

# A summary for the program of $\mathcal{R}_0$

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## Abstract

This note is to summary how to compute the basic reproduction ratios numerically.

## 1 Introduction

How to calculate  $\mathcal{R}_0$  numerically is a very important problem. For the autonomous models of ODEs, Van den Driessche and Watmough [5] found that the principal eigenvalue of the next generation matrix admits  $\mathcal{R}_0$ . For time-periodic compartmental ODEs models, Wang and Zhao [6] proposed a numerical method to compute  $\mathcal{R}_0$  by solving the unique root of a critical equation(see [6, Theorem 2.1]). We call it Root method. An alternative method to deal with the problem goes to Posny and Wang [4]. They transformed the problem into a matrix eigenvalue problem. For the infinite-dimensional cases, motivated by [5], Wang and Zhao presented a numerical method to approximate the principal eigenvalue of the next generation operator for the autonomous models of reaction-diffusion equations. Combined with the Root method and the principal eigenvalue of positive operators, a numerical method was developed by Liang, Zhang and Zhao [3]. This method can be applied to various kinds of periodic models, including ODEs, reaction-diffusion equations, nonlocal dispersal equations with or without time delay. In [8], Yang and Zhang proposed a direct method to compute  $\mathcal{R}_0$  by computing the spectral radius of the corresponding operator, which can be regarded as a generalized method in [4]. We also refer to [2, 1].

The following part of this note is to describe the algorithm to compute  $\mathcal{R}_0$ . In the next section, we show a generalized Power Method to compute the spectral radius of a positive operator. In section 3, we show a numerical method to calculate  $\mathcal{R}_0$  by approximating to the spectral radius of the

associated operator for autonomous cases. In section 4, we show the Root method to compute  $\mathcal{R}_0$ . In section 5, we give ten examples to show how to compute  $\mathcal{R}_0$  in Matlab. In section 6, we present a short summary for these methods.

## 2 Compute the spectral radius

Now we introduce a generalized Power Method to compute the spectral radius of a positive operator  $L$  numerically.

**Lemma 1.** *Assume that  $(E, E_+)$  is an ordered Banach space with  $E_+$  being normal and  $\text{Int}(E_+) \neq \emptyset$ . Let  $L$  be a positive bounded linear operator. Then  $r(L) = \lim_{n \rightarrow +\infty} \|L^n e\|_E^{\frac{1}{n}}$ ,  $\forall e \in \text{Int}(E_+)$ .*

**Lemma 2.** *Assume that  $(E, E_+)$  is an ordered Banach space with  $E_+$  being normal and  $\text{Int}(E_+) \neq \emptyset$ , which is equipped with the norm  $\|\cdot\|_E$ . Let  $L$  be a positive bounded linear operator. Choose  $v_0 \in \text{Int}(E_+)$  and define  $a_n = \|Lv_{n-1}\|_E$ ,  $v_n = \frac{Lv_{n-1}}{a_n}$ ,  $\forall n \geq 1$ . If  $\lim_{n \rightarrow +\infty} a_n$  exists, then  $r(L) = \lim_{n \rightarrow +\infty} a_n$ .*

**Remark 1.** *The algorithm to compute  $r(L)$ :*

1.  $v_0 = 1$ .
2.  $v_n = \frac{Lv_{n-1}}{\|Lv_{n-1}\|}$ ,  $\forall n \geq 1$ .
3. Let  $a_n = \|Lv_{n-1}\|$ ,  $n \geq 1$

(a) By Lemma 1,  $r(L) = \lim_{n \rightarrow +\infty} (\prod_{k=1}^n a_k)^{\frac{1}{n}}$ . This is because

$$\|L^n v_0\| = \prod_{k=1}^n a_k. \quad (1)$$

(b) In particular, if  $\lim_{n \rightarrow +\infty} a_n$  exists, then  $r(L) = \lim_{n \rightarrow +\infty} a_n$ .

