Microeconomic Theory
(Mas-Colell et al., 1995):
Definitions etc.

Abstract

This document compiles definitions, propositions, and lemmas from *Microeconomic Theory* by Mas-Colell et al., 1995. All numberings correspond to those in the book. The appendices are not included.

Contents

Part I

Individual Decision Making

Chapter 1

Preference and Choice

1.B Preference Relations

Definition 1.B.1. The preference relation $\succsim$ is *rational* if it possesses the following two properties:

(i) *Completeness*: for all $x, y \in X$ we have that $x \succsim y$ or $y \succsim x$ (or both).

(ii) *Transitivity*: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Proposition 1.B.1. If $\succsim$ is rational, then

(i) $\succ$ is both *irreflexive* ($x \succ x$ never holds) and *transitive* (if $x \succ y$ and $y \succ z$, then $x \succ z$).

(ii) $\sim$ is *reflexive* ($x \sim x$ for all x), *transitive* (if $x \sim y$ and $y \sim z$, then $x \sim z$), and *symmetric* (if $x \sim y$, then $y \sim x$).

(iii) If $x \succ y \succsim z$ then $x \succ z$.

Definition 1.B.2. A function $u : X \to \mathbb{R}$ is a *utility function representing* $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y).$$

Proposition 1.B.2. A preference relation $\succsim$ can be represented by a utility function only if it is rational.

1.C Choice Rules

Definition 1.C.1. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* if the following property holds:

If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

Definition 1.C.2. Given a choice structure $(\mathscr{B}, C(\cdot))$ the *revealed preference relation* $\succsim^*$ is defined by

$x \succsim^* y \iff$ there is some $B \in \mathscr{B}$ such that $x, y \in B$ and $x \in C(B)$.

1.D The Relationship between Preference Relations and Choice Rules

Proposition 1.D.1. Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$ satisfies the weak axiom.

Definition 1.D.1. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalises* $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B, \succsim)$$

for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

Proposition 1.D.2. If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) the weak axiom is satisfied,

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalises $C(\cdot)$ relative to $\mathscr{B}$; that is, $C(B) = C^*(B, \succsim)$, for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the *only* preference relation that does.

Chapter 2

Consumer Choice

2.D Competitive Budgets

Definition 2.D.1. The *Walrasian, or competitive budget set* $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

2.E Demand Functions and Comparative Statics

Definition 2.E.1. The Walrasian demand correspondence $x(p, w)$ is *homogeneous of degree zero* if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

Definition 2.E.2. The Walrasian demand correspondence $x(p, w)$ satisfies Walras' law, if for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p, w)$.

Proposition 2.E.1. If the Walrasian demand function $x(p, w)$ is homogeneous of degree zero, then for all p and w:

$$\sum_{k=1}^{L} \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} w = 0 \text{ for } \ell = 1, \ldots, L.$$

In matrix notation, this is expressed as

$$D_p x(p,w)p + D_w x(p,w)w = 0.$$

Proposition 2.E.2. If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p,w)}{\partial p_k} + x_k(p,w) = 0 \text{ for } k = 1, \ldots, L$$

or, written in matrix notion,

$$p \cdot D_p x(p,w) + x(p,w)^T = 0^T.$$

Proposition 2.E.3. If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} \frac{\partial x_\ell(p,w)}{\partial w} = 1,$$

or, written in matrix notation,

$$p \cdot D_w x(p, w) = 1.$$

2.F The Weak Axiom of Revealed Preference and the Law of Demand

Definition 2.F.1. The Walrasian demand function $x(p, w)$ satisfies the *weak axiom of revealed preference* (the WA) if the following property holds for any two price wealth situations (p, w) and (p', w'):

$$\text{If } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \text{ then } p' \cdot x(p, w) > w'.$$

Proposition 2.F.1. Suppose the Walrasian demand function $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom if and only if the following property holds:

For any compensated price change from an initial situation p, w to a new price wealth pair $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0,$$

with strict inequality whenever $x(p, w) \neq x(p', w')$.

Proposition 2.F.2. If a differentiable Walrasian demand function $x(p, w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any (p, w), the Slutsky (subsitution) matrix $S(p, w)$ satisfies $v \cdot S(p, w)v \leq 0$ and any $v \in \mathbb{R}^L$.

Proposition 2.F.3. Suppose that the Walrasian demand function $x(p, w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $p \cdot S(p, w) = 0$ and $S(p, w)p = 0$ for any (p, w).

Chapter 3

Classical Demand Theory

3.B Preference Relations: Basic Properties

Definition 3.B.1. The preference relation $\succsim$ is *rational* if it possesses the following two properties:

(i) *Completeness*: for all $x, y \in X$ we have that $x \succsim y$ or $y \succsim x$ (or both).

(ii) *Transitivity*: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Definition 3.B.2. The preference relation $\succsim$ on X is *monotone* if $x \in X$ and $y \gg x$ implies $y \succ x$. It is *strongly monotone* if $y \geq x$ and $y \neq x$ imply that $y \succ x$.

Definition 3.B.3. The preference relation $\succsim$ on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $||y - x|| \leq \varepsilon$ and $y \succ x$.

Definition 3.B.4. The preference relation $\succsim$ on X is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

Definition 3.B.5. The preference relation $\succsim$ on X is strictly convex if for every x, we have that $y \succsim x, z \succsim x$, and $y \neq z$ implies $\alpha y + (1 - \alpha)z \succ x$ for all $\alpha \in (0, 1)$.

Definition 3.B.6. A monotone preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}^{L-1}_{+}$ is *quasilinear* with respect to commodity 1 (called, in this case, the *numeraire* commodity) if

(i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.

(ii) Good 1 is desirable; that is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

3.C Preference and Utility

Definition 3.C.1. The preference relation $\succsim$ on X is *continuous* if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.

Proposition 3.C.1. Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

3.D The Utility Maximisation Problem

Proposition 3.D.1. If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximisation problem has a solution.

Proposition 3.D.2. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on a consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

(i) *Homogeneity of degree zero in* (p, w)*:* $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar α.

(ii) *Walras' law:* $p \cdot x = w$ for all $x \in x(p, w)$.

(iii) *Convexity/uniqueness:* If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is *strictly convex*, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Proposition 3.D.3. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p, w)$ is

(i) Homogeneous of degree zero.

(ii) Strictly increasing in w and nonincreasing in p_ℓ for and ℓ.

(iii) Quasiconvex; that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

(iv) Continuous in p and w.

3.E The Expenditure Minimisation Problem

Proposition 3.E.1. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have

(i) If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimised expenditure level in this EMP is exactly w.

(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximised utility level in this UMP is exactly u.

Proposition 3.E.2. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the expenditure function $e(p, u)$ is

(i) Homogeneous of degree one in p.

(ii) Strictly increasing in u and nondecreasing in p_ℓ for any ℓ.

(iii) Concave in p.

(iv) Continuous in p and u.

Proposition 3.E.3. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

(i) *Homogeneity of degree zero in p*: $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.

(ii) *No excess utility:* For any $x \in h(p, u), u(x) = u$.

(iii) *Convexity/uniqueness:* If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is *strictly* convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p, u)$.

Proposition 3.E.4. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p, u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p, u)$ satisfies the compensated law of demand: For all p' and p'',

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0.$$

3.F Duality: A Mathematical Introduction

Definition 3.F.1. For any nonempty closed set $K \subset \mathbb{R}^L$, the *support function* of K is defined for any $p \in \mathbb{R}^L$ to be

$$\mu_K(p) = \inf\{p \cdot x : x \in K\}.$$

Proposition 3.F.1 (The Duality Theorem)**.** Let K be a nonempty closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{p}$. Moreover, in this case,

$$\nabla \mu_K(\bar{p}) = \bar{x}.$$

3.G Relationships between Demand, Indirect Utility, and Expenditure Functions

Proposition 3.G.1. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. For all p and u, the Hicksian demand $h(p, u)$ is the derivatve vector of the expenditure function with respect to prices:

$$h(p, u) = \nabla_p e(p, u).$$

That is, $h_\ell(p, u) = \partial e(p, u)/\partial p_\ell$ for all $\ell = 1, \ldots, L$.

Proposition 3.G.2. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot, u)$ is continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix by $D_p h(p, u)$. Then

(i) $D_p h(p, u) = D_p^2 e(p, u)$.

(ii) $D_p h(p, u)$ is a negative semidefinite matrix.

(iii) $D_p h(p, u)$ is a symmetric matrix.

(iv) $D_p h(p, u) p = 0$.

Proposition 3.G.3 (The Slutsky Equation)**.** Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p, w), and $u = v(p, w)$, we have

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{x_\ell(p,w)}{p_k} + \frac{x_\ell(p,w)}{\partial w} x_k(p,w) \quad \text{for all } \ell, k$$

or equivalently, in matrix notation,

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T.$$

Proposition 3.G.4 (Roy's Identity)**.** Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and stricly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$. Then

$$x(\bar{p},\bar{w}) = -\frac{1}{\nabla_w v(\bar{p},\bar{w})} \nabla_p v(\bar{p},\bar{w}).$$

That is, for every $\ell = 1, \ldots, L$:

$$x_\ell(\bar{p},\bar{w}) = -\frac{\partial v(\bar{p},\bar{w})/\partial p_\ell}{\partial v(\bar{p},\bar{w})/\partial w}.$$

Proposition 3.G.5. Suopose that $e(p, u)$ is strictly increasing in u and is continuous, increasing, homogeneous of degree one, concave, and differentiable in p. Then, for every utility level u, $e(p, u)$ is the expenditure function associated with the at-least-as-good-as set

$$V_u = \{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p,u) \text{ for all } p \gg 0\}$$

3.H Welfare Evaluation of Economic Changes

Proposition 3.H.1. Suppose that the consumer has a locally nonsatiated rational preference relation $\succsim$. If $({}^1{-}p^0) \cdot x^0 < 0$, then the consumer is strictly better off under price wealth situation (p^1, w) than under (p^0, w).

