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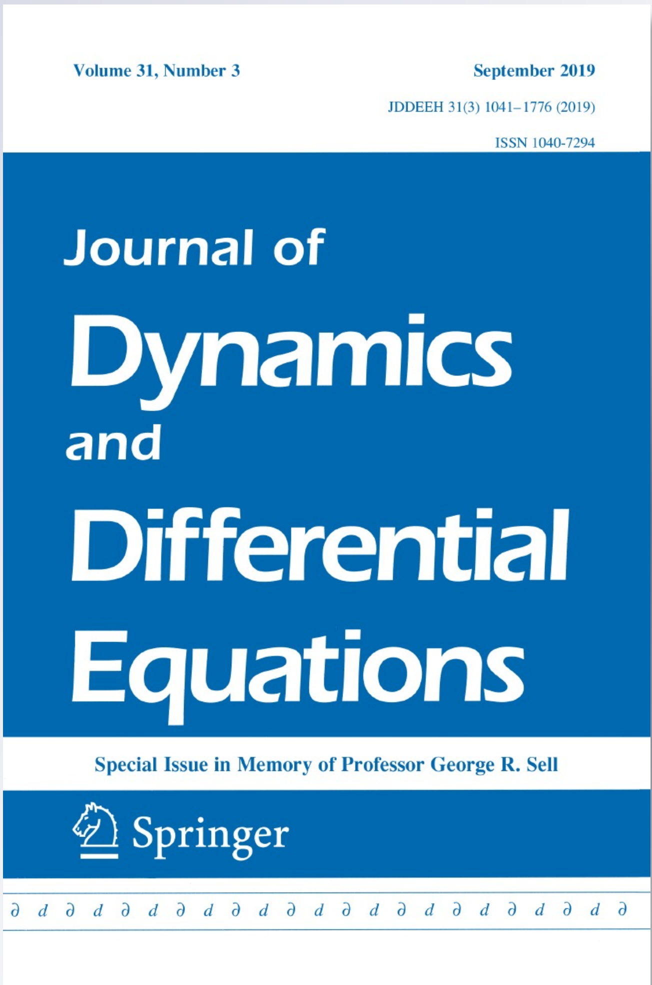

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Basic Reproduction Ratios for Periodic Abstract Functional Differential Equations (with Application to a Spatial Model for Lyme Disease)

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In Memory of Professor George Sell

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Abstract In this paper, we develop the theory of basic reproduction ratios $\mathcal{R}_0$ for abstract functional differential systems in a time-periodic environment. It is proved that $\mathcal{R}_0 - 1$ has the same sign as the exponential growth bound of an associated linear system. Then we apply it to a time-periodic Lyme disease model with time-delay and obtain a threshold type result on its global dynamics in terms of $\mathcal{R}_0$.

Keywords Basic reproduction ratio · Abstract functional differential system · Periodic solution · Lyme disease · Threshold dynamics

AMS Subject Classification 34K20 · 35K57 · 37B55 · 92D30

1 Introduction

The basic reproduction ratio $\mathcal{R}_0$ is an important parameter in population biology. In epidemiology, $\mathcal{R}_0$ is the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. Diekmann et al. [9] first introduced the next generation matrix approach to $\mathcal{R}_0$ and then van den Driessche and Watmough [35] established the theory of $\mathcal{R}_0$ for the autonomous models of ordinary differential equations(ODEs) with compartment structure. These two works have been applied extensively to various infectious disease models. For population models in a time-periodic environment, Bacaër and Guernaoui [4] presented a general definition of $\mathcal{R}_0$, that is, $\mathcal{R}_0$ is the spectral radius of an integral operator defined on the space of continuous periodic functions. Wang and Zhao [37] developed the theory of $\mathcal{R}_0$ to time-periodic compartmental ODEs models and proved that

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$\mathcal{R}_0$ is a threshold value for the local stability of the disease-free periodic solution. Further, Thieme [34] obtained the relationship between spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity. Bacaër and Dads [2,3] also gave a more biological interpretation of $\mathcal{R}_0$ for periodic models and showed that $\mathcal{R}_0$ is the asymptotic ratio of total infections in two successive generations of the infection tree. Inaba [19] introduced the concept of a generation evolution operator to give a new definition of $\mathcal{R}_0$ for structured populations in heterogeneous environments, which unifies two definitions in [4,9] and has intuitively clear biological meaning. More recently, Zhao [45] established the theory of basic reproduction ratios for a large class of time-delayed compartmental population models in a periodic environment.

For population growth models, the basic reproduction ratio $\mathcal{R}_0$ can be regard as the total number of the expected offspring an average individual would produce during its lifetime. Accordingly, if $\mathcal{R}_0 > 1$, each individual, on average, can replace itself, which means the population persists; if $\mathcal{R}_0 < 1$, an average individual cannot replace itself, and the population becomes extinct. Mathematically, this implies that $\mathcal{R}_0$ is a threshold value to determine the local stability of the zero steady state. The theory in [34] can be applied to autonomous reaction-diffusion models with or without time-delay (see, e.g., [14,22,26,41]), time-periodic reaction-diffusion models without delay (see, e.g., [38,43]), and almost periodic compartment ODE models (see, e.g., [36]). However, it is hard to apply the abstract results to periodic and time-delay models. The recent work in [45] can also be employed for the periodic and time-delayed reaction-diffusion models whose solution maps are eventually compact (see, e.g., [40,42]). The purpose of this paper is to extend such a theory to abstract functional differential equations whose solution maps may be non-compact. Motivated by the works of [37,45], we introduce a one-parameter family of positive linear operators and study the relationship between their spectral radius and basic reproduction ratios with the help of the principal eigenvalues for the associated positive operators. We also propose a numerical method to compute $\mathcal{R}_0$, which seems to be more effective (see Remark 3.2).

The remaining part of this paper is organized as follows. In the next section, we introduce notations, give the definition of the principal eigenvalue and present some preliminary results on positive operators. In Sect. 3, we develop the theory of basic reproduction ratios for abstract functional differential equations and provide a numerical method to compute $\mathcal{R}_0$. In Sect. 4, we apply the developed theory to a time-periodic Lyme disease model with time-delay and establish a threshold type result on its global dynamics in terms of $\mathcal{R}_0$. A brief discussion section then completes the paper.

2 Preliminaries

Let (E, E_+) be an ordered Banach space. We use $\geq$($>$ and $\gg$) to represent the (strict and strong) order relation induced by the cone E_+. Letting G be another Banach space, we denote by $\mathcal{L}(E, G)$ the Banach space of all continuous linear operators from E to G which is equipped with the operator norm $\|\cdot\|_{E,G}$. For convenience, we write $\mathcal{L}(E) = \mathcal{L}(E, E)$ and $\|\cdot\|_E = \|\cdot\|_{E,E}$. Let $T > 0$ be a given number. A family of bounded linear operators $\Theta(t, s)$ on E, $t, s \in \mathbb{R}$ with $t \geq s$, is called T- periodic evolutionary system provided that

$$\Theta(s, s) = I, \quad \Theta(t, r)\Theta(r, s) = \Theta(t, s), \quad \Theta(t + T, s + T) = \Theta(t, s),$$

for all $t, s, r \in \mathbb{R}$ with $t \geq r \geq s$, and for each $e \in E$, $\Theta(t, s)e$ is a continuous function of (t, s), $t \geq s$. The exponential growth bound of evolution family $\{\Theta(t, s) : t \geq s\}$ is defined as

$$\omega(\Theta) = \inf\{\tilde{\omega} : \exists M \geq 1 : \forall s \in \mathbb{R},\ t \geq s : \|\Theta(t,s)\| \leq Me^{\tilde{\omega}(t-s)}\}.$$

The following lemma shows the relationship between $\omega(\Theta)$ and $r(\Theta(T,0))$, where $r(\Theta(T,0))$ is the spectral radius of $\Theta(T,0)$ on E.

Lemma 2.1 ([34, Propostion A.2]) *Let E be a Banach space and $\{\Theta(t,s) : t \geq s\}$ be a evolution family on E. Then $\omega(\Theta) = \frac{\ln r(\Theta(T,0))}{T} = \frac{\ln r(\Theta(T+\xi,\xi))}{T}$, $\forall \xi \in [0,T]$.*

Throughout this paper, we use the following definition of the principal eigenvalue.

Definition 2.2 Assume that (E, E_+) is an ordered Banach space, and L is a positive bounded linear operator on E. The spectral radius $r(L)$ of L is called the principal eigenvalue if $r(L)$ is an eigenvalue of L with a positive eigenvector. The principal eigenvalue is said to be isolated if $r(L)$ is isolated in the spectrum of L. The isolated principal eigenvalue with finite multiplicity means that the $r(L)$ is isolated principal eigenvalue and the null space of $r(L)I - L$ is finite dimensional, where I is the identity map.

Next, we recall a generalized Ascoli–Arzelà theorem(see, e.g., [21, Chapter 7] or [8, Section 7.4]).

Lemma 2.3 (Generalized Ascoli-Arzelà Theorem) *Let $a < b$ be two real numbers and $\mathcal{X}$ be a complete metric space. Assume that a sequence of functions $\{f_n\}$ in $C([a,b],\mathcal{X})$ satisfies the following conditions:*

(i) The family $\{f_n(s)\}_{n\geq 1}$ is uniformly bounded on $[a,b]$;
(ii) For each $s \in [a,b]$, the set $\{f_n(s) : n \geq 1\}$ is precompact in $\mathcal{X}$;
(iii) The family $\{f_n(s)\}_{n\geq 1}$ is equi-continuous on $[a,b]$.

Then $\{f_n\}$ has a convergent subsequence in $C([a,b],\mathcal{X})$, that is, there exists a subsequence of functions $\{f_{n_k}(s)\}$ which converges in $\mathcal{X}$ uniformly for $s \in [a,b]$

The following result is about the spectral radius of positive linear operators (see, e.g., [32, Section 2]). For completeness, here we provide an elementary proof.

Lemma 2.4 *Assume that (E, E_+) is an ordered Banach space with E_+ being normal and $\mathrm{Int}(E_+) \neq \emptyset$. Let L be a positive bounded linear operator. Then $r(L) = \lim_{n\to+\infty} \|L^n e\|_E^{\frac{1}{n}}$, $\forall e \in \mathrm{Int}(E_+)$.*

Proof Let $\|\cdot\|_E$ be the norm in E. We fix an $e \in \mathrm{Int}(E_+)$ and define

$$\|x\|_e = \inf\{\alpha : -\alpha e \leq x \leq \alpha e\},\ \forall x \in E.$$

Then $\|\cdot\|_e$ is an equivalent norm on E(see, e.g., [1, Theroem 2.4]). Next, we assert that $\|L\|_e = \|Le\|_e$, where $\|L\|_e$ denotes the operator norm of L which is defined by $\|L\|_e = \sup_{\|x\|_e=1} \|Lx\|_e$. Clearly, $\|L\|_e \geq \|Le\|_e$ due to $\|e\|_e = 1$. We now consider the opposite inequality. It is easy to see that $-e \leq x \leq e$, for all $x \in X$ with $\|x\|_e \leq 1$. This implies that

$$-\|Le\|_e e \leq -Le \leq Lx \leq Le \leq \|Le\|_e e.$$

Thus, $\|Lx\|_e \leq \|Le\|_e$, and hence, $\|L\|_e \leq \|Le\|_e$. Therefore, $\|L\|_e = \|Le\|_e$. By induction arguments, we obtain that $\|L^n\|_e = \|L^n e\|_e$, $\forall n \geq 1$. It then follows from the Gelfand's formula that

$$r(L) = \lim_{n\to+\infty} \|L^n\|_e^{\frac{1}{n}} = \lim_{n\to+\infty} \|L^n e\|_e^{\frac{1}{n}} = \lim_{n\to+\infty} \|L^n e\|_E^{\frac{1}{n}}.$$

Here the last equality is because $\|\cdot\|_e$ is an equivalent norm on E. □

Lemma 2.5 *Assume that (E, E_+) is an ordered Banach space with E_+ being normal and $\mathrm{Int}(E_+) \neq \emptyset$, which is equipped with the norm $\|\cdot\|_E$. Let L be a positive bounded linear operator. Choose $v_0 \in \mathrm{Int}(E_+)$ and define $a_n = \|Lv_{n-1}\|_E$, $v_n = \frac{Lv_{n-1}}{a_n}$, $\forall n \geq 1$. If $\lim\limits_{n\to+\infty} a_n$ exists, then $r(L) = \lim\limits_{n\to+\infty} a_n$.*

Proof By an induction argument, it easily follows that $v_n = \frac{L^n v_0}{\prod_{i=1}^n a_i}$, $\forall n \geq 1$. Thus, $\|L^n v_0\|_E = \prod_{i=1}^n a_i$ due to $\|v_n\|_E = 1$ for any $n \geq 1$. Then Lemma 2.4 implies that $r(L) = \lim\limits_{n\to+\infty} (\prod_{i=1}^n a_i)^{\frac{1}{n}}$. It is easy to see that $\lim\limits_{n\to+\infty} (\prod_{i=1}^n a_i)^{\frac{1}{n}} = \lim\limits_{n\to+\infty} a_n$ whenever $\lim\limits_{n\to+\infty} a_n$ exists. □

Lemma 2.6 *Assume that (E, E_+) is an ordered Banach space with E_+ being normal and $\mathrm{Int}(E_+) \neq \emptyset$, which is equipped with the norm $\|\cdot\|_E$. Let L be a positive bounded linear operator. If λ is an eigenvalue of L with an strongly positive eigenvector, then $\lambda = r(L)$.*

Proof Let f be the strongly positive eigenvector of L corresponding to the eigenvalue λ. Then $Lf = \lambda f \geq 0$, and hence, $\lambda \geq 0$. Since $L^n f = \lambda^n f$, $\forall n \geq 1$. It follows from Lemma 2.4 that

$$r(L) = \lim_{n\to+\infty} \|L^n f\|_E^{\frac{1}{n}} = \lim_{n\to+\infty} \|\lambda^n f\|_E^{\frac{1}{n}} = \lambda.$$

This completes the proof. □

3 Basic Reproduction Ratio

Let X be a Banach space with a normal and reproducing cone X_+(see, e.g., [8, Section 19.1]), and $\tilde{X}$ be a Banach space with $\tilde{X} \hookrightarrow X$. We denote their norms $\|\cdot\|_X$ and $\|\cdot\|_{\tilde{X}}$, respectively. Let $\tau \geq 0$ be a given number, and $\mathcal{C} = C([-\tau, 0], X)$, which is equipped with the maximum norm $\|\cdot\|_{\mathcal{C}}$ and the positive cone $\mathcal{C}_+ = C([-\tau, 0], X_+)$.

Let $(V(t))_{0\leq t\leq T}$ be a family of T-periodic closed linear operators with the following properties:

(i) $D(V(t)) = \tilde{X}$, $\forall t \in [0, T]$.
(ii) There is some $\lambda_0 \in \mathbb{R}$ such that$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \lambda_0\} \subset \rho(-V(t))$, $\forall t \in [0, T]$ and $\|(\lambda + V(t))^{-1}\|_X \leq \frac{C}{1+|\lambda|}$, $\forall \lambda \in \mathbb{C}$ with$\mathrm{Re}\lambda \geq \lambda_0$, $\forall t \in [0, T]$.
(iii) $V(\cdot) : [0, T] \to \mathcal{L}(\tilde{X}, X)$ is Hölder continuous.

Assume that $F(\cdot) : \mathbb{R} \to \mathcal{L}(\mathcal{C}, X)$ is T-periodic, $F(t)\phi$ is continuous jointly in $(t, \phi) \in \mathbb{R} \times \mathcal{C}$ and the operator norm of $F(t)$ is uniformly bounded for all $t \in [0, T]$. For a continuous function $u : [-\tau, \varsigma) \to X$ with $\varsigma > 0$, we define $u_t \in \mathcal{C}$ by

$$u_t(\theta) = u(t + \theta), \quad \forall \theta \in [-\tau, 0],$$

for any $t \in [0, \varsigma)$. We consider a linear and T-periodic functional differential system:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(t)u_t - V(t)u(t), \quad t \geq 0. \tag{3.1}$$

The internal evolution of individuals is governed by the following linear differential system:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -V(t)u(t). \tag{3.2}$$

According to [7, Section 2], system (3.2) admits the evolution family $\{\Phi(t, s) : t \geq s\}$. It follows from [24, Corollary 4] that for any $s \in \mathbb{R}$ and $\phi \in \mathcal{C}$ system (3.1) has a unique solution $u(t, s, \phi)$ on $[s, +\infty)$ with $u_s = \phi$. We define the evolution family $\{U(t, s) : t \geq s\}$ on $\mathcal{C}$ associated with (3.1) as

$$U(t, s)\phi = u_t(s, \phi), \quad \forall \phi \in \mathcal{C}, \ t \geq s, \ s \in \mathbb{R},$$

where $u_t(s, \phi) = u(t + \theta, s, \phi), \quad \theta \in [-\tau, 0]$.

In order to introduce the basic reproduction ratio for system (3.1), we assume that

(H1) Each operator $F(t) : \mathcal{C} \to X$ is positive in the sense that $F(t)\mathcal{C}_+ \subset X_+$.

(H2) Each operator $-V(t)$ is resolvent positive in the sense that there is $\gamma \in \mathbb{R}$ such that $(\gamma, +\infty) \subset \rho(-V(t))$ and $(\lambda + V(t))^{-1}$ is positive $\lambda > \gamma$, and $\omega(\Phi) < 0$.

