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REMARKS ON BASIC REPRODUCTION RATIOS FOR PERIODIC ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we extend the theory of basic reproduction ratios $\mathcal{R}_0$ in [Liang, Zhang, Zhao, JDDE], which concerns with abstract functional differential systems in a time-periodic environment. We prove the threshold dynamics, that is, the sign of $\mathcal{R}_0 - 1$ determines the dynamics of the associated linear system. We also propose a direct and efficient numerical method to calculate $\mathcal{R}_0$.

1. **Introduction.** The basic reproduction ratio $\mathcal{R}_0$ is an important number in population biology. In epidemiology, $\mathcal{R}_0$ is the expected number of secondary cases produced by a typical infective individual in a disease-free environment. Two critical problems about basic reproduction ratios are raised. The first one is how to define the basic reproduction ratio and prove the threshold dynamics. The threshold dynamics can be stated: the disease can invade if $\mathcal{R}_0 > 1$, whereas it cannot if $\mathcal{R}_0 < 1$ (see, e.g.,[7]). The other one is how to calculate $\mathcal{R}_0$ exactly and numerically.

For autonomous models, Diekmann and his collaborators [7] firstly introduced the next generation operators, and called its radius as $\mathcal{R}_0$. Van den Driessche and Watmough [17] established the theory of $\mathcal{R}_0$ for autonomous models of ordinary differential equations(ODEs) with compartment structure. They gave the definition of $\mathcal{R}_0$ and proved the threshold dynamics. For time-periodic models, Bacaër and Guernaoui [3] presented a general definition of $\mathcal{R}_0$. Wang and Zhao [19] developed the theory of $\mathcal{R}_0$ to time-periodic compartmental ODEs models. Especially, they proved that $\mathcal{R}_0$ is a threshold value for the local stability of the disease-free periodic solution. For infinite-dimensional population structure and time heterogeneity, Thieme [16] obtained the relationship between spectral bound and reproduction

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number. This work can be applied to autonomous reaction-diffusion models with or without time-delay (see, e.g., [8, 12, 14, 21]), time-periodic reaction-diffusion models without delay (see, e.g., [20, 22]), and almost periodic compartment ODE models (see, e.g., [18]). However, for time-periodic models with time-delay, it is invalid. More recently, Zhao and his collaborators [23, 11] established the theory of basic reproduction ratios for abstract time-delayed compartmental population models in a time-periodic environment. Their theory can be applied to various models with time-delay, including some non-compact models. It is interesting to prove the same results in [23, 11] under conditions that are easier to verify. There are also some intuitively clear biological interpretations of $\mathcal{R}_0$ in a time-periodic environment, see, e.g., [1, 2, 9].

With the development of the theory of basic reproduction ratios, some numerical approaches to calculate $\mathcal{R}_0$ appeared. For the autonomous models of ODEs, Van den Driessche and Watmough [17] found that the principal eigenvalue of the next generation matrix admits $\mathcal{R}_0$. For time-periodic compartmental ODEs models, Wang and Zhao [19] proposed a numerical method to compute $\mathcal{R}_0$ by solving the unique root of a critical parameter equation (see [19, Theorem 2.1]). We call it Root method. An alternative method to deal with the problem goes to Posny and Wang [15]. They transformed the problem into a matrix eigenvalue problem. For the infinite-dimensional cases, motivated by [17], Wang and Zhao presented a numerical method to approximate the principal eigenvalue of the next generation operator for the autonomous models of reaction-diffusion equations. Combine Root method with the principal eigenvalue of positive operators, a numerical method was developed by Liang, Zhang, and Zhao [11]. This method can be applied to various kinds of periodic models, including ODEs, reaction-diffusion equations, nonlocal dispersal equations with or without time delay. In this paper, we introduce a numerical method to calculate basic reproduction ratios directly by the definition of $\mathcal{R}_0$.

The first purpose of the paper is to extend the theory in [23, 11]. In [23], Zhao established the theory of basic reproduction ratios for time-delayed compartmental population models in a periodic environment. In [11], Liang, Zhang and Zhao developed the theory of abstract functional differential equations whose solution maps may be non-compact. In this paper, we prove the same conclusions under weaker conditions when the cone is solid and normal. The second purpose of the paper is to propose a new numerical approach to $\mathcal{R}_0$. The idea is to approximate the spectral radius of the next generation operator directly. Compared with Root method, our method is valid and more efficient.

The remaining part of this paper is organized as follows. In the next section, we present some preliminary results on positive operators and bounded operators. In section 3, we develop the theory of basic reproduction ratios for abstract functional differential equations. In section 4, we provide a numerical method to calculate $\mathcal{R}_0$ and make comparisons with Root method to validate our method.

2. **Preliminaries.** In this section, We show some properties for positive operators and bounded operators.