Proposition 3.H.2. Suppose that the consumer has a differentiable expenditure function. Then if $({}^1{-}p^0) \cdot x^0 > 0$, there is a sufficiently small $\bar{\alpha} \in (0, 1)$ such that for all $\alpha < \bar{\alpha}$, we have $e((1-\alpha)p^0 + \alpha p^1, u^0) > w$, and so the consumer is strictly better off under price wealth situation (p^0, w) than under $((1-\alpha)p^0 + \alpha p^1, w)$.

3.I The Strong Axiom of Revealed Preference

Definition 3.I.1. The market demand function $x(p, w)$ satisfies the *strong axiom of revealed preference* (the SA) if for any list

$$(p^1, w^1), \ldots, (p^N, w^N)$$

with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n < N - 1$, we have $p^N \cdot x(p^1, w^1) > w^N$ whenever $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N - 1$.

Chapter 4

Aggregate Demand

4.B Aggregate Demand and Aggregate Wealth

Proposition 4.B.1. A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences admit indirect utility functions of the Gorman form with the coefficients on w_i the same for every consumer i. That is:

$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

4.C Aggregate Demand and the Weak Axiom

Definition 4.C.1. The aggregate demand function $x(p, w)$ satisfies the weak axiom (WA) if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$ imply $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').

Definition 4.C.2. The individual demand function $x_i(p, w_i)$ satisfies the *uncompensated law of demand (ULD)* property if

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0$$

for any p, p', and w_i, with strict inequality if $x_i(p', w_i) \neq x_i(p, w_i)$. The analogous definition applies to the aggregate demand function $x(p, w)$.

Proposition 4.C.1. If every consumer's Walrasian demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property, so does the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$. As a consequence, the aggregate demand $x(p, w)$ satisfies the weak axiom.

Proposition 4.C.2. If $\succsim_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property.

Proposition 4.C.3. Suppose that $\succsim_i$ is defined on the consumption set $X = \mathbb{R}^L_+$ and is representable by a twice continuously differentiable concave function $u_i(\cdot)$. If

$$-\frac{x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4 \quad \text{for all } x_i,$$

then $x_i(p, w_i)$ satisfies the unrestricted law of demand (ULD) property.

Proposition 4.C.4. Suppose that all consumers have identical preferences $\succsim$ defined on $\mathbb{R}^L_+$ [with individual demand functions denoted by $\tilde{x}(p, w)$] and that individual wealth is uniformly distributed on an interval $[0, \bar{w}]$ (strictly speaking this requires a continuum of consumers). Then the aggregate (rigorously, the average) demand function

$$x(p) = \int_0^{\bar{w}} \tilde{x}(p, w) dw$$

satisfies the unrestricted law of demand (ULD) property.

4.D Aggregate Demand and the Existence of a Representative Consumer

Definition 4.D.1. A *positive representative consumer* exists of there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p, w)$ is precisely the Walrasian demand function generated by this preference relation. That is, $x(p, w) \succ x$ whenever $x \neq x(p, w)$ and $p \cdot x \leq w$.

Definition 4.D.2. A *(Berson-Samuelson) social welfare function* is a function $W : \mathbb{R}^I \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, \ldots, u_I) \in \mathbb{R}^I$ of utility levels for the I consumers in the economy.

Proposition 4.D.1. Suppose that for each level of prices p and aggregate wealth w, the wealth distribution $w_1(p, w), \ldots, w_I(p, w)$ solves

$$\begin{aligned} \max_{w_1, \ldots, w_I} \; & W\left(v_1(p, w_1), \ldots, v_I(p, w_I)\right) \\ \text{s.t.} \; & \sum_{i=1}^{I} w_i \leq w. \end{aligned} \tag{4.D.1}$$

Then the value function $v(p, w)$ of problem (4.D.1) is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$.

Definition 4.D.3. The positive representative consumer $\succsim$ for the aggregate demand $x(p, w) = \sum_i x_i(p, w_i(p, w))$ is a *normative representative consumer* relative to the social welfare function $W(\cdot)$ if for every (p, w), the distribution of wealth $w_1(p, w), \ldots, w_I(p, w)$ solves problem (4.D.1) and, therefore, the value function of problem (4.D.1) is an indirect utility function for $\succsim$.

Chapter 5

Production

5.B Production Sets

Proposition 5.B.1. The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.

Proposition 5.B.2. For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$.

5.C Profit Maximisation and Cost Minimisation

Proposition 5.C.1. Suppose that $\pi(\cdot)$ is the profit function of the production set Y and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies the free disposal property. Then

(i) $\pi(\cdot)$ is homogeneous of degree one.

(ii) $\pi(\cdot)$ is convex.

(iii) If Y is convex, then $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$.

(iv) $y(\cdot)$ is homogeneous of degree zero.

(v) If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty).

(vi) (*Hotelling's lemma*) If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.

(vii) If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.

Proposition 5.C.2. Suppose that $c(p, w)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $z(w, q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then

(i) $c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.

(ii) $c(\cdot)$ is a concave function of w.

(iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q)$ for all $w \gg 0\}$.

(iv) $z(\cdot)$ is homogeneous of degree zero in w.

(v) If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(p, w)$ is single-valued.

(vi) (*Shephard's lemma*) If $z(\bar{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w at $\bar{w}$ and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

(vii) If $z(\cdot)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ is a symmetric and negative semidefinite matrix with $D_w z(\bar{w}, q)\bar{w} = 0$.

(viii) If $f(\cdot)$ is homogeneous of degree one (i.e. exhibits constant returns to scale), then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.

(ix) If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q (in particular, marginal costs are nondecreasing in q).

5.E Aggregation

Proposition 5.E.1. For all $p \gg 0$, we have

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$ $(= \{\sum_j y_j : y_j \in y_j(p) \text{ for every } j\})$.

5.F Efficient Production

Definition 5.F.1. A production vetor $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Proposition 5.F.1. If $y \in Y$ is profit maximising for some $p \gg 0$, then y is efficient.

Proposition 5.F.2. Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximising production for some nonzero price vector $p \geq 0$.

Chapter 6

Choice Under Uncertainty

6.B Expected Utility Theory

Definition 6.B.1. A *simple lottery* L is a list $L = (p_1, \ldots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

Definition 6.B.2. Given K simple lotteries $L_k = (p_1^k, \ldots, p_N^k)$, $k = 1, \ldots, K$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the *compound lottery* $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \ldots, K$.

Definition 6.B.3. The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is *continuous* if for any $L, L', L'' \in \mathscr{L}$, the sets

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$$

and

$$\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed.

Definition 6.B.4. The preference relation $\succsim$ on the space simple lotteries $\mathscr{L}$ satisfies the *independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

Definition 6.B.5. The utility function $U : \mathscr{L} \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathscr{L}$ we have

$$U(L) = u_1 p_1 + \cdots + u_N p_N.$$

A utility function $U : \mathscr{L} \to \mathbb{R}$ with the expected utility form is called a *von Neumann-Morgenstern (v.N-M) expected utility function.*

Proposition 6.B.1. A utility function $U : \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \ldots, K$, and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0, \sum_k \alpha_k = 1$.

Proposition 6.B.2. Suppose that $U : \mathscr{L} \to \mathbb{R}$ is a v.N-M expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\tilde{U} : \mathscr{L} \to \mathbb{R}$ is another v.N-M utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.

Proposition 6.B.3 (Expected Utility Theorem)**.** Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n-1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n.$$

6.C Money Lotteries and Risk Aversion

Definition 6.C.1. A decision maker is a *risk averse* (or exhibits *risk aversion*) if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x dF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself. If the decision maker is always [i.e. for any $F(\cdot)$] indifferent between these two lotteries, we say that he is *risk neutral.* Finally, we say that he is *strictly risk averse* if indifference holds only when the two lotteries are the same [i.e. when $F(\cdot)$ is degenerate].

Definition 6.C.2. Given a Bernoulli utility function $u(\cdot)$ we defined the following concepts:

(i) The *certainty equivalent of* $F(\cdot)$, denoted $c(F,u)$, is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F,u)$; that is

$$u\left(c(F,u)\right) = \int u(x) dF(x).$$

(ii) For any fixed amount of money x and positive number ε, the *probability premium* denoted by $\pi(x,\varepsilon,u)$, is the excess on winning the probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x+\varepsilon$ and $x-\varepsilon$. That is

$$u(x) = \left(\frac{1}{2} + \pi(x,\varepsilon,u)\right) u(x+\varepsilon) + \left(\frac{1}{2} - \pi(x,\varepsilon,u)\right) u(x-\varepsilon).$$

Proposition 6.C.1. Suppose a decision maker is an expected utility maximiser with a Bernoulli utility function $u(\cdot)$ on amounts of money. Then the following properties are equivalent:

(i) The decision maker is risk averse.

