A summary for the program of $\mathcal{R}_0$

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Abstract

This note is to summary how to compute the basic reproduction ratios numerically.

1 Introduction

How to calculate $\mathcal{R}_0$ numerically is a very important problem. For the autonomous models of ODEs, Van den Driessche and Watmough [5] found that the principal eigenvalue of the next generation matrix admits $\mathcal{R}_0$. For time-periodic compartmental ODEs models, Wang and Zhao [6] proposed a numerical method to compute $\mathcal{R}_0$ by solving the unique root of a critical equation(see [6, Theorem 2.1]). We call it Root method. An alternative method to deal with the problem goes to Posny and Wang [4]. They transformed the problem into a matrix eigenvalue problem. For the infinite-dimensional cases, motivated by [5], Wang and Zhao presented a numerical method to approximate the principal eigenvalue of the next generation operator for the autonomous models of reaction-diffusion equations. Combined with the Root method and the principal eigenvalue of positive operators, a numerical method was developed by Liang, Zhang and Zhao [3]. This method can be applied to various kinds of periodic models, including ODEs, reaction-diffusion equations, nonlocal dispersal equations with or without time delay. In [8], Yang and Zhang proposed a direct method to compute $\mathcal{R}_0$ by computing the spectral radius of the corresponding operator, which can be regarded as a generalized method in [4]. We also refer to [2, 1].

The following part of this note is to describe the algorithm to compute $\mathcal{R}_0$. In the next section, we show a generalized Power Method to compute the spectral radius of a positive operator. In section 3, we show a numerical method to calculate $\mathcal{R}_0$ by approximating to the spectral radius of the associated operator for autonomous cases. In section 4, we show the Root method to compute $\mathcal{R}_0$. In section 5, we give ten examples to show how to compute $\mathcal{R}_0$ in Matlab. In section 6, we present a short summary for these methods.

2 Compute the spectral radius

Now we introduce a generalized Power Method to compute the spectral radius of a positive operator L numerically.

Lemma 1. *Assume that (E, E_+) is an ordered Banach space with E_+ being normal and $\mathrm{Int}(E_+) \neq \emptyset$. Let L be a positive bounded linear operator. Then $r(L) = \lim_{n\to+\infty} \|L^n e\|_E^{\frac{1}{n}}$, $\forall e \in \mathrm{Int}(E_+)$.*

Lemma 2. *Assume that (E, E_+) is an ordered Banach space with E_+ being normal and $\mathrm{Int}(E_+) \neq \emptyset$, which is equipped with the norm $\|\cdot\|_E$. Let L be a positive bounded linear operator. Choose $v_0 \in \mathrm{Int}(E_+)$ and define $a_n = \|Lv_{n-1}\|_E$, $v_n = \frac{Lv_{n-1}}{a_n}$, $\forall n \geq 1$. If $\lim\limits_{n\to+\infty} a_n$ exists, then $r(L) = \lim\limits_{n\to+\infty} a_n$.*

Remark 1. *The algorithm to compute $r(L)$:*

1. $v_0 = 1$.
2. $v_n = \frac{Lv_{n-1}}{\|Lv_{n-1}\|}$, $\forall n \geq 1$.
3. *Let* $a_n = \|Lv_{n-1}\|$, $n \geq 1$
   (a) *By Lemma 1,* $r(L) = \lim_{n\to+\infty} (\Pi_{k=1}^n a_k)^{\frac{1}{n}}$. *This is because*
   $$\|L^n v_0\| = \Pi_{k=1}^n a_k. \tag{1}$$
   (b) *In particular, if* $\lim\limits_{n\to+\infty} a_n$ *exists, then* $r(L) = \lim\limits_{n\to+\infty} a_n$.

Remark 2. *For a non-negative matrix L, it is known that*

1. *If L^{k_0} is strongly positive for some $k_0 > 0$, then a_n in Lemma 2 converges.*
2. *If L is irreducible, then there exists some k_0 such that $(L + I)^{k_0}$ is positive.*

3 Autonomous Cases

Now, we present a numerical method to compute $\mathcal{R}_0$ for a autonomous model. We refer to [5, 7].

Let X be a Banach space with the positive cone X_+. Assume that

(H1) F is a bounded positive operator on X.

(H2) $-V$ is a resolvent positive operator on X and $s(-V) < 0$.

Consider the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (F - V)u \tag{2}$$

So $\mathcal{L}$, $\mathcal{R}_0$ can be defined by

$$\mathcal{L}u = FV^{-1}u,\ \mathcal{R}_0 = r(\mathcal{L}). \tag{3}$$

Remark 3. *1. It is easy to see that*

$$r(FV^{-1}) = r(V^{-1}F).$$

2. In Matlab, eigenvalues of an matrix can be computed directly.

3. We can also compute the spectral radius of $r(\mathcal{L})$ by Remark 1 with $L = \mathcal{L}$.

4. Operators V and F can be approximated by discretization.

4 Periodic cases

Next, we introduce the Root method. This method is proposed in [6] and developed in [3].

