

*Microeconomic Theory*  
(Mas-Colell et al., 1995):  
Definitions etc.

### **Abstract**

This document compiles definitions, propositions, corollaries, and lemmas from *Microeconomic Theory* by Mas-Colell et al., 1995. All numberings correspond to those in the book. The appendices are not included.

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**Part I**

**Individual Decision Making**

# Chapter 1

## Preference and Choice

### 1.B Preference Relations

**Definition 1.B.1.** The preference relation  $\succsim$  is *rational* if it possesses the following two properties:

- (i) *Completeness*: for all  $x, y \in X$  we have that  $x \succsim y$  or  $y \succsim x$  (or both).
- (ii) *Transitivity*: For all  $x, y, z \in X$ , if  $x \succsim y$  and  $y \succsim z$ , then  $x \succsim z$ .

**Proposition 1.B.1.** If  $\succsim$  is rational, then

- (i)  $\succ$  is both *irreflexive* ( $x \succ x$  never holds) and *transitive* (if  $x \succ y$  and  $y \succ z$ , then  $x \succ z$ ).
- (ii)  $\sim$  is *reflexive* ( $x \sim x$  for all  $x$ ), *transitive* (if  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ ), and *symmetric* (if  $x \sim y$ , then  $y \sim x$ ).
- (iii) If  $x \succ y \succsim z$  then  $x \succ z$ .

**Definition 1.B.2.** A function  $u : X \rightarrow \mathbb{R}$  is a *utility function representing*  $\succsim$  if, for all  $x, y \in X$ ,

$$x \succsim y \iff u(x) \geq u(y).$$

**Proposition 1.B.2.** A preference relation  $\succsim$  can be represented by a utility function only if it is rational.

### 1.C Choice Rules

**Definition 1.C.1.** The choice structure  $(\mathcal{B}, C(\cdot))$  satisfies the *weak axiom of revealed preference* if the following property holds:

If for some  $B \in \mathcal{B}$  with  $x, y \in B$  we have  $x \in C(B)$ , then for any  $B' \in \mathcal{B}$  with  $x, y \in B'$  and  $y \in C(B')$ , we must also have  $x \in C(B')$ .

**Definition 1.C.2.** Given a choice structure  $(\mathcal{B}, C(\cdot))$  the *revealed preference relation*  $\succsim^*$  is defined by

$$x \succsim^* y \iff \text{there is some } B \in \mathcal{B} \text{ such that } x, y \in B \text{ and } x \in C(B).$$

## 1.D The Relationship between Preference Relations and Choice Rules

**Proposition 1.D.1.** Suppose that  $\succsim$  is a rational preference relation. Then the choice structure generated by  $\succsim$ ,  $(\mathcal{B}, C^*(\cdot, \succsim))$  satisfies the weak axiom.

**Definition 1.D.1.** Given a choice structure  $(\mathcal{B}, C(\cdot))$ , we say that the rational preference relation  $\succsim$  *rationalises*  $C(\cdot)$  relative to  $\mathcal{B}$  if

$$C(B) = C^*(B, \succsim)$$

for all  $B \in \mathcal{B}$ , that is, if  $\succsim$  generates the choice structure  $(\mathcal{B}, C(\cdot))$ .

**Proposition 1.D.2.** If  $(\mathcal{B}, C(\cdot))$  is a choice structure such that

- (i) the weak axiom is satisfied,
- (ii)  $\mathcal{B}$  includes all subsets of  $X$  of up to three elements,

then there is a rational preference relation  $\succsim$  that rationalises  $C(\cdot)$  relative to  $\mathcal{B}$ ; that is,  $C(B) = C^*(B, \succsim)$ , for all  $B \in \mathcal{B}$ . Furthermore, this rational preference relation is the *only* preference relation that does.



## Chapter 2

# Consumer Choice

### 2.D Competitive Budgets

**Definition 2.D.1.** The *Walrasian, or competitive budget set*  $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$  is the set of all feasible consumption bundles for the consumer who faces market prices  $p$  and has wealth  $w$ .

### 2.E Demand Functions and Comparative Statics

**Definition 2.E.1.** The Walrasian demand correspondence  $x(p, w)$  is *homogeneous of degree zero* if  $x(\alpha p, \alpha w) = x(p, w)$  for any  $p, w$  and  $\alpha > 0$ .

**Definition 2.E.2.** The Walrasian demand correspondence  $x(p, w)$  satisfies Walras' law, if for every  $p \gg 0$  and  $w > 0$ , we have  $p \cdot x = w$  for all  $x \in x(p, w)$ .

**Proposition 2.E.1.** If the Walrasian demand function  $x(p, w)$  is homogeneous of degree zero, then for all  $p$  and  $w$ :

$$\sum_{k=1}^L \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} w = 0 \text{ for } \ell = 1, \dots, L.$$

In matrix notation, this is expressed as

$$D_p x(p, w) p + D_w x(p, w) w = 0.$$

**Proposition 2.E.2.** If the Walrasian demand function  $x(p, w)$  satisfies Walras' law, then for all  $p$  and  $w$ :

$$\sum_{\ell=1}^L p_\ell \frac{\partial x_\ell(p, w)}{\partial p_k} + x_k(p, w) = 0 \text{ for } k = 1, \dots, L$$

or, written in matrix notion,

$$p \cdot D_p x(p, w) + x(p, w)^T = 0^T.$$

**Proposition 2.E.3.** If the Walrasian demand function  $x(p, w)$  satisfies Walras' law, then for all  $p$  and  $w$ :

$$\sum_{\ell=1}^L \frac{\partial x_\ell(p, w)}{\partial w} = 1,$$

or, written in matrix notation,

$$p \cdot D_w x(p, w) = 1.$$

## 2.F The Weak Axiom of Revealed Preference and the Law of Demand

**Definition 2.F.1.** The Walrasian demand function  $x(p, w)$  satisfies the *weak axiom of revealed preference* (the WA) if the following property holds for any two price wealth situations  $(p, w)$  and  $(p', w')$ :

$$\text{If } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \text{ then } p' \cdot x(p, w) > w'.$$

**Proposition 2.F.1.** Suppose the Walrasian demand function  $x(p, w)$  is homogeneous of degree zero and satisfies Walras' law. Then  $x(p, w)$  satisfies the weak axiom if and only if the following property holds:

For any compensated price change from an initial situation  $p, w$  to a new price wealth pair  $(p', w') = (p', p' \cdot x(p, w))$ , we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0,$$

with strict inequality whenever  $x(p, w) \neq x(p', w')$ .

**Proposition 2.F.2.** If a differentiable Walrasian demand function  $x(p, w)$  satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any  $(p, w)$ , the Slutsky (substitution) matrix  $S(p, w)$  satisfies  $v \cdot S(p, w)v \leq 0$  and any  $v \in \mathbb{R}^L$ .

**Proposition 2.F.3.** Suppose that the Walrasian demand function  $x(p, w)$  is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then  $p \cdot S(p, w) = 0$  and  $S(p, w)p = 0$  for any  $(p, w)$ .

## Chapter 3

# Classical Demand Theory

### 3.B Preference Relations: Basic Properties

**Definition 3.B.1.** The preference relation  $\succsim$  is *rational* if it possesses the following two properties:

- (i) *Completeness*: for all  $x, y \in X$  we have that  $x \succsim y$  or  $y \succsim x$  (or both).
- (ii) *Transitivity*: For all  $x, y, z \in X$ , if  $x \succsim y$  and  $y \succsim z$ , then  $x \succsim z$ .

**Definition 3.B.2.** The preference relation  $\succsim$  on  $X$  is *monotone* if  $x \in X$  and  $y \gg x$  implies  $y \succ x$ . It is *strongly monotone* if  $y \geq x$  and  $y \neq x$  imply that  $y \succ x$ .

**Definition 3.B.3.** The preference relation  $\succsim$  on  $X$  is *locally nonsatiated* if for every  $x \in X$  and every  $\varepsilon > 0$ , there is  $y \in X$  such that  $\|y - x\| \leq \varepsilon$  and  $y \succ x$ .

**Definition 3.B.4.** The preference relation  $\succsim$  on  $X$  is *convex* if for every  $x \in X$ , the upper contour set  $\{y \in X : y \succsim x\}$  is convex; that is, if  $y \succsim x$  and  $z \succsim x$ , then  $\alpha y + (1 - \alpha)z \succsim x$  for any  $\alpha \in [0, 1]$ .

**Definition 3.B.5.** The preference relation  $\succsim$  on  $X$  is strictly convex if for every  $x$ , we have that  $y \succ x, z \succ x$ , and  $y \neq z$  implies  $\alpha y + (1 - \alpha)z \succ x$  for all  $\alpha \in (0, 1)$ .

**Definition 3.B.6.** A monotone preference relation  $\succsim$  on  $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$  is *quasilinear* with respect to commodity 1 (called, in this case, the *numeraire* commodity) if

- (i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if  $x \sim y$ , then  $(x + \alpha e_1) \sim (y + \alpha e_1)$  for  $e_1 = (1, 0, \dots, 0)$  and any  $\alpha \in \mathbb{R}$ .
- (ii) Good 1 is desirable; that is,  $x + \alpha e_1 \succ x$  for all  $x$  and  $\alpha > 0$ .

### 3.C Preference and Utility

**Definition 3.C.1.** The preference relation  $\succsim$  on  $X$  is *continuous* if it is preserved under limits. That is, for any sequence of pairs  $\{(x^n, y^n)\}_{n=1}^\infty$  with  $x^n \succsim y^n$  for all  $n$ ,  $x = \lim_{n \rightarrow \infty} x^n$ , and  $y = \lim_{n \rightarrow \infty} y^n$ , we have  $x \succsim y$ .

**Proposition 3.C.1.** Suppose that the rational preference relation  $\succsim$  on  $X$  is continuous. Then there is a continuous utility function  $u(x)$  that represents  $\succsim$ .

### 3.D The Utility Maximisation Problem

**Proposition 3.D.1.** If  $p \gg 0$  and  $u(\cdot)$  is continuous, then the utility maximisation problem has a solution.

**Proposition 3.D.2.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated preference relation  $\succsim$  defined on a consumption set  $X = \mathbb{R}_+^L$ . Then the Walrasian demand correspondence  $x(p, w)$  possesses the following properties:

- (i) *Homogeneity of degree zero in  $(p, w)$ :*  $x(\alpha p, \alpha w) = x(p, w)$  for any  $p, w$  and scalar  $\alpha$ .
- (ii) *Walras' law:*  $p \cdot x = w$  for all  $x \in x(p, w)$ .
- (iii) *Convexity/uniqueness:* If  $\succsim$  is convex, so that  $u(\cdot)$  is quasiconcave, then  $x(p, w)$  is a convex set. Moreover, if  $\succsim$  is *strictly convex*, so that  $u(\cdot)$  is strictly quasiconcave, then  $x(p, w)$  consists of a single element.

**Proposition 3.D.3.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . The indirect utility function  $v(p, w)$  is

- (i) Homogeneous of degree zero.
- (ii) Strictly increasing in  $w$  and nonincreasing in  $p_\ell$  for any  $\ell$ .
- (iii) Quasiconvex; that is, the set  $\{(p, w) : v(p, w) \leq \bar{v}\}$  is convex for any  $\bar{v}$ .
- (iv) Continuous in  $p$  and  $w$ .

### 3.E The Expenditure Minimisation Problem

**Proposition 3.E.1.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally non-satiated preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$  and that the price vector is  $p \gg 0$ . We have

- (i) If  $x^*$  is optimal in the UMP when wealth is  $w > 0$ , then  $x^*$  is optimal in the EMP when the required utility level is  $u(x^*)$ . Moreover, the minimised expenditure level in this EMP is exactly  $w$ .
- (ii) If  $x^*$  is optimal in the EMP when the required utility level is  $u > u(0)$ , then  $x^*$  is optimal in the UMP when wealth is  $p \cdot x^*$ . Moreover, the maximised utility level in this UMP is exactly  $u$ .

**Proposition 3.E.2.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally non-satiated preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . Then the expenditure function  $e(p, u)$  is

- (i) Homogeneous of degree one in  $p$ .
- (ii) Strictly increasing in  $u$  and nondecreasing in  $p_\ell$  for any  $\ell$ .
- (iii) Concave in  $p$ .
- (iv) Continuous in  $p$  and  $u$ .

**Proposition 3.E.3.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally non-satiated preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . Then for any  $p \gg 0$ , the Hicksian demand correspondence  $h(p, u)$  possesses the following properties:

- (i) *Homogeneity of degree zero in  $p$ :*  $h(\alpha p, u) = h(p, u)$  for any  $p, u$  and  $\alpha > 0$ .
- (ii) *No excess utility:* For any  $x \in h(p, u)$ ,  $u(x) = u$ .
- (iii) *Convexity/uniqueness:* If  $\succsim$  is convex, then  $h(p, u)$  is a convex set; and if  $\succsim$  is strictly convex, so that  $u(\cdot)$  is strictly quasiconcave, then there is a unique element in  $h(p, u)$ .

**Proposition 3.E.4.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally non-satiated preference relation  $\succsim$  and that  $h(p, u)$  consists of a single element for all  $p \gg 0$ . Then the Hicksian demand function  $h(p, u)$  satisfies the compensated law of demand: For all  $p'$  and  $p''$ ,

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0.$$

### 3.F Duality: A Mathematical Introduction

**Definition 3.F.1.** For any nonempty closed set  $K \subset \mathbb{R}^L$ , the *support function* of  $K$  is defined for any  $p \in \mathbb{R}^L$  to be

$$\mu_K(p) = \inf\{p \cdot x : x \in K\}.$$

**Proposition 3.F.1** (The Duality Theorem). Let  $K$  be a nonempty closed set, and let  $\mu_K(\cdot)$  be its support function. Then there is a unique  $\bar{x} \in K$  such that  $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$  if and only if  $\mu_K(\cdot)$  is differentiable at  $\bar{p}$ . Moreover, in this case,

$$\nabla \mu_K(\bar{p}) = \bar{x}.$$

### 3.G Relationships between Demand, Indirect Utility, and Expenditure Functions

**Proposition 3.G.1.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated and strictly convex preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . For all  $p$  and  $u$ , the Hicksian demand  $h(p, u)$  is the derivative vector of the expenditure function with respect to prices:

$$h(p, u) = \nabla_p e(p, u).$$

That is,  $h_\ell(p, u) = \partial e(p, u) / \partial p_\ell$  for all  $\ell = 1, \dots, L$ .