(ii) $u(\cdot)$ is concave.

(iii) $c(F,u) \leq \int x dF(x)$ for all $F(\cdot)$.

(iv) $\pi(x,\varepsilon,u) \geq 0$ for all x, ε.

Definition 6.C.3. Given a (twice differentiable) Bernoulli utility function $u(\cdot)$ for money, the *Arrow Pratt coefficient of absolute risk aversion* at x is defined as $r_A(x) = -u''(x)/u'(x)$.

Definition (More-risk-averse-than)**.** Given two Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$, when can we say that $u_2(\cdot)$ is unambiguously *more risk averse than* $u_1(\cdot)$? Several possible approaches to a definition seem plausible:

(i) $r_A(x, u_2) \geq r_A(x, u_1)$ for every x.

(ii) There exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$ at all x; that is, $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$. [In other words, $u_2(\cdot)$ is "more concave" than $u_1(\cdot)$.]

(iii) $c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$.

(iv) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

(v) Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds $F(\cdot)$ at least as good as $\bar{x}$. That is, $\int u_2(x)dF(x) \geq u_2(\bar{x})$ implies $\int u_1(x)dF(x) \geq u_1(\bar{x})$ for any $F(\cdot)$ and $\bar{x}$.

Proposition 6.C.2. Definitions (i) to (v) of the *more-risk-averse-than* relation are equivalent.

Definition 6.C.4. The Bernoulli utility function $u(\cdot)$ for money exhibits *decreasing absolute risk aversion* if $r_A(x, u)$ is a decreasing function of x.

Proposition 6.C.3. The following properties are equivalent:

(i) The Bernoulli utility function $u(\cdot)$ exhibits decreasing absolute risk aversion.

(ii) Whenever $x_2 < x_1, u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.

(iii) For any risk $F(z)$, the certainty equivalent of the lottery formed adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x + z)dF(z)$, is such that $(x - c_x)$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.

(iv) The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.

(v) For any $F(z)$, if $\int u(x_2 + z)dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1 + z)dF(z) \geq u(x_1)$.

Definition 6.C.5. Given a Bernoulli utility function $u(\cdot)$, the *coefficient of relative risk aversion at* x is $r_R(x, u) = -xu''(x)/u'(x)$.

Proposition 6.C.4. The following conditions for a Bernoulli utility function $u(\cdot)$ on amounts of money are equivalent:

(i) $r_R(x, u)$ is decreasing in x.

(ii) Whenever $x_2 < x_1$, $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.

(iii) Given any risk $F(t)$ on $t > 0$, the certainty equivalent $\bar{c}_x$ defined by $u(\bar{c}_x) = \int u(tx)dF(t)$ is such that $x/\bar{c}_x$ is decreasing in x.

6.D Comparison of Payoff Distributions in Terms of Return and Risk

Definition 6.D.1. The distribution $F(\cdot)$ *first-order stochastically dominates* $G(\cdot)$ if, for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition 6.D.1. The distribution of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution $G(\cdot)$ if and only if $F(x) \le G(x)$ for every x.

Definition 6.D.2. For any two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ *second-order stochastically dominates* (or *is less risky than*) $G(\cdot)$ if for every nondecreasing concave function $u : \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int u(x)dF(x) \ge \int u(x)dG(x).$$

Proposition 6.D.2. Consider two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean. Then the following statements are equivalent:

(i) $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.

(ii) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.

(iii) Property 6.D.2 holds.

6.E State-Dependent Utility

Definition 6.E.1. A *random variable* is a function $g : S \to \mathbb{R}_+$ that maps states into monetary outcomes.

Definition 6.E.2. The preference relation $\succsim$ has an *extended expected utility representation* if for every $s \in S$, there is a function $u_s : \mathbb{R}_+ \to \mathbb{R}$ such that for any $(x_1, \ldots, x_S) \in \mathbb{R}_+^S$ and $(x'_1, \ldots, x'_S) \in \mathbb{R}_+^S$,

$$(x_1, \ldots, x_S) \succ (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u_s(x_s) \ge \sum_s \pi_s u_s(x'_s).$$

Definition 6.E.3. The preference relation $\succsim$ on $\mathscr{L}$ satisfies the *extended independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0, 1)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

Proposition 6.E.1 (Extended Expected Utility Theorem)**.** Suppose that the preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and extended independence axioms. Then we can assign a utility function $u_s(\cdot)$ for money in every state s such that for any $L = (F_1, \ldots, F_S)$ and $L' = (F'_1, \ldots, F'_S)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_s \left(\int u_s(x_s)dF_s(x_s) \right) \ge \sum_s \left(\int u_s(x_s)dF'_s(x_s) \right).$$

Definition 6.E.4. The preference relation $\succsim$ satisfies the *sure-thing axiom* if, for any subset of states $E \subset S$ (E is called an *event*), whenever $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ differ only in the entries corresponding to E (so that $x'_s = x_s$ for $s \notin E$), the preference ordering between $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ is independent of the particular (common) payoffs for states not in E. Formally, suppose that $(x_1, \ldots, x_S), (x'_1, \ldots, x'_S), (\bar{x}_1, \ldots, \bar{x}_S)$, and $(\bar{x}'_1, \ldots, \bar{x}'_S)$ are such that

$$\begin{aligned} &\text{For all } s \notin E: \quad x_s = x'_s \quad \text{and} \quad \bar{x}_s = \bar{x}'_s. \\ &\text{For all } s \in E: \quad x_s = \bar{x}_s \quad \text{and} \quad x'_s = \bar{x}'_s. \end{aligned}$$

Then $(x_1, \ldots, x_S) \succsim (\bar{x}'_1, \ldots, \bar{x}'_S)$ if and only if $(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S)$.

Proposition 6.E.2. Suppose that there are at least three states and that the preferences $\succsim$ on $\mathbb{R}_+^S$ are continuous and satisfy the sure-thing axiom. Then $\succsim$ admits and extended expected utility representation.

6.F Subjective Probability Theory

Definition 6.F.1. The state preferences $(\succsim_1, \ldots, \succsim_S)$ on state lotteries are *state uniform* if $\succsim_s = \succsim'_s$ for any s and s'.

Proposition 6.F.1 (Subjective Expected Utility Theorem)**.** Suppose that the preference relation $\succsim$ on $\mathscr{L}$ satisfies the continuity and extended independence axioms. Suppose, in addition, that the derived state preferences are state uniform. Then there are probabilities $(\pi_1, \ldots, \pi_S) \gg 0$ and a utility function $u(\cdot)$ on amounts of money such that for any $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ we have

$$(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u(x_s) \geq \sum_s \pi_s u(x'_s).$$

Part II

Game Theory

Chapter 7

Basic Elements of Noncooperative Games

Chapter 8

Simultaneous-Move Games

Chapter 9

Dynamic Games

Part III

Market Equilibrium and Market Failure

Chapter 10

Competitive Markets

10.B Pareto Optimality and Competitive Equilibria

Definition 10.B.1. An *economic allocation* $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $k_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. The allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *feasible* if

$$\sum_{i=1}^{I} x_{\ell i} \leq w_\ell + \sum_{j=1}^{J} y_{\ell j} \quad \text{for } \ell = 1, \ldots, L.$$

Definition 10.B.2. A feasible allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $u_i(x'_i) \geq u_i(x_i)$ for all $i = 1, \ldots, I$ and $u_i(x'_i) > u_i(x_i)$ for some i.

Definition 10.B.3. The allocation $(x^*_1, \ldots, x^*_I, y^*_1, \ldots, y^*_J)$ and price vector $p^* \in \mathbb{R}^L$ constitute a *competitive* (or *Walrasian*) *equilibrium* if the following conditions are satisfied:

(i) *Profit maximisation:* For each firm j, y^*_j solves

$$\max_{y_j \in Y_i j} p^* \cdot y_j. \tag{10.B.1}$$

(ii) *Utility maximisation:* For each consumer i, x^*_i solves

$$\begin{aligned} &\max_{x_i \in X_i} u_i(x_i) \\ &\quad \text{s.t. } p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot y^*_j). \end{aligned} \tag{10.B.2}$$

(iii) *Market clearing:* For each good $\ell = 1, \ldots, L$,

$$\sum_{i=1}^{I} x^*_{\ell i} = \omega_\ell + \sum_{j=1}^{J} y^*_{\ell j}. \tag{10.B.3}$$

Lemma 10.B.1. If the allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ and price vector $p \gg 0$ satisfy the market clearing condition (Definition 10.B.3) for all goods $\ell \neq k$, and if every consumer's budget constraint is satisfied with equality, so that $p \cdot x_i = p \cdot w_i + \sum_j \theta_{ij} p \cdot y_j$ for all i, then the market for good k also clears.

10.D The Fundamental Welfare Theorems in a Partial Equilibrium Context

Proposition 10.D.1 (The First Fundamental Theorem of Welfare Economics)**.** If the prive p^* and allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ constitutes a competitive equilibrium, then this allocation is Pareto optimal.

