Notes on length-2 relative Hilbert scheme of Lefschetz fibration

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Abstract

In this note we show that the length-2 relative Hilbert scheme of an A_1-Milnor fiber is smooth and in fact a Morse-Bott-Lefschetz fibration, generalizing the discussion of [Ran04] which restricts to nodal degeneration of curves. There should be references on the same topic somewhere and we rewrite a proof here.

1 Length-2 relative Hilbert scheme

Definition 1. *Let M be some projective scheme, the length-m Hilbert scheme $\mathrm{Hilb}^m(M)$ parametrizes length-m subschemes of M.*

It is worth noting that $\mathrm{Hilb}^2(M)$ is the same as $\mathrm{Sym}^2(M)$ after complex blow- up along the diagonal. Thus a length-2 subscheme supported at the same point $p \in M$ is parametrized by p and $v \in PT_pM$. In particular, $\mathrm{Hilb}^2(M)$ is smooth if M is smooth.

Definition 2. *Given a Lefschetz fibration $\pi : E \to \mathbb{C}$, the relative Hilbert scheme $\mathrm{Hilb}^m(E, \mathbb{C}, \pi)$ parametrizes length-m subschemes contained in the fibers of $\mathrm{Spec}(E) \to \mathrm{Spec}(\mathbb{C}) = \mathbb{C}$.*

Intuitively, the set of m disjoint points on the same fiber of $E \to B$ is an open subset of $\mathrm{Hilb}^m(E, \mathbb{C}, \pi)$. There is also an induced fibration $\pi_* : \mathrm{Hilb}^m(E, \mathbb{C}, \pi) \to \mathbb{C}$. Our goal is then to prove that π_* is a Morse-Bott type Lefschetz fibration for $m = 2$.

Since the discussion is totally local, we assume $E = \mathbb{C}^{n+1}$ and

$$\begin{aligned} \pi : E &\to \mathbb{C} \\ (x_1, ..., x_{n+1}) &\to x_1^2 + ... + x_{n+1}^2, \end{aligned} \tag{1}$$

where there exists a nodal degeneration at the origin.

Now we study the neighbourhood of each point in $\mathrm{Hilb}^2(E, \mathbb{C}, \pi)$. We first have the following easy observation and ignore its proof:

Lemma 1. *Suppose $(x_1, x_2) \in \mathrm{Hilb}^2(E, \mathbb{C}, \pi)$ where x_1 is the origin and x_2 is a disjoint point over $0 \in \mathbb{C}$, then the neighbourhood of (x_1, x_2) is the product of a nodal degeneration and a trivial fibration, that is,*

$$\begin{aligned} \pi_* : nb(x_1, x_2) &\to \mathbb{C} \\ (x_1, ..., x_{n+1}, y_1, ..., y_n) &\to x_1^2 + ... + x_{n+1}^2, \end{aligned} \tag{2}$$

in some local coordinates. Moreover, if both x_1, x_2 are disjoint from 0, then it is locally a trivial fibration.

It remains to consider length-2 subschemes supported at $0 \in \mathbb{C}^{n+1}$, denoted by $\mathrm{Hilb}_0^2(\mathbb{C}^{n+1})$. Let R be the localization of the ring $\mathbb{C}[x_1, ..., x_{n+1}]/(x_1^2 + ... + x_{n+1}^2)$ at the origin, then we are considering ideals I of R with colength 2. By the discussion below Definition 1, $\mathrm{Hilb}_0^2(\mathbb{C}^{n+1})$ is homeomorphic to $\mathbb{C}P^n$. Specifically we see that:

Lemma 2. $\mathrm{Hilb}_0^2(\mathbb{C}^{n+1})$ *is homeomorphic to $\mathbb{C}P^n$. Moreover, each element of $\mathrm{Hilb}_0^2(\mathbb{C}^{n+1})$ can be expressed by the ideal $(u_1^2, u_2, ..., u_{n+1}) \subset R$, where $\boldsymbol{x} = G\boldsymbol{u}$ for some $G \in PGL_{n+1}(\mathbb{C})$, parametrized by the first column $g_{11}, ..., g_{(n+1)1}$ of G.*

Proof. Clearly $R/(u_1^2, u_2, ..., u_{n+1})$ is a R-module with bases $(1, u_1)$ and thus $(u_1^2, u_2, ..., u_{n+1}) \in \mathrm{Hilb}_0^2(\mathbb{C}^{n+1})$.