**Proposition 3.G.2.** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated and strictly convex preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . Suppose also that  $h(\cdot, u)$  is continuously differentiable at  $(p, u)$ , and denote its  $L \times L$  derivative matrix by  $D_p h(p, u)$ . Then

- (i)  $D_p h(p, u) = D_p^2 e(p, u)$ .
- (ii)  $D_p h(p, u)$  is a negative semidefinite matrix.
- (iii)  $D_p h(p, u)$  is a symmetric matrix.
- (iv)  $D_p h(p, u)p = 0$ .

**Proposition 3.G.3** (The Slutsky Equation). Suppose that  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated and strictly convex preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . Then for all  $(p, w)$ , and  $u = v(p, w)$ , we have

$$\frac{\partial h_\ell(p, u)}{\partial p_k} = \frac{x_\ell(p, w)}{p_k} + \frac{x_\ell(p, w)}{\partial w} x_k(p, w) \quad \text{for all } \ell, k$$

or equivalently, in matrix notation,

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) x(p, w)^T.$$

**Proposition 3.G.4** (Roy's Identity). Suppose that  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated and strictly convex preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . Suppose also that the indirect utility function is differentiable at  $(\bar{p}, \bar{w}) \gg 0$ . Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w}).$$

That is, for every  $\ell = 1, \dots, L$ :

$$x_\ell(\bar{p}, \bar{w}) = -\frac{\partial v(\bar{p}, \bar{w}) / \partial p_\ell}{\partial v(\bar{p}, \bar{w}) / \partial w}.$$

**Proposition 3.G.5.** Suppose that  $e(p, u)$  is strictly increasing in  $u$  and is continuous, increasing, homogeneous of degree one, concave, and differentiable in  $p$ . Then, for every utility level  $u$ ,  $e(p, u)$  is the expenditure function associated with the at-least-as-good-as set

$$V_u = \{x \in \mathbb{R}_+^L : p \cdot x \geq e(p, u) \text{ for all } p \gg 0\}$$

### 3.H Welfare Evaluation of Economic Changes

**Proposition 3.H.1.** Suppose that the consumer has a locally nonsatiated rational preference relation  $\succsim$ . If  $(p^1 - p^0) \cdot x^0 < 0$ , then the consumer is strictly better off under price wealth situation  $(p^1, w)$  than under  $(p^0, w)$ .

**Proposition 3.H.2.** Suppose that the consumer has a differentiable expenditure function. Then if  $(p^1 - p^0) \cdot x^0 > 0$ , there is a sufficiently small  $\bar{\alpha} \in (0, 1)$  such that for all  $\alpha < \bar{\alpha}$ , we have  $e((1 - \alpha)p^0 + \alpha p^1, u^0) > w$ , and so the consumer is strictly better off under price wealth situation  $(p^0, w)$  than under  $((1 - \alpha)p^0 + \alpha p^1, w)$ .

### 3.I The Strong Axiom of Revealed Preference

**Definition 3.I.1.** The market demand function  $x(p, w)$  satisfies the *strong axiom of revealed preference* (the SA) if for any list

$$(p^1, w^1), \dots, (p^N, w^N)$$

with  $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$  for all  $n < N - 1$ , we have  $p^N \cdot x(p^1, w^1) > w^N$  whenever  $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$  for all  $n \leq N - 1$ .

## Chapter 4

# Aggregate Demand

### 4.B Aggregate Demand and Aggregate Wealth

**Proposition 4.B.1.** A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector  $p$  is that preferences admit indirect utility functions of the Gorman form with the coefficients on  $w_i$  the same for every consumer  $i$ . That is:

$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

### 4.C Aggregate Demand and the Weak Axiom

**Definition 4.C.1.** The aggregate demand function  $x(p, w)$  satisfies the weak axiom (WA) if  $p \cdot x(p', w') \leq w$  and  $x(p, w) \neq x(p', w')$  imply  $p' \cdot x(p, w) > w'$  for any  $(p, w)$  and  $(p', w')$ .

**Definition 4.C.2.** The individual demand function  $x_i(p, w_i)$  satisfies the *uncompensated law of demand (ULD)* property if

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0$$

for any  $p, p'$ , and  $w_i$ , with strict inequality if  $x_i(p', w_i) \neq x_i(p, w_i)$ . The analogous definition applies to the aggregate demand function  $x(p, w)$ .

**Proposition 4.C.1.** If every consumer's Walrasian demand function  $x_i(p, w_i)$  satisfies the uncompensated law of demand (ULD) property, so does the aggregate demand  $x(p, w) = \sum_i x_i(p, \alpha_i w)$ . As a consequence, the aggregate demand  $x(p, w)$  satisfies the weak axiom.

**Proposition 4.C.2.** If  $\succsim_i$  is homothetic, then  $x_i(p, w_i)$  satisfies the uncompensated law of demand (ULD) property.

**Proposition 4.C.3.** Suppose that  $\succsim_i$  is defined on the consumption set  $X = \mathbb{R}_+^L$  and is representable by a twice continuously differentiable concave function  $u_i(\cdot)$ . If

$$-\frac{x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4 \quad \text{for all } x_i,$$

then  $x_i(p, w_i)$  satisfies the unrestricted law of demand (ULD) property.

**Proposition 4.C.4.** Suppose that all consumers have identical preferences  $\succsim$  defined on  $\mathbb{R}_+^L$  [with individual demand functions denoted by  $\tilde{x}(p, w)$ ] and that individual wealth is uniformly distributed on an interval  $[0, \bar{w}]$  (strictly speaking this requires a continuum of consumers). Then the aggregate (rigorously, the average) demand function

$$x(p) = \int_0^{\bar{w}} \tilde{x}(p, w) dw$$

satisfies the unrestricted law of demand (ULD) property.

## 4.D Aggregate Demand and the Existence of a Representative Consumer

**Definition 4.D.1.** A *positive representative consumer* exists if there is a rational preference relation  $\succsim$  on  $\mathbb{R}_+^L$  such that the aggregate demand function  $x(p, w)$  is precisely the Walrasian demand function generated by this preference relation. That is,  $x(p, w) \succ x$  whenever  $x \neq x(p, w)$  and  $p \cdot x \leq w$ .

**Definition 4.D.2.** A (*Berson-Samuelson*) *social welfare function* is a function  $W : \mathbb{R}^I \rightarrow \mathbb{R}$  that assigns a utility value to each possible vector  $(u_1, \dots, u_I) \in \mathbb{R}^I$  of utility levels for the  $I$  consumers in the economy.

**Proposition 4.D.1.** Suppose that for each level of prices  $p$  and aggregate wealth  $w$ , the wealth distribution  $w_1(p, w), \dots, w_I(p, w)$  solves

$$\begin{aligned} \max_{w_1, \dots, w_I} \quad & W(v_1(p, w_1), \dots, v_I(p, w_I)) \\ \text{s.t.} \quad & \sum_{i=1}^I w_i \leq w. \end{aligned} \tag{4.D.1}$$

Then the value function  $v(p, w)$  of problem (4.D.1) is an indirect utility function of a positive representative consumer for the aggregate demand function  $x(p, w) = \sum_i x_i(p, w_i(p, w))$ .

**Definition 4.D.3.** The positive representative consumer  $\succsim$  for the aggregate demand  $x(p, w) = \sum_i x_i(p, w_i(p, w))$  is a *normative representative consumer* relative to the social welfare function  $W(\cdot)$  if for every  $(p, w)$ , the distribution of wealth  $w_1(p, w), \dots, w_I(p, w)$  solves problem (4.D.1) and, therefore, the value function of problem (4.D.1) is an indirect utility function for  $\succsim$ .



# Chapter 5

## Production

### 5.B Production Sets

**Proposition 5.B.1.** The production set  $Y$  is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.

**Proposition 5.B.2.** For any convex production set  $Y \subset \mathbb{R}^L$  with  $0 \in Y$ , there is a constant returns, convex production set  $Y' \subset \mathbb{R}^{L+1}$  such that  $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$ .

### 5.C Profit Maximisation and Cost Minimisation

**Proposition 5.C.1.** Suppose that  $\pi(\cdot)$  is the profit function of the production set  $Y$  and that  $y(\cdot)$  is the associated supply correspondence. Assume also that  $Y$  is closed and satisfies the free disposal property. Then

- (i)  $\pi(\cdot)$  is homogeneous of degree one.
- (ii)  $\pi(\cdot)$  is convex.
- (iii) If  $Y$  is convex, then  $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$ .
- (iv)  $y(\cdot)$  is homogeneous of degree zero.
- (v) If  $Y$  is convex, then  $y(p)$  is a convex set for all  $p$ . Moreover, if  $Y$  is strictly convex, then  $y(p)$  is single-valued (if nonempty).
- (vi) (*Hotelling's lemma*) If  $y(\bar{p})$  consists of a single point, then  $\pi(\cdot)$  is differentiable at  $\bar{p}$  and  $\nabla \pi(\bar{p}) = y(\bar{p})$ .
- (vii) If  $y(\cdot)$  is a function differentiable at  $\bar{p}$ , then  $Dy(\bar{p}) = D^2 \pi(\bar{p})$  is a symmetric and positive semidefinite matrix with  $Dy(\bar{p})\bar{p} = 0$ .

**Proposition 5.C.2.** Suppose that  $c(p, w)$  is the cost function of a single-output technology  $Y$  with production function  $f(\cdot)$  and that  $z(w, q)$  is the associated conditional factor demand correspondence. Assume also that  $Y$  is closed and satisfies the free disposal property. Then

- (i)  $c(\cdot)$  is homogeneous of degree one in  $w$  and nondecreasing in  $q$ .
- (ii)  $c(\cdot)$  is a concave function of  $w$ .

- (iii) If the sets  $\{z \geq 0 : f(z) \geq q\}$  are convex for every  $q$ , then  $Y = \{(-z, q) : w \cdot z \geq c(w, q) \text{ for all } w \gg 0\}$ .
- (iv)  $z(\cdot)$  is homogeneous of degree zero in  $w$ .
- (v) If the set  $\{z \geq 0 : f(z) \geq q\}$  is convex, then  $z(w, q)$  is a convex set. Moreover, if  $\{z \geq 0 : f(z) \geq q\}$  is a strictly convex set, then  $z(p, w)$  is single-valued.
- (vi) (*Shephard's lemma*) If  $z(\bar{w}, q)$  consists of a single point, then  $c(\cdot)$  is differentiable with respect to  $w$  at  $\bar{w}$  and  $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$ .
- (vii) If  $z(\cdot)$  is differentiable at  $\bar{w}$ , then  $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$  is a symmetric and negative semi-definite matrix with  $D_w z(\bar{w}, q) \bar{w} = 0$ .
- (viii) If  $f(\cdot)$  is homogeneous of degree one (i.e. exhibits constant returns to scale), then  $c(\cdot)$  and  $z(\cdot)$  are homogeneous of degree one in  $q$ .
- (ix) If  $f(\cdot)$  is concave, then  $c(\cdot)$  is a convex function of  $q$  (in particular, marginal costs are nondecreasing in  $q$ ).

## 5.E Aggregation

**Proposition 5.E.1.** For all  $p \gg 0$ , we have

- (i)  $\pi^*(p) = \sum_j \pi_j(p)$
- (ii)  $y^*(p) = \sum_j y_j(p)$  ( $= \{\sum_j y_j : y_j \in y_j(p) \text{ for every } j\}$ ).

## 5.F Efficient Production

**Definition 5.F.1.** A production vector  $y \in Y$  is *efficient* if there is no  $y' \in Y$  such that  $y' \geq y$  and  $y' \neq y$ .

**Proposition 5.F.1.** If  $y \in Y$  is profit maximising for some  $p \gg 0$ , then  $y$  is efficient.

**Proposition 5.F.2.** Suppose that  $Y$  is convex. Then every efficient production  $y \in Y$  is a profit-maximising production for some nonzero price vector  $p \geq 0$ .

## Chapter 6

# Choice Under Uncertainty

### 6.B Expected Utility Theory

**Definition 6.B.1.** A *simple lottery*  $L$  is a list  $L = (p_1, \dots, p_N)$  with  $p_n \geq 0$  for all  $n$  and  $\sum_n p_n = 1$ , where  $p_n$  is interpreted as the probability of outcome  $n$  occurring.

**Definition 6.B.2.** Given  $K$  simple lotteries  $L_k = (p_1^k, \dots, p_N^k)$ ,  $k = 1, \dots, K$ , and probabilities  $\alpha_k \geq 0$  with  $\sum_k \alpha_k = 1$ , the *compound lottery*  $(L_1, \dots, L_K; \alpha_1, \dots, \alpha_K)$  is the risky alternative that yields the simple lottery  $L_k$  with probability  $\alpha_k$  for  $k = 1, \dots, K$ .

**Definition 6.B.3.** The preference relation  $\succsim$  on the space of simple lotteries  $\mathcal{L}$  is *continuous* if for any  $L, L', L'' \in \mathcal{L}$ , the sets

$$\{\alpha \in [0, 1] : \alpha L + (1 - \alpha)L' \succsim L''\} \subset [0, 1]$$

and

$$\{\alpha \in [0, 1] : L'' \succsim \alpha L + (1 - \alpha)L'\} \subset [0, 1]$$

are closed.

**Definition 6.B.4.** The preference relation  $\succsim$  on the space simple lotteries  $\mathcal{L}$  satisfies the *independence axiom* if for all  $L, L', L'' \in \mathcal{L}$  and  $\alpha \in (0, 1)$  we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

**Definition 6.B.5.** The utility function  $U : \mathcal{L} \rightarrow \mathbb{R}$  has an *expected utility form* if there is an assignment of numbers  $(u_1, \dots, u_N)$  to the  $N$  outcomes such that for every simple lottery  $L = (p_1, \dots, p_N) \in \mathcal{L}$  we have

$$U(L) = u_1 p_1 + \dots + u_N p_N.$$

A utility function  $U : \mathcal{L} \rightarrow \mathbb{R}$  with the expected utility form is called a *von Neumann-Morgenstern (v.N-M) expected utility function*.

**Proposition 6.B.1.** A utility function  $U : \mathcal{L} \rightarrow \mathbb{R}$  has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U \left( \sum_{k=1}^K \alpha_k L_k \right) = \sum_{k=1}^K \alpha_k U(L_k)$$

for any  $K$  lotteries  $L_k \in \mathcal{L}$ ,  $k = 1, \dots, K$ , and probabilities  $(\alpha_1, \dots, \alpha_K) \geq 0, \sum_k \alpha_k = 1$ .

**Proposition 6.B.2.** Suppose that  $U : \mathcal{L} \rightarrow \mathbb{R}$  is a v.N-M expected utility function for the preference relation  $\succsim$  on  $\mathcal{L}$ . Then  $\tilde{U} : \mathcal{L} \rightarrow \mathbb{R}$  is another v.N-M utility function for  $\succsim$  if and only if there are scalars  $\beta > 0$  and  $\gamma$  such that  $\tilde{U}(L) = \beta U(L) + \gamma$  for every  $L \in \mathcal{L}$ .