Proposition 10.D.2 (The Second Fundamental Theorem of Welfare Economics)**.** For any Pareto optimal levels of utility $(u_1^*, \ldots, u_I^*)$, there are transfers of the numeraire commodity $(T_1, \ldots, T_I)$ satisfying $\sum_i T_i = 0$, such that a competitive equilibrium reached from the endowments $\omega_{m1} + T_1, \ldots, \omega_{mI} + T_I$ yields precisely the utilities $(u_1^*, \ldots, u_I^*)$.

10.F Free Entry and Long-Run Competitive Equilibria

Definition 10.F.1. Given an aggregate demand function $x(p)$ and a cost function $c(q)$ for each potentially active firm having $c(0) = 0$, a triple (p^*, q^*, J^*) is a *long-run competitive equilibrium* if

(i) q^* solves $\max_{q>0} p^* q - c(q)$ (Profit maximisation)

(ii) $x(p^*) = J^* q^*$ (Demand = supply)

(iii) $p^* q^* - c(q^*) = 0$ (Free Entry Condition).

Chapter 11

Externalities and Public Goods

11.B A Simple Bilateral Externality

Definition 11.B.1. An *externality* is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of another agent in the economy.

11.C Public Goods

Definition 11.C.1. A *public good* is a commodity for which use of a unit of the good by one agent does not preclude use by other agents.

Chapter 12

Market Power

12.C Static Models of Oligopoly

Proposition 12.C.1. There is a unique Nash equilibrium (p_1^*, p_2^*) in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost: $p_1^* = p_2^* = c$.

Proposition 12.C.2. In any Nash equilibrium of the Cournot duopoly model with cost $c > 0$ per unit for the two firms and an inverse demand function $p(\cdot)$ satisfying $p'(q) < 0$ for all $q \geq 0$ and $p(0) > c$, the market price is greater than c (the competitive price) and smaller than the monopoly price.

12.D Repeated Interaction

Proposition 12.D.1. The strategies

$$p_{jt}(H_{t-1}) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ equal } (p^m, p^m) \text{ or } t = 1 \\ c & \text{otherwise} \end{cases}$$

constitute a subgame perfect Nash equilibrium (SPNE) of the infinitely repeated Betrand duopoly game if and only if $\delta \geq \frac{1}{2}$ in the firms optimisation problem

$$\max \sum_{t=1}^{\infty} \delta^{t-1} \pi_{jt}, \quad \delta < 1.$$

Proposition 12.D.2. In the infinitely repeated Betrand duopoly game, when $\delta \geq \frac{1}{2}$ repeated choice of any price $p \in [c, p^m]$ can be supported as a subgame perfect Nash equilibrium outcome path using Nash reversion strategies. By contrast, when $\delta < \frac{1}{2}$, any subgame perfect Nash equilibrium outcome path must have all sales occurring at a price equal to c in every period.

12.E Entry

Proposition 12.E.1. Suppose that conditions

(A1) $Jq_J \geq J'q_{J'}$ whenever $J > J'$;

(A2) $q_J \leq q_{J'}$ whenever $J > J'$;

(A3) $p(Jq_J) - c'(q_J) \geq 0$ for all J

are satisfied by the post-entry oligopoly game, that $p'(\cdot) < 0$, and that $c''(\cdot) \geq 0$. Then the equilibrium number of entrants J^*, is at least $J^\circ - 1$, where J° is the socially optimal number of entrants.

12.F The Competitive Limit

Proposition 12.F.1. As the market size grows, the price in any subgame perfect Nash equilibrium of the two-stage Cournot entry model converges to the level of minimum average cost (the "competitive" price). Formally,

$$\max_{p_\alpha \in P_\alpha} |p_\alpha - \bar{c}| \to 0 \text{ as } \alpha \to \infty.$$

Chapter 13

Adverse Selection, Signaling, and Screening

13.B Informational Asymmetries and Adverse Selection

Definition 13.B.1. In the competitive labour market model with unobservable worker productivity levels, a *competitive equilibrium* is a wage rate w^* and a set Θ^* of worker types who accept employment such that

$$\Theta^* = \{\theta : r(\theta) \leq w^*\}$$

and

$$w^* = E[\theta | \theta \in \Theta^*].$$

Proposition 13.B.1. Let W^* denote the set of competitive equilibrium wages for the adverse selection labour market model, and let $W^* = \max\{w : w \in W^*\}$.

(i) If $w^* > r(\underline{\theta})$ and there is an $\varepsilon > 0$ such that $E[\theta | r(\theta) < w'] > w'$ for all $w' \in (w^* - \varepsilon, w^*)$, then there is a unique pure strategy SPNE of the two-stage game-theoretic model. In this SPNE, employed workers receive a wage of w^*, and workers with types in the set $\Theta(w^*) = \{\theta : r(\theta) \leq w^*\}$ accept employment in firms.

(ii) If $w^* = r(\underline{\theta})$, then there are multiple pure strategy SPNEs. However, in every pure strategy SPNE each agent's payoff exactly equals her payoff in the highest-wage competitive equilibrium.

Proposition 13.B.2. In the adverse selection labour market model (where $r(\cdot)$ is strictly increasing with $r(\theta) \leq \theta$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$ and $F(\cdot)$ has an associated density $f(\cdot)$ with $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$), the highest-wage competitive equilibrium is a constrained Pareto optimum.

13.C Signaling

Lemma 13.C.1. In any separating perfect Bayesian equilibrium, $w^*(e^*(\theta_H)) = \theta_H$ and $w^*(e^*(\theta_L)) = \theta_L$; that is, each worker type receives a wage equal to her productivity level.

Lemma 13.C.2. In any separating perfect Bayesian equilibrium, $e^*(\theta_L) = 0$; that is, a low-ability worker chooses to get no education.

13.D Screening

Proposition 13.D.1. In any SPNE of the screening game with observable worker types, a type θ_i worker accepts contract $(w_i^*, t_i^*) = (\theta_i, 0)$, and firms earn zero profits.

Lemma 13.D.1. In any equilibrium, whether pooling or separating, both firms must earn zero profits.

Lemma 13.D.2. No pooling equilibria exist.

Lemma 13.D.3. If (w_L, t_L) and (w_H, t_H) are the contracts signed by the low- and high-ability workers in a separating equilibrium, then both contracts yield zero profits; that is, $w_L = \theta_L$ and $w_H = \theta_H$.

Lemma 13.D.4. In any separating equilibrium, the low-ability workers accept contract $(\theta_L, 0)$; that is, they receive the same contract as when no informational imperfections are present in the market.

Lemma 13.D.5. In any separating equilibrium, the high-ability workers accept contract $(\theta_H, \hat{t}_H)$, where $\hat{t}_H$ satisfies $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$.

Proposition 13.D.2. In any subgame perfect Nash equilibrium of the screening game, low-ability workers accept contract $(\theta_L, 0)$, and high-ability workers accept contract $(\theta_H, \hat{t}_H)$, where $\hat{t}_H$ satisfies $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$.

Chapter 14

The Principal-Agent Problem

14.B Hidden Actions (Moral Hazard)

Proposition 14.B.1. In the principal-agent model with observable managerial effort, an optimal contract specifies that the manager chooses the effort e^* that maximises $\left[\int \pi f(\pi|e)d\pi - v^{-1}(\bar{u} + g(e))\right]$ and pays the manager a fixed ware $w^* = v^{-1}(\bar{u} + g(e^*))$. This is the uniquely optimal contract if $v''(w) < 0$ at all w.

Proposition 14.B.2. In the principal-agent model with unobservable managerial effort and a risk-neutral manager, an optimal contract generates the same effort choice and expected utilities for the manager and the owner as when effort is observable.

Lemma 14.B.1. In any solution to the problem

$$\begin{aligned} \min_{w(\pi)} & \int w(\pi) f(\pi|e) d\pi \\ \text{s.t. } & \text{(i) } \int v\left(w(\pi)\right) f(\pi|e) d\pi - g(e) \geq \bar{u} \\ & \text{(ii) } e \text{ solves } \max_{\tilde{e}} \int v\left(w(\pi)\right) f(\pi|\tilde{e}) d\pi - g(\tilde{e}) \end{aligned}$$

with $e = e_H$, both $\gamma > 0$ and $\mu > 0$.

Proposition 14.B.3. In the principal-agent model with unobservable manager effort, a risk-averse manager, and two possible effort choices, the optimal compensation scheme for implementing e_H satisfies

$$\frac{1}{v'\left(w(\pi)\right)} = \gamma + \mu\left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right],$$

gives the manager expected utility $\tilde{u}$, and involves a larger expected wage payment than is required when effort is observable. The optimal compensation scheme for implementing e_L involves the same fixed wage payment as if effort were observable. Whenever the optimal effort level with observable effort would be e_H, nonobservability causes a welfare loss.

14.C Hidden Information (and Monopolistic Screening)

Proposition 14.C.1. In the principal-agent model with an observable state variable θ, the optimal contract involves an effort level e_i^* in state θ_i such that $\pi(e_i^*) = g_e(e_i^*, \theta)$ and fully insures the manager, setting his wage in each state θ_i at the level w_i^* such that $v\left(w_i^* - g(e_i^*, \theta_i)\right) = \bar{u}$.

Proposition 14.C.2 (The Revelation Principle)**.** Denote the set of possible states by Θ. In searching for an optimal contract, the owner can without loss restrict himself to contracts of the following form:

(i) After the state θ is realised, the manager is required to announce which state has occurred.

