Accurate and Warm-Startable Linear Cutting-Plane Relaxations for ACOPF*

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Abstract. We present a linear cutting-plane relaxation approach that rapidly proves tight lower bounds for the Alternating Current Optimal Power Flow Problem (ACOPF). Our method leverages outer-envelope linear cuts for well-known second-order cone relaxations for ACOPF along with modern cut management techniques. These techniques prove effective on a broad family of ACOPF instances, including the largest ones publicly available, quickly and robustly yielding sharp bounds. Our primary focus concerns the (frequent) case where an ACOPF instance is considered following a small or moderate change in problem data, e.g., load changes and generator or branch shut-offs. We provide significant computational evidence that the cuts computed on the prior instance provide an effective warm-start for our algorithm. Please refer to [1] for the full version of this paper.

Keywords: ACOPF · Cutting planes · Second-order cones

1 Introduction

The Alternating-Current Optimal Power Flow (ACOPF) problem [2] is a well-known challenging computational task. It is strongly NP-hard, nonlinear, non-convex and with feasible region that may be disconnected; see [3–7]. Some interior point methods are empirically successful at computing excellent solutions. Strong lower bounds follow from (challenging) SOCP relaxations [8–10, 1, 11]. Techniques such as spatial-branch-and-bound methods applied to QCQP formulations for ACOPF tend to yield poor performance.

Here we present a fast cutting-plane method to obtain tight relaxations by approximating the SOC relaxations. As we will show herein, our approach is fast, robust, accurate, and warm-startable; the latter feature is important in the practice because, quite frequently, problems need to be resolved after limited data changes. This is our central focus.

* This work was supported by an ARPA-E GO Competition Grant. We would like to thank Erling Andersen, Bob Fourer, Ed Klotz and Richard Waltz.

Our contributions (1) We describe very tight linearly constrained relaxations for ACOPF. The relaxations can be constructed and solved robustly and quickly via a cutting-plane algorithm that relies on proper cut management. On medium to (very) large instances our algorithm is competitive or better, from scratch, with what was previously possible using nonlinear relaxations, both in terms of bound quality and solution speed. (2) We provide a theoretical justification for the tightness of the SOC relaxation for ACOPF as well as for the use of our linear relaxations. (3) As a main contribution we demonstrate, through extensive numerical testing, that the warm-start feature for our cutting-plane algorithm yields tight bounds far faster than otherwise possible. It is worth noting that this capability stands in contrast to what is possible using nonlinear (convex) solvers (cf. Table 1).

2 ACOPF Problem Formulation and Relaxations

Let $\mathcal{B}$ denote the set of buses, $\mathcal{E}$ the set of branches and $\mathcal{G}$ the set of generators; for each bus $k \in \mathcal{B}$, $\mathcal{G}_k \subseteq \mathcal{G}$ is the generators at k. Each bus k has fixed active load $P_k^d \geq 0$ and reactive load Q_k^d, and lower $V_k^{\min} \geq 0$ and upper $V_k^{\max} \geq 0$ voltage limits. For each branch $\{k,m\}$ we are given a thermal limit $0 \leq U_{km} \leq +\infty$, and maximum angle-difference $|\varDelta_{km}| \leq \pi$. The so-called *polar representation* of ACOPF uses variables $|V_k|$, θ_k at each bus k, and P^g and Q^g for every generator g, and is given by the following optimization problem:

$$[\text{ACOPF}]: \qquad \min \qquad \sum_{k \in \mathcal{G}} F_k(P_k^g) \tag{1a}$$

subject to:

$$\forall\, k \in \mathcal{B}: \quad (V_k^{\min})^2 \leq v_k^{(2)} \leq (V_k^{\max})^2, \tag{1b}$$

$$\sum_{\{k,m\} \in \delta(k)} P_{km} = \sum_{\ell \in \mathcal{G}_k} P_\ell^g - P_k^d, \qquad \sum_{\{k,m\} \in \delta(k)} Q_{km} = \sum_{\ell \in \mathcal{G}_k} Q_\ell^g - Q_k^d, \tag{1c}$$