We remark that $\Phi(t, s)X_+ \subset X_+$ and $U(t, s)\mathcal{C}_+ \subset \mathcal{C}_+$ for any $t \geq s$ (see, e.g., [24, Corollary 5]). Following [45], in view of the periodic environment, we suppose that $v(t)$, T-periodic in t, is the initial distribution of infectious individuals. For any given $s \geq 0$, $F(t - s)v_{t-s}$ is the distribution of newly infected individuals at time $t - s$, which produced by the infectious individuals who were introduced over the time interval $[t - s - \tau, \ t - s]$. Then $\Phi(t, t-s)F(t-s)v_{t-s}$ is the distribution of those infected individuals who were newly infected at time $t - s$ and remain in the infected compartments at time t. It follows that

$$\int_0^{+\infty} \Phi(t, t - s)F(t - s)v_{t-s}\mathrm{d}s = \int_0^{+\infty} \Phi(t, t - s)F(t - s)v(t - s + \cdot)\mathrm{d}s$$

is the distribution of accumulative new infections at time t produced by all those infectious individuals introduced at previous times to t. Note that for any given $s \geq 0$, $\Phi(t, t - s)v(t - s)$ gives the distribution of those infectious individuals at time $t - s$ and remain in the infected compartments at time t, and hence, $\int_0^{+\infty} \Phi(t, t - s)v(t - s)\mathrm{d}s$ is the distribution of accumulative infectious individuals who were introduced at all previous times to t and remain in the infected compartments at time t. Thus, distribution of newly infected individuals at time t is $F(t) \int_0^{+\infty} \Phi(t + \cdot, t - s + \cdot)v(t - s + \cdot)\mathrm{d}s$.

Let $\mathbb{X}$ be the ordered Banach space of all continuous and T-periodic functions from $\mathbb{R}$ to X, which is equipped with the maximum norm and the positive cone

$$\mathbb{X}_+ = \{v \in \mathbb{X} : v(t) \geq 0 \text{ in } X, \ t \in \mathbb{R}\}.$$

Then we can define two linear operators on $\mathbb{X}$ by

$$[\mathfrak{L}v](t) = \int_0^{+\infty} \Phi(t, t - s)F(t - s)v(t - s + \cdot)\mathrm{d}s, \quad \forall t \in \mathbb{R}, \ v \in \mathbb{X},$$

and

$$[\tilde{\mathfrak{L}}v](t) = F(t) \int_0^{+\infty} \Phi(t + \cdot, t - s + \cdot)v(t - s + \cdot)\mathrm{d}s, \quad \forall t \in \mathbb{R}, \ v \in \mathbb{X}.$$

Clearly,

$$[\mathfrak{L}v](t) = \int_{-\infty}^{t} \Phi(t, s)F(s)v(s + \cdot)\mathrm{d}s, \quad \forall t \in \mathbb{R}, \ v \in \mathbb{X}.$$

Let A and B be two bounded linear operators on $\mathbb{X}$ defined by

$$[Av](t) = \int_0^{+\infty} \Phi(t, t - s)v(t - s)\mathrm{d}s, \ [Bv](t) = F(t)v_t, \quad \forall t \in \mathbb{R}, \ v \in \mathbb{X}. \tag{3.3}$$

It then follows that $\mathfrak{L} = A \circ B$ and $\tilde{\mathfrak{L}} = B \circ A$, and hence $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ have the same spectral radius. Motivated by the concept of next generation operators (see, e.g., [4,9,34, 35,37,38,45]), we define the spectral radius of $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ as the basic reproduction ratio $\mathcal{R}_0 = r(\mathfrak{L}) = r(\tilde{\mathfrak{L}})$ for periodic system (3.1).

For any given $\lambda \in [0, +\infty)$, we consider the following linear and periodic system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \lambda F(t)u_t - V(t)u(t), \quad t \geq 0. \tag{3.4}$$

Let $\{U(t, s, \lambda) : t \geq s\}$ be the evolution family on $\mathcal{C}$ determined by system (3.4), and write $R(\lambda) := r(\lambda\mathfrak{L}) = \lambda\mathcal{R}_0$, $\forall \lambda \in [0, +\infty)$. Here $R(\lambda)$ can also be regarded as the basic reproduction ratio of system (3.4) and $U(T, 0, 1) = U(T, 0)$. Then we have the following result.

Lemma 3.1 *Assume that* (H1) and (H2) *hold. Then for any given* $t > 0$, $U(t, 0, \lambda)$ *is continuous in the uniform operator topology with respect to* $\lambda \in [0, +\infty)$.

Proof We denote by $v(t, \lambda, \phi)$ the solution of (3.4) with initial data $v_0(\lambda, \phi) = \phi$. For any given $\lambda \in [0, +\infty)$, by the constant-variation formula, we have

$$v(t, \lambda, \phi) = \Phi(t, 0)[\phi(0)] + \lambda \int_0^t \Phi(t, s)F(s)v_s(\lambda, \phi)\mathrm{d}s, \quad \forall \phi \in \mathcal{C}.$$

We first consider $t \in [0, \tau]$. Fix $\mu \in \mathbb{R}$ and $\Lambda > \mu$. Noting that $\mathcal{C}_+$ is normal and reproducing, we can obtain that $\|L_1\|_{\mathcal{C}} \leq M_1\|L_2\|_{\mathcal{C}}$ for some M_1 whenever L_1 and L_2 are positive bounded linear operators with $L_1\psi \leq L_2\psi$ in $\mathcal{C}$ for all $\psi \in \mathcal{C}_+$(see, e.g., [32, Lemma 3]). Clearly,

$$U(s, 0, \Lambda)\psi = v_s(\Lambda, \psi) \geq v_s(\lambda, \psi) = U(s, 0, \lambda)\psi$$

in $\mathcal{C}$ for any $s \in [0, t]$, $\lambda \in [0, \Lambda]$. It then follows that

$$\|U(s, 0, \lambda)\|_{\mathcal{C}} \leq M_1\|U(s, 0, \Lambda)\|_{\mathcal{C}}, \ \forall \lambda \in [0, \Lambda], \quad s \in [0, t].$$

Let $M_2 = \max_{s\in[0,t]} \|U(s, 0, \Lambda)\|_{\mathcal{C}}$ and $D = \{(s_1, s_2) \in \mathbb{R}^2 : -t \leq s_1 \leq 0, \ 0 \leq s_2 \leq s_1 + t\}$. It is easy to see that

$$\|v_s(\lambda, \phi)\|_{\mathcal{C}} = \|U(s, 0, \lambda)\phi\|_{\mathcal{C}} \leq M_1M_2\|\phi\|_{\mathcal{C}}, \quad \forall s \in [0, t], \lambda \in [0, \Lambda], \ \phi \in \mathcal{C},$$

and

$$\|F(s)\|_{\mathcal{C},X} \leq M_3, \quad \forall s \in [0, t], \quad \|\Phi(t + \theta, s)\|_X \leq M_4, \quad \forall (\theta, s) \in D,$$

for some positive numbers M_3 and M_4. We fix $\lambda \in [0, \Lambda]$ to obtain

$$\begin{aligned}
&\|v(t, \mu, \phi) - v(t, \lambda, \phi)\|_X \\
&= \|\mu \int_0^t \Phi(t, s)F(s)v_s(\mu, \phi)\mathrm{d}s - \lambda \int_0^t \Phi(t, s)F(s)v_s(\lambda, \phi)\mathrm{d}s\|_X \\
&= \|(\mu - \lambda) \int_0^t \Phi(t, s)F(s)v_s(\mu, \phi)\mathrm{d}s\|_X \\
&\quad + \|\lambda \int_0^t \Phi(t, s)F(s)[v_s(\mu, \phi) - v_s(\lambda, \phi)]\mathrm{d}s\|_X \\
&\leq |\mu - \lambda|tM_4M_3M_2M_1\|\phi\|_{\mathcal{C}} + |\lambda| \int_0^t M_4M_3\|v_s(\mu, \phi) - v_s(\lambda, \phi)\|_{\mathcal{C}}\mathrm{d}s.
\end{aligned}$$

Letting $C_1 = M_4 M_3 M_2 M_1$ and $C_2 = \Lambda M_4 M_3$, and repeating the above arguments, we conclude that

$$\|v(t+\theta,\mu,\phi) - v(t+\theta,\lambda,\phi)\|_X \le |\mu-\lambda| C_1 t \|\phi\|_{\mathcal{C}} + \int_0^t C_2 \|v_s(\mu,\phi) - v_s(\lambda,\phi)\|_{\mathcal{C}} \mathrm{d}s,$$

for all $\theta \in [-t, 0]$. Clearly,

$$\|v(t+\theta,\mu,\phi) - v(t+\theta,\lambda,\phi)\|_X = 0, \quad \forall \theta \in [-\tau, -t].$$

This implies that

$$\|v_t(\mu,\phi) - v_t(\lambda,\phi)\|_{\mathcal{C}} \le |\mu-\lambda| C_1 t \|\phi\|_{\mathcal{C}} + \int_0^t C_2 \|v_s(\mu,\phi) - v_s(\lambda,\phi)\|_{\mathcal{C}} \mathrm{d}s.$$

By Gröwnwall's inequality, it then follows that

$$\|v_t(\mu,\phi) - v_t(\lambda,\phi)\|_{\mathcal{C}} \le |\mu-\lambda| C_1 t e^{C_2 t} \|\phi\|_{\mathcal{C}}.$$

It is worth pointing out that C_1 and C_2 depend only on t. We then conclude that $U(t, 0, \lambda)$ is continuous in uniform operator topology with respect to $\lambda \in [0, +\infty)$ for any $t \in [0, \tau]$. The same conclusion can be proved by modifying the above arguments when $t > \tau$. □

In order to establish the relationship between the sign of $\mathcal{R}_0 - 1$ and $r(U(T, 0))$, we make the following assumptions:

(H3) The positive linear operator $\mathfrak{L}$ possesses the principal eigenvalue.
(H4) The positive linear operators $U(T, 0, \lambda)$ possesses the isolated principal eigenvalue with finite multiplicity for any $\lambda \in [0, +\infty)$ whenever $r(U(T, 0, \lambda)) \ge 1$.
(H5) Either the principal eigenvalue of $\mathfrak{L}$ is isolated, or there exists an integer $n_0 > 0$ such that $\mathfrak{L}^{n_0}$ is strongly positive.

Remark 3.1 If $r(\mathfrak{L}) > r_e(\mathfrak{L})$, then $r(\mathfrak{L})$ is the isolated principal eigenvalue of L with finite multiplicity, where $r_e(\mathfrak{L})$ is the essential spectral radius of $\mathfrak{L}$(see, e.g., [27, Corollary 2.2]).

To prove our main result in this section, we need a series of lemmas.

Lemma 3.2 *Assume that* (H1), (H2) and (H4) *hold. If* $r(U(T, 0)) = 1$, *then* $\mathcal{R}_0 \ge 1$.

Proof Since $r(U(T, 0)) = 1$, it follows from the assumption (H4) that 1 is an eigenvalue of $U(T, 0)$ corresponding an eigenvector $\phi^* \in \mathcal{C}_+ \setminus \{0\}$. This implies that (3.4) possesses a T-periodic solution $u^*(t)$ with $u^*(t) = [U(t, 0)\phi^*](0)$, $\forall t \in [0, T]$. By the constant-variation formula, we obtain

$$u^*(t) = \Phi(t, r)u^*(r) + \int_r^t \Phi(t, s)F(s)u_s^* \mathrm{d}s, \quad \forall t \ge r, \ r \in \mathbb{R}.$$

It is worth to pointing out that $u^* \ne 0$. Otherwise, $u^*(t) = 0$ for any $t \in \mathbb{R}$, which is impossible since $u^*(\theta) = \phi(\theta)$, $\forall \theta \in [-\tau, 0]$. Letting $r \to -\infty$, we then have

$$u^*(t) = \int_{-\infty}^t \Phi(t, s)F(s)u_s^* \mathrm{d}s = [Lu^*](t), \quad \forall t \in \mathbb{R},$$

that is, $\mathfrak{L}u^* = u^*$. Therefore, $\mathcal{R}_0 = r(L) \ge 1$. □

Lemma 3.3 *Assume that* (H1), (H2) and (H4) *hold. If* $r(U(T, 0)) > 1$, *then* $\mathcal{R}_0 > 1$.

Proof We first define a set $S = \{\lambda \in [0, 1] : r(U(T, 0, \lambda)) \geq 1\}$ and $\lambda^* = \inf S$. Clearly, $S \neq \emptyset$. According to [20, Section IV.3.5] and Lemma 3.1, we see that $r(U(T, 0, \lambda))$ is continuous with respect to $\lambda \in S$. It is worthy to point out that $r(U(T, 0, 0)) = r(\Phi(T, 0)) < 1$ due to Lemma 2.1. Since $r(U(T, 0, \lambda))$ is nondecreasing with respect to $\lambda \in [0, 1]$(see, e.g. [5, Theorem 1.1]), it then follows that $S = [\lambda^*, 1]$. Thus, $0 < \lambda^* < 1$.

We now show $r(U(T, 0, \lambda^*)) = 1$. Assume, by contradiction, that $r(U(T, 0, \lambda^*)) > 1$. Then there is an $\epsilon > 0$ such that $r(U(T, 0, \lambda^* - \epsilon)) \geq 1$ due to [20, Section IV.3.5]. This implies that $\lambda^* - \epsilon \in S$, which is impossible. Applying Lemma 3.2 again to $U(T, 0, \lambda^*)$, we have $R(\lambda^*) \geq 1$, and hence, $\mathcal{R}_0 = (\lambda^*)^{-1} R(\lambda^*) \geq (\lambda^*)^{-1} > 1$. □

Lemma 3.4 *Assume that* (H1), (H2) and (H4) *hold. If* $\mathcal{R}_0 = 0$*, then* $r(U(T, 0, \lambda)) < 1$ *for any* $\lambda \in [0, +\infty)$.

Proof Assume, by contradiction, that there is some $\tilde{\lambda} \geq 0$ such that $r(U(T, 0, \tilde{\lambda})) \geq 1$. Clearly, $\tilde{\lambda} > 0$ due to $r(U(T, 0, 0)) < 1$. Applying Lemmas 3.2 and 3.3 to $U(T, 0, \tilde{\lambda})$, we obtain that $R(\tilde{\lambda}) \geq 1$. This implies that $\mathcal{R}_0 = (\tilde{\lambda})^{-1} R(\tilde{\lambda}) > 0$, which contradicts $\mathcal{R}_0 = 0$.□

Lemma 3.5 *Assume that* (H1)–(H3) *hold. If* $\mathcal{R}_0 = 1$*, then* $r(U(T, 0)) \geq 1$.

Proof Since $r(\mathfrak{L}) = \mathcal{R}_0 = 1$, there exists $u^* \in \mathbb{X}_+$ such that $Lu^* = u^*$ due to assumption (H3). It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}(Lu^*)(t) = F(t)u_t^* - V(t)[(Lu^*)(t)], \quad \forall t \in \mathbb{R},$$

that is, $u^*(t)$ satisfies (3.1). This implies that $U(T, 0)\phi^* = \phi^*$, where $\phi^*(\theta) = u^*(\theta)$, $\forall \theta \in [-\tau, 0]$. Thus, we have $r(U(T, 0)) \geq 1$. □

Lemma 3.6 *Assume that* (H1)–(H3) and (H5) *hold. If* $\mathcal{R}_0 > 1$*, then* $r(U(T, 0)) > 1$.

Proof It is easy to see that $R(\mathcal{R}_0^{-1}) = 1$ since $R(1) = \mathcal{R}_0 > 1$. This implies that $r(U(T, 0, \mathcal{R}_0^{-1})) \geq 1$ due to Lemma 3.5. Since $r(U(T, 0, \lambda))$ is nondecreasing in $\lambda \in [0, +\infty)$, we have $r(U(T, 0, 1)) \geq r(U(T, 0, \mathcal{R}_0^{-1})) \geq 1$.

Suppose, by contradiction, that $r(U(T, 0)) = 1$. Then $r(U(T, 0, \lambda)) = 1$ for all $\lambda \in [\mathcal{R}_0^{-1}, 1]$. From the proof of Lemma 3.2, we see that 1 is an eigenvalue of $\lambda\mathfrak{L}$ with positive eigenvector for all $\lambda \in [\mathcal{R}_0^{-1}, 1]$. That is, λ^{-1} is an eigenvalue of $\mathfrak{L}$ with positive eigenvector for all $\lambda \in [\mathcal{R}_0^{-1}, 1]$. In the case where the principal eigenvalue of $\mathfrak{L}$ is isolated, then $\mathcal{R}_0$ is an isolated eigenvalue, which is a contradiction. In the case where $\mathfrak{L}^{n_0}$ is strongly positive, it then follows that $\mathfrak{L}^{n_0}$ can not have two eigenvalues corresponding positive eigenvectors (see, e.g., [10, Theorem 1.2]). But λ^{-n_0} is an eigenvalue of $\mathfrak{L}^{n_0}$ with positive eigenvector for all $\lambda \in [\mathcal{R}_0^{-1}, 1]$, which is impossible. This shows that $r(U(T, 0)) > 1$. □

Now we are in a position to prove the main result of this section.

Theorem 3.7 *Assume that* (H1)–(H5) *hold. Then the following statements are valid:*

(i) $\mathcal{R}_0 > 1$ *if and only if* $r(U(T, 0)) > 1$.
(ii) $\mathcal{R}_0 = 1$ *if and only if* $r(U(T, 0)) = 1$.
(iii) $\mathcal{R}_0 < 1$ *if and only if* $r(U(T, 0)) < 1$.

Proof Statements (i) and (ii) can be derived by using Lemmas 3.2, 3.3, 3.5 and 3.6. Hence, statement (iii) is a straightforward consequence of (i) and (ii). □

According to Lemma 2.1 and Theorem 3.7, we conclude that $\mathcal{R}_0 - 1$ has the same sign as $\omega(U)$, where $\omega(U)$ is the exponential growth bound of evolution family $\{U(t,s) : t \geq s\}$.