(ii) The contract specifies an outcome $[w(\hat{\theta}), e(\hat{\theta})]$ for each possible announcement $\hat{\theta} \in \Theta$.

(iii) In every state $\theta \in \Theta$, the manager finds is optimal to report the state *truthfully.*

Lemma 14.C.1. In the problem

$$\begin{aligned} \max_{w_H, e_H \geq 0, w_L, e_L > 0} \; & \lambda[\pi(e_H) - w_H] + (1-\lambda)[\pi(e_L) - w_L] \\ \text{s.t. (i) } & w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}) \\ \text{(ii) } & w_H - g(e_H, \theta_H) \geq v^{-1}(\bar{u}) \\ & (\textit{reservation utility} \text{ (or } \textit{individual rationality}\text{) } \textit{constraint}) \\ \text{(iii) } & w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H) \\ \text{(iv) } & w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L) \\ & (\textit{incentive compatibility} \text{ (or } \textit{truth-telling} \text{ or } \textit{self-selection}\text{) } \textit{constraints}) \end{aligned}$$

we can ignore constraint (ii). That is, a contract is a solution to the problem if and only if it is the solution to the problem derived from it by dropping (ii).

Lemma 14.C.2. An optimal contract in the problem given in Lemma 14.C.1 must have $w_L - g(e_L, \theta_L) = v^{-1}(\bar{u})$.

Lemma 14.C.3. In any optimal contract:

(i) $e_L \leq e_L^*$; that is, the manager's effort level in state θ_L is no more than the level that would arise if θ were observable.

(ii) $e_H = e_H^*$; that is, the manager's effort level in state θ_H is exactly equal to the level that arise if θ were observable.

Lemma 14.C.4. In any optimal contract, $e_L < e_L^*$; that is, the effort level in state θ_L is necessarily *strictly* below the level that would arise in state θ_L if θ were observable.

Proposition 14.C.3. In the hidden information principal-agent model with an infinitely risk-averse manager the optimal contract sets the level of effort in state θ_H at its first-best (full observability) level e_H^*. The effort level in state θ_L is distorted downward from its first-best level e_L^*. In addition, the manager is inefficiently insured, receiving a utility greater than $\bar{u}$ in state θ_H and a utility equal to $\bar{u}$ in state θ_L. The owner's expected payoff is strictly lower than the expected payoff he receives when θ is observable, while the infinitely risk-averse manager's expected utility is the same as when θ is observable (it equals $\bar{u}$).

Part IV

General Equilibrium

Chapter 15

General Equilibrium Theory: Some Examples

15.B Pure Exchange: The Edgeworth Box

Definition 15.B.1. A *Walrasian* (or *competitve*) *equilibrium* for an Edgeworth box economy is a price vector p^* and an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$,

$$x_i^* \succsim_i x_i' \text{ for all } x_i' \in B_i(p^*).$$

Definition 15.B.2. An allocation x in the Edgeworth box is *Pareto optimal* (or *Pareto efficient*) if there is no other allocation x' in the Edgeworth box with $x_i' \succsim_i x_i$ for $i = 1, 2$ and $x_i' \succsim_i x_i$ for some i.

Definition 15.B.3. An allocation x^* in the Edgeworth box is supportable as an *equilibrium with transfers* if there is a price system p^* and wealth transfers T_1 and T_2 satisfying $T_1 + T_2 = 0$, such that for each consumer i we have

$$x_i^* \succsim_i x_i' \text{ for all } x_i' \in \mathbb{R}_+^2 \text{ such that } p^* \cdot x_i' \leq p^* \cdot \omega_i + T_i.$$

15.D The 2 x 2 Production Model

Definition 15.D.1. The production of good 1 is *relatively more intensive in factor* 1 than is production of good 2 if

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

at *all* factor prices $w = (w_1, w_2)$.

Chapter 16

Equilibrium and Its Basic Welfare Properties

16.B The Basic Model and Definitions

Definition 16.B.1. An *allocation* $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X$ for each consumer $i = 1, \ldots, I$ and a production vector $y_i \in Y$ for each firm $j = 1, \ldots, J$. An allocation (x, y) is *feasible* if $\sum_i x_{\ell i} = \ddot{\omega}_\ell + \sum_j y_{\ell j}$ for every commodity ℓ. That is, if

$$\sum_i x_i = \bar{\omega} + \sum_j y_j. \tag{16.B.1}$$

We denote the set of feasible allocations by

$$A = \left\{ (x, y) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J : \sum_i x_i = \bar{\omega} + \sum_j y_j \right\} \subset \mathbb{R}^{L(I+J)}.$$

Definition 16.B.2. A feasible allocation (x, y) is *Pareto optimal* (or *Pareto efficient*) if there is no other allocation $(x', y') \in A$ that *Pareto dominates* it, that is, if there is no feasible allocation (x', y') such that $x'_i \succsim_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

Definition 16.B.3. Given a private ownership economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J\right), \{(\omega_i, 0_{i1}, \ldots, 0_{iJ})\}_{i=1}^I\big)$, an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitutes *Walrasian* (or *competitive*) *equilibrium* if:

(i) For every j, y^*, j maximises profits in Y_j; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\left\{ x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^* \right\}.$$

(iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

Definition 16.B.4. Given an economy specified by $\big(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}), \bar{\omega}\big)$ an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a *price equilibrium with transfers* of there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y^*, j maximises profits in Y_j; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq \omega\}.$$

(iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

16.C The First Fundamental Theorem of Welfare Economics

Definition 16.C.1. The preference relation $\succsim$ on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $||y - x|| \leq \varepsilon$ and $y \succ x$.

Proposition 16.C.1 (The First Fundamental Theorem of Welfare Economics)**.** If the prive p^* and allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ constitutes a competitive equilibrium, then this allocation is Pareto optimal.

16.D The Second Fundamental Theorem of Welfare Economics

Definition 16.D.1. Given an economy specified by $\big(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\omega}\big)$ an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L) \neq 0$ constitute a *price quasiequilibrium with transfers* if there is an assignment of wealth levels $(w_1, \ldots w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y^*, j maximises profits in Y_j; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, if $x_i \succ x_i^*$ then $p \cdot x_i \geq w_i$.

(iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

Proposition 16.D.1 (The Second Fundamental Theorem of Welfare Economics)**.** Consider an economy specified by $\big(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}), \bar{\omega}\big)$, and suppose that every Y_j is convex and every preference relation $\succsim_i$ is convex [i.e., the set $\{x_i' \in X_i : x_i' \succsim_i x_i\}$ is convex for every $x_i \in X$] and locally nonsatiated. Then, for every Pareto optimal allocation (x^*, y^*), there is a price vector $p = (p_1, \ldots, p_L) \neq 0$ such that (x^*, y^*, p) is a price quasiequilibrium with transfers.

Proposition 16.D.2. Assume that X_i is convex and $\succsim_i$ is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p, and the wealth level w_i are such that $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq w_i$. Then, if there is a consumption vector $x_i' \in X_i$ such that $p \cdot x_i' < w_i$ [a *cheaper consumption* for (p, w_i)], it follows that $x_i \succ_i x_i^*$ implies $p \cdot x_i > w_i$.

Proposition 16.D.3. Suppose that for every i, X_i is convex, $0 \in X_i$, and $\succsim_i$ is continuous. Then any price quasiequilibrium with transfers that has $(w_1, \ldots, w_I) \gg 0$ is a price equilibrium with transfers.

16.E Pareto Optimality and Social Welfare Optima

Proposition 16.E.1. A feasible allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a Pareto optimum if and only if $(u_1(x_1), \ldots, u_I(x_I)) \in UP$, where $UP = \{u_1, \ldots, u_I \in U\colon$ there is no $(u_1', \ldots, u') \in U$ such that $u_i' \geq u_i$ for all i and $u_i' > u_i$ for some $i\}$ and $U = \{(u_1, \ldots, u_I) \in \mathbb{R}^I\colon$ there is a feasible allocation (x, y) such that $u_i \leq u_i(x_i)$ for $i = 1, \ldots, I\}$.

Proposition 16.E.2. If $u^* = (u_1^*, \ldots u_I^*)$ is a solution to the social welfare maximisation problem $\max_{u \in U} \lambda \cdot u$ with $\lambda \gg 0$, then $u^* \in UP$; that is, u^* is the utility vector of a Pareto optimal allocation. Moreover, if the utility possibility set U is convex, then for any $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_I) \in UP$, there is a vector of welfare weights $\lambda = (\lambda_1, \ldots, \lambda_I) \geq 0, \lambda \neq 0$, such that $\lambda \cdot \tilde{u} \geq \lambda \cdot u$ for all $u \in U$, that is, such that $\tilde{u}$ is a solution to the social welfare maximisation problem.

16.F First-Order Conditions for Pareto Optimality

Proposition 16.F.1. Under the assumptions made about the economy [in particular, the concavity of every $u_i(\cdot)$ and the convexity of ever $F_j(\cdot)$], every Pareto optimal allocation (and, hence, every price equilibrium with transfers) maximises a weighted sum of utilities subject to the resource and technological constraints. Moreover, the weight λ_i of the utility of the ith consumer equals the reciprocal of consumer i's marginal utility or wealth evaluated at the supporting prices and imputed wealth.