$$v_k^{(2)} = |V_k|^2, \tag{1d}$$

$$\forall \{k,m\} \in \mathcal{E}: \quad \theta_{km} = \theta_k - \theta_m,$$

$$P_{km} = G_{kk} v_k^{(2)} + G_{km} c_{km} + B_{km} s_{km}, \tag{1e}$$

$$P_{mk} = G_{mm} v_m^{(2)} + G_{mk} c_{km} - B_{mk} s_{km}, \tag{1f}$$

$$Q_{km} = -B_{kk} v_k^{(2)} + B_{km} c_{km} - G_{km} s_{km}, \tag{1g}$$

$$Q_{mk} = -B_{mm} v_m^{(2)} + B_{mk} c_{km} + G_{mk} s_{km}, \tag{1h}$$

$$c_{km} = |V_k||V_m| \cos(\theta_{km}),\ s_{km} = |V_k||V_m| \sin(\theta_{km}), \tag{1i}$$

$$|\theta_{km}| \leq \bar{\varDelta}_{km}, \quad \max\left\{P_{km}^2 + Q_{km}^2\ ,\ P_{mk}^2 + Q_{mk}^2\right\} \leq U_{km}^2, \tag{1j}$$

$$\forall k \in \mathcal{G}: \quad P_k^{\min} \leq P_k^g \leq P_k^{\max},\ Q_k^{\min} \leq Q_k^g \leq Q_k^{\max}. \tag{1k}$$

Above, $G_{kk}, B_{kk}, G_{km}, B_{km}, G_{mk}, B_{mk}, G_{mm}$ and B_{mm} are physical parameters of branch $\{k,m\}$ and are used in (1e)-(1h); (1j) amount to flow capacity constraints; (1b) and (1k) impose operational limits; and (1c) impose active and reactive power balance at each bus k. For each $k \in \mathcal{G}$, the functions $F_k : \mathbb{R} \to \mathbb{R}$ in (1a) are usually convex and piecewise-linear or quadratic. Often constraint $|\theta_{km}| \leq \bar{\Delta}_{km}$ in (1j) is either not present or concerns angle limits $\bar{\Delta}_{km}$ that are *small* (smaller than $\pi/2$); hence we will not consider it in our relaxations.

Please refer to the surveys [12] and [13] for equivalent ACOPF formulations.

2.1 Brief review on prior work on convex relaxations

The simplest relaxations use an equivalent rectangular formulation of the ACOPF problem yielding a QCQP and rely on the (linear) McCormick [14] reformulation; this relaxation is known to provide very weak bounds.

The SOC relaxation in [8], known as the Jabr relaxation (see next subsection), is very effective as a lower bounding technique – though in the case of large ACOPF instances, the SOCs prove challenging for the best solvers. A wide variety of techniques have been proposed to strengthen the Jabr relaxation; see [9, 10]. An SDP relaxation based on the *Shor relaxation* [15] for non-convex QCQPs is presented in [16]. This formulation is at least as tight as the Jabr relaxation at the expense of even higher computational cost [11]. In [11] a reformulation of the rank-one constraints in the semidefinite programming formulation for ACOPF is proposed. Overall, experiments for all of these nonlinear relaxations have been limited to small and medium-sized cases.

Next we review linear relaxations for ACOPF. [17, 18] introduces the so-called active-power loss linear inequalities which state that on any branch the active power loss is nonnegative, yielding good lower bounds. In a similar same vein, [19] propose a relaxation comprised of the active-power loss and additional sparse linear inequalities that lower bound net reactive power losses in appropriate cases. See [20] for a relaxation which enforces a (valid) linear relationship between active and reactive power losses. A linear ϵ-approximation for ACOPF, based on the Jabr relaxation, is used in [21]. See also [17] for mixed integer linear ϵ-approximation for ACOPF.

