$, and use Formula (1) to compute $e^S$ and compare it to `expm(S)`. Furthermore, since we are seeking to demonstrate that our closed form formula for $e^S$ performs better than `expm`, we will also compare our formula with the exponential $e^{10^k S}$, where $k$ takes on integer values from zero to 20.

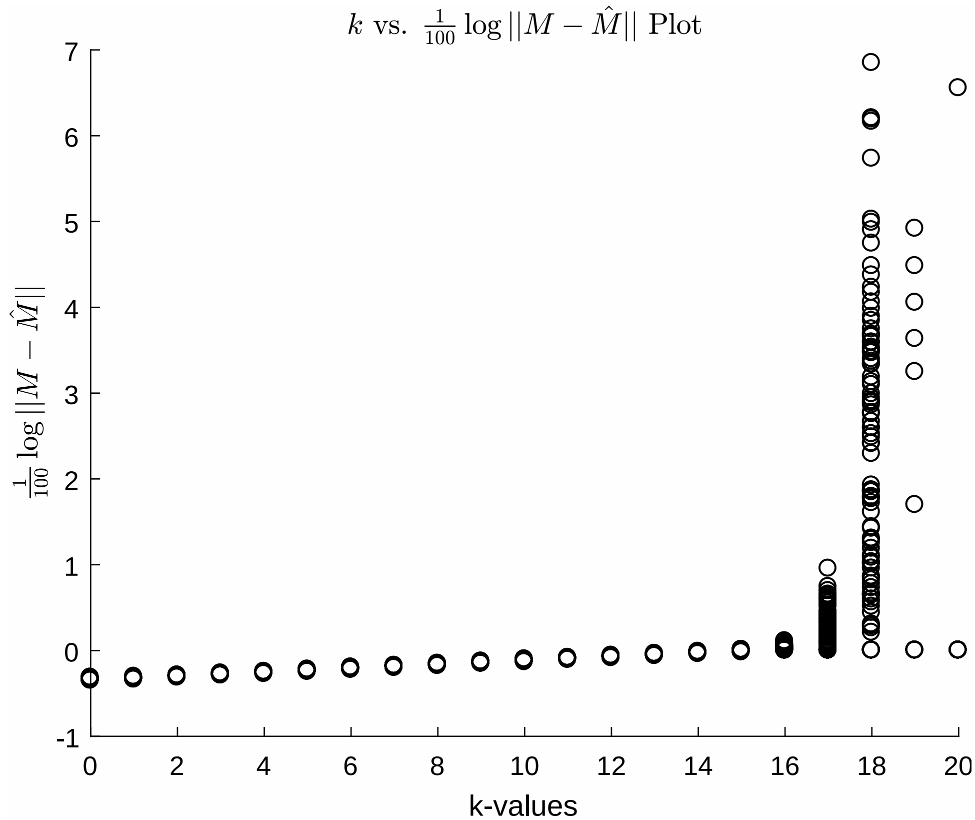
The beauty of Formula (*) is that if instead of $e^S$, we desire to compute $e^{10^k S}$, then we can use the fact that if $\lambda$ is an eigenvalue of $S$, then $r\lambda$ is an eigenvalue of $rS$ for any scalar $r$ and simplify the coefficients of $S$. Therefore, if we use Formula (*) to compute $e^{10^k S}$ and cancel the expressions $10^k$ from each numerator and denominator of each power of $10^k S$, we obtain