4.1 Periodic cases without time-delay

Let X be a Banach space with the positive cone X_+, and

$$\mathbb{X} = \{u \in C(\mathbb{R}, X) : u(t) = u(t + T)\}$$

with the maximum norm and the positive cone

$$\mathbb{X}_+ = \{u \in C(\mathbb{R}, X_+) : u(t) = u(t + T)\}$$

Assume that

(H1) $F(t)$ is positive on X and T-periodic for all $t \in \mathbb{R}$.

(H2) Let $\{\Phi(t,s) : t \geq s\}$ be the T-periodic evolution family of

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -V(t)u(t),\ t \geq 0. \tag{4}$$

on X. $\Phi(t,s)$ is positive on X for all $t \geq s$ and $\omega(\Phi) < 0$.

Consider the equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(t)u(t) - V(t)u(t),\ t \geq 0. \tag{5}$$

So $\mathcal{L}$, $\mathcal{R}_0$ can be defined by

$$\mathcal{L}u(t) = \int_0^{+\infty} \Phi(t, t-s)F(t-s)u(t-s)\mathrm{d}s,\ u \in \mathbb{X}, \tag{6}$$

and

$$\mathcal{R}_0 = r(\mathcal{L}).$$

Let $\{U(t,s;\mu) : t \geq s\}$ be evolution family on X of the following system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \frac{1}{\mu}F(t)u(t) - V(t)u(t),\ t \geq 0. \tag{7}$$

Proposition 3. *Under some conditions, $\mu = \mathcal{R}_0$ is the unique solution of $r(U(T,0;\mu)) = 1$.*

We also compute the spectral radius of $r(U(T,0;\mu))$ by Remark 1 with $L = U(T,0;\mu)$ and search the solution of the equation $r(U(T,0;\mu)) = 1$ by the bisection method.

Remark 4. *The algorithm to compute $r(L)$:*

1. *$v_0 = 1$ is the initial data.*
2. *$v_n = \frac{Lv_{n-1}}{\|Lv_{n-1}\|}$, $\forall n \geq 1$. Here $Lv_{n-1} = U(T,0;\mu)v_{n-1}$ can be numerically computed by standard numerical method of differential equations.*
3. *Let $a_n = \|Lv_{n-1}\|$, $n \geq 1$. Usually, $U(T,0,\mu)$ is eventually positive. Thus, $\lim\limits_{n\to+\infty} a_n$ exists, and $r(L) = \lim\limits_{n\to+\infty} a_n$.*

4.2 Periodic cases with time-delay

Let X be a Banach space with the positive cone X_+, and

$$\mathbb{X} = \{u \in C(\mathbb{R}, X) : u(t) = u(t+T)\},\ \mathcal{X} = C([-\tau, 0], X)$$

with the maximum norm and the positive cone

$$\mathbb{X}_+ = \{u \in C(\mathbb{R}, X_+) : u(t) = u(t+T)\},\ \mathcal{X}_+ = C([-\tau, 0], X_+).$$

Assume that

(H1) $F(t)$ is positive from $\mathcal{X}$ to X and T-periodic for all $t \in \mathbb{R}$.

(H2) Let $\{\Phi(t,s) : t \geq s\}$ be the T-periodic evolution family of

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -V(t)u(t),\ t \geq 0. \tag{8}$$

on X. $\Phi(t,s)$ is positive on X for all $t \geq s$ and $\omega(\Phi) < 0$.

Consider the equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(t)u_t - V(t)u(t),\ t \geq 0. \tag{9}$$

So $\mathcal{L}$, $\mathcal{R}_0$ can be defined by

$$\mathcal{L}u(t) = \int_0^{+\infty} \Phi(t, t-s)F(t-s)u_{t-s}\mathrm{d}s, u \in \mathbb{X}, \tag{10}$$

and

$$\mathcal{R}_0 = r(\mathcal{L}).$$

Let $\{U(t,s;\mu) : t \geq s\}$ be evolution family on $\mathcal{X}$ of the following system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \frac{1}{\mu}F(t)u_t - V(t)u(t),\ t \geq 0. \tag{11}$$

Proposition 4. *Under some conditions, $\mu = \mathcal{R}_0$ is the unique solution of $r(U(T,0;\mu)) = 1$.*

We also compute the spectral radius of $r(U(T,0;\mu))$ by Remark 1 with $L = U(T,0;\mu)$ and search the unique solution of the equation $r(U(T,0;\mu)) = 1$ by the bisection method.