Moreover, [22, 23] propose successive linear programming (SLP) algorithms for finding locally optimal AC solutions. One of the algorithms in [23] is an SLP method focusing on the Jabr relaxation, and thus yielding a linear relaxation for ACOPF. We remark that the well-known Direct Current Optimal Power Flow (DCOPF) may prove a poor approximation to ACOPF (see [24]).

We refer the reader to the surveys [12, 13, 25] for additional material on convex relaxations for ACOPF.

2.2 Two Convex Relaxations for ACOPF

The Jabr SOCP A well-known convex relaxation of ACOPF is the *Jabr relaxation* [8]. A simple derivation is as follows: For any line $\{k, m\}$, squaring and adding the equations (1i) yields $c_{km}^2 + s_{km}^2 = v_k^{(2)} v_m^{(2)}$, which in [8] is relaxed into the convex inequality

$$c_{km}^2 + s_{km}^2 \leq v_k^{(2)} v_m^{(2)}. \tag{2}$$

This is a rotated-cone inequality hence it can be represented as a second-order cone constraint. Therefore, the Jabr relaxation can be obtained from formulation (1) by (i) adding the convex inequalities (2), and dropping (1d) and (1i).

The i2 SOCP Recall that complex power injected into branch $\{k, m\} \in \mathcal{E}$ at bus $k \in \mathcal{B}$ is defined by $S_{km} := V_k I^*_{km}$, hence, $|S_{km}|^2 = |V_k|^2 |I_{km}|^2$ holds. Moreover, since complex power can be decomposed into active and reactive power as $S_{km} = P_{km} + jQ_{km}$, letting $i^{(2)}_{km} := |I_{km}|^2$, we have $P^2_{km} + Q^2_{km} = v^{(2)}_k i^{(2)}_{km}$. By relaxing the former equality we obtain the rotated-cone inequality [10, 26]

$$P^2_{km} + Q^2_{km} \leq v^{(2)}_k i^{(2)}_{km}. \tag{3}$$

Since the variable $i^{(2)}_{km}$ can be defined linearly in terms of $v^{(2)}_k$, $v^{(2)}_m$, c_{km}, and s_{km} (c.f. [31]), i.e,

$$i^2_{km} = \alpha_{km} v^{(2)}_k + \beta_{km} v^{(2)}_m + \gamma_{km} c_{km} + \zeta_{km} s_{km} \tag{4}$$

where $\alpha_{km}, \beta_{km}, \gamma_{km}$ and ζ_{km} are constants dependent on branch parameters, we obtain an alternative SOC relaxation. This formulation, which we call the *i2 relaxation*, differs from the Jabr relaxation in the linear definition of $i^{(2)}_{km}$ (4) and the rotated-cone inequalities (3).

Proposition 1. *The Jabr and the i2 relaxations are equivalent if $i^{(2)}$ is not upper bounded, and otherwise the i2 relaxation can be stronger.*

Proof. Sufficiency was proven in [10]. Also see [27, 28]. An instance where the i2 relaxation is stronger than Jabr is 1354pegase, see [1].

Our computational experiments corroborate this fact; we have found that linear outer approximation cuts for the rotated-cone inequalities (2) and (3) have significantly different impact in lower bounding ACOPF (c.f. 3.2).

3 Our work

In this paper we develop an algorithm that iteratively approximates the i2 relaxation of ACOPF by adding cuts (valid linear inequalities) that are be outer-envelope approximations to (3), (2) and (1j).

We note that direct solution of the Jabr and i2 relaxations of ACOPF, for large instances, is computationally prohibitive and often results in non-convergence (c.f. Table 1). Empirical evidence further shows that outer approximation of the rotated-cone inequalities (in either case) requires a large number of cuts in order to achieve a tight relaxation value. Moreover, employing such large families of cuts yields a relaxation that, while linearly constrained, still proves challenging – both from the perspective of running time and numerical tractability.