Suppose $\boldsymbol{x} = G\boldsymbol{u} = H\boldsymbol{v}$ with $G, H \in PGL_{n+1}(\mathbb{C})$. We show that $(u_1^2, u_2, ..., u_{n+1}) = (v_1^2, v_2, ..., v_{n+1})$ in R if and only if $(g_{11}, ..., g_{(n+1)1}) = (h_{11}, ..., h_{(n+1)1})$ in $\mathbb{C}P^n$. Since

$$\begin{pmatrix} g_{11} & \cdots & g_{1(n+1)} \\ \vdots & & \vdots \\ g_{(n+1)1} & \cdots & g_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} h_{11} & \cdots & h_{1(n+1)} \\ \vdots & & \vdots \\ h_{(n+1)1} & \cdots & h_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n+1} \end{pmatrix}, \tag{3}$$

we can assume $g_{11} = h_{11} \neq 0$.

Now if $(g_{11}, ..., g_{(n+1)1}) = (h_{11}, ..., h_{(n+1)1})$ in $\mathbb{C}P^n$, then

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{h_1^T} \\ 0 & \\ \vdots & H_1 \\ 0 & \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n+1} \end{pmatrix}, \tag{4}$$

that is,

$$\begin{pmatrix} u_2 \\ \vdots \\ u_{n+1} \end{pmatrix} = H_1 \begin{pmatrix} v_2 \\ \vdots \\ v_{n+1} \end{pmatrix} \text{ and } u_1 = v_1 + \boldsymbol{h_1^T} \begin{pmatrix} v_2 \\ \vdots \\ v_{n+1} \end{pmatrix}, \tag{5}$$

which implies $(u_1^2, u_2, ..., u_{n+1}) \subset (v_1^2, v_2, ..., v_{n+1})$. The other direction is similar, so $(u_1^2, u_2, ..., u_{n+1}) = (v_1^2, v_2, ..., v_{n+1})$ in R.

If $(g_{11}, ..., g_{(n+1)1}) \neq (h_{11}, ..., h_{(n+1)1})$ in $\mathbb{C}P^n$, then without loss of generality we have

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} * & \\ \vdots & H_1 \\ * & \\ 1 & \boldsymbol{h_2^T} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n+1} \end{pmatrix}, \tag{6}$$

where $u_{n+1} = v_1 + (v_2, ..., v_{n+1})\boldsymbol{h_2}$, so $u_{n+1} \notin (v_1^2, v_2, ..., v_{n+1})$ and thus $(u_1^2, u_2, ..., u_{n+1}) \neq (v_1^2, v_2, ..., v_{n+1})$ in R. $\square$

Therefore, we can always assume

$$\boldsymbol{x} = G\boldsymbol{u} = \begin{pmatrix} g_{11} & 0 & \dots & 0 \\ g_{21} & 1 & & \\ \vdots & & \ddots & \\ g_{(n+1)1} & & & 1 \end{pmatrix} \boldsymbol{u}, \tag{7}$$

where $g_{11} \neq 0$. We then consider the neighbourhood of $I_0 = (u_1^2, u_2, ..., u_{n+1})$ in $\mathrm{Hilb}^2(E, \mathbb{C}, \pi)$, which contains ideals I generated by

$$u_1^2 - a_1 u_1 - b_1, \tag{8}$$

$$u_i - a_i u_1 - b_i,\ i = 2, ..., n+1 \tag{9}$$

$$x_1^2 + ... + x_{n+1}^2 - t, \tag{10}$$

where t is the coordinate of the base $\mathbb{C}$. There are $2n+3$ parameters above and $\mathrm{Hilb}^2(E, \mathbb{C}, \pi)$ is expected to have dimension $2n+1$, so we need two relations from (8)(9)(10) and prove the following:

Lemma 3. *$\pi_* : \mathrm{Hilb}^2(E, \mathbb{C}, \pi) \to \mathbb{C}$ is a Morse-Bott type Lefschetz fibration. The singular subset $\mathrm{Sing}(\pi)$ contains two parts: Those supported at two distinct points are described by Lemma 1; those supported at $0 \in \mathbb{C}^{n+1}$ are the projective surface $S := \{x_1^2 + ... + x_{n+1}^2 = 0\} \subset \mathbb{C}P^n$.*