Lemma 2.1. (see [11, Lemma 2.6]) *Assume that (E, E_+) is an ordered Banach space with E_+ being normal and $\mathrm{Int}(E_+) \neq \emptyset$, which is equipped with the norm $\|\cdot\|_E$. Let L be a positive bounded linear operator. If λ is an eigenvalue of L corresponding to a strongly positive eigenvector, then $\lambda = r(L)$.*

Lemma 2.2. (see [11, Lemma 2.4]) *Assume that (E, E_+) is an ordered Banach space with E_+ being normal and $\mathrm{Int}(E_+) \neq \emptyset$. Let L be a positive bounded linear operator. Then $r(L) = \lim_{n\to+\infty} \|L^n e\|_E^{\frac{1}{n}}$, $\forall e \in \mathrm{Int}(E_+)$.*

Lemma 2.3. *Assume that F is a Banach space with the norm $\|\cdot\|_F$. Let $E \subseteq F$ be a Banach space with the norm $\|\cdot\|_E$, there exists $c_0 > 0$ such that*

$$\|u\|_F \leq c_0\|u\|_E, \ \forall u \in E.$$

Let L be a bounded linear operator from F to E. Then L is a bounded operator on E and F. Moreover, $r(L)_E = r(L)_F$, where $r(L)_E$ and $r(L)_F$ are the spectral radius of L on E and F, respectively.

Proof. Since L is a bounded operator from F to E, there exists $c_1 > 0$ such that

$$\|Lu\|_E \leq c_1\|u\|_F, \ \forall u \in F.$$

This implies that

$$\|Lu\|_E \leq c_1\|u\|_F \leq c_0c_1\|u\|_E, \ \forall u \in E,$$

and

$$\|Lu\|_F \leq c_0\|Lu\|_E \leq c_0c_1\|u\|_F, \ \forall u \in F.$$

Hence, L is a bounded operator on E and F. For any $u \in E$ with $u \neq 0$, it is easy to see that

$$\frac{\|L^n u\|_E}{\|u\|_E} \leq c_0c_1\frac{\|L^{n-1}u\|_F}{\|u\|_F}, \ n \geq 1.$$

This implies that $\|L^n\|_E \leq c_0c_1\|L^{n-1}\|_F$, $\forall n \geq 1$, where $\|L\|_E$ and $\|L\|_F$ are operator norms of L on E and F, respectively. Therefore, $r(L)_E \leq r(L)_F$, by Gelfand's formula.

Note that for any $n \geq 1$ and $u \in F$, we have $L^n u \in E$. It follows that for any $u \in F$ with $u \neq 0$,

$$\frac{\|L^n u\|_F}{\|u\|_F} \leq c_0c_1\frac{\|L^n u\|_E}{\|Lu\|_E}, \ n \geq 1.$$

This implies that $\|L^n\|_F \leq c_0c_1\|L^{n-1}\|_E$, $\forall n \geq 1$. Thus, $r(L)_F \leq r(L)_E$, and hence, the equality holds. □

3. Theoretic results on $\mathcal{R}_0$.

Let X be a Banach space with a cone X_+ being normal (see, e.g., [6, Section 19.1]) and solid($\mathrm{Int}(X_+) \neq \emptyset$), and $\tilde{X}$ be a Banach space with $\tilde{X} \overset{d}{\hookrightarrow} X$, that is, $\tilde{X}$ is continuously embedded in X and a dense subset of X. We denote their norms by $\|\cdot\|_X$ and $\|\cdot\|_{\tilde{X}}$, respectively. Let $\tau \geq 0$ be a given number, and $\mathcal{X} = C([-\tau, 0], X)$, which is equipped with the maximum norm $\|\cdot\|_{\mathcal{X}}$ and the positive cone $\mathcal{X}_+ = C([-\tau, 0], X_+)$.

Let $(V(t))_{0\leq t\leq T}$ be a family of T-periodic closed linear operators on X with the following properties:

(i) $D(V(t)) = \tilde{X}$, $\forall t \in [0, T]$.

(ii) There is some $\lambda_0 \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \lambda_0\} \subseteq \rho(-V(t))$, $\forall t \in [0, T]$ and $\|(\lambda + V(t))^{-1}\|_X \leq \frac{C}{1+|\lambda|}$, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq \lambda_0$, $\forall t \in [0, T]$.

(iii) $V(\cdot) : [0, T] \to \mathcal{L}(\tilde{X}, X)$ is Hölder continuous.

Here, the internal evolution of individuals is governed by the following linear differential system:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -V(t)u(t). \tag{1}$$

According to [5, Section 2], system (1) admits the evolution family $\{\Phi(t,s) : t \geq s\}$.