**Remark 2.** *For a non-negative matrix  $L$ , it is known that*

1. If  $L^{k_0}$  is strongly positive for some  $k_0 > 0$ , then  $a_n$  in Lemma 2 converges.
2. If  $L$  is irreducible, then there exists some  $k_0$  such that  $(L + I)^{k_0}$  is positive.

### 3 Autonomous Cases

Now, we present a numerical method to compute  $\mathcal{R}_0$  for a autonomous model. We refer to [5, 7].

Let  $X$  be a Banach space with the positive cone  $X_+$ . Assume that

(H1)  $F$  is a bounded positive operator on  $X$ .

(H2)  $-V$  is a resolvent positive operator on  $X$  and  $s(-V) < 0$ .

Consider the equation

$$\frac{du}{dt} = (F - V)u \quad (2)$$

So  $\mathcal{L}$ ,  $\mathcal{R}_0$  can be defined by

$$\mathcal{L}u = FV^{-1}u, \quad \mathcal{R}_0 = r(\mathcal{L}). \quad (3)$$

**Remark 3.** 1. *It is easy to see that*

$$r(FV^{-1}) = r(V^{-1}F).$$

2. *In Matlab, eigenvalues of an matrix can be computed directly.*

3. *We can also compute the spectral radius of  $r(\mathcal{L})$  by Remark 1 with  $L = \mathcal{L}$ .*

4. *Operators  $V$  and  $F$  can be approximated by discretization.*

### 4 Periodic cases

Next, we introduce the Root method. This method is proposed in [6] and developed in [3].

#### 4.1 Periodic cases without time-delay

Let  $X$  be a Banach space with the positive cone  $X_+$ , and

$$\mathbb{X} = \{u \in C(\mathbb{R}, X) : u(t) = u(t + T)\}$$

with the maximum norm and the positive cone

$$\mathbb{X}_+ = \{u \in C(\mathbb{R}, X_+) : u(t) = u(t + T)\}$$

Assume that

(H1)  $F(t)$  is positive on  $X$  and  $T$ -periodic for all  $t \in \mathbb{R}$ .

(H2) Let  $\{\Phi(t, s) : t \geq s\}$  be the  $T$ -periodic evolution family of

$$\frac{du(t)}{dt} = -V(t)u(t), \quad t \geq 0. \quad (4)$$

on  $X$ .  $\Phi(t, s)$  is positive on  $X$  for all  $t \geq s$  and  $\omega(\Phi) < 0$ .

Consider the equation

$$\frac{du(t)}{dt} = F(t)u(t) - V(t)u(t), \quad t \geq 0. \quad (5)$$

So  $\mathcal{L}$ ,  $\mathcal{R}_0$  can be defined by

$$\mathcal{L}u(t) = \int_0^{+\infty} \Phi(t, t-s)F(t-s)u(t-s)ds, \quad u \in \mathbb{X}, \quad (6)$$

and

$$\mathcal{R}_0 = r(\mathcal{L}).$$

Let  $\{U(t, s; \mu) : t \geq s\}$  be evolution family on  $X$  of the following system

$$\frac{du(t)}{dt} = \frac{1}{\mu}F(t)u(t) - V(t)u(t), \quad t \geq 0. \quad (7)$$

**Proposition 3.** *Under some conditions,  $\mu = \mathcal{R}_0$  is the unique solution of  $r(U(T, 0; \mu)) = 1$ .*

We also compute the spectral radius of  $r(U(T, 0; \mu))$  by Remark 1 with  $L = U(T, 0; \mu)$  and search the solution of the equation  $r(U(T, 0; \mu)) = 1$  by the bisection method.

**Remark 4.** *The algorithm to compute  $r(L)$ :*

1.  $v_0 = 1$  is the initial data.
2.  $v_n = \frac{Lv_{n-1}}{\|Lv_{n-1}\|}$ ,  $\forall n \geq 1$ . Here  $Lv_{n-1} = U(T, 0; \mu)v_{n-1}$  can be numerically computed by standard numerical method of differential equations.
3. Let  $a_n = \|Lv_{n-1}\|$ ,  $n \geq 1$ . Usually,  $U(T, 0, \mu)$  is eventually positive. Thus,  $\lim_{n \rightarrow +\infty} a_n$  exists, and  $r(L) = \lim_{n \rightarrow +\infty} a_n$ .