Proof. From (7)(8)(9)(10) we get

$$\begin{aligned}
t =& \boldsymbol{x}^T \boldsymbol{x} \\
=& \boldsymbol{u}^T G^T G \boldsymbol{u} \\
=& c u_1^2 + u_2^2 + ... + u_{n+1}^2 + 2u_1(g_{21}u_2 + ... + g_{(n+1)1}u_{n+1}) \\
=& \left(c + \sum_{i=2}^{n+1} a_i^2 + 2 \sum_{i=2}^{n+1} g_{i1} a_i \right) u_1^2 + \left(2 \sum_{i=2}^{n+1} (a_i + g_{i1}) b_i \right) u_1 + \sum_{i=2}^{n+1} b_i^2 \\
=& \left[a_1 \left(c + \sum_{i=2}^{n+1} a_i^2 + 2 \sum_{i=2}^{n+1} g_{i1} a_i \right) + \sum_{i=2}^{n+1} (a_i + g_{i1}) b_i \right] u_1 \\
&+ \left[b_1 \left(c + \sum_{i=2}^{n+1} a_i^2 + 2 \sum_{i=2}^{n+1} g_{i1} a_i \right) + \sum_{i=2}^{n+1} b_i^2 \right],
\end{aligned} \tag{11}$$

where $c = g_{11}^2 + ... + g_{(n+1)1}^2$. Comparing the coefficients there are two relations

$$\begin{cases} a_1 \left(c + \sum_{i=2}^{n+1} a_i^2 + 2 \sum_{i=2}^{n+1} g_{i1} a_i \right) + \sum_{i=2}^{n+1} (a_i + g_{i1}) b_i = 0 \\ b_1 \left(c + \sum_{i=2}^{n+1} a_i^2 + 2 \sum_{i=2}^{n+1} g_{i1} a_i \right) + \sum_{i=2}^{n+1} b_i^2 = t \end{cases}. \tag{12}$$

If $c \neq 0$, i.e. $(g_{11}, ..., g_{(n+1)1}) \notin S$, then for small $a_2, ..., a_{n+1}$,

$$r := c + \sum_{i=2}^{n+1} a_i^2 + 2 \sum_{i=2}^{n+1} g_{i1} a_i \neq 0, \tag{13}$$

so a_1, b_1 are uniquely determined by (12) and $(a_2, ..., a_{n+1}, b_2, ..., b_{n+1}, t)$ are all free parameters. Therefore, locally the fibration is trivial:

$$\pi_* : (a_2, ..., a_{n+1}, b_2, ..., b_{n+1}, t) \longmapsto t. \tag{14}$$

If $c = 0$, i.e. $(g_{11}, ..., g_{(n+1)1}) \in S$, we can assume both $g_{11} \neq 0$ and $g_{(n+1)1} \neq 0$. For small $a_2, ..., a_{n+1}$, the first equation of (12) determines $b_{(n+1)1}$ since $a_{n+1} + g_{(n+1)1} \neq 0$; the second equation of (12) then becomes

$$\begin{aligned} t &= rs + \sum_{i=2}^{n} b_i^2 + \frac{\left(\sum_{i=2}^{n} (a_i + g_{i1}) b_i\right)^2}{(a_{n+1} + g_{(n+1)1})^2} \\ &= rs + \boldsymbol{b}^T (1 + K) \boldsymbol{b}, \end{aligned} \tag{15}$$

where

$$s \coloneqq b_1 + \frac{a_1^2 r + 2a_1 \sum_{i=2}^{n} (a_i + g_{i1}) b_i}{(a_{n+1} + g_{(n+1)1})^2}, \tag{16}$$

$$\boldsymbol{b} \coloneqq (b_2, ..., b_n)^T, \tag{17}$$

$$k_{ij} \coloneqq \frac{(a_i + g_{i1})(a_j + g_{j1})}{(a_{n+1} + g_{(n+1)1})^2}. \tag{18}$$

Here k_{ij} is the $(i-1, j-1)$-entry of K. Observe that K is of rank 1, so $n-2$ of its $n-1$ eigenvalues are 0 and the last one is

$$\lambda = tr(K) = \sum_{i=2}^{n} \frac{(a_i + g_{i1})^2}{(a_{n+1} + g_{(n+1)1})^2}. \tag{19}$$

Now assume $a_2 = ... = a_{n+1} = 0$, then

$$\lambda = \frac{\sum_{i=2}^{n} g_{i1}^2}{g_{(n+1)1}^2} = -1 - \left(\frac{g_{11}}{g_{(n+1)1}} \right)^2 \neq -1. \tag{20}$$

Therefore, $1+K$ has $n-2$ eigenvalues of 1 and one eigenvalue of $-(\frac{g_{11}}{g_{(n+1)1}})^2 \neq 0$, which says $1+K$ is nondegenerate. For small a_i, $i = 2, ..., n+1$, $1+K$ is still nondegenerate. By (15) and Implicit Function Theorem, we see that π_* is locally like

$$\pi_* : (w_1, ..., w_{n+1}, z_1, ..., z_n) \longmapsto w_1^2 + ... + w_{n+1}^2, \tag{21}$$

that is, a Morse-Bott type Lefschetz fibration. □

References

[Ran04] Ziv Ran. *A note on Hilbert schemes of nodal curves*. 2004. arXiv: math/0410037 [math.AG].