Assume that $F(\cdot) : \mathbb{R} \to \mathcal{L}(\mathcal{X}, X)$ is T-periodic, $F(t)\phi$ is continuous jointly in $(t,\phi) \in \mathbb{R} \times \mathcal{X}$ and the operator norm of $F(t)$ is uniformly bounded for all $t \in [0,T]$. For a continuous function $u : [-\tau, \varsigma) \to X$ with $\varsigma > 0$, we define $u_t \in \mathcal{X}$ by

$$u_t(\theta) = u(t+\theta),\ \forall \theta \in [-\tau, 0],$$

for any $t \in [0, \varsigma)$. We consider a linear and T-periodic functional differential system:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(t)u_t - V(t)u(t),\ t \geq 0. \tag{2}$$

It follows from [13, Corollary 4] that for any $s \in \mathbb{R}$ and $\phi \in \mathcal{X}$, system (2) has a unique solution $u(t,s,\phi)$ on $[s, +\infty)$ with $u_s = \phi$. We define the evolution family $\{U(t,s) : t \geq s\}$ on $\mathcal{X}$ associated with (2) as

$$U(t,s)\phi = u_t(s,\phi),\ \forall \phi \in \mathcal{X},\ t \geq s,\ s \in \mathbb{R},$$

where $u_t(s,\phi) = u(t+\theta, s, \phi),\ \theta \in [-\tau, 0]$.

In order to introduce the basic reproduction ratio for system (2), we assume that

(A1) Each operator $F(t) : \mathcal{X} \to X$ is positive in the sense that $F(t)\mathcal{X}_+ \subseteq X_+$, $\forall t \in \mathbb{R}$.

(A2) $\Phi(t,s)X_+ \subseteq X_+$, $\forall t \geq s$; $\Phi(t,s)\mathrm{Int}(X_+) \subseteq \mathrm{Int}(X_+)$, $\forall t \geq s$ and $\omega(\Phi) < 0$.

According to [13, Corollary 5], we have $U(t,s)\mathcal{X}_+ \subseteq \mathcal{X}_+$ for any $t \geq s$. Let $\mathbb{X}$ be an ordered Banach space of all continuous and T-periodic functions from $\mathbb{R}$ to X, which is equipped with the maximum norm and the positive cone

$$\mathbb{X}_+ = \{v \in \mathbb{X} : v(t) \geq 0 \text{ in } X,\ t \in \mathbb{R}\}.$$

Then we define two linear operators on $\mathbb{X}$ by

$$[\mathfrak{L}v](t) = \int_0^{+\infty} \Phi(t, t-s)F(t-s)v(t-s+\cdot)\mathrm{d}s,\ \forall t \in \mathbb{R},\ v \in \mathbb{X},$$

and

$$[\tilde{\mathfrak{L}}v](t) = F(t)\int_0^{+\infty} \Phi(t+\cdot, t-s+\cdot)v(t-s+\cdot)\mathrm{d}s,\ \forall t \in \mathbb{R},\ v \in \mathbb{X}.$$

Clearly,

$$[\mathfrak{L}v](t) = \int_{-\infty}^{t} \Phi(t,s)F(s)v(s+\cdot)\mathrm{d}s,\ \forall t \in \mathbb{R},\ v \in \mathbb{X}. \tag{3}$$

Let A and B be two bounded linear operators on $\mathbb{X}$ defined by

$$[Av](t) = \int_0^{+\infty} \Phi(t, t-s)v(t-s)\mathrm{d}s,\ [Bv](t) = F(t)v_t,\ \forall t \in \mathbb{R},\ v \in \mathbb{X}. \tag{4}$$

It then follows that $\mathfrak{L} = A \circ B$ and $\tilde{\mathfrak{L}} = B \circ A$, and hence $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ have the same spectral radius. Motivated by the concept of next generation operators (see, e.g., [7, 17, 3, 19, 16, 20, 23]), we define the spectral radius of $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ as the basic reproduction ratio $\mathcal{R}_0 = r(\mathfrak{L}) = r(\tilde{\mathfrak{L}})$ for periodic system (2).

The main purpose of this paper is to prove the following properties.

(S) $\mathcal{R}_0 - 1$ and $r(U(T,0)) - 1$ have the same sign.

This is equivalent to the following properties.

(i) $\mathcal{R}_0 > 1$ if and only if $r(U(T,0)) > 1$.

(ii) $\mathcal{R}_0 = 1$ if and only if $r(U(T,0)) = 1$.

(iii) $\mathcal{R}_0 < 1$ if and only if $r(U(T,0)) < 1$.

To obtain these properties, we will introduce some other assumptions. For any given $\lambda \in [0, +\infty)$, we consider the following linear and periodic system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \lambda F(t)u_t - V(t)u(t),\ t \geq 0. \tag{5}$$

Let $\{U(t, s, \lambda) : t \geq s\}$ be the evolution family on $\mathcal{X}$ determined by system (5), and write $R(\lambda) := r(\lambda\mathfrak{L}) = \lambda\mathcal{R}_0,\ \forall\lambda \in [0, +\infty)$. Here $R(\lambda)$ can also be regarded as the basic reproduction ratio of system (5) and $U(T, 0, 1) = U(T, 0)$.

(A3) The positive linear operators $U(T, 0, \lambda)$ possess the isolated principal eigenvalue with strongly positive eigenvector and finite multiplicity for any $\lambda \in [0, +\infty)$ whenever $r(U(T, 0, \lambda)) \geq 1$.

Now, we present our main result.

Theorem 3.1. *Assume that* (A1)–(A3) *hold. Then* (S) *holds.*

Remark 1. Here, assumption (A1) admits (H1) in [11]. Assumptions (A2) and (A3) are stronger than (H2) and (H4) in [11]. But no more assumptions are needed.