However, as we show, adequate cut management proves successful, yielding a procedure that is (a) rapid, (b) numerically stable, and (c) constitutes a very tight relaxation (c.f. Table 1). The critical ingredients in this procedure are: (1) quick cut separation; (2) appropriate violated cut selection; and (3) effective dynamic cut management, including rejection of *nearly-parallel* cuts and removal of *expired* cuts, i.e., previously added cuts that are slack (cf. 3.2).

Our procedure possesses efficient warm-starting capabilities – this is a central goal of our work. Cuts computed for a certain instance can be reused in runs of *related* instances, reducing computational effort. In 3.3 we justify this feature and Table 1 presents numerical evidence of its performance relative to solving SOCPs 'from scratch' (Please see [1] for more computational results). Adequate cut management is critical towards this feature for large ACOPF instances.

3.1 Cuts

We present a theoretical justification for using an outer approximation cutting-plane framework on the Jabr and i2 relaxations, as well as computationally efficient cut separation procedures. We also present brief intuition on the complementarity of the Jabr and i2 outer-envelope cuts. See [1] for proofs of propositions 2, 3, 4.

Losses and Outer-Envelope Cuts For transmission lines with $G_{kk} > 0 > G_{km} = G_{mk} \geq -G_{kk}$ and $B_{km} = B_{mk}$, in particular lines with no transformer nor shunt elements, active-power loss inequalities are implied by the Jabr inequalities, and also by the definition of the $i^{(2)}$ variable. If negative losses are present total generation is smaller than total loads plus positive losses – negative losses amount to a source of free generation thus yielding a lower objective value than feasible. This follows from a flow decomposition argument showing that every unit of demand and (positive) loss is matched by a corresponding unit of generation *or* negative loss. See [30] and [31] for numerical examples showing the impact of negative losses – we remind the reader that in standard ACOPF the objective function accounts for generation. We begin with two simple technical observations.

First, a rotated cone inequality $x^2 + y^2 \leq wz$ is equivalent to $(2x)^2 + (2y)^2 \leq (w + z)^2 - (w - z)^2$. Hence,

$$x^2 + y^2 \leq wz \iff ||(2x, 2y, w - z)^\top||_2 \leq w + z. \tag{5}$$

Next, let $\lambda \in \mathbb{R}^3$ satisfy $||\lambda||_2 = 1$. Then, by (5),

$$\begin{aligned}(2x, 2y, w - z)\lambda &\leq ||(2x, 2y, w - z)^\top||_2 \, ||\lambda||_2 \\ &\leq w + z.\end{aligned} \tag{6}$$

Inequality (6) provides a generic recipe to obtain outer-envelope inequalities for a rotated cone. As a result:

Proposition 2. *For a transmission line $\{k,m\} \in \mathcal{E}$ with $G_{kk} > 0 > G_{km} = G_{mk} \geq -G_{kk}$ and $B_{km} = B_{mk}$, the Jabr inequality $c_{km}^2 + s_{km}^2 \leq v_k^{(2)} v_m^{(2)}$ implies, as an outer envelope approximation inequality, that $P_{km} + P_{mk} \geq 0$.*

See [31] for examples where removing a *single* Jabr inequality from the SOC formulation results in a strictly weaker relaxation – this arises because on that branch we will have a negative loss, which acts as cost-free generation. See the discussion above on flow decompositions in [30]. Moreover, it is known that for transmission lines with no transformers nor shunt elements the definition of the variable $i^{(2)}$ implies the active-power loss inequalities [27, 28].