## 4.2 Periodic cases with time-delay

Let  $X$  be a Banach space with the positive cone  $X_+$ , and

$$\mathbb{X} = \{u \in C(\mathbb{R}, X) : u(t) = u(t + T)\}, \quad \mathcal{X} = C([-\tau, 0], X)$$

with the maximum norm and the positive cone

$$\mathbb{X}_+ = \{u \in C(\mathbb{R}, X_+) : u(t) = u(t + T)\}, \quad \mathcal{X}_+ = C([-\tau, 0], X_+).$$

Assume that

(H1)  $F(t)$  is positive from  $\mathcal{X}$  to  $X$  and  $T$ -periodic for all  $t \in \mathbb{R}$ .

(H2) Let  $\{\Phi(t, s) : t \geq s\}$  be the  $T$ -periodic evolution family of

$$\frac{du(t)}{dt} = -V(t)u(t), \quad t \geq 0. \quad (8)$$

on  $X$ .  $\Phi(t, s)$  is positive on  $X$  for all  $t \geq s$  and  $\omega(\Phi) < 0$ .

Consider the equation

$$\frac{du(t)}{dt} = F(t)u_t - V(t)u(t), \quad t \geq 0. \quad (9)$$

So  $\mathcal{L}$ ,  $\mathcal{R}_0$  can be defined by

$$\mathcal{L}u(t) = \int_0^{+\infty} \Phi(t, t-s)F(t-s)u_{t-s}ds, \quad u \in \mathbb{X}, \quad (10)$$

and

$$\mathcal{R}_0 = r(\mathcal{L}).$$

Let  $\{U(t, s; \mu) : t \geq s\}$  be evolution family on  $\mathcal{X}$  of the following system

$$\frac{du(t)}{dt} = \frac{1}{\mu}F(t)u_t - V(t)u(t), \quad t \geq 0. \quad (11)$$

**Proposition 4.** *Under some conditions,  $\mu = \mathcal{R}_0$  is the unique solution of  $r(U(T, 0; \mu)) = 1$ .*

We also compute the spectral radius of  $r(U(T, 0; \mu))$  by Remark 1 with  $L = U(T, 0; \mu)$  and search the unique solution of the equation  $r(U(T, 0; \mu)) = 1$  by the bisection method.

## 5 Example

Next, we give some examples for the above methods. For convenience, we omit the corresponding domain when we give the definition of the operator  $V(t)$  and  $F(t)$ .

**Example 1** (Autonomous, ODEs, 3D). *Let*

$$V := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, F := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

*Then*

$$\mathcal{L} = FV^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \mathcal{R}_0 = r(\mathcal{L}) = \left(\frac{1}{6}\right)^{\frac{1}{3}}.$$

**Example 2** (Periodic, ODEs, 1D). *Let  $V(t) := m(t)$ ,  $F(t) := f(t)$ , where  $T = 12$  and*

$$m(t) = 0.2(1 + 0.2 \cos(2\pi \frac{t}{T})), \quad f(t) = 0.35(1 + 0.2 \cos(2\pi \frac{t}{T})).$$

*In this case,  $\mathcal{R}_0 = \frac{\int_0^T f(t)dt}{\int_0^T m(t)dt} = 1.75$ .*

**Example 3** (Periodic, ODEs, 2D). *Let*

$$V(t) := \begin{pmatrix} m_1(t) & 0 \\ 0 & m_2(t) \end{pmatrix}, \quad F(t) := \begin{pmatrix} 0 & f_{12}(t) \\ f_{21}(t) & 0 \end{pmatrix},$$

*where  $T = 12$  and*

$$\begin{aligned} m_1(t) &= 0.2(1 + 0.8 \cos(2\pi \frac{t}{T})), \quad m_2(t) = 0.3(1 + 0.8 \cos(2\pi \frac{t}{T})), \\ f_{12}(t) &= 0.35(1 + 0.8 \cos(2\pi \frac{t}{T})), \quad f_{21}(t) = 0.5(1 + 0.8 \sin(2\pi \frac{t}{T})). \end{aligned}$$