Theorem 3.8 *Assume that* (H1)–(H5) *hold. If* $\mathcal{R}_0 > 0$, *then* $\lambda = \mathcal{R}_0^{-1}$ *is the unique solution of* $r(U(T,0,\lambda)) = 1$.

Proof Our arguments are motivated by those in [45, Theorem 2.2]. In view of Theorem 3.7, we have

$$sign(R(\lambda) - 1) = sign(r(U(T,0,\lambda)) - 1), \quad \forall \lambda \in [0,+\infty). \tag{3.5}$$

Letting $\lambda = \mathcal{R}_0^{-1} > 0$ in the above equation, we then obtain $r(U(T,0,\mathcal{R}_0^{-1})) = 1$. It remains to show that $r(U(T,0,\lambda)) = 1$ has at most one positive solution for λ. Assume, by contradiction, that $r(U(T,0,\lambda)) = 1$ has two positive solution $\lambda_1 < \lambda_2$. Since $r(U(T,0,\lambda))$ is nondecreasing with respect to $\lambda \in [0,+\infty)$, we have $r(U(T,0,\lambda)) = 1, \ \forall \lambda \in [\lambda_1,\lambda_2]$. It follows from (3.5) that $R(\lambda) = 1, \ \forall \lambda \in [\lambda_1,\lambda_2]$, which is impossible since $R(\lambda) = \lambda\mathcal{R}_0, \ \forall \lambda \in [0,+\infty)$. □

Remark 3.2 With Theorem 3.8, we can use the bisection method to obtain the numerical solution λ_0 to $r(U(T,0,\lambda)) = 1$, and hence, $\mathcal{R}_0 = \frac{1}{\lambda_0}$. Note that for each $\lambda \in [0,+\infty)$, $r(U(T,0,\lambda))$ can be computed numerically via Lemma 2.5.

Under the assumptions (H1) and (H2), we are able to show that the following condition is sufficient for (H3)–(H5) to hold:

(H6) Each operator $\Phi(t,s)$ is compact on X for $t > s$.

Proposition 3.9 *Assume that* (H1), (H2) *and* (H6) *hold and* $\mathcal{R}_0 > 0$. *Then* (H1)–(H5) *are valid.*

Proof In order to prove (H3) and (H5), we start with the following claim.
Claim 1. $\mathfrak{L}$ is compact on $\mathbb{X}$.

Let D be a given bounded subset of $\mathbb{X}$. According to Lemma 2.3, it is suffices to show that (i) for each $t \in [0,T]$, the set $\{[\mathfrak{L}u](t) : u \in D\}$ is precompact on X; and (ii) the set $\mathfrak{L}D$ is equicontinuous in $t \in [0,T]$.

Now we prove statement (i) by modifying the arguments in [46, Theorem 3.5.1]. Let $t \in [0,T]$ be given. It is easy to see that there exists $K > 0$ such that $\|\Phi(t,s)F(s)v_s\|_X \leq K$ for all $v \in D$, $s \in [-T,t]$. For any $\sigma \in (0,t)$, we define two sets

$$D_1^\sigma = \left\{\int_{t-\sigma}^t \Phi(t,s)F(s)v_s ds : \ v \in D\right\},$$
$$D_2^\sigma = \left\{\int_{-\infty}^{t-\sigma} \Phi(t,s)F(s)v_s ds, \ v \in D\right\}.$$

Clearly,

$$\int_{-\infty}^{t-\sigma} \Phi(t,s)F(s)v_s ds = \Phi(t,t-\sigma)\int_{-\infty}^{t-\sigma} \Phi(t-\sigma,s)F(s)v_s ds, \quad \forall\, v \in \mathbb{X}.$$

Let α be the Kuratowski measure of noncompactness in X. Since $\Phi(t,t-\sigma)$ is compact, it follows that

$$\alpha(\{[\mathfrak{L}u](t) : u \in D\}) \leq \alpha(D_1^\sigma) + \alpha(D_2^\sigma) \leq 2K\sigma + 0 \leq 2K\sigma.$$

Letting $\sigma \to 0^+$, we obtain $\alpha(\{[\mathfrak{L}u](t) : u \in D\}) = 0$, which implies that the set $\{[\mathfrak{L}u](t) : u \in D\}$ is precompact on X.

Motivated by [45, Lemma 2.1], we prove statement (ii). It is easy to see that

$$[\mathfrak{L}v](t)=\int_{-T}^{t}\Phi(t,s)F(s)v_s\mathrm{d}s+\int_{-\infty}^{-T}\Phi(t,s)F(s)v_s\mathrm{d}s,\ \forall t\in[0,T],\quad v\in\mathbb{X}.$$

Thus, by [7, I.(5.10)], there exists $\gamma\in(0,1)$ and $\tilde{K}(\gamma,\Phi)>0$ such that

$$\left\|\int_{-T}^{t_1}\Phi(t_1,s)F(s)v_s\mathrm{d}s-\int_{-T}^{t_2}\Phi(t_2,s)F(s)v_s\mathrm{d}s\right\|_X\le\tilde{K}(t_1-t_2)^{\gamma}\max_{s\in[0,T]}\|F(s)\|_{\mathcal{C},X}\|v\|_{\mathbb{X}},$$

for all $t_1,t_2\in[0,T]$, $v\in\mathbb{X}$. We also have

$$\begin{aligned}&\left\|\int_{-\infty}^{-T}\Phi(t_1,s)F(s)v_s\mathrm{d}s-\int_{-\infty}^{-T}\Phi(t_2,s)F(s)v_s\mathrm{d}s\right\|_X\\&\le\|\Phi(t_1,-T)-\Phi(t_2,-T)\|_X\int_{-\infty}^{-T}\|\Phi(-T,s)\|_X\max_{s\in[0,T]}\|F(s)\|_{\mathcal{C},X}\|v\|_{\mathbb{X}}\mathrm{d}s,\end{aligned}$$

for all $t_1,t_2\in[0,T]$, $v\in\mathbb{X}$. Note that for any $\epsilon>0$, there exists $\delta\in(0,\epsilon^{\frac{1}{\gamma}})$ such that

$$\|\Phi(t_1,-T)-\Phi(t_2,-T)\|_X\le\epsilon,\ \text{for any } t_1,\ t_2\in[0,T] \text{ with } |t_1-t_2|\le\delta,$$

and $\omega(\Phi)<0$. Therefore,

$$\begin{aligned}&\|[\mathfrak{L}v](t_1)-[\mathfrak{L}v](t_2)\|_X\\&\le\tilde{K}\max_{s\in[0,T]}\|F(s)\|_{\mathcal{C},X}\|v\|_{\mathbb{X}}\epsilon+\epsilon\int_{-\infty}^{-T}\|\Phi(-T,s)\|_X\max_{s\in[0,T]}\|F(s)\|_{\mathcal{C},X}\|v\|_{\mathbb{X}}\mathrm{d}s,\end{aligned}$$

for any $t_1,\ t_2\in[0,T]$ with $|t_1-t_2|\le\delta$. It follows that the set $\mathfrak{L}D$ is equicontinuous in $t\in[0,T]$ in view of the above estimates, and hence, Claim 1 is proved. Therefore, assumptions (H3) and (H5) hold true by the celebrated Krein-Rutman theorem (see, e.g., [8, Theorem 19.2]) due to $\mathcal{R}_0>0$. In order to verify (H4), we need the following claim.

Claim 2. For any $\lambda\in[0,+\infty)$, $U(t,s,\lambda)$ is compact on $\mathcal{C}$ for all $t>s+\tau$.

We first fix a $t>\tau$ and $s=0$ and show that $U(t,0,\lambda)$ is compact on $\mathcal{C}$ (the other cases can be addressed by using similar arguments). Define

$$(Q_1\phi)(\theta)=\Phi(t+\theta,0)[\phi(0)],\quad\forall\theta\in[-\tau,0],\ \phi\in\mathcal{C}.$$

Let $\tilde{D}$ be a given bounded subset of $\mathcal{C}$. We proceed with two cases.

Case 1. $\lambda=0$.

It is easy to see that $U(t,0,0)=Q_1$. By virtue of Lemma 2.3, it suffices to prove that (iii) for each $\theta\in[-\tau,0]$, the set $\{(Q_1\phi)(\theta):\phi\in\tilde{D}\}$ is precompact in X; and (iv) the set $Q_1\tilde{D}$ is equi-continuous in $\theta\in[-\tau,0]$. Statement (iii) is a straightforward consequence of assumption (H6). Since $t>\tau$, statement (iv) can be derived by estimates

$$\|\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t+\theta,0)\|_X\le C_1,\quad\forall\theta\in[-\tau,0].$$

where $C_1>0$ is a constant(see, e.g., [7, Theorem 2.6]). This implies that Q_1, and hence, $U(t,0,0)$ is compact on $\mathcal{C}$.

Case 2. $\lambda>0$.

We fix a $\lambda\in(0,+\infty)$ and let $v(t,\lambda,\phi)$ be the mild solution of (3.4) with initial date ϕ. Define

$$(Q_2v)(\theta)=\lambda\int_0^{t+\theta}\Phi(t+\theta,r)F(r)v_r(\lambda,\phi)\mathrm{d}r,\ \theta\in[-\tau,0],\quad\forall\phi\in\mathcal{C}.$$

By the constant-variation formula, we have

$$v(t,\lambda,\phi) = \Phi(t,0)[\phi(0)] + \lambda \int_0^t \Phi(t,r)F(r)v_r(\lambda,\phi)\mathrm{d}r, \quad \forall \phi \in \mathcal{C},$$

and hence, $U(t,0,\lambda) = Q_1 + Q_2$. From the proof of Case 1, it is known that Q_1 is compact on $\mathcal{C}$. In view of Lemma 2.3, it suffices to prove that (v) for each $\theta \in [-\tau, 0]$, the set $\{(Q_2\phi)(\theta) : \phi \in \tilde{D}\}$ is precompact in X; and (vi) the set $Q_2\tilde{D}$ is equi-continuous in $\theta \in [-\tau, 0]$. The statements (v) can be derived by the similar arguments to those in (i). The statement (vi) is a straightforward consequence of

$$\begin{aligned}&\left\| \int_0^{t+\theta_1} \Phi(t+\theta_1, r)F(r)v_r(\lambda,\phi)\mathrm{d}r - \int_0^{t+\theta_2} \Phi(t+\theta_2, r)F(r)v_r(\lambda,\phi)\mathrm{d}r \right\|_X \\ &\le C|\theta_1 - \theta_2|^\gamma \max_{r\in[0,T]} \|F(r)\|_{\mathcal{C},X} \max_{r\in[0,t]} \|U(r,0,\lambda)\|_{\mathcal{C}} \|\phi\|_{\mathcal{C}}, \quad \forall \phi \in \mathcal{C},\end{aligned}$$

where γ is a constant in (0, 1) and C is a constant dependent on the system (3.2), time t and constant γ. The estimate can be found in [7](see I.(5.10)). Thus, $Q_2\tilde{D}$ is precompact on $\mathcal{C}$, and hence, we finished the proof of Case 2 and Claim 2. Therefore, (H4) holds true by the Krein-Rutman Theorem(see, e.g., [8, Theorem 19.2]). □

In the case where X is a finite dimensional real vector space, the assumption (H6) holds automatically, and hence, Theorems 3.7 and 3.8 reduce to Theorems 2.1 and 2.2 in [45], respectively, due to Proposition 3.9.

Proposition 3.10 *Assume that* (H1), (H2) and (H4) *hold. If* $\mathcal{R}_0 > 0$ *and there exists an integer* $n_0 > 0$ *such that* $\mathfrak{L}^{n_0}$ *is strongly positive, then* (H1)–(H5) *are valid.*

Proof We only need to verify the assumption (H3). Since $\mathcal{R}_0 > 0$, it follows from Lemma 3.4 and the proof of Lemma 3.3 that there exists $\lambda^* > 0$ such that $r(U(T,0,\lambda^*)) = 1$. By the arguments similar to those in Lemma 3.2, $\lambda^*\mathfrak{L}$ has an eigenvalue 1 with positive eigenvector u^*. Since $\mathfrak{L}^{n_0}$ is strongly positive, we have $u^* \in \mathrm{Int}(\mathbb{X}_+)$. Thus, Lemma 2.6 implies that $r(\mathfrak{L}) = \dfrac{1}{\lambda^*}$ is the principal eigenvalue of $\mathfrak{L}$, that is, (H3) holds. □

4 A Spatial Model for Lyme Disease

In this section, we apply the theory developed in Section 3 to a time-periodic Lyme disease model with time delay. There have been quite a few investigations on the Lyme disease, see, e.g., [6,25,28,29,31,39,41,43]. Here we consider a time-periodic version of the autonomous spatial model for Lyme disease proposed in [41] since the parameters may be affected by the seasonality, varying temperature and humidity. Let Ω be a domain in $\mathbb{R}^2$ with the smooth boundary $\partial\Omega$, and ν be the unit normal vector on $\partial\Omega$. Let $\Gamma(x, y, t, s, D)$ be the Green function associated with the linear parabolic equation:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla \cdot (D(x,t)\nabla u), \quad && x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} &= 0, && x \in \partial\Omega, t > 0.\end{aligned}$$

Then $\int_\Omega \Gamma(x, y, t, s, D)\phi(y)\mathrm{d}y$ denotes the distribution at time t through the diffusion with the given initial distribution $\phi(x)$ from time s. For convenience, we omit the domain throughout this section if we consider the equation on $x \in \Omega,\ t > 0$.

Table 1 Biological interpretations of parameters

α	Attack rate, juvenile ticks on mice.
γ	Attack rate, tick nymphs on humans.
ξ	Coefficient of an adult tick to attach to deer.
δ_A	Coefficient of self-regulation for adult ticks.
τ_l	Feeding duration of tick larvae on mice.
τ_n	Feeding duration of tick nymphs on mice.
τ_a	Feeding duration of adult ticks on deer.
$r_M(t)$	Maximum individual birth rate in mice at time t.
$r(t)$	Maximum individual birth rate in ticks at time t.
$r_H(t)$	Birth rate of deer at time t.
$\mu_M(t)$	Mortality rate per mouse at time t.
$\mu_L(t)$	Mortality rate per tick larvae at time t.
$\mu_N(t)$	Mortality rate per tick nymph at time t.
$\mu_A(t)$	Mortality rate per adult tick at time t.
$\mu_H(t)$	Mortality rate per deer at time t.
$D_M(x,t)$	Diffusion coefficient for mice at location x and time t.
$D_H(x,t)$	Diffusion coefficient for deer at location x and time t.
$K_M(x,t)$	Carrying capacity for mice at location x and time t.
$\beta(x,t)$	Susceptibility to infection in mice at location x and time t.
$\beta_T(x,t)$	Susceptibility to infection in ticks at location x and time t.

Following [41], we let $M(x,t)$ and $m(x,t)$ be the densities of susceptible and pathogen-infected mice, $L(x,t)$ be the density of questing tick larvae, $N(x,t)$ and $n(x,t)$ be the densities of susceptible and infectious questing tick nymphs, $A(x,t)$ and $a(x,t)$ be the densities of uninfected and pathogen-infected questing adult ticks, and $H(x,t)$ be the density of deer, at location x and time t. The parameters are shown as in Table 1. Here we assume, for simplicity, that the ticks have the same ability to attack hosts while hosts have the different ability to immune to be infected, which depends on the environment elements(e.g., temperature and humidity). Accordingly, all the attack rates are constant while the susceptibilities to infection depend on both time and space.