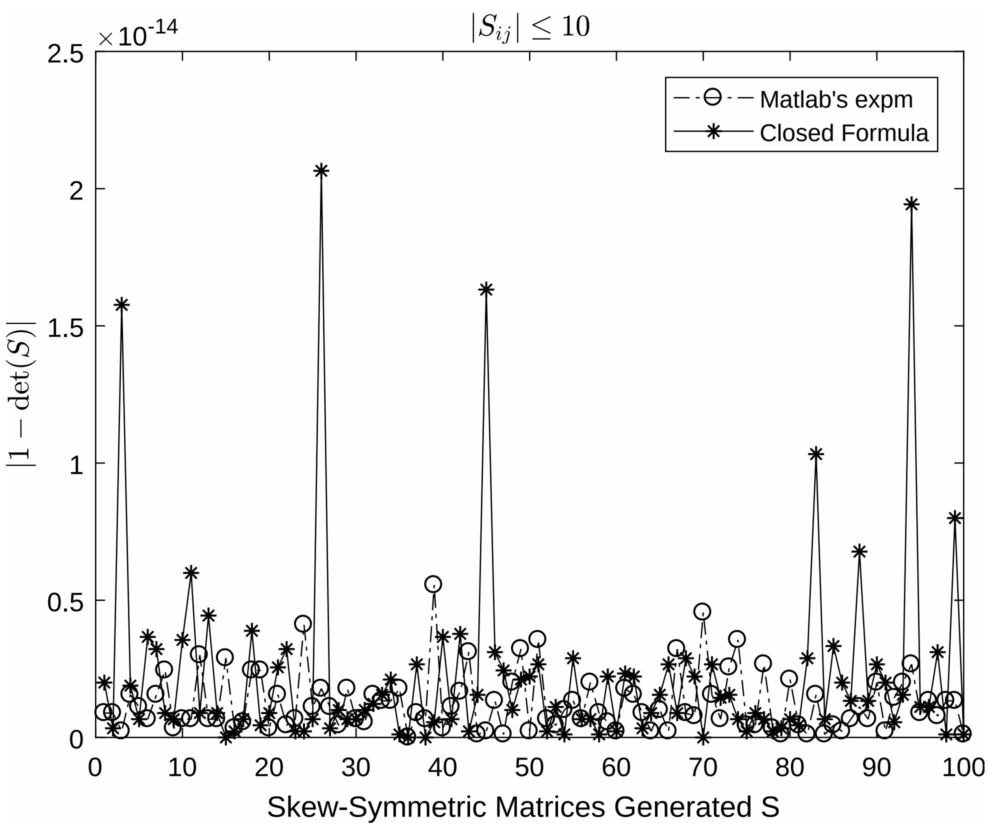
$$
\begin{aligned}
e^{10^k S} =& I_5 + \frac{\theta_2^3 \sin 10^k\theta_1 - \theta_1^3 \sin 10^k\theta_2}{\sqrt{a^2b-4b^2}} S + \frac{\theta_2^4(1-\cos 10^k\theta_1) - \theta_1^4(1-\cos 10^k\theta_2)}{b\sqrt{a^2-4b}} S^2 \\
&+ \frac{\theta_2 \sin 10^k\theta_1 - \theta_1 \sin 10^k\theta_2}{\sqrt{a^2b-4b^2}} S^3 + \frac{\theta_2^2(1-\cos 10^k\theta_1) - \theta_1^2(1-\cos 10^k\theta_2)}{b\sqrt{a^2-4b}} S^4.
\end{aligned} \tag{2}
$$

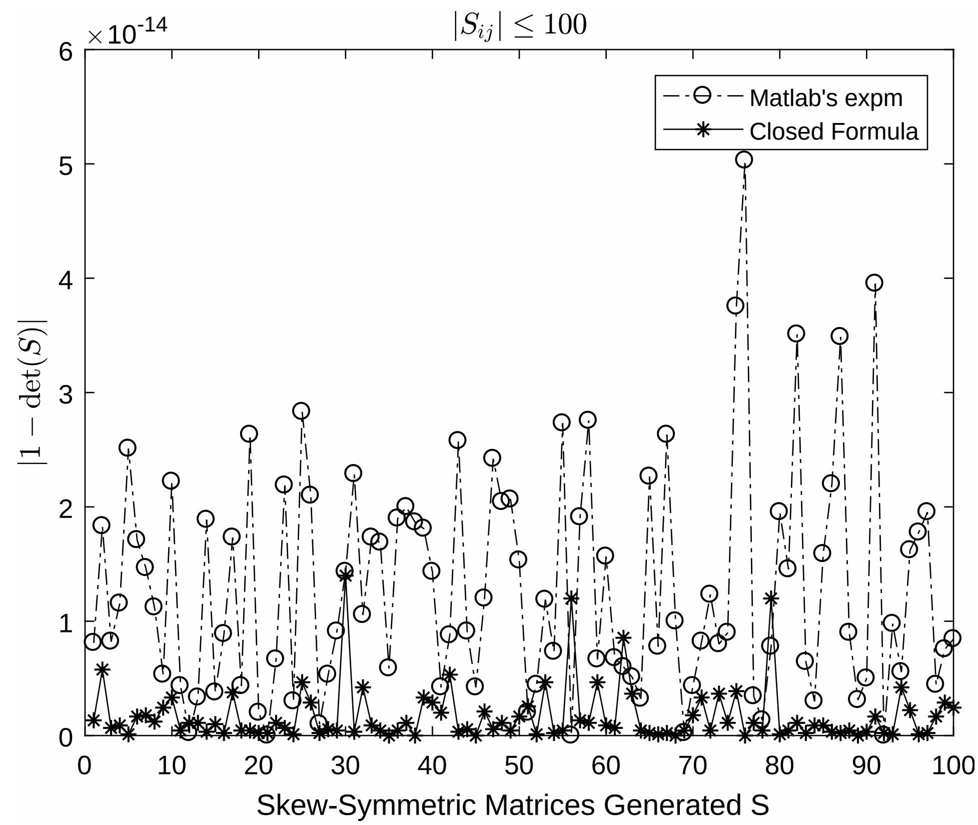
We code Formula (2) into Matlab to compute the exponentials of $10^k S$ and compare them against `expm(10^k*S)`. Observe that we are relying on Matlab's values of sines and cosines of very large values (e.g., $\cos 10^{20}$).