16.G Some Applications

Definition 16.G.1. A *Lindahl equilibrium* for the public goods economy is a price equilibrium with transfers for the artificial economy with personalised commodities. That is, an allocation $(x_1^*), \ldots, x_I^*, q^*, z^* \in \mathbb{R}^{2I} \times \mathbb{R} \times \mathbb{R}$ and a price system $(p_1, p_{21}, \ldots, p_{2I}) \in \mathbb{R}^{I+1}$ constitutes a Lindahl equilibrium if there is a set of wealth levels $(w_1, \ldots, w_I)$ satisfying $\sum_i w_i = \sum_i p_1 x_{1i}^* + (\sum_i p_{2i}) q^* - p_1 z^*$ and such that

(i) $q^* \leq f(z^*)$ and $(\sum_i p_{2i}) q^* - p_1 z^* \geq (\sum_i p_{2i}) q - p_1 z$ for all (q, z) with $z \geq 0$ and $q \leq f(z)$.

(ii) For every i, $x_i^* = (x_{1i}^*, x_{2i}^*)$ is maximal for $\succsim_i$ in the set $\{(x_{1i}, x_{2i}) \in X_i : p_1 x_{1i} + p_2 x_{2i} \leq w_i\}$.

(iii) $\sum_i x_{1i}^* + z^* = \bar{\omega}_1$ and $x_{2i}^* = q^*$ for every i.

Proposition 16.G.1. Suppose that the basic assumptions of Section 16.F hold and that, in addition, all consumers have convex preferences (so utility functions are quasiconcave). If (x^*, y^*) is Pareto optimal, then there exists a price vector $p = (p_1, \ldots, p_L)$ and wealth levels $w = (w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that:

(i) For any firm j, we have
$$p = \gamma_j \nabla F_j(y_j^*) \quad \text{for some } \gamma_j > 0.$$

(ii) For any i, x_i^* is maximal for $\succsim_i$ in the budget set
$$\{x_i \in X : p \cdot x_i \leq w_i\}.$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Chapter 17

The Positive Theory of Equilibrium

17.B Equilibrium: Definitions and Basic Equations

Definition 17.B.1. Given a private ownership economy specified by

$$\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}, \ldots, \theta_{iJ})\}_{i=1}^I\right),$$

an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a *Walrasian* (or *competitive*, or *market*, or *price-taking*) equilibrium if

(i) For every j, $y_j^* \in Y_j$ maximises profits in Y_j; that is

$$p \cdot y_j \leq y \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, $x_i^* \in X_i$ is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii) $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$.

Proposition 17.B.1. In a pure exchange economy in which consumer preferences are continuous, strictly convex and locally nonsatiated, $p \geq 0$ is a Walrasian equilibrium price vector if and only if:

$$\sum_i (x_i(p, p \cdot \omega_i) - \omega_i) \leq 0.$$

Definition 17.B.2. The *excess demand function of consumer i* is

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i,$$

where $x_i(p, p \cdot \omega_i)$ is consumer i's Walrasian demand function. The (*aggregate*) *excess demand function* of the economy is

$$z(p) = \sum_i z_i(p).$$

The domain of this function is a set of nonnegative price vectors that includes all strictly positive price vectors.

Proposition 17.B.2. Suppose that, for every consumer i, $X_i = \mathbb{R}^L_+$ and $\succsim_i$ is continuous, strictly convex, and strongly monotone. Suppose also that $\sum_i \omega_i \gg 0$. Then the aggregate excess demand function $z(p)$, defined for all price vectors $p \gg 0$, satisfies the properties:

(i) $z(\cdot)$ is continuous.

(ii) $z(\cdot)$ is homogeneous of degree zero.

(iii) $p \cdot z(\cdot) = 0$ for all p (*Walras' law*).

(iv) There is an $s > 0$ such that $z_\ell(p) > -s$ for every commodity ℓ and all p.

(v) If $p^n \to p$, where $p \neq 0$ and $p_\ell = 0$ for some ℓ, then

$$\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty.$$

17.C Existence of Walrasian Equilibrium

Proposition 17.C.1. Suppose that $z(p)$ is a function defined for all strictly positive price vectors $p \in \mathbb{R}^L_{++}$ and satisfying conditions (i) to (v) of Proposition 17.B.2. Then the system of euqations $z(p) = 0$ has a solution. Hence, a Walrasian equilibrium exists in any pure exchange economy in which $\sum_i \omega_i \gg 0$ and every consumer has continuous, strictly convex, and strongly monotone preferences.

Proposition 17.C.2. Suppose that $z(p)$ is a function defined for all nonzero, nonnegative price vectors $p \in \mathbb{R}^L_+$ and satisfying conditions (i) to (iii) of Proposition 17.B.2 (i.e. continuity homogeneity of degree zero and Walras' law). Then there is a price vector p^* such that $z(p^*) \leq 0$.

17.D Local Uniqueness and the Index Theorem

Definition 17.D.1. An equilibrium price vector $p = (p_1, \ldots, p_{L-1})$ is *regular* if the $(L-1)\times(L-1)$ matrix of price effects $D\hat{z}(p)$ is nonsingular, that is, has rank $L-1$. If every normalised equilibrium price vector is regular, we say that the *economy is regular*.

Proposition 17.D.1. Any regular (normalised) equilibrium price vector

$$p = (p_1, \ldots, p_{L-1}, 1)$$

is *locally isolated* (or *locally unique*). That is, there is an $\varepsilon > 0$ such that if $p' \neq p, p'_L = p_L = 1$, and $||p' - p|| < \varepsilon$, then $z(p') \neq 0$. Moreoever, of the economy is regular, then the number of normalised equilibrium price vectors is finite.

Definition 17.D.2. Suppose that $p = (p_1, \ldots, p_{L-1}, 1)$ is a regular equilibrium of the economy. Then we denote

$$\text{index } p = (-1)^{L-1} \text{sign}|D\hat{z}(p)|,$$

where $|D\hat{z}(p)|$ is the determinant of the $(L-1) \times (L-1)$ matrix $D\hat{z}(p)$.

Proposition 17.D.2 (The Index Theorem)**.** For any regular economy, we have

$$\sum_{\{p: z(p)=0, p_L=1\}} \text{index } p = +1.$$

Definition 17.D.3. The system of M equations in N unknowns $f(v) = 0$ is *regular* if rank $Df(v) = M$ whenever $f(v) = 0$.

Proposition 17.D.3 (The Transversality Theorem)**.** If the $M \times (N + S)$ matrix $Df(v; q)$ has rank M whenever $f(v; q) = 0$ then for almost every q, the $M \times N$ matrix $D_v f(v; q)$ has rank M whenever $f(v; q) = 0$.

Proposition 17.D.4. For any p and ω, rank $D_\omega \hat{z}(p; \omega) = L - 1$.

Proposition 17.D.5. For almost every vector of initial endowments $(\omega_1, \ldots, \omega_I) \in \mathbb{R}^{LI}_{++}$, the economy defined by $\{(\succsim_i, \omega_i)\}_{i=1}^{I}$ is regular.

17.E Anything Goes: The Sonnenschein-Mantel-Debreu Theorem

Proposition 17.E.1. Suppose that $I < L$. Then for any equilibrium price vector p there is some direction of price change $dp \neq 0$ such that $p \cdot dp = 0$ (hence dp is not proportional to p) and $dp \cdot Dz(p) dp \leq 0$.

Proposition 17.E.2. Given a price vector p, let $z \in \mathbb{R}^L$ be an arbitrary vector and A an arbitrary $L \times L$ matrix satisfying $p \cdot z = 0$, $Ap = 0$ and $p \cdot A = -z$. Then there is a collection of L consumers generating an aggregate excess demand function $z(\cdot)$ such that $z(p) = z$ and $Dz(p) = A$.

Proposition 17.E.3. Suppose that $z(\cdot)$ is a continuous function defined on

$$P_\varepsilon = \{p \in \mathbb{R}^L_+ : p_\ell / p_{\ell'} \geq \varepsilon \text{ for every } \ell \text{ and } \ell'\}$$

and with values in $\mathbb{R}^L$. Assume that, in addition, $z(\cdot)$ is homogeneous of degree zero and satisfies Walras' law. Then there is an economy of L consumers whose aggregate excess demand function coincides with $z(p)$ in the domain of P_ε.

Proposition 17.E.4. For any $N \geq 1$, suppose that we assign to each $n = 1, \ldots, N$ a price vector p^n, normalised to $||p^n|| = 1$, and an $L \times L$ matrix A_n of rank $L - 1$, satisfying $A_n p^n = 0$ and $p^n \cdot A_n = 0$. Suppose that, in addition, the index formula $\sum_n (-1)^{L-1} \text{sign}|\hat{A}_n| = +1$ holds. If $L = 2$, assume also that positive and negative index equilibria alternate.

Then there is an economy with L consumers such that the aggregate excess demand $z(\cdot)$ has the properties:

(i) $z(p) = 0$ for $||p|| = 1$ if and only if $p = p^n$ for some n.

(ii) $Dz(p^n) = A_n$ for every n.

17.F Uniqueness of Equilibria

Proposition 17.F.1. Given an economy specified by the constant returns technology Y and the aggregate excess demand function of the consumers $z(\cdot)$, a price vector p is a Walrasian equilibrium price vector if and only if

(i) $p \cdot y \leq 0$ for every $y \in Y$, and

(ii) $z(p)$ is a feasible production; that is, $z(p) \in Y$.