**Proposition 6.B.3** (Expected Utility Theorem). Suppose that the rational preference relation  $\succsim$  on the space of lotteries  $\mathcal{L}$  satisfies the continuity and independence axioms. Then  $\succsim$  admits a utility representation of the expected utility form. That is, we can assign a number  $u_n$  to each outcome  $n = 1, \dots, N$  in such a manner that for any two lotteries  $L = (p_1, \dots, p_N)$  and  $L' = (p'_1, \dots, p'_N)$  we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n=1}^N u_n p_n \geq \sum_{n=1}^N u_n p'_n.$$

## 6.C Money Lotteries and Risk Aversion

**Definition 6.C.1.** A decision maker is a *risk averse* (or exhibits *risk aversion*) if for any lottery  $F(\cdot)$ , the degenerate lottery that yields the amount  $\int x dF(x)$  with certainty is at least as good as the lottery  $F(\cdot)$  itself. If the decision maker is always [i.e. for any  $F(\cdot)$ ] indifferent between these two lotteries, we say that he is *risk neutral*. Finally, we say that he is *strictly risk averse* if indifference holds only when the two lotteries are the same [i.e. when  $F(\cdot)$  is degenerate].

**Definition 6.C.2.** Given a Bernoulli utility function  $u(\cdot)$  we defined the following concepts:

- (i) The *certainty equivalent* of  $F(\cdot)$ , denoted  $c(F, u)$ , is the amount of money for which the individual is indifferent between the gamble  $F(\cdot)$  and the certain amount  $c(F, u)$ ; that is

$$u(c(F, u)) = \int u(x) dF(x).$$

- (ii) For any fixed amount of money  $x$  and positive number  $\varepsilon$ , the *probability premium* denoted by  $\pi(x, \varepsilon, u)$ , is the excess on winning the probability over fair odds that makes the individual indifferent between the certain outcome  $x$  and a gamble between the two outcomes  $x + \varepsilon$  and  $x - \varepsilon$ . That is

$$u(x) = \left( \frac{1}{2} + \pi(x, \varepsilon, u) \right) u(x + \varepsilon) + \left( \frac{1}{2} - \pi(x, \varepsilon, u) \right) u(x - \varepsilon).$$

**Proposition 6.C.1.** Suppose a decision maker is an expected utility maximiser with a Bernoulli utility function  $u(\cdot)$  on amounts of money. Then the following properties are equivalent:

- (i) The decision maker is risk averse.
- (ii)  $u(\cdot)$  is concave.
- (iii)  $c(F, u) \leq \int x dF(x)$  for all  $F(\cdot)$ .
- (iv)  $\pi(x, \varepsilon, u) \geq 0$  for all  $x, \varepsilon$ .

**Definition 6.C.3.** Given a (twice differentiable) Bernoulli utility function  $u(\cdot)$  for money, the *Arrow Pratt coefficient of absolute risk aversion* at  $x$  is defined as  $r_A(x) = -u''(x)/u'(x)$ .

**Definition** (More-risk-averse-than). Given two Bernoulli utility functions  $u_1(\cdot)$  and  $u_2(\cdot)$ , when can we say that  $u_2(\cdot)$  is unambiguously *more risk averse than*  $u_1(\cdot)$ ? Several possible approaches to a definition seem plausible:

- (i)  $r_A(x, u_2) \geq r_A(x, u_1)$  for every  $x$ .
- (ii) There exists an increasing concave function  $\psi(\cdot)$  such that  $u_2(x) = \psi(u_1(x))$  at all  $x$ ; that is,  $u_2(\cdot)$  is a concave transformation of  $u_1(\cdot)$ . [In other words,  $u_2(\cdot)$  is “more concave” than  $u_1(\cdot)$ .]
- (iii)  $c(F, u_2) \leq c(F, u_1)$  for any  $F(\cdot)$ .
- (iv)  $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$  for any  $x$  and  $\varepsilon$ .
- (v) Whenever  $u_2(\cdot)$  finds a lottery  $F(\cdot)$  at least as good as a riskless outcome  $\bar{x}$ , then  $u_1(\cdot)$  also finds  $F(\cdot)$  at least as good as  $\bar{x}$ . That is,  $\int u_2(x)dF(x) \geq u_2(\bar{x})$  implies  $\int u_1(x)dF(x) \geq u_1(\bar{x})$  for any  $F(\cdot)$  and  $\bar{x}$ .

**Proposition 6.C.2.** Definitions (i) to (v) of the *more-risk-averse-than* relation are equivalent.

**Definition 6.C.4.** The Bernoulli utility function  $u(\cdot)$  for money exhibits *decreasing absolute risk aversion* if  $r_A(x, u)$  is a decreasing function of  $x$ .

**Proposition 6.C.3.** The following properties are equivalent:

- (i) The Bernoulli utility function  $u(\cdot)$  exhibits decreasing absolute risk aversion.
- (ii) Whenever  $x_2 < x_1$ ,  $u_2(z) = u(x_2 + z)$  is a concave transformation of  $u_1(z) = u(x_1 + z)$ .
- (iii) For any risk  $F(z)$ , the certainty equivalent of the lottery formed adding risk  $z$  to wealth level  $x$ , given by the amount  $c_x$  at which  $u(c_x) = \int u(x + z)dF(z)$ , is such that  $(x - c_x)$  is decreasing in  $x$ . That is, the higher  $x$  is, the less is the individual willing to pay to get rid of the risk.
- (iv) The probability premium  $\pi(x, \varepsilon, u)$  is decreasing in  $x$ .
- (v) For any  $F(z)$ , if  $\int u(x_2 + z)dF(z) \geq u(x_2)$  and  $x_2 < x_1$ , then  $\int u(x_1 + z)dF(z) \geq u(x_1)$ .

**Definition 6.C.5.** Given a Bernoulli utility function  $u(\cdot)$ , the *coefficient of relative risk aversion* at  $x$  is  $r_R(x, u) = -xu''(x)/u'(x)$ .

**Proposition 6.C.4.** The following conditions for a Bernoulli utility function  $u(\cdot)$  on amounts of money are equivalent:

- (i)  $r_R(x, u)$  is decreasing in  $x$ .
- (ii) Whenever  $x_2 < x_1$ ,  $\tilde{u}_2(t) = u(tx_2)$  is a concave transformation of  $\tilde{u}_1(t) = u(tx_1)$ .
- (iii) Given any risk  $F(t)$  on  $t > 0$ , the certainty equivalent  $\bar{c}_x$  defined by  $u(\bar{c}_x) = \int u(tx)dF(t)$  is such that  $x/\bar{c}_x$  is decreasing in  $x$ .

## 6.D Comparison of Payoff Distributions in Terms of Return and Risk

**Definition 6.D.1.** The distribution  $F(\cdot)$  *first-order stochastically dominates*  $G(\cdot)$  if, for every nondecreasing function  $u : \mathbb{R} \rightarrow \mathbb{R}$ , we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

**Proposition 6.D.1.** The distribution of monetary payoffs  $F(\cdot)$  first-order stochastically dominates the distribution  $G(\cdot)$  if and only if  $F(x) \leq G(x)$  for every  $x$ .

**Definition 6.D.2.** For any two distributions  $F(\cdot)$  and  $G(\cdot)$  with the same mean,  $F(\cdot)$  *second-order stochastically dominates* (or *is less risky than*)  $G(\cdot)$  if for every nondecreasing concave function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we have

$$\int u(x) dF(x) \geq \int u(x) dG(x).$$

**Proposition 6.D.2.** Consider two distributions  $F(\cdot)$  and  $G(\cdot)$  with the same mean. Then the following statements are equivalent:

- (i)  $F(\cdot)$  second-order stochastically dominates  $G(\cdot)$ .
- (ii)  $G(\cdot)$  is a mean-preserving spread of  $F(\cdot)$ .
- (iii) Property 6.D.2 holds.

## 6.E State-Dependent Utility

**Definition 6.E.1.** A *random variable* is a function  $g : S \rightarrow \mathbb{R}_+$  that maps states into monetary outcomes.

**Definition 6.E.2.** The preference relation  $\succsim$  has an *extended expected utility representation* if for every  $s \in S$ , there is a function  $u_s : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that for any  $(x_1, \dots, x_S) \in \mathbb{R}_+^S$  and  $(x'_1, \dots, x'_S) \in \mathbb{R}_+^S$ ,

$$(x_1, \dots, x_S) \succ (x'_1, \dots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x'_s).$$

**Definition 6.E.3.** The preference relation  $\succsim$  on  $\mathcal{L}$  satisfies the *extended independence axiom* if for all  $L, L', L'' \in \mathcal{L}$  and  $\alpha \in (0, 1)$  we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

**Proposition 6.E.1** (Extended Expected Utility Theorem). Suppose that the preference relation  $\succsim$  on the space of lotteries  $\mathcal{L}$  satisfies the continuity and extended independence axioms. Then we can assign a utility function  $u_s(\cdot)$  for money in every state  $s$  such that for any  $L = (F_1, \dots, F_S)$  and  $L' = (F'_1, \dots, F'_S)$ , we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_s \left( \int u_s(x_s) dF_s(x_s) \right) \geq \sum_s \left( \int u_s(x_s) dF'_s(x_s) \right).$$

**Definition 6.E.4.** The preference relation  $\succsim$  satisfies the *sure-thing axiom* if, for any subset of states  $E \subset S$  ( $E$  is called an *event*), whenever  $(x_1, \dots, x_S)$  and  $(x'_1, \dots, x'_S)$  differ only in the entries corresponding to  $E$  (so that  $x'_s = x_s$  for  $s \notin E$ ), the preference ordering between  $(x_1, \dots, x_S)$  and  $(x'_1, \dots, x'_S)$  is independent of the particular (common) payoffs for states not in  $E$ . Formally, suppose that  $(x_1, \dots, x_S), (x'_1, \dots, x'_S), (\bar{x}_1, \dots, \bar{x}_S)$ , and  $(\bar{x}'_1, \dots, \bar{x}'_S)$  are such that

$$\begin{aligned} \text{For all } s \notin E : \quad & x_s = x'_s \quad \text{and} \quad \bar{x}_s = \bar{x}'_s. \\ \text{For all } s \in E : \quad & x_s = \bar{x}_s \quad \text{and} \quad x'_s = \bar{x}'_s. \end{aligned}$$

Then  $(x_1, \dots, x_S) \succsim (\bar{x}_1, \dots, \bar{x}_S)$  if and only if  $(x_1, \dots, x_S) \succsim (x'_1, \dots, x'_S)$ .

**Proposition 6.E.2.** Suppose that there are at least three states and that the preferences  $\succsim$  on  $\mathbb{R}_+^S$  are continuous and satisfy the sure-thing axiom. Then  $\succsim$  admits an extended expected utility representation.

## 6.F Subjective Probability Theory

**Definition 6.F.1.** The state preferences  $(\succsim_1, \dots, \succsim_S)$  on state lotteries are *state uniform* if  $\succsim_s = \succsim_{s'}$  for any  $s$  and  $s'$ .

**Proposition 6.F.1** (Subjective Expected Utility Theorem). Suppose that the preference relation  $\succsim$  on  $\mathcal{L}$  satisfies the continuity and extended independence axioms. Suppose, in addition, that the derived state preferences are state uniform. Then there are probabilities  $(\pi_1, \dots, \pi_S) \gg 0$  and a utility function  $u(\cdot)$  on amounts of money such that for any  $(x_1, \dots, x_S)$  and  $(x'_1, \dots, x'_S)$  we have

$$(x_1, \dots, x_S) \succsim (x'_1, \dots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u(x_s) \geq \sum_s \pi_s u(x'_s).$$

**Part II**

**Game Theory**



## Chapter 7

# Basic Elements of Noncooperative Games

### 7.C The Extensive Form Representation of a Game

**Definition 7.C.1.** A game is one of *perfect information* if each information set contains a single decision node. Otherwise, it is a game of *imperfect information*.

### 7.D Strategies and the Normal Form Representation of a Game

**Definition 7.D.1.** Let  $\mathcal{H}_i$  denote the collection of player  $i$ 's information sets,  $\mathcal{A}$  the set of possible actions in the game, and  $C(H) \subset \mathcal{A}$  the set of actions possible at information set  $H$ . A *strategy* for player  $i$  is a function  $s_i : \mathcal{H}_i \rightarrow \mathcal{A}$  such that  $s_i(H) \in C(H)$  for all  $H \in \mathcal{H}_i$ .

**Definition 7.D.2.** For a game with  $I$  players, the *normal form representation*  $\Gamma_N$  specifies for each player  $i$  a set of strategies  $S_i$  (with  $s_i \in S_i$ ) and a payoff function  $u_i(s_1, \dots, s_I)$  giving the von Neumann-Morgenstern utility levels associated with the (possibly random) outcome arising from strategies  $s_1, \dots, s_I$ . Formally, we write  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ .

### 7.E Randomized Choices

**Definition 7.E.1.** Given player  $i$ 's (finite) pure strategy set  $S_i$ , a *mixed strategy* for player  $i$ ,  $\sigma_i : S_i \rightarrow [0, 1]$ , assigns to each pure strategy  $s_i \in S_i$  a probability  $\sigma_i(s_i) \geq 0$  that it will be played, where  $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$ .

**Definition 7.E.2.** Given an extensive form game  $\Gamma_E$ , a *behaviour strategy* for player  $i$  specifies, for every information set  $H \in \mathcal{H}_i$  and action  $a \in C(H)$ , a probability  $\lambda_i(a, H) \geq 0$ , with  $\sum_{a \in C(H)} \lambda_i(a, H) = 1$  for all  $H \in \mathcal{H}_i$ .

## Chapter 8

# Simultaneous-Move Games

### 8.B Dominant and Dominated Strategies

**Definition 8.B.1.** A strategy  $s_i \in S_i$  is a *strictly dominant strategy* for player  $i$  in game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if for all  $s'_i \neq s_i$ , we have

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

for all  $s_{-i} \in S_{-i}$ .

**Definition 8.B.2.** A strategy  $s_i \in S_i$  is a *strictly dominated* for player  $i$  in game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if there exists another strategy  $s'_i \in S_i$  such that for all  $s_{-i} \in S_{-i}$ ,

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}).$$

In this case, we say that strategy  $s'_i$  *strictly dominates* strategy  $s_i$ .