# Discussion and Results

Let $S$ be a randomly generated real skew-symmetric matrix with coefficients $|S_{ij}| \leq 5$. We denote the matrix exponential $e^{10^k S}$ computed with Formula (2) by $M$, and the matrix exponential of $10^k S$ computed with Matlab's `expm` by $\hat{M}$. We will be evaluating the relative error $||M - \hat{M}||$


$k$ vs. $\frac{1}{100}\log||M-\hat{M}||$ Plot
$\frac{1}{100}\log||M-\hat{M}||$
k-values


$|S_{ij}| \leq 10$
$\times 10^{-14}$
$|1-\det(S)|$
Matlab's expm
Closed Formula
Skew-Symmetric Matrices Generated S


$|S_{ij}| \leq 100$
$\times 10^{-14}$
Matlab's expm
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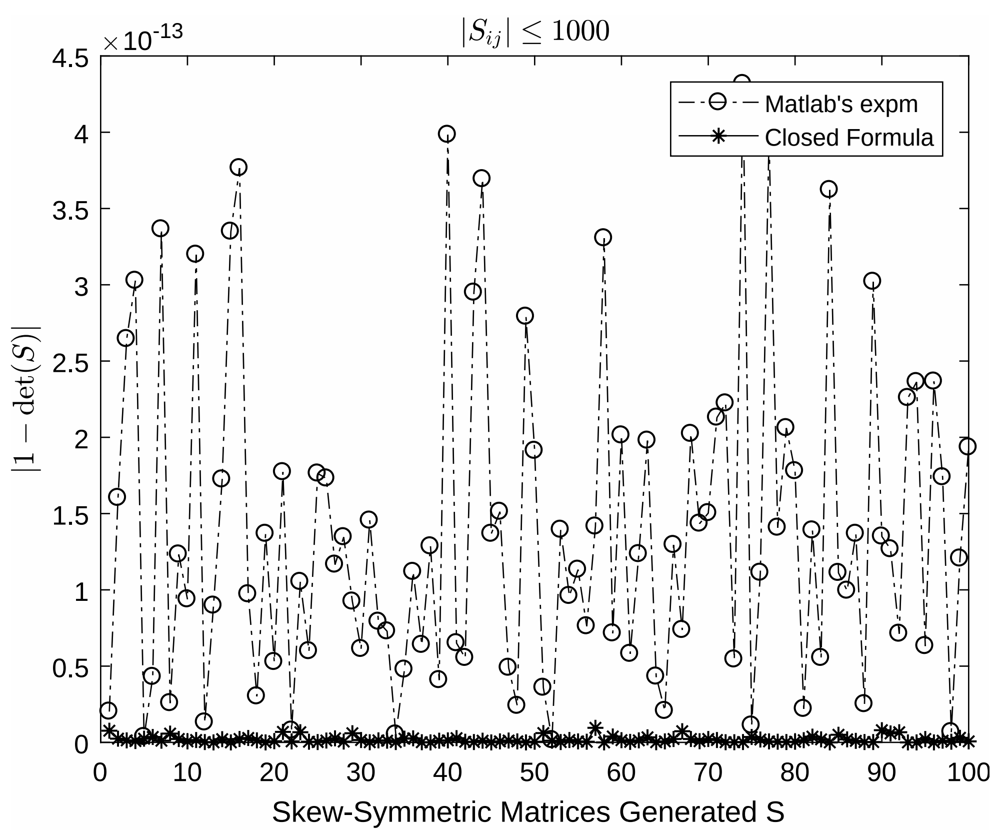
$|1 - \det(S)|$
Skew-Symmetric Matrices Generated S


$|S_{ij}| \leq 1000$
$\times 10^{-13}$
Matlab's expm
Closed Formula
$|1 - \det(S)|$
Skew-Symmetric Matrices Generated S

# Source Code

```
function [MyExponential, MatlabExponential, rel_error] = expskew5(r,k)
% r is a 1x10 vector, the terms correspond to the entries above the
% diagonal of the skew-symmetric matrix S. Use k=0 to get the value e^S.
format long
S = [ 0     r(1) r(2)  r(3)   r(4);...
     -r(1)   0   r(5)  r(6)   r(7);...
     -r(2) -r(5)  0    r(8)   r(9);...
     -r(3) -r(6) -r(8) 0      r(10);...
     -r(4) -r(7) -r(9) -r(10)  0    ];
k=10^k;
MatlabExponential = expm(k*S);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computing my exponential
A = charpoly(S); a = A(3); b = A(5);
t1 = sqrt((a-sqrt(a^2-4*b))/2);
t2 = sqrt((a+sqrt(a^2-4*b))/2);

c1 = (t2^3*sin(k*t1)-t1^3*sin(k*t2))/(t1*t2*(t2^2-t1^2));
c2 = (t2^4*(1-cos(k*t1))-t1^4*(1-cos(k*t2)))/(t1^2*t2^2*(t2^2-t1^2));
c3 = (t2*sin(k*t1)-t1*sin(k*t2))/(t1*t2*(t2^2-t1^2));
c4 = (t2^2*(1-cos(k*t1))-t1^2*(1-cos(k*t2)))/(t1^2*t2^2*(t2^2-t1^2));

MyExponential = eye(5) + c1*S + c2*S^2 + c3*S^3 + c4*S^4;
E = MyExponential-MatlabExponential;
rel_error = norm(E);
```