The drop-off rate of adult ticks from deer after blood meals is given by

$$\begin{aligned} L_b(x,t,H,A,a) = P_a(t)\int_\Omega &\Gamma(x,y,t,t-\tau_a,D_H)[A(y,t-\tau_a) \\ &+a(y,t-\tau_a)]H(y,t-\tau_a)\mathrm{d}y, \end{aligned}$$

where $P_a(t) = r(t)\xi e^{-\int_{t-\tau_a}^t[\mu_A(s)+\mu_H(s)]\mathrm{d}s}$. Let N_b, n_b, A_b, a_b be the drop-off rate of susceptible and infected larvae, as well as susceptible and infected adult ticks from mice, respectively, which are defined by

$$\begin{aligned} N_b(x,t,M,L,m) &= P_l(t)\int_\Omega \Gamma(x,y,t,t-\tau_l,D_M)[M(y,t-\tau_l) \\ &\quad +(1-\beta_T(y,t))m(y,t-\tau_l)]L(y,t-\tau_l)\mathrm{d}y, \\ A_b(x,t,M,N,m) &= P_n(t)\int_\Omega \Gamma(x,y,t,t-\tau_n,D_M)N(y,t-\tau_n) \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad [M(y,t-\tau_n)+(1-\beta_T(y,t))m(y,t-\tau_n)]\mathrm{d}y,\\
n_b(x,t,L,m) &= P_l(t)\int_\Omega \Gamma(x,y,t,t-\tau_l,D_M)\beta_T(y,t)m(y,t-\tau_l)L(y,t-\tau_l)\mathrm{d}y,\\
a_b(x,t,M,N,m,n) &= P_n(t)\int_\Omega \Gamma(x,y,t,t-\tau_a,D_M)[(M(y,t-\tau_n)\\
&\quad +m(y,t-\tau_n))n(y,t-\tau_n)\\
&\quad +\beta_T(y,t)m(y,t-\tau_n)N(y,t-\tau_n)]\mathrm{d}y,
\end{aligned}
$$

where $P_l(t) = \alpha e^{-\int_{t-\tau_l}^t[\mu_L(s)+\mu_M(s)]\mathrm{d}s}$, $P_n(t) = \alpha e^{-\int_{t-\tau_n}^t[\mu_L(s)+\mu_M(s)]\mathrm{d}s}$. The per capita birth rate B_M of mice is taken in [39] as the negative exponential function :

$$
B_M(x,t,M+m) = r_M(t)\exp(-\frac{M+m}{K_M(x,t)}),
$$

Accordingly, the model in [39,41] can be modified as

$$
\begin{aligned}
\frac{\partial H}{\partial t} &= \nabla\cdot(D_H(x,t)\nabla H) + r_H(t) - \mu_H(t)H,\\
\frac{\partial M}{\partial t} &= \nabla\cdot(D_M(x,t)\nabla M) + (M+m)B_M(x,t,M+m)\\
&\quad -\mu_M(t)M - \alpha\beta(x,t)Mn,\\
\frac{\partial L}{\partial t} &= L_b(x,t,H,A,a) - \mu_L(t)L - \alpha L(M+m),\\
\frac{\partial N}{\partial t} &= N_b(x,t,M,L,m) - [\gamma+\alpha(M+m)+u_N(t)]N,\\
\frac{\partial A}{\partial t} &= A_b(x,t,M,N,m) - (\mu_A(t)+\xi H)A - \delta_A(A+a)A,\\
\frac{\partial m}{\partial t} &= \nabla\cdot(D_M(x,t)\nabla m) - \mu_M(t)m + \alpha\beta(x,t)Mn,\\
\frac{\partial n}{\partial t} &= n_b(x,t,L,m) - [\gamma+\alpha(M+m)+u_N(t)]n,\\
\frac{\partial a}{\partial t} &= a_b(x,t,M,N,m,n) - (\mu_A(t)+\xi H)a - \delta_A(A+a)a,
\end{aligned}
\tag{4.1}
$$

where H, M and m are subject to Neumann boundary condition in the sense that

$$
\frac{\partial H}{\partial \nu} = \frac{\partial M}{\partial \nu} = \frac{\partial m}{\partial \nu} = 0, \quad \forall x\in\partial\Omega,\ t>0.
$$

Throughout this section, we assume that all parameters are positive and continuously differentiable in Table 1; all time-dependent parameters are T-periodic; $0<\beta(x,t)\le 1$, $\forall (x,t)\in \overline{\Omega}\times\mathbb{R}$. For any given $1\le i\le 8$, we introduce the a series of Banach spaces. Let $Y^i := C(\overline{\Omega},\mathbb{R}^i)$ be the ordered Banach space with norm $\|\cdot\|_{Y_i}$ and the positive cone Y^i_+. We denote by

$$
\mathbb{Y}^i := \{u\in C(\mathbb{R},Y^i) : u(t) = u(t+T),\quad \forall t\in\mathbb{R}\},
$$

which is equipped with the maximum norm defined by $\|u\|_{\mathbb{Y}^i} = \max_{t\in[0,T]}\|u(t)\|_{Y_i}$, $\forall u\in \mathbb{Y}^i$ and the positive cone

$$
\mathbb{Y}^i_+ := \{u\in C(\mathbb{R},Y^i_+) : u(t) = u(t+T),\quad \forall t\in\mathbb{R}\}.
$$

We choose $\hat{\tau} = \max\{\tau_l, \tau_n, \tau_a\}$. Let $\mathcal{Y}^i := C([-\hat{\tau}, 0], Y^i)$ be a Banach space which is equipped with the maximum norm defined by $\|\phi\|_{\mathcal{Y}^i} = \max_{\theta\in[-\hat{\tau},0]} \|\phi(\theta)\|_{Y_i}$, $\forall\phi \in \mathbb{Y}^i$ and the positive cone $\mathcal{Y}^i_+ := C([-\hat{\tau}, 0], Y^i_+)$. For a continuous function $u : [-\hat{\tau}, \varsigma) \to Y^i$ with $\varsigma > 0$, we define $u_t \in \mathcal{Y}^i$ by

$$u_t(\theta) = u(t+\theta), \quad \forall\theta \in [-\hat{\tau}, 0],$$

for any $t \in [0, \varsigma)$.

4.1 Disease-Free Dynamics

In this subsection, we study the disease-free T-periodic solution and its global attractivity. Without the infection of Lyme disease, system (4.1) reduces to

$$\begin{aligned}
\frac{\partial H}{\partial t} &= \nabla\cdot(D_H(x,t)\nabla H) + r_H(t) - \mu_H(t)H,\\
\frac{\partial M}{\partial t} &= \nabla\cdot(D_M(x,t)\nabla M) + MB_M(x,t,M) - \mu_M(t)M,\\
\frac{\partial L}{\partial t} &= L_b(x,t,H,A,0) - \mu_L(t)L - \alpha ML,\\
\frac{\partial N}{\partial t} &= N_b(x,t,M,L,0) - [\gamma + \alpha M + u_N(t)]N,\\
\frac{\partial A}{\partial t} &= A_b(x,t,M,N,0) - (\mu_A(t) + \xi H)A - \delta_A A^2,
\end{aligned} \tag{4.2}$$

and H and M are subject to the Neumann boundary condition:

$$\frac{\partial H}{\partial \nu} = \frac{\partial M}{\partial \nu} = 0, \quad \forall x \in \partial\Omega,\ t > 0.$$

We first consider

$$\begin{aligned}
\frac{\partial H}{\partial t} &= \nabla\cdot(D_H(x,t)\nabla H) + r_H(t) - \mu_H(t)H, && x\in\Omega,\ t>0,\\
\frac{\partial H}{\partial \nu} &= 0, && x\in\partial\Omega,\ t>0.
\end{aligned} \tag{4.3}$$

System (4.3) admits a positive periodic solution $H^*(t)$ which is globally attractive on $Y^1_+\backslash\{0\}$. Now, we assume that

(A1) $\int_0^T r_M(s)\mathrm{d}s > \int_0^T \mu_M(s)\mathrm{d}s$.

By a standard convergence result on the logistic type reaction-diffusion equations(see, e.g., [46, Theorems 2.3.4 and 3.1.6]), it then follows that the following reaction-diffusion system:

$$\begin{aligned}
\frac{\partial M}{\partial t} &= \nabla\cdot(D_M(x,t)\nabla M) + MB_M(x,t,M) - \mu_M(t)M, && x\in\Omega,\ t>0,\\
\frac{\partial M}{\partial \nu} &= 0, && x\in\partial\Omega,\ t>0.
\end{aligned} \tag{4.4}$$

admits a globally T-periodic stable positive solution M^* in $Y^1_+ \setminus \{0\}$. Thus, we should study the global dynamics of the following limiting system:

$$
\begin{aligned}
\frac{\partial L}{\partial t} &= L_b(x,t,H^*,A,0) - [\mu_L(t) + \alpha M^*(x,t)]L, \\
\frac{\partial N}{\partial t} &= N_b(x,t,M^*,L,0) - [\gamma + \alpha M^*(x,t) + u_N(t)]N, \\
\frac{\partial A}{\partial t} &= A_b(x,t,M^*,N,0) - (\mu_A(t) + \xi H^*(t))A - \delta_A A^2.
\end{aligned} \tag{4.5}
$$

Lemma 4.1 *Assume that* (A1) *holds. For any $\phi \in \mathcal{Y}^3_+$, system* (4.5) *admits a unique solution $v(x,t,\phi)$ on $[0,+\infty)$ with initial data $v_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \overline{\Omega} \times [-\hat{\tau}, 0]$. Moreover, the solutions are uniformly bounded on Y^3_+.*

Proof By [24, Corollary 5] and the arguments similar to those in [41, Section 2], it follows that system (4.5) admits a unique nonnegative continuous solution $v(x,t,\phi)$ on $[0, t_\phi)$ with initial data $v_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \overline{\Omega} \times [-\hat{\tau}, 0]$ and the comparison principle holds for upper and lower solutions of system (4.5).

Next, we show that the solutions are uniformly bounded on Y^3_+ by modifying the arguments in [41, Section 2]. Note that there exists a positive vector $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ such that

$$
\overline{P_a H^*}\eta_3 - \underline{\mu_L}\eta_1 = 0, \quad \overline{P_L M^*}\eta_1 - (\gamma + \underline{\mu_N})\eta_2 = 0, \quad \overline{P_N M^*}\eta_2 - \delta_A \eta_3^2 \le 0.
$$

Here,

$$
\begin{aligned}
\overline{P_a} &= \max_{t\in[0,T]} P_a(t), \ \overline{P_l} = \max_{t\in[0,T]} P_l(t), \ \overline{P_n} = \max_{t\in[0,T]} P_n(t), \\
\overline{M^*} &= \max_{(x,t)\in\overline{\Omega}\times[0,T]} M^*(x,t), \ \underline{\mu_L} = \min_{t\in[0,T]} \mu_L(t), \ \underline{\mu_N} = \min_{t\in[0,T]} \mu_N(t).
\end{aligned}
$$

Then it is easy to see that for any $k \ge 1$, $k\eta$ is an super-solution of system (4.5). This implies that the solution of system (4.5) is uniformly bounded on Y^3_+, and hence $t_\phi = +\infty$.

Let $v(x,t,\phi)$ be the mild solution of system (4.5) with initial data $v_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \Omega \times [-\hat{\tau}, 0]$. We denote by $[S_t\phi](\theta) = v(\cdot, t+\theta, \phi)$, $\forall \theta \in [-\hat{\tau}, 0]$. Lemma 4.1 implies that $S_t B$ is bounded on $\mathcal{Y}^3_+$ for any bounded $B \subset \mathcal{Y}^3_+$. By the arguments similar to those in [41, Lemma 2.1], we have following result due to Lemma 2.3 and [7, Remark 2.4].

Lemma 4.2 *Assume that* (A1) *holds. For any $\phi \in \mathcal{Y}^3_+$, the discrete forward orbit $\gamma^+(\phi) = \{S_{nT}\phi : n \ge 1\}$ is asymptotically compact in the sense that for any $n_k \to +\infty$, there exists a subsequence $n_{k_j} \to +\infty$ such that $S_{n_{k_j}T}\phi$ converges in $\mathcal{Y}^3$ as $j \to +\infty$.*

Now, we should introduce an ordered Banach space

$$
\mathcal{Z}^3 = C([-\tau_l, 0], \ Y^1) \times C([-\tau_n, 0], \ Y^1) \times C([-\tau_a, 0], \ Y^1),
$$

which is equipped with the maximum norm and the positive cone

$$
\mathcal{Z}^3_+ = C([-\tau_l, 0], \ Y^1_+) \times C([-\tau_n, 0], \ Y^1_+) \times C([-\tau_a, 0], \ Y^1_+).
$$

We define an extension operator E from $\mathcal{Z}^3$ to $\mathcal{Y}^3$ by

$$
E\psi = (E_1\psi_1, E_2\psi_2, E_3\psi_3), \quad \forall \psi = (\psi_1, \psi_2, \psi_3) \in \mathcal{Z}^3,
$$

where

$$
[E_i\psi_i](\theta) = \begin{cases} \psi_i(\theta), & -\tau_i \le \theta \le 0, \\ \psi_i(-\tau_i), & \hat{\tau} \le \theta \le -\tau_i, \end{cases}
$$

for any $1 \leq i \leq 3$. Here, $\tau_1 = \tau_l$, $\tau_2 = \tau_n$, $\tau_3 = \tau_a$. On the other hand, we can define a restriction operator R from $\mathcal{Y}^3$ to $\mathcal{Z}^3$ by

$$R\phi = (R_1\phi_1, R_2\phi_2, R_3\phi_3), \ \forall \phi = (\phi_1, \phi_2, \phi_3),$$

where

$$[R_i\phi_i](\theta) = \phi_i(\theta), \quad \tau_i \leq \theta \leq 0, \ 1 \leq i \leq 3.$$

Let $\{V(t, s) : t \geq s\}$ be the evolution family determined by the following linearized system of (4.5) at (0, 0, 0) on $\mathcal{Y}^3$

$$\begin{aligned} \frac{\partial L}{\partial t} &= L_b(x, t, H^*, A, 0) - [\mu_L(t) + \alpha M^*(x, t)]L, \\ \frac{\partial N}{\partial t} &= N_b(x, t, M^*, L, 0) - [\gamma + \alpha M^*(x, t) + u_N(t)]N, \\ \frac{\partial A}{\partial t} &= A_b(x, t, M^*, N, 0) - (\mu_A(t) + \xi H^*(t))A. \end{aligned} \tag{4.6}$$

Write $\mathbf{v}(\cdot, t, \psi) = v(\cdot, t, E\psi)$ for any $\psi \in \mathcal{Z}^3_+$. We remark that $\mathbf{v}(\cdot, t, \psi)$ is a unique solution of system (4.5) with initial data ψ in $\mathcal{Z}^3_+$. According to [24, Theorem 3], $\mathbf{v}(\cdot, t, \psi)$ is strongly positive on Y^3 for any $t \geq 3\hat{\tau}$. Now, we write $\mathbf{S}_t = RS_tE$ and $\mathbf{V}(t, s) = RV_tE$ on $\mathcal{Z}^3$. It then follows that $\mathbf{S}_t$ and $\mathbf{V}(t, 0)$ is strongly positive for any $t \geq 4\hat{\tau}$.

Lemma 4.3 *Assume that* (A1) *holds. If* $\omega(V) \geq 0$, *then* $V(T, 0)$ *has the principal eigenvalue with a strongly positive eigenvector on* $\mathcal{Y}^3$. *Moreover,* $\omega(V) = \omega(\mathbf{V})$.

Proof Let $\hat{v}(x, t, \phi)$ be the solution of linear system (4.6) with initial data $\hat{v}_0(x, \theta, \phi) = \phi(x, \theta)$, $\forall (x, \theta) \in \overline{\Omega} \times [-\hat{\tau}, 0]$. It easily follows from Lemma 2.1 that $r(V(T, 0)) \geq 1$. We proceed with two steps.

Step 1. $r_e(V(T, 0)) < 1$.

Let $\{\Lambda(t, s) : t \geq s\}$ be the evolution family on Y^3 of the following linear system

$$\begin{aligned} \frac{\partial L}{\partial t} &= -[\mu_L(t) + \alpha M^*(x, t)]L, \\ \frac{\partial N}{\partial t} &= -[\gamma + \alpha M^*(x, t) + u_N(t)]N, \\ \frac{\partial A}{\partial t} &= -(\mu_A(t) + \xi H^*(t))A. \end{aligned}$$

It is easy to choose a $K > 0$ such that $KT > \hat{\tau}$ and

$$\|\Lambda(KT + \theta, 0)\|_{Y^3} \leq \frac{1}{2}, \quad \forall \theta \in [-\hat{\tau}, 0].$$

We define

$$J(t, \psi) = (J_1(t, \psi), J_2(t, \psi), J_3(t, \psi)), \quad \forall t > 0, \ \psi = (\psi_1, \psi_2, \psi_3) \in \mathcal{Y}^3,$$

where

$$J_1(t, \psi)(x) = P_a(t) \int_\Omega \Gamma(x, y, t, t - \tau_a, D_H)H^*(y, t - \tau_a)\psi_3(y, t - \tau_a)\mathrm{d}y, \quad \forall x \in \overline{\Omega},$$

$$J_2(t, \psi)(x) = P_l(t) \int_\Omega \Gamma(x, y, t, t - \tau_l, D_M)M^*(y, t - \tau_l)\psi_1(y, t - \tau_l)\mathrm{d}y, \quad \forall x \in \overline{\Omega},$$

$$J_3(t, \psi)(x) = P_n(t) \int_\Omega \Gamma(x, y, t, t - \tau_n, D_M)M^*(y, t - \tau_n)\psi_2(y, t - \tau_n)\mathrm{d}y, \quad \forall x \in \overline{\Omega}.$$

By the constant-variation formula, we have

$$\hat{v}(x,t,\phi) = \int_0^t \Lambda(t,s)J(s,\hat{v}_s(\phi))\mathrm{d}s + \Lambda(t,0)[\phi(\cdot,0)], \quad \forall t > 0.$$

Then we can define

$$\begin{aligned}
(Q_1\psi)(\theta) &= \Lambda(KT+\theta,0)[\psi(0)], \quad \theta \in [-\hat{\tau},0],\ \psi \in \mathcal{Y}^3,\\
(Q_2\psi)(\theta) &= \int_0^{KT+\theta} \Lambda(KT+\theta,s)J(s,\hat{v}_s(\psi))\mathrm{d}s, \quad \theta \in [-\hat{\tau},0],\ \psi \in \mathcal{Y}^3.
\end{aligned}$$

Let B be a given bounded subset of $\mathcal{Y}^3$, we then have the following claim.

Claim. Q_2B is precompact on $\mathcal{Y}^3$.

Let $\{\Lambda_1(t,s) : t \geq s\}$ be the evolution family on Y^1 of the following linear system

$$\frac{\partial L}{\partial t} = -[\mu_L(t) + \alpha M^*(x,t)]L,$$

and define

$$[P\psi](\theta) = \int_0^{KT+\theta} \Lambda_1(KT+\theta,s)J_1(s,\hat{v}_s(\psi))\mathrm{d}s, \quad \forall\theta \in [-\hat{\tau},0],\ \psi \in \mathcal{Y}^3.$$

It is easy to see that P is the first component of Q_2. Therefore, we only need to prove that PB is precompact on $\mathcal{Y}^1$ since the other two components can be proved in a similar way. In view of Lemma 2.3, it suffices to show that (i) for each $\theta \in [-\hat{\tau},0]$, the set $\{[P\phi](\theta) : \phi \in B\}$ is precompact in X; and (ii) the set PB is equi-continuous in $\theta \in [-\hat{\tau},0]$.