To prove Theorem 3.1, we need a series of lemmas.

Lemma 3.2. *Assume that* (A1) and (A2) *hold. Then for any given $t > 0$, $U(t, 0, \lambda)$ is continuous in the operator norm with respect to $\lambda \in [0, +\infty)$.*

Now, we introduce assumption (A4). We will prove statement (S) under (A1)–(A4) and then remove (A4) by showing that it can be obtained by (A1)–(A3).

(A4) The positive linear operator $\mathfrak{L}$ possesses the principal eigenvalue.

Lemma 3.3. *Assume that* (A1)–(A3) *hold. If $r(U(T, 0)) = 1$, then $\mathcal{R}_0 = 1$ and* (A4) *holds.*

Proof. Since $r(U(T, 0)) = 1$, it follows from assumption (A3) that 1 is an eigenvalue of $U(T, 0)$ corresponding to a strongly positive eigenvector $\phi^* \in \mathcal{X}_+$. This implies that (2) possesses a T-periodic solution $u^*(t)$ with $u^*(t) = [U(t, 0)\phi^*](0),\ \forall t \in [0, T]$. It is easy to see that $u^*(0) \gg 0$ in X. By the constant-variation formula, we obtain

$$u^*(t) = \Phi(t, r)u^*(r) + \int_r^t \Phi(t, s)F(s)u_s^*\mathrm{d}s,\ \forall t \geq r,\ r \in \mathbb{R}.$$

It is worth pointing out that $u^* \neq 0$. Otherwise, $u^*(t) = 0$ for any $t \in \mathbb{R}$, which is impossible since $u^*(\theta) = \phi(\theta),\ \forall\theta \in [-\tau, 0]$. Note that $u^*(t) \geq \Phi(t, 0)u^*(0),\ \forall t \geq 0$ in X. By assumption (A2), $u^*(t) \gg 0$ in X for all $t > 0$. Thus, $u^* \gg 0$ in $\mathbb{X}$. Letting $r \to -\infty$, we then have

$$u^*(t) = \int_{-\infty}^t \Phi(t, s)F(s)u_s^*\mathrm{d}s = [Lu^*](t),\ \forall t \in \mathbb{R},$$

that is, $\mathfrak{L}u^* = u^*$. Therefore, $\mathcal{R}_0 = r(L) = 1$ due to Lemma 2.1. □

By Lemmas 3.3, 3.4 and 3.5 in [11], we obtain the following results.

Lemma 3.4. *Assume that* (A1)–(A3) *hold. If $r(U(T, 0)) > 1$, then $\mathcal{R}_0 > 1$.*

Lemma 3.5. *Assume that* (A1)–(A3) *hold. If $\mathcal{R}_0 = 0$, then $r(U(T, 0, \lambda)) < 1$ for any $\lambda \in [0, +\infty)$.*

Lemma 3.6. *Assume that* (A1), (A2) *and* (A4) *hold. If $\mathcal{R}_0 = 1$, then $r(U(T, 0)) \geq 1$.*

Lemma 3.7. *Assume that* (A1)–(A4) *hold. If* $\mathcal{R}_0 > 1$, *then* $r(U(T,0)) > 1$.

Proof. It is easy to see that $R(\mathcal{R}_0^{-1}) = 1$ since $R(1) = \mathcal{R}_0 > 1$. This implies that $r(U(T,0,\mathcal{R}_0^{-1})) \geq 1$ due to Lemma 3.6. Since $r(U(T,0,\lambda))$ is nondecreasing in $\lambda \in [0,+\infty)$, we have $r(U(T,0,1)) \geq r(U(T,0,\mathcal{R}_0^{-1})) \geq 1$.

Suppose, by contradiction, that $r(U(T,0)) = 1$. Then $r(U(T,0,\lambda)) = 1$ for all $\lambda \in [\mathcal{R}_0^{-1}, 1]$. From the proof of Lemma 3.3, we see that 1 is an eigenvalue of $\lambda\mathfrak{L}$ with strongly positive eigenvector for all $\lambda \in [\mathcal{R}_0^{-1}, 1]$. That is, λ^{-1} is an eigenvalue of $\mathfrak{L}$ with strongly positive eigenvector for all $\lambda \in [\mathcal{R}_0^{-1}, 1]$. Lemma 2.1 implies that $\mathcal{R}_0^{-1} = 1$, which is a contradiction. This indicates that $r(U(T,0)) > 1$. □

Indeed, we now can prove (S) under (A1)–(A4). In the remaining parts of this section, we will remove assumption (A4).

Lemma 3.8. *Assume that* (A1)–(A3) *hold. If* $\mathcal{R}_0 > 0$, *then* (A1)–(A4) *are valid.*

Proof. We only need to verify (A4). Now we have the following claim.

Claim. *There exists some* $\lambda^* > 0$ *such that* $r(U(T,0,\lambda^*)) = 1$.

Assume that the above claim holds, by Lemma 3.3, (A4) holds.