Two Simple Cut Procedures The following two propositions give us inexpensive computational procedures for separating rotated-cone inequalities and (1j)

$$c_{km}^2 + s_{km}^2 \leq v_k^{(2)} v_m^{(2)}, \qquad P_{km}^2 + Q_{km}^2 \leq v_k^{(2)} i_{km}^{(2)}, \tag{7}$$

$$P_{km}^2 + Q_{km}^2 \leq U_{km}^2, \qquad P_{mk}^2 + Q_{mk}^2 \leq U_{km}^2. \tag{8}$$

Proposition 3. *Consider the second-order cone $C := \{(x,s) \in \mathbb{R}^n \times \mathbb{R}_+ : ||x||_2 \leq s\}$. Suppose $(\overline{x},\overline{s}) \notin C$ with $\overline{s} > 0$. Then the cut for C which achieves the maximum violation by $(\overline{x},\overline{s})$ is given by $\overline{x}^\top x \leq s||\overline{x}||$.*

Proposition 4. *Consider the Euclidean ball in $\mathbb{R}^2$ of radius r, $S_r := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$, and let $(\overline{x},\overline{y}) \notin S_r$. Then the cut that attains the maximum violation by $(\overline{x},\overline{y})$ is given by $(\overline{x})^\top x + (\overline{y})^\top y \leq r||(\overline{x},\overline{y})^\top||$.*

On the Complementarity of the Jabr and i2 cuts If $\{k,m\}$ is a transmission line with no transformer nor shunt elements, then $i_{km}^{(2)} = (1/(r_{km}^2 + x_{km}^2))(v_k^{(2)} + v_m^{(2)} - 2c_{km})$ where r_{km} and x_{km} denote line's $\{k,m\}$ resistance and reactance (c.f. [31]). Suppose that $i_{km}^{(2)}$ is upper-bounded by some constant H_{km} and that the line $\{k,m\}$ has a small resistance, e.g., on the order of 10^{-5} (p.u.). Since x_{km} is usually an order of magnitude larger than r_{km}, the coefficient $(r_{km}^2 + x_{km}^2)H_{km}$ can be fairly small, hence we have

$$v_k^{(2)} + v_m^{(2)} - 2c_{km} \leq (r_{km}^2 + x_{km}^2)H_{km} \approx 0 \tag{9}$$

Since $v_k^{(2)} + v_m^{(2)} - 2c_{km} \geq 0$ is a Jabr outer-envelope cut (c.f. proof Proposition 2 in [1]), (9) is enforcing our solutions to be on the surface of the rotated-cone $c_{km}^2 + s_{km}^2 \leq v_k^{(2)} v_m^{(2)}$.

3.2 Basic Algorithm and Cut Management

In what follows we describe our cutting-plane algorithm. We define the linearly constrained model M_0 as model (1) with only linear constraints, i.e., without (1d), (1i), and (1j). In every round of our procedure, linear constraints will

Algorithm 1 Cutting-Plane Algorithm

procedure CUTPLANE
    Initialize $r \leftarrow 0$, $M \leftarrow M_0$, $z_0 \leftarrow +\infty$
    while $t < T$ and $r < T_{ftol}$ **do**
        $z \leftarrow \min M$ and $\bar{x} \leftarrow \operatorname{argmin} M$
        Check for violated inequalities by solution $\overline{x}$
        Sort inequalities by violation
        Compute cuts for the most violated inequalities
        Add cuts if they are not ϵ-parallel to cuts in M
        Drop cuts of age $\geq T_{age}$ whose slack is $\geq \epsilon_j$
        if $z - z_0 < z_0 \cdot \epsilon_{ftol}$ **then**
            $r \leftarrow r + 1$
        else
            $r \leftarrow 0$
        end if
        $z_0 \leftarrow z$
    end while
end procedure

be added to and removed from M_0. The exact manner in how this will be done is described below. We will denote by M our dynamic relaxation at some iteration of our algorithm.

Given a feasible solution $\bar{x}$ to M, and letting $f_{km}(x) \leq 0$ be some valid convex inequality (7) or (8), our measure of *cut-quality* is the amount $\max\{f_{km}(\overline{x}), 0\}$ by which the solution $\overline{x}$ violates the valid convex inequality. Let $\epsilon > 0$, then for each type $\tau \in \{\text{Jabr}, \text{i2}, \text{limit}\}$ of inequality, i.e., Jabr and i2 rotated-cones and thermal limits, we sort the branches from highest to lowest violation strictly greater than ϵ, and pick as τ-candidates branches, for which cuts will be added to M, the top p_τ percentage of the most violated branches.