**Definition 8.B.3.** A strategy  $s_i \in S_i$  is a *weakly dominated* for player  $i$  in game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if there exists another strategy  $s'_i \in S_i$  such that for all  $s_{-i} \in S_{-i}$ ,

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}),$$

with strict inequality for some  $s_{-i}$ . In this case, we say that strategy  $s'_i$  *weakly dominates* strategy  $s_i$ . A strategy is a *weakly dominant strategy* for player  $i$  in game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if it weakly dominates every other strategy in  $S_i$ .

**Definition 8.B.4.** A strategy  $\sigma_i \in \Delta(S_i)$  is *strictly dominated* for player  $i$  in game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if there exists another strategy  $\sigma'_i \in \Delta(S_i)$  such that for all  $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$ ,

$$u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

In this case, we say that strategy  $\sigma'_i$  *strictly dominates* strategy  $\sigma_i$ . A strategy  $\sigma_i$  is a *strictly dominant strategy* for player  $i$  in game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$  if it strictly dominates every other strategy in  $\Delta(S_i)$ .

**Proposition 8.B.1.** Player  $i$ 's pure strategy  $s_i \in S_i$  is strictly dominated in game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$  if and only if there exists another strategy  $\sigma'_i \in \Delta(S_i)$  such that

$$u_i(\sigma'_i, s_{-i}) > u_i(s_i, s_{-i})$$

for all  $s_{-i} \in S_{-i}$ .

## 8.C Rationalisable Strategies

**Definition 8.C.1.** In game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ , a strategy  $\sigma_i$  is a *best response* for player  $i$  to his rivals' strategies  $\sigma_{-i}$  if

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma'_i, \sigma_{-i})$$

for all  $\sigma'_i \in \Delta(S_i)$ . Strategy  $\sigma_i$  is *never a best response* if there is no  $\sigma_{-i}$  for which  $\sigma_i$  is a best response.

**Definition 8.C.2.** In game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ , the strategies in  $\Delta(S_i)$  that survive the iterated removal of strategies that are never a best response are known as player  $i$ 's *rationalisable strategies*.

## 8.D Nash Equilibrium

**Definition 8.D.1.** A strategy profile  $s = (s_1, \dots, s_I)$  constitutes a *Nash equilibrium* of game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if for every  $i = 1, \dots, I$ ,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

for all  $s'_i \in S_i$ .

**Definition 8.D.2.** A mixed strategy profile  $\sigma = \sigma_1, \dots, \sigma_I$  constitutes a *Nash equilibrium* of game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$  if for every  $i = 1, \dots, I$ ,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$$

for all  $\sigma'_i \in \Delta(S_i)$ .

**Proposition 8.D.1.** Let  $S_i^+ \subset S_i$  denote the set of pure strategies that player  $i$  plays with positive probability in mixed strategy profile  $\sigma = \sigma_1, \dots, \sigma_I$ . Strategy profile  $\sigma$  is a Nash equilibrium in game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$  if and only if for all  $i = 1, \dots, I$ ,

- (i)  $u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i})$  for all  $s_i, s'_i \in S_i^+$
- (ii)  $u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})$  for all  $s_i \in S_i^+$  and all  $s'_i \notin S_i^+$ .

**Corollary 8.D.1.** Pure strategy profile  $s = (s_1, \dots, s_I)$  is a Nash equilibrium of game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if and only if it is a (degenerate) mixed strategy Nash equilibrium game of  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ .

**Proposition 8.D.2.** Every game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$  in which sets  $S_1, \dots, S_I$  have a finite number of elements has a mixed strategy Nash equilibrium.

**Proposition 8.D.3.** A Nash equilibrium exists in game  $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$  if for all  $i = 1, \dots, I$ ,

- (i)  $S_i$  is a nonempty, convex, and compact subset of some Euclidean space  $\mathbb{R}^M$ .
- (ii)  $u_i(s_1, \dots, s_I)$  is continuous in  $(s_1, \dots, s_I)$  and quasiconcave in  $s_i$ .

## 8.E Games of Incomplete Information: Bayesian Nash Equilibrium

**Definition 8.E.1.** A (pure strategy) *Bayesian Nash equilibrium* for the Bayesian game  $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$  is a profile of decision rules  $(s_1(\cdot), \dots, s_I(\cdot))$  that constitutes a Nash equilibrium of game  $\Gamma_N = [I, \{\mathcal{L}_i\}, \{\tilde{u}_i(\cdot)\}]$ . That is, for every  $i = 1, \dots, I$ ,

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s'_i(\cdot), s_{-i}(\cdot))$$

for all  $s'_i(\cdot) \in \mathcal{L}_i$ , where

$$\tilde{u}_i(s_1(\cdot), \dots, s_I(\cdot)) = E_\theta [u_i(s_1(\theta_1), \dots, s_I(\theta_I), \theta_i)].$$

**Proposition 8.E.1.** A profile of decision rules  $(s_1(\cdot), \dots, s_I(\cdot))$  is a Bayesian Nash equilibrium in Bayesian game  $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$  if and only if for all  $i$  and all  $\bar{\theta}_i \in \Theta_i$  occurring with positive probability

$$E_{\theta_{-i}} [u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) | \bar{\theta}_i] \geq E_{\theta_{-i}} [u_i(s'_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) | \bar{\theta}_i]$$

for all  $s'_i \in S_i$ , where the expectation is taken over realisations of the other players' random variables conditional on player  $i$ 's realisation of his signal  $\bar{\theta}_i$ .

## 8.F The Possibility of Mistakes: Trembling-Hand Perfection

**Definition 8.F.1.** A Nash equilibrium  $\sigma$  of a game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$  is (*normal form*) *trembling-hand perfect* if there is *some* sequence of perturbed games  $\{\Gamma_{\varepsilon^k}\}_{k=1}^\infty$  that converges to  $\Gamma_N$  [in the sense that  $\lim_{k \rightarrow \infty} \varepsilon_i^k(s_i) = 0$  for all  $i$  and  $s_i \in S_i$ ], for which there is *some* associated sequence of Nash equilibria  $\{\sigma^k\}_{k=1}^\infty$  that converges to  $\sigma$  (i.e., such that  $\lim_{k \rightarrow \infty} \sigma^k = \sigma$ ).

**Proposition 8.F.1.** A Nash equilibrium  $\sigma$  of a game  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$  is (normal form) trembling-hand perfect if and only if there is some sequence of totally mixed strategies  $\{\sigma^k\}_{k=1}^\infty$  such that  $\lim_{k \rightarrow \infty} \sigma^k = \sigma$  and  $\sigma_i$  is the best response to every element of sequence  $\{\sigma_{-i}^k\}_{k=1}^\infty$  for all  $i = 1, \dots, I$ .

**Proposition 8.F.2.** If  $\sigma = (\sigma_1, \dots, \sigma_I)$  is a (normal form) trembling-hand perfect Nash equilibrium, then  $\sigma_i$  is not a weakly dominated strategy for any  $i = 1, \dots, I$ . Hence, in any (normal form) trembling-hand perfect Nash equilibrium, no weakly dominated pure strategy can be played with positive probability.

## Chapter 9

# Dynamic Games

## Part III

# Market Equilibrium and Market Failure

# Chapter 10

## Competitive Markets

### 10.B Pareto Optimality and Competitive Equilibria

**Definition 10.B.1.** An *economic allocation*  $(x_1, \dots, x_I, y_1, \dots, y_J)$  is a specification of a consumption vector  $x_i \in X_i$  for each consumer  $i = 1, \dots, I$  and a production vector  $y_j \in Y_j$  for each firm  $j = 1, \dots, J$ . The allocation  $(x_1, \dots, x_I, y_1, \dots, y_J)$  is *feasible* if

$$\sum_{i=1}^I x_{\ell i} \leq w_{\ell} + \sum_{j=1}^J y_{\ell j} \quad \text{for } \ell = 1, \dots, L.$$

**Definition 10.B.2.** A feasible allocation  $(x_1, \dots, x_I, y_1, \dots, y_J)$  is *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation  $(x'_1, \dots, x'_I, y'_1, \dots, y'_J)$  such that  $u_i(x'_i) \geq u_i(x_i)$  for all  $i = 1, \dots, I$  and  $u_i(x'_i) > u_i(x_i)$  for some  $i$ .

**Definition 10.B.3.** The allocation  $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$  and price vector  $p^* \in \mathbb{R}^L$  constitute a *competitive* (or *Walrasian*) *equilibrium* if the following conditions are satisfied:

(i) *Profit maximisation:* For each firm  $j$ ,  $y_j^*$  solves

$$\max_{y_j \in Y_{ij}} p^* \cdot y_j. \quad (10.B.1)$$

(ii) *Utility maximisation:* For each consumer  $i$ ,  $x_i^*$  solves

$$\begin{aligned} \max_{x_i \in X_i} u_i(x_i) \\ \text{s.t. } p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^J \theta_{ij}(p^* \cdot y_j^*). \end{aligned} \quad (10.B.2)$$

(iii) *Market clearing:* For each good  $\ell = 1, \dots, L$ ,

$$\sum_{i=1}^I x_{\ell i}^* = \omega_{\ell} + \sum_{j=1}^J y_{\ell j}^*. \quad (10.B.3)$$

**Lemma 10.B.1.** If the allocation  $(x_1, \dots, x_I, y_1, \dots, y_J)$  and price vector  $p \gg 0$  satisfy the market clearing condition (Definition 10.B.3) for all goods  $\ell \neq k$ , and if every consumer's budget constraint is satisfied with equality, so that  $p \cdot x_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j$  for all  $i$ , then the market for good  $k$  also clears.

## 10.D The Fundamental Welfare Theorems in a Partial Equilibrium Context

**Proposition 10.D.1** (The First Fundamental Theorem of Welfare Economics). If the price  $p^*$  and allocation  $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$  constitutes a competitive equilibrium, then this allocation is Pareto optimal.

**Proposition 10.D.2** (The Second Fundamental Theorem of Welfare Economics). For any Pareto optimal levels of utility  $(u_1^*, \dots, u_I^*)$ , there are transfers of the numeraire commodity  $(T_1, \dots, T_I)$  satisfying  $\sum_i T_i = 0$ , such that a competitive equilibrium reached from the endowments  $\omega_{m1} + T_1, \dots, \omega_{mI} + T_I$  yields precisely the utilities  $(u_1^*, \dots, u_I^*)$ .

## 10.F Free Entry and Long-Run Competitive Equilibria

**Definition 10.F.1.** Given an aggregate demand function  $x(p)$  and a cost function  $c(q)$  for each potentially active firm having  $c(0) = 0$ , a triple  $(p^*, q^*, J^*)$  is a *long-run competitive equilibrium* if

- (i)  $q^*$  solves  $\max_{q \geq 0} p^* q - c(q)$  (Profit maximisation)
- (ii)  $x(p^*) = J^* q^*$  (Demand = supply)
- (iii)  $p^* q^* - c(q^*) = 0$  (Free Entry Condition).



## Chapter 11

# Externalities and Public Goods

### 11.B A Simple Bilateral Externality

**Definition 11.B.1.** An *externality* is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of another agent in the economy.

### 11.C Public Goods

**Definition 11.C.1.** A *public good* is a commodity for which use of a unit of the good by one agent does not preclude use by other agents.

# Chapter 12

## Market Power

### 12.C Static Models of Oligopoly

**Proposition 12.C.1.** There is a unique Nash equilibrium  $(p_1^*, p_2^*)$  in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost:  $p_1^* = p_2^* = c$ .

**Proposition 12.C.2.** In any Nash equilibrium of the Cournot duopoly model with cost  $c > 0$  per unit for the two firms and an inverse demand function  $p(\cdot)$  satisfying  $p'(q) < 0$  for all  $q \geq 0$  and  $p(0) > c$ , the market price is greater than  $c$  (the competitive price) and smaller than the monopoly price.

### 12.D Repeated Interaction

**Proposition 12.D.1.** The strategies

$$p_{jt}(H_{t-1}) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ equal } (p^m, p^m) \text{ or } t = 1 \\ c & \text{otherwise} \end{cases}$$

constitute a subgame perfect Nash equilibrium (SPNE) of the infinitely repeated Bertrand duopoly game if and only if  $\delta \geq \frac{1}{2}$  in the firms optimisation problem

$$\max \sum_{t=1}^{\infty} \delta^{t-1} \pi_{jt}, \quad \delta < 1.$$

**Proposition 12.D.2.** In the infinitely repeated Bertrand duopoly game, when  $\delta \geq \frac{1}{2}$  repeated choice of any price  $p \in [c, p^m]$  can be supported as a subgame perfect Nash equilibrium outcome path using Nash reversion strategies. By contrast, when  $\delta < \frac{1}{2}$ , any subgame perfect Nash equilibrium outcome path must have all sales occurring at a price equal to  $c$  in every period.

### 12.E Entry

**Proposition 12.E.1.** Suppose that conditions

(A1)  $Jq_J \geq J'q_{J'}$  whenever  $J > J'$ ;

(A2)  $q_J \leq q_{J'}$  whenever  $J > J'$ ;

(A3)  $p(Jq_J) - c'(q_J) \geq 0$  for all  $J$

are satisfied by the post-entry oligopoly game, that  $p'(\cdot) < 0$ , and that  $c''(\cdot) \geq 0$ . Then the equilibrium number of entrants  $J^*$ , is at least  $J^\circ - 1$ , where  $J^\circ$  is the socially optimal number of entrants.

## 12.F The Competitive Limit

**Proposition 12.F.1.** As the market size grows, the price in any subgame perfect Nash equilibrium of the two-stage Cournot entry model converges to the level of minimum average cost (the “competitive” price). Formally,

$$\max_{p_\alpha \in P_\alpha} |p_\alpha - \bar{c}| \rightarrow 0 \text{ as } \alpha \rightarrow \infty.$$

## Chapter 13

# Adverse Selection, Signaling, and Screening

### 13.B Informational Asymmetries and Adverse Selection

**Definition 13.B.1.** In the competitive labour market model with unobservable worker productivity levels, a *competitive equilibrium* is a wage rate  $w^*$  and a set  $\Theta^*$  of worker types who accept employment such that

$$\Theta^* = \{\theta : r(\theta) \leq w^*\}$$

and

$$w^* = E[\theta | \theta \in \Theta^*].$$

**Proposition 13.B.1.** Let  $W^*$  denote the set of competitive equilibrium wages for the adverse selection labour market model, and let  $W^* = \max\{w : w \in W^*\}$ .

- (i) If  $w^* > r(\underline{\theta})$  and there is an  $\varepsilon > 0$  such that  $E[\theta | r(\theta) < w'] > w'$  for all  $w' \in (w^* - \varepsilon, w^*)$ , then there is a unique pure strategy SPNE of the two-stage game-theoretic model. In this SPNE, employed workers receive a wage of  $w^*$ , and workers with types in the set  $\Theta(w^*) = \{\theta : r(\theta) \leq w^*\}$  accept employment in firms.
- (ii) If  $w^* = r(\underline{\theta})$ , then there are multiple pure strategy SPNEs. However, in every pure strategy SPNE each agent's payoff exactly equals her payoff in the highest-wage competitive equilibrium.