Definition 17.F.1 (The Weak Axiom for Excess Demand Functions)**.** The excess demand function $z(\cdot)$ satisfies the weak axiom of revealed preferences (WA) if for any pair of price vectors p and p', we have

$$z(p) \neq z(p') \text{ and } p \cdot z(p') \leq 0 \text{ implies } p' \cdot z(p) \geq 0.$$

Proposition 17.F.2. Suppose that the excess demand function $z(\cdot)$ is such that, for any constant returns technology Y, the economy formed by $z(\cdot)$ and Y has a unique (normalised) equilibrium price vector. Then $z(\cdot)$ satisfies the weak axiom. Conversely, if $z(\cdot)$ satisfies the weak axiom then, for any constant returns convex technology Y, the set of equilibrium price vectors is convex (and so, if the set of normalised price equilibria is finite, there can be at most one normalised price equilibrium).

Definition 17.F.2. The function $z(\cdot)$ has the *gross substitute* (GS) property if whenever p' and p are such that, for some ℓ, $p'_\ell > p_\ell$ and $p'_k > p_k$ for $k \neq \ell$, we have $z_k(p') > z_k(p)$ for $k \neq \ell$.

Proposition 17.F.3. An aggregate excess demand function $z(\cdot)$ that satisfies the gross substitute property has at most one exchange equilibrium; that is, $z(p) = 0$ has at most one (normalised) solution.

Proposition 17.F.4. If $z(\cdot)$ is an aggregate excess demand function, $z(p) = 0$, and $Dz(p)$ has the gross substitute sign pattern, then we also have $dp \cdot Dz(p)dp < 0$ whenever $dp \neq 0$ is not proportional to p.

Proposition 17.F.5. Suppose that the initial endowment allocation $(\omega_1, \ldots, \omega_I)$ constitutes a Walrasian equilibrium allocation for an exchange economy with strictly convex and strongly monotone consumer preferences (i.e., no-trade is an equilibrium). Then this is the unique equilibrium allocation.

17.G Comparative Statics Analysis

Proposition 17.G.1. Given any price vector $\bar{p}$, endowments for the first consumer of the first $L-1$ commodities $\hat{\bar{\omega}}_1 = (\bar{\omega}_{11}, \ldots, \bar{\omega}_{L-1,1})$, and a $(L-1) \times (L-1)$ nonsingular matrix B, there is an exchange economy formed by $L+1$ consumers in which the first consumer has the prescribed endowments of the first $L-1$ commodities, $\hat{z}(\bar{p}, \hat{\bar{\omega}}_1) = 0$, $\hat{z}(\cdot, \hat{\bar{\omega}}_1) = 0$ is regular at $\bar{p}$ and $Dp(\hat{\bar{\omega}}_1) = B$.

Proposition 17.G.2. Suppose that $\hat{z}(\bar{p}; \bar{q}) = 0$, where $\hat{z}(\cdot)$ is differentiable. If $D_q\hat{z}(\bar{p}; \bar{q})$ is negative definite, then

$$(D_q\hat{z}(\bar{p}; \bar{q})dq) \cdot (Dp(\bar{q})dq) \geq 0 \text{ for any } dq.$$

Proposition 17.G.3. Suppose that $\hat{z}(\bar{p}; \bar{q}) = 0$, where $\hat{z}(\cdot; \cdot)$ is differentiable. If the $L \times L$ matrix $D_p z(\bar{p}; \bar{q})$ has negative diagonal entries and positive off-diagonal entries, then $[D_p z(\bar{p}; \bar{q})]^{-1}$ has all its entries negative.

17.H Tâtonnement Stability

Proposition 17.H.1. Suppose that $z(p^*) = 0$ and $p^* \cdot z(p) > 0$ for every p not proportional to p^*. Then the relative prices of any solution trajectory of the differential equation

$$\frac{dp_\ell}{dt} = c_\ell z_\ell(p) \quad \text{for every } \ell$$

converge to the relative prices of p^*.

Definition 17.H.1. We say that the differentiable trajectory $y(t) \in Y$ is *admissible* if $p(y(t)) \cdot (dy(t)/dt) \geq 0$ for every t, with equality only if $y(t)$ is profit maximising for $p(y(t))$ (in which case we could say that we are at a long-run equilibrium).

Proposition 17.H.2. If there is a single strictly convex consumer, then any admissible trajectory converges to the (unique) equilibrium.

Chapter 18

Some Foundations for Competitive Equilibria

18.B Core and Equilibria

Definition 18.B.1. A coalition $S \subset I$ *improves upon*, or *blocks*, the feasible allocation $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_+^{LI}$ if for every $i \in S$ we can find a consumption $x_i \geq 0$ with the properties:

(i) $x_i \succ_i x_i^*$ for every $i \in S$

(ii) $\sum_{i \in S} x_i \in Y + \left\{\sum_{i \in S} \omega_i\right\}$.

Definition 18.B.2. We say that a feasible allocation $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_+^{LI}$ has the *core property* if there is no coalition of consumers $S \subset I$ that can improve upon x^*. The *core* is the set of allocations that have the core property.

Proposition 18.B.1. Any Walrasian equilibrium allocation has the core property.

Proposition 18.B.2. Denoting by hn the nth individual of type h, suppose that the allocation

$$x^* = (x_{11}^*, \dots, x_{1n}^*, \dots, x_{1N}^*, \dots, x_{H1}^*, \dots, x_{Hn}^*, \dots, x_{HN}^*) \in \mathbb{R}_+^{LHN}$$

belongs to the core of the N-replica economy. Then x^* has the *equal-treatment property*, that is, all consumers of the same type get the same consumption bundle:

$$x_{hm}^* = x_{hn}^* \quad \text{for all } 1 \leq m, n \leq N \text{ and } 1 \leq h \leq H.$$

Proposition 18.B.3. If the feasible type allocation $x^* = (x_1^*, \dots, x_H^*) \in \mathbb{R}_+^{LH}$ has the core property for all $N = 1, 2, \dots$, that is, $x^* \in C_N$ for all N, then x^* is a Walrasian equilibrium allocation.

18.C Noncooperative Foundations of Walrasian Equilibria

Definition 18.C.1. The profiles of actions $a^* = (a_1^*, \dots, a_I^*) \in A_1 \times \cdots \times A_I$ is a *trading equilibrium* if, for every i,

$$u_i\left(g\left(a_i^*; p(a^*)\right) + \omega_i\right) \geq u_i\left(g\left(a_i; p(a_i; a_{-i}^*)\right) + \omega_i\right) \quad \text{for all } a_i \in A_i.$$

18.D The Limits to Redistribution

Definition 18.D.1. The feasible allocation $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LI}$ is *self-selective* (or *anonymous*, or *envy-free in net trades*) if there is a set of net trades $B \subset \mathbb{R}^L$, to be called a *generalised budget set*, or a *tax system*, such that, for every i, $z_i^* = x_i^* - \omega_i$ solves the problem

$$\begin{aligned} &\max u_i(z_i + \omega_i) \\ &\text{s.t. } z_i \in B, \\ &\qquad z_i + \omega_i \geq 0. \end{aligned}$$

Proposition 18.D.1. Suppose we have an exchange economy with a continuum of consumer types. Assume:

(i) The preferences of all consumers are representable by differentiable utility functions.

(ii) The set of characteristics of consumers present in the economy cannot be split into two disconnected classes. Formally, if $(u(\cdot), \omega), (u'(\cdot), \omega')$ are two preferences-endowment pairs present in the economy then there is a continuous function $(u(\cdot; t), \omega(t))$ of $t \in [0, 1]$ such that
$$(u(\cdot; 0), \omega(0)) = (u(\cdot, \omega)), (u(\cdot; 1), \omega(1)) = (u'(\cdot), \omega),$$
and $(u(\cdot; t), \omega(t))$ is present in the economy for every t.

Then any allocation $x^* = \{x_i^*\}_{i \in I}$ that is Pareto optimal, self-selective, and interior (i.e., $x_i^* \gg 0$ for all i) must be a Walrasian equilibrium allocation. Here I is an infinite set of names.

18.E Equilibrium and the Marginal Productivity Principle

Definition 18.E.1. Given a continuum population $\mu = (\mu_1, \ldots, \mu_H) \in \mathbb{R}_+^H$ a feasible allocation $(x_1^*, \ldots, x_H^*)$ is a *marginal product*, or *no-surplus*, *allocation* if

$$u_h(x_h^*) = \frac{\partial v(\mu)}{\partial \mu_h} \quad \text{for all } h.$$

In words: at a no-surplus allocation everyone is getting exactly what she contributes on the margin.

Proposition 18.E.1. For any *continuum* population $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_H) \gg 0$ a feasible allocation $(x_1^*, \ldots, x_H^*) \gg 0$ is a marginal product allocation if and only if it is a Walrasian equilibrium allocation.

Chapter 19

General Equilibrium Under Uncertainty

19.B A Market Economy with Contingent Commodities: Description

Definition 19.B.1. For every physical commodity $\ell = 1, \ldots, L$ and states $s = 1, \ldots, S$, a unit of *(state-)contingent commodity* ℓs is a title to receive a unit of physical good ℓ if and only if s occurs. Accordingly, a *(state-)contingent commodity vector* is specified by

$$x = (x_{11}, \ldots, x_{L1}, \ldots, x_{1S}, \ldots, x_{LS}) \in \mathbb{R}^{LS},$$

and is understood as an entitlement to receive the commodity vector $x = (x_{1s}, \ldots, x_{Ls})$ if state s occurs.