For each list of τ-candidates, we compute cuts for the corresponding branches using the procedures in 3.1. Candidate cuts will be rejected if they are *too parallel* to incumbent cuts in M [32, 33]. Given $\epsilon_{par} > 0$, we say that two inequalities $c^t x \leq 0$ and $d^t x \leq 0$ are ϵ_{par}*-parallel* if the cosine of the angle between c and d is strictly more that $1 - \epsilon_{par}$.

We describe a heuristic for *cleaning-up* our formulation. For each added cut, we keep track of its current *cut-age*, i.e., the difference between the current round and the round in which it was added. Then, in every iteration, if a cut $c^\top x \leq d$ has age greater or equal than a fixed parameter T_{age}, and it is ϵ*-slack*, i.e., $d - c^\top \overline{x} > \epsilon$, then we remove it from M.

Other input parameters for our procedure are: a time limit $T > 0$; the number of admissible iterations without sufficient objective improvement $T_{ftol} \in \mathbb{N}$; and a threshold for objective relative improvement $\epsilon_{ftol} > 0$.

3.3 Computational Results

We ran our experiments on an Intel(R) Xeon(R) Linux64 machine CPU E5-2687W v3 3.10GHz with 20 physical cores, 40 logical processors, and 256 GB RAM. We used three state-of-the-art commercial solvers: Gurobi 10.0.1 [34], Artelys Knitro 13.2.0 [35], and Mosek 10.0.43 [36] For the SOCP and ACOPF we wrote AMPL modfiles and we ran them with a Python 3 script. We note that unlike Gurobi and Knitro, Mosek-AMPL does not detect that a constraint like $x^2 + y^2 \leq z^2$ or $x^2 + y^2 \leq wz$ is actually a conic constraint, therefore we had to reformulate the SOCP to a format Mosek-AMPL was able to read. With respect to parameter specifications for each solver, we set barrier convergence tolerance, as well as primal and dual feasibility tolerances to 10^{-6} (see [1] for detailed solver specifications).

Our algorithm in 3.2 was implemented in Python 3 and calls Gurobi as a subroutine for solving an LP or convex QP. All of our reported experiments were obtained with the following parameter setup: $\epsilon = 10^{-5}$, $p_{Jabr} = 0.55$, $p_{i2} = 0.15$, $p_{limit} = 1$, $T_{age} = 5$, $\epsilon_{par} = 10^{-5}/2$, $\epsilon_{ftol} = 10^{-5}$, and $T_{ftol} = 5$. Our codes and AMPL model files can be downloaded from www.github.com/matias-vm.

We report extensive numerical experiments on instances with at least 9000 buses from the data sets: Pan European Grid Advanced Simulation and State Estimation (PEGASE) project [37, 38], ACTIVSg synthetic cases developed as part of the US ARPA-E GRID DATA research project [39, 40], and the largest instances from the Power Grid Library for Benchmarking AC Optimal Power Flow Algorithms [41].

We set a time limit of $1,000$ seconds for all of our SOCP experiments. We did not set a time limit for computing ACOPF primal bounds, and for our cutting-plane algorithm we enforced the $1,000$ seconds time limit before *starting* a new round. The character "$-$" denotes that the solver did not converge, while "TLim" means that the solver did not converge within our time limit. By convergence we mean that the solver *declares* to have obtained an *optimal* solution, within the previously defined tolerances. We remark though that Gurobi and Knitro provide control of absolute primal and dual feasibility and optimality tolerances, while Mosek only allows controlling normalized (by the RHS of the constraints) primal and dual feasibility tolerances. The string "INF" means that the instance was declared infeasible by the solver, while "LOC INF" that the instance might be locally infeasible. Moreover, if Gurobi declares *numerical trouble* while solving our LPs or convex QPs at some iteration of our algorithm, we report the objective value of the previous iteration followed by "$*$".