Now we prove the statement (i). It is worth to pointing out that $J_1(t,v_t(\phi)),\ \forall\phi \in B$ can be regarded as a solution of

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla\cdot(D_H(x,t)\nabla u), && x \in \Omega,\ t > 0,\\
\frac{\partial u}{\partial \nu} &= 0, && x \in \partial\Omega, t > 0.
\end{aligned}$$

with initial data $u(\cdot,t-\tau_a) = H^*(\cdot,t-\tau_a)\hat{v}_3(\cdot,t-\tau_a,\phi)$. Clearly,

$$\|H^*(\cdot,t-\tau_a)\hat{v}_3(\cdot,t-\tau_a,\phi)\| \leq C_0,\ \forall t \in [0,KT],\ \phi \in B,$$

for some $C_0 > 0$. According to [7, Remark 2.4], there exists $C_1 > 0$ such that

$$\|J_1(t,v_t(\phi))\|_{Y^1} \leq C_1C_0, \quad \left\|\frac{\mathrm{d}J_1(t,v_t(\phi))}{\mathrm{d}x}\right\|_{Y^1} \leq C_1C_0,\ \forall t \in [0,KT],\ \phi \in B.$$

By the intermediate value theorem,

$$|J_1(t,v_t(\phi))(x_1) - J_1(t,v_t(\phi))(x_2)| \leq C_1C_0|x_1-x_2|,\ \forall t \in [0,KT],\ \forall x_1,x_2 \in \overline{\Omega},\ \phi \in B.$$

For any fixed $x \in \overline{\Omega}$, let $\{\mathcal{O}_x(t,s) : t \geq s\}$ be the evolution family on $\mathbb{R}$ of the following linear system

$$\frac{\partial L}{\partial t} = -[\mu_L(t) + \alpha M^*(x,t)]L.$$

It is easy to see that

$$\mathcal{O}_x(t,s)[u(x)] = [\Lambda_1(t,s)u](x), \quad \forall u \in Y^1, x \in \overline{\Omega},\ 0 \leq s \leq t \leq KT.$$

By the dependence on the parameters for ODE(see, e.g., [16, Chapter V]), there exists some $C_2 > 0$

$$\|\mathcal{O}_{x_1}(t,s) - \mathcal{O}_{x_2}(t,s)\|_{\mathbb{R}} \le C_2|x_1 - x_2|, \quad \forall x_1, x_2 \in \overline{\Omega},\ 0 \le s \le t \le KT.$$

It is easy to see that

$$\|\mathcal{O}_x(t,s)\|_{\mathbb{R}} \le C_3, \quad \forall x \in \overline{\Omega},\ 0 \le s \le t \le KT,$$

for some $C_3 > 0$. Therefore,

$$\begin{aligned}
&\left|\int_0^{KT+\theta} \Lambda_1(KT+\theta,s)J_1(s,\hat{v}_s(\phi))\mathrm{d}s(x_1) - \int_0^{KT+\theta} \Lambda_1(KT+\theta,s)J_1(s,\hat{v}_s(\phi))\mathrm{d}s(x_2)\right| \\
&= \left|\int_0^{KT+\theta} \mathcal{O}_{x_1}(KT+\theta,s)[J_1(s,\hat{v}_s(\phi))(x_1)]\mathrm{d}s - \int_0^{KT+\theta} \mathcal{O}_{x_2}(KT+\theta,s)[J_1(s,\hat{v}_s(\phi))(x_2)]\mathrm{d}s\right| \\
&\le \int_0^{KT+\theta} \left|\mathcal{O}_{x_1}(KT+\theta,s)[J_1(s,\hat{v}_s(\phi))(x_1)] - \mathcal{O}_{x_2}(KT+\theta,s)[J_1(s,\hat{v}_s(\phi))(x_1)]\right| \mathrm{d}s \\
&\quad + \int_0^{KT+\theta} |\mathcal{O}_{x_2}(KT+\theta,s)[J_1(s,\hat{v}_s(\phi))(x_1)] - \mathcal{O}_{x_2}(KT+\theta,s)[J_1(s,\hat{v}_s(\phi))(x_2)]|\mathrm{d}s \\
&\le C_2C_1C_0KT|x_1-x_2| + C_3C_1C_0KT|x_1-x_2|, \quad \theta \in [-\hat{\tau},0],\ \forall x_1,x_2 \in \overline{\Omega},\ \phi \in B.
\end{aligned}$$

This implies that statement (i) holds true. Next, we prove (ii) by modifying the arguments in [46, Section 3.5]. Define $D = \{(\theta,s) : -\hat{\tau} \le \theta \le 0,\ 0 \le s \le KT+\theta\}$. Clearly,

$$\|\Lambda_1(KT+\theta,s)\|_{Y^1} = \max_{x\in\overline{\Omega}} \|\mathcal{O}_x(KT+\theta,s)\|_{\mathbb{R}} \le C_3, \quad \forall (\theta,s) \in D.$$

For any $\epsilon > 0$, there exists $\delta \in (0,\epsilon)$ such that

$$\|\Lambda_1(KT+\theta_1,s) - \Lambda_1(KT+\theta_2,s)\|_{\mathbb{R}} \le \epsilon,\ \forall (\theta_1,s), (\theta_2,s) \in D \text{ with } |\theta_1-\theta_2| \le \delta.$$

Therefore,

$$\begin{aligned}
&\|[P\phi](\theta_1) - [P\phi](\theta_2)\|_{Y^1} \\
&= \left\|\int_0^{KT+\theta_1} \Lambda_1(KT+\theta_1,s)J_1(s,\hat{v}_s(\phi))\mathrm{d}s - \int_0^{KT+\theta_2} \Lambda_1(KT+\theta_2,s)J_1(s,\hat{v}_s(\phi))\mathrm{d}s\right\|_{Y^1} \\
&\le \left\|\int_{KT+\theta_2}^{KT+\theta_1} \Lambda_1(KT+\theta_1,s)J_1(s,\hat{v}_s(\phi))\mathrm{d}s\right\|_{Y^1} \\
&\quad + \left\|\int_0^{KT+\theta_2} [\Lambda_1(KT+\theta_1,s)J_1(s,\hat{v}_s(\phi)) - \Lambda_1(KT+\theta_2,s)J_1(s,\hat{v}_s(\phi))]\mathrm{d}s\right\|_{Y^1} \\
&\le C_3C_1C_0\epsilon + KTC_1C_0\epsilon,\ \forall (\theta_1,s), (\theta_2,s) \in D \text{ with } |\theta_1-\theta_2| \le \delta.
\end{aligned}$$

It then follows that statement (ii), and hence, the Claim holds true. It is worthy to point out that $V(KT,0) = Q_1 + Q_2$. This implies that $r_e(V(KT,0)) = r_e(Q_1)$ due to [33, Theorem 7.27], where $r_e(Q_1)$ is the essential spectral radius of Q_1. Note that there exists some $C > 0$ such that $\|\frac{\mathrm{d}}{\mathrm{d}\theta}\Lambda(KT+\theta,0)\|_X \le C$ for all $\theta \in [-\hat{\tau},0]$ due to $KT > \tau$. According to [8, Section 7.4], we have

$$\alpha(Q_1B) = \max_{\theta\in[-\hat{\tau},0]} \alpha((Q_1B)(\theta)) = \max_{\theta\in[-\hat{\tau},0]} \alpha(\Lambda(kT+\theta,0)[B(0)]),$$

where α is the Kuratowski non-compactness measure, $(Q_1B)(\theta) := \{(Q\phi)(\theta) : \phi \in B\}$ and $B(0) := \{\phi(0) : \phi \in B\}$. It easily follows from the proof of [8, Section 7.4] that $\alpha(B(0)) \le \alpha(B)$. Then we have

$$\alpha(Q_1 B) \leq \max_{\theta\in[-\hat{\tau},0]} \|\Lambda(KT+\theta,0)\|_X \alpha(B(0)) \leq \frac{1}{2}\alpha(B).$$

This implies that $r_e(Q_1) \leq \frac{1}{2}$ due to [8, Theorem 9.9], and hence, $r_e(V(KT,0)) \leq \frac{1}{2}$ as well as $r_e(V(KT,0)) \leq (\frac{1}{2})^{\frac{1}{K}} < 1$.

Step 2. Completion of the proof.

By Step 1 and the generalized Krein-Rutman theorem (see, e.g., [27, Corollary 2.2]), $r(V(T,0))$ is the principal eigenvalue of $V(T,0)$ with the eigenvector $\hat{\phi} \in \mathcal{Y}^3$. Write $\hat{\mathbf{v}}(\cdot,t,\psi) = \hat{v}(\cdot,t,E\psi)$ for any $\psi \in \mathcal{Z}^3_+$. We will prove that $\hat{\phi} \in \mathrm{Int}(\mathcal{Y}^3_+)$ by modifying the arguments in [23, Lemma 3.8]. Letting $\hat{\psi} = R\hat{\phi}$, we first show that $\hat{\psi} \neq 0$. Otherwise, $\hat{v}(\cdot,t,\hat{\phi}) = \hat{\mathbf{v}}(\cdot,t,\hat{\psi}) = 0$ for any $t \geq 0$. Therefore,

$$r(V(T,0))\hat{\phi} = V(T,0)\hat{\phi} = 0$$

implies that $\hat{\phi} = 0$, which is impossible. Since $\hat{v}(\cdot,t,\hat{\phi}) = \hat{\mathbf{v}}(\cdot,t,\hat{\psi})$ is strongly positive on Y^3 for any $t \geq 3\hat{\tau}$, it then follows that $\hat{\phi}$ is strongly positive on $\mathcal{Y}^3$ by [24, Theorem 3].

Finally, $\omega(V) = \omega(\mathbf{V})$ can be proved by the arguments similar to those in [40, Lemma 3.8]. □

Lemma 4.4 *Assume that* (A1) *holds. Then the following statements are valid.*

(i) If $\omega(V) < 0$, then $(0,0,0)$ is globally attractive for solutions of system (4.5) on Y^3 with initial data in $\mathcal{Y}^3_+$.

(ii) If $\omega(V) > 0$, then system (4.5) admits a unique T-periodic positive solution $\mathbb{V}(x,t) = (L^*(x,t), N^*(x,t), A^*(x,t))$, and $\mathbb{V}(\cdot,t)$ is globally attractive for solutions of system (4.5) on Y^3 with initial data in $\mathrm{Int}(\mathcal{Y}^3_+)$.

Proof In the case where $\omega(V) < 0$, we first let $\hat{v}(x,t,\phi)$ be the solution of linear system (4.6) with initial data $\hat{v}_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \overline{\Omega} \times [-\hat{\tau},0]$. It is easy to see that $\hat{v}(x,t,\phi)$ is a super-solution of system (4.5). By the comparison principle, we then have $v(x,t,\phi) \leq \hat{v}(x,t,\phi)$, $\forall x \in \overline{\Omega}$, $t \geq 0$. This implies that $\|S_t\phi\|_{\mathcal{Y}^3} \leq \|V(t,0)\phi\|_{\mathcal{Y}^3}$. It then follows from $\omega(V) < 0$ that $\|V(t,0)\phi\|_{\mathcal{Y}^3} \to 0$, and hence, $\|S_t\phi\|_{\mathcal{Y}^3} \to 0$ as $t \to +\infty$.

In the case where $\omega(V) > 0$, we proceed with three steps.

Step 1. S_T has at most one fixed point in $\mathrm{Int}(\mathcal{Y}^3_+)$.

We fix an integer $n_1 > 0$ such that $n_1 T \geq 4\hat{\tau}$. It is easy to see that $\mathbf{S}_{n_1T}$ is strongly monotone and strictly subhomogeneous on $\mathcal{Z}^3$ (see, e.g.,[13, Theorem 2.2]). By [46, Lemma 2.3.1] or [44, Lemma 1], $\mathbf{S}_{n_1T}$, and hence, $\mathbf{S}_T$ has at most one fixed point in $\mathrm{Int}(\mathcal{Z}^3_+)$. Then system (4.5) has at most one positive T-periodic solution. It follows that S_T has at most one fixed point in $\mathrm{Int}(\mathcal{Y}^3_+)$.

Step 2. S_T possesses a fixed point in $\mathrm{Int}(\mathcal{Y}^3_+)$.

Let $\{V_\epsilon(t,s) : t \geq s\}$ be the evolution family on $\mathcal{Y}^3$ determined by the following linear system

$$\begin{aligned}
\frac{\partial L}{\partial t} &= L_b(x,t,H^*,A,0) - [\mu_L(t) + \alpha M^*(x,t)]L,\\
\frac{\partial N}{\partial t} &= N_b(x,t,M^*,L,0) - [\gamma + \alpha M^*(x,t) + u_N(t)]N,\\
\frac{\partial A}{\partial t} &= A_b(x,t,M^*,N,0) - (\mu_A(t) + \xi H^*(t))A - \delta_A \epsilon A.
\end{aligned}$$

It is easy to see that $r(V(T,0)) > 1$ due to Lemma 2.1 and $\omega(V) > 0$. Thus, there exists $\epsilon_0 > 0$ such that $r(V_\epsilon(T,0)) > 1$ for any $\epsilon \in (0,\epsilon_0)$(see, e.g., [20, Section IV.3.5]). Let ϵ be a given number with $\epsilon \in (0,\epsilon_0)$. By repeating the arguments in Lemma 4.3, we obtain that $V_\epsilon(T,0)$ possesses the principal eigenvalue with strongly positive eigenvector $\hat{\phi}_\epsilon$ on $\mathcal{Y}^3$. Let $\lambda_\epsilon = -\frac{\ln r(V_\epsilon(T,0))}{T} < 0$ and $\tilde{v}_\epsilon(t) = [e^{\lambda_\epsilon t} V_\epsilon(t,0)\hat{\phi}_\epsilon](0)$. It is worth to pointing out that $\tilde{v}_\epsilon(t) = \tilde{v}_\epsilon(t+T), \forall t \in \mathbb{R}$. Without loss of generality, we can assume that $\|\tilde{v}_\epsilon(t)\|_{Y^3} \le \epsilon$ for any $t \in [0,T]$(otherwise, multiply $\hat{\phi}_\epsilon$ by a small enough positive number). It is easy to see that $\tilde{v}_\epsilon$ satisfy

$$
\begin{aligned}
\frac{\partial L}{\partial t} &= L_b^\epsilon(x,t,A) - [\mu_L(t) + \alpha M^*(x,t)]L + \lambda_\epsilon L,\\
\frac{\partial N}{\partial t} &= N_b^\epsilon(x,t,L) - [\gamma + \alpha M^*(x,t) + u_N(t)]N + \lambda_\epsilon N,\\
\frac{\partial A}{\partial t} &= A_b^\epsilon(x,t,N) - (\mu_A(t) + \xi H^*(t))A - \delta_A \epsilon A + \lambda_\epsilon A.
\end{aligned}
$$

where

$$
\begin{aligned}
L_b^\epsilon(x,t,A) &= P_a(t)\int_\Omega \Gamma(x,y,t,t-\tau_a,D_H)e^{\lambda_\epsilon \tau_a} H^*(t-\tau_a)A(y,t-\tau_a)\mathrm{d}y,\\
N_b^\epsilon(x,t,L) &= P_l(t)\int_\Omega \Gamma(x,y,t,t-\tau_l,D_M)e^{\lambda_\epsilon \tau_l} M^*(t,t-\tau_l)L(y,t-\tau_l)\mathrm{d}y,\\
A_b^\epsilon(x,t,N) &= P_n(t)\int_\Omega \Gamma(x,y,t,t-\tau_n,D_M)e^{\lambda_\epsilon \tau_n} M^*(y,t-\tau_n)N(y,t-\tau_n)\mathrm{d}y.
\end{aligned}
$$

Note that $e^{\lambda_\epsilon \tau_a} < 1, e^{\lambda_\epsilon \tau_n} < 1, e^{\lambda_\epsilon \tau_l} < 1$ due to $\lambda_\epsilon < 0$. We then obtain that $\tilde{v}_\epsilon$ satisfy

$$
\begin{aligned}
\frac{\partial L}{\partial t} &\le L_b(x,t,H^*,A,0) - [\mu_L(t) + \alpha M^*(x,t)]L,\\
\frac{\partial N}{\partial t} &\le N_b(x,t,M^*,L,0) - [\gamma + \alpha M^*(x,t) + u_N(t)]N,\\
\frac{\partial A}{\partial t} &\le A_b(x,t,M^*,N,0) - (\mu_A(t) + \xi H^*(t))A - \delta_A A^2.
\end{aligned}
$$

This implies that $\tilde{v}_\epsilon$ is a sub-solution of system (4.5). By the comparison principle, it then follows that $v(x,t,\hat{\phi}_\epsilon) \ge \tilde{v}_\epsilon(x,t,\hat{\phi}_\epsilon),\ \forall (x,t) \in \overline{\Omega} \times [0,T]$. Thus, $S_T\hat{\phi}_\epsilon \ge \hat{\phi}_\epsilon$ in $\mathcal{Y}^3$. By induction, we then obtain that $S_{(n+1)T}\hat{\phi}_\epsilon \ge S_{nT}\hat{\phi}_\epsilon,\ \forall n \ge 0$. Clearly, the discrete orbit $\gamma^+(\hat{\phi}_\epsilon) = \{S_{nT}\hat{\phi}_\epsilon : n \ge 1\}$ is asymptotically compact in $\mathcal{Y}^3$ due to Lemma 4.2. Then its omega limit sets $\omega(\hat{\phi}_\epsilon)$ is nonempty and compact. By a standard monotone iteration scheme (see, e.g., [17, Lemma 1.1]), there is $\phi^* \in \mathcal{Y}_+^3$ such that $S_{nT}\hat{\phi}_\epsilon \to \phi^*$ in $\mathcal{Y}^3$ as $n \to +\infty$ and $S_T\phi^* = \phi^*$. It follows from $\phi^* \ge \hat{\phi}_\epsilon$ that $\phi^* \in \mathrm{Int}(\mathcal{Y}_+^3)$.

