Microeconomic Theory (Mas-Colell et al., 1995): Definitions etc.

Preface

This document compiles definitions, propositions, corollaries, and lemmas from *Microeconomic Theory* by Mas-Colell et al., 1995. Sections that contain none, and the appendices are not included. All numberings correspond to those in the book.

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Part I

Individual Decision Making

Chapter 1

Preference and Choice

1.B Preference Relations

Definition 1.B.1. The preference relation $\succsim$ is *rational* if it possesses the following two properties:

(i) *Completeness*: for all $x, y \in X$ we have that $x \succsim y$ or $y \succsim x$ (or both).

(ii) *Transitivity*: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Proposition 1.B.1. If $\succsim$ is rational, then

(i) $\succ$ is both *irreflexive* ($x \succ x$ never holds) and *transitive* (if $x \succ y$ and $y \succ z$, then $x \succ z$).

(ii) $\sim$ is *reflexive* ($x \sim x$ for all x), *transitive* (if $x \sim y$ and $y \sim z$, then $x \sim z$), and *symmetric* (if $x \sim y$, then $y \sim x$).

(iii) If $x \succ y \succsim z$ then $x \succ z$.

Definition 1.B.2. A function $u : X \to \mathbb{R}$ is a *utility function representing* $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y).$$

Proposition 1.B.2. A preference relation $\succsim$ can be represented by a utility function only if it is rational.

1.C Choice Rules

Definition 1.C.1. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* if the following property holds:

If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

Definition 1.C.2. Given a choice structure $(\mathscr{B}, C(\cdot))$ the *revealed preference relation* $\succsim^*$ is defined by

$x \succsim^* y \iff$ there is some $B \in \mathscr{B}$ such that $x, y \in B$ and $x \in C(B)$.

1.D The Relationship between Preference Relations and Choice Rules

Proposition 1.D.1. Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$ satisfies the weak axiom.

Definition 1.D.1. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalises* $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B, \succsim)$$

for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

Proposition 1.D.2. If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) the weak axiom is satisfied,

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalises $C(\cdot)$ relative to $\mathscr{B}$; that is, $C(B) = C^*(B, \succsim)$, for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the *only* preference relation that does.

Chapter 2

Consumer Choice

2.D Competitive Budgets

Definition 2.D.1. The *Walrasian, or competitive budget set* $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

2.E Demand Functions and Comparative Statics

Definition 2.E.1. The Walrasian demand correspondence $x(p, w)$ is *homogeneous of degree zero* if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

Definition 2.E.2. The Walrasian demand correspondence $x(p, w)$ satisfies Walras' law, if for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p, w)$.

Proposition 2.E.1. If the Walrasian demand function $x(p, w)$ is homogeneous of degree zero, then for all p and w:

$$\sum_{k=1}^{L} \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} w = 0 \text{ for } \ell = 1, \ldots, L.$$

In matrix notation, this is expressed as

$$D_p x(p, w) p + D_w x(p, w) w = 0.$$

Proposition 2.E.2. If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p, w)}{\partial p_k} + x_k(p, w) = 0 \text{ for } k = 1, \ldots, L$$

or, written in matrix notion,

$$p \cdot D_p x(p, w) + x(p, w)^T = 0^T.$$

Proposition 2.E.3. If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} \frac{\partial x_\ell(p, w)}{\partial w} = 1,$$

or, written in matrix notation,

$$p \cdot D_w x(p, w) = 1.$$

2.F The Weak Axiom of Revealed Preference and the Law of Demand

Definition 2.F.1. The Walrasian demand function $x(p, w)$ satisfies the *weak axiom of revealed preference* (the WA) if the following property holds for any two price wealth situations (p, w) and (p', w'):

$$\text{If } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \text{ then } p' \cdot x(p, w) > w'.$$

Proposition 2.F.1. Suppose the Walrasian demand function $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom if and only if the following property holds:

For any compensated price change from an initial situation p, w to a new price wealth pair $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0,$$

with strict inequality whenever $x(p, w) \neq x(p', w')$.

Proposition 2.F.2. If a differentiable Walrasian demand function $x(p, w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any (p, w), the Slutsky (subsitution) matrix $S(p, w)$ satisfies $v \cdot S(p, w)v \leq 0$ and any $v \in \mathbb{R}^L$.

Proposition 2.F.3. Suppose that the Walrasian demand function $x(p, w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $p \cdot S(p, w) = 0$ and $S(p, w)p = 0$ for any (p, w).

Chapter 3

Classical Demand Theory

3.B Preference Relations: Basic Properties

Definition 3.B.1. The preference relation $\succsim$ is *rational* if it possesses the following two properties:

(i) *Completeness*: for all $x, y \in X$ we have that $x \succsim y$ or $y \succsim x$ (or both).

(ii) *Transitivity*: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Definition 3.B.2. The preference relation $\succsim$ on X is *monotone* if $x \in X$ and $y \gg x$ implies $y \succ x$. It is *strongly monotone* if $y \geq x$ and $y \neq x$ imply that $y \succ x$.

Definition 3.B.3. The preference relation $\succsim$ on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $||y - x|| \leq \varepsilon$ and $y \succ x$.

Definition 3.B.4. The preference relation $\succsim$ on X is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

Definition 3.B.5. The preference relation $\succsim$ on X is strictly convex if for every x, we have that $y \succsim x, z \succsim x$, and $y \neq z$ implies $\alpha y + (1 - \alpha)z \succ x$ for all $\alpha \in (0, 1)$.

Definition 3.B.6. A monotone preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}^{L-1}_{+}$ is *quasilinear* with respect to commodity 1 (called, in this case, the *numeraire* commodity) if

(i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.

(ii) Good 1 is desirable; that is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

3.C Preference and Utility

Definition 3.C.1. The preference relation $\succsim$ on X is *continuous* if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.

Proposition 3.C.1. Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

3.D The Utility Maximisation Problem

Proposition 3.D.1. If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximisation problem has a solution.

Proposition 3.D.2. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on a consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

(i) *Homogeneity of degree zero in (p, w):* $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar α.

(ii) *Walras' law:* $p \cdot x = w$ for all $x \in x(p, w)$.

(iii) *Convexity/uniqueness:* If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is *strictly convex*, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Proposition 3.D.3. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p, w)$ is

(i) Homogeneous of degree zero.

(ii) Strictly increasing in w and nonincreasing in p_ℓ for and ℓ.

(iii) Quasiconvex; that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

(iv) Continuous in p and w.

3.E The Expenditure Minimisation Problem

Proposition 3.E.1. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have

(i) If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimised expenditure level in this EMP is exactly w.

(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximised utility level in this UMP is exactly u.

Proposition 3.E.2. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the expenditure function $e(p, u)$ is

(i) Homogeneous of degree one in p.

(ii) Strictly increasing in u and nondecreasing in p_ℓ for any ℓ.

(iii) Concave in p.

(iv) Continuous in p and u.

Proposition 3.E.3. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

(i) *Homogeneity of degree zero in* p: $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.

(ii) *No excess utility:* For any $x \in h(p, u), u(x) = u$.

(iii) *Convexity/uniqueness:* If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is *strictly* convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p, u)$.

Proposition 3.E.4. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p, u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p, u)$ satisfies the compensated law of demand: For all p' and p'',

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0.$$

3.F Duality: A Mathematical Introduction

Definition 3.F.1. For any nonempty closed set $K \subset \mathbb{R}^L$, the *support function* of K is defined for any $p \in \mathbb{R}^L$ to be

$$\mu_K(p) = \inf\{p \cdot x : x \in K\}.$$

Proposition 3.F.1 (The Duality Theorem)**.** Let K be a nonempty closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{p}$. Moreover, in this case,

$$\nabla \mu_K(\bar{p}) = \bar{x}.$$

3.G Relationships between Demand, Indirect Utility, and Expenditure Functions

Proposition 3.G.1. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. For all p and u, the Hicksian demand $h(p, u)$ is the derivatve vector of the expenditure function with respect to prices:

$$h(p, u) = \nabla_p e(p, u).$$

That is, $h_\ell(p, u) = \partial e(p, u)/\partial p_\ell$ for all $\ell = 1, \ldots, L$.

Proposition 3.G.2. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot, u)$ is continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix by $D_p h(p, u)$. Then

(i) $D_p h(p, u) = D^2_p e(p, u)$.

(ii) $D_p h(p, u)$ is a negative semidefinite matrix.

(iii) $D_p h(p, u)$ is a symmetric matrix.

(iv) $D_p h(p, u) p = 0$.

Proposition 3.G.3 (The Slutsky Equation)**.** Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p, w), and $u = v(p, w)$, we have

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{x_\ell(p,w)}{p_k} + \frac{x_\ell(p,w)}{\partial w} x_k(p,w) \quad \text{for all } \ell, k$$

or equivalently, in matrix notation,

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T.$$

Proposition 3.G.4 (Roy's Identity)**.** Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and stricly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$. Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w}).$$

That is, for every $\ell = 1, \ldots, L$:

$$x_\ell(\bar{p}, \bar{w}) = -\frac{\partial v(\bar{p}, \bar{w})/\partial p_\ell}{\partial v(\bar{p}, \bar{w})/\partial w}.$$

Proposition 3.G.5. Suopose that $e(p, u)$ is strictly increasing in u and is continuous, increasing, homogeneous of degree one, concave, and differentiable in p. Then, for every utility level u, $e(p, u)$ is the expenditure function associated with the at-least-as-good-as set

$$V_u = \{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p,u) \text{ for all } p \gg 0\}$$

3.H Welfare Evaluation of Economic Changes

Proposition 3.H.1. Suppose that the consumer has a locally nonsatiated rational preference relation $\succsim$. If $(p^1 - p^0) \cdot x^0 < 0$, then the consumer is strictly better off under price wealth situation (p^1, w) than under (p^0, w).

Proposition 3.H.2. Suppose that the consumer has a differentiable expenditure function. Then if $(p^1 - p^0) \cdot x^0 > 0$, there is a sufficiently small $\bar{\alpha} \in (0, 1)$ such that for all $\alpha < \bar{\alpha}$, we have $e((1 - \alpha)p^0 + \alpha p^1, u^0) > w$, and so the consumer is strictly better off under price wealth situation (p^0, w) than under $((1 - \alpha)p^0 + \alpha p^1, w)$.

3.I The Strong Axiom of Revealed Preference

Definition 3.I.1. The market demand function $x(p, w)$ satisfies the *strong axiom of revealed preference* (the SA) if for any list

$$(p^1, w^1), \ldots, (p^N, w^N)$$

with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n < N - 1$, we have $p^N \cdot x(p^1, w^1) > w^N$ whenever $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N - 1$.

Chapter 4

Aggregate Demand

4.B Aggregate Demand and Aggregate Wealth

Proposition 4.B.1. A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences admit indirect utility functions of the Gorman form with the coefficients on w_i the same for every consumer i. That is:

$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

4.C Aggregate Demand and the Weak Axiom

Definition 4.C.1. The aggregate demand function $x(p, w)$ satisfies the weak axiom (WA) if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$ imply $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').

Definition 4.C.2. The individual demand function $x_i(p, w_i)$ satisfies the *uncompensated law of demand (ULD)* property if

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0$$

for any p, p', and w_i, with strict inequality if $x_i(p', w_i) \neq x_i(p, w_i)$. The analogous definition applies to the aggregate demand function $x(p, w)$.

Proposition 4.C.1. If every consumer's Walrasian demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property, so does the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$. As a consequence, the aggregate demand $x(p, w)$ satisfies the weak axiom.

Proposition 4.C.2. If $\succsim_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property.

Proposition 4.C.3. Suppose that $\succsim_i$ is defined on the consumption set $X = \mathbb{R}^L_+$ and is representable by a twice continuously differentiable concave function $u_i(\cdot)$. If

$$-\frac{x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4 \quad \text{for all } x_i,$$

then $x_i(p, w_i)$ satisfies the unrestricted law of demand (ULD) property.

Proposition 4.C.4. Suppose that all consumers have identical preferences $\succsim$ defined on $\mathbb{R}^L_+$ [with individual demand functions denoted by $\tilde{x}(p, w)$] and that individual wealth is uniformly distributed on an interval $[0, \bar{w}]$ (strictly speaking this requires a continuum of consumers). Then the aggregate (rigorously, the average) demand function

$$x(p) = \int_0^{\bar{w}} \tilde{x}(p, w) dw$$

satisfies the unrestricted law of demand (ULD) property.

4.D Aggregate Demand and the Existence of a Representative Consumer

Definition 4.D.1. A *positive representative consumer* exists of there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p, w)$ is precisely the Walrasian demand function generated by this preference relation. That is, $x(p, w) \succ x$ whenever $x \neq x(p, w)$ and $p \cdot x \leq w$.

Definition 4.D.2. A *(Berson-Samuelson) social welfare function* is a function $W : \mathbb{R}^I \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, \ldots, u_I) \in \mathbb{R}^I$ of utility levels for the I consumers in the economy.

Proposition 4.D.1. Suppose that for each level of prices p and aggregate wealth w, the wealth distribution $w_1(p, w), \ldots, w_I(p, w)$ solves

$$\begin{aligned} \max_{w_1, \ldots, w_I} \; & W\left(v_1(p, w_1), \ldots, v_I(p, w_I)\right) \\ \text{s.t.} \; & \sum_{i=1}^{I} w_i \leq w. \end{aligned} \tag{4.D.1}$$

Then the value function $v(p, w)$ of problem (4.D.1) is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$.

Definition 4.D.3. The positive representative consumer $\succsim$ for the aggregate demand $x(p, w) = \sum_i x_i(p, w_i(p, w))$ is a *normative representative consumer* relative to the social welfare function $W(\cdot)$ if for every (p, w), the distribution of wealth $w_1(p, w), \ldots, w_I(p, w)$ solves problem (4.D.1) and, therefore, the value function of problem (4.D.1) is an indirect utility function for $\succsim$.

Chapter 5

Production

5.B Production Sets

Proposition 5.B.1. The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.

Proposition 5.B.2. For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$.

5.C Profit Maximisation and Cost Minimisation

Proposition 5.C.1. Suppose that $\pi(\cdot)$ is the profit function of the production set Y and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies the free disposal property. Then

(i) $\pi(\cdot)$ is homogeneous of degree one.

(ii) $\pi(\cdot)$ is convex.

(iii) If Y is convex, then $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$.

(iv) $y(\cdot)$ is homogeneous of degree zero.

(v) If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty).

(vi) (*Hotelling's lemma*) If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.

(vii) If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.

Proposition 5.C.2. Suppose that $c(p, w)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $z(w, q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then

(i) $c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.

(ii) $c(\cdot)$ is a concave function of w.

(iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q)$ for all $w \gg 0\}$.

(iv) $z(\cdot)$ is homogeneous of degree zero in w.

(v) If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(p, w)$ is single-valued.

(vi) (*Shephard's lemma*) If $z(\bar{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w at $\bar{w}$ and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

(vii) If $z(\cdot)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ is a symmetric and negative semi-definite matrix with $D_w z(\bar{w}, q)\bar{w} = 0$.

(viii) If $f(\cdot)$ is homogeneous of degree one (i.e. exhibits constant returns to scale), then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.

(ix) If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q (in particular, marginal costs are nondecreasing in q).

5.E Aggregation

Proposition 5.E.1. For all $p \gg 0$, we have

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$ $(= \{\sum_j y_j : y_j \in y_j(p)$ for every $j\})$.

5.F Efficient Production

Definition 5.F.1. A production vetor $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Proposition 5.F.1. If $y \in Y$ is profit maximising for some $p \gg 0$, then y is efficient.

Proposition 5.F.2. Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximising production for some nonzero price vector $p \geq 0$.

Chapter 6

Choice Under Uncertainty

6.B Expected Utility Theory

Definition 6.B.1. A *simple lottery* L is a list $L = (p_1, \ldots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

Definition 6.B.2. Given K simple lotteries $L_k = (p_1^k, \ldots, p_N^k)$, $k = 1, \ldots, K$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the *compound lottery* $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \ldots, K$.