We remark that, to the best of our knowledge, this is the first computational study which compares the performance of three leading commercial solvers on the Jabr SOCP using a common framework (AMPL). We evaluate the solvers on Jabr SOCP, and compare our warm-started formulations on this formulation instead of the i2 SOCP because Jabr is numerically better behaved from the solvers' perspective. See [1] for additional computational results on: cut computations – used to warm-start instances; solvers' performance on the previously mentioned data sets; and more warm-started computations.

Warm-Starts In power engineering practice, it is often the case that a power flow problem is solved on data that reflects a recent, and likely limited, update on a case that was previously handled. In power engineering language, a 'prior solution' was computed, and the problem is not solved 'from scratch.' In the context of our type of algorithm, this feature opens the door for the use of *warm-started formulations*. In this subsection we present this warm-starting feature of our algorithm and show via numerical experiments its appealing lower bounding capabilities.

The inequalities (7) and (8), based on which we are dynamically adding cuts, do not depend on input data such as loads. Any such inequality remains valid and can be used if the associated branch remains operational.

Due to space limitations, here we present only one class of perturbed instances – instances were the load of each bus was perturbed by a Gaussian $(\mu, \sigma) = (0.01 \cdot P_d, 0.01 \cdot P_d)$, where P_d denotes the original load, subject to the newly perturbed load being non negative. Table 1 summarizes our warm-started experiments on perturbed instances and compare to solvers' performance on the Jabr SOCP. "First Round" reports the objective value and running time of the relaxation M_0 loaded with pre-computed cuts, i.e., our warm-started relaxation. "Last Round" presents the objective value and running time of the last iteration of our cutting-algorithm (on the warm-started relaxation). "Jabr SOCP" and "Primal bound" report, respectively, on the objective value and running time of the Jabr SOCP for the three solvers, and ACOPF primal solutions. We stress the comparison between the running time for our first round, and the solvers' running time.

For most of the instances, our procedure proves very tight lower bounds in less than 25 seconds ("First Round" column). Our procedure stands out in quickly lower bounding the largest cases, e.g., a very sharp bound for ACTIVSg70k is obtained in 102.25 seconds, taking less than half of the time it takes the fastest SOCP solver to converge. Similar performance is achieved on the largest epigrids cases where our method is 3x to 5x faster. Moreover, our relaxations are able to quickly prove infeasible some of these perturbed cases.

An interesting empirical fact is that our cuts are robust with respect to load perturbations. Indeed, our evidence shows that there is not a considerable improvement from the "First Round" to the "Last Round" objectives. This means that the pre-computed cuts loaded to M_0 in the first iteration are accurately outer approximating the SOC relaxations.

4 Future Work

Our work paves the way for promising new research directions. For instance, since our relaxations are linear they could be deployed for practical pricing schemes in energy markets which could increase welfare and mitigate biasedness in price signals [42]. Moreover, we believe our relaxation is a natural candidate to supersede the well-known DC linear approximation in harder problems such as the Unit-Commitment problem or Security Constrained ACOPF (SCOPF), hence it would be interesting to evaluate its performance on these challenging problems.

Table 1. Warm-Started Relaxations, Loads perturbed by Gaussian $(\mu, \sigma) = (0.01 \cdot P_d, 0.01 \cdot P_d)$