Next, we prove the claim. The proof is motivated by Lemma 3.3 in [11]. We first define a set $S = \{\lambda \geq 0 : r(U(T,0,\lambda)) \geq 1\}$ and $\lambda^* = \inf S$. By $\mathcal{R}_0 > 0$ and Lemma 3.5, there exists $\lambda^0 > 0$ such that $r(U(T,0,\lambda^0)) \geq 1$. Thus, $S \neq \emptyset$. We have the following three facts.

Fact 1: $r(U(T,0,\lambda))$ is continuous with respect to $\lambda \in S$.

Fact 2: $r(U(T,0,\lambda))$ is nondecreasing with respect to $\lambda \in \mathbb{R}_+$.

Fact 3: $r(U(T,0,0)) = r(\Phi(T,0)) < 1$.

Fact 1 can be derived from assumption (A3), Lemma 3.2 and the perturbation theory of an isolated eigenvalue (see, e.g.,[10, Section IV.3.5]). Fact 2 is a straightforward result of [4, Theorem 1.1]. Fact 3 is derived thanks to [16, Propostion A.2].

By these facts, it is easy to see that $\lambda^* > 0$. We only need to show that $r(U(T,0,\lambda^*)) = 1$. If, on the contrary, we suppose that $r(U(T,0,\lambda^*)) > 1$. By Fact 2 and the perturbation theory of an isolated eigenvalue (see, e.g.,[10, Section IV.3.5]), there is an $\epsilon > 0$ such that $r(U(T,0,\lambda^* - \epsilon)) \geq 1$. This implies that $\lambda^* - \epsilon \in S$, which is impossible. □

Now we are in a position to prove the main result of this section.

Proof of Theorem 3.1. In the case where $\mathcal{R}_0 = 0$, we have $r(U(T,0)) < 1$. In the case where $\mathcal{R}_0 > 0$, by Lemma 3.8, (A1)–(A4) hold. It suffices to prove the equivalent statements (i)-(iii). Statements (i) and (ii) can be derived by Lemmas 3.3, 3.4, 3.6 and 3.7. Hence, statement (iii) is a straightforward consequence of (i) and (ii). □

4. **Numerical method to compute $\mathcal{R}_0$.** The aim of this section is to present a new numerical method to compute $\mathcal{R}_0$. This method can be applied to systems with and without time-delay. We use the same notations X, $\mathbb{X}$, $\mathcal{X}$, $V(t)$, $F(t)$, $\Phi(t,s)$, $U(t,s)$, A, B as those in section 3.

Define a closed operator C on $\mathbb{X}$:

$$[Cu](t) = -V(t)u(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t),\ t \in \mathbb{R},\ u \in \mathbb{X}.$$

By section 5 in Thieme's work[16] and (A2), it follows that $s(C) < 0$. We will discuss more properties about the operator C. For any $v \in \mathbb{X}$, let $u = -C^{-1}v$. Thus, u satisfies the following equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -V(t)u(t) + v(s),\ t \in \mathbb{R},\ v \in \mathbb{X}. \tag{6}$$

The system (6) admits a unique periodic solution which has the following form:

$$u(t) = \Phi(t,0)\varphi + \int_0^t \Phi(t,s)v(s)\mathrm{d}s,\ \forall t \in [0,T],\ v \in \mathbb{X}, \tag{7}$$

where

$$\varphi := u(0) = (I - \Phi(T,0))^{-1} \int_0^T \Phi(T,s)v(s)\mathrm{d}s. \tag{8}$$

Remark 2. Here, u can also be written as:

$$u(t) = \int_{-\infty}^t \Phi(t,s)v(s)\mathrm{d}s,\ \forall t \in [0,T],\ v \in \mathbb{X}.$$

We conclude that $A = -C^{-1}$ due to (4).

Now, we present our numerical algorithm to approximate $r(\mathfrak{L}) = r(AB)$.

Step 1 Let $w^1(t) = 1$, $t \in [0,T]$ be the initial value of the interation.

Step 2 For any $i \geq 1$,

(1) Compute $v^i = Bw^i$.

(2) Obtain $u^i = Av^i$ by the following two steps:

- Compute $u^i(0)$ by formula (8).
- For any $t \in [0,T]$, obtain $u^i(t)$ by formula (7).

(3) Let $w^{i+1}(t) = \frac{u^i(t)}{\|u^i\|}$, $t \in [0,T]$ and $R^i = \|u^i\|$.

The iteration ends until $(\Pi_{i\geq 1}R^i)^{\frac{1}{i}}$ converges.

By similar arguments to those in [11, Theorem 3.8], we have the following numerical method to compute $\mathcal{R}_0$.

Theorem 4.1. *Assume that* (A1)–(A3) *hold. If* $\mathcal{R}_0 > 0$, *then* $\lambda = \mathcal{R}_0^{-1}$ *is the unique solution of* $r(U(T,0,\lambda)) = 1$.

Remark 3. The key of this method is to solve the parameter equation $r(U(T,0,\lambda)) = 1$, so we call it Root method. The method proposed in this paper can be called as Direct method since it approximates the spectral radius of $\mathfrak{L}$ directly.