Definition 6.B.3. The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is *continuous* if for any $L, L', L'' \in \mathscr{L}$, the sets

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$$

and

$$\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed.

Definition 6.B.4. The preference relation $\succsim$ on the space simple lotteries $\mathscr{L}$ satisfies the *independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

Definition 6.B.5. The utility function $U : \mathscr{L} \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathscr{L}$ we have

$$U(L) = u_1 p_1 + \cdots + u_N p_N.$$

A utility function $U : \mathscr{L} \to \mathbb{R}$ with the expected utility form is called a *von Neumann-Morgenstern (v.N-M) expected utility function.*

Proposition 6.B.1. A utility function $U : \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \ldots, K$, and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0, \sum_k \alpha_k = 1$.

Proposition 6.B.2. Suppose that $U : \mathscr{L} \to \mathbb{R}$ is a v.N-M expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\tilde{U} : \mathscr{L} \to \mathbb{R}$ is another v.N-M utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.

Proposition 6.B.3 (Expected Utility Theorem)**.** Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n-1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n.$$

6.C Money Lotteries and Risk Aversion

Definition 6.C.1. A decision maker is a *risk averse* (or exhibits *risk aversion*) if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x dF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself. If the decision maker is always [i.e. for any $F(\cdot)$] indifferent between these two lotteries, we say that he is *risk neutral.* Finally, we say that he is *strictly risk averse* if indifference holds only when the two lotteries are the same [i.e. when $F(\cdot)$ is degenerate].

Definition 6.C.2. Given a Bernoulli utility function $u(\cdot)$ we defined the following concepts:

(i) The *certainty equivalent of* $F(\cdot)$, denoted $c(F, u)$, is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F, u)$; that is

$$u\left(c(F, u)\right) = \int u(x) dF(x).$$

(ii) For any fixed amount of money x and positive number ε, the *probability premium* denoted by $\pi(x, \varepsilon, u)$, is the excess on winning the probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$. That is

$$u(x) = \left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u(x + \varepsilon) + \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u(x - \varepsilon).$$

Proposition 6.C.1. Suppose a decision maker is an expected utility maximiser with a Bernoulli utility function $u(\cdot)$ on amounts of money. Then the following properties are equivalent:

(i) The decision maker is risk averse.

(ii) $u(\cdot)$ is concave.

(iii) $c(F, u) \leq \int x dF(x)$ for all $F(\cdot)$.

(iv) $\pi(x, \varepsilon, u) \geq 0$ for all x, ε.

Definition 6.C.3. Given a (twice differentiable) Bernoulli utility function $u(\cdot)$ for money, the *Arrow Pratt coefficient of absolute risk aversion* at x is defined as $r_A(x) = -u''(x)/u'(x)$.

Definition (More-risk-averse-than)**.** Given two Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$, when can we say that $u_2(\cdot)$ is unambiguously *more risk averse than* $u_1(\cdot)$? Several possible approaches to a definition seem plausible:

(i) $r_A(x, u_2) \geq r_A(x, u_1)$ for every x.

(ii) There exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$ at all x; that is, $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$. [In other words, $u_2(\cdot)$ is "more concave" than $u_1(\cdot)$.]

(iii) $c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$.

(iv) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

(v) Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds $F(\cdot)$ at least as good as $\bar{x}$. That is, $\int u_2(x)dF(x) \geq u_2(\bar{x})$ implies $\int u_1(x)dF(x) \geq u_1(\bar{x})$ for any $F(\cdot)$ and $\bar{x}$.

Proposition 6.C.2. Definitions (i) to (v) of the *more-risk-averse-than* relation are equivalent.

Definition 6.C.4. The Bernoulli utility function $u(\cdot)$ for money exhibits *decreasing absolute risk aversion* if $r_A(x, u)$ is a decreasing function of x.

Proposition 6.C.3. The following properties are equivalent:

(i) The Bernoulli utility function $u(\cdot)$ exhibits decreasing absolute risk aversion.

(ii) Whenever $x_2 < x_1, u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.

(iii) For any risk $F(z)$, the certainty equivalent of the lottery formed adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x + z)dF(z)$, is such that $(x - c_x)$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.

(iv) The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.

(v) For any $F(z)$, if $\int u(x_2 + z)dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1 + z)dF(z) \geq u(x_1)$.

Definition 6.C.5. Given a Bernoulli utility function $u(\cdot)$, the *coefficient of relative risk aversion at x* is $r_R(x, u) = -xu''(x)/u'(x)$.

Proposition 6.C.4. The following conditions for a Bernoulli utility function $u(\cdot)$ on amounts of money are equivalent:

(i) $r_R(x, u)$ is decreasing in x.

(ii) Whenever $x_2 < x_1$, $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.

(iii) Given any risk $F(t)$ on $t > 0$, the certainty equivalent $\bar{c}_x$ defined by $u(\bar{c}_x) = \int u(tx)dF(t)$ is such that $x/\bar{c}_x$ is decreasing in x.

6.D Comparison of Payoff Distributions in Terms of Return and Risk

Definition 6.D.1. The distribution $F(\cdot)$ *first-order stochastically dominates* $G(\cdot)$ if, for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition 6.D.1. The distribution of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution $G(\cdot)$ if and only if $F(x) \leq G(x)$ for every x.

Definition 6.D.2. For any two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ *second-order stochastically dominates* (or *is less risky than*) $G(\cdot)$ if for every nondecreasing concave function $u : \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition 6.D.2. Consider two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean. Then the following statements are equivalent:

(i) $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.

(ii) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.

(iii) Property 6.D.2 holds.

6.E State-Dependent Utility

Definition 6.E.1. A *random variable* is a function $g : S \to \mathbb{R}_+$ that maps states into monetary outcomes.

Definition 6.E.2. The preference relation $\succsim$ has an *extended expected utility representation* if for every $s \in S$, there is a function $u_s : \mathbb{R}_+ \to \mathbb{R}$ such that for any $(x_1, \ldots, x_S) \in \mathbb{R}_+^S$ and $(x'_1, \ldots, x'_S) \in \mathbb{R}_+^S$,

$$(x_1, \ldots, x_S) \succ (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x'_s).$$

Definition 6.E.3. The preference relation $\succsim$ on $\mathscr{L}$ satisfies the *extended independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0, 1)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

Proposition 6.E.1 (Extended Expected Utility Theorem)**.** Suppose that the preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and extended independence axioms. Then we can assign a utility function $u_s(\cdot)$ for money in every state s such that for any $L = (F_1, \ldots, F_S)$ and $L' = (F'_1, \ldots, F'_S)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_s \left(\int u_s(x_s)dF_s(x_s) \right) \geq \sum_s \left(\int u_s(x_s)dF'_s(x_s) \right).$$

Definition 6.E.4. The preference relation $\succsim$ satisfies the *sure-thing axiom* if, for any subset of states $E \subset S$ (E is called an *event*), whenever $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ differ only in the entries corresponding to E (so that $x'_s = x_s$ for $s \notin E$), the preference ordering between $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ is independent of the particular (common) payoffs for states not in E. Formally, suppose that $(x_1, \ldots, x_S), (x'_1, \ldots, x'_S), (\bar{x}_1, \ldots, \bar{x}_S)$, and $(\bar{x}'_1, \ldots, \bar{x}'_S)$ are such that

$$\begin{aligned} &\text{For all } s \notin E: \quad x_s = x'_s \quad \text{and} \quad \bar{x}_s = \bar{x}'_s. \\ &\text{For all } s \in E: \quad x_s = \bar{x}_s \quad \text{and} \quad x'_s = \bar{x}'_s. \end{aligned}$$

Then $(x_1, \ldots, x_S) \succsim (\bar{x}'_1, \ldots, \bar{x}'_S)$ if and only if $(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S)$.

Proposition 6.E.2. Suppose that there are at least three states and that the preferences $\succsim$ on $\mathbb{R}_+^S$ are continuous and satisfy the sure-thing axiom. Then $\succsim$ admits and extended expected utility representation.

6.F Subjective Probability Theory

Definition 6.F.1. The state preferences $(\succsim_1, \ldots, \succsim_S)$ on state lotteries are *state uniform* if $\succsim_s = \succsim'_s$ for any s and s'.

Proposition 6.F.1 (Subjective Expected Utility Theorem)**.** Suppose that the preference relation $\succsim$ on $\mathscr{L}$ satisfies the continuity and extended independence axioms. Suppose, in addition, that the derived state preferences are state uniform. Then there are probabilities $(\pi_1, \ldots, \pi_S) \gg 0$ and a utility function $u(\cdot)$ on amounts of money such that for any $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ we have

$$(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u(x_s) \geq \sum_s \pi_s u(x'_s).$$

Part II

Game Theory

Chapter 7

Basic Elements of Noncooperative Games

7.C The Extensive Form Representation of a Game

Definition 7.C.1. A game is one of *perfect information* if each information set contains a single decision node. Otherwise, it is a game of *imperfect information.*

7.D Strategies and the Normal Form Representation of a Game

Definition 7.D.1. Let $\mathscr{H}_i$ denote the collection of player i's information sets, $\mathscr{A}$ the set of possible actions in the game, and $C(H) \subset \mathscr{A}$ the set of actions possible at information set H. A *strategy* for player i is a function $s_i : \mathscr{H}_i \to \mathscr{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathscr{H}_i$.

Definition 7.D.2. For a game with I players, the *normal form representation* Γ_N specifies for each player i a set of strategies S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1, \dots, s_I)$ giving the von Neumann-Morgenstern utility levels associated with the (possibly random) outcome arising from strategies $s_1, \dots, s_I$. Formally, we write $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$.

7.E Randomized Choices

Definition 7.E.1. Given player i's (finite) pure strategy set S_i, a *mixed strategy* for player i, $\sigma_i : S_i \to [0, 1]$, assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

Definition 7.E.2. Given an extensive form game Γ_E, a *behaviour strategy* for player i specifies, for every information set $H \in \mathscr{H}_i$ and action $a \in C(H)$, a probability $\lambda_i(a, H) \geq 0$, with $\sum_{a \in C(H)} \lambda_i(a, H) = 1$ for all $H \in \mathscr{H}_i$.

Chapter 8

Simultaneous-Move Games

8.B Dominant and Dominated Strategies

Definition 8.B.1. A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$.

Definition 8.B.2. A strategy $s_i \in S_i$ is a *strictly dominated* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

In this case, we say that strategy s_i' *strictly dominates* strategy s_i.

Definition 8.B.3. A strategy $s_i \in S_i$ is a *weakly dominated* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),$$

with strict inequality for some s_{-i}. In this case, we say that strategy s_i' *weakly dominates* strategy s_i. A strategy is a *weakly dominant strategy* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it weakly dominates every other strategy in S_i.

Definition 8.B.4. A strategy $\sigma_i \in \Delta(S_i)$ is *strictly dominated* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$,

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

In this case, we say that strategy σ_i' *strictly dominates* strategy σ_i. A strategy σ_i is a *strictly dominant strategy* for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in $\Delta(S_i)$.

Proposition 8.B.1. Player i's pure strategy $s_i \in S_i$ is strictly dominated in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

8.C Rationalisable Strategies

Definition 8.C.1. In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, a strategy σ_i is a *best response* for player i to his rivals' strategies σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$. Strategy σ_i is *never a best response* if there is no σ_{-i} for which σ_i is a best response.

Definition 8.C.2. In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player i's *rationalisable strategies*.

8.D Nash Equilibrium

Definition 8.D.1. A strategy profile $s = (s_1, \dots, s_I)$ constitutes a *Nash equilibrium* of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for every $i = 1, \dots, I$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$.

Definition 8.D.2. A mixed strategy profile $\sigma = \sigma_1, \dots, \sigma_I$ constitutes a *Nash equilibrium* of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if for every $i = 1, \dots, I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$.

Proposition 8.D.1. Let $S_i^+ \subset S_i$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\sigma = \sigma_1, \dots, \sigma_I$. Strategy profile σ is a Nash equilibrium in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if for all $i = 1, \dots, I$,

(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ for all $s_i, s_i' \in S_i^+$

(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all $s_i \in S_i^+$ and all $s_i' \notin S_i^+$.

Corollary 8.D.1. Pure strategy profile $s = (s_1, \dots, S_I)$ is a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if it is a (degenerate) mixed strategy Nash equilibrium game of $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$.

Proposition 8.D.2. Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which sets $S_1, \dots, S_I$ have a finite number of elements has a mixed strategy Nash equilibrium.

Proposition 8.D.3. A Nash equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $i = 1, \dots, I$,

(i) S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$.

(ii) $u_i(s_1, \dots, s_I)$ is continuous in $(s_1, \dots, s_I)$ and quasiconcave in s_i.

8.E Games of Incomplete Information: Bayesian Nash Equilibrium

Definition 8.E.1. A (pure strategy) *Bayesian Nash equilibrium* for the Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ that constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\mathscr{L}_i\}, \{\tilde{u}_i(\cdot)\}]$. That is, for every $i = 1, \ldots, I$,

$$\tilde{u}_i\left(s_i(\cdot), s_{-i}(\cdot)\right) \geq \tilde{u}_i\left(s_i'(\cdot), s_{-i}(\cdot)\right)$$

for all $s_i'(\cdot) \in \mathscr{L}_i$, where

$$\tilde{u}_i\left(s_1(\cdot), \ldots, s_I(\cdot)\right) = E_\theta\left[u_i\left(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i\right)\right].$$

Proposition 8.E.1. A profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian Nash equilibrium in Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ if and only if for all i and all $\bar{\theta}_i \in \Theta_i$ occurring with positive probability

$$E_{\theta_{-i}}\left[u_i\left(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i\right) | \bar{\theta}_i\right] \geq E_{\theta_{-i}}\left[u_i\left(s_i'(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i\right) | \bar{\theta}_i\right]$$

for all $s_i' \in S_i$, where the expectation is taken over realisations of the other players' random variables conditional on player i's realisation of his signal $\bar{\theta}_i$.

8.F The Possibility of Mistakes: Trembling-Hand Perfection

Definition 8.F.1. A Nash equilibrium σ of a game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (*normal form*) *trembling-hand perfect* if there is *some* sequence of perturbed games $\{\Gamma_{\varepsilon^k}\}_{k=1}^{\infty}$ that converges to Γ_N [in the sense that $\lim_{k\to\infty} \varepsilon_i^k(s_i) = 0$ for all i and $s_i \in S_i$], for which there is *some* associated sequence of Nash equilibria $\{\sigma^k\}_{k=1}^{\infty}$ that converges to σ (i.e., such that $\lim_{k\to\infty} \sigma^k = \sigma$).

Proposition 8.F.1. A Nash equilibrium σ of a game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (normal form) trembling-hand perfect if and only if there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \sigma^k = \sigma$ and σ_i is the best response to every element of sequence $\{\sigma_{-i}^k\}_{k=1}^{\infty}$ for all $i = 1, \ldots, I$.

Proposition 8.F.2. If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a (normal form) trembling-hand perfect Nash equilibrium, then σ_i is not a weakly dominated strategy for any $i = 1, \ldots, I$. Hence, in any (normal form) trembling-hand perfect Nash equilibrium, no weakly dominated pure strategy can be played with positive probability.

Chapter 9

Dynamic Games

9.B Sequential Rationality, Backward Induction, and Subgame Perfection

Proposition 9.B.1 (Zermelo's Theorem)**.** Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

Definition 9.B.1. A *subgame* of an extensive form game Γ_E is a subset of the game having the following properties:

(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* these nodes.

(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x. (That is, there are no "broken" information sets.)

Definition 9.B.2. A profile of strategies $\sigma = (\sigma_1, \dots, \sigma_I)$ in an I-player extensive form game Γ_E is a *subgame perfect Nash equilibrium* (SPNE) if it induces a Nash equilibrium in every subgame of Γ_E.

Proposition 9.B.2. Every finite game of perfect information Γ_E has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.