Step 3. Completion of the proof.

For any $\phi \in \mathrm{Int}(\mathcal{Y}_+^3)$, there is $0 < \rho_1 < 1 < \rho_2$ such that $\rho_1\phi^* \le \phi \le \rho_2\phi^*$. By using the same arguments in [39, Proposition 3.7], we can prove that $S_{nT}[\rho_1\phi^*] \to \phi^*$ and $S_{nT}[\rho_2\phi^*] \to \phi^*$ in $\mathcal{Y}^3$ as $n \to +\infty$. Thus, $S_{nT}\phi \to \phi^*$ in $\mathcal{Y}^3$ as $n \to +\infty$. Write $\mathbb{V}(\cdot,t) = [S_t\phi^*](0)$. Therefore, $\mathbb{V}(\cdot,t)$ is globally attractive in Y^3 for any positive solution of system (4.5). □

Remark 4.1 In Lemma 4.4, if $\omega(V) > 0$, $\mathbb{V}(x,t)$ is globally attractive for all solutions of system (4.5) on Y^3 with initial data in $\phi \in \mathcal{Y}_+^3 \setminus \{0\}$ with $R\phi \ne 0$ and the solutions with initial data $\phi \in \mathcal{Y}_+^3 \setminus \{0\}$ with $R\phi = 0$ will always be 0. This is because $v(\cdot,t,\phi)$ is strongly positive when $t \ge 4\hat{\tau}$ if $R\phi \ne 0$ and $v(\cdot,t,\phi) = 0$ when $t > 0$ if $R\phi = 0$.

We are in a position to show the main result of this subsection.

Lemma 4.5 *Assume that* (A1) *holds. Then the following statements are valid.*

(i) If $\omega(V) < 0$, then $(H^*(x,t), M^*(x,t), 0, 0, 0)$ is globally attractive for the solutions of system (4.5) *in Y^5 with initial data in $\mathcal{Y}^5_+$.*
(ii) If $\omega(V) > 0$, then the system admits a unique T-periodic positive solution $\mathbb{U}^5(x,t) = (H^*(x,t), M^*(x,t), L^*(x,t), N^*(x,t), A^*(x,t))$ which is globally attractive for the solutions of system (4.5) *in Y^5 with initial data in* $\mathrm{Int}(\mathcal{Y}^5_+)$.

Proof For any $\phi \in \mathcal{Y}^5_+$, system (4.2) admits a unique mild solution $w(x,t,\phi)$ with initial data $w_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \overline{\Omega} \times [-\hat{\tau}, 0]$. For each $t \geq 0$, let $(Q_t\phi)(\theta) = w(\cdot, t+\theta, \phi)$, $\forall \theta \in [-\hat{\tau}, 0]$ from $\mathcal{Y}^5_+$ to $\mathcal{Y}^5_+$. Let $\phi \in \mathcal{Y}^5_+ \setminus \{0\}$ be given and denote by $w(x,t,\phi) = (H(x,t), M(x,t), L(x,t), N(x,t), A(x,t))$, $\forall (x,t) \in \overline{\Omega} \times [-\hat{\tau}, +\infty)$ and $w_t(\phi) = (H_t, M_t, L_t, N_t, A_t)$, $\forall t \geq 0$. It is easy to see that

$$\lim_{t\to+\infty} \|(H(\cdot,t), M(\cdot,t)) - (H^*(t), M^*(\cdot,t))\|_{Y^2} = 0,$$

and

$$\lim_{t\to+\infty} \|(H_t, M_t) - (H^*_t, M^*_t)\|_{\mathcal{Y}^2} = 0,$$

where $H^*_t(\theta) = H^*(t+\theta)$ and $M^*_t(\theta) = M^*(\cdot, t+\theta)$ for all $t \geq 0$ and $\theta \in [-\hat{\tau}, 0]$. By Lemma 2.3 and the similar arguments to those in [18, Lemma 5.2], it follows that the discrete forward orbit $\gamma(\phi) = \{Q_{nT}\phi : n \geq 0\}$ is asymptotically compact. Thus its omega limit set $\omega(\phi)$ is a compact, invariant, internally chain transitive set for the Poincaré map Q_T. Therefore,

$$\omega(\phi) = (H^*_0, M^*_0) \times \tilde{\omega}$$

where $\tilde{\omega}$ is a subset of $\mathcal{Y}^3_+$. In the case where $\omega(V) < 0$, Lemma 4.4 and [46, Theorem 1.2.1] imply that $\tilde{\omega} = (0,0,0)$, and hence, $\omega(\phi) = (H^*_0, M^*_0, 0, 0, 0)$.

In the case where $\omega(V) > 0$, let $\phi \in \mathrm{Int}(\mathcal{Y}^5_+)$. It then follows from Lemma 4.4, Remark 4.1 and [46, Theorem 1.2.2] that either $\tilde{\omega} = (0,0,0)$ or $\tilde{\omega} = (L^*_0, N^*_0, A^*_0)$. Now, we let $\{V_\epsilon(t,s) : t \geq s\}$ be the evolution family on $\mathcal{Y}^3$ determined by the following linear system

$$\begin{aligned}
\frac{\partial L}{\partial t} &= L_b(x,t,H^* - \epsilon, A, 0) - [\mu_L(t) + \alpha(M^*(x,t) + \epsilon)]L, \\
\frac{\partial N}{\partial t} &= N_b(x,t,M^* - \epsilon, L, 0) - [\gamma + \alpha(M^*(x,t) + \epsilon) + u_N(t)]N, \\
\frac{\partial A}{\partial t} &= A_b(x,t,M^* - \epsilon, N, 0) - (\mu_A(t) + \xi(H^*(t) + \epsilon))A - \delta_A \epsilon A.
\end{aligned} \tag{4.7}$$

It is easy to see that $r(V(T,0)) > 1$ due to Lemma 2.1 and $\omega(V) > 0$. Thus, there exists $\epsilon_0 > 0$ such that $r(V_\epsilon(T,0)) > 1$ for any $\epsilon \in (0, \epsilon_0)$(see, e.g., [20, Section IV.3.5]). Fix an $\epsilon \in (0, \epsilon_0)$. By the same arguments as those in Lemma 4.3, it then follows that $V_\epsilon(T,0)$ possess the principal eigenvalue with strongly positive eigenvector $\hat{\phi}_\epsilon$ on $\mathcal{Y}^3$. Assume, by contradiction, that $\tilde{\omega} = (0,0,0)$. Then $\omega(\phi) = (H^*_0, M^*_0, 0, 0, 0)$. Without loss of generality, we can assume that $\|(H(\cdot,t), M(\cdot,t)) - (H^*(t), M^*(\cdot,t))\|_{Y^2} \leq \epsilon$ and $\|(L(\cdot,t), N(\cdot,t), A(\cdot,t)) - (0,0,0)\|_{Y^3} \leq \epsilon$ for any $t \geq 0$. Hence, $(L(x,t), N(x,t), A(x,t))$ satisfy the following inequities

$$\frac{\partial L}{\partial t} \geq L_b(x, t, H^* - \epsilon, A, 0) - [\mu_L(t) + \alpha(M^*(x, t) + \epsilon)]L,$$
$$\frac{\partial N}{\partial t} \geq N_b(x, t, M^* - \epsilon, L, 0) - [\gamma + \alpha(M^*(x, t) + \epsilon) + u_N(t)]N,$$
$$\frac{\partial A}{\partial t} \geq A_b(x, t, M^* - \epsilon, N, 0) - (\mu_A(t) + \xi(H^*(t) + \epsilon))A - \delta_A \epsilon A.$$

That is, (L, N, A) is a super-solution of linear system (4.7). It is easy to choose a small enough number $\eta > 0$ such that $(L_0, N_0, A_0) \geq \eta\hat{\phi}_\epsilon$ in $\mathcal{Y}^3$. By the comparison principle, we conclude that $(L(\cdot, t), N(\cdot, t), A(\cdot, t)) \geq V_\epsilon(t, 0)\eta\hat{\phi}_\epsilon$ in Y^3 for any $t \geq 0$, which is impossible since $V_\epsilon(t, 0)\eta\hat{\phi}_\epsilon$ is unbounded in Y^3 as $t \to +\infty$. It follows that $\tilde{\omega} = (L_0^*, N_0^*, A_0^*)$, and hence, $\omega(\phi) = (H_0^*, M_0^*, L_0^*, N_0^*, A_0^*)$. Thus, $\lim_{t\to+\infty} \|w(\cdot, t, \phi) - \mathbb{U}^5(\cdot, t)\|_{Y^5} = 0$. □

4.2 Global Dynamics

In this subsection, we introduce the basic reproduction ratio for model (4.1) and study the global dynamics of Lyme disease invasion. Throughout of this subsection, we assume that (A1) holds and $\omega(V) > 0$, where the evolution family is determined by system (4.6).

According to Lemma 4.5, we see that the system (4.2) admits a globally attractive T-periodic positive solution $\mathbb{U}^5(x, t) = (H^*(x, t), M^*(x, t), L^*(x, t), N^*(x, t), A^*(x, t))$. That is, system (4.1) has a unique disease-free T-periodic solution

$$\mathbb{E} = (H^*(x, t), M^*(x, t), L^*(x, t), N^*(x, t), A^*(x, t), 0, 0, 0).$$

We then consider the linearized system of infective compartments for system (4.1) at $\mathbb{E}$.

$$\begin{aligned}
\frac{\partial m}{\partial t} &= \nabla \cdot (D_M(x, t)\nabla m) - \mu_M(t)m + \alpha\beta(x, t)M^*(x, t)n,\\
\frac{\partial n}{\partial t} &= n_b(x, t, L^*, m) - [\gamma + \alpha M^*(x, t) + u_N(t)]n,\\
\frac{\partial a}{\partial t} &= a_b(x, t, M^*, N^*, m, n) - (\mu_A(t) + \xi H^*(t) - \delta_A A^*(x, t))a,
\end{aligned} \tag{4.8}$$

where m is subject to the Neumann boundary condition. Note that the third equation of system (4.8) is decoupled from the first two equations. Thus, the basic reproduction ratio can be defined for model (4.1) only using the first two equations of system (4.8).

$$\begin{aligned}
\frac{\partial m}{\partial t} &= \nabla \cdot (D_M(x, t)\nabla m) - \mu_M(t)m + \alpha\beta(x, t)M^*(x, t)n,\\
\frac{\partial n}{\partial t} &= n_b(x, t, L^*, m) - [\gamma + \alpha M^*(x, t) + u_N(t)]n,
\end{aligned} \tag{4.9}$$

where m is subject to the Neumann boundary condition. Hence, we can choose $X = Y^2$, $\mathbb{X} = \mathbb{Y}^2$, $\mathcal{C} = \mathcal{Y}^2$. Let $v = (v_1, v_2) \in \mathbb{X}$ be the spatial distribution. Let $\{U(t, s) : t \geq s\}$ and $\{\Phi(t, s) : t \geq s\}$ are the evolution family, respectively, on $\mathcal{C}$ and X determined by (4.9) and

$$\begin{aligned}
\frac{\partial m}{\partial t} &= \nabla \cdot (D_M(x, t)\nabla m) - \mu_M(t)m,\\
\frac{\partial n}{\partial t} &= -[\gamma + \alpha M^*(x, t) + u_N(t)]n,
\end{aligned}$$

where m is subject to the Neumann boundary condition. We define the operator $F(t)$ from $\mathcal{C}$ to X by

$$F(t)\phi = (F_1(t)\phi_2, F_2(t)\phi_1), \quad \forall \phi \in \mathcal{C},\ t \in \mathbb{R},$$

where $F_1(t)\phi_2 = \alpha\beta(\cdot, t)M^*(\cdot, t)\phi_2(\cdot, t)$ and $F_2(t)\phi_1 = n_b(x, t, L^*, \phi_1)$. Hence,

$$Bv = (B_1 v_2, B_2 v_1), \quad \forall v \in \mathbb{X}.$$

Here $B_1 v_2(t) = F_1(t)(v_2)_t$ and $B_2 v_1(t) = F_2(t)(v_1)_t$ for any $t \in \mathbb{R}$, where $((v_1)_t, (v_2)_t) = v_t \in \mathcal{C}$ with $v_t(\theta) = v(t + \theta),\ \theta \in [-\hat{\tau}, 0]$.

Let $\{\Phi_1(t, s) : t \geq s\}$ and $\{\Phi_2(t, s) : t \geq s\}$ be the evolution families on Y^1, respectively, determined by two linear equations

$$\frac{\partial m}{\partial t} = \nabla \cdot (D_M(x, t)\nabla m) - \mu_M(t)m$$

subject to the Neumann boundary condition, and

$$\frac{\partial n}{\partial t} = -[\gamma + \alpha M^*(x, t) + u_N(t)]n.$$

Thus,

$$Av = (A_1 v_1, A_2 v_2), \quad \forall v \in \mathbb{X},$$

where $[A_1 v_1](t) = \int_0^{+\infty} \Phi_1(t, t-s)v_1(t-s)\mathrm{d}s$ and $[A_2 v_2](t) = \int_0^{+\infty} \Phi_2(t, t-s)v_2(t-s)\mathrm{d}s$ for any $t \in \mathbb{R}$. We then have

$$\mathfrak{L}v = ABv = (A_1 B_1 v_2, A_2 B_2 v_1)$$

and the basic reproduction ratio can be defined by $\mathcal{R}_0 = r(\mathfrak{L})$, which is the spectral radius of $\mathfrak{L}$ on $\mathbb{X}$. By the arguments similar to those in Lemma 4.3, we have the following result.

Lemma 4.6 *Assume that* (A1) *holds and* $\omega(V) > 0$. *If* $r(U(T, 0)) \geq 1$, *then* $r(U(T, 0))$ *is the principal eigenvalue.*

Lemma 4.7 *Assume that* (A1) *holds and* $\omega(V) > 0$. *Then* $\mathcal{R}_0 - 1$ *has the same sign as* $\omega(U)$ *and* $r(U(T, 0)) - 1$.

Proof It is easy to see that (H1) and (H2) hold. Next, we show that (H3) and (H5) are valid in two steps.
Step 1. $\mathfrak{L}^2$ is compact on $\mathbb{X}$.

It is easy to see that A_1, B_1, A_2, B_2 are bounded on $\mathbb{Y}^1$. By repeating the arguments for Claim 1 in Proposition 3.9, we obtain that A_1 is compact on $\mathbb{Y}^1$. Clearly,

$$\mathfrak{L}^2 v = ABABv = (A_1 B_1 A_2 B_2 v_1, A_2 B_2 A_1 B_1 v_2).$$

This implies that $\mathfrak{L}^2$ is compact on $\mathbb{X}$, since $A_1 B_1 A_2 B_2$ and $A_2 B_2 A_1 B_1$ are compact on $\mathbb{Y}^1$.
Step 2. $\mathfrak{L}$ admits the principal eigenvalue.

It is easy to see that A_1, B_1, A_2, B_2 are strictly positive and map $\mathrm{Int}(\mathcal{Y}_+^1)$ to $\mathrm{Int}(\mathcal{Y}_+^1)$. Note that $\Phi_1(t, s)$ is strongly positive on Y^1 for any $t > s$. For any $v_1 \in \mathbb{Y}_+^1 \setminus \{0\}$, we obtain that $\int_0^{+\infty} \Phi_1(t, t - s)v_1(t - s)\mathrm{d}s,\ t \in \mathbb{R}$ is strongly positive on Y^1. This implies that A_1 is strongly positive on $\mathbb{Y}^1$, and hence, $A_1 B_1 A_2 B_2$ are strongly positive on $\mathbb{Y}^1$.

Next, we will show that $r(\mathfrak{L}) > 0$. For a fixed $v_1 \in \mathbb{Y}_+^1 \setminus \{0\}$, there is a $r > 0$ such that $A_1 B_1 A_2 B_2 v_1 \geq r v_1$ in Y^1. This implies that $\mathfrak{L}^2 v \geq rv$, where $v = (v_1, 0)$, and hence, $r(\mathfrak{L}^2) > 0$ by the Gelfand's formula(see, e.g., [30, Theorem VI.6]). By the Krein-Rutman theorem(see, e.g., [8, Theorem 19.2]), $\mathfrak{L}^2$ possesses the principal eigenvalue with an eigenvector in $\tilde{v} \in \mathbb{X}_+ \setminus \{0\}$. Noting that $r^2(\mathfrak{L}) = r(\mathfrak{L}^2)$ and $(r^2(\mathfrak{L}) - \mathfrak{L}^2)\tilde{v} = 0$, we have $(r(\mathfrak{L}) - \mathfrak{L})\hat{v} = 0$, where $\hat{v} = (r(\mathfrak{L}) + \mathfrak{L})\tilde{v} \in \mathbb{X}_+$. This implies that $\mathfrak{L}$ possesses the principal eigenvalue with positive eigenvector in $\mathbb{X}_+$. Therefore, (H3) and (H5) hold true. It remains to show (H4). Let $\{U(t,s,\lambda) : t \geq s\}$ be the evolution family on $\mathcal{C}$ determined by system

$$\begin{aligned}
\frac{\partial m}{\partial t} &= \nabla \cdot (D_M(x,t)\nabla m) - \mu_M(t)m + \lambda\alpha\beta(x,t)M^*(x,t)n,\\
\frac{\partial n}{\partial t} &= \lambda n_b(x,t,L^*,m) - [\gamma + \alpha M^*(x,t) + u_N(t)]n,
\end{aligned}$$

where m is subject to the Neumann boundary condition. We repeat the arguments in Lemma 4.6 to obtain that $r(U(T,0,\lambda))$ is the principal eigenvalue whenever $r(U(T,0,\lambda)) \geq 1$. Then (H4) holds true, and hence, the desired statement follows from Theorem 3.7.