**Example 4** (Periodic, DDEs, 2D). *Let*

$$V(t)u(t) := \begin{pmatrix} m_1(t)u_1(t) \\ m_2(t)u_2(t) \end{pmatrix}, \quad F(t)u_t := \begin{pmatrix} f_{12}(t)u_2(t - \tau_2) \\ f_{21}(t)u_1(t - \tau_1) \end{pmatrix},$$

*where  $T = 12$ ,  $\tau_1 = 3$ ,  $\tau_2 = 2$  and*

$$\begin{aligned} m_1(t) &= 0.2(1 + 0.8 \cos(2\pi \frac{t}{T})), \quad m_2(t) = 0.3(1 + 0.8 \cos(2\pi \frac{t}{T})), \\ f_{12}(t) &= 0.35(1 + 0.8 \cos(2\pi \frac{t}{T})), \quad f_{21}(t) = 0.5(1 + 0.8 \sin(2\pi \frac{t}{T})). \end{aligned}$$

**Example 5** (Periodic, DDEs with periodic delay, 2D). *Let*

$$V(t)u(t) := \begin{pmatrix} m_1(t)u_1(t) \\ m_2(t)u_2(t) \end{pmatrix}, F(t)u_t := \begin{pmatrix} f_{12}(t)u_2(t - \tau_2(t)) \\ f_{21}(t)u_1(t - \tau_1(t)) \end{pmatrix},$$

where  $T = 12$ ,  $\tau_1 = 1.8 \cos(2\pi \frac{t}{T}) + 2$ ,  $\tau_2 = 1.8 \sin(2\pi \frac{t}{T}) + 2$  and

$$m_1(t) = 0.2(1 + 0.8 \cos(2\pi \frac{t}{T})), \quad m_2(t) = 0.3(1 + 0.8 \cos(2\pi \frac{t}{T})), \\ f_{12}(t) = 0.35(1 + 0.8 \cos(2\pi \frac{t}{T})), \quad f_{21}(t) = 0.5(1 + 0.8 \sin(2\pi \frac{t}{T})).$$

**Example 6** (Autonomous, PDEs, scalar equation). *Let  $\bar{\Omega} = [0, 1]$ ,*

$$[Fu](x) := f(x)u(x), \quad [Vu](x) := -[d\Delta u(x) - m(x)u(x)]$$

with Neumann boundary condition,

$$m(x) = 1 + 0.5 \cos(\frac{\pi}{2}x), \quad f(x) = 1 + 0.5 \sin(\frac{\pi}{2}x)$$

and  $d = 0.01$ . Then

$$\mathcal{L} = FV^{-1}, \quad \mathcal{R}_0 = r(\mathcal{L}).$$

**Example 7** (Autonomous, PDEs, two equations). *Let  $\bar{\Omega} = [0, 1]$ ,*

$$[Fu](x) := \begin{pmatrix} f_{12}(x)u_2(x) \\ f_{21}(x)u_1(x) \end{pmatrix}, [Vu](x) := - \begin{pmatrix} d_1\Delta u_1(x) - m_1(x)u_1(x) \\ d_2\Delta u_2(x) - m_2(x)u_2(x) \end{pmatrix}$$

with Neumann boundary condition and

$$f_{12}(x) = 1 + 0.5 \cos(\frac{\pi}{2}x), \quad f_{21}(x) = 1 + 0.5 \sin(\frac{\pi}{2}x),$$

$$m_1(x) = 1 + 0.5 \cos(\frac{\pi}{2}x), \quad m_2(x) = 1 + 0.5 \sin(\frac{\pi}{2}x),$$

and  $d_1 = 0.01$ ,  $d_2 = 0.02$ . Then

$$\mathcal{L} = FV^{-1}, \quad \mathcal{R}_0 = r(\mathcal{L}).$$

In this case,  $\mathcal{R}_0 = 1$ .