**Proposition 13.B.2.** In the adverse selection labour market model (where  $r(\cdot)$  is strictly increasing with  $r(\theta) \leq \theta$  for all  $\theta \in [\underline{\theta}, \bar{\theta}]$  and  $F(\cdot)$  has an associated density  $f(\cdot)$  with  $f(\theta) > 0$  for all  $\theta \in [\underline{\theta}, \bar{\theta}]$ ), the highest-wage competitive equilibrium is a constrained Pareto optimum.

### 13.C Signaling

**Lemma 13.C.1.** In any separating perfect Bayesian equilibrium,  $w^*(e^*(\theta_H)) = \theta_H$  and  $w^*(e^*(\theta_L)) = \theta_L$ ; that is, each worker type receives a wage equal to her productivity level.

**Lemma 13.C.2.** In any separating perfect Bayesian equilibrium,  $e^*(\theta_L) = 0$ ; that is, a low-ability worker chooses to get no education.

## 13.D Screening

**Proposition 13.D.1.** In any SPNE of the screening game with observable worker types, a type  $\theta_i$  worker accepts contract  $(w_i^*, t_i^*) = (\theta_i, 0)$ , and firms earn zero profits.

**Lemma 13.D.1.** In any equilibrium, whether pooling or separating, both firms must earn zero profits.

**Lemma 13.D.2.** No pooling equilibria exist.

**Lemma 13.D.3.** If  $(w_L, t_L)$  and  $(w_H, t_H)$  are the contracts signed by the low- and high-ability workers in a separating equilibrium, then both contracts yield zero profits; that is,  $w_L = \theta_L$  and  $w_H = \theta_H$ .

**Lemma 13.D.4.** In any separating equilibrium, the low-ability workers accept contract  $(\theta_L, 0)$ ; that is, they receive the same contract as when no informational imperfections are present in the market.

**Lemma 13.D.5.** In any separating equilibrium, the high-ability workers accept contract  $(\theta_H, \hat{t}_H)$ , where  $\hat{t}_H$  satisfies  $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$ .

**Proposition 13.D.2.** In any subgame perfect Nash equilibrium of the screening game, low-ability workers accept contract  $(\theta_L, 0)$ , and high-ability workers accept contract  $(\theta_H, \hat{t}_H)$ , where  $\hat{t}_H$  satisfies  $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$ .

## Chapter 14

# The Principal-Agent Problem

### 14.B Hidden Actions (Moral Hazard)

**Proposition 14.B.1.** In the principal-agent model with observable managerial effort, an optimal contract specifies that the manager chooses the effort  $e^*$  that maximises  $[\int \pi f(\pi|e) d\pi - v^{-1}(\bar{u} + g(e))]$  and pays the manager a fixed wage  $w^* = v^{-1}(\bar{u} + g(e^*))$ . This is the uniquely optimal contract if  $v''(w) < 0$  at all  $w$ .

**Proposition 14.B.2.** In the principal-agent model with unobservable managerial effort and a risk-neutral manager, an optimal contract generates the same effort choice and expected utilities for the manager and the owner as when effort is observable.

**Lemma 14.B.1.** In any solution to the problem

$$\begin{aligned} \min_{w(\pi)} & \int w(\pi) f(\pi|e) d\pi \\ \text{s.t. (i)} & \int v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \bar{u} \\ \text{(ii)} & e \text{ solves } \max_{\tilde{e}} \int v(w(\pi)) f(\pi|\tilde{e}) d\pi - g(\tilde{e}) \end{aligned}$$

with  $e = e_H$ , both  $\gamma > 0$  and  $\mu > 0$ .

**Proposition 14.B.3.** In the principal-agent model with unobservable manager effort, a risk-averse manager, and two possible effort choices, the optimal compensation scheme for implementing  $e_H$  satisfies

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left[ 1 - \frac{f(\pi|e_L)}{f(\pi|e_H)} \right],$$

gives the manager expected utility  $\tilde{u}$ , and involves a larger expected wage payment than is required when effort is observable. The optimal compensation scheme for implementing  $e_L$  involves the same fixed wage payment as if effort were observable. Whenever the optimal effort level with observable effort would be  $e_H$ , nonobservability causes a welfare loss.

### 14.C Hidden Information (and Monopolistic Screening)

**Proposition 14.C.1.** In the principal-agent model with an observable state variable  $\theta$ , the optimal contract involves an effort level  $e_i^*$  in state  $\theta_i$  such that  $\pi(e_i^*) = g_e(e_i^*, \theta)$  and fully insures the

manager, setting his wage in each state  $\theta_i$  at the level  $w_i^*$  such that  $v(w_i^* - g(e_i^*, \theta_i)) = \bar{u}$ .

**Proposition 14.C.2** (The Revelation Principle). Denote the set of possible states by  $\Theta$ . In searching for an optimal contract, the owner can without loss restrict himself to contracts of the following form:

- (i) After the state  $\theta$  is realised, the manager is required to announce which state has occurred.
- (ii) The contract specifies an outcome  $[w(\hat{\theta}), e(\hat{\theta})]$  for each possible announcement  $\hat{\theta} \in \Theta$ .
- (iii) In every state  $\theta \in \Theta$ , the manager finds it optimal to report the state *truthfully*.

**Lemma 14.C.1.** In the problem

$$\begin{aligned} \max_{w_H, e_H \geq 0, w_L, e_L > 0} & \lambda[\pi(e_H) - w_H] + (1 - \lambda)[\pi(e_L) - w_L] \\ \text{s.t. (i)} & w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}) \\ \text{(ii)} & w_H - g(e_H, \theta_H) \geq v^{-1}(\bar{u}) \\ & \text{(reservation utility (or individual rationality) constraint)} \\ \text{(iii)} & w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H) \\ \text{(iv)} & w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L) \\ & \text{(incentive compatibility (or truth-telling or self-selection) constraints)} \end{aligned}$$

we can ignore constraint (ii). That is, a contract is a solution to the problem if and only if it is the solution to the problem derived from it by dropping (ii).

**Lemma 14.C.2.** An optimal contract in the problem given in Lemma 14.C.1 must have  $w_L - g(e_L, \theta_L) = v^{-1}(\bar{u})$ .

**Lemma 14.C.3.** In any optimal contract:

- (i)  $e_L \leq e_L^*$ ; that is, the manager's effort level in state  $\theta_L$  is no more than the level that would arise if  $\theta$  were observable.
- (ii)  $e_H = e_H^*$ ; that is, the manager's effort level in state  $\theta_H$  is exactly equal to the level that arise if  $\theta$  were observable.

**Lemma 14.C.4.** In any optimal contract,  $e_L < e_L^*$ ; that is, the effort level in state  $\theta_L$  is necessarily *strictly* below the level that would arise in state  $\theta_L$  if  $\theta$  were observable.

**Proposition 14.C.3.** In the hidden information principal-agent model with an infinitely risk-averse manager the optimal contract sets the level of effort in state  $\theta_H$  at its first-best (full observability) level  $e_H^*$ . The effort level in state  $\theta_L$  is distorted downward from its first-best level  $e_L^*$ . In addition, the manager is inefficiently insured, receiving a utility greater than  $\bar{u}$  in state  $\theta_H$  and a utility equal to  $\bar{u}$  in state  $\theta_L$ . The owner's expected payoff is strictly lower than the expected payoff he receives when  $\theta$  is observable, while the infinitely risk-averse manager's expected utility is the same as when  $\theta$  is observable (it equals  $\bar{u}$ ).

Part IV

General Equilibrium



## Chapter 15

# General Equilibrium Theory: Some Examples

### 15.B Pure Exchange: The Edgeworth Box

**Definition 15.B.1.** A *Walrasian* (or *competitive*) *equilibrium* for an Edgeworth box economy is a price vector  $p^*$  and an allocation  $x^* = (x_1^*, x_2^*)$  in the Edgeworth box such that for  $i = 1, 2$ ,

$$x_i^* \succsim_i x'_i \text{ for all } x'_i \in B_i(p^*).$$

**Definition 15.B.2.** An allocation  $x$  in the Edgeworth box is *Pareto optimal* (or *Pareto efficient*) if there is no other allocation  $x'$  in the Edgeworth box with  $x'_i \succsim_i x_i$  for  $i = 1, 2$  and  $x'_i \succ_i x_i$  for some  $i$ .

**Definition 15.B.3.** An allocation  $x^*$  in the Edgeworth box is supportable as an *equilibrium with transfers* if there is a price system  $p^*$  and wealth transfers  $T_1$  and  $T_2$  satisfying  $T_1 + T_2 = 0$ , such that for each consumer  $i$  we have

$$x_i^* \succsim_i x'_i \text{ for all } x'_i \in \mathbb{R}_+^2 \text{ such that } p^* \cdot x'_i \leq p^* \cdot \omega_i + T_i.$$

### 15.D The 2 x 2 Production Model

**Definition 15.D.1.** The production of good 1 is *relatively more intensive in factor 1* than is production of good 2 if

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

at *all* factor prices  $w = (w_1, w_2)$ .

## Chapter 16

# Equilibrium and Its Basic Welfare Properties

### 16.B The Basic Model and Definitions

**Definition 16.B.1.** An *allocation*  $(x, y) = (x_1, \dots, x_I, y_1, \dots, y_J)$  is a specification of a consumption vector  $x_i \in X$  for each consumer  $i = 1, \dots, I$  and a production vector  $y_j \in Y$  for each firm  $j = 1, \dots, J$ . An allocation  $(x, y)$  is *feasible* if  $\sum_i x_{\ell i} = \bar{\omega}_\ell + \sum_j y_{\ell j}$  for every commodity  $\ell$ . That is, if

$$\sum_i x_i = \bar{\omega} + \sum_j y_j. \quad (16.B.1)$$

We denote the set of feasible allocations by

$$A = \left\{ (x, y) \in X_1 \times \dots \times X_I \times Y_1 \times \dots \times Y_J : \sum_i x_i = \bar{\omega} + \sum_j y_j \right\} \subset \mathbb{R}^{L(I+J)}.$$

**Definition 16.B.2.** A feasible allocation  $(x, y)$  is *Pareto optimal* (or *Pareto efficient*) if there is no other allocation  $(x', y') \in A$  that *Pareto dominates* it, that is, if there is no feasible allocation  $(x', y')$  such that  $x'_i \succsim_i x_i$  for all  $i$  and  $x'_i \succ_i x_i$  for some  $i$ .

**Definition 16.B.3.** Given a private ownership economy specified by  $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J), \{(\omega_i, 0_{i1}, \dots, 0_{iJ})\}_{i=1}^I)$ , an allocation  $(x^*, y^*)$  and a price vector  $p = (p_1, \dots, p_L)$  constitutes *Walrasian* (or *competitive*) *equilibrium* if:

- (i) For every  $j$ ,  $y^*, j$  maximises profits in  $Y_j$ ; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

- (ii) For every  $i$ ,  $x_i^*$  is maximal for  $\succsim_i$  in the budget set

$$\left\{ x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^* \right\}.$$

- (iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

**Definition 16.B.4.** Given an economy specified by  $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$  an allocation  $(x^*, y^*)$  and a price vector  $p = (p_1, \dots, p_L)$  constitute a *price equilibrium with transfers* if there is an assignment of wealth levels  $(w_1, \dots, w_I)$  with  $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$  such that

- (i) For every  $j$ ,  $y_j^*$  maximises profits in  $Y_j$ ; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

- (ii) For every  $i$ ,  $x_i^*$  is maximal for  $\succsim_i$  in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq w_i\}.$$

- (iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

## 16.C The First Fundamental Theorem of Welfare Economics

**Definition 16.C.1.** The preference relation  $\succsim$  on  $X$  is *locally nonsatiated* if for every  $x \in X$  and every  $\varepsilon > 0$ , there is  $y \in X$  such that  $\|y - x\| \leq \varepsilon$  and  $y \succ x$ .

**Proposition 16.C.1** (The First Fundamental Theorem of Welfare Economics). If the price  $p^*$  and allocation  $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$  constitutes a competitive equilibrium, then this allocation is Pareto optimal.

## 16.D The Second Fundamental Theorem of Welfare Economics

**Definition 16.D.1.** Given an economy specified by  $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$  an allocation  $(x^*, y^*)$  and a price vector  $p = (p_1, \dots, p_L) \neq 0$  constitute a *price quasiequilibrium with transfers* if there is an assignment of wealth levels  $(w_1, \dots, w_I)$  with  $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$  such that

- (i) For every  $j$ ,  $y_j^*$  maximises profits in  $Y_j$ ; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

- (ii) For every  $i$ , if  $x_i \succ x_i^*$  then  $p \cdot x_i \geq w_i$ .

- (iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

**Proposition 16.D.1** (The Second Fundamental Theorem of Welfare Economics). Consider an economy specified by  $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$ , and suppose that every  $Y_j$  is convex and every preference relation  $\succsim_i$  is convex [i.e., the set  $\{x'_i \in X_i : x'_i \succsim_i x_i\}$  is convex for every  $x_i \in X_i\}$  and locally nonsatiated. Then, for every Pareto optimal allocation  $(x^*, y^*)$ , there is a price vector  $p = (p_1, \dots, p_L) \neq 0$  such that  $(x^*, y^*, p)$  is a price quasiequilibrium with transfers.

**Proposition 16.D.2.** Assume that  $X_i$  is convex and  $\succsim_i$  is continuous. Suppose also that the consumption vector  $x_i^* \in X_i$ , the price vector  $p$ , and the wealth level  $w_i$  are such that  $x_i \succ_i x_i^*$  implies  $p \cdot x_i \geq w_i$ . Then, if there is a consumption vector  $x_i' \in X_i$  such that  $p \cdot x_i' < w_i$  [a *cheaper consumption* for  $(p, w_i)$ ], it follows that  $x_i \succ_i x_i^*$  implies  $p \cdot x_i > w_i$ .

**Proposition 16.D.3.** Suppose that for every  $i$ ,  $X_i$  is convex,  $0 \in X_i$ , and  $\succsim_i$  is continuous. Then any price quasiequilibrium with transfers that has  $(w_1, \dots, w_I) \gg 0$  is a price equilibrium with transfers.