5 Example

Next, we give some examples for the above methods. For convenience, we omit the corresponding domain when we give the definition of the operator $V(t)$ and $F(t)$.

Example 1 (Autonomous, ODEs, 3D)**.** *Let*

$$V := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, F := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$\mathcal{L} = FV^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \ \mathcal{R}_0 = r(\mathcal{L}) = (\frac{1}{6})^{\frac{1}{3}}.$$

Example 2 (Periodic, ODEs, 1D)**.** *Let $V(t) := m(t)$, $F(t) := f(t)$, where $T = 12$ and*

$$m(t) = 0.2(1 + 0.2\cos(2\pi\frac{t}{T})), \ f(t) = 0.35(1 + 0.2\sin(2\pi\frac{t}{T})).$$

In this case, $\mathcal{R}_0 = \frac{\int_0^T f(t)\mathrm{d}t}{\int_0^T m(t)\mathrm{d}t} = 1.75$.

Example 3 (Periodic, ODEs, 2D)**.** *Let*

$$V(t) := \begin{pmatrix} m_1(t) & 0 \\ 0 & m_2(t) \end{pmatrix}, \ F(t) := \begin{pmatrix} 0 & f_{12}(t) \\ f_{21}(t) & 0 \end{pmatrix},$$

where $T = 12$ and

$$m_1(t) = 0.2(1 + 0.8\cos(2\pi\frac{t}{T})), \ m_2(t) = 0.3(1 + 0.8\cos(2\pi\frac{t}{T})),$$
$$f_{12}(t) = 0.35(1 + 0.8\cos(2\pi\frac{t}{T})), \ f_{21}(t) = 0.5(1 + 0.8\sin(2\pi\frac{t}{T})).$$

Example 4 (Periodic, DDEs, 2D)**.** *Let*

$$V(t)u(t) := \begin{pmatrix} m_1(t)u_1(t) \\ m_2(t)u_2(t) \end{pmatrix}, F(t)u_t := \begin{pmatrix} f_{12}(t)u_2(t-\tau_2) \\ f_{21}(t)u_1(t-\tau_1) \end{pmatrix},$$

where $T = 12$, $\tau_1 = 3$, $\tau_2 = 2$ and

$$m_1(t) = 0.2(1 + 0.8\cos(2\pi\frac{t}{T})), \ m_2(t) = 0.3(1 + 0.8\cos(2\pi\frac{t}{T})),$$
$$f_{12}(t) = 0.35(1 + 0.8\cos(2\pi\frac{t}{T})), \ f_{21}(t) = 0.5(1 + 0.8\sin(2\pi\frac{t}{T})).$$

Example 5 (Periodic, DDEs with periodic delay, 2D)**.** *Let*

$$V(t)u(t) := \begin{pmatrix} m_1(t)u_1(t) \\ m_2(t)u_2(t) \end{pmatrix}, F(t)u_t := \begin{pmatrix} f_{12}(t)u_2(t-\tau_2(t)) \\ f_{21}(t)u_1(t-\tau_1(t)) \end{pmatrix},$$

where $T = 12$, $\tau_1 = 1.8\cos(2\pi\frac{t}{T}) + 2$, $\tau_2 = 1.8\sin(2\pi\frac{t}{T}) + 2$ *and*

$$m_1(t) = 0.2(1 + 0.8\cos(2\pi\frac{t}{T})),\ m_2(t) = 0.3(1 + 0.8\cos(2\pi\frac{t}{T})),$$
$$f_{12}(t) = 0.35(1 + 0.8\cos(2\pi\frac{t}{T})),\ f_{21}(t) = 0.5(1 + 0.8\sin(2\pi\frac{t}{T})).$$

Example 6 (Autonomous, PDEs, scalar equation)**.** *Let* $\overline{\Omega} = [0, 1]$,

$$[Fu](x) := f(x)u(x),\ [Vu](x) := -[d\Delta u(x) - m(x)u(x)]$$

with Neumann boundary condition,

$$m(x) = 1 + 0.5\cos(\frac{\pi}{2}x),\ f(x) = 1 + 0.5\sin(\frac{\pi}{2}x)$$

and $d = 0.01$. *Then*

$$\mathcal{L} = FV^{-1},\ \mathcal{R}_0 = r(\mathcal{L}).$$

Example 7 (Autonomous, PDEs, two equations)**.** *Let* $\overline{\Omega} = [0, 1]$,

$$[Fu](x) := \begin{pmatrix} f_{12}(x)u_2(x) \\ f_{21}(x)u_1(x) \end{pmatrix}, [Vu](x) := -\begin{pmatrix} d_1\Delta u_1(x) - m_1(x)u_1(x) \\ d_2\Delta u_2(x) - m_2(x)u_2(x) \end{pmatrix}$$