In order to show the global dynamics of system (4.1), we let $\mathcal{M} = M + m$, $\mathcal{N} = N + n$, $\mathcal{A} = A + a$. Then system (4.1) is equivalent the following system:

$$\begin{aligned}
\frac{\partial H}{\partial t} &= \nabla \cdot (D_H(x,t)\nabla H) + r_H(t) - \mu_H(t)H,\\
\frac{\partial \mathcal{M}}{\partial t} &= \nabla \cdot (D_M(x,t)\nabla \mathcal{M}) + \mathcal{M}B_M(x,t,\mathcal{M}) - \mu_M(t)\mathcal{M},\\
\frac{\partial L}{\partial t} &= L_b(x,t,H,\mathcal{A},0) - \mu_L(t)L - \alpha L\mathcal{M},\\
\frac{\partial \mathcal{N}}{\partial t} &= N_b(x,t,\mathcal{M},L,0) - [\gamma + \alpha\mathcal{M} + u_N(t)]\mathcal{N},\\
\frac{\partial \mathcal{A}}{\partial t} &= A_b(x,t,\mathcal{M},\mathcal{N},0) - (\mu_A(t) + \xi H)\mathcal{A} - \delta_A\mathcal{A}^2,\\
\frac{\partial m}{\partial t} &= \nabla \cdot (D_M(x,t)\nabla m) - \mu_M(t)m + \alpha\beta(x,t)(\mathcal{M} - m)n,\\
\frac{\partial n}{\partial t} &= n_b(x,t,L,m) - [\gamma + \alpha\mathcal{M} + u_N(t)]n,\\
\frac{\partial a}{\partial t} &= a_b(x,t,\mathcal{M} - m,\mathcal{N} - n,m,n) - (\mu_A(t) + \xi H)a - \delta_A\mathcal{A}a,
\end{aligned} \tag{4.10}$$

where H, $\mathcal{M}$, m are subject to the Neumann boundary condition. In view of Lemma 4.5, $\mathbb{U}^5(x,t) = (H^*(x,t), M^*(x,t), L^*(x,t), N^*(x,t), A^*(x,t))$ is a globally attractive periodic positive solution of system (4.2), which is exactly the same as the first five equations of system (4.10). So we need to consider the following system.

$$\begin{aligned}
\frac{\partial m}{\partial t} &= \nabla \cdot (D_M(x,t)\nabla m) - \mu_M(t)m + \alpha\beta(x,t)(\mathcal{M}^*(x,t) - m)n,\\
\frac{\partial n}{\partial t} &= n_b(x,t,L^*,m) - [\gamma + \alpha\mathcal{M}^*(x,t) + u_N(t)]n,\\
\frac{\partial a}{\partial t} &= a_b(x,t,\mathcal{M}^* - m,\mathcal{N}^* - n,m,n) - (\mu_A(t) + \xi H^*(t))a - \delta_A\mathcal{A}^*(x,t)a,
\end{aligned} \tag{4.11}$$

where m is subject to the Neumann boundary condition. Note that the first two equations are decoupled with third one, so we consider the following system:

$$\begin{aligned}
\frac{\partial m}{\partial t} &= \nabla \cdot (D_M(x,t)\nabla m) - \mu_M(t)m + \alpha\beta(x,t)(\mathcal{M}^*(x,t) - m)n,\\
\frac{\partial n}{\partial t} &= n_b(x,t,L,m) - [\gamma + \alpha\mathcal{M}^*(x,t) + u_N(t)]n,
\end{aligned} \tag{4.12}$$

where m is subject to the Neumann boundary condition. Let

$$\mathcal{C}_{M^*} := \{\phi = (\phi_1, \phi_2) \in \mathcal{Y}_+^2 : \phi(x,\theta) \le M^*(x,\theta), \quad \forall (x,\theta) \in \overline{\Omega} \times [-\hat{\tau}, 0]\}.$$

and

$$\mathcal{C}_{M^*}^{++} := \{\phi = (\phi_1, \phi_2) \in \mathcal{Y}_+^2 : 0 < \phi(x,\theta) \le M^*(x,\theta), \quad \forall (x,\theta) \in \overline{\Omega} \times [-\hat{\tau}, 0]\}.$$

Lemma 4.8 *Assume that* (A1) *holds and* $\omega(V) > 0$. *For any* $\phi \in \mathcal{C}_{M^*}$, *system* (4.12) *admits a unique solution* $u(x,t,\phi)$ *on* $[0,+\infty)$ *with initial data* $u_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \Omega \times [-\hat{\tau}, 0]$. *Moreover, solutions are uniformly bounded on* $\mathcal{Y}_+^2$.

Proof According to [24, Corollary 5], system (4.12) admits a unique nonnegative continuous solution $u(x,t,\phi)$ on $[0,t_\phi)$ with initial data $u_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \Omega \times [-\hat{\tau}, 0]$ and the comparison principle holds for upper and lower solutions of system (4.12).

Clearly,

$$\frac{\partial M^*}{\partial t} \ge \nabla \cdot (D_M(x,t)\nabla M^*) - \mu_M(t) M^*(x,t),$$

and there is $\tilde{N} > 0$ such that

$$\frac{\partial \hat{N}}{\partial t} \ge n_b(x,t,L,M^*) - [\gamma + \alpha M^*(x,t) + u_N(t)]\hat{N},$$

for any $\hat{N} \ge \tilde{N}$. This implies that $(M^*(x,t), \hat{N}) \in \mathcal{C}_{M^*}$ is a super-solution of system (4.12), and hence, $t_\phi = +\infty$. $\square$

Lemma 4.9 *Assume that* (A1) *holds and* $\omega(V) > 0$. *Then the following statements are valid.*

(i) If $\mathcal{R}_0 < 1$, *then* $(0,0)$ *is globally attractive for solutions of system* (4.12) *on* Y^2 *with initial data in* $\mathcal{C}_{M^*}$.

(ii) If $\mathcal{R}_0 > 1$, *then system* (4.12) *admits a unique* T*-periodic positive solution* $\mathbb{W}(x,t) = (\overline{m}(x,t), \overline{n}(x,t))$, *and* $\mathbb{W}(\cdot,t)$ *is globally attractive for solutions of system* (4.12) *on* Y^2 *with initial data in* $\mathcal{C}_{M^*}^{++}$.

Proof This lemma can be proved by the arguments similar to those in Lemma 4.5. It is worth to pointing out that $\rho_2\phi^*$ should be replaced by the $(M_0^*, \overline{N}_0)$ in the Step 3 of Lemma 4.5, where $M_0^*(x,\theta) = M^*(x,\theta)$, $\overline{N}_0(\theta) = \tilde{N}$, $\forall \theta \in [-\hat{\tau}, 0]$ with initial data $\phi \le (M_0^*, \overline{N}_0)$ in $\mathcal{C}_{M^*}$.

Since the first seven equations in system (4.10) do not depend on the variable a, we write them as $(*)$. Next, it is necessary to introduce the following spaces:

$$\begin{aligned}
\mathcal{W}_+^7 &= \{\phi = (\phi_1, \phi_2, \ldots, \phi_7) \in \mathcal{Y}_+^7 : \phi_2 \ge \phi_6 \text{ in } \mathcal{Y}^1\},\\
\mathcal{W}_+^8 &= \{\phi = (\phi_1, \phi_2, \ldots, \phi_8) \in \mathcal{Y}_+^8 : \phi_2 \ge \phi_6 \text{ in } \mathcal{Y}^1\},\\
\mathcal{W}_{++}^7 &= \{\phi = (\phi_1, \phi_2, \ldots, \phi_7) \in \mathrm{Int}(\mathcal{Y}_+^7) : \phi_2 \ge \phi_6 \text{ in } \mathcal{Y}^1\},
\end{aligned}$$

and

$$\mathcal{W}_{++}^8 = \{\phi = (\phi_1, \phi_2, \ldots, \phi_8) \in \mathrm{Int}(\mathcal{Y}_+^8) : \phi_2 \ge \phi_6 \text{ in } \mathcal{Y}^1\}.$$

Then we have the following result.

Lemma 4.10 *Assume that* (A1) *holds and* $\omega(V) > 0$. *Then the following statements are valid.*

(i) If $\mathcal{R}_0 < 1$, *then* $\mathbb{E}^7(x,t) = (H^*(t), M^*(x,t), L^*(x,t), N^*(x,t), A^*(x,t), 0, 0)$ *is globally attractive for solutions of system* $(*)$ *on* Y^7 *with initial data in* $\mathcal{W}^7_+$.

(ii) If $\mathcal{R}_0 > 1$, *then system* $(*)$ *admits a unique* T*-periodic positive solution* $\mathbb{U}^7(x,t) = (H^*(t), M^*(x,t), L^*(x,t), N^*(x,t), A^*(x,t), \overline{m}(x,t), \overline{n}(x,t))$, *and* $\mathbb{U}^7(\cdot,t)$ *is globally attractive for solutions of system* $(*)$ *on* Y^7 *with initial data in* $\mathcal{W}^7_{++}$.

Proof For any $\phi \in \mathcal{W}^7_+$, system $(*)$ admits a unique mild solution $w(x,t,\phi)$ with initial data $w_0(x,\theta,\phi) = \phi(x,\theta)$, $\forall (x,\theta) \in \overline{\Omega} \times [-\hat{\tau}, 0]$. For each $t \geq 0$, denote $(Q_t\phi)(\theta) = w(\cdot, t+\theta, \phi)$, $\forall \theta \in [-\hat{\tau}, 0]$ on $\mathcal{W}^7_+$. Let $\phi \in \mathcal{W}^7_+$ be given and write

$$w(x,t,\phi) = (H(x,t), \mathcal{M}(x,t), L(x,t), \mathcal{N}(x,t), \mathcal{A}(x,t), m(x,t), n(x,t)),$$

for any $(x,t) \in \overline{\Omega} \times [-\hat{\tau}, +\infty)$ and $w_t(\phi) = (H_t, \mathcal{M}_t, L_t, \mathcal{N}_t, \mathcal{A}_t, m_t, n_t)$, $\forall t \geq 0$. It is easy to see that

$$\lim_{t \to +\infty} \|(H_t, \mathcal{M}_t, L_t, \mathcal{N}_t, \mathcal{A}_t) - (H^*_t, M^*_t, L^*_t, N^*_t, A^*_t)\|_{\mathcal{Y}^5} = 0,$$

where $H^*_t(\theta) = H^*(t+\theta)$, $M^*_t(\theta) = M^*(\cdot, t+\theta)$, $L^*_t(\theta) = L^*(\cdot, t+\theta)$, $N^*_t(\theta) = N^*(\cdot, t+\theta)$ and $A^*_t(\theta) = A^*(t+\theta)$ and for all $t \geq 0$ and $\theta \in [-\hat{\tau}, 0]$. By Lemma 2.3 and the similar arguments to those in [18, Lemma 5.2], it follows that the discrete forward orbit $\gamma(\phi) = \{Q_{nT}\phi : n \geq 0\}$ is asymptotically compact. Thus, its omega limit set $\omega(\phi)$ is a compact, invariant, internally chain transitive set for the Poincaré map Q_T. It then follows that

$$\omega(\phi) = (H^*_0, M^*_0, L^*_0, N^*_0, A^*_0) \times \tilde{\omega}.$$

By the comparison arguments, we have $\mathcal{M}(\cdot,t) \geq m(\cdot,t)$ for any $t \geq 0$. For any $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_7) \in \omega(\phi)$, there exists a sequence $k_j \to +\infty$ such that $Q_{k_j}\phi \to \varphi$ as $j \to +\infty$. Hence $\mathcal{M}_{k_jT} \geq m_{k_jT}$ in $\mathcal{Y}^1$. Letting $j \to +\infty$, we obtain that $M^*_0 = \varphi_2 \geq \varphi_6$ in $\mathcal{Y}^1$. By the comparison arguments, we conclude that $M^*_t \geq (Q_t[\varphi])_6$, where $(Q_t[\varphi])_6$ is the sixth component of $Q_t[\varphi]$. Thus, $\tilde{\omega} \subset \mathcal{C}_{M^*}$. By the arguments similar to those in Lemma 4.5, it follows that statements (i) and (ii) hold true. □

By repeating the above arguments, we obtain the following result.

Lemma 4.11 *Assume that* (A1) *holds and* $\omega(V) > 0$. *Then the following statements are valid.*

(i) If $\mathcal{R}_0 < 1$, *then* $\mathbb{E}^8(x,t) = (H^*(t), M^*(x,t), L^*(x,t), N^*(x,t), A^*(x,t), 0, 0, 0)$ *is globally attractive for solutions of system* (4.10) *on* Y^8 *with initial data in* $\mathcal{W}^8_+$.

(ii) If $\mathcal{R}_0 > 1$, *then system* (4.10) *admits a unique* T*-periodic positive solution* $\mathbb{U}^8(x,t) = (H^*(t), M^*(x,t), L^*(x,t), N^*(x,t), A^*(x,t), \overline{m}(x,t), \overline{n}(x,t), \overline{a}(x,t))$, *and* $\mathbb{U}^8(\cdot,t)$ *is globally attractive for solutions of system* (4.10) *on* Y^8 *with initial data in* $\mathcal{W}^8_{++}$.

We are in a position to prove the main result of this section.

Theorem 4.12 *Assume that* (A1) *holds and* $\omega(V) > 0$. *Then the following statements are valid.*

(i) If $\mathcal{R}_0 < 1$, *then* $\mathbb{E}(x,t) = (H^*(t), M^*(x,t), L^*(x,t), N^*(x,t), A^*(x,t), 0, 0, 0)$ *is globally attractive for solutions of system* (4.1) *on* Y^8 *with initial data in* $\mathcal{Y}^8_+$.

(ii) If $\mathcal{R}_0 > 1$*, then system* (4.1) *admits a unique* T*-periodic positive solution* $\mathbb{U}(x,t) = (H^*(t), M^*(x,t) - \overline{m}(x,t), L^*(x,t), N^*(x,t) - \overline{n}(x,t), A^*(x,t) - \overline{a}(x,t), \overline{m}(x,t), \overline{n}(x,t), \overline{a}(x,t))$*, and* $\mathbb{U}(\cdot,t)$ *is globally attractive for solutions of system* (4.1) *on* Y^8 *with initial data in* $\mathrm{Int}(\mathcal{Y}^8_+)$.

Proof For any $\phi = (\phi_1, \phi_2, \cdots, \phi_8) \in \mathcal{Y}^8_+$, let $\psi = (\psi_1, \psi_2, \cdots, \psi_8) \in \mathcal{W}^8_+$, where $\psi_i = \phi_i, \forall i = 1, 3, 6, 7, 8$ and $\psi_2 = \phi_2 + \phi_6,\ \psi_4 = \phi_4 + \phi_7, \psi_5 = \phi_5 + \phi_8$. It is easy to see that $\psi \in \mathcal{W}^8_{++}$ if $\phi \in \mathrm{Int}(\mathcal{Y}^8_+)$. Hence, the threshold type result follows from Lemma 4.11.

It remains to show that $\overline{M} = M^* - \overline{m}$, $\overline{N} = N^* - \overline{n}$, $\overline{A} = A^* - \overline{a}$ are nonzero in the case where $\mathcal{R}_0 > 1$. We consider the following equation

$$\frac{\partial M}{\partial t} = \nabla\cdot(D_M(x,t)\nabla M) + M^*(x,t)B_M(x,t,M^*) - (\mu_M(t) + \alpha\beta(x,t)\overline{n}(x,t))M, \quad (4.13)$$

where M is subject to the Neumann boundary condition. It is easy to see that $\overline{M}$ and 0 are solution and sub-solution of system (4.13), respectively. Let $M(x,t,\varphi)$ be the solution of system (4.13) with initial data $M(x,0,\varphi) = \varphi(x),\ \forall x \in \Omega$. By the comparison principle, we conclude that $M(x,t,0) \le M(x,t,\overline{M}(\cdot,0)) = \overline{M}(x,t),\ \forall x \in \overline{\Omega},\ t \ge 0$. We claim that

Claim. $M(x,t,0) > 0,\ \forall x \in \overline{\Omega},\ t > 0$.

Considering the strong maximum principle and $M(x,t,0) \ge 0$, we need to show that $M(\cdot,t,0) > 0$ in Y^1 when $t > 0$. Assume, by contradiction, that there exists $t_0 > 0$ such that $M(\cdot,t,0) = 0$ for all $t \in [0,t_0]$ and $M(\cdot,t,0) > 0$ in Y^1 for all $t > t_0$. Therefore, $M(x,t,0) > 0,\ \forall x \in \overline{\Omega},\ t > t_0$ due to the strong maximum principle. Since 0 isn't the solution equation (4.13), it then follows that $t_0 < +\infty$. On the one hand,

$$\frac{\partial M}{\partial t}(x,t_0) = \lim_{t \to t_0^-} \frac{\partial M}{\partial t}(x,t) = 0, \quad \forall x \in \Omega.$$

On the other hand,

$$\frac{\partial M}{\partial t}(x,t_0) = M^*(x,t_0)B_M(x,t_0,M^*) > 0, \quad \forall x \in \Omega.$$

This contradiction shows that $M(\cdot,t,0) > 0$ in Y^1 for all $t > 0$, and hence, $M(x,t,0) > 0$ and $\overline{M}(x,t,0) > 0,\ \forall x \in \overline{\Omega},\ t > 0$.

By analogous arguments, we can show that $\overline{N}(x,t) > 0$ and $\overline{A}(x,t) > 0$ for all $x \in \overline{\Omega}$ and $t > 0$.