	Cutting-Plane				Jabr SOCP							
	First Round		Last Round		Objective			Time (s)			Primal bound	
Case	Objective	Time (s)	Objective	Time (s)	Gurobi	Knitro	Mosek	Gurobi	Knitro	Mosek	Objective	Time (s)
9241pegase	309288.32	13.78	309299.97	160.28	-	309302.67	-	73.12	32.21	36.04	315979.53	101.48
9241pegase-api	INF	23.10	INF	23.10	-	-	-	134.53	TLim	72.96	LOC INF	1845.92
9241pegase-sad	6153913.91	16.18	6154117.59	136.78	-	6096743.03	-	97.51	26.07	83.43	6333763.92	43.71
9591goc-api	1343642.47	11.06	1343670.62	56.36	1343767.43	1345384.57	1343190.29	39.36	25.36	35.30	1571582.59	54.16
9591goc-sad	1058124.48	12.62	1058157.44	65.37	1058337.76	1061275.83	1057323.31	51.85	34.04	37.52	1178895.53	29.53
ACTIVSg10k	2475041.43	9.52	2475078.69	50.51	-	2466383.20	-	42.31	21.75	29.33	2484093.15	57.24
10000goc-api	2502049.28	8.51	2502098.01	36.48	2501946.30	2507074.78	2499373.75	31.91	43.44	32.33	LOC INF	1677.21
10000goc-sad	1388833.86	8.70	1388859.09	44.50	1388824.91	1390230.41	1387588.17	25.96	29.31	23.67	1493481.44	93.72
10192epigrids-api	1848085.36	10.27	1848133.48	45.84	-	1848285.26	1847120.93	65.38	41.17	25.99	LOC INF	1458.35
10192epigrids-sad	1672358.89	10.33	1672398.61	53.37	-	1672533.02	1671364.67	73.64	28.61	35.66	1717429.36	23.89
10480goc-api	2704157.29	12.43	2704252.95	58.45	-	2704373.73	2703432.85	197.17	27.57	55.92	2868495.28	36.89
10480goc-sad	2294908.37	12.81	2294990.69	70.93	-	2294080.35	2292830.56	185.22	35.90	58.31	2322198.81	27.34
13659pegase	379742.62	60.74	379794.51	426.88	379799.37	379804.43	-	34.21	43.17	32.75	386765.25	370.23
13659pegase-api	9253539.07	21.25	9253773.43	109.20	9181205.93	9181269.20	-	97.11	30.41	118.31	9368277.57	62.20
13659pegase-sad	8865733.59	21.28	8865892.49	113.04	8824442.20	8824486.03	-	86.49	33.19	102.59	9039904.52	40.02
19402goc-api	2452185.69	23.55	2452270.83	120.10	-	2452448.33	2451708.50	146.87	120.39	103.32	LOC INF	4440.99
19402goc-sad	1956255.19	23.28	1956313.91	113.89	-	1956570.60	1955018.07	231.90	172.82	102.19	1986936.95	66.02
20758epigrids-api	3043006.76	22.34	3043076.56	104.06	-	-	3032919.24	134.60	TLim	78.32	LOC INF	12425.89
20758epigrids-sad	2610197.53	20.46	2610261.88	93.09	-	-	2608090.26	143.69	TLim	72.19	2635892.81	49.25
24464goc-api	2561680.14	26.28	INF	50.38	-	LOC INF	-	223.07	573.37	118.6	-	19444.54
24464goc-sad	2606391.76	26.78	2606473.78	133.54	-	-	2604708.86	423.12	TLim	128.84	2655942.01	72.48
ACTIVSg25k	5988886.18	28.24	5989016.75	198.58	5952404.50	5960068.30	5949381.04	138.01	73.75	109.39	6013477.05	57.87
30000goc-api	1527412.96	25.35	1527487.45	151.75	-	1528338.73	1525625.64	243.61	369.83	119.92	LOC INF	3407.47
30000goc-sad	-	46.33	-	46.33	-	-	1132715.53	257.94	TLim	75.20	1318389.55	620.27
ACTIVSg70k	16316572.42	102.25	16317886.35	536.51	-	16210682.53	16206290.43	498.80	309.56	229.07	16428367.50	243.84
78484epigrids-api	15862318.24	115.76	15865624.98	883.93	-	-	15859950.52	757.64	TLim	642.24	-	8113.53
78484epigrids-sad	15176866.00	151.77	15180592.27	1118.02	15182602.75	-	15174716.43	420.56	TLim	589.46	15316872.94	353.13

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