Proposition 9.B.3. Consider an extensive form game Γ_E and some subgame S of Γ_E. Suppose that strategy profile σ^S is an SPNE in subgame S, and let $\hat{\Gamma}_E$ be the reduced game formed by replacing subgame S by a terminal node with payoffs equal to those arising from play of σ^S. Then:

(i) In any SPNE σ of Γ_E in which σ^S is the play in subgame S, players' moves at information sets outside subgame S must constitute an SPNE of reduced game $\hat{\Gamma}_E$.

(ii) If $\hat{\sigma}$ is an SPNE of $\hat{\Gamma}_E$, then the strategy profile σ that specifies the moves in σ^S at information sets in subgame S and that specifies the moves in $\hat{\sigma}$ at information sets not in S is an SPNE of Γ_E.

Proposition 9.B.4. Consider an I-player extensive form game Γ_E involving successive play of T simultaneous-move games, $\Gamma_N^t = [I, \{\Delta(S_i^t)\}, \{u_i^t(\cdot)\}]$ for $t = 1, \ldots, T$, with the players observing the pure strategies played in each game immediately after its play is concluded. Assume that each player's payoff is equal to the sum of her payoffs in the plays of the T games. If there is a unique Nash equilibrium in each game Γ_N^t, say $\sigma^t = (\sigma_1^t, \ldots, \sigma_I^t)$, then there is a unique SPNE of Γ_E and it consists of each player i playing strategy σ_i^t in each game Γ_N^t regardless of what has happened previously.

9.C Beliefs and Sequential Rationality

Definition 9.C.1. A *system of beliefs* μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that

$$\sum_{x \in H} \mu(x) = 1$$

for all information sets H.

Definition 9.C.2. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ in extensive form game Γ_E is *sequentially rational at information set H given a system of beliefs* μ if, denoting by $\iota(H)$ the player who moves at information set H, we have

$$E[u_{\iota(H)}|H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)}|H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile σ satisfies this condition for *all* information sets H, then we say that σ is *sequentially rational given belief system* μ.

Definition 9.C.3. A profile of strategies and system of beliefs (σ, μ) is a *weak perfect Bayesian equilibrium* (weak PBE) in extensive form game Γ_E if it has the following properties:

(i) The strategy profile σ is sequentially rational given belief system μ.

(ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible. That is, for any information set H such that $\text{Prob}(H|\sigma) > 0$ (read as "the probability of reaching information set H is positive under strategies σ"), we must have

$$\mu(x) = \frac{\text{Prob}(x|\sigma)}{\text{Prob}(H|\sigma)} \quad \text{for all } x \in H.$$

Proposition 9.C.1. A strategy profile σ is a Nash equilibrium of extensive form game Γ_E if and only if there exists a system of beliefs μ such that

(i) The strategy profile σ is sequentially rational given belief system μ *at all information sets H such that $Prob(H|\sigma) > 0$.*

(ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible.

Definition 9.C.4. A strategy profile and system of beliefs (σ, μ) is a *sequential equilibrium* of extensive form game Γ_E if it has the following properties:

(i) Strategy profile σ is sequentially rational given belief system μ.

(ii) There exists a sequence of completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$, with $\lim_{k\to\infty}\sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty}\mu^k$, where μ^k denotes the beliefs derived from strategy profile σ^k using Bayes' rule.

Proposition 9.C.2. In every sequential equilibrium (σ, μ) of an extensive form game Γ_E, the equilibrium strategy profile σ constitutes a subgame perfect Nash equilibrium of Γ_E.

Part III

Market Equilibrium and Market Failure

Chapter 10

Competitive Markets

10.B Pareto Optimality and Competitive Equilibria

Definition 10.B.1. An *economic allocation* $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $k_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. The allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *feasible* if

$$\sum_{i=1}^{I} x_{\ell i} \leq w_\ell + \sum_{j=1}^{J} y_{\ell j} \quad \text{for } \ell = 1, \ldots, L.$$

Definition 10.B.2. A feasible allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation $(x_1', \ldots, x_I', y_1', \ldots, y_J')$ such that $u_i(x_i') \geq u_i(x_i)$ for all $i = 1, \ldots, I$ and $u_i(x_i') > u_i(x_i)$ for some i.

Definition 10.B.3. The allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and price vector $p^* \in \mathbb{R}^L$ constitute a *competitive* (or *Walrasian*) *equilibrium* if the following conditions are satisfied:

(i) *Profit maximisation:* For each firm j, y_j^* solves

$$\max_{y_j \in Y_i j} p^* \cdot y_j. \tag{10.B.1}$$

(ii) *Utility maximisation:* For each consumer i, x_i^* solves

$$\begin{aligned} &\max_{x_i \in X_i} u_i(x_i) \\ &\text{s.t. } p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot y_j^*). \end{aligned} \tag{10.B.2}$$

(iii) *Market clearing:* For each good $\ell = 1, \ldots, L$,

$$\sum_{i=1}^{I} x_{\ell i}^* = \omega_\ell + \sum_{j=1}^{J} y_{\ell j}^*. \tag{10.B.3}$$

Lemma 10.B.1. If the allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ and price vector $p \gg 0$ satisfy the market clearing condition (Definition 10.B.3) for all goods $\ell \neq k$, and if every consumer's budget constraint is satisfied with equality, so that $p \cdot x_i = p \cdot w_i + \sum_j \theta_{ij} p \cdot y_j$ for all i, then the market for good k also clears.

10.D The Fundamental Welfare Theorems in a Partial Equilibrium Context

Proposition 10.D.1 (The First Fundamental Theorem of Welfare Economics)**.** If the prive p^* and allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ constitutes a competitive equilibrium, then this allocation is Pareto optimal.

Proposition 10.D.2 (The Second Fundamental Theorem of Welfare Economics)**.** For any Pareto optimal levels of utility $(u_1^*, \ldots, u_I^*)$, there are transfers of the numeraire commodity $(T_1, \ldots, T_I)$ satisfying $\sum_i T_i = 0$, such that a competitive equilibrium reached from the endowments $\omega_{m1} + T_1, \ldots, \omega_{mI} + T_I$ yields precisely the utilities $(u_1^*, \ldots, u_I^*)$.

10.F Free Entry and Long-Run Competitive Equilibria

Definition 10.F.1. Given an aggregate demand function $x(p)$ and a cost function $c(q)$ for each potentially active firm having $c(0) = 0$, a triple (p^*, q^*, J^*) is a *long-run competitive equilibrium* if

(i) q^* solves $\max_{q>0} p^*q - c(q)$ (Profit maximisation)

(ii) $x(p^*) = J^*q^*$ (Demand = supply)

(iii) $p^*q^* - c(q^*) = 0$ (Free Entry Condition).

Chapter 11

Externalities and Public Goods

11.B A Simple Bilateral Externality

Definition 11.B.1. An *externality* is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of another agent in the economy.

11.C Public Goods

Definition 11.C.1. A *public good* is a commodity for which use of a unit of the good by one agent does not preclude use by other agents.

Chapter 12

Market Power

12.C Static Models of Oligopoly

Proposition 12.C.1. There is a unique Nash equilibrium (p_1^*, p_2^*) in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost: $p_1^* = p_2^* = c$.

Proposition 12.C.2. In any Nash equilibrium of the Cournot duopoly model with cost $c > 0$ per unit for the two firms and an inverse demand function $p(\cdot)$ satisfying $p'(q) < 0$ for all $q \geq 0$ and $p(0) > c$, the market price is greater than c (the competitive price) and smaller than the monopoly price.

12.D Repeated Interaction

Proposition 12.D.1. The strategies

$$p_{jt}(H_{t-1}) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ equal } (p^m, p^m) \text{ or } t = 1 \\ c & \text{otherwise} \end{cases}$$

constitute a subgame perfect Nash equilibrium (SPNE) of the infinitely repeated Betrand duopoly game if and only if $\delta \geq \frac{1}{2}$ in the firms optimisation problem

$$\max \sum_{t=1}^{\infty} \delta^{t-1} \pi_{jt}, \quad \delta < 1.$$

Proposition 12.D.2. In the infinitely repeated Betrand duopoly game, when $\delta \geq \frac{1}{2}$ repeated choice of any price $p \in [c, p^m]$ can be supported as a subgame perfect Nash equilibrium outcome path using Nash reversion strategies. By contrast, when $\delta < \frac{1}{2}$, any subgame perfect Nash equilibrium outcome path must have all sales occurring at a price equal to c in every period.

12.E Entry

Proposition 12.E.1. Suppose that conditions

(A1) $Jq_J \geq J'q_{J'}$ whenever $J > J'$;

(A2) $q_J \leq q_{J'}$ whenever $J > J'$;

(A3) $p(Jq_J) - c'(q_J) \geq 0$ for all J

are satisfied by the post-entry oligopoly game, that $p'(\cdot) < 0$, and that $c''(\cdot) \geq 0$. Then the equilibrium number of entrants J^*, is at least $J^\circ - 1$, where J° is the socially optimal number of entrants.

12.F The Competitive Limit

Proposition 12.F.1. As the market size grows, the price in any subgame perfect Nash equilibrium of the two-stage Cournot entry model converges to the level of minimum average cost (the "competitive" price). Formally,

$$\max_{p_\alpha \in P_\alpha} |p_\alpha - \bar{c}| \to 0 \text{ as } \alpha \to \infty.$$

Chapter 13

Adverse Selection, Signaling, and Screening

13.B Informational Asymmetries and Adverse Selection

Definition 13.B.1. In the competitive labour market model with unobservable worker productivity levels, a *competitive equilibrium* is a wage rate w^* and a set Θ^* of worker types who accept employment such that

$$\Theta^* = \{\theta : r(\theta) \leq w^*\}$$

and

$$w^* = E[\theta | \theta \in \Theta^*].$$

Proposition 13.B.1. Let W^* denote the set of competitive equilibrium wages for the adverse selection labour market model, and let $W^* = \max\{w : w \in W^*\}$.

(i) If $w^* > r(\underline{\theta})$ and there is an $\varepsilon > 0$ such that $E[\theta | r(\theta) < w'] > w'$ for all $w' \in (w^* - \varepsilon, w^*)$, then there is a unique pure strategy SPNE of the two-stage game-theoretic model. In this SPNE, employed workers receive a wage of w^*, and workers with types in the set $\Theta(w^*) = \{\theta : r(\theta) \leq w^*\}$ accept employment in firms.

(ii) If $w^* = r(\underline{\theta})$, then there are multiple pure strategy SPNEs. However, in every pure strategy SPNE each agent's payoff exactly equals her payoff in the highest-wage competitive equilibrium.

Proposition 13.B.2. In the adverse selection labour market model (where $r(\cdot)$ is strictly increasing with $r(\theta) \leq \theta$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$ and $F(\cdot)$ has an associated density $f(\cdot)$ with $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$), the highest-wage competitive equilibrium is a constrained Pareto optimum.

13.C Signaling

Lemma 13.C.1. In any separating perfect Bayesian equilibrium, $w^*(e^*(\theta_H)) = \theta_H$ and $w^*(e^*(\theta_L)) = \theta_L$; that is, each worker type receives a wage equal to her productivity level.

Lemma 13.C.2. In any separating perfect Bayesian equilibrium, $e^*(\theta_L) = 0$; that is, a low-ability worker chooses to get no education.

13.D Screening

Proposition 13.D.1. In any SPNE of the screening game with observable worker types, a type θ_i worker accepts contract $(w_i^*, t_i^*) = (\theta_i, 0)$, and firms earn zero profits.

Lemma 13.D.1. In any equilibrium, whether pooling or separating, both firms must earn zero profits.

Lemma 13.D.2. No pooling equilibria exist.

Lemma 13.D.3. If (w_L, t_L) and (w_H, t_H) are the contracts signed by the low- and high-ability workers in a separating equilibrium, then both contracts yield zero profits; that is, $w_L = \theta_L$ and $w_H = \theta_H$.

Lemma 13.D.4. In any separating equilibrium, the low-ability workers accept contract $(\theta_L, 0)$; that is, they receive the same contract as when no informational imperfections are present in the market.

Lemma 13.D.5. In any separating equilibrium, the high-ability workers accept contract $(\theta_H, \hat{t}_H)$, where $\hat{t}_H$ satisfies $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$.

Proposition 13.D.2. In any subgame perfect Nash equilibrium of the screening game, low-ability workers accept contract $(\theta_L, 0)$, and high-ability workers accept contract $(\theta_H, \hat{t}_H)$, where $\hat{t}_H$ satisfies $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$.

Chapter 14

The Principal-Agent Problem

14.B Hidden Actions (Moral Hazard)

Proposition 14.B.1. In the principal-agent model with observable managerial effort, an optimal contract specifies that the manager chooses the effort e^* that maximises $\left[\int \pi f(\pi|e)d\pi - v^{-1}(\bar{u} + g(e))\right]$ and pays the manager a fixed ware $w^* = v^{-1}(\bar{u} + g(e^*))$. This is the uniquely optimal contract if $v''(w) < 0$ at all w.

Proposition 14.B.2. In the principal-agent model with unobservable managerial effort and a risk-neutral manager, an optimal contract generates the same effort choice and expected utilities for the manager and the owner as when effort is observable.

Lemma 14.B.1. In any solution to the problem

$$\begin{aligned}
&\min_{w(\pi)} \int w(\pi) f(\pi|e) d\pi \\
&\quad \text{s.t. (i)} \int v\left(w(\pi)\right) f(\pi|e) d\pi - g(e) \geq \bar{u} \\
&\qquad\quad \text{(ii) } e \text{ solves } \max_{\tilde{e}} \int v\left(w(\pi)\right) f(\pi|\tilde{e}) d\pi - g(\tilde{e})
\end{aligned}$$

with $e = e_H$, both $\gamma > 0$ and $\mu > 0$.

Proposition 14.B.3. In the principal-agent model with unobservable manager effort, a risk-averse manager, and two possible effort choices, the optimal compensation scheme for implementing e_H satisfies

$$\frac{1}{v'\left(w(\pi)\right)} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right],$$

gives the manager expected utility $\tilde{u}$, and involves a larger expected wage payment than is required when effort is observable. The optimal compensation scheme for implementing e_L involves the same fixed wage payment as if effort were observable. Whenever the optimal effort level with observable effort would be e_H, nonobservability causes a welfare loss.

14.C Hidden Information (and Monopolistic Screening)

Proposition 14.C.1. In the principal-agent model with an observable state variable θ, the optimal contract involves an effort level e_i^* in state θ_i such that $\pi(e_i^*) = g_e(e_i^*, \theta)$ and fully insures the manager, setting his wage in each state θ_i at the level w_i^* such that $v\left(w_i^* - g(e_i^*, \theta_i)\right) = \bar{u}$.

Proposition 14.C.2 (The Revelation Principle)**.** Denote the set of possible states by Θ. In searching for an optimal contract, the owner can without loss restrict himself to contracts of the following form:

(i) After the state θ is realised, the manager is required to announce which state has occurred.

(ii) The contract specifies an outcome $[w(\hat{\theta}), e(\hat{\theta})]$ for each possible announcement $\hat{\theta} \in \Theta$.

(iii) In every state $\theta \in \Theta$, the manager finds is optimal to report the state *truthfully.*

Lemma 14.C.1. In the problem

$$\begin{aligned}
\max_{w_H, e_H \geq 0, w_L, e_L > 0} \; & \lambda[\pi(e_H) - w_H] + (1-\lambda)[\pi(e_L) - w_L] \\
\text{s.t. (i) } & w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}) \\
\text{(ii) } & w_H - g(e_H, \theta_H) \geq v^{-1}(\bar{u}) \\
& (\textit{reservation utility} \text{ (or } \textit{individual rationality}) \textit{ constraint}) \\
\text{(iii) } & w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H) \\
\text{(iv) } & w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L) \\
& (\textit{incentive compatibility} \text{ (or } \textit{truth-telling} \text{ or } \textit{self-selection}) \textit{ constraints})
\end{aligned}$$

we can ignore constraint (ii). That is, a contract is a solution to the problem if and only if it is the solution to the problem derived from it by dropping (ii).