with Neumann boundary condition and

$$f_{12}(x) = 1 + 0.5\cos(\frac{\pi}{2}x),\ f_{21}(x) = 1 + 0.5\sin(\frac{\pi}{2}x),$$

$$m_1(x) = 1 + 0.5\cos(\frac{\pi}{2}x),\ m_2(x) = 1 + 0.5\sin(\frac{\pi}{2}x),$$

and $d_1 = 0.01$, $d_2 = 0.02$. *Then*

$$\mathcal{L} = FV^{-1},\ \mathcal{R}_0 = r(\mathcal{L}).$$

In this case, $\mathcal{R}_0 = 1$.

Example 8 (Periodic, PDEs, scalar equation). *Let* $\overline{\Omega} = [0, 1]$,

$$[F(t)u](x) := f(x,t)u(x),$$

$$[V(t)u](x) := -[d\Delta u(x) - m(x,t)u(x)],$$

with Neumann boundary condition,

$$m(x,t) = (1 + 0.5\cos(\frac{\pi}{2}x))(1 + 0.5\cos(2\pi\frac{t}{T})),$$

$$f(x,t) = (1 + 0.5\sin(\frac{\pi}{2}x))(1 + 0.5\cos(2\pi\frac{t}{T})),$$

$d = 0.01$ *and* $T = 12$.

Example 9 (Periodic, PDEs, two equations). *Let* $\overline{\Omega} = [0, 1]$,

$$[F(t)u](x) := \begin{pmatrix} f_{12}(x,t)u_2(x) \\ f_{21}(x,t)u_1(x) \end{pmatrix},$$

$$[V(t)u](x) := -\begin{pmatrix} d_1\Delta u_1(x) - m_1(x,t)u_1(x) \\ d_2\Delta u_2(x) - m_2(x,t)u_2(x) \end{pmatrix}$$

with Neumann boundary condition,

$$m_1(x,t) = (1 + 0.5\cos(\frac{\pi}{2}x))(1 + 0.5\cos(2\pi\frac{t}{T})),$$

$$m_2(x,t) = (1 + 0.5\sin(\frac{\pi}{2}x))(1 + 0.5\cos(2\pi\frac{t}{T})),$$

$$f_{12}(x,t) = (1 + 0.5\cos(\frac{\pi}{2}x))(1 + 0.5\sin(2\pi\frac{t}{T})),$$

$$f_{21}(x,t) = (1 + 0.5\sin(\frac{\pi}{2}x))(1 + 0.5\sin(2\pi\frac{t}{T})),$$

$d_1 = 0.01$, $d_2 = 0.02$ *and* $T = 12$.

Example 10 (Periodic, PDEs with time delay, two equations). *Let* $\overline{\Omega} = [0, 1]$,

$$[F(t)\phi](x) := \begin{pmatrix} f_{12}(x,t)\phi_2(-\tau)(x) \\ f_{21}(x,t)\phi_1(0)(x) \end{pmatrix},$$

$$[V(t)u](x) := -\begin{pmatrix} d_1\Delta u_1(x) - m_1(x,t)u_1(x) \\ d_2\Delta u_2(x) - m_2(x,t)u_2(x) \end{pmatrix}$$

with Neumann boundary condition,

$$m_1(x,t) = (1 + 0.5\cos(\frac{\pi}{2}x))(1 + 0.5\cos(2\pi\frac{t}{T})),$$

$$m_2(x,t) = (1 + 0.5\sin(\frac{\pi}{2}x))(1 + 0.5\cos(2\pi\frac{t}{T})),$$

$$f_{12}(x,t) = (1 + 0.5\cos(\frac{\pi}{2}x))(1 + 0.5\sin(2\pi\frac{t}{T})),$$

$$f_{21}(x,t) = (1 + 0.5\sin(\frac{\pi}{2}x))(1 + 0.5\sin(2\pi\frac{t}{T})),$$

$d_1 = 0.01$*,* $d_2 = 0.02$*,* $T = 12$ *and* $\tau = 0.6$*.*

6 Summary

For an autonomous model, $\mathcal{R}_0$ can be calculated numerically by computing the spectral radius of the corresponding operator. For a time-periodic model, $\mathcal{R}_0$ can be calculated by transfering to solve another problem, which is called Root method. In [8], we proposed a direct method to compute $\mathcal{R}_0$ by computing the spectral radius of the corresponding operator, which can be regarded as a generalized method in [4]. But it costs too much memory and it is difficult to run on a personal computer for a complex model. So we don't introduce it in this note.

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