Next, we validate the convergence of Direct method by an ODE.

Example 1. Convergence

Firstly, the convergence property is investigated. Consider the following ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -V(t)u(t) + F(t)u(t),$$

where $V(t) = \beta(1 + c\cos(\frac{2\pi t}{T}))$, $F(t) = \gamma(1 + c\cos(\frac{2\pi t}{T}))$.

Parameters used in this test are $\beta = 0.2$, $\gamma = 0.35$, $T = 12$. The interval $[0,T]$ is divided into m partitions. The classical SIR model admits system in this form. It is not hard to obtain that $\mathcal{R}_0 = \frac{\gamma}{\beta} = 1.75$ for all $c \in [0,1]$. We use the single degree numerical method to simulate the solution of (1) and calculate (8) and (7). Mean values and relative errors are computed by an arithmetic sequence c from 0 to 0.95 with 100 partitions. Results under different partitions are shown in Table 1.

TABLE 1. Mean values and relative errors under different partitions.

m	Mean numerical value	Relative error(%)
500	1.7599	0.5681
1000	1.7550	0.2838
2000	1.7525	0.1419
8000	1.7506	0.0351

Next, we will test the efficiency of this method when compared with Root method. We always divide $[0, T]$ into m partitions. Consider the model without time-delay:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -V(t)u(t) + F(t)u(t).$$

For convenience, we take the system with two species. For any $t \in \mathbb{R}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in X$, let

$$V(t)v = \begin{pmatrix} V_{11}(t) & 0 \\ 0 & V_{22}(t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad F(t)v = \begin{pmatrix} 0 & F_{12}(t) \\ F_{21}(t) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Many models can be described in this form, such as Dengue fever model, Malaria model, and Lyme disease model.

Example 2. ODEs

For any $t \in \mathbb{R}$, we let

$$V_{11}(t) = \beta_1(1 + c\cos(\frac{2\pi t}{T})), \quad V_{22}(t) = \beta_2(1 + c\cos(\frac{2\pi t}{T})),$$

$$F_{12}(t) = \gamma_1(1 + c\cos(\frac{2\pi t}{T})), \quad F_{21}(t) = \gamma_2(1 + c\sin(\frac{2\pi t}{T})),$$

and c is a parameter. Here, we choose $\beta_1 = 0.2$, $\beta_2 = 0.3$, $\gamma_1 = 0.35$, $\gamma_2 = 0.5$, $T = 12$, $m = 8000$. The sequence c varies from 0 to 0.95 with 100 partitions. From Fig. 1, we see that both methods achieve similar results. However, as to the efficiency, it takes $409s$ by Root method, while it takes only $131s$ by Direct method.

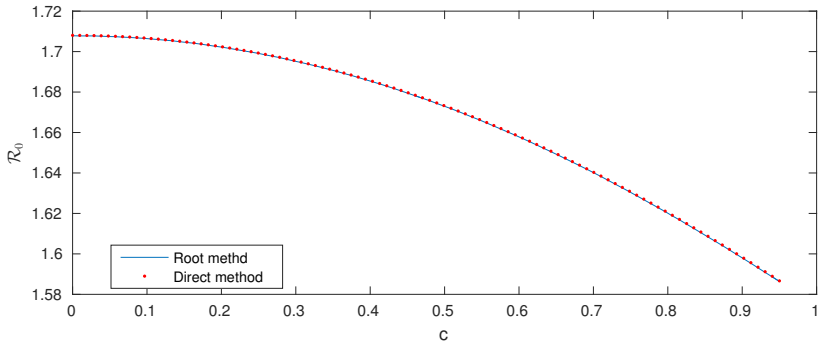

FIGURE 1. Comparison of results using two methods by ODEs.

Example 3. Reaction-Diffusion systems

We consider the following system.

$$\begin{aligned}
&\frac{\partial v_1}{\partial t}(x,t) = D_1\Delta v_1(x,t) - m_1(x,t)v_1(x,t) + r_1(x,t)v_2(x,t), \quad x\in(0,1),\ t>0,\\
&\frac{\partial v_2}{\partial t}(x,t) = D_2\Delta v_2(x,t) - m_2(x,t)v_2(x,t) + r_2(x,t)v_1(x,t), \quad x\in(0,1),\ t>0,\\
&\frac{\partial v_1}{\partial x}(0,t) = 0,\ \frac{\partial v_1}{\partial x}(1,t) = 0,\ \frac{\partial v_2}{\partial x}(0,t) = 0,\ \frac{\partial v_2}{\partial x}(1,t) = 0, \qquad t>0.
\end{aligned} \tag{9}$$