Lemma 14.C.2. An optimal contract in the problem given in Lemma 14.C.1 must have $w_L - g(e_L, \theta_L) = v^{-1}(\bar{u})$.

Lemma 14.C.3. In any optimal contract:

(i) $e_L \leq e_L^*$; that is, the manager's effort level in state θ_L is no more than the level that would arise if θ were observable.

(ii) $e_H = e_H^*$; that is, the manager's effort level in state θ_H is exactly equal to the level that arise if θ were observable.

Lemma 14.C.4. In any optimal contract, $e_L < e_L^*$; that is, the effort level in state θ_L is necessarily *strictly* below the level that would arise in state θ_L if θ were observable.

Proposition 14.C.3. In the hidden information principal-agent model with an infinitely risk-averse manager the optimal contract sets the level of effort in state θ_H at its first-best (full observability) level e_H^*. The effort level in state θ_L is distorted downward from its first-best level e_L^*. In addition, the manager is inefficiently insured, receiving a utility greater than $\bar{u}$ in state θ_H and a utility equal to $\bar{u}$ in state θ_L. The owner's expected payoff is strictly lower than the expected payoff he receives when θ is observable, while the infinitely risk-averse manager's expected utility is the same as when θ is observable (it equals $\bar{u}$).

Part IV

General Equilibrium

Chapter 15

General Equilibrium Theory: Some Examples

15.B Pure Exchange: The Edgeworth Box

Definition 15.B.1. A *Walrasian* (or *competitve*) *equilibrium* for an Edgeworth box economy is a price vector p^* and an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$,

$$x_i^* \succsim_i x_i' \text{ for all } x_i' \in B_i(p^*).$$

Definition 15.B.2. An allocation x in the Edgeworth box is *Pareto optimal* (or *Pareto efficient*) if there is no other allocation x' in the Edgeworth box with $x_i' \succsim_i x_i$ for $i = 1, 2$ and $x_i' \succsim_i x_i$ for some i.

Definition 15.B.3. An allocation x^* in the Edgeworth box is supportable as an *equilibrium with transfers* if there is a price system p^* and wealth transfers T_1 and T_2 satisfying $T_1 + T_2 = 0$, such that for each consumer i we have

$$x_i^* \succsim_i x_i' \text{ for all } x_i' \in \mathbb{R}_+^2 \text{ such that } p^* \cdot x_i' \leq p^* \cdot \omega_i + T_i.$$

15.D The 2 x 2 Production Model

Definition 15.D.1. The production of good 1 is *relatively more intensive in factor* 1 than is production of good 2 if

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

at *all* factor prices $w = (w_1, w_2)$.

Chapter 16

Equilibrium and Its Basic Welfare Properties

16.B The Basic Model and Definitions

Definition 16.B.1. An *allocation* $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X$ for each consumer $i = 1, \ldots, I$ and a production vector $y_i \in Y$ for each firm $j = 1, \ldots, J$. An allocation (x, y) is *feasible* if $\sum_i x_{\ell i} = \ddot{\omega}_\ell + \sum_j y_{\ell j}$ for every commodity ℓ. That is, if

$$\sum_i x_i = \bar{\omega} + \sum_j y_j. \tag{16.B.1}$$

We denote the set of feasible allocations by

$$A = \left\{ (x, y) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J : \sum_i x_i = \bar{\omega} + \sum_j y_j \right\} \subset \mathbb{R}^{L(I+J)}.$$

Definition 16.B.2. A feasible allocation (x, y) is *Pareto optimal* (or *Pareto efficient*) if there is no other allocation $(x', y') \in A$ that *Pareto dominates* it, that is, if there is no feasible allocation (x', y') such that $x'_i \succsim_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

Definition 16.B.3. Given a private ownership economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J), \{(\omega_i, 0_{i1}, \ldots, 0_{iJ})\}_{i=1}^I\right)$, an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitutes *Walrasian* (or *competitive*) *equilibrium* if:

(i) For every j, y^*, j maximises profits in Y_j; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\left\{ x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^* \right\}.$$

(iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

Definition 16.B.4. Given an economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J), \bar{\omega}\right)$ an allocation (x^*, y^*) and a price vector $p = (p_1, \dots, p_L)$ constitute a *price equilibrium with transfers* of there is an assignment of wealth levels $(w_1, \dots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y^*, j maximises profits in Y_j; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq \omega\}.$$

(iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

16.C The First Fundamental Theorem of Welfare Economics

Definition 16.C.1. The preference relation $\succsim$ on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $||y - x|| \leq \varepsilon$ and $y \succ x$.

Proposition 16.C.1 (The First Fundamental Theorem of Welfare Economics)**.** If the prive p^* and allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ constitutes a competitive equilibrium, then this allocation is Pareto optimal.

16.D The Second Fundamental Theorem of Welfare Economics

Definition 16.D.1. Given an economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega}\right)$ an allocation (x^*, y^*) and a price vector $p = (p_1, \dots, p_L) \neq 0$ constitute a *price quasiequilibrium with transfers* if there is an assignment of wealth levels $(w_1, \dots w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y^*, j maximises profits in Y_j; that is

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, if $x_i \succ x_i^*$ then $p \cdot x_i \geq w_i$.

(iii)

$$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*.$$

Proposition 16.D.1 (The Second Fundamental Theorem of Welfare Economics)**.** Consider an economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J), \bar{\omega}\right)$, and suppose that every Y_j is convex and every preference relation $\succsim_i$ is convex [i.e., the set $\{x_i' \in X_i : x_i' \succsim_i x_i\}$ is convex for every $x_i \in X$] and locally nonsatiated. Then, for every Pareto optimal allocation (x^*, y^*), there is a price vector $p = (p_1, \dots, p_L) \neq 0$ such that (x^*, y^*, p) is a price quasiequilibrium with transfers.

Proposition 16.D.2. Assume that X_i is convex and $\succsim_i$ is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p, and the wealth level w_i are such that $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq w_i$. Then, if there is a consumption vector $x_i' \in X_i$ such that $p \cdot x_i' < w_i$ [a *cheaper consumption* for (p, w_i)], it follows that $x_i \succ_i x_i^*$ implies $p \cdot x_i > w_i$.

Proposition 16.D.3. Suppose that for every i, X_i is convex, $0 \in X_i$, and $\succsim_i$ is continuous. Then any price quasiequilibrium with transfers that has $(w_1, \ldots, w_I) \gg 0$ is a price equilibrium with transfers.

16.E Pareto Optimality and Social Welfare Optima

Proposition 16.E.1. A feasible allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a Pareto optimum if and only if $(u_1(x_1), \ldots, u_I(x_I)) \in UP$, where $UP = \{u_1, \ldots, u_I \in U$: there is no $(u_1', \ldots, u') \in U$ such that $u_i' \geq u_i$ for all i and $u_i' > u_i$ for some $i\}$ and $U = \{(u_1, \ldots, u_I) \in \mathbb{R}^I$: there is a feasible allocation (x, y) such that $u_i \leq u_i(x_i)$ for $i = 1, \ldots, I\}$.

Proposition 16.E.2. If $u^* = (u_1^*, \ldots u_I^*)$ is a solution to the social welfare maximisation problem $\max_{u \in U} \lambda \cdot u$ with $\lambda \gg 0$, then $u^* \in UP$; that is, u^* is the utility vector of a Pareto optimal allocation. Moreover, if the utility possibility set U is convex, then for any $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_I) \in UP$, there is a vector of welfare weights $\lambda = (\lambda_1, \ldots, \lambda_I) \geq 0, \lambda \neq 0$, such that $\lambda \cdot \tilde{u} \geq \lambda \cdot u$ for all $u \in U$, that is, such that $\tilde{u}$ is a solution to the social welfare maximisation problem.

16.F First-Order Conditions for Pareto Optimality

Proposition 16.F.1. Under the assumptions made about the economy [in particular, the concavity of every $u_i(\cdot)$ and the convexity of ever $F_j(\cdot)$], every Pareto optimal allocation (and, hence, every price equilibrium with transfers) maximises a weighted sum of utilities subject to the resource and technological constraints. Moreover, the weight λ_i of the utility of the ith consumer equals the reciprocal of consumer i's marginal utility or wealth evaluated at the supporting prices and imputed wealth.

16.G Some Applications

Definition 16.G.1. A *Lindahl equilibrium* for the public goods economy is a price equilibrium with transfers for the artificial economy with personalised commodities. That is, an allocation $(x_1^*), \ldots, x_I^*, q^*, z^* \in \mathbb{R}^{2I} \times \mathbb{R} \times \mathbb{R}$ and a price system $(p_1, p_{21}, \ldots, p_{2I}) \in \mathbb{R}^{I+1}$ constitutes a Lindahl equilibrium if there is a set of wealth levels $(w_1, \ldots, w_I)$ satisfying $\sum_i w_i = \sum_i p_1 x_{1i}^* + (\sum_i p_{2i}) q^* - p_1 z^*$ and such that

(i) $q^* \leq f(z^*)$ and $(\sum_i p_{2i}) q^* - p_1 z^* \geq (\sum_i p_{2i}) q - p_1 z$ for all (q, z) with $z \geq 0$ and $q \leq f(z)$.

(ii) For every i, $x_i^* = (x_{1i}^*, x_{2i}^*)$ is maximal for $\succsim_i$ in the set $\{(x_{1i}, x_{2i}) \in X_i : p_1 x_{1i} + p_2 x_{2i} \leq w_i\}$.

(iii) $\sum_i x_{1i}^* + z^* = \bar{\omega}_1$ and $x_{2i}^* = q^*$ for every i.

Proposition 16.G.1. Suppose that the basic assumptions of Section 16.F hold and that, in addition, all consumers have convex preferences (so utility functions are quasiconcave). If (x^*, y^*) is Pareto optimal, then there exists a price vector $p = (p_1, \ldots, p_L)$ and wealth levels $w = (w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that:

(i) For any firm j, we have

$$p = \gamma_j \nabla F_j(y_j^*) \quad \text{for some } \gamma_j > 0.$$

(ii) For any i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X : p \cdot x_i \leq w_i\}.$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Chapter 17

The Positive Theory of Equilibrium

17.B Equilibrium: Definitions and Basic Equations

Definition 17.B.1. Given a private ownership economy specified by

$$\left(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{(\omega_i, \theta_{i1}, \ldots, \theta_{iJ})\}_{i=1}^{I}\right),$$

an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a *Walrasian* (or *competitive*, or *market*, or *price-taking*) equilibrium if

(i) For every j, $y_j^* \in Y_j$ maximises profits in Y_j; that is

$$p \cdot y_j \leq y \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, $x_i^* \in X_i$ is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii) $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$.

Proposition 17.B.1. In a pure exchange economy in which consumer preferences are continuous, strictly convex and locally nonsatiated, $p \geq 0$ is a Walrasian equilibrium price vector if and only if:

$$\sum_i \left(x_i(p, p \cdot \omega_i) - \omega_i\right) \leq 0.$$

Definition 17.B.2. The *excess demand function of consumer* i is

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i,$$

where $x_i(p, p \cdot \omega_i)$ is consumer i's Walrasian demand function. The (*aggregate*) *excess demand function* of the economy is

$$z(p) = \sum_i z_i(p).$$

The domain of this function is a set of nonnegative price vectors that includes all strictly positive price vectors.

Proposition 17.B.2. Suppose that, for every consumer i, $X_i = \mathbb{R}^L_+$ and $\succsim_i$ is continuous, strictly convex, and strongly monotone. Suppose also that $\sum_i \omega_i \gg 0$. Then the aggregate excess demand function $z(p)$, defined for all price vectors $p \gg 0$, satisfies the properties:

(i) $z(\cdot)$ is continuous.

(ii) $z(\cdot)$ is homogeneous of degree zero.

(iii) $p \cdot z(\cdot) = 0$ for all p (*Walras' law*).

(iv) There is an $s > 0$ such that $z_\ell(p) > -s$ for every commodity ℓ and all p.

(v) If $p^n \to p$, where $p \neq 0$ and $p_\ell = 0$ for some ℓ, then

$$\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty.$$

17.C Existence of Walrasian Equilibrium

Proposition 17.C.1. Suppose that $z(p)$ is a function defined for all strictly positive price vectors $p \in \mathbb{R}^L_{++}$ and satisfying conditions (i) to (v) of Proposition 17.B.2. Then the system of euqations $z(p) = 0$ has a solution. Hence, a Walrasian equilibrium exists in any pure exchange economy in which $\sum_i \omega_i \gg 0$ and every consumer has continuous, strictly convex, and strongly monotone preferences.

Proposition 17.C.2. Suppose that $z(p)$ is a function defined for all nonzero, nonnegative price vectors $p \in \mathbb{R}^L_+$ and satisfying conditions (i) to (iii) of Proposition 17.B.2 (i.e. continuity homogeneity of degree zero and Walras' law). Then there is a price vector p^* such that $z(p^*) \leq 0$.

17.D Local Uniqueness and the Index Theorem

Definition 17.D.1. An equilibrium price vector $p = (p_1, \ldots, p_{L-1})$ is *regular* if the $(L-1)\times(L-1)$ matrix of price effects $D\hat{z}(p)$ is nonsingular, that is, has rank $L-1$. If every normalised equilibrium price vector is regular, we say that the *economy is regular*.

Proposition 17.D.1. Any regular (normalised) equilibrium price vector

$$p = (p_1, \ldots, p_{L-1}, 1)$$

is *locally isolated* (or *locally unique*). That is, there is an $\varepsilon > 0$ such that if $p' \neq p, p'_L = p_L = 1$, and $||p' - p|| < \varepsilon$, then $z(p') \neq 0$. Moreoever, of the economy is regular, then the number of normalised equilibrium price vectors is finite.

Definition 17.D.2. Suppose that $p = (p_1, \ldots, p_{L-1}, 1)$ is a regular equilibrium of the economy. Then we denote

$$\text{index } p = (-1)^{L-1} \text{sign} |D\hat{z}(p)|,$$

where $|D\hat{z}(p)|$ is the determinant of the $(L-1) \times (L-1)$ matrix $D\hat{z}(p)$.

Proposition 17.D.2 (The Index Theorem)**.** For any regular economy, we have

$$\sum_{\{p: z(p)=0, p_L=1\}} \text{index } p = +1.$$

Definition 17.D.3. The system of M equations in N unknowns $f(v) = 0$ is *regular* if rank $Df(v) = M$ whenever $f(v) = 0$.

Proposition 17.D.3 (The Transversality Theorem)**.** If the $M \times (N + S)$ matrix $Df(v; q)$ has rank M whenever $f(v; q) = 0$ then for almost every q, the $M \times N$ matrix $D_v f(v; q)$ has rank M whenever $f(v; q) = 0$.

Proposition 17.D.4. For any p and ω, rank $D_\omega \hat{z}(p; \omega) = L - 1$.

Proposition 17.D.5. For almost every vector of initial endowments $(\omega_1, \ldots, \omega_I) \in \mathbb{R}^{LI}_{++}$, the economy defined by $\{(\succsim_i, \omega_i)\}_{i=1}^I$ is regular.

17.E Anything Goes: The Sonnenschein-Mantel-Debreu Theorem

Proposition 17.E.1. Suppose that $I < L$. Then for any equilibrium price vector p there is some direction of price change $dp \neq 0$ such that $p \cdot dp = 0$ (hence dp is not proportional to p) and $dp \cdot Dz(p)dp \leq 0$.

Proposition 17.E.2. Given a price vector p, let $z \in \mathbb{R}^L$ be an arbitrary vector and A an arbitrary $L \times L$ matrix satisfying $p \cdot z = 0$, $Ap = 0$ and $p \cdot A = -z$. Then there is a collection of L consumers generating an aggregate excess demand function $z(\cdot)$ such that $z(p) = z$ and $Dz(p) = A$.

Proposition 17.E.3. Suppose that $z(\cdot)$ is a continuous function defined on

$$P_\varepsilon = \{p \in \mathbb{R}^L_+ : p_\ell / p_{\ell'} \geq \varepsilon \text{ for every } \ell \text{ and } \ell'\}$$

and with values in $\mathbb{R}^L$. Assume that, in addition, $z(\cdot)$ is homogeneous of degree zero and satisfies Walras' law. Then there is an economy of L consumers whose aggregate excess demand function coincides with $z(p)$ in the domain of P_ε.