## 16.E Pareto Optimality and Social Welfare Optima

**Proposition 16.E.1.** A feasible allocation  $(x, y) = (x_1, \dots, x_I, y_1, \dots, y_J)$  is a Pareto optimum if and only if  $(u_1(x_1), \dots, u_I(x_I)) \in UP$ , where  $UP = \{u_1, \dots, u_I \in U: \text{there is no } (u'_1, \dots, u'_I) \in U \text{ such that } u'_i \geq u_i \text{ for all } i \text{ and } u'_i > u_i \text{ for some } i\}$  and  $U = \{(u_1, \dots, u_I) \in \mathbb{R}^I: \text{there is a feasible allocation } (x, y) \text{ such that } u_i \leq u_i(x_i) \text{ for } i = 1, \dots, I\}$ .

**Proposition 16.E.2.** If  $u^* = (u_1^*, \dots, u_I^*)$  is a solution to the social welfare maximisation problem  $\max_{u \in U} \lambda \cdot u$  with  $\lambda \gg 0$ , then  $u^* \in UP$ ; that is,  $u^*$  is the utility vector of a Pareto optimal allocation. Moreover, if the utility possibility set  $U$  is convex, then for any  $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_I) \in UP$ , there is a vector of welfare weights  $\lambda = (\lambda_1, \dots, \lambda_I) \geq 0, \lambda \neq 0$ , such that  $\lambda \cdot \tilde{u} \geq \lambda \cdot u$  for all  $u \in U$ , that is, such that  $\tilde{u}$  is a solution to the social welfare maximisation problem.

## 16.F First-Order Conditions for Pareto Optimality

**Proposition 16.F.1.** Under the assumptions made about the economy [in particular, the concavity of every  $u_i(\cdot)$  and the convexity of every  $F_j(\cdot)$ ], every Pareto optimal allocation (and, hence, every price equilibrium with transfers) maximises a weighted sum of utilities subject to the resource and technological constraints. Moreover, the weight  $\lambda_i$  of the utility of the  $i$ th consumer equals the reciprocal of consumer  $i$ 's marginal utility or wealth evaluated at the supporting prices and imputed wealth.

## 16.G Some Applications

**Definition 16.G.1.** A *Lindahl equilibrium* for the public goods economy is a price equilibrium with transfers for the artificial economy with personalised commodities. That is, an allocation  $(x_1^*, \dots, x_I^*, q^*, z^* \in \mathbb{R}^{2I} \times \mathbb{R} \times \mathbb{R}$  and a price system  $(p_1, p_{21}, \dots, p_{2I}) \in \mathbb{R}^{I+1}$  constitutes a Lindahl equilibrium if there is a set of wealth levels  $(w_1, \dots, w_I)$  satisfying  $\sum_i w_i = \sum_i p_1 x_{1i}^* + (\sum_i p_{2i}) q^* - p_1 z^*$  and such that

- (i)  $q^* \leq f(z^*)$  and  $(\sum_i p_{2i}) q^* - p_1 z^* \geq (\sum_i p_{2i}) q - p_1 z$  for all  $(q, z)$  with  $z \geq 0$  and  $q \leq f(z)$ .
- (ii) For every  $i$ ,  $x_i^* = (x_{1i}^*, x_{2i}^*)$  is maximal for  $\succsim_i$  in the set  $\{(x_{1i}, x_{2i}) \in X_i : p_1 x_{1i} + p_2 x_{2i} \leq w_i\}$ .
- (iii)  $\sum_i x_{1i}^* + z^* = \bar{\omega}_1$  and  $x_{2i}^* = q^*$  for every  $i$ .

**Proposition 16.G.1.** Suppose that the basic assumptions of Section 16.F hold and that, in addition, all consumers have convex preferences (so utility functions are quasiconcave). If  $(x^*, y^*)$  is Pareto optimal, then there exists a price vector  $p = (p_1, \dots, p_L)$  and wealth levels  $w = (w_1, \dots, w_I)$  with  $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$  such that:

(i) For any firm  $j$ , we have

$$p = \gamma_j \nabla F_j(y_j^*) \quad \text{for some } \gamma_j > 0.$$

(ii) For any  $i$ ,  $x_i^*$  is maximal for  $\succsim_i$  in the budget set

$$\{x_i \in X : p \cdot x_i \leq w_i\}.$$

(iii)  $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$ .

## Chapter 17

# The Positive Theory of Equilibrium

### 17.B Equilibrium: Definitions and Basic Equations

**Definition 17.B.1.** Given a private ownership economy specified by

$$\left( \{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}, \dots, \theta_{iJ})\}_{i=1}^I \right),$$

an allocation  $(x^*, y^*)$  and a price vector  $p = (p_1, \dots, p_L)$  constitute a *Walrasian* (or *competitive*, or *market*, or *price-taking*) equilibrium if

- (i) For every  $j$ ,  $y_j^* \in Y_j$  maximises profits in  $Y_j$ ; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

- (ii) For every  $i$ ,  $x_i^* \in X_i$  is maximal for  $\succsim_i$  in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

- (iii)  $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$ .

**Proposition 17.B.1.** In a pure exchange economy in which consumer preferences are continuous, strictly convex and locally nonsatiated,  $p \geq 0$  is a Walrasian equilibrium price vector if and only if:

$$\sum_i (x_i(p, p \cdot \omega_i) - \omega_i) \leq 0.$$

**Definition 17.B.2.** The *excess demand function* of consumer  $i$  is

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i,$$

where  $x_i(p, p \cdot \omega_i)$  is consumer  $i$ 's Walrasian demand function. The (*aggregate*) *excess demand function* of the economy is

$$z(p) = \sum_i z_i(p).$$

The domain of this function is a set of nonnegative price vectors that includes all strictly positive price vectors.

**Proposition 17.B.2.** Suppose that, for every consumer  $i$ ,  $X_i = \mathbb{R}_+^L$  and  $\succsim_i$  is continuous, strictly convex, and strongly monotone. Suppose also that  $\sum_i \omega_i \gg 0$ . Then the aggregate excess demand function  $z(p)$ , defined for all price vectors  $p \gg 0$ , satisfies the properties:

- (i)  $z(\cdot)$  is continuous.
- (ii)  $z(\cdot)$  is homogeneous of degree zero.
- (iii)  $p \cdot z(\cdot) = 0$  for all  $p$  (*Walras' law*).
- (iv) There is an  $s > 0$  such that  $z_\ell(p) > -s$  for every commodity  $\ell$  and all  $p$ .
- (v) If  $p^n \rightarrow p$ , where  $p \neq 0$  and  $p_\ell = 0$  for some  $\ell$ , then

$$\max\{z_1(p^n), \dots, z_L(p^n)\} \rightarrow \infty.$$

## 17.C Existence of Walrasian Equilibrium

**Proposition 17.C.1.** Suppose that  $z(p)$  is a function defined for all strictly positive price vectors  $p \in \mathbb{R}_{++}^L$  and satisfying conditions (i) to (v) of Proposition 17.B.2. Then the system of equations  $z(p) = 0$  has a solution. Hence, a Walrasian equilibrium exists in any pure exchange economy in which  $\sum_i \omega_i \gg 0$  and every consumer has continuous, strictly convex, and strongly monotone preferences.

**Proposition 17.C.2.** Suppose that  $z(p)$  is a function defined for all nonzero, nonnegative price vectors  $p \in \mathbb{R}_+^L$  and satisfying conditions (i) to (iii) of Proposition 17.B.2 (i.e. continuity homogeneity of degree zero and Walras' law). Then there is a price vector  $p^*$  such that  $z(p^*) \leq 0$ .

## 17.D Local Uniqueness and the Index Theorem

**Definition 17.D.1.** An equilibrium price vector  $p = (p_1, \dots, p_{L-1})$  is *regular* if the  $(L-1) \times (L-1)$  matrix of price effects  $D\hat{z}(p)$  is nonsingular, that is, has rank  $L-1$ . If every normalised equilibrium price vector is regular, we say that the *economy is regular*.

**Proposition 17.D.1.** Any regular (normalised) equilibrium price vector

$$p = (p_1, \dots, p_{L-1}, 1)$$

is *locally isolated* (or *locally unique*). That is, there is an  $\varepsilon > 0$  such that if  $p' \neq p$ ,  $p'_L = p_L = 1$ , and  $\|p' - p\| < \varepsilon$ , then  $z(p') \neq 0$ . Moreover, if the economy is regular, then the number of normalised equilibrium price vectors is finite.

**Definition 17.D.2.** Suppose that  $p = (p_1, \dots, p_{L-1}, 1)$  is a regular equilibrium of the economy. Then we denote

$$\text{index } p = (-1)^{L-1} \text{sign}|D\hat{z}(p)|,$$

where  $|D\hat{z}(p)|$  is the determinant of the  $(L-1) \times (L-1)$  matrix  $D\hat{z}(p)$ .

**Proposition 17.D.2** (The Index Theorem). For any regular economy, we have

$$\sum_{\{p: z(p)=0, p_L=1\}} \text{index } p = +1.$$

**Definition 17.D.3.** The system of  $M$  equations in  $N$  unknowns  $f(v) = 0$  is *regular* if  $\text{rank } Df(v) = M$  whenever  $f(v) = 0$ .

**Proposition 17.D.3** (The Transversality Theorem). If the  $M \times (N + S)$  matrix  $Df(v; q)$  has rank  $M$  whenever  $f(v; q) = 0$  then for almost every  $q$ , the  $M \times N$  matrix  $D_v f(v; q)$  has rank  $M$  whenever  $f(v; q) = 0$ .

**Proposition 17.D.4.** For any  $p$  and  $\omega$ ,  $\text{rank } D_\omega \hat{z}(p; \omega) = L - 1$ .

**Proposition 17.D.5.** For almost every vector of initial endowments  $(\omega_1, \dots, \omega_I) \in \mathbb{R}_{++}^{LI}$ , the economy defined by  $\{(\succsim_i, \omega_i)\}_{i=1}^I$  is regular.

## 17.E Anything Goes: The Sonnenschein-Mantel-Debreu Theorem

**Proposition 17.E.1.** Suppose that  $I < L$ . Then for any equilibrium price vector  $p$  there is some direction of price change  $dp \neq 0$  such that  $p \cdot dp = 0$  (hence  $dp$  is not proportional to  $p$ ) and  $dp \cdot Dz(p)dp \leq 0$ .

**Proposition 17.E.2.** Given a price vector  $p$ , let  $z \in \mathbb{R}^L$  be an arbitrary vector and  $A$  an arbitrary  $L \times L$  matrix satisfying  $p \cdot z = 0$ ,  $Ap = 0$  and  $p \cdot A = -z$ . Then there is a collection of  $L$  consumers generating an aggregate excess demand function  $z(\cdot)$  such that  $z(p) = z$  and  $Dz(p) = A$ .

**Proposition 17.E.3.** Suppose that  $z(\cdot)$  is a continuous function defined on

$$P_\varepsilon = \{p \in \mathbb{R}_+^L : p_\ell/p_{\ell'} \geq \varepsilon \text{ for every } \ell \text{ and } \ell'\}$$

and with values in  $\mathbb{R}^L$ . Assume that, in addition,  $z(\cdot)$  is homogeneous of degree zero and satisfies Walras' law. Then there is an economy of  $L$  consumers whose aggregate excess demand function coincides with  $z(\cdot)$  in the domain of  $P_\varepsilon$ .

**Proposition 17.E.4.** For any  $N \geq 1$ , suppose that we assign to each  $n = 1, \dots, N$  a price vector  $p^n$ , normalised to  $\|p^n\| = 1$ , and an  $L \times L$  matrix  $A_n$  of rank  $L - 1$ , satisfying  $A_n p^n = 0$  and  $p^n \cdot A_n = 0$ . Suppose that, in addition, the index formula  $\sum_n (-1)^{L-1} \text{sign} |\hat{A}_n| = +1$  holds. If  $L = 2$ , assume also that positive and negative index equilibria alternate.

Then there is an economy with  $L$  consumers such that the aggregate excess demand  $z(\cdot)$  has the properties:

- (i)  $z(p) = 0$  for  $\|p\| = 1$  if and only if  $p = p^n$  for some  $n$ .
- (ii)  $Dz(p^n) = A_n$  for every  $n$ .

## 17.F Uniqueness of Equilibria

**Proposition 17.F.1.** Given an economy specified by the constant returns technology  $Y$  and the aggregate excess demand function of the consumers  $z(\cdot)$ , a price vector  $p$  is a Walrasian equilibrium price vector if and only if

- (i)  $p \cdot y \leq 0$  for every  $y \in Y$ , and
- (ii)  $z(p)$  is a feasible production; that is,  $z(p) \in Y$ .



**Definition 17.F.1** (The Weak Axiom for Excess Demand Functions). The excess demand function  $z(\cdot)$  satisfies the weak axiom of revealed preferences (WA) if for any pair of price vectors  $p$  and  $p'$ , we have

$$z(p) \neq z(p') \text{ and } p \cdot z(p') \leq 0 \text{ implies } p' \cdot z(p) \geq 0.$$

**Proposition 17.F.2.** Suppose that the excess demand function  $z(\cdot)$  is such that, for any constant returns technology  $Y$ , the economy formed by  $z(\cdot)$  and  $Y$  has a unique (normalised) equilibrium price vector. Then  $z(\cdot)$  satisfies the weak axiom. Conversely, if  $z(\cdot)$  satisfies the weak axiom then, for any constant returns convex technology  $Y$ , the set of equilibrium price vectors is convex (and so, if the set of normalised price equilibria is finite, there can be at most one normalised price equilibrium).

**Definition 17.F.2.** The function  $z(\cdot)$  has the *gross substitute* (GS) property if whenever  $p'$  and  $p$  are such that, for some  $\ell$ ,  $p'_\ell > p_\ell$  and  $p'_k > p_k$  for  $k \neq \ell$ , we have  $z_k(p') > z_k(p)$  for  $k \neq \ell$ .

**Proposition 17.F.3.** An aggregate excess demand function  $z(\cdot)$  that satisfies the gross substitute property has at most one exchange equilibrium; that is,  $z(p) = 0$  has at most one (normalised) solution.

**Proposition 17.F.4.** If  $z(\cdot)$  is an aggregate excess demand function,  $z(p) = 0$ , and  $Dz(p)$  has the gross substitute sign pattern, then we also have  $dp \cdot Dz(p)dp < 0$  whenever  $dp \neq 0$  is not proportional to  $p$ .

**Proposition 17.F.5.** Suppose that the initial endowment allocation  $(\omega_1, \dots, \omega_I)$  constitutes a Walrasian equilibrium allocation for an exchange economy with strictly convex and strongly monotone consumer preferences (i.e., no-trade is an equilibrium). Then this is the unique equilibrium allocation.