**Example 8** (Periodic, PDEs, scalar equation). *Let  $\overline{\Omega} = [0, 1]$ ,*

$$[F(t)u](x) := f(x, t)u(x),$$

$$[V(t)u](x) := -[d\Delta u(x) - m(x, t)u(x)],$$

*with Neumann boundary condition,*

$$m(x, t) = (1 + 0.5 \cos(\frac{\pi}{2}x))(1 + 0.5 \cos(2\pi \frac{t}{T})),$$

$$f(x, t) = (1 + 0.5 \sin(\frac{\pi}{2}x))(1 + 0.5 \cos(2\pi \frac{t}{T})),$$

*$d = 0.01$  and  $T = 12$ .*

**Example 9** (Periodic, PDEs, two equations). *Let  $\overline{\Omega} = [0, 1]$ ,*

$$[F(t)u](x) := \begin{pmatrix} f_{12}(x, t)u_2(x) \\ f_{21}(x, t)u_1(x) \end{pmatrix},$$

$$[V(t)u](x) := - \begin{pmatrix} d_1\Delta u_1(x) - m_1(x, t)u_1(x) \\ d_2\Delta u_2(x) - m_2(x, t)u_2(x) \end{pmatrix}$$

*with Neumann boundary condition,*

$$m_1(x, t) = (1 + 0.5 \cos(\frac{\pi}{2}x))(1 + 0.5 \cos(2\pi \frac{t}{T})),$$

$$m_2(x, t) = (1 + 0.5 \sin(\frac{\pi}{2}x))(1 + 0.5 \cos(2\pi \frac{t}{T})),$$

$$f_{12}(x, t) = (1 + 0.5 \cos(\frac{\pi}{2}x))(1 + 0.5 \sin(2\pi \frac{t}{T})),$$

$$f_{21}(x, t) = (1 + 0.5 \sin(\frac{\pi}{2}x))(1 + 0.5 \sin(2\pi \frac{t}{T})),$$

*$d_1 = 0.01$ ,  $d_2 = 0.02$  and  $T = 12$ .*

**Example 10** (Periodic, PDEs with time delay, two equations). *Let  $\overline{\Omega} = [0, 1]$ ,*

$$[F(t)\phi](x) := \begin{pmatrix} f_{12}(x, t)\phi_2(-\tau)(x) \\ f_{21}(x, t)\phi_1(0)(x) \end{pmatrix},$$

$$[V(t)u](x) := - \begin{pmatrix} d_1\Delta u_1(x) - m_1(x, t)u_1(x) \\ d_2\Delta u_2(x) - m_2(x, t)u_2(x) \end{pmatrix}$$



with Neumann boundary condition,

$$m_1(x, t) = (1 + 0.5 \cos(\frac{\pi}{2}x))(1 + 0.5 \cos(2\pi \frac{t}{T})),$$

$$m_2(x, t) = (1 + 0.5 \sin(\frac{\pi}{2}x))(1 + 0.5 \cos(2\pi \frac{t}{T})),$$

$$f_{12}(x, t) = (1 + 0.5 \cos(\frac{\pi}{2}x))(1 + 0.5 \sin(2\pi \frac{t}{T})),$$

$$f_{21}(x, t) = (1 + 0.5 \sin(\frac{\pi}{2}x))(1 + 0.5 \sin(2\pi \frac{t}{T})),$$

$d_1 = 0.01$ ,  $d_2 = 0.02$ ,  $T = 12$  and  $\tau = 0.6$ .

## 6 Summary

For an autonomous model,  $\mathcal{R}_0$  can be calculated numerically by computing the spectral radius of the corresponding operator. For a time-periodic model,  $\mathcal{R}_0$  can be calculated by transferring to solve another problem, which is called Root method. In [8], we proposed a direct method to compute  $\mathcal{R}_0$  by computing the spectral radius of the corresponding operator, which can be regarded as a generalized method in [4]. But it costs too much memory and it is difficult to run on a personal computer for a complex model. So we don't introduce it in this note.

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