Proposition 17.E.4. For any $N \geq 1$, suppose that we assign to each $n = 1, \ldots, N$ a price vector p^n, normalised to $||p^n|| = 1$, and an $L \times L$ matrix A_n of rank $L - 1$, satisfying $A_n p^n = 0$ and $p^n \cdot A_n = 0$. Suppose that, in addition, the index formula $\sum_n (-1)^{L-1} \text{sign}|\hat{A}_n| = +1$ holds. If $L = 2$, assume also that positive and negative index equilibria alternate.

Then there is an economy with L consumers such that the aggregate excess demand $z(\cdot)$ has the properties:

(i) $z(p) = 0$ for $||p|| = 1$ if and only if $p = p^n$ for some n.

(ii) $Dz(p^n) = A_n$ for every n.

17.F Uniqueness of Equilibria

Proposition 17.F.1. Given an economy specified by the constant returns technology Y and the aggregate excess demand function of the consumers $z(\cdot)$, a price vector p is a Walrasian equilibrium price vector if and only if

(i) $p \cdot y \leq 0$ for every $y \in Y$, and

(ii) $z(p)$ is a feasible production; that is, $z(p) \in Y$.

Definition 17.F.1 (The Weak Axiom for Excess Demand Functions)**.** The excess demand function $z(\cdot)$ satisfies the weak axiom of revealed preferences (WA) if for any pair of price vectors p and p', we have

$$z(p) \neq z(p') \text{ and } p \cdot z(p') \leq 0 \text{ implies } p' \cdot z(p) \geq 0.$$

Proposition 17.F.2. Suppose that the excess demand function $z(\cdot)$ is such that, for any constant returns technology Y, the economy formed by $z(\cdot)$ and Y has a unique (normalised) equilibrium price vector. Then $z(\cdot)$ satisfies the weak axiom. Conversely, if $z(\cdot)$ satisfies the weak axiom then, for any constant returns convex technology Y, the set of equilibrium price vectors is convex (and so, if the set of normalised price equilibria is finite, there can be at most one normalised price equilibrium).

Definition 17.F.2. The function $z(\cdot)$ has the *gross substitute* (GS) property if whenever p' and p are such that, for some ℓ, $p'_\ell > p_\ell$ and $p'_k > p_k$ for $k \neq \ell$, we have $z_k(p') > z_k(p)$ for $k \neq \ell$.

Proposition 17.F.3. An aggregate excess demand function $z(\cdot)$ that satisfies the gross substitute property has at most one exchange equilibrium; that is, $z(p) = 0$ has at most one (normalised) solution.

Proposition 17.F.4. If $z(\cdot)$ is an aggregate excess demand function, $z(p) = 0$, and $Dz(p)$ has the gross substitute sign pattern, then we also have $dp \cdot Dz(p)dp < 0$ whenever $dp \neq 0$ is not proportional to p.

Proposition 17.F.5. Suppose that the initial endowment allocation $(\omega_1, \dots, \omega_I)$ constitutes a Walrasian equilibrium allocation for an exchange economy with strictly convex and strongly monotone consumer preferences (i.e., no-trade is an equilibrium). Then this is the unique equilibrium allocation.

17.G Comparative Statics Analysis

Proposition 17.G.1. Given any price vector $\bar{p}$, endowments for the first consumer of the first $L-1$ commodities $\hat{\bar{\omega}}_1 = (\bar{\omega}_{11}, \dots, \bar{\omega}_{L-1,1})$, and a $(L-1) \times (L-1)$ nonsingular matrix B, there is an exchange economy formed by $L+1$ consumers in which the first consumer has the prescribed endowments of the first $L-1$ commodities, $\hat{z}(\bar{p}, \hat{\bar{\omega}}_1) = 0$, $\hat{z}(\cdot, \hat{\bar{\omega}}_1) = 0$ is regular at $\bar{p}$ and $Dp(\hat{\bar{\omega}}_1) = B$.

Proposition 17.G.2. Suppose that $\hat{z}(\bar{p}; \bar{q}) = 0$, where $\hat{z}(\cdot)$ is differentiable. If $D_q\hat{z}(\bar{p}; \bar{q})$ is negative definite, then

$$(D_q\hat{z}(\bar{p}; \bar{q})dq) \cdot (Dp(\bar{q})dq) \geq 0 \text{ for any } dq.$$

Proposition 17.G.3. Suppose that $\hat{z}(\bar{p}; \bar{q}) = 0$, where $\hat{z}(\cdot; \cdot)$ is differentiable. If the $L \times L$ matrix $D_p z(\bar{p}; \bar{q})$ has negative diagonal entries and positive off-diagonal entries, then $[D_p z(\bar{p}; \bar{q})]^{-1}$ has all its entries negative.

17.H Tâtonnement Stability

Proposition 17.H.1. Suppose that $z(p^*) = 0$ and $p^* \cdot z(p) > 0$ for every p not proportional to p^*. Then the relative prices of any solution trajectory of the differential equation

$$\frac{dp_\ell}{dt} = c_\ell z_\ell(p) \quad \text{for every } \ell$$

converge to the relative prices of p^*.

Definition 17.H.1. We say that the differentiable trajectory $y(t) \in Y$ is *admissible* if $p(y(t)) \cdot (dy(t)/dt) \geq 0$ for every t, with equality only if $y(t)$ is profit maximising for $p(y(t))$ (in which case we could say that we are at a long-run equilibrium).

Proposition 17.H.2. If there is a single strictly convex consumer, then any admissible trajectory converges to the (unique) equilibrium.

Chapter 18

Some Foundations for Competitive Equilibria

18.B Core and Equilibria

Definition 18.B.1. A coalition $S \subset I$ *improves upon*, or *blocks*, the feasible allocation $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_+^{LI}$ if for every $i \in S$ we can find a consumption $x_i \geq 0$ with the properties:

(i) $x_i \succ_i x_i^*$ for every $i \in S$

(ii) $\sum_{i \in S} x_i \in Y + \left\{\sum_{i \in S} \omega_i\right\}$.

Definition 18.B.2. We say that a feasible allocation $x^* = (x_1^*, \dots, x_I^*) \in \mathbb{R}_+^{LI}$ has the *core property* if there is no coalition of consumers $S \subset I$ that can improve upon x^*. The *core* is the set of allocations that have the core property.

Proposition 18.B.1. Any Walrasian equilibrium allocation has the core property.

Proposition 18.B.2. Denoting by hn the nth individual of type h, suppose that the allocation

$$x^* = (x_{11}^*, \dots, x_{1n}^*, \dots, x_{1N}^*, \dots, x_{H1}^*, \dots, x_{Hn}^*, \dots, x_{HN}^*) \in \mathbb{R}_+^{LHN}$$

belongs to the core of the N-replica economy. Then x^* has the *equal-treatment property*, that is, all consumers of the same type get the same consumption bundle:

$$x_{hm}^* = x_{hn}^* \quad \text{for all } 1 \leq m, n \leq N \text{ and } 1 \leq h \leq H.$$

Proposition 18.B.3. If the feasible type allocation $x^* = (x_1^*, \dots, x_H^*) \in \mathbb{R}_+^{LH}$ has the core property for all $N = 1, 2, \dots$, that is, $x^* \in C_N$ for all N, then x^* is a Walrasian equilibrium allocation.

18.C Noncooperative Foundations of Walrasian Equilibria

Definition 18.C.1. The profiles of actions $a^* = (a_1^*, \dots, a_I^*) \in A_1 \times \dots \times A_I$ is a *trading equilibrium* if, for every i,

$$u_i\left(g\left(a_i^*; p(a^*)\right) + \omega_i\right) \geq u_i\left(g\left(a_i; p(a_i; a_{-i}^*)\right) + \omega_i\right) \quad \text{for all } a_i \in A_i.$$

18.D The Limits to Redistribution

Definition 18.D.1. The feasible allocation $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LI}$ is *self-selective* (or *anonymous*, or *envy-free in net trades*) if there is a set of net trades $B \subset \mathbb{R}^L$, to be called a *generalised budget set*, or a *tax system*, such that, for every i, $z_i^* = x_i^* - \omega_i$ solves the problem

$$\begin{aligned} &\max u_i(z_i + \omega_i) \\ &\text{s.t. } z_i \in B, \\ &\qquad z_i + \omega_i \geq 0. \end{aligned}$$

Proposition 18.D.1. Suppose we have an exchange economy with a continuum of consumer types. Assume:

(i) The preferences of all consumers are representable by differentiable utility functions.

(ii) The set of characteristics of consumers present in the economy cannot be split into two disconnected classes. Formally, if $(u(\cdot), \omega), (u'(\cdot), \omega')$ are two preferences-endowment pairs present in the economy then there is a continuous function $(u(\cdot; t), \omega(t))$ of $t \in [0, 1]$ such that

$$(u(\cdot; 0), \omega(0)) = (u(\cdot, \omega)), (u(\cdot; 1), \omega(1)) = (u'(\cdot), \omega),$$

and $(u(\cdot; t), \omega(t))$ is present in the economy for every t.

Then any allocation $x^* = \{x_i^*\}_{i \in I}$ that is Pareto optimal, self-selective, and interior (i.e., $x_i^* \gg 0$ for all i) must be a Walrasian equilibrium allocation. Here I is an infinite set of names.

18.E Equilibrium and the Marginal Productivity Principle

Definition 18.E.1. Given a continuum population $\mu = (\mu_1, \ldots, \mu_H) \in \mathbb{R}_+^H$ a feasible allocation $(x_1^*, \ldots, x_H^*)$ is a *marginal product*, or *no-surplus*, *allocation* if

$$u_h(x_h^*) = \frac{\partial v(\mu)}{\partial \mu_h} \quad \text{for all } h.$$

In words: at a no-surplus allocation everyone is getting exactly what she contributes on the margin.

Proposition 18.E.1. For any *continuum* population $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_H) \gg 0$ a feasible allocation $(x_1^*, \ldots, x_H^*) \gg 0$ is a marginal product allocation if and only if it is a Walrasian equilibrium allocation.

Chapter 19

General Equilibrium Under Uncertainty

19.B A Market Economy with Contingent Commodities: Description

Definition 19.B.1. For every physical commodity $\ell = 1, \ldots, L$ and states $s = 1, \ldots, S$, a unit of (*state-*)*contingent commodity* ℓs is a title to receive a unit of physical good ℓ if and only if s occurs. Accordingly, a (*state-*)*contingent commodity vector* is specified by

$$x = (x_{11}, \ldots, x_{L1}, \ldots, x_{1S}, \ldots, x_{LS}) \in \mathbb{R}^{LS},$$

and is understood as an entitlement to receive the commodity vector $x = (x_{1s}, \ldots, x_{Ls})$ if state s occurs.

19.C Arrow-Debreu Equilibrium

Definition 19.C.1. An allocation

$$(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J \subset \mathbb{R}^{LS(I+J)}$$

and a system of prices for the contingent commodities $p = (p_{11}, \ldots, p_{LS}) \in \mathbb{R}^{LS}$ constitute an *Arrow-Debreu equilibrium* if:

(i) For every j, y_j^* satisfies $p \cdot y_j^* \geq p \cdot y_j$ for all $y_j \in Y_j$.

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\}.$$

(iii) $\sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i$.

19.D Sequential Trade

Definition 19.D.1. A collection formed by a price vector $q = (q_1, \ldots, q_S) \in \mathbb{R}^S$ for contingent first good commodities at $t = 0$, a spot price vector

$$p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$$

for every s, and, for every consumer i, consumption plans $z_i^* = (z_{1i}^*, \ldots, z_{Si}^*) \in \mathbb{R}^S$ at $t = 0$ and $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ constitute a *Radner equilibrium* if:

(i) For every i, the consumption plans z_i^*, x_i^* solve the problem

$$\max_{\substack{(x_{1i},\ldots,x_{Si}) \in \mathbb{R}_+^{LS} \\ (z_{1i},\ldots,z_{Si}) \in \mathbb{R}^S}} U_i(x_{1i}, \ldots, x_{Si})$$

$$\text{s.t. (i) } \sum_s q_s z_{si} \leq 0,$$

$$\text{(ii) } p_s \cdot x_{si} \leq p_s \omega_{si} + p_{1s} z_{si} \quad \text{for every } s.$$

(ii) $\sum_i z_{si}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_{si}$ for every s.

Proposition 19.D.1. We have:

(i) If the allocation $x^* \in \mathbb{R}^{LSI}$ and the contingent commodities price vector $(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute and Arrow-Debreu equilibrium, then there are prices $q \in \mathbb{R}_{++}^S$ for contingent first good commodities and consumption plans for these commodities $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{SI}$ such that the consumption plans x^*, z^*, the prices q, and the spot prices $(p_1, \ldots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$, $z^* \in \mathbb{R}^{SI}$ and prices $q \in \mathbb{R}_{++}^S$, $(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute a Radner equilibrium, then there are multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}_{++}^S$ such that the allocation x^* and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s.)

19.E Asset Markets

Definition 19.E.1. A unit of an *asset*, or *security*, is a title to receive an amount r_s of good 1 at date $t = 1$ if state s occurs. An asset is therefore characterised by its *return vector* $r = (r_1, \ldots, r_S) \in \mathbb{R}^S$.

Definition 19.E.2. A collection formed by a price vector $q = (q_1, \ldots, q_K) \in \mathbb{R}^K$ for assets traded at $t = 0$, a spot price vector $p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$ for every s, and, for every consumer i, portfolio plans $z_i^* = (z_{1i}^*, \ldots, z_{Ki}^*) \in \mathbb{R}^K$ at $t = 0$ and consumption plans $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ constitutes a *Radner equilibrium* if:

(i) For every i, the consumption plans z_i^*, x_i^* solve the problem

$$\max_{\substack{(x_{1i},\ldots,x_{Si}) \in \mathbb{R}_+^{LS} \\ (z_{1i},\ldots,z_{Ki}) \in \mathbb{R}^K}} U_i(x_{1i}, \ldots, x_{Si})$$

$$\text{s.t. (i) } \sum_k q_k z_{ki} \leq 0,$$

$$\text{(ii) } p_s \cdot x_{si} \leq p_s \omega_{si} + \sum_k p_{1s} z_{ki} r_{sk} \quad \text{for every } s.$$

(ii) $\sum_i z_{ki}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_{si}$ for every k and s.

Proposition 19.E.1. Assume that every return vector is nonnegative and nonzero; that is, $r_k \geq 0$ and $r_k \neq 0$ for all k. Then, for every (column) vector $q \in \mathbb{R}^K$ of asset prices arising in a Radner equilibrium, we can find multipliers $\mu = (\mu_1, \ldots, \mu_S) \geq 0$, such that $q_k = \sum_s \mu_s r_{sk}$ for all k (in matrix notation, $q^T = \mu \cdot R$).

Definition 19.E.3. An asset structure with an $S \times K$ return matrix R is *complete* of rank$R = S$, that is, if there is some subset of S assets with linearly independent returns.

Proposition 19.E.2. Suppose that the asset structure is complete. Then:

(i) If the consumption plans $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}$ and the price vector

$$(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$$

constitute an Arrow-Debreu equilibrium, then there are asset prices $q \in \mathbb{R}_{++}^K$ and portfolio plans $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ such that the consumption plans x^*, portfolio plans z^*, asset prices q, and spot prices $(p_1, \ldots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$, portfolio plans $z^* \in \mathbb{R}^{KI}$, and prices $q \in \mathbb{R}_{++}^K, (p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute a Radner equilibrium, then there are multipliers $\mu = (\mu_1, \ldots, \mu_S) \in \mathbb{R}_{++}^S$ such that consumption plans x^* and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s; recall that $p_{1s} = 1$.)