## 17.G Comparative Statics Analysis

**Proposition 17.G.1.** Given any price vector  $\bar{p}$ , endowments for the first consumer of the first  $L - 1$  commodities  $\hat{\omega}_1 = (\hat{\omega}_{11}, \dots, \hat{\omega}_{L-1,1})$ , and a  $(L - 1) \times (L - 1)$  nonsingular matrix  $B$ , there is an exchange economy formed by  $L + 1$  consumers in which the first consumer has the prescribed endowments of the first  $L - 1$  commodities,  $\hat{z}(\bar{p}, \hat{\omega}_1) = 0$ ,  $\hat{z}(\cdot, \hat{\omega}_1) = 0$  is regular at  $\bar{p}$  and  $Dp(\hat{\omega}_1) = B$ .

**Proposition 17.G.2.** Suppose that  $\hat{z}(\bar{p}; \bar{q}) = 0$ , where  $\hat{z}(\cdot)$  is differentiable. If  $D_q \hat{z}(\bar{p}; \bar{q})$  is negative definite, then

$$(D_q \hat{z}(\bar{p}; \bar{q})dq) \cdot (Dp(\bar{q})dq) \geq 0 \text{ for any } dq.$$

**Proposition 17.G.3.** Suppose that  $\hat{z}(\bar{p}; \bar{q}) = 0$ , where  $\hat{z}(\cdot; \cdot)$  is differentiable. If the  $L \times L$  matrix  $D_p z(\bar{p}; \bar{q})$  has negative diagonal entries and positive off-diagonal entries, then  $[D_p z(\bar{p}; \bar{q})]^{-1}$  has all its entries negative.

## 17.H Tâtonnement Stability

**Proposition 17.H.1.** Suppose that  $z(p^*) = 0$  and  $p^* \cdot z(p) > 0$  for every  $p$  not proportional to  $p^*$ . Then the relative prices of any solution trajectory of the differential equation

$$\frac{dp_\ell}{dt} = c_\ell z_\ell(p) \quad \text{for every } \ell$$

converge to the relative prices of  $p^*$ .

**Definition 17.H.1.** We say that the differentiable trajectory  $y(t) \in Y$  is *admissible* if  $p(y(t)) \cdot (dy(t)/dt) \geq 0$  for every  $t$ , with equality only if  $y(t)$  is profit maximising for  $p(y(t))$  (in which case we could say that we are at a long-run equilibrium).

**Proposition 17.H.2.** If there is a single strictly convex consumer, then any admissible trajectory converges to the (unique) equilibrium.

## Chapter 18

# Some Foundations for Competitive Equilibria

### 18.B Core and Equilibria

**Definition 18.B.1.** A coalition  $S \subset I$  *improves upon*, or *blocks*, the feasible allocation  $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_+^{LI}$  if for every  $i \in S$  we can find a consumption  $x_i \geq 0$  with the properties:

- (i)  $x_i \succ_i x_i^*$  for every  $i \in S$
- (ii)  $\sum_{i \in S} x_i \in Y + \{\sum_{i \in S} \omega_i\}$ .

**Definition 18.B.2.** We say that a feasible allocation  $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_+^{LI}$  has the *core property* if there is no coalition of consumers  $S \subset I$  that can improve upon  $x^*$ . The *core* is the set of allocations that have the core property.

**Proposition 18.B.1.** Any Walrasian equilibrium allocation has the core property.

**Proposition 18.B.2.** Denoting by  $hn$  the  $n$ th individual of type  $h$ , suppose that the allocation

$$x^* = (x_{11}^*, \dots, x_{1n}^*, \dots, x_{1N}^*, \dots, x_{H1}^*, \dots, x_{Hn}^*, \dots, x_{HN}^*) \in \mathbb{R}_+^{LHN}$$

belongs to the core of the  $N$ -replica economy. Then  $x^*$  has the *equal-treatment property*, that is, all consumers of the same type get the same consumption bundle:

$$x_{hm}^* = x_{hn}^* \quad \text{for all } 1 \leq m, n \leq N \text{ and } 1 \leq h \leq H.$$

**Proposition 18.B.3.** If the feasible type allocation  $x^* = (x_1^*, \dots, x_H^*) \in \mathbb{R}_+^{LH}$  has the core property for all  $N = 1, 2, \dots$ , that is,  $x^* \in C_N$  for all  $N$ , then  $x^*$  is a Walrasian equilibrium allocation.

### 18.C Noncooperative Foundations of Walrasian Equilibria

**Definition 18.C.1.** The profiles of actions  $a^* = (a_1^*, \dots, a_I^*) \in A_1 \times \dots \times A_I$  is a *trading equilibrium* if, for every  $i$ ,

$$u_i(g(a_i^*; p(a^*)) + \omega_i) \geq u_i(g(a_i; p(a_i; a_{-i}^*)) + \omega_i) \quad \text{for all } a_i \in A_i.$$

## 18.D The Limits to Redistribution

**Definition 18.D.1.** The feasible allocation  $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_+^{LI}$  is *self-selective* (or *anonymous*, or *envy-free in net trades*) if there is a set of net trades  $B \subset \mathbb{R}^L$ , to be called a *generalised budget set*, or a *tax system*, such that, for every  $i$ ,  $z_i^* = x_i^* - \omega_i$  solves the problem

$$\begin{aligned} & \max u_i(z_i + \omega_i) \\ & \text{s.t. } z_i \in B, \\ & \quad z_i + \omega_i \geq 0. \end{aligned}$$

**Proposition 18.D.1.** Suppose we have an exchange economy with a continuum of consumer types. Assume:

- (i) The preferences of all consumers are representable by differentiable utility functions.
- (ii) The set of characteristics of consumers present in the economy cannot be split into two disconnected classes. Formally, if  $(u(\cdot), \omega), (u'(\cdot), \omega')$  are two preferences-endowment pairs present in the economy then there is a continuous function  $(u(\cdot; t), \omega(t))$  of  $t \in [0, 1]$  such that

$$(u(\cdot; 0), \omega(0)) = (u(\cdot), \omega), (u(\cdot; 1), \omega(1)) = (u'(\cdot), \omega'),$$

and  $(u(\cdot; t), \omega(t))$  is present in the economy for every  $t$ .

Then any allocation  $x^* = \{x_i^*\}_{i \in I}$  that is Pareto optimal, self-selective, and interior (i.e.,  $x_i^* \gg 0$  for all  $i$ ) must be a Walrasian equilibrium allocation. Here  $I$  is an infinite set of names.

## 18.E Equilibrium and the Marginal Productivity Principle

**Definition 18.E.1.** Given a continuum population  $\mu = (\mu_1, \dots, \mu_H) \in \mathbb{R}_+^H$  a feasible allocation  $(x_1^*, \dots, x_H^*)$  is a *marginal product*, or *no-surplus*, allocation if

$$u_h(x_h^*) = \frac{\partial v(\mu)}{\partial \mu_h} \quad \text{for all } h.$$

In words: at a no-surplus allocation everyone is getting exactly what she contributes on the margin.

**Proposition 18.E.1.** For any *continuum* population  $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_H) \gg 0$  a feasible allocation  $(x_1^*, \dots, x_H^*) \gg 0$  is a marginal product allocation if and only if it is a Walrasian equilibrium allocation.

## Chapter 19

# General Equilibrium Under Uncertainty

### 19.B A Market Economy with Contingent Commodities: Description

**Definition 19.B.1.** For every physical commodity  $\ell = 1, \dots, L$  and states  $s = 1, \dots, S$ , a unit of *(state-)contingent commodity  $\ell s$*  is a title to receive a unit of physical good  $\ell$  if and only if  $s$  occurs. Accordingly, a *(state-)contingent commodity vector* is specified by

$$x = (x_{11}, \dots, x_{L1}, \dots, x_{1S}, \dots, x_{LS}) \in \mathbb{R}^{LS},$$

and is understood as an entitlement to receive the commodity vector  $x = (x_{1s}, \dots, x_{Ls})$  if state  $s$  occurs.

### 19.C Arrow-Debreu Equilibrium

**Definition 19.C.1.** An allocation

$$(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*) \in X_1 \times \dots \times X_I \times Y_1 \times \dots \times Y_J \subset \mathbb{R}^{LS(I+J)}$$

and a system of prices for the contingent commodities  $p = (p_{11}, \dots, p_{LS}) \in \mathbb{R}^{LS}$  constitute an *Arrow-Debreu equilibrium* if:

- (i) For every  $j$ ,  $y_j^*$  satisfies  $p \cdot y_j^* \geq p \cdot y_j$  for all  $y_j \in Y_j$ .
- (ii) For every  $i$ ,  $x_i^*$  is maximal for  $\succsim_i$  in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

- (iii)  $\sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i$ .

## 19.D Sequential Trade

**Definition 19.D.1.** A collection formed by a price vector  $q = (q_1, \dots, q_S) \in \mathbb{R}^S$  for contingent first good commodities at  $t = 0$ , a spot price vector

$$p_s = (p_{1s}, \dots, p_{Ls}) \in \mathbb{R}^L$$

for every  $s$ , and, for every consumer  $i$ , consumption plans  $z_i^* = (z_{1i}^*, \dots, z_{Si}^*) \in \mathbb{R}^S$  at  $t = 0$  and  $x_i^* = (x_{1i}^*, \dots, x_{Si}^*) \in \mathbb{R}^{LS}$  at  $t = 1$  constitute a *Radner equilibrium* if:

- (i) For every  $i$ , the consumption plans  $z_i^*, x_i^*$  solve the problem

$$\begin{aligned} \max_{\substack{(x_{1i}, \dots, x_{Si}) \in \mathbb{R}_+^{LS} \\ (z_{1i}, \dots, z_{Si}) \in \mathbb{R}^S}} U_i(x_{1i}, \dots, x_{Si}) \\ \text{s.t. (i) } \sum_s q_s z_{si} \leq 0, \\ \text{(ii) } p_s \cdot x_{si} \leq p_s \omega_{si} + p_{1s} z_{si} \quad \text{for every } s. \end{aligned}$$

- (ii)  $\sum_i z_{si}^* \leq 0$  and  $\sum_i x_{si}^* \leq \sum_i \omega_{si}$  for every  $s$ .

**Proposition 19.D.1.** We have:

(i) If the allocation  $x^* \in \mathbb{R}^{LSI}$  and the contingent commodities price vector  $(p_1, \dots, p_S) \in \mathbb{R}_{++}^{LS}$  constitute an Arrow-Debreu equilibrium, then there are prices  $q \in \mathbb{R}_{++}^S$  for contingent first good commodities and consumption plans for these commodities  $z^* = (z_1^*, \dots, z_I^*) \in \mathbb{R}^{SI}$  such that the consumption plans  $x^*, z^*$ , the prices  $q$ , and the spot prices  $(p_1, \dots, p_S)$  constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans  $x^* \in \mathbb{R}^{LSI}$ ,  $z^* \in \mathbb{R}^{SI}$  and prices  $q \in \mathbb{R}_{++}^S$ ,  $(p_1, \dots, p_S) \in \mathbb{R}_{++}^{LS}$  constitute a Radner equilibrium, then there are multipliers  $(\mu_1, \dots, \mu_S) \in \mathbb{R}_{++}^S$  such that the allocation  $x^*$  and the contingent commodities price vector  $(\mu_1 p_1, \dots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$  constitute an Arrow-Debreu equilibrium. (The multiplier  $\mu_s$  is interpreted as the value, at  $t = 0$ , of a dollar at  $t = 1$  and state  $s$ .)

## 19.E Asset Markets

**Definition 19.E.1.** A unit of an *asset*, or *security*, is a title to receive an amount  $r_s$  of good 1 at date  $t = 1$  if state  $s$  occurs. An asset is therefore characterised by its *return vector*  $r = (r_1, \dots, r_S) \in \mathbb{R}^S$ .

**Definition 19.E.2.** A collection formed by a price vector  $q = (q_1, \dots, q_K) \in \mathbb{R}^K$  for assets traded at  $t = 0$ , a spot price vector  $p_s = (p_{1s}, \dots, p_{Ls}) \in \mathbb{R}^L$  for every  $s$ , and, for every consumer  $i$ , portfolio plans  $z_i^* = (z_{1i}^*, \dots, z_{Ki}^*) \in \mathbb{R}^K$  at  $t = 0$  and consumption plans  $x_i^* = (x_{1i}^*, \dots, x_{Si}^*) \in \mathbb{R}^{LS}$  at  $t = 1$  constitutes a *Radner equilibrium* if:

- (i) For every  $i$ , the consumption plans  $z_i^*, x_i^*$  solve the problem

$$\begin{aligned} \max_{\substack{(x_{1i}, \dots, x_{Si}) \in \mathbb{R}_+^{LS} \\ (z_{1i}, \dots, z_{Ki}) \in \mathbb{R}^K}} U_i(x_{1i}, \dots, x_{Si}) \\ \text{s.t. (i) } \sum_k q_k z_{ki} \leq 0, \\ \text{(ii) } p_s \cdot x_{si} \leq p_s \omega_{si} + \sum_k p_{1s} z_{ki} r_{sk} \quad \text{for every } s. \end{aligned}$$

(ii)  $\sum_i z_{ki}^* \leq 0$  and  $\sum_i x_{si}^* \leq \sum_i \omega_{si}$  for every  $k$  and  $s$ .

**Proposition 19.E.1.** Assume that every return vector is nonnegative and nonzero; that is,  $r_k \geq 0$  and  $r_k \neq 0$  for all  $k$ . Then, for every (column) vector  $q \in \mathbb{R}^K$  of asset prices arising in a Radner equilibrium, we can find multipliers  $\mu = (\mu_1, \dots, \mu_S) \geq 0$ , such that  $q_k = \sum_s \mu_s r_{sk}$  for all  $k$  (in matrix notation,  $q^T = \mu \cdot R$ ).

**Definition 19.E.3.** An asset structure with an  $S \times K$  return matrix  $R$  is *complete* of rank  $R = S$ , that is, if there is some subset of  $S$  assets with linearly independent returns.

**Proposition 19.E.2.** Suppose that the asset structure is complete. Then:

(i) If the consumption plans  $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}^{LSI}$  and the price vector

$$(p_1, \dots, p_S) \in \mathbb{R}_{++}^{LS}$$

constitute an Arrow-Debreu equilibrium, then there are asset prices  $q \in \mathbb{R}_{++}^K$  and portfolio plans  $z^* = (z_1^*, \dots, z_I^*) \in \mathbb{R}^{KI}$  such that the consumption plans  $x^*$ , portfolio plans  $z^*$ , asset prices  $q$ , and spot prices  $(p_1, \dots, p_S)$  constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans  $x^* \in \mathbb{R}^{LSI}$ , portfolio plans  $z^* \in \mathbb{R}^{KI}$ , and prices  $q \in \mathbb{R}_{++}^K, (p_1, \dots, p_S) \in \mathbb{R}_{++}^{LS}$  constitute a Radner equilibrium, then there are multipliers  $\mu = (\mu_1, \dots, \mu_S) \in \mathbb{R}_{++}^S$  such that consumption plans  $x^*$  and the contingent commodities price vector  $(\mu_1 p_1, \dots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$  constitute an Arrow-Debreu equilibrium. (The multiplier  $\mu_s$  is interpreted as the value, at  $t = 0$ , of a dollar at  $t = 1$  and state  $s$ ; recall that  $p_{1s} = 1$ .)