19.C Arrow-Debreu Equilibrium

Definition 19.C.1. An allocation

$$(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J \subset \mathbb{R}^{LS(I+J)}$$

and a system of prices for the contingent commodities $p = (p_{11}, \ldots, p_{LS}) \in \mathbb{R}^{LS}$ constitute an *Arrow-Debreu equilibrium* if:

(i) For every j, y_j^* satisfies $p \cdot y_j^* \geq p \cdot y_j$ for all $y_j \in Y_j$.

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii) $\sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i$.

19.D Sequential Trade

Definition 19.D.1. A collection formed by a price vector $q = (q_1, \ldots, q_S) \in \mathbb{R}^S$ for contingent first good commodities at $t = 0$, a spot price vector

$$p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$$

for every s, and, for every consumer i, consumption plans $z_i^* = (z_{1i}^*, \ldots, z_{Si}^*) \in \mathbb{R}^S$ at $t = 0$ and $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ constitute a *Radner equilibrium* if:

(i) For every i, the consumption plans z_i^*, x_i^* solve the problem

$$\begin{aligned} \max_{\substack{(x_{1i},\ldots,x_{Si})\in\mathbb{R}_+^{LS} \\ (z_{1i},\ldots,z_{Si})\in\mathbb{R}^S}} & U_i(x_{1i}, \ldots, x_{Si}) \\ \text{s.t. (i)} & \sum_s q_s z_{si} \le 0, \\ \text{(ii)} & p_s \cdot x_{si} \le p_s \omega_{si} + p_{1s} z_{si} \quad \text{for every } s. \end{aligned}$$

(ii) $\sum_i z_{si}^* \le 0$ and $\sum_i x_{si}^* \le \sum_i \omega_{si}$ for every s.

Proposition 19.D.1. We have:

(i) If the allocation $x^* \in \mathbb{R}^{LSI}$ and the contingent commodities price vector $(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute and Arrow-Debreu equilibrium, then there are prices $q \in \mathbb{R}_{++}^S$ for contingent first good commodities and consumption plans for these commodities $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{SI}$ such that the consumption plans x^*, z^*, the prices q, and the spot prices $(p_1, \ldots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$, $z^* \in \mathbb{R}^{SI}$ and prices $q \in \mathbb{R}_{++}^S$, $(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute a Radner equilibrium, then there are multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}_{++}^S$ such that the allocation x^* and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s.)

19.E Asset Markets

Definition 19.E.1. A unit of an *asset*, or *security*, is a title to receive an amount r_s of good 1 at date $t = 1$ if state s occurs. An asset is therefore characterised by its *return vector* $r = (r_1, \ldots, r_S) \in \mathbb{R}^S$.

Definition 19.E.2. A collection formed by a price vector $q = (q_1, \ldots, q_K) \in \mathbb{R}^K$ for assets traded at $t = 0$, a spot price vector $p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$ for every s, and, for every consumer i, portfolio plans $z_i^* = (z_{1i}^*, \ldots, z_{Ki}^*) \in \mathbb{R}^K$ at $t = 0$ and consumption plans $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ constitutes a *Radner equilibrium* if:

(i) For every i, the consumption plans z_i^*, x_i^* solve the problem

$$\begin{aligned} \max_{\substack{(x_{1i},\ldots,x_{Si})\in\mathbb{R}_+^{LS} \\ (z_{1i},\ldots,z_{Si})\in\mathbb{R}^K}} & U_i(x_{1i}, \ldots, x_{Si}) \\ \text{s.t. (i)} & \sum_k q_k z_{ki} \le 0, \\ \text{(ii)} & p_s \cdot x_{si} \le p_s \omega_{si} + \sum_k p_{1s} z_{ki} r_{sk} \quad \text{for every } s. \end{aligned}$$

(ii) $\sum_i z^*_{ki} \leq 0$ and $\sum_i x^*_{si} \leq \sum_i \omega_{si}$ for every k and s.

Proposition 19.E.1. Assume that every return vector is nonnegative and nonzero; that is, $r_k \geq 0$ and $r_k \neq 0$ for all k. Then, for every (column) vector $q \in \mathbb{R}^K$ of asset prices arising in a Radner equilibrium, we can find multipliers $\mu = (\mu_1, \dots, \mu_S) \geq 0$, such that $q_k = \sum_s \mu_s r_{sk}$ for all k (in matrix notation, $q^T = \mu \cdot R$).

Definition 19.E.3. An asset structure with an $S \times K$ return matrix R is *complete* of $\text{rank}R = S$, that is, if there is some subset of S assets with linearly independent returns.

Proposition 19.E.2. Suppose that the asset structure is complete. Then:

(i) If the consumption plans $x^* = (x^*_1, \dots, x^*_I) \in \mathbb{R}^{LSI}$ and the price vector

$$(p_1, \dots, p_S) \in \mathbb{R}^{LS}_{++}$$

constitute an Arrow-Debreu equilibrium, then there are asset prices $q \in \mathbb{R}^K_{++}$ and portfolio plans $z^* = (z^*_1, \dots, z^*_I) \in \mathbb{R}^{KI}$ such that the consumption plans x^*, portfolio plans z^*, asset prices q, and spot prices $(p_1, \dots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$, portfolio plans $z^* \in \mathbb{R}^{KI}$, and prices $q \in \mathbb{R}^K_{++}, (p_1, \dots, p_S) \in \mathbb{R}^{LS}_{++}$ constitute a Radner equilibrium, then there are multipliers $\mu = (\mu_1, \dots, \mu_S) \in \mathbb{R}^S_{++}$ such that consumption plans x^* and the contingent commodities price vector $(\mu_1 p_1, \dots, \mu_S p_S) \in \mathbb{R}^{LS}_{++}$ constitute an Arrow-Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s; recall that $p_{1s} = 1$.)

Proposition 19.E.3. Suppose that the asset price vector $q \in \mathbb{R}^K$, the spot prices $p = (p_1, \dots, p_S) \in \mathbb{R}^{LS}$, the consumption plans $x^* = (x^*_1, \dots, x^*_I) \in \mathbb{R}^{LSI}_+$, and the portfolio plans $(z^*_1, \dots, z^*_I) \in \mathbb{R}^{KI}$ constitute a Radner equilibrium for an asset structure with $S \times K$ return matrix R. Let R' be the $S \times K'$ return matrix of a second asset structure. If $\text{range}R' = \text{range}R$, then x^* is still the consumption allocation of a Radner equilibrium in the economy with the second asset structure.

19.F Incomplete Markets

Definition 19.F.1. The asset allocation $(z_1, \dots, z_I) \in \mathbb{R}^{KI}$ is constrained Pareto optimal if it is feasible (i.e. $\sum_i z_i \leq 0$) and if there is no other feasible asset allocation $(z'_1, \dots, z'_I) \in \mathbb{R}^{KI}$ such that

$$U^*_i(z'_1, \dots, z'_I) \geq U^*_i(z_1, \dots, z_I) \quad \text{for every } j,$$

with at least one inequality strict.

Proposition 19.F.1. Suppose that there are two periods and only one consumption good in the second period. Then any Radner equilibrium is *constrained Pareto optimal* in the sense that there is no possible redistribution of assets in the first period that leaves every consumer as well off and at least one consumer strictly better off.

19.G Firm Behaviour in General Equilibrium Models under Uncertainty

Definition 19.G.1. A set $A \subset \mathbb{R}^S$ of random variables is *spanned* by a given asset structure of every $a \in A$ is in the range of the return matrix R of the asset structure, that is, if every $a \in A$ can be expressed as a linear combination of the available asset returns.

19.H Imperfect Information

Definition 19.H.1. The signal function $\sigma' : S \to \mathbb{R}$ is *at least as informative* as $\sigma : S \to \mathbb{R}$ if $\sigma(s) \neq \sigma(s')$ implies $\sigma'(s) \neq \sigma'(s')$ for any pair s, s'. It is *more informative* if, in addition, $\sigma'(s) \neq \sigma'(s')$ for some pair s, s' with $\sigma(s) = \sigma(s')$.

Proposition 19.H.1. In the single-consumer problem, if the signal function $\sigma'(\cdot)$ is at least as informative as the signal function $\sigma(\cdot)$, then the ex ante utility derived from $\sigma'(\cdot)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma'(\cdot)})$, is at least as large as the ex ante utility derived from $\sigma(\cdot)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma(\cdot)})$.

Definition 19.H.2. The price function $p(\cdot)$ is a *rational expectations equilibrium price function* if, for every s, $p(s)$ clears the spot market when every consumer i knows that $s \in E_{p(s),\sigma_i(s)}$ and, therefore, evaluates commodity bundles $x_i \in \mathbb{R}^2$ according to the updated utility function

$$\sum_s \left(\pi_{s'i} | p(s), \sigma_i(s)\right) u_{s'i}(x).$$

Chapter 20

Equilibrium and Time

Part V

Welfare Economics and Incentives

Chapter 21

Social Choice Theory

Chapter 22

Elements of Welfare Economics and Axiomatic Bargaining

Chapter 23

Incentives and Mechanism Design

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