Proposition 19.E.3. Suppose that the asset price vector $q \in \mathbb{R}^K$, the spot prices $p = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$, the consumption plans $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LSI}$, and the portfolio plans $(z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ constitute a Radner equilibrium for an asset structure with $S \times K$ return matrix R. Let R' be the $S \times K'$ return matrix of a second asset structure. If rangeR' = range$R$, then x^* is still the consumption allocation of a Radner equilibrium in the economy with the second asset structure.

19.F Incomplete Markets

Definition 19.F.1. The asset allocation $(z_1, \ldots, z_I) \in \mathbb{R}^{KI}$ is constrained Pareto optimal if it is feasible (i.e. $\sum_i z_i \leq 0$) and if there is no other feasible asset allocation $(z_1', \ldots, z_I') \in \mathbb{R}^{KI}$ such that

$$U_i^*(z_1', \ldots, z_I') \geq U_i^*(z_1, \ldots, z_I) \quad \text{for every } j,$$

with at least one inequality strict.

Proposition 19.F.1. Suppose that there are two periods and only one consumption good in the second period. Then any Radner equilibrium is *constrained Pareto optimal* in the sense that there is no possible redistribution of assets in the first period that leaves every consumer as well off and at least one consumer strictly better off.

19.G Firm Behaviour in General Equilibrium Models under Uncertainty

Definition 19.G.1. A set $A \subset \mathbb{R}^S$ of random variables is *spanned* by a given asset structure of every $a \in A$ is in the range of the return matrix R of the asset structure, that is, if every $a \in A$ can be expressed as a linear combination of the available asset returns.

19.H Imperfect Information

Definition 19.H.1. The signal function $\sigma' : S \to \mathbb{R}$ is *at least as informative* as $\sigma : S \to \mathbb{R}$ if $\sigma(s) \neq \sigma(s')$ implies $\sigma'(s) \neq \sigma'(s')$ for any pair s, s'. It is *more informative* if, in addition, $\sigma'(s) \neq \sigma'(s')$ for some pair s, s' with $\sigma(s) = \sigma(s')$.

Proposition 19.H.1. In the single-consumer problem, if the signal function $\sigma'(\cdot)$ is at least as informative as the signal function $\sigma(\cdot)$, then the ex ante utility derived from $\sigma'(\cdot)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma'(\cdot)})$, is at least as large as the ex ante utility derived from $\sigma(\cdot)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma(\cdot)})$.

Definition 19.H.2. The price function $p(\cdot)$ is a *rational expectations equilibrium price function* if, for every s, $p(s)$ clears the spot market when every consumer i knows that $s \in E_{p(s),\sigma_i(s)}$ and, therefore, evaluates commodity bundles $x_i \in \mathbb{R}^2$ according to the updated utility function

$$\sum_s (\pi_{s'i} | p(s), \sigma_i(s))\, u_{s'i}(x).$$

Chapter 20

Equilibrium and Time

20.C Intertemporal Production and Efficiency

Definition 20.C.1. The list $(y_0, y_1, \dots, y_t, \dots)$ is a *production path*, or *trajectory*, or *program*, if $y_t \in Y \subset \mathbb{R}^{2L}$ for every t.

Definition 20.C.2. The production path $(y_0, \dots, y_t, \dots)$ is *efficient* if there is no other production path $(y'_0, \dots, y'_t, \dots)$ such that

$$y_{a,t-1} + y_{bt} \leq y'_{a,t-1} + y'_{bt} \quad \text{for all } t,$$

and equality does not hold for at least one t.

Definition 20.C.3. The production path $(y_0, \dots, y_t, \dots)$ is *myopically*, or *short-run*, *profit maximising for the price sequence* $(p_0, \dots, p_t, \dots)$ if for every t we have

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \geq p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at} \quad \text{for all } y'_t \in Y.$$

Proposition 20.C.1. Suppose that the production path $(y_0, \dots, y_t, \dots)$ is myopically profit maximising with respect to the price sequence $(p_0, \dots, p_t, \dots) \gg 0$. Suppose also that the production path and the price sequence satisfy the *transversality condition* $p_{t+1} \cdot y_{at} \to 0$. Then the path $(y_0, \dots, y_t, \dots)$ is efficient.

20.D Equilibrium: The One-Consumer Case

Definition 20.D.1. The (bounded) production path $(y^*_0, \dots, y^*_t, \dots), y^*_t \in Y$, and the (bounded) price sequence $p = (p_0, \dots, p_t, \dots)$ constitute a *Walrasian* (or *competitive*) equilibrium if:

(i) $c^*_t = y^*_{a,t-1} + y^*_{bt} + \omega_t \geq 0$ for all t.

(ii) For every t,

$$\pi_t = p_t \cdot y^*_{bt} + p_{t+1} \cdot y^*_{at} \geq p_t \cdot y_b + p_{t+1} \cdot y_a$$

for all $y = (y_b, y_a) \in Y$.

(iii) The consumption sequence $(c^*_0, \dots, c^*_t, \dots) \geq 0$ solves the problem

$$\begin{aligned} \max & \sum_t \delta^t u(c_t) \\ \text{s.t.} & \sum_t p_t \cdot c_t \leq \sum_t \pi_t + \sum_t p_t \cdot \omega_t. \end{aligned}$$

Proposition 20.D.1. Suppose that the (bounded) production path $(y_0^*, \dots, y_t^*, \dots)$ and the (bounded) price sequence $(p_0, \dots, p_t, \dots)$ constitute a Walrasian equilibrium. Then the transversality condition $p_{t+1} \cdot y_{at}^* \to 0$ holds.

Definition 20.D.2. We say the consumption stream $(c_0, \dots, c_t, \dots) \gg 0$ is *myopically*, or *short-run, utility maximising* in the budget set determined by $(p_0, \dots, p_t, \dots)$ and $w < \infty$ if utility cannot be increased by a new consumption stream that merely transfers purchasing power between some two consecutive periods.

Proposition 20.D.2. If the consumption stream $(c_0, \dots, c_t, \dots)$ satisfies $\sum_t p_t \cdot c_t = w < \infty$ and $\lambda p_t = \delta^t \nabla u(c_t)$ for some λ and all t, then it is utility maximising in the budget set determined by $(p_0, \dots, p_t, \dots)$ and w.

Proposition 20.D.3. Any Walrasian equilibrium path $(y_0^*, \dots, y_t^*, \dots)$ solves the planning problem

$$\begin{aligned} &\max \sum_t \delta^t u(c_t) \\ &\quad \text{s.t. } c_t = y_{a,t-1} + y_{bt} + \omega_t \geq 0 \quad \text{and} \quad y_t \in Y \text{ for all } t. \end{aligned} \tag{20.D.7}$$

Proposition 20.D.4. Suppose that the (bounded) path $(y_0^*, \dots, y_t^*, \dots)$ solves the planning problem (20.D.7) and that it yields strictly positive consumption (in the sense that, for some $\varepsilon > 0, c_{\ell t} = y_{\ell a,t-1}^* + y_{\ell bt}^* + \omega_{\ell t} > \varepsilon$ for all ℓ and t). Then the path is a Walrasian equilibrium with respect to some price sequence $(p_0, \dots, p_t, \dots)$.

Proposition 20.D.5. Suppose that there is a uniform bound on the consumption streams generated by all the feasible paths. Then the planning problem (20.D.7) attains a maximum; that is, there is a feasible path that yields utility at least as large as the utility corresponding to any other feasible path.

Proposition 20.D.6. The planning problem (20.D.7) has at most one consumption stream solution.

Proposition 20.D.7. Suppose that the path $(\bar{k}_0, \dots, k_t, \dots)$ is bounded, is strictly interior, and satisfies the Euler equations

$$\nabla_2 u(k_{t-1}, k_t) + \delta \nabla_1 u(k_t, k_{t+1}) = 0 \quad \text{for every } t \geq 1$$

to the planning problem

$$\begin{aligned} &\max \sum_t \delta^t u(k_{t-1}, k_t) \\ &\quad \text{s.t. } (k_{t-1}, k_t) \in A \text{ for every } t, \text{ and } k_0 = \bar{k}_0. \end{aligned}$$

Then it solves this optimisation problem.

20.E Stationary Paths, Interest Rates, and Golden Rules

Definition 20.E.1. A production path $(y_0, \dots, y_t, \dots)$ is *stationary*, or a *steady state*, if there is a production plan $\bar{y} = (\bar{y}_b, \bar{y}_a) \in Y$ such that $y_t = y$ for all $t > 0$.

Proposition 20.E.1. Suppose that $\bar{y} \in Y$ defines a stationary and efficient path. Then, there is a price vector $p_0 \in \mathbb{R}^L$ and an $\alpha > 0$ such that the path is myopically profit maximising for the price sequence $(p_0, \alpha p_0, \dots, \alpha^t p_0, \dots)$.

Proposition 20.E.2. Suppose that the stationary path $(\bar{y}, \dots, \bar{y}, \dots), \bar{y} \in Y$, is myopically supported by proportional prices with rate of interest r, then the path is efficient if $r > 0$ and inefficient if $r < 0$.

Definition 20.E.2. A stationary production path that is myopically supported by proportional prices $p_t = \alpha^t p_0$ with $\alpha = \delta$ is called a *modified golden rule path.* A stationary production path myopically supported by constant prices $p_t = p_0$ is called a *golden rule path.*

20.G Equilibrium: Several Consumers

Definition 20.G.1. The (bounded) production path $(y_0^*, \dots, y_t^*, \dots), y_t^* \in Y$, the (bounded) price sequence $(p_0, \dots, p_t, \dots) \geq 0$, and consumption streams $(c_{0i}^*, \dots, c_{ti}^*, \dots) \geq 0$, $i = 1, \dots I$, constitute a *Walrasian* (or *competitive*) equilibrium if:

(i)
$$\sum_i c_{ti}^* = y_{a,t-1}^* + y_{bt}^* + \sum_i \omega_{ti}, \quad \text{for all } t.$$

(ii) For every t,
$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \geq p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}$$
for all $y = (y_{bt}, y_{at} \in Y)$.

(iii) For every i, the consumption stream $(c_{0i}^*, \dots, c_{ti}^*, \dots) \geq 0$ solves the problem
$$\max \sum_t \delta_i^t u_i(c_i)$$
$$\text{s.t.} \ \sum_t p_t \cdot c_{ti} \leq \sum_t \theta_{ti}\pi_t + \sum_t p_t \cdot \omega_{ti} = w_i,$$
where θ_{ti} is consumer i's given share of period t's profits.

Proposition 20.G.1. A Walrasian equilibrium allocation if Pareto optimal.

Proposition 20.G.2. Suppose that $(y_0^*, \dots, y_t^*, \dots)$ is the production path and $(p_0, \dots, p_t, \dots)$ is the price sequence of a Walrasian equilibrium of an economy with I consumers. Then there are weights $(\eta_1, \dots, \eta_I) \gg 0$ such that $(y_0^*, \dots, y_t^*, \dots)$ and $(p_0, \dots, p_t, \dots)$ constitute a Walrasian equilibrium for the one-consumer economy defined by the utility $\sum_t \delta^t u(c_t)$, where $u(c_t)$ is the solution to $\max \sum_i \eta_i u_i(c_{ti})$ s.t. $\sum_i c_{ti} \leq c_t$.

20.H Overlapping Generations

Definition 20.H.1. A sequence of prices $(p_0, \dots, p_t, \dots)$, an $M \geq 0$, and a family of consumptions $\{(c_{bt}^*, c_{at}^*)\}_{t=0}^{\infty}$ constitutes a *Walrasian* (or *competitive*) *equilibrium* if:

(i) Every (c_{bt}^*, c_{at}^*) solves the individual utility maximisation problem subject to the budget constraints $p_t c_{bt} + p_{t+1} c_{at} \leq (1-\varepsilon)p_t$ for $t > 0$, and $p_0 c_{b0} + p_1 c_{a0} \leq (1-\varepsilon)p_0 + \varepsilon\left(\sum_t p_t\right) + M$ for $\varepsilon > 0$.

(ii) The feasibility requirement $(c_{a,t-1}^* + c_{bt}^* = 1)$ is satisfied for all $t \geq 0$ (we put $c_{a,-1}^* = 0$).

Proposition 20.H.1. Any Walrasian equilibrium $(p_0, \dots, p_t, \dots)$, $\{(c_{bt}^*, c_{at}^*)\}_{t=0}^{\infty}$, with $\sum_t p_t < \infty$ is a Pareto optimum; that is, there are no feasible consumptions $\{(c_{bt}, c_{at})\}_{t=0}^{\infty}$ such that $u(c_{bt}, c_{at}) \geq u(c_{bt}^*, c_{at}^*)$ for all $t \geq 0$, with strict inequality for some t.

Proposition 20.H.2. Suppose that at an equilibrium we have $\sum_t p_t < \infty$. Then $M = 0$.

Part V

Welfare Economics and Incentives

Chapter 21

Social Choice Theory

21.B A Special Case: Social Preferences over Two Alternatives

Definition 21.B.1. A *social welfare functional* (or *social welfare aggregator*) is a rule $F(\alpha_1, \dots, \alpha_I)$ that assigns a social preference, that is, $F(\alpha_1, \dots, \alpha_I) \in \{-1, 0, 1\}$, to every possible profile of individual preferences $(\alpha_1, \dots, \alpha_I) \in \{-1, 0, 1\}^I$.

Definition 21.B.2. The social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is *Paretian*, or has the *Pareto property*, if it respects unanimity of strict preferences on the part of the agents, that is, $F(1, \dots, 1) = 1$ and $F(-1, \dots, -1) = -1$.

Definition 21.B.3. The social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is *symmetric among agents* (or *anonymous*) if the names of the agents do not matter, that is, if a permutation of preferences across agents does not alter the social preference. Precisely, let $\pi : \{1, \dots, I\} \to \{1, \dots, I\}$ be an onto function (i.e., a function with the property that for any i there is h such that $\pi(h) = i$). Then for any profile $(\alpha_1, \dots, \alpha_I)$ we have $F(\alpha_1, \dots, \alpha_I) = F(\alpha_{\pi(1)}, \dots, \alpha_{\pi(I)})$.

Definition 21.B.4. The social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is *neutral between alternatives* if $F(\alpha_1, \dots, \alpha_I) = -F(-\alpha_1, \dots, -\alpha_I)$ for every profile $(\alpha_1, \dots, \alpha_I)$, that is, if the social preference is reversed when we reverse the preferences of all agents.

Definition 21.B.5. The social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is *positively responsive* if, whenever $(\alpha_1, \dots, \alpha_I) \geq (\alpha'_1, \dots, \alpha'_I), (\alpha_1, \dots, \alpha_I) \neq (\alpha'_1, \dots, \alpha'_I)$, and $F(\alpha'_1, \dots, \alpha'_I) \geq 0$ we have $F(\alpha_1, \dots, \alpha_I) = +1$. That is, if x is socially preferred or indifferent to y and some agents raise their consideration of x, then x becomes socially preferred.

Proposition 21.B.1 (May's Theorem)**.** A social welfare functional $F(\alpha_1, \dots, \alpha_I)$ is a majority voting social welfare functional if and only if it is symmetric among agents, neutral between alternatives, and positive responsive.

21.C The General Case: Arrow's Impossibility Theorem

Definition 21.C.1. A *social welfare functional* (or *social welfare aggregator*) defined on a given subset $\mathscr{A} \subset \mathscr{R}^I$ is a rule $F : \mathscr{A} \to \mathscr{R}$ that assigns a rational preference relation $F(\succsim_1, \dots, \succsim_I) \in \mathscr{R}$, interpreted as the social preference relation, to any profile of individual rational preference relations $(\succsim_1, \dots, \succsim_I)$ in the admissible domain $\mathscr{A} \subset \mathscr{R}^I$.

Definition 21.C.2. The *social welfare functional* $F : \mathscr{A} \to \mathscr{R}$ is *Paretian* if, for any pair of alternatives $\{x, y\} \subset X$ and any preference profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, we have that x is socially preferred to y, that is, $xF_p(\succsim_1, \ldots, \succsim_I)y$, whenever $x \succ_i y$ for every i.