**Proposition 19.E.3.** Suppose that the asset price vector  $q \in \mathbb{R}^K$ , the spot prices  $p = (p_1, \dots, p_S) \in \mathbb{R}^{LS}$ , the consumption plans  $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_{++}^{LSI}$ , and the portfolio plans  $(z_1^*, \dots, z_I^*) \in \mathbb{R}^{KI}$  constitute a Radner equilibrium for an asset structure with  $S \times K$  return matrix  $R$ . Let  $R'$  be the  $S \times K'$  return matrix of a second asset structure. If  $\text{range } R' = \text{range } R$ , then  $x^*$  is still the consumption allocation of a Radner equilibrium in the economy with the second asset structure.

## 19.F Incomplete Markets

**Definition 19.F.1.** The asset allocation  $(z_1, \dots, z_I) \in \mathbb{R}^{KI}$  is constrained Pareto optimal if it is feasible (i.e.  $\sum_i z_i \leq 0$ ) and if there is no other feasible asset allocation  $(z'_1, \dots, z'_I) \in \mathbb{R}^{KI}$  such that

$$U_i^*(z'_1, \dots, z'_I) \geq U_i^*(z_1, \dots, z_I) \quad \text{for every } i,$$

with at least one inequality strict.

**Proposition 19.F.1.** Suppose that there are two periods and only one consumption good in the second period. Then any Radner equilibrium is *constrained Pareto optimal* in the sense that there is no possible redistribution of assets in the first period that leaves every consumer as well off and at least one consumer strictly better off.

## 19.G Firm Behaviour in General Equilibrium Models under Uncertainty

**Definition 19.G.1.** A set  $A \subset \mathbb{R}^S$  of random variables is *spanned* by a given asset structure if every  $a \in A$  is in the range of the return matrix  $R$  of the asset structure, that is, if every  $a \in A$  can be expressed as a linear combination of the available asset returns.

## 19.H Imperfect Information

**Definition 19.H.1.** The signal function  $\sigma' : S \rightarrow \mathbb{R}$  is *at least as informative* as  $\sigma : S \rightarrow \mathbb{R}$  if  $\sigma(s) \neq \sigma(s')$  implies  $\sigma'(s) \neq \sigma'(s')$  for any pair  $s, s'$ . It is *more informative* if, in addition,  $\sigma'(s) \neq \sigma'(s')$  for some pair  $s, s'$  with  $\sigma(s) = \sigma(s')$ .

**Proposition 19.H.1.** In the single-consumer problem, if the signal function  $\sigma'(\cdot)$  is at least as informative as the signal function  $\sigma(\cdot)$ , then the ex ante utility derived from  $\sigma'(\cdot)$ ,  $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma'(\cdot)})$ , is at least as large as the ex ante utility derived from  $\sigma(\cdot)$ ,  $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma(\cdot)})$ .

**Definition 19.H.2.** The price function  $p(\cdot)$  is a *rational expectations equilibrium price function* if, for every  $s$ ,  $p(s)$  clears the spot market when every consumer  $i$  knows that  $s \in E_{p(s), \sigma_i(s)}$  and, therefore, evaluates commodity bundles  $x_i \in \mathbb{R}^2$  according to the updated utility function

$$\sum_s (\pi_{s'i} | p(s), \sigma_i(s)) u_{s'i}(x).$$



## Chapter 20

# Equilibrium and Time

### 20.C Intertemporal Production and Efficiency

**Definition 20.C.1.** The list  $(y_0, y_1, \dots, y_t, \dots)$  is a *production path*, or *trajectory*, or *program*, if  $y_t \in Y \subset \mathbb{R}^{2L}$  for every  $t$ .

**Definition 20.C.2.** The production path  $(y_0, \dots, y_t, \dots)$  is *efficient* if there is no other production path  $(y'_0, \dots, y'_t, \dots)$  such that

$$y_{a,t-1} + y_{bt} \leq y'_{a,t-1} + y'_{bt} \quad \text{for all } t,$$

and equality does not hold for at least one  $t$ .

**Definition 20.C.3.** The production path  $(y_0, \dots, y_t, \dots)$  is *myopically*, or *short-run*, *profit maximising for the price sequence*  $(p_0, \dots, p_t, \dots)$  if for every  $t$  we have

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \geq p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at} \quad \text{for all } y'_t \in Y.$$

**Proposition 20.C.1.** Suppose that the production path  $(y_0, \dots, y_t, \dots)$  is myopically profit maximising with respect to the price sequence  $(p_0, \dots, p_t, \dots) \gg 0$ . Suppose also that the production path and the price sequence satisfy the *transversality condition*  $p_{t+1} \cdot y_{at} \rightarrow 0$ . Then the path  $(y_0, \dots, y_t, \dots)$  is efficient.

### 20.D Equilibrium: The One-Consumer Case

**Definition 20.D.1.** The (bounded) production path  $(y_0^*, \dots, y_t^*, \dots)$ ,  $y_t^* \in Y$ , and the (bounded) price sequence  $p = (p_0, \dots, p_t, \dots)$  constitute a *Walrasian* (or *competitive*) equilibrium if:

(i)  $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t \geq 0$  for all  $t$ .

(ii) For every  $t$ ,

$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \geq p_t \cdot y_b + p_{t+1} \cdot y_a$$

for all  $y = (y_b, y_a) \in Y$ .

(iii) The consumption sequence  $(c_0^*, \dots, c_t^*, \dots) \geq 0$  solves the problem

$$\begin{aligned} \max \quad & \sum_t \delta^t u(c_t) \\ \text{s.t.} \quad & \sum_t p_t \cdot c_t \leq \sum_t \pi_t + \sum_t p_t \cdot \omega_t. \end{aligned}$$

**Proposition 20.D.1.** Suppose that the (bounded) production path  $(y_0^*, \dots, y_t^*, \dots)$  and the (bounded) price sequence  $(p_0, \dots, p_t, \dots)$  constitute a Walrasian equilibrium. Then the transversality condition  $p_{t+1} \cdot y_{at}^* \rightarrow 0$  holds.

**Definition 20.D.2.** We say the consumption stream  $(c_0, \dots, c_t, \dots) \gg 0$  is *myopically*, or *short-run, utility maximising* in the budget set determined by  $(p_0, \dots, p_t, \dots)$  and  $w < \infty$  if utility cannot be increased by a new consumption stream that merely transfers purchasing power between some two consecutive periods.

**Proposition 20.D.2.** If the consumption stream  $(c_0, \dots, c_t, \dots)$  satisfies  $\sum_t p_t \cdot c_t = w < \infty$  and  $\lambda p_t = \delta^t \nabla u(c_t)$  for some  $\lambda$  and all  $t$ , then it is utility maximising in the budget set determined by  $(p_0, \dots, p_t, \dots)$  and  $w$ .

**Proposition 20.D.3.** Any Walrasian equilibrium path  $(y_0^*, \dots, y_t^*, \dots)$  solves the planning problem

$$\begin{aligned} \max \sum_t \delta^t u(c_t) \\ \text{s.t. } c_t = y_{a,t-1} + y_{bt} + \omega_t \geq 0 \quad \text{and} \quad y_t \in Y \text{ for all } t. \end{aligned} \tag{20.D.7}$$

**Proposition 20.D.4.** Suppose that the (bounded) path  $(y_0^*, \dots, y_t^*, \dots)$  solves the planning problem (20.D.7) and that it yields strictly positive consumption (in the sense that, for some  $\varepsilon > 0$ ,  $c_{\ell t} = y_{\ell a,t-1}^* + y_{\ell bt}^* + \omega_{\ell t} > \varepsilon$  for all  $\ell$  and  $t$ ). Then the path is a Walrasian equilibrium with respect to some price sequence  $(p_0, \dots, p_t, \dots)$ .

**Proposition 20.D.5.** Suppose that there is a uniform bound on the consumption streams generated by all the feasible paths. Then the planning problem (20.D.7) attains a maximum; that is, there is a feasible path that yields utility at least as large as the utility corresponding to any other feasible path.

**Proposition 20.D.6.** The planning problem (20.D.7) has at most one consumption stream solution.

**Proposition 20.D.7.** Suppose that the path  $(\bar{k}_0, \dots, k_t, \dots)$  is bounded, is strictly interior, and satisfies the Euler equations

$$\nabla_2 u(k_{t-1}, k_t) + \delta \nabla_1 u(k_t, k_{t+1}) = 0 \quad \text{for every } t \geq 1$$

to the planning problem

$$\begin{aligned} \max \sum_t \delta^t u(k_{t-1}, k_t) \\ \text{s.t. } (k_{t-1}, k_t) \in A \text{ for every } t, \text{ and } k_0 = \bar{k}_0. \end{aligned}$$

Then it solves this optimisation problem.

## 20.E Stationary Paths, Interest Rates, and Golden Rules

**Definition 20.E.1.** A production path  $(y_0, \dots, y_t, \dots)$  is *stationary*, or a *steady state*, if there is a production plan  $\bar{y} = (\bar{y}_b, \bar{y}_a) \in Y$  such that  $y_t = \bar{y}$  for all  $t > 0$ .

**Proposition 20.E.1.** Suppose that  $\bar{y} \in Y$  defines a stationary and efficient path. Then, there is a price vector  $p_0 \in \mathbb{R}^L$  and an  $\alpha > 0$  such that the path is myopically profit maximising for the price sequence  $(p_0, \alpha p_0, \dots, \alpha^t p_0, \dots)$ .

**Proposition 20.E.2.** Suppose that the stationary path  $(\bar{y}, \dots, \bar{y}, \dots)$ ,  $\bar{y} \in Y$ , is myopically supported by proportional prices with rate of interest  $r$ , then the path is efficient if  $r > 0$  and inefficient if  $r < 0$ .

**Definition 20.E.2.** A stationary production path that is myopically supported by proportional prices  $p_t = \alpha^t p_0$  with  $\alpha = \delta$  is called a *modified golden rule path*. A stationary production path myopically supported by constant prices  $p_t = p_0$  is called a *golden rule path*.

## 20.G Equilibrium: Several Consumers

**Definition 20.G.1.** The (bounded) production path  $(y_0^*, \dots, y_t^*, \dots)$ ,  $y_t^* \in Y$ , the (bounded) price sequence  $(p_0, \dots, p_t, \dots) \geq 0$ , and consumption streams  $(c_{0i}^*, \dots, c_{ti}^*, \dots) \geq 0$ ,  $i = 1, \dots, I$ , constitute a *Walrasian* (or *competitive*) equilibrium if:

(i)

$$\sum_i c_{ti}^* = y_{a,t-1}^* + y_{bt}^* + \sum_i \omega_{ti}, \quad \text{for all } t.$$

(ii) For every  $t$ ,

$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \geq p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}$$

for all  $y = (y_{bt}, y_{at} \in Y)$ .

(iii) For every  $i$ , the consumption stream  $(c_{0i}^*, \dots, c_{ti}^*, \dots) \geq 0$  solves the problem

$$\begin{aligned} \max \quad & \sum_t \delta_i^t u_i(c_i) \\ \text{s.t.} \quad & \sum_t p_t \cdot c_{ti} \leq \sum_t \theta_{ti} \pi_t + \sum_t p_t \cdot \omega_{ti} = w_i, \end{aligned}$$

where  $\theta_{ti}$  is consumer  $i$ 's given share of period  $t$ 's profits.

**Proposition 20.G.1.** A Walrasian equilibrium allocation is Pareto optimal.

**Proposition 20.G.2.** Suppose that  $(y_0^*, \dots, y_t^*, \dots)$  is the production path and  $(p_0, \dots, p_t, \dots)$  is the price sequence of a Walrasian equilibrium of an economy with  $I$  consumers. Then there are weights  $(\eta_1, \dots, \eta_I) \gg 0$  such that  $(y_0^*, \dots, y_t^*, \dots)$  and  $(p_0, \dots, p_t, \dots)$  constitute a Walrasian equilibrium for the one-consumer economy defined by the utility  $\sum_t \delta^t u(c_t)$ , where  $u(c_t)$  is the solution to  $\max \sum_i \eta_i u_i(c_{ti})$  s.t.  $\sum_i c_{ti} \leq c_t$ .

## 20.H Overlapping Generations

**Definition 20.H.1.** A sequence of prices  $(p_0, \dots, p_t, \dots)$ , an  $M \geq 0$ , and a family of consumptions  $\{(c_{bt}^*, c_{at}^*)\}_{t=0}^\infty$  constitutes a *Walrasian* (or *competitive*) equilibrium if:

(i) Every  $(c_{bt}^*, c_{at}^*)$  solves the individual utility maximisation problem subject to the budget constraints  $p_t c_{bt} + p_{t+1} c_{at} \leq (1 - \varepsilon) p_t$  for  $t > 0$ , and  $p_0 c_{b0} + p_1 c_{a0} \leq (1 - \varepsilon) p_0 + \varepsilon (\sum_t p_t) + M$  for  $\varepsilon > 0$ .

(ii) The feasibility requirement  $(c_{a,t-1}^* + c_{bt}^* = 1)$  is satisfied for all  $t \geq 0$  (we put  $c_{a,-1}^* = 0$ ).

**Proposition 20.H.1.** Any Walrasian equilibrium  $(p_0, \dots, p_t, \dots)$ ,  $\{(c_{bt}^*, c_{at}^*)\}_{t=0}^\infty$ , with  $\sum_t p_t < \infty$  is a Pareto optimum; that is, there are no feasible consumptions  $\{(c_{bt}, c_{at})\}_{t=0}^\infty$  such that  $u(c_{bt}, c_{at}) \geq u(c_{bt}^*, c_{at}^*)$  for all  $t \geq 0$ , with strict inequality for some  $t$ .

**Proposition 20.H.2.** Suppose that at an equilibrium we have  $\sum_t p_t < \infty$ . Then  $M = 0$ .

## **Part V**

# **Welfare Economics and Incentives**

## Chapter 21

# Social Choice Theory

## Chapter 22

# Elements of Welfare Economics and Axiomatic Bargaining

## **Chapter 23**

# **Incentives and Mechanism Design**

# Bibliography

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