Here,

$$\begin{aligned}
m_1(x,t) &= \beta_1(1+c\cos(2\pi x))(1+c\cos(\frac{2\pi t}{T})), \quad x\in[0,1],\ t\in\mathbb{R},\\
m_2(x,t) &= \beta_2(1+c\sin(2\pi x))(1+c\cos(\frac{2\pi t}{T})), \quad x\in[0,1],\ t\in\mathbb{R},\\
r_1(x,t) &= \gamma_1(1+c\cos(\frac{2\pi t}{T})), \qquad x\in[0,1],\ t\in\mathbb{R},\\
r_2(x,t) &= \gamma_2(1+c\sin(\frac{2\pi t}{T})), \qquad x\in[0,1],\ t\in\mathbb{R},
\end{aligned} \tag{10}$$

where c is a parameter. In this example, $X = C([0,1],\mathbb{R}^2)$. Let $Y = C([0,1])$. Define $V_{11}(t)$, $V_{22}(t)$ by

$$\begin{aligned}
[V_{11}(t)w](x) &:= -D_1\Delta w(x) + m_1(x,t)w(x),\ x\in(0,1),\ t\in\mathbb{R},\ w\in Y,\\
[V_{22}(t)w](x) &:= -D_2\Delta w(x) + m_2(x,t)w(x),\ x\in(0,1),\ t\in\mathbb{R},\ w\in Y,
\end{aligned} \tag{11}$$

with Neumann boundary condition. Let

$$[F_{12}(t)w](x) := r_1(x,t)w(x),\ x\in(0,1),\ t\in\mathbb{R},\ w\in Y,$$

$$[F_{21}(t)w](x) := r_2(x,t)w(x),\ x\in(0,1),\ t\in\mathbb{R},\ w\in Y.$$

The interval $[0,1]$ is divided into l partitions. We choose $D_1 = 0.01$, $D_2 = 0.02$, $\beta_1 = 0.2$, $\beta_2 = 0.3$, $\gamma_1 = 0.35$, $\gamma_2 = 0.5$, $T = 12$, $m = 8000$ and $l = 80$. The sequence c increases from 0 to 0.95 with 100 partitions. From Fig. 2, we can see that both methods lead to similar numerical results. The running time of Root method is $25038s$. However, it takes only $12479s$ by Direct method.

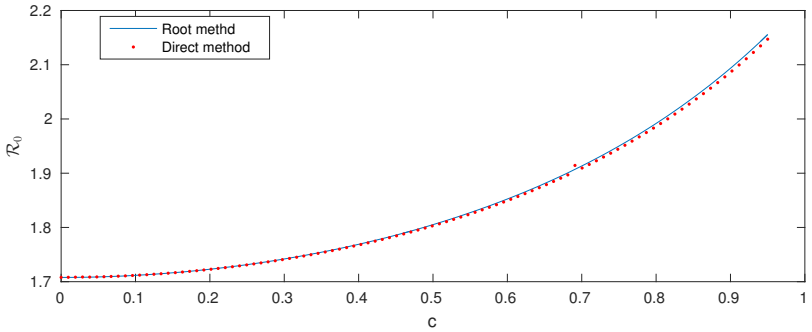

FIGURE 2. Comparison of results using two methods by Reaction-Diffusion systems.

Remark 4. In Example 3, we calculate $\mathcal{R}_0$ numerically for Reaction-Diffusion systems with Neumann boundary condition. According to Lemma 2.3, for Reaction-Diffusion systems with Dirichlet boundary condition, $\mathcal{R}_0$ can also be calculated by Root method and Direct method.

Consider the models with time-delay

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -V(t)u(t) + F(t)u_t.$$

For convenience, we take the system with two variables. Let

$$V(t)v = \begin{pmatrix} V_{11}(t) & 0 \\ 0 & V_{22}(t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ F(t)\psi = \begin{pmatrix} 0 & F_{12}(t) \\ F_{21}(t) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad t \in \mathbb{R},$$

where $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in X$, and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{X}$.

Example 4. DDEs

In this example, $X = \mathbb{R}^2$. Let $Y = \mathbb{R}$ and $\mathcal{Y} = C([-\tau, 0])$. Define

$$V_{11}(t)w = \beta_1 w, \ V_{22}(t)w = \beta_2 w, \ t \in \mathbb{R}, \ w \in Y,$$

and

$$F_{12}(t)\phi = \gamma_1(1 + 0.8\cos(\frac{2\pi t}{T}))\phi(0), \ t \in \mathbb{R}, \ \phi \in \mathcal{Y},$$
$$F_{21}(t)\phi = \gamma_2(1 + 0.8\sin(\frac{2\pi t}{T}))\phi(-\tau), \ t \in \mathbb{R}, \ \phi \in \mathcal{Y}.$$

Let $\tau = c$ be a parameter. We choose $\beta_1 = 0.2$, $\beta_2 = 0.3$, $\gamma_1 = 0.35$, $\gamma_2 = 0.5$, $T = 12$, $m = 8000$. The sequence c varies from 0 to 0.95 with 60 partitions. Similar numerical results are derived from both Direct and Root methods, as depicted in Fig. 3. In addition, time cost of Direct method is less than Root method: It takes $1062s$ by Root method, while it takes only $78s$ by Direct method. This indicates that Direct method has superior efficiency compared with Root method.