Definition 21.C.3. The *social welfare functional* $F : \mathscr{A} \to \mathscr{R}$ defined on the domain $\mathscr{A}$ satisfies the *pairwise independence condition* (or the *independence of irrelevant alternatives condition*) if the social preference between any two alternatives $\{x, y\} \subset X$ depends only on the profile of individual preferences over the same alternatives. Formally, for any pair of alternatives $\{x, y\} \subset X$, and for any pair of preference profiles $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ and $(\succsim_1', \ldots, \succsim_I') \in \mathscr{A}$ with the property that, for every i,

$$x \succsim_i y \iff x \succsim_i' y \quad \text{and} \quad y \succsim_i x \iff y \succsim_i' x,$$

we have that

$$xF(\succsim_1, \ldots, \succsim_I)y \iff xF(\succsim_1', \ldots, \succsim_I')y$$

and

$$yF(\succsim_1, \ldots, \succsim_I)x \iff yF(\succsim_1', \ldots, \succsim_I')x.$$

Proposition 21.C.1 (Arrow's Impossibility Theorem)**.** Suppose that the number of alternatives is at least three and that the domain of admissible individual profiles, denoted $\mathscr{A}$, is either $\mathscr{A} = \mathscr{R}^I$ or $\mathscr{A} = \mathscr{P}^I$. Then every social welfare function $F : \mathscr{A} \to \mathscr{R}$ that is Paretian and satisfies the pairwise independence condition is *dictatorial* in the following sense: There is an agent h such that, for any $\{x, y\} \subset X$ and any profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, we have that x is socially preferred to y, that is, $xF_p(\succsim_1, \ldots, \succsim_I)y$, whenever $x \succ_h y$.

Definition 21.C.4. Given $F(\cdot)$, we say that a subset of agents $S \subset I$ is:

(i) *Decisive for x over y* if whenever every agent in S prefers x to y *and* every agent not in S prefers y to x, x is socially preferred to y.

(ii) *Decisive* if, for any pair $\{x, y\} \subset X$, S is decisive for x over y.

(iii) *Completely decisive for x over y* if whenever every agent in S prefers x to y, x is socially preferred to y.

21.D Some Possibility Results: Restricted Domains

Definition 21.D.1. Suppose that the preference relation $\succsim$ on X is reflexive and complete. We that then that:

(i) $\succsim$ is *quasitransitive* if the strict preference $\succ$ induced by $\succsim$ (i.e. $x \succ y \iff x \succsim y$ but not $y \succsim x$) is transitive.

(ii) $\succsim$ is *acyclic* if $\succsim$ has a maximal element in every finite subset $X' \subset X$, that is, $\{x \in X' : x \succsim y \text{ for all } y \in X'\} \neq \emptyset$.

Definition 21.D.2. A binary relation $\geq$ on the set of alternatives X is a *linear order* on X if it is *reflexive* (i.e., $x \geq x$ for every $x \in X$), transitive (i.e., $x \geq y$ and $y \geq z$ implies $x \geq z$) and *total* (i.e., for any distinct $x, y \in X$, we have that either $x \geq y$ or $y \geq x$, but not both).

Definition 21.D.3. The rational preference relation $\succsim$ is *single peaked* with respect to the linear order $\geq$ on X if there is an alternative $x \in X$ with the property that $\succsim$ is increasing with respect to $\geq$ on $\{y \in X : x \geq y\}$ and decreasing with respect to $\geq$ on $\{y \in X : y \geq x\}$. That is,

$$\text{If } x \geq z > y \quad \text{then} \quad z \succ y$$

and

$$\text{If } y > z \geq x \quad \text{then} \quad z \succ y.$$

Definition 21.D.4. Given a linear order $\geq$ on X, we denote $\mathscr{R}_{\geq} \subset \mathscr{R}$ the collection of all rational preference relations that are single peaked with respect to $\geq$.

Definition 21.D.5. Agent $h \in I$ is a *median agent for the profile* $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}_{\geq}^I$ if

$$\#\{i \in I : x_i \geq x_h\} \geq \frac{I}{2} \quad \text{and} \quad \#\{i \in I : x_h \geq x_i\} \geq \frac{I}{2}.$$

Proposition 21.D.1. Suppose that $\geq$ is a linear order on X and consider a profile of preferences $(\succsim_1, \ldots, \succsim_I)$ where, for every i, $\succsim_i$ is single peaked with respect to $\geq$. Let $h \in I$ be a median agent. Then $x_h \hat{F}(\succsim_1, \ldots, \succsim_I) y$ for every $y \in X$. That is, the peak x_h of the median agent cannot be defeated by majority voting by any other alternative. Any alternative having this property is called a *Condorcet winner*. Therefore, a Condorcet winner exists whenever the preferences of all agents are single-peaked with respect to the same linear order.

Proposition 21.D.2. Suppose that I is odd and that $\geq$ is a linear order on X. Then pairwise majority voting generates a well-defined social welfare functional $F : \mathscr{P}_{\geq}^I \to \mathscr{R}$. That is, on the domain of preferences that are single-peaked with respect to $\geq$ and, moreover, have the property that no two distinct alternatives are indifferent, we can conclude that the social relation $\hat{F}(\succsim_1, \ldots, \succsim_I)$ generated by pairwise majority voting is complete and transitive.

21.E Social Choice Functions

Definition 21.E.1. Given any subset $\mathscr{A} \subset \mathscr{R}^I$, a *social choice function* $f : \mathscr{A} \to X$ defined on $\mathscr{A}$ assigns a chosen element $f(\succsim_1, \ldots, \succsim_I) \in X$ to every profile of individual preferences in $\mathscr{A}$.

Definition 21.E.2. The social choice function $f : \mathscr{A} \to X$ defined on $\mathscr{A} \subset \mathscr{R}^I$ is *weakly Paretian* if for any profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ the choice $f(\succsim_1, \ldots, \succsim_I) \in X$ is a weak Pareto optimum. That is, if for some pair $\{x, y\} \in X$ we have that $x \succ_i y$ for every i, then $y \neq f(\succsim_1, \ldots, \succsim_I)$.

Definition 21.E.3. The alternative $x \in X$ *maintains its position from* the profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I$ to the profile $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{R}^I$ if

$$x \succsim_i y \quad \text{implies} \quad x \succsim'_i y$$

for every i and every $y \in X$.

Definition 21.E.4. The social choice function $f : \mathscr{A} \to X$ defined on $\mathscr{A} \subset \mathscr{R}^I$ is *monotonic* of for any two profiles $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{A}$ with the property that the chosen alternatives $x = f(\succsim_1, \ldots, \succsim_I)$ maintains its position from $(\succsim_1, \ldots, \succsim_I)$ to $(\succsim'_1, \ldots, \succsim'_I)$, we have that $f(\succsim'_1, \ldots, \succsim'_I) = x$.

Definition 21.E.5. An agent $h \in I$ is a *dictator* for the social choice function $f : \mathscr{A} \to X$ if, for every profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, $f(\succsim_1, \ldots, \succsim_I)$ is a most preferred alternative for $\succsim_h$ in X; that is,

$$f(\succsim_1, \ldots, \succsim_I) = \{x \in X : x \succsim_h y \text{ for every } y \in X\}.$$

A social choice function that admits a dictator is called *dictatorial*.

Proposition 21.E.1. Suppose that the number of alternatives is at least three and that the domain of admissible preference profiles is either $\mathscr{A} = \mathscr{R}^I$ or $\mathscr{A} = \mathscr{P}^I$. Then every weakly Paretian and monotonic social choice function $f : \mathscr{A} \to X$ is dictatorial.

Definition 21.E.6. Given a finite subset $X' \subset X$ and a profile $(\succsim_1, \dots, \succsim_I) \in \mathscr{R}^I$, we say that the profile $(\succsim_1', \dots, \succsim_I')$ *takes* X' *to the top from* $(\succsim_1, \dots, \succsim_I)$ if, for every i,

$$\begin{array}{ll} x \succ_i y & \text{for } x \in X' \text{ and } y \notin X', \\ x \succsim_i y \iff x \succsim_i' y & \text{for all } x, y \in X'. \end{array}$$

Proposition 21.E.2. Suppose that the number of alternatives is at least three and that $f : \mathscr{P}^I \to X$ is a social choice function that is weakly Paretian and satisfies the following *no-incentive-to-misrepresent* condition:

$$f(\succsim_1, \dots, \succsim_{h-1}, \succsim_h, \succsim_{h+1}, \cdots \succsim_I) \succsim_h f(\succsim_1, \dots, \succsim_{h-1}, \succsim_h', \succsim_{h+1}, \cdots \succsim_I)$$

for every agent h, every $\succsim_h \in \mathscr{P}$, and every profile $(\succsim_1, \dots, \succsim_I) \in \mathscr{P}^I$. Then $f(\cdot)$ is dictatorial.

Chapter 22

Elements of Welfare Economics and Axiomatic Bargaining

22.B Utility Possibility Sets

Definition 22.B.1. The *utility possibility set* (UPS) is the set

$$U = \{(u_1, \dots, u_I) \in \mathbb{R}^I : u_1 \leq u_I(x), \dots, u_I \leq u_I(x) \text{ for some } x \in X\} \subset \mathbb{R}^L.$$

The *Pareto frontier* of U is formed by the utility vectors $u = (u_1, \dots, u_I) \in U$ for which there is no other $u' = (u'_1, \dots, u'_I) \in U$ with $u'_i \geq u_i$ for every i and $u'_i > u_i$ for some i.

22.D Invariance Properties of Social Welfare Functions

Definition 22.D.1. Given a set X of alternatives, a *social welfare functional* $F : \mathscr{U}^I \to \mathscr{R}$ is a rule that assigns a rational preference relation $F(\tilde{u}_1, \dots, \tilde{u}_I)$ among the alternatives in the domain X to every possible profile of individual utility functions $(\tilde{u}_1(\cdot), \dots, \tilde{u}_I(\cdot))$ defined on X. The strict preference relation derived from $F(\tilde{u}_1, \dots, \tilde{u}_I)$ is denoted $F_p(\tilde{u}_1, \dots, \tilde{u}_I)$.

Definition 22.D.2. The social welfare functional $F : \mathscr{U}^I \to \mathscr{R}$ satisfies the (weak) *Pareto property*, or is *Paretian*, if, for any profile $(\tilde{u}_1, \dots, \tilde{u}_I) \in \mathscr{U}^I$ and any pair $x, y \in X$, we have that $\tilde{u}_i(x) \geq \tilde{u}_i(y)$ for all i implies $xF(\tilde{u}_1, \dots, \tilde{u}_I)y$, and also that $\tilde{u}_i(x) > \tilde{u}_i(y)$ for all i implies $xF_p(\tilde{u}_1, \dots, \tilde{u}_I)y$.

Definition 22.D.3. The social welfare functional $F : \mathscr{U}^I \to \mathscr{R}$ satisfies the *pairwise independence condition* if, whenever $x, y \in X$ are two alternatives and $(\tilde{u}_1, \dots, \tilde{u}_I) \in \mathscr{U}^I, (\tilde{u}'_1, \dots, \tilde{u}'_I) \in \mathscr{U}^I$ are two utility function profiles with $\tilde{u}_i(x) = \tilde{u}'_i(x)$ and $\tilde{u}_i(y) = \tilde{u}'_i(y)$ for all i, we have

$$xF(\tilde{u}_1, \dots, \tilde{u}_I)y \iff xF(\tilde{u}'_1, \dots, \tilde{u}'_I)y.$$

Proposition 22.D.1. Suppose that there are at least three alternatives in X and that the Paretian social welfare functional $F : \mathscr{U}^I \to \mathscr{R}$ satisfies the pairwise independence condition. Then there is a rational preference relation $\succsim$ defined on $\mathbb{R}^I$ [that is, on profiles $(u_1, \dots, u_I) \in \mathbb{R}^I$ of individual utility values] that generates $F(\cdot)$. In other words, for every profile of utility functions $(\tilde{u}_1, \dots, \tilde{u}_I) \in \mathscr{U}^I$ and for every pair of alternatives $x, y \in X$ we have

$$xF(\tilde{u}_1, \dots, \tilde{u}_I)y \iff (\tilde{u}_1(x), \dots, \tilde{u}_I(x)) \succsim (\tilde{u}_1(y), \dots, \tilde{u}_I(y)).$$

Definition 22.D.4. We say that the social welfare functional $F : \mathscr{U}^I \to \mathscr{R}$ is *invariant to common cardinal transformations* if $F(\tilde{u}_1, \ldots, \tilde{u}_I) = F(\tilde{u}'_1, \ldots, \tilde{u}'_I)$ whenever the profiles of utility functions $(\tilde{u}_1, \ldots, \tilde{u}_I)$ and $(\tilde{u}'_1, \ldots, \tilde{u}'_I)$ differ only by a common change of origin and units, that is, whenever there are numbers $\beta > 0$ and α such that $\tilde{u}_i(x) = \beta \tilde{u}'_i(x) + \alpha$ for all i and $x \in X$. If the invariance is only with respect to common changes of origin (i.e., we require $\beta = 1$) or of units (i.e. we require $\alpha = 0$), then we say that $F(\cdot)$ is *invariant to common changes of origin* or *of units*, respectively.

Proposition 22.D.2. Suppose that the social welfare functional $F : \mathscr{U}^I \to \mathscr{R}$ is generated from a continuous and increasing social welfare function. Suppose also that $F(\cdot)$ is invariant to common changes of origin. Then the social welfare functional can be generated from a social welfare function of the form

$$W(u_1, \ldots, u_I) = \bar{u} - g(u_1 - \bar{u}, \ldots, u_I - \bar{u}),$$

where $\bar{u} = (1/I) \sum_i u_i$.

Moreover, if $F(\cdot)$ is also independent of common changes of units, that is, fully invariant to common cardinal transformations, then $g(\cdot)$ is homogeneous of degree one on its domain: $\{s \in \mathbb{R}^I : \sum_i s_i = 0\}$.

Definition 22.D.5. The social welfare functional $F : \mathscr{U}^I \to \mathscr{R}$ *does not allow interpersonal comparison of utility* if $F(\tilde{u}_1, \ldots, \tilde{u}_I) = F(\tilde{u}'_1, \ldots, \tilde{u}'_I)$ whenever there are numbers β_i and α_i such that $\tilde{u}_i(x) = \beta_i \tilde{u}'_i(x) + \alpha_i$ for all i and x. If the invariance is only with respect to independent changes of origin (i.e., we require $\beta_i = 1$ for all i), or only with respect to independent changes of units (i.e., we require that $\alpha_i = 0$ for all i), then we say that $F(\cdot)$ is *invariant to changes of origins* or *of units*, respectively.

Proposition 22.D.3. Suppose that the social welfare functional $F : \mathscr{U}^I \to \mathscr{R}$ can be generated from an increasing, continuous social welfare function. If $F(\cdot)$ is invariant to independent changes of origins, then $F(\cdot)$ can be generated from a social welfare function $W(\cdot)$ of the purely utilitarian (but possibly nonsymmetric) form. That is, there are constants $b_i \geq 0$, not all zero, such that

$$W(u_1, \ldots, u_I) = \sum_i b_i u_i \quad \text{for all } i.$$

Moreover, if $F(\cdot)$ is also invariant to independent changes of units [i.e., if $F(\cdot)$ does not allow for interpersonal comparisons of utility], then F is dictatorial: There is an agent h such that, for every pair $x, y \in X$, $\tilde{u}_h(x) > \tilde{u}_h(y)$ implies $x F_p(\tilde{u}_1, \ldots, \tilde{u}_I) y$.

22.E The Axiomatic Bargaining Approach

Definition 22.E.1. A *bargaining solution* is a rule that assigns a solution vector $f(U, u^*) \in U$ to every bargaining problem (U, u^*).

Definition 22.E.2. The bargaining solution $f(\cdot)$ is *independent of utility origins* (IUO), or *invariant to independent changes of origins*, if for any $\alpha = (\alpha_1, \ldots, \alpha_I) \in \mathbb{R}^I$ we have

$$f_i(U', u^* + \alpha) = f_i(U, u^*) + \alpha_i \quad \text{for every } i$$

whenever $U' = \{(u_1 + \alpha_1, \ldots, u_I + \alpha_I) : u \in U\}$.