At the end of this section, we explore the possible measure to control the Lyme disease by numerically computing $\mathcal{R}_0$ using the method suggested in Remark 3.2. For convenience, we choose $T = 12$ and $\overline{\Omega} = [0,1]$. The baseline parameter are $\mu_M = 0.02$, $\mu_L = 0.09$, $\mu_N = 0.06$, $\mu_A = 0.03$, $D_H = 0.1$, $D_M = 0.01$, as derived from [6], and $\tau_l = \frac{3}{30.4}$, $\tau_n = \frac{5}{30.4}$, $\tau_a = \frac{10}{30.4}$, as the same in [29], $\xi = 0.1$, $\mu_H = 0.01$, $\delta_A = 0.065$, $\beta(x,t) = 0.6(1 + 0.9\cos(\pi x))$ and $\beta_T(x,t) = 0.6(1 + 0.9\cos(\pi x))$, $H^* = 0.042$, $\alpha = 0.02$, $\gamma = 0.005$, $r_m = 0.06$.

The first strategy for the control of the disease transmission is to reduce the tick population. Here, we discuss the effect of the birth rate of ticks. By fixing $K_M(x) = 3(1 + \cos(2\pi x))$ and letting $r(t) = c(1 + 0.9(\cos(2\pi t/12)))$, we can see that $\mathcal{R}_0$ is a increasing function of c and the gradient of $\mathcal{R}_0$ with respect to c may be a large constant (Fig. 1). This implies that the maximum birth rate of ticks plays an important pole in the process of disease spread. Reducing the birth rate of ticks is probably an effective measure to control the Lyme disease.

The second strategy is to decrease the number of mice. Fixing $r(t) = 0.35(1 + 0.9(\cos(\frac{2\pi t}{12})))$ and letting $K_M(x) = c(1 + \cos(2\pi x))$, we can observe that the $\mathcal{R}_0$ is initially

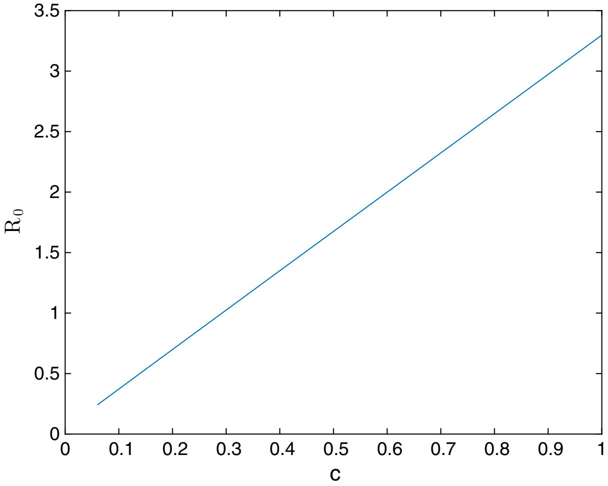

Fig. 1 $\mathcal{R}_0$ is increasing with respect to c. Here $K_M(x) = 3(1 + 0.8\cos(2\pi x))$ and $r(t) = c(1 + 0.9(\cos(2\pi t/12)))$

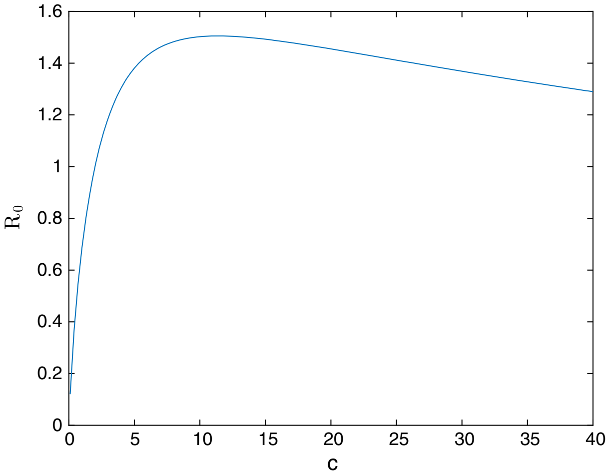

Fig. 2 $\mathcal{R}_0$ initially increases and then decreases with respect to c. Here $r(t) = 0.35(1 + 0.9(\cos(\frac{2\pi t}{12})))$ and $K_M(x) = c(1 + 0.8\cos(2\pi x))$

increases and attained its maximum point in Fig. 2. Clearly, reducing the number of mice is ineffective near the maximum value point. The gradient $\mathcal{R}_0$ with respect to c is negative when the number of mice is large enough, which suggests that $\mathcal{R}_0$ might have paradoxical effects with decreased population of mice. Here we should point out that a similar figure can be found in [31, Fig. 2]. Next, we present more illustrations. It is interesting to see that the population of L initially increases and then decreases with respect to c from Fig. 3, which may be an essential reason that the curve of $\mathcal{R}_0$ versus c have the similar qualitative property. The number of N and A are still increasing with respect to c and so is total population of ticks by observing Fig. 4. This implies that there may be an outbreak of the Lyme disease after a mouse plague.

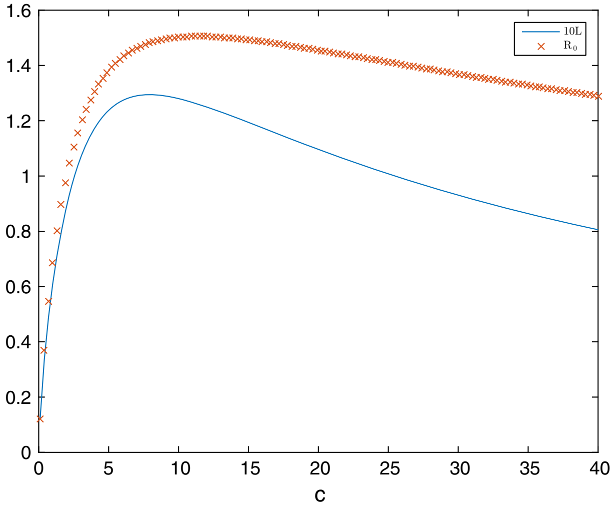

Fig. 3 The parameters are the same as those in Fig. 2, and $10L$ means the ten times of the average amount of L at the disease-free state

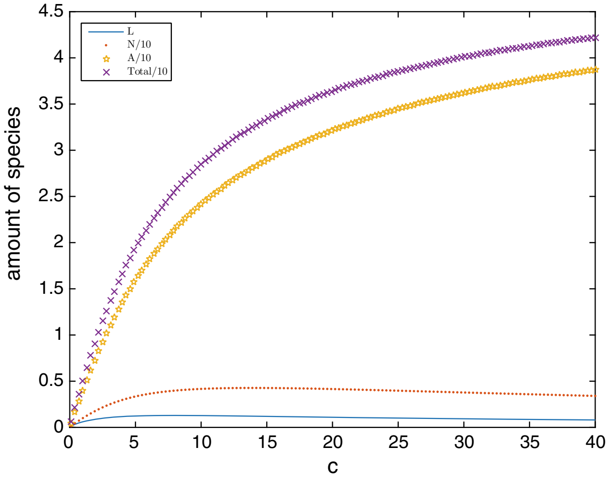

Fig. 4 The parameters are the same as those in Fig. 2. L is the average number of larvae and $N/10$, $A/10$ and $Total/10$ means one-tenth of the average number of nymphs, adults and total ticks at the disease-free state, respectively

5 Discussion

In this paper, we have established the theory of basic reproduction ratios for periodic abstract functional differential systems whose solution maps are not necessarily eventually compact. A numerical method to compute $\mathcal{R}_0$ has also been proposed in Remark 3.2, which seems to be more effective for infinite dimensional time-periodic systems. As an application, we have applied the developed theory to a time-periodic spatial model for Lyme disease. In order to obtain the global dynamics of this model system, we have also studied the principal eigenvalue problem for the associated linear systems whose evolution families are not compact.

The time-periodic Lyme disease model is a natural extension of the autonomous model presented in [41], as the parameters may be affected by the seasonality, varying temperature and humidity. Indeed, the basic biological assumptions for this model are the same as those in [6]. The first assumption is that the ticks have no effect on the mice and deer from which they take their blood meals. The second assumption is that Lyme disease does not affect its host population in any way: susceptible and infectious hosts reproduce, die and diffusion in exactly the same way. The first assumption allows the equations for the mice and deer population to decouple from the equation for the tick population in its various stages(larva, nymph, adult). The second assumption allows the equations for various combined susceptible and infectious host populations to decouple from the disease dynamics. With these two assumptions, we are able to obtain a complete description of the disease-free dynamics and the global dynamics of the nonlocal model system although the associated solution maps are not compact. However, there is some evidence that mice and deer have an immune response to ticks which may effect the fertility of the ticks feeding on them (see, e.g., [29]). This type of feedback can destroy the global stability even in the models of ticks population (see, e.g., [11,12]). It is an interesting but challenging problem to consider the spatial models without these two assumptions, and we leave it for future investigation.

For the tick population, growth is limited by density-dependent morality rates. We have assumed that the self-regulation process is mainly due to the carrying capacity of hosts and some density-dependent death terms, which was discussed in [6]. These terms may also be modified into these of other types. The assumption on the linear tick birth rate makes the associated subsystems admit the comparison principle so that the powerful theory and methods of monotone systems can be employed. When the birth rate term is nonlinear, it is a highly nontrivial problem to analyze the global dynamics (see, e.g., [39, Appendix]).

For the spatial model of Lyme disease, we have assumed that the tick birth rate is positive throughout the year. For regions with colder climate, we may assume that this rate is above a sufficiently small positive number. In the case where the birth rate vanishes somewhere, the global dynamics cannot be established by using the same arguments as in this paper, since the associated evolution families are not eventually strongly monotone. Thus, a more careful analysis is necessary for the global dynamics in this degenerate case.

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References

1. Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces. SIAM Rev. **18**, 620–709 (1976)
2. Bacaër, N., Dads, E.H.A.: Genealogy with seasonality, the basic reproduction number, and the influenza pandemic. J. Math. Biol. **62**, 741–762 (2011)
3. Bacaër, N., Dads, E.H.A.: On the biological interpretation of a definition for the parameter R_0 in periodic population models. J. Math. Biol. **65**, 601–621 (2012)
4. Bacaër, N., Guernaoui, S.: The epidemic threshold of vector-borne diseases with seasonality. J. Math. Biol. **53**, 421–436 (2006)

5. Burlando, L.: Monotonicity of spectral radius for positive operators on ordered banach spaces. Arch. Math. (Basel) **56**, 49–57 (1991)
6. Caraco, T., Glavanakov, S., Chen, G., Flaherty, J.E., Ohsumi, T.K., Szymanski, B.K.: Stage-structured infection transmission and a spatial epidemic: a model for Lyme disease. Am. Nat. **160**, 348–359 (2002)
7. Daners, D., Medina, P.K.: Abstract Evolution Equations, Periodic Problems and Applications. Pitman Research Notes in Mathematics Series, vol. 279. Longman Scientific & Technical, Harlow, UK (1992).
8. Deimling, K.: Nonlinear Functional Analysis. Springer-Verlag, Berlin, Heidelberg (1985)
9. Diekmann, O., Heesterbeek, J., Metz, J.A.: On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations. J. Math. Biol. **28**, 365–382 (1990)
10. Du, Y.: Order structure and topological methods in nonlinear partial differential equations. In: Maximum Principles and Applications, vol. 1, World Scientific, New Jersey (2006)
11. Fan, G., Lou, Y., Thieme, H., Wu, J.: Stability and persistence in ODE models for populations with many stages. Math. Biosci. Eng. **12**, 661–686 (2015)
12. Fan, G., Thieme, H.R., Zhu, H.: Delay differential systems for tick population dynamics. J. Math. Biol. **71**, 1017–1048 (2015)
13. Freedman, H., Zhao, X.-Q.: Global asymptotics in some quasimonotone reaction–diffusion systems with delays. J. Differ. Equ. **137**, 340–362 (1997)
14. Guo, Z., Wang, F.-B., Zou, X.: Threshold dynamics of an infective disease model with a fixed latent period and non-local infections. J. Math. Biol. **65**, 1387–1410 (2012)
15. Hartemink, N., Randolph, S., Davis, S., Heesterbeek, J.: The basic reproduction number for complex disease systems: defining R_0 for tick-borne infections. Am. Nat. **171**, 743–754 (2008)
16. Hartman, P.: Ordinary Differential Equations. Classics in Applied Mathematics, vol. 38. SIAM, Philadelphia (2002)
17. Hess, P.: Periodic-Parabolic Boundary Value Problems and Positivity. Longman Scientific & Technical, Harlow, UK (1991)
18. Hsu, S.-B., Wang, F.-B., Zhao, X.-Q.: Dynamics of a periodically pulsed bio-reactor model with a hydraulic storage zone. J. Dyn. Differ. Equ. **23**, 817–842 (2011)
19. Inaba, H.: On a new perspective of the basic reproduction number in heterogeneous environments. J. Math. Biol. **65**, 309–348 (2012)
20. Kato, T.: Perturbation Theory for Linear Operators. Classics in Mathematics [Reprint of the 1980 edition]. Springer-Verlag, Berlin, Heidelberg (1995)
21. Kelley, J.L.: General Topology, vol. 27. Springer-Verlag, New York, Heidelberg, Berlin (1975)
22. Lou, Y., Zhao, X.-Q.: A reaction–diffusion malaria model with incubation period in the vector population. J. Math. Biol. **62**, 543–568 (2011)
23. Lou, Y., Zhao, X.-Q.: A theoretical approach to understanding population dynamics with seasonal developmental durations. J. Nonlinear Sci. **27**, 573–603 (2017)
24. Martin, R., Smith, H.: Abstract functional-differential equations and reaction-diffusion systems. Trans. Am. Math. Soc. **321**, 1–44 (1990)
25. Matser, A., Hartemink, N., Heesterbeek, H., Galvani, A., Davis, S.: Elasticity analysis in epidemiology: an application to tick-borne infections. Ecol. Lett. **12**, 1298–1305 (2009)
26. Mckenzie, H., Jin, Y., Jacobsen, J., Lewis, M.: R_0 analysis of a spatiotemporal model for a stream population. SIAM J. Appl. Dyn. Syst. **11**, 567–596 (2012)
27. Nussbaum, R.: Eigenvectors of nonlinear positive operators and the linear Krein-Rutman theorem. In: Fadell, E., Fournier, G. (eds.), Fixed Point Theory, Lecture Notes in Math, vol. 886, pp. 309–330 (1981)
28. Ogden, N., Bigras-Poulin, M., O'callaghan, C., Barker, I., Lindsay, L., Maarouf, A., Smoyer-Tomic, K., Waltner-Toews, D., Charron, D.: A dynamic population model to investigate effects of climate on geographic range and seasonality of the tick ixodes scapularis. Int. J. Parasitol. **35**, 375–389 (2005)
29. Ogden, N., Bigras-Poulin, M., O'callaghan, C., Barker, I., Kurtenbach, K., Lindsay, L., Charron, D.: Vector seasonality, host infection dynamics and fitness of pathogens transmitted by the tick ixodes scapularis. Parasitology **134**, 209–227 (2007)
30. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, vol. 1. Functional Analysis. Academic, New York (1980)
31. Rosa, R., Pugliese, A.: Effects of tick population dynamics and host densities on the persistence of tick-borne infections. Math. Biosci. **208**, 216–240 (2007)
32. Schaefer, H.: Some spectral properties of positive linear operators. Pac. J. Math. **10**, 1009–1019 (1960)
33. Schechter, M.: Principles of Functional Analysis, vol. 2. Academic, New York (1971)
34. Thieme, H.R.: Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity. SIAM J. Appl. Math. **70**, 188–211 (2009)
35. Van den Driessche, P., Watmough, J.: Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Biosci. **180**, 29–48 (2002)

36. Wang, B.-G., Zhao, X.-Q.: Basic reproduction ratios for almost periodic compartmental epidemic models. J. Dyn. Differ. Equ. **25**, 535–562 (2013)
37. Wang, W., Zhao, X.-Q.: Threshold dynamics for compartmental epidemic models in periodic environments. J. Dyn. Differ. Equ. **20**, 699–717 (2008)
38. Wang, W., Zhao, X.-Q.: Basic reproduction numbers for reaction–diffusion epidemic models. SIAM J. Appl. Dyn. Syst. **11**, 1652–1673 (2012)
39. Wang, W., Zhao, X.-Q.: Spatial invasion threshold of Lyme disease. SIAM J. Appl. Math. **75**, 1142–1170 (2015)
40. Wang, X., Zhao, X.-Q.: A periodic vector-bias malaria model with incubation period. SIAM J. Appl. Math. **77**, 181–201 (2017)
41. Yu, X., Zhao, X.-Q.: A nonlocal spatial model for Lyme disease. J. Differ. Equ. **261**, 340–372 (2016)
42. Zhang, L., Wang, Z.-C., Zhao, X.-Q.: Threshold dynamics of a time periodic reaction–diffusion epidemic model with latent period. J. Differ. Equ. **258**, 3011–3036 (2015)
43. Zhang, Y., Zhao, X.-Q.: A reaction–diffusion Lyme disease model with seasonality. SIAM J. Appl. Math. **73**, 2077–2099 (2013)
44. Zhao, X.-Q.: Global attractivity and stability in some monotone discrete dynamical systems. Bull. Aust. Math. Soc. **53**, 305–324 (1996)
45. Zhao, X.-Q.: Basic reproduction ratios for periodic compartmental models with time delay. J. Dyn. Differ. Equ. **29**, 67–82 (2017)
46. Zhao, X.-Q.: Dynamical Systems in Population Biology, 2nd edn. Springer, New York (2017)