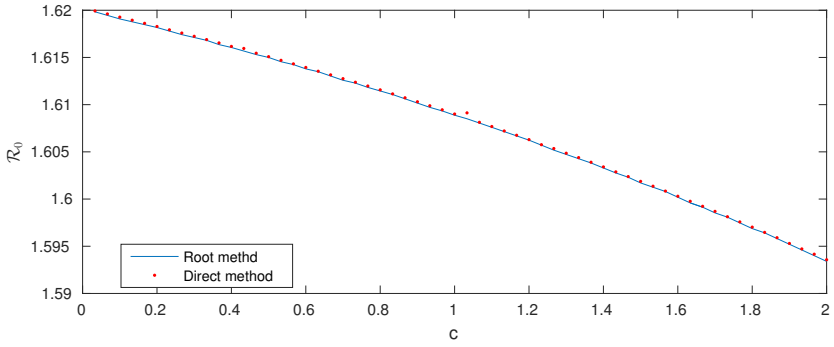

FIGURE 3. Comparison of results using two methods by DDEs.

Example 5. Reaction-Diffusion system with time-delay

Finally, we consider the following system:

$$\frac{\partial v_1}{\partial t}(x,t) = D_1\Delta v_1(x,t) - m_1(x,t)v_1(x,t) + r_1(x,t)v_2(x,t), \qquad x \in (0,1), \ t > 0,$$
$$\frac{\partial v_2}{\partial t}(x,t) = D_2\Delta v_2(x,t) - m_2(x,t)v_2(x,t) + r_2(x,t)v_1(x,t-\tau), \quad x \in (0,1), \ t > 0,$$

$$\frac{\partial v_1}{\partial x}(0,t)=0,\ \frac{\partial v_1}{\partial x}(1,t)=0,\ \frac{\partial v_2}{\partial x}(0,t)=0,\ \frac{\partial v_2}{\partial x}(1,t)=0,\quad t>0, \tag{12}$$

where m_1, m_2, r_1, r_2 are the same as those in (10).

In this example, $X=C([0,1],\mathbb{R}^2)$. Let $Y=C([0,1])$ and $\mathcal{Y}=C([-\tau,0],Y)$. The operators $V_{11}(t)$, $V_{22}(t)$ can also be defined by (11). Let

$$[F_{12}(t)\phi](x):=r_1(x,t)[\phi(0)](x),\ x\in(0,1),\ t\in\mathbb{R},\ \phi\in\mathcal{Y},$$

$$[F_{21}(t)\phi](x):=r_2(x,t)[\phi(-\tau)](x),\ x\in(0,1),\ t\in\mathbb{R},\ \phi\in\mathcal{Y}.$$

The interval $[0,1]$ is divided into l partitions. Other parameters are $D_1=0.01$, $D_2=0.02$, $\beta_1=0.2$, $\beta_2=0.3$, $\gamma_1=0.35$, $\gamma_2=0.5$, $T=12$, $m=8000$, $l=80$ and $\tau=0.6$. The sequence c increases from 0 to 0.95 with 60 partitions. Numerical results of $\mathcal{R}_0$ using Direct method and Root method are depicted in Fig. 4. Similar to former examples, the results obtained using both methods are very close to each other. Compared with the running time of Root method, whose time cost is $44567s$, Direct method only takes $14612s$. This indicates a significant improvement in the efficiency.

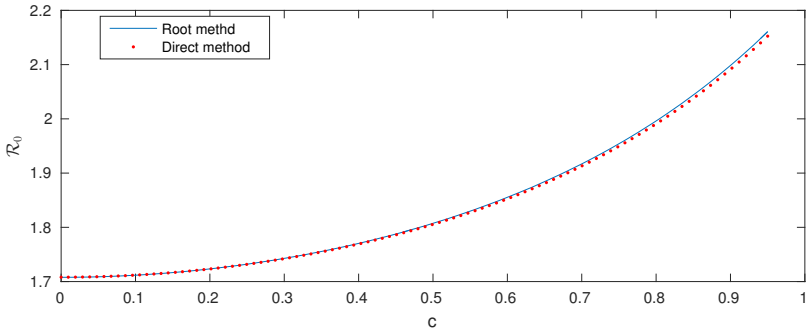

FIGURE 4. Comparison of results using two methods by Reaction-Diffusion systems with time-delay.

5. **Discussion.** The purpose of this paper is to extend the theory of basic reproduction ratios established in [11]. We prove the threshold dynamics under conditions that are easier to implement. Assumption (H3) (in [11]) is necessary to prove the threshold dynamics, and it admits (A4) in this paper. However, it is not easy to verify this assumption for a noncompact system. The reason is that it is difficult to analyze the spectrum of $\mathfrak{L}$ when it is non-compact. In this paper, we prove (A4) under assumptions (A1)–(A3) and then remove this assumption.

We also propose a new numerical method, which approximates the spectral radius of the next generation operator directly. So it can be called as Direct method. By comparison, we find it is much faster than Root method. Some fundamental problems in numerical analysis raise naturally. For example, how to estimate the error for these two methods? And also, under what conditions can stability be obtained?

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