Definition 22.E.3. The bargaining solution $f(\cdot)$ is *independent of utility units* (IUU), of *invariant to independence changes of units*, if for any $\beta = (\beta_1, \ldots, \beta_I) \in \mathbb{R}^I$ with $\beta_i > 0$ for all i, we have

$$f_i(U') = \beta_i f_i(U) \quad \text{for every } i$$

whenever $U' = \{(\beta_1 u_1, \ldots, \beta_I u_i) : u \in U\}$.

Definition 22.E.4. The bargaining solution $f(\cdot)$ satisfies the *Pareto* property (P), or is *Paretian*, if, for every U, $f(U)$ is a (weak) Pareto optimum, that is, there is no $u \in U$ such that $u_i > f_i(U)$ for every i.

Definition 22.E.5. The bargaining solution $f(\cdot)$ satisfies the property of *symmetry* (S) if whenever $U \subset \mathbb{R}^I$ is a symmetric set (i.e. U remains unaltered under permutations of the axes), we have that all entries of $f(U)$ are equal.

Definition 22.E.6. The bargaining solution $f(\cdot)$ satisfies the property of *individual rationality* (IR) if $f(U) \geq 0$.

Definition 22.E.7. The bargaining solution satisfies the property of *independence of irrelevant alternatives* (IIA) if, whenever $U' \subset U$ and $f(U) \in U'$, it follows that $f(U') = f(U)$.

Proposition 22.E.1. The Nash solution is the only bargaining solution that is independent of utility origins and units, Paretian, symmetric, and independent of irrelevant alternatives.

22.F Coalition Bargaining: The Shapley Value

Definition 22.F.1. Given the set of agents I, a *cooperative solution* $f(\cdot)$ is a rule that assigns to every game $v(\cdot)$ in characteristic form a utility allocation $f(v) \in \mathbb{R}^I$ that is feasible for the entire group, that is, such that $\sum_i f_i(v) \leq v(I)$.

Definition 22.F.2. The cooperative solution $f(\cdot)$ is *independent of utility origins and common changes of utility units* if, whenever we have two characteristic forms $v(\cdot)$ and $v'(\cdot)$ such that $v(S) = \beta v'(S) + \sum_{i \in S} \alpha_i$ for every $S \subset I$ and some numbers $\alpha_1, \ldots, \alpha_I$, and $\beta > 0$, it follows that $f(v) = \beta f(v') + (\alpha_1, \ldots, \alpha_I)$.

Definition 22.F.3. The cooperative solution $f(\cdot)$ is *Paretian* if $\sum_i f_i(v) = v(I)$, for every characteristic form $v(\cdot)$.

Definition 22.F.4. The cooperative solution $f(\cdot)$ is *symmetric* if the following property holds: Suppose that two characteristic forms, $v(\cdot)$ and $v'(\cdot)$ differ only by a permutation $\pi : I \to I$ of the names of the agents; that is, $v'(S) = v(\pi(S))$ for all $S \subset I$. Then the solution also differs only by this permutation; that is, $f_i(v') = f_{\pi(i)}(v)$ for all i.

Definition 22.F.5. The cooperative solution $f(\cdot)$ satisfies the *dummy axiom* if, for all games $v(\cdot)$ and all agents i such that $v(S \cup i) = v(S)$ for all $S \subset I$, we have $f_i(v) = v(i)(= 0)$. In words: If agent i is a *dummy* (i.e. does not contribute anything to any coalition), then agent i does not receive any share of the surplus.

Definition 22.F.6. The Shapley value solution $f_s(\cdot)$ is defined by

$$f_{si}(v) = \frac{1}{I!} \sum_{\pi} g_{v,\pi}(i) \quad \text{for every } i.$$

Chapter 23

Incentives and Mechanism Design

23.B The Mechanism Design Problem

Definition 23.B.1. A *social choice function* is a function $f : \Theta_1 \times \cdots \times \Theta_I \to X$ that, for each possible profile of agents' types $(\theta_1, \ldots, \theta_I)$, assigns a collective choice $f(\theta_1, \ldots, \theta_I) \in X$.

Definition 23.B.2. The social choice function $f : \Theta_1 \times \cdots \times \Theta_I \to X$ is *ex post efficient* (or *Paretian*) if for no profile $\theta = (\theta_1, \ldots, \theta_I)$ is there an $x \in X$ such that $u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i)$ for every i, and $u_i(x, \theta_i) > u_i(f(\theta), \theta_i)$ for some i.

Definition 23.B.3. A *mechanism* $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ is a collection of I strategy sets $(S_1, \ldots, S_I)$ and an outcome function $g : S_1 \times \cdots \times S_I \to X$.

Definition 23.B.4. The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ *implements* social choice function $f(\cdot)$ if there is an equilibrium strategy profile $(s_1^*(\cdot), \ldots, s_I^*(\cdot))$ of the game induced by Γ such that $g(s_1^*(\theta_1), \ldots, s_I^*(\theta_I)) = f(\theta_1, \ldots, \theta_I)$ for all $(\theta_1, \ldots, \theta_I) \in \Theta_1 \times \cdots \times \Theta_I$.

Definition 23.B.5. A *direct revelation mechanism* is a mechanism in which $S_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$ for all $\theta \in \Theta_1 \times \cdots \times \Theta_I$.

Definition 23.B.6. The social choice function $f(\cdot)$ is *truthfully implementable* (or *incentive compatible*) if the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$ has an equilibrium $(s_1^*(\cdot), \ldots, s_I^*(\cdot))$ in which $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and all $i = 1, \ldots, I$; that is, if truth telling by each agent i constitutes an equilibrium of $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$.

23.C Dominant Strategy Implementation

Definition 23.C.1. The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_I^*(\cdot))$ is a *dominant strategy equilibrium* of mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ if, for all i and all $\theta_i \in \Theta_i$,

$$u_i\left(g\left(s_i^*(\theta_i), s_{-i}\right), \theta_i\right) \geq u_i\left(g\left(s_i', s_{-i}\right), \theta_i\right)$$

for all $s_i' \in S_i$ and all $s_{-i} \in S_{-i}$.

Definition 23.C.2. The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ *implements the social choice function $f(\cdot)$ in dominant strategies* if there exists a dominant strategy equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_I^*(\cdot))$, such that $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

Definition 23.C.3. The social choice function $f(\cdot)$ is *truthfully implementable in dominant strategies* (or *dominant strategy compatible*, or *strategy-proof*, or *straightforward*) if $s_i^*(\theta_i) = \theta_i$ for every $\theta_i \in \Theta_i$ and $i = 1, \ldots, I$ is a dominant strategy equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$. That is, if for all i and $\theta_i \in \Theta_i$,

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)$$

for all $\hat{\theta}_i \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.

Proposition 23.C.1 (The Revelation Principle for Dominant Strategies)**.** Suppose that there exists a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ that implements the social choice function $f(\cdot)$ in dominant strategies. Then $f(\cdot)$ is truthfully implementable in dominant strategies.

Proposition 23.C.2. The social choice function $f(\cdot)$ is truthfully implementable in dominant strategies if and only if for all i, all $\theta_{-i} \in \Theta_{-i}$, and all pairs of types for agent i, θ_i' and $\theta_i'' \in \Theta_i$, we have

$$f(\theta_i'', \theta_{-i}) \in L_i(f(\theta_i', \theta_{-i}), \theta_i') \quad \text{and} \quad f(\theta_i', \theta_{-i}) \in L_i(f(\theta_i'', \theta_{-i}), \theta_i'').$$

Definition 23.C.4. The social choice function $f(\cdot)$ is *dictatorial* if there is an agent i such that, for all $\theta = (\theta_1, \ldots, \theta_I) \in \Theta$,

$$f(\theta) \in \{x \in X : u_i(x, \theta_i) \geq u_i(y, \theta_i) \text{ for all } y \in X\}.$$

Definition 23.C.5. The social choice function $f(\cdot)$ is *monotonic* if, for any θ, if θ' is such that $L_i(f(\theta), \theta_i) \subset L_i(f(\theta), \theta_i')$ for all i [i.e., if $L_i(f(\theta), \theta_i)$ is weakly included in $L_i(f(\theta), \theta_i')$ for all i], then $f(\theta') = f(\theta)$.

Proposition 23.C.3 (The Gibbard Satterthwaite Theorem)**.** Suppose that X is finite and contains at least three elements, that $\mathscr{R}_i = \mathscr{P}$ for all i, and that $f(\Theta) = X$. Then the social choice function $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial.

Corollary 23.C.1. Suppose that X is finite and contains at least three elements, that $\mathscr{P} \subset \mathscr{R}_i$ for all i, and that $f(\Theta) = X$. Then the social choice function $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial.

Definition 23.C.6. The social choice function $f(\cdot)$ is *dictatorial on set* $\hat{X} \subset X$ if there exists an agent i such that, for all $\theta = (\theta_1, \ldots, \theta_I) \in \Theta$, $f(\theta) \in \{x \in \hat{X} : u_i(x, \theta_i) \geq u_i(y, \theta_i) \text{ for all } y \in \hat{X}\}$.

Corollary 23.C.2. Suppose that X is finite, that the number of elements in $f(\Theta)$ is at least three, and that $\mathscr{P} \subset \mathscr{R}_i$ for all $i = 1, \ldots, I$. Then $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial on the set $f(\Theta)$.

Proposition 23.C.4. Let $k^*(\cdot)$ be a function satisfying

$$\sum_{i=1}^{I} v_i(k(\theta), \theta_i) \geq \sum_{i=1}^{I} v_i(k, \theta_i) \quad \text{for all } k \in K. \tag{23.C.7}$$

The social choice function $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ is truthfully implementable in dominant strategies if, for all $i = 1, \ldots, I$,

$$t_i(\theta) = \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] + h_i(\theta_{-i}) \tag{23.C.8}$$

where $h_i(\cdot)$ is an arbitrary function of θ_{-i}.

Proposition 23.C.5. Suppose that for each agent $i = 1, \dots, I$, $\{v_i(\cdot, \theta_i) : \theta_i \in \Theta_i\} = \mathscr{V}$; that is, every possible valuation function from K to $\mathbb{R}$ arises for some $\theta_i \in \Theta_i$. Then a social choice function $f(\cdot) = (k^*(\cdot), t_1(\cdot), \dots, t_I(\cdot))$ in which $k^*(\cdot)$ satisfies 23.C.7 is truthfully implementable in dominant strategies only if $t_i(\cdot)$ satisfies 23.C.8 for all $i = 1, \dots, I$.

Proposition 23.C.6. Suppose that for each agent $i = 1, \dots, I$, $\{v_i(\cdot, \theta_i) : \theta_i \in \Theta_i\} = \mathscr{V}$; that is, every possible valuation function from K to $\mathbb{R}$ arises for some $\theta_i \in \Theta_i$. Then there is no social choice function $f(\cdot) = (k^*(\cdot), t_1(\cdot), \dots, t_I(\cdot))$ that is truthfully implementable in dominant strategies and is ex post efficient, that is, that satisfies 23.C.7 and

$$\sum_i t_i(\theta) = 0 \quad \text{for all } \theta \in \Theta.$$

23.D Bayesian Implementation

Definition 23.D.1. The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \dots, s_I^*(\cdot))$ is a *Bayesian Nash equilibrium* of mechanism $\Gamma = (S_1, \dots, S_I, g(\cdot))$ if, for all i and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(g(\hat{s}_i(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i]$$

for all $\hat{s}_i \in S_i$.

Definition 23.D.2. The mechanism $\Gamma = (S_1, \dots, S_I, g(\cdot))$ *implements the social choice function $f(\cdot)$ in Bayesian Nash equilibrium* if there is a Bayesian Nash equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), \dots, s_I^*(\cdot))$, such that $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

Definition 23.D.3. The social choice function $f(\cdot)$ is *truthfully implementable in Bayesian Nash equilibrium* (or *Bayesian incentive compatible*) if $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i = 1, \dots, I$ is a Bayesian Nash equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \dots, \Theta_I, f(\cdot))$. That is, if for all $i = 1, \dots, I$ and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i]$$

for all $\hat{\theta}_i \in \Theta_i$.

Proposition 23.D.1 (The Revelation Principle for Bayesian Nash Equilibrium)**.** Suppose that there exists a mechanism $\Gamma = (S_1, \dots, S_I, g(\cdot))$ that implements the social choice function $f(\cdot)$ in Bayesian Nash equilibrium. Then $f(\cdot)$ is truthfully implementable in Bayesian Nash equilibrium.

Proposition 23.D.2. The social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \dots, t_I(\cdot))$ is Bayesian incentive compatible if and only if, for all $i = 1, \dots, I$,

(i) $\bar{v}_i(\cdot)$ is nondecreasing.

(ii) $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s)ds$ for all θ_i.

Proposition 23.D.3 (The Revenue Equivalence Theorem)**.** Consider an auction setting with I risk-neutral buyers, in which buyer i's valuation is drawn from an interval $[\underline{\theta}_i, \theta_i]$ with $\underline{\theta}_i \neq \theta_i$ and a strictly positive density $\phi(\cdot) > 0$, and in which buyers' types are statistically independent. Suppose that a given pair of Bayesian Nash equilibria of two different auction procedures are such that for every buyer i: (i) For each possible realisation of $(\theta_i, \dots, \theta_I)$, buyer i has an identical probability of getting the good in the two auctions; and (ii) Buyer i has the same expected utility level on the two auctions when his valuation for the object is at its lowest possible level. Then these equilibria of the two auction generate the same expected revenue for the seller.

23.E Participation Constraints

Proposition 23.E.1 (The Myerson Satterthwaite Theorem)**.** Consider a bilateral trade setting in which the buyer and seller are risk neutral, the valuations θ_1 and θ_2 are independently drawn from the intervals $[\underline{\theta}_1, \bar{\theta}_1] \subset \mathbb{R}$ and $[\underline{\theta}_2, \bar{\theta}_2] \subset \mathbb{R}$ with strictly positive densities, and $(\underline{\theta}_1, \bar{\theta}_1) \cap (\underline{\theta}_2, \bar{\theta}_2) \neq \emptyset$. Then there is no Bayesian incentive compatible social choice function that is ex post efficient and gives every buyer type and every seller type nonnegative expected gains from participation.

23.F Optimal Bayesian Mechanisms

Definition 23.F.1. Given any set of feasible social choice functions F, the social choice function $f(\cdot) \in F$ is *ex ante efficient in* F if there is no $\hat{f}(\cdot) \in F$ having the property that $U_i(\hat{f}) \geq U_i(f)$ for all $i = 1, \dots, I$, and $U_i(\hat{f}) > U_i(f)$ for some i.

Definition 23.F.2. Given any set of feasible social choice functions F, the social choice function $f(\cdot) \in F$ is *interim efficient in* F if there is no $\hat{f}(\cdot) \in F$ having the property that $U_i(\theta_i|\hat{f}) \geq U_i(\theta_i|f)$ for all $\theta_i \in \Theta_i$ and all $i = 1, \dots, I$, and $U_i(\theta_i|\hat{f}) > U_i(\theta_i|f)$ for some i and $\theta_i \in \Theta_i$.

Proposition 23.F.1. Given any set of feasible social functions F, if the social choice function $f(\cdot) \in F$ is ex ante efficient in F, then it is also interim efficient in F.

Definition 23.F.3. Given any set of feasible social choice functions F, the social choice function $f(\cdot) \in F$ is *ex post efficient in* F if there is no $\hat{f}(\cdot) \in F$ having the property that $u_i(\hat{f}(\theta), \theta_i) \geq u_i(f(\theta), \theta_i)$ for all $i = 1, \dots, I$ and all $\theta \in \Theta$, and $u_i(\hat{f}(\theta), \theta_i) > u_i(f(\theta), \theta_i)$ for some i and $\theta \in \Theta$